NUMERICAL SPDE SCHEMES FOR MIXED MONTE-CARLO PDE METHOD

KAUSTAV DAS[†‡], IVAN GUO[†‡], AND GRÉGOIRE LOEPER[†‡§]

1. Introduction

This document details the numerical SPDE schemes utilised in the python code for the article 'On Stochastic Partial Differential Equations and their applications to derivative pricing through a conditional Feynman-Kac formula'. For more in depth details, please refer to the article, in particular Section 6.

Fix a finite time horizon $T > 0$. Let W and B be one-dimensional Brownian motions on a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, with deterministic time-dependent instantaneous correlation $(\rho_t)_{t\in[0,T]}$. In the following, we consider the diffusion process (X, V) taking values in $\mathbb{R}^2$ and given by

$$\begin{aligned} \mathrm{d}X_t &= \mu(t, X_t, V_t)\mathrm{d}t + \sigma(t, X_t, V_t)\mathrm{d}W_t, \quad X_0 = x, \\ \mathrm{d}V_t &= \alpha(t, V_t)\mathrm{d}t + \beta(t, V_t)\mathrm{d}B_t, \quad V_0 = v_0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t. \end{aligned} \tag{1.1}$$

Here $\mu, \sigma : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $\alpha, \beta : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Borel measurable and deterministic. The system eq. (1.1) can be rewritten as

$$\begin{aligned} \mathrm{d}X_t &= \mu(t, X_t, V_t)\mathrm{d}t + \rho_t\sigma(t, X_t, V_t)\mathrm{d}B_t + \varrho_t\sigma(t, X_t, V_t)\mathrm{d}\hat{B}_t, \quad X_0 = x, \\ \mathrm{d}V_t &= \alpha(t, V_t)\mathrm{d}t + \beta(t, V_t)\mathrm{d}B_t \quad V_0 = v_0, \end{aligned} \tag{1.2}$$

where $\hat{B}$ is a one-dimensional Brownian motion independent of B, and $\varrho_t := \sqrt{1 - \rho_t^2}$. Here $w := (B, \hat{B})$ is a standard two-dimensional Brownian motion, and we denote its natural filtration by $(\mathcal{F}_t^w)_{t\in[0,T]}$, which satisfies the usual conditions.

Let $\bar{u}(t, x) = \mathbb{E}\big[\varphi(X_T)|X_t = x, \bar{\mathcal{F}}_{t,T}^{V,B}\big]$, where $\mathcal{F}_{t,T}^{V,B} = \sigma(B_v - B_u, t \le u < v \le T) \vee \sigma(V_t)$. Then $\bar{u}(t, x)$ solves the informal SPDE

$$\begin{aligned} -\mathrm{d}u(t, x) &= (\mathcal{L}_t^x - \mathcal{C}_t^x)\, u(t, x)\mathrm{d}t + \mathcal{B}_t^x u(t, x)\overleftarrow{\mathrm{d}}B_t, \\ u(T, x) &= \varphi(x), \end{aligned} \tag{1.3}$$

[†]School of Mathematics, Monash University, Victoria, 3800 Australia.

[‡]Centre for Quantitative Finance and Investment Strategies, Monash University, Victoria, 3800 Australia.

[§]BNP Paribas Global Markets, Paris, France.

E-mail addresses: kaustav.das@monash.edu, ivan.guo@monash.edu, gregoire.loeper@bnpparibas.com.

where we have the (stochastic) differential operators

$$\mathcal{L}_t^x := \frac{1}{2}\sigma^2(t,x,V_t)\partial_x^2 + \mu(t,x,V_t)\partial_x,$$
$$\mathcal{B}_t^x := \rho_t\sigma(t,x,V_t)\partial_x,$$
$$\mathcal{C}_t^x := \rho_t\beta(t,V_t)\sigma_y(t,x,V_t)\partial_x.$$

Moreover, the time t price of a European derivative with payoff φ and deterministic interest rate $(\mathfrak{r}_r)_{t\in[0,T]}$ is given by $H_t = e^{-\int_t^T \mathfrak{r}_r \mathrm{d}r}\mathbb{E}\big[\bar{u}(t,X_t)|X_t,V_t\big]$.

