Άσκησεις

Αλγόριθμοι & Πολυπλοκότητα

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Ασκηση 2

$\cdot \log^*(n) = i$ ①

$f(n) = \log(\log^*(n)) \overset{①}{=} \log(i)$ $\quad \vartheta(i) > \vartheta(\log(i))$

$g(n) = \log^*(\log(n)) = i+1$

Αρα $O(g(n)) > O(f(n))$

Ασκηση 3

a) $T(n) = 4\,T(n/8) + \Omega(n^{3/4})$ $\quad a=4$

θετω $T'(n) = 4T(n/8) + n^{3/4}\,(n^{9/12})$ $\quad b=8$

$n^{\log_b a} = n^{\log_8 4} = n^{\frac{2}{3}} = n^{8/12}$

Case 3

$f(n) = \Omega\left(n^{\frac{2}{3}+\varepsilon}\right), \quad \varepsilon \in (0, 1/12)$

$a f\left(\frac{n}{b}\right) \leq c f(n) \Leftrightarrow 4\left(\frac{n}{8}\right)^{\frac{3}{4}} \leq c\, n^{\frac{3}{4}} \Leftrightarrow 4\left(\frac{1}{8}\right)^{\frac{3}{4}} \cancel{n^{\frac{3}{4}}} \leq c\, \cancel{n^{\frac{3}{4}}} \Leftrightarrow$

$\Leftrightarrow 0{,}8408 \leq c$ με $c<1 \rightarrow$ ισχύει

Αρα $T(n) = \vartheta(f(n)) = \vartheta(\Omega(n^{\frac{3}{4}})) = \Omega(n^{\frac{3}{4}})$

$a = 1$
$b = \frac{5}{3}$

β) $T(n) = T(\lfloor 3n/5 \rfloor) + \theta(n^{1/(1-\ln 2)})$

έστω $T'(n) = T(3n/5) + n^{1/(1-\ln 2)} = T(3n/5) + n^{3,258}$

$n^{\log_b a} = n^{\log_{\frac{5}{3}} 1} = n^0 = 1$

$f(n) = O(n^{0+\varepsilon})$, $\varepsilon = 1/(1-\ln 2) \simeq 3{,}258$

Άρα $T(n) = \theta(f(n)) = \theta(\theta(n^{1/(1-\ln 2)})) = \theta(n^{1/(1-\ln 2)})$

$af(\frac{n}{b}) \leq cf(n) \Leftrightarrow \left(\frac{3n}{5}\right)^{\frac{1}{1-\ln 2}} \leq c\, n^{1/(1-\ln 2)} \Leftrightarrow$

$\Leftrightarrow \left(\frac{3}{5}\right)^{\frac{1}{1-\ln 2}} \cancel{n^{\frac{1}{1-\ln 2}}} \leq c \cdot \cancel{n^{1/1-\ln 2}} \Leftrightarrow$

$\Leftrightarrow 0{,}1892 \leq c$ με $c < 1 \rightarrow$ ισχύει

γ) $T(n) = 8T(n/16) + O(n^{3/4})$

$T'(n) = 8T(n/16) + n^{\frac{3}{4}}$

$n^{\log_b a} = n^{\log_{16} 8} = n^{\frac{3}{4}}$

Case 2

$f(n) = \theta(n^{\frac{3}{4}} \log^0(n))$, $k = 0$ $(k \geq 0)$

Άρα $T(n) = O(n^{\frac{3}{4}} \log n)$

δ) $T(n) = 27T(\lceil n/3 \rceil) + \theta(n^3 \log^2 n)$

$T'(n) = 27T(n/3) + n^3 \log^2 n$

$n^{\log_b a} = n^{\log_3 27} = n^3$

Case 2

$f(n) = \theta(n^3 \log^2 n)$, $k = 2$

Άρα $T(n) = \theta(n^3 \log^3 n)$

ε) $T(n) = 3T(n/2) + \Theta(n\log n)$

$$T'(n) = 3T(n/2) + n\log n$$

$$n^{\log_b a} = n^{\log_2 3} = n^{1{,}584}$$

$$\log n^k = O(n^k)$$

Έστω $k=\frac{1}{2}$ $\log n^{\frac{1}{2}} = O(n^{\frac{1}{2}}) \Leftrightarrow \frac{1}{2}\log n = O(n^{\frac{1}{2}}) \Leftrightarrow$

