Assignment 4

Question 1.

Ans To prove :

$$R\left(f(x), \hat{f}_n(x)\right) = \text{bias}^2 + \text{Var}$$

Risk function is the expected value of loss function.

$$E\left[\left(f(x) - \hat{f}_n(x)\right)^2\right] = E\left[\left(f^2(x) - 2\hat{f}_n(x)f(x) + \hat{f}_n^2(x)\right]\right.$$

$$\text{LHS} = f^2(x) - 2E\left(\hat{f}_n(x)f(x)\right) + E\left(\hat{f}_n^2(x)\right)$$

Where expectations are taken with $\hat{f}_n(x)$ random variable.

$f(x)$ is not a random variable

$\therefore$ It is equal to its own expectation with respect to any distribution.

$$\text{RHS} = \text{Bias}^2 + \text{Var}$$

$$\text{Bias}^2 = \left(E\left(\hat{f}_n(x)\right) - f(x)\right)^2$$

$$= \left[E\left(\hat{f}_n(x)\right)\right]^2 - 2E\left(\hat{f}_n(x)\right)f(x) + \left(f(x)\right)^2$$

$$\text{Var}\left(\hat{f}_n(x)\right) = E\left(\hat{f}_n(x)^2\right) - \left(E\left(\hat{f}_n(x)\right)\right)^2$$

$$\text{RHS} = E\left(\hat{f}_n(x)^2\right) - 2E\left(\hat{f}_n(x)\right)f(x) + \left(f(x)\right)^2$$

$$= E\left[\left(f(x) - \hat{f}_n(x)\right)^2\right]$$

$$= \text{LHS}$$

$$\text{LHS} = \text{RHS}.$$

Question 2

Ans

$$R(f, \hat{f}) = E(D(f, \hat{f}))$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}, \qquad \hat{\theta} = \bar{x}$$

$$\therefore \hat{f}(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\bar{x})^2/2}$$

$$E\left(\int_{-\infty}^{\infty} f(x) \log \frac{f(x)}{\hat{f}(x)} dx \right)$$

$$= E\left(\int_{-\infty}^{\infty} f(x) \log \frac{e^{-(x-\theta)^2/2}}{e^{-(x-\bar{x})^2/2}} dx \right)$$

$$= E\left(\int_{-\infty}^{\infty} f(x) \frac{(x-\bar{x})^2 - (x-\theta)^2}{2} dx \right)$$

$$= E_f \left[\frac{1}{2} (\bar{x} + 2(\theta - \bar{x})x - \theta^2 \right]$$

$$= \frac{1}{2} \left[\bar{x}^2 + 2(\theta - \bar{x}) E_f(x) - \theta^2 \right]$$

$$= \frac{1}{2} (\bar{x}^2 + 2(\theta - \bar{x})\theta - \theta^2)$$

$$= \frac{1}{2} (\bar{x}^2 + 2\bar{x}\theta + \theta^2) = \left(\frac{(\bar{x}-\theta)^2}{2} \right)$$