1.1. **Numerical SPDE schemes.** Consider a time grid $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ and space grid $\{x_{\min} < \cdots < x_{\max}\}$, with $\Delta t := t_{i+1} - t_i$ and $\Delta x := x_{j+1} - x_j$. Let $u^{i,j} \equiv u(t_i, x_j)$ and $\Delta B_i := B_{t_{i+1}} - B_{t_i}$. Define the following:

$$\mathcal{L}_i^j[u] := \frac{1}{2}(\sigma^{i,j})^2\left(\frac{u^{i,j+1} - 2u^{i,j} + u^{i,j-1}}{(\Delta x)^2}\right) + \mu^{i,j}\left(\frac{u^{i,j+1} - u^{i,j-1}}{2\Delta x}\right),$$
$$\mathcal{B}_i^j[u] := \rho_i\sigma^{i,j}\left(\frac{u^{i,j+1} - u^{i,j-1}}{2\Delta x}\right),$$
$$\mathcal{C}_i^j[u] := \rho_i\beta^i\sigma_y^{i,j}\left(\frac{u^{i,j+1} - u^{i,j-1}}{2\Delta x}\right).$$

Here it is clear that for example, $f^{i,j} \equiv f(t_i, x_j, V_{t_i})$. The SPDE eq. (1.3) yields the following numerical schemes:

- Semi-implicit:

$$u^{i,j} = u^{i+1,j} + (\mathcal{L}_i^j - \mathcal{C}_i^j)[u]\Delta t + \mathcal{B}_{i+1}^j[u]\Delta B_i, \quad u^{n,j} = \varphi(x_j). \tag{1.4}$$

- Crank-Nicolson:

$$u^{i,j} = u^{i+1,j} + \frac{1}{2}\big((\mathcal{L}_i^j + \mathcal{L}_{i+1}^j)[u] - (\mathcal{C}_i^j + \mathcal{C}_{i+1}^j)[u]\big)\Delta t + \mathcal{B}_{i+1}^j[u]\Delta B_i, \quad u^{n,j} = \varphi(x_j). \tag{1.5}$$

Note that one must take the right end point when discretising the backward stochastic integral.

We will now define coefficients which will be utilised in the above numerical schemes. For each scheme, there will exist matrices A_i and B_i both in $\mathbb{R}^{(m+1)\times(m+1)}$ such that $A_i U_{i,\bullet} = B_{i+1}U_{i+1,\bullet}$ for $i \in \{0,\dots,n-1\}$. Define the following:

$$a_{i,j} := \left(\frac{1}{2(\Delta x)^2}(\sigma^{i,j})^2 - \frac{1}{2\Delta x}\mu^{i,j} + \frac{1}{2\Delta x}\rho_i\beta^i\sigma_y^{i,j}\right)\Delta t,$$
$$b_{i,j}^{(k)} := k - \left[\frac{1}{(\Delta x)^2}(\sigma^{i,j})^2\right]\Delta t,$$
$$c_{i,j} := \left[\frac{1}{2(\Delta x)^2}(\sigma^{i,j})^2 + \frac{1}{2\Delta x}\mu^{i,j} - \frac{1}{2\Delta x}\rho_i\beta^i\sigma_y^{i,j}\right]\Delta t.$$

The following are their 'stochastic' counterparts for backwards differencing:

$$\begin{aligned}
\tilde{a}_{i,j}^{(l)} &:= a_{i,j} + l\frac{1}{2\Delta x}\rho_i \sigma^{i,j}\Delta B_{i-1},\\
\tilde{b}_{i,j}^{(k)} &:= b_{i,j}^{(k)},\\
\tilde{c}_{i,j}^{(l)} &:= c_{i,j} + \frac{l}{2\Delta x}\rho_i \sigma^{i,j}\Delta B_{i-1},\\
\tilde{d}_{i,j}^{(l)} &:= \frac{l}{2\Delta x}\rho_i \sigma^{i,j}\Delta B_{i-1}.
\end{aligned}$$

1.2. **Matrix forms.** Expanding the above numerical schemes and writing in matrix form yields:

- Semi-implicit:

$$-a_{i,j}u^{i,j-1} - b_{i,j}^{(-1)}u^{i,j} - c_{i,j}u^{i,j+1} = \tilde{d}_{i+1,j-1}^{(-1)}u^{i+1,j-1} + u^{i+1,j} + \tilde{d}_{i+1,j}^{(1)}u^{i+1,j+1}.$$