$\Leftrightarrow \frac{1}{2} n \log n = O(n^{3/2}) \Leftrightarrow n\log n = O(n^{3/2}) = O(n^{1{,}58-\varepsilon})$

$, \varepsilon = \cancel{[illegible]} \frac{3}{2} - 1{,}58$

case 1

Άρα $T(n) = \Theta(n^{1{,}584})$

στ) $T(n) = 4T(n/2) + \Theta(n^2\sqrt{n})$

$T'(n) = 4T(n/2) + n^2\sqrt{n} = 4T(n/2) + \sqrt{n^4 \cdot n} = 4T(n/2) + n^{\frac{5}{2}}$ $\quad n^{2{,}5}$

$n^{\log_b a} = n^{\log_2 4} = n^2$

case 3

$f(n) = n^{2{,}5} = \Omega(n^{2+\varepsilon})$, $\varepsilon = 0{,}5 > 0$

$af(\frac{n}{b}) \leq c f(n) \Leftrightarrow 4 \cdot (\frac{n}{2})^{\frac{5}{2}} \leq c \cdot n^{\frac{5}{2}} \Leftrightarrow$

$\Leftrightarrow 4 \cdot (\frac{1}{2})^{\frac{5}{2}} \cdot \cancel{n^{\frac{5}{2}}} \leq c\, \cancel{n^{\frac{5}{2}}} \Leftrightarrow 0{,}707 \leq c$ με $c < 1$

ισχύει

Άρα $T(n) = \Theta(n^{\frac{5}{2}})$

ζ) $T(n) = 2T(n/2) + T(n/3) + \Theta(n^2)$

$T(n) = 2T(n/2) + T(n/3) + \theta(n^2)$

cn^2

(βαθύ) μεγαλύτερη ρίζα

(ρηχό) μικρότερη ρίζα

$c\left(\frac{n}{2}\right)^2 \quad c\left(\frac{n}{2}\right)^2 \quad c\left(\frac{n}{3}\right)^2 \rightarrow cn^2\left(\frac{11}{18}\right)$

$c\left(\frac{n}{4}\right)^2 \quad c\left(\frac{n}{4}\right)^2 \quad c\left(\frac{n}{6}\right)^2 \quad c\left(\frac{n}{6}\right)^2 \quad c\left(\frac{n}{6}\right)^2 \quad c\left(\frac{n}{9}\right)^2 \rightarrow cn^2\left(\frac{11}{18}\right)^2$

στην i ρίζα θα έχουμε $cn^2\left(\frac{11}{18}\right)^i$

$\left(\frac{1}{2}\right)^1 n^2 \qquad \left(\frac{1}{3}\right)^1 n^2$

$\left(\frac{1}{2}\right)^2 n^2 \qquad \left(\frac{1}{3}\right)^2 n^2$

$\left(\frac{1}{2}\right)^3 n^2 \qquad \left(\frac{1}{3}\right)^3 n^2$

$\vdots \qquad \vdots$

$\left(\frac{1}{2}\right)^i n^2$

$\vdots \qquad \left(\frac{1}{3}\right)^{h_{min}} n^2$

$\left(\frac{1}{2}\right)^{h_{max}} n^2$

$\left(\frac{1}{2}\right)^{h_{max}} n^2 = 1 \Leftrightarrow \boxed{h_{max} = \log_2 n^2}$

$\left(\frac{1}{4}\right)^{h_{min}} n^2 = 1 \Leftrightarrow \boxed{h_{min} = \log_4 n^2}$

$$\sum_{i=0}^{h_{min}} \left(\frac{11}{18}\right)^i cn^2 \leq T(n) \leq \sum_{i=0}^{h_{max}} \left(\frac{11}{18}\right)^i cn^2$$

$$\sum_{i=0}^{\log_2 n^2} \left(\frac{11}{18}\right)^i cn^2 = cn^2 \sum_{i=0}^{\log_2 n^2} \left(\frac{11}{18}\right)^i = cn^2 \frac{\left(\frac{11}{18}\right)^{\log_2 n^2 + 1} - 1}{\frac{11}{18} - 1} \; \left(= -\frac{7}{18}\right) =$$