  This gives $A_iU_{i,\bullet} = B_{i+1}U_{i+1,\bullet}$, $i \in \{0, n-1\}$, where

$$A_i = \begin{pmatrix}
1 & 0 & 0 & \dots & \dots & 0\\
-a_{i,1} & -b_{i,1}^{(-1)} & -c_{i,1} & 0 & \dots & 0\\
0 & -a_{i,2} & -b_{i,2}^{(-1)} & -c_{i,2} & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & 0 & -a_{i,m-1} & -b_{i,m-1}^{(-1)} & -c_{i,m-1}\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

  and

$$B_i = \begin{pmatrix}
1 & 0 & 0 & \dots & \dots & 0\\
\tilde{d}_{i,1}^{(-1)} & 1 & \tilde{d}_{i,1}^{(1)} & 0 & \dots & 0\\
0 & \tilde{d}_{i,2}^{(-1)} & 1 & \tilde{d}_{i,2}^{(2)} & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & 0 & \tilde{d}_{i,m-1}^{(-1)} & 1 & \tilde{d}_{i,m-1}^{(1)}\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

  are tridiagonal.

- Crank-Nicolson:

$$-a_{i,j}u^{i,j-1} - b_{i,j}^{(-2)}u^{i,j} - c_{i,j}u^{i,j+1} = \tilde{a}_{i+1,j}^{(2)}u^{i+1,j-1} + \tilde{b}_{i+1,j}^{(2)}u^{i+1,j} + \tilde{c}_{i+1,j}^{(2)}u^{i+1,j+1}.$$

  This gives $A_iU_{i,\bullet} = B_{i+1}U_{i+1,\bullet}$, $i \in \{0, n-1\}$, where

$$A_i = \begin{pmatrix}
1 & 0 & 0 & \dots & \dots & 0\\
-a_{i,1} & -b_{i,1}^{(-2)} & -c_{i,1} & 0 & \dots & 0\\
0 & -a_{i,2} & -b_{i,2}^{(-2)} & -c_{i,2} & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & 0 & -a_{i,m-1} & -b_{i,m-1}^{(-2)} & -c_{i,m-1}\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and

$$B_i = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ \tilde{a}_{i,1} & \tilde{b}^{(2)}_{i,1} & \tilde{c}^{(2)}_{i,1} & 0 & \dots & 0 \\ 0 & \tilde{a}_{i,2} & \tilde{b}^{(2)}_{i,2} & \tilde{c}^{(2)}_{i,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \tilde{a}_{i,m-1} & \tilde{b}^{(2)}_{i,m-1} & \tilde{c}^{(2)}_{i,m-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are tridiagonal.

1.3. **Example models.** In the following examples we consider the pricing of a European option where the strike of the contract is K and the underlying spot process is given by a stochastic process S.

Example 1.1 (Inverse-Gamma model)**.** Consider the Inverse-Gamma model:

$$\begin{aligned} \mathrm{d}X_t &= \left(\mathfrak{r}_t - \frac{1}{2}V_t^2\right)\mathrm{d}t + \rho_t V_t \mathrm{d}B_t + \varrho_t V_t \mathrm{d}\hat{B}_t, \quad X_0 = \ln(S_0/K), \\ \mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0, \end{aligned}$$

where $X_t = \ln(S_t/K)$ denotes the log-spot on strike. Then we have

$$\begin{aligned} \mu^{i,j} &= \mathfrak{r}_i - \frac{1}{2}V_{t_i}^2, \\ \sigma^{i,j} &= V_{t_i}, \\ \sigma_y^{i,j} &= 1, \\ \beta^{i,j} &= \lambda_{t_i} V_{t_i}. \end{aligned}$$

Example 1.2 (Heston model)**.** Consider the Heston model:

$$\begin{aligned} \mathrm{d}X_t &= \left(\mathfrak{r}_t - \frac{1}{2}V_t\right)\mathrm{d}t + \rho_t \sqrt{V_t} \mathrm{d}B_t + \varrho_t \sqrt{V_t} \mathrm{d}\hat{B}_t, \quad X_0 = \ln(S_0/K), \\ \mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0, \end{aligned}$$

where $X_t = \ln(S_t/K)$ denotes the log-spot on strike. Then we have

$$\begin{aligned} \mu^{i,j} &= \mathfrak{r}_i - \frac{1}{2}V_{t_i}, \\ \sigma^{i,j} &= \sqrt{V_{t_i}}, \\ \sigma_y^{i,j} &= \frac{1}{2\sqrt{V_{t_i}}}, \\ \beta^{i,j} &= \lambda_{t_i} \sqrt{V_{t_i}}. \end{aligned}$$