$$= -\frac{18}{7} cn^2 \left(\left(\frac{11}{18}\right)^{\log_2 n^2} \frac{11}{18} - 1\right) = -\frac{18}{7} cn^2 \left(n^{2\log_2\left(\frac{11}{18}\right) = -0,71} \frac{11}{18} - 1\right) =$$

$$= -\frac{11}{7} c n^{0,58} + \frac{18}{7} cn^2 = O(n^2)$$

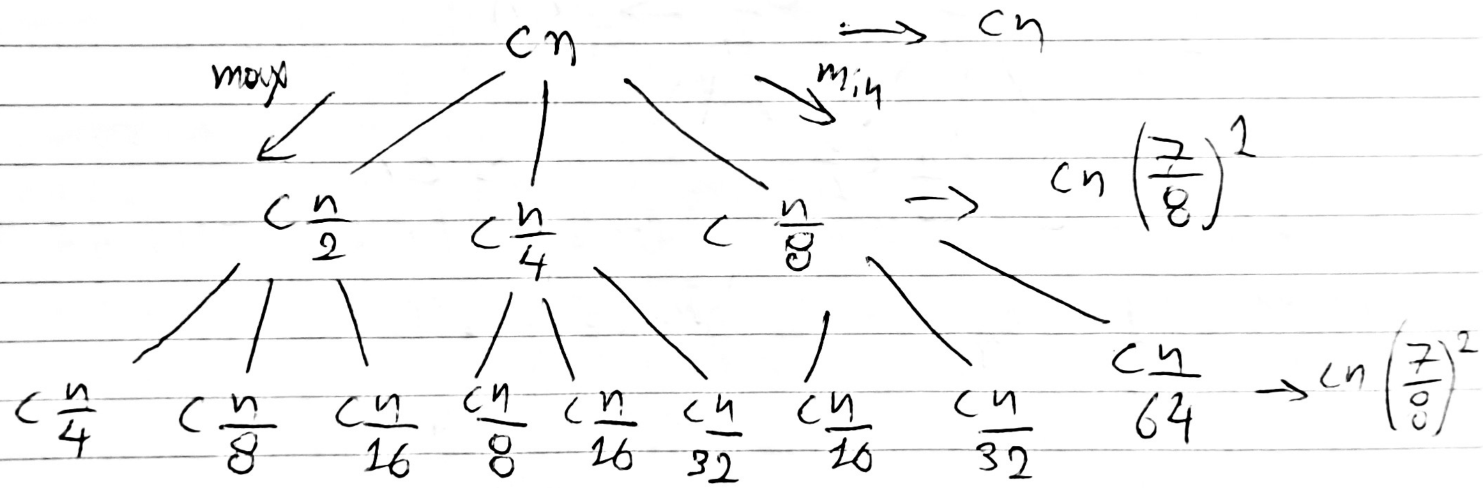
• $\sum_{i=0}^{\log_4 n} \left(\frac{11}{18}\right)^i cn^2 = -\frac{18}{7} cn^2 \left(n^{\log_4\left(\frac{11}{18}\right)} \frac{11}{18} - 1\right) =$

$= -\frac{11}{7} c n^{1,28688} + \frac{18}{7} cn^2 = \underline{O}(n^2)$

$\underline{O}(n^2) \leq T(n) \leq O(n^2)$

Άρα $T(n) = \Theta(n^2)$

n) $T(n) = T(n/2) + T(n/4) + T(n/8) + \Theta(n)$



$cn\left(\frac{7}{8}\right)^i$

$\left(\frac{1}{2}\right)^{h_{max}} n \qquad \left(\frac{1}{8}\right)^{h_{min}} n$

$h_{max} = \log_2 n \qquad h_{min} = \log_8 n$

$\sum_{i=0}^{h_{min}} \left(\frac{7}{8}\right)^i cn \leq T(n) \leq \sum_{i=0}^{h_{max}} \left(\frac{7}{8}\right)^i cn$

• $\sum_{i=0}^{h_{max}} \left(\frac{7}{8}\right)^i cn = cn \sum_{i=0}^{\log_2 n} \left(\frac{7}{8}\right)^i = cn \dfrac{\left(\frac{7}{8}\right)^{\log_2 n + 1} - 1}{\frac{7}{8} - 1 \ \left(= -\frac{1}{8}\right)} =$

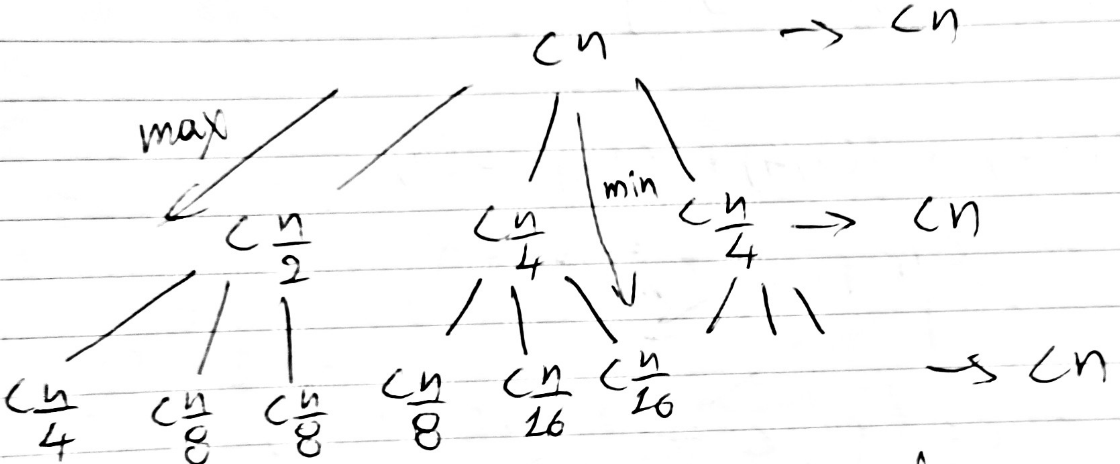
$= -8cn\left(\left(\frac{7}{8}\right)^{\log_2 n} \frac{7}{8} - 1\right) = -8cn\left(n^{\overbrace{\log_2\left(\frac{7}{8}\right)}^{=-0,192}} \frac{7}{8} - 1\right) =$

$= -\cancel{8}7 c \cdot n^{0,807} + 8cn = O(n)$

• $\sum_{i=0}^{h_{min}} \left(\frac{7}{8}\right)^i cn = -8cn\left(n^{\log_8\left(\frac{7}{8}\right)} \frac{7}{8} - 1\right) = -7cn^{0,9358} + 8cn = \Omega(n)$

Άρα $T(n) = \Theta(n)$

θ) $T(n) = T(n/2) + 2T(n/4) + \Theta(n)$



$\left(\frac{1}{2}\right)^{h_{max}} n \qquad \left(\frac{1}{4}\right)^{h_{min}} n \qquad h_{max} = \log_2 n$

$h_{min} = \log_4 n$

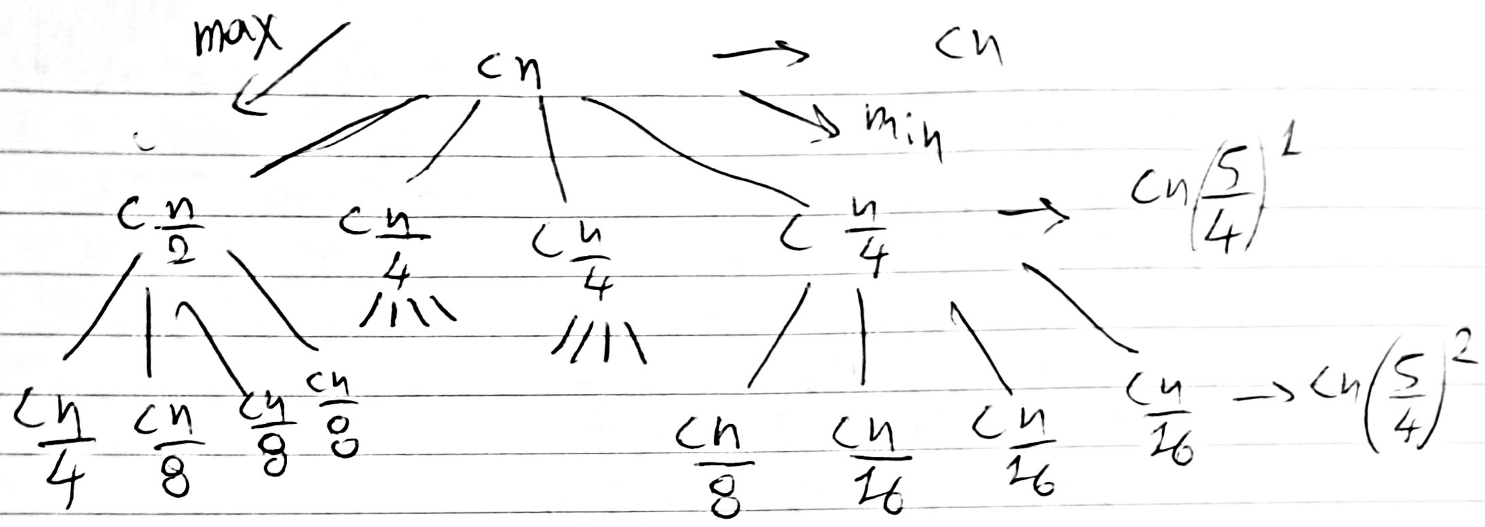
$cn \sum_{i=0}^{\log_4 n} 1 \le T(n) \le cn \sum_{i=0}^{\log_2 n} 1$

$cn(\log_4 n + 1) \le T(n) \le cn(\log_2 n + 1)$

$\Leftrightarrow \Omega(n\log n) \le T(n) \le O(n\log n)$

Άρα $T(n) = \Theta(n\log n)$

ι) $T(n) = T(n/2) + 3T(n/4) + \Theta(n)$



$\left(\frac{1}{2}\right)^{h_{max}} n \qquad \left(\frac{1}{4}\right)^{h_{min}} n$

$$cn\sum_{i=0}^{\log_4 n}\left(\frac{5}{4}\right)^i \le T(n) \le cn\sum_{i=0}^{\log_2 n}\left(\frac{5}{4}\right)^i$$

$$cn\sum_{i=0}^{\log_2 n}\left(\frac{5}{4}\right)^i = cn\,\frac{\left(\frac{5}{4}\right)^{\log_2 n+1}-1}{\frac{5}{4}-1} =$$

$$= 4cn\left(\left(\frac{5}{4}\right)^{\log_2 n}\cdot\frac{5}{4}-1\right) = 4cn\left(n^{\log_2\left(\frac{5}{4}\right)}\cdot\frac{5}{4}-1\right) =$$

$$= 5cn^{1,321} - 4cn = O\left(n^{1,321}\right)$$

$$\bullet\ cn\sum_{i=0}^{\log_4 n}\left(\frac{5}{4}\right)^i = 4cn\left(n^{\overset{=0,16}{\log_4\left(\frac{5}{4}\right)}}\cdot\frac{5}{4}-1\right) = 5cn^{1,16}-4cn = \underline{O}\left(n^{1,16}\right)$$

Άρα $\quad T(n) = O\left(n^{1,321}\right)$

$\qquad T(n) = \underline{O}\left(n^{1,16}\right)$

Άσκηση 4

L	M	R

$\frac{2m}{3}$ $\quad \frac{m}{3}$ $\quad 0$

$$X = X_L \cdot 2^{2m/3} + X_m 2^{m/3} + X_R \cdot 2^0$$

$$Y = Y_L\, 2^{2m/3} + Y_m\, 2^{m/3} + Y_R \cdot 2^0$$

$$X \cdot Y = (X_L 2^{2m/3} + X_m 2^{m/3} + X_R)(Y_L 2^{2m/3} + Y_m 2^{m/3} + Y_R) =$$

$$= X_L \cdot Y_L\, 2^{4m/3} + X_L Y_m\, 2^{m} + X_L Y_R\, 2^{2m/3} + X_m Y_L \cdot 2^{m} + X_m \cdot Y_m \cdot 2^{2m/3} + X_m \cdot Y_R \cdot 2^{m/3} +$$

$$+ X_R Y_L \cdot 2^{2m/3} + X_R Y_m \cdot 2^{m/3} + X_R \cdot Y_R =$$

$$= X_L Y_L\, 2^{4m/3} + (X_L Y_m + X_m Y_L) 2^{m} + (X_L Y_R + X_m Y_m + X_R Y_L) 2^{2m/3} +$$

$$+ (X_m \cdot Y_R + X_R Y_m) 2^{m/3} + X_R Y_R$$

Θέτω $\quad Z_1 = X_R Y_R \;,\; Z_2 = X_L Y_L \;,\; Z_3 = X_m Y_m$

$$Z_4 = (X_L + X_m)(Y_m + Y_L) \;,\; Z_5 = (X_L + X_R)(Y_R + Y_L)$$

$$Z_6 = (X_m + X_R)(Y_m + Y_R)$$

Αλλάζω τα $\quad$ • $X_L Y_m + X_m Y_L = (X_L + X_m)(Y_m + Y_L) - X_L Y_L - X_m Y_m =$

$$= Z_4 - Z_2 - Z_3$$

• $X_L Y_R + X_m Y_m + X_R Y_L = (X_L + X_R)(Y_R + Y_L) - X_L Y_L - X_R Y_R + X_m Y_m =$

$$= Z_5 - Z_2 - Z_1 + Z_3$$

• $X_m Y_R + X_R Y_m = (X_m + X_R)(Y_m + Y_R) - X_m Y_m - X_R Y_R = Z_6 - Z_3 - Z_1$

Η παραλλαγή αυτή του Karatsuba που διαιρεί τους αριθμούς σε 3 τμήματα αντί για δύο και διαιρεί το πρόβλημα σε 6 υποπροβλήματα (αντί για 4)

Άρα $T(n) = 6\,T(\frac{n}{3}) + \theta(n)$

$T'(n) = 6\,T(\frac{n}{3}) + n$

$n^{\log_b a} = n^{\log_3 6} = n^{1{,}63}$

case 1

$f(n) = O(n^{2-\varepsilon})$, $\varepsilon = 0{,}63$

Άρα $T(n) = \theta(\theta(n^{1{,}63})) = \theta(n^{1{,}63})$

Άσκηση 6

```
int mergeSortCount(int A[], int first, int last, int swaps){
    if (first < last){
        int mid = (first + last)/2 ;
        swaps += mergeSortCount(A, first, mid, swaps);
        swaps += mergeSortCount(A, mid+1, last, swaps);
        swaps += merge(A, first, mid+1, last);
    }
    return swaps;
}
```

```
int merge(int A[], int firstL, int firstR, int last){
    int swaps=0;
    int pos=;
    int leftpos=firstL;
    int rightpos=firstR;
    int total= last-firstL+1;
    int results[total];
    while (leftpos <firstR && rightpos ≤ last){
        if(A[leftpos] >A[rightpos]){
            results[pos++]= A[rightpos+1];
            swaps += (firstR-1) -leftpos +1;
        }
        else {
            results[pos++] =A[ leftpos ++];

        }

    }
    while ( leftpos < firstRight){
            results[ pos++] = A[leftpos ++];
    }
    while( rightpos≤ last){
            results[pos++]= A[rightpos++];

    }
    for(int i=0; i< total; i++){
        A[firstL+i] = results[i];
    }
    return swaps;

}
```

Άσκηση 5

```
int maxCrossingSum (int a[], int l, int m, int h) {
    int sum=0;
    int left_sum = -∞
    for (int i=m; i>=l; i--) {
        sum+= a[i];
        if (sum > left_sum)
            left_sum = sum;
    }
    sum=0
    int right_sum = -∞;
    for (int i=m+1; i<=h; i++) {
        sum += a[i];
        if (sum > right_sum)
            right_sum = sum;
    }
    return max(left_sum+right_sum, left_sum, right_sum);
}

int maxSubArraySum (int a[], int l, int h) {
    if (l==h)
        return a[l];

    int m= (l+h)/2;

    return max(maxSubArraySum(a, l, m), maxSubArraySum(a, m+1, h),
               maxCrossingSum (a, l, m, h));
}
```

$T(n) = 2\,T(n/2) + \Theta(n)$ $\quad n^{\log_b a} = n^1 = n$

case 2 $\quad f(n) = \Theta(n \cdot \log^0 n)$, $k=0$ $\quad$ Άρα $T(n) = \Theta(n \log n)$

ε)
```
int max SubArraySum (int a[], int size){
    int max_so_far = -∞;
    int max_ending_here = 0;
    for (int i=0; i<size; i++){
        max_ending_here += a[i];
        if(max_so_far < max_ending_here)
            max_so_far = max_ending_here;
        if (max_ending_here < 0)
            max_ending_here = 0;
    }
    return max_so_far;
}
```

Ο παραπάνω αλγόριθμος τρέχει όλα τα στοιχεία του πίνακα και αποθηκεύει το μεγαλύτερο συνεχόμενο άθροισμα και το επιστρέφει στο τέλος. Επομένως μπορούμε να δούμε ότι έχει πολυπλοκότητα $\Theta(n)$

Άσκηση 1

$$S = \sum_{n=1}^{k^2-1} \lfloor \sqrt{n} \rfloor$$

Η $\lfloor\sqrt{n}\rfloor$ είναι μονότονα αύξουσα συνάρτηση

Επομένως $$\int_0^{k^2-1} \lfloor\sqrt{x}\rfloor \, dx \le \sum_{n=1}^{k^2-1}\lfloor\sqrt{n}\rfloor \le \int_1^{k^2}\lfloor\sqrt{x}\rfloor \, dx \quad (3)$$

με $\sqrt{x} - 1 \le \lfloor\sqrt{x}\rfloor \le \sqrt{x}$

- $\sqrt{x} - 1 \le \lfloor \sqrt{x}\rfloor \Leftrightarrow \int_0^{k^2-1}(\sqrt{x} - 1)\, dx \le \int_0^{k^2-1}\lfloor\sqrt{x}\rfloor \, dx \quad (1)$

- $\lfloor\sqrt{x}\rfloor \le \sqrt{x} \Leftrightarrow \int_1^{k^2}\lfloor\sqrt{x}\rfloor \, dx \le \int_1^{k^2}\sqrt{x}\, dx \quad (2)$

$$(3) \xrightarrow{(1),(2)} \int_0^{k^2-1}(\sqrt{x}-1)\, dx \le \sum_{n=1}^{k^2-1}\lfloor\sqrt{n}\rfloor \le \int_1^{k^2}\sqrt{x}\, dx$$

$$\int_0^{k^2-1}(\sqrt{x}-1)\, dx = \int_0^{k^2-1} x^{\frac{1}{2}} - 1 \, dx =$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^{k^2-1} - \left[x\right]_0^{k^2-1} = \frac{2}{3}(k^2-1)^{\frac{3}{2}} - (k^2-1) + 0 =$$

$$= \frac{2}{3}\sqrt{(k^2-1)^3} - (k^2-1) =$$

$$= \frac{2}{3}\sqrt{(k^2)^3 - 3(k^2)^2 \cdot 1 + 3k^2 \cdot 1^2 - 1^3} - (k^2-1) =$$

$$= \frac{2}{3}\sqrt{k^6 - 3k^4 + 3k^2 - 1} - (k^2+1) = \frac{2}{3}k^3\sqrt{1 - \frac{3}{k^2} + \frac{3}{k^4} - \frac{1}{k^6}} - k^2 + 1 = \Theta(k^3)$$

- Άρα $\sum_{n=1}^{k^2-1}\lfloor\sqrt{n}\rfloor = \Omega(k^3)$ (5)

$$\int_1^{k^2} \sqrt{x}\,dx = \frac{2}{3}\left[x^{3/2} \right]_1^{k^2} =$$

$$= \frac{2}{3}\left(k^3 - 1 \right) = \frac{2}{3}k^3 - \frac{2}{3} = \Theta(k^3)$$

Αρα $\sum_{n=1}^{k^2-1} \lfloor \sqrt{n} \rfloor = \Theta(k^3)$ (4)

Επομένως από (4), (5) $\Rightarrow \sum_{n=1}^{k^2-1} \lfloor \sqrt{n} \rfloor = \Theta(k^3)$

$f(n) = k^3$