Infinite-dimensional α-divergence minimisation for Variational Inference

Kamélia Daudel

University of Oxford
kamelia.daudel@stats.ox.ac.uk

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Joint work with Randal Douc and François Portier

Outline

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Bayesian statistics

- Compute / sample from the posterior density of the latent variables y given the data $\mathscr{D}$
$$p(y|\mathscr{D}) = \frac{p(\mathscr{D}, y)}{p(\mathscr{D})} \; .$$
- Problem : for many important models, we can only evaluate $p(y|\mathscr{D})$ up to the constant $p(\mathscr{D})$.

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Variational Inference in a nutshell

→ Variational Inference : inference is seen as an optimisation problem.

❶ Posit a *simpler* variational family $\mathcal{Q}$, where $q \in \mathcal{Q}$.

❷ Fit q to obtain the best approximation to the posterior density

$$\inf_{q \in \mathcal{Q}} D(\mathbb{Q}||\mathbb{P}_{|\mathscr{D}}) \ ,$$

where D is a measure of dissimilarity between the variational distribution $\mathbb{Q}$ and the posterior distribution $\mathbb{P}_{|\mathscr{D}}$

→ Typically, D : exclusive Kullback-Leibler (KL) divergence and $\mathcal{Q}$: parametric family (e.g. Mean-field)

$$\begin{cases} D_{KL}(\mathbb{Q}||\mathbb{P}) = \int_{\mathsf{Y}} \log\left(\frac{q(y)}{p(y)}\right) q(y)\nu(\mathrm{d}y) \\ \mathcal{Q} = \{q : y \mapsto k(\theta, y) \ : \ \theta \in \mathsf{T}\} \end{cases}$$

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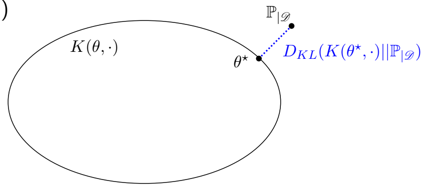

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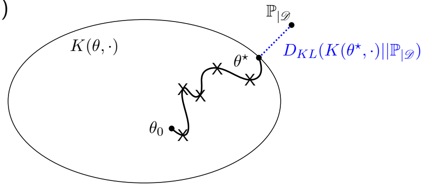

Core question in Variational Inference

$$\inf_{q \in \mathcal{Q}} D(\mathbb{Q}||\mathbb{P}_{|\mathscr{D}})$$

Question : How to choose D and $\mathcal{Q}$?

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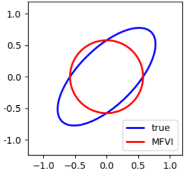

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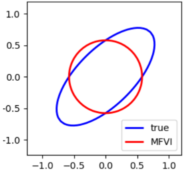

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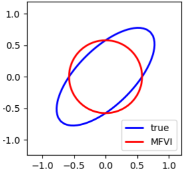

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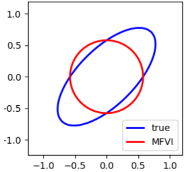

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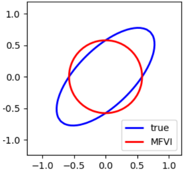

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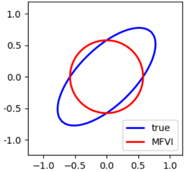

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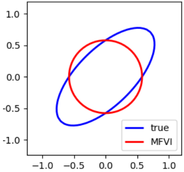

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Variational Inference with the α-divergence family

$(\mathsf{Y}, \mathcal{Y}, \nu)$: measured space, ν is a σ-finite measure on $(\mathsf{Y}, \mathcal{Y})$.
$\mathbb{Q}$ and $\mathbb{P}$: $\mathbb{Q} \preceq \nu$, $\mathbb{P} \preceq \nu$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\nu} = q$, $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\nu} = p$.

α-divergence between $\mathbb{Q}$ and $\mathbb{P}$

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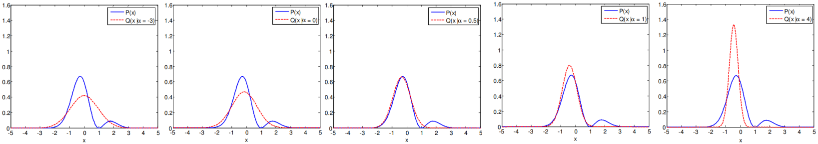
❶ A flexible family of divergences...

Figure: In red, the Gaussian which minimises $D_\alpha(\mathbb{Q}||\mathbb{P})$ for a varying α



$\alpha = -3$ $\alpha = 0$ $\alpha = 0.5$ $\alpha = 1$ $\alpha = 4$

Adapted from V. Cevher's lecture notes (2008) `https://www.ece.rice.edu/~vc3/elec633/AlphaDivergence.pdf`

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Mixture weights optimisation for Alpha-Divergence Variational Inference.
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Idea : Extend the traditional variational parametric family

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by putting a prior on the variational parameter θ

$$\mathcal{Q} = \left\{ q : y \mapsto \mu k(y) := \int_{\mathsf{T}} \mu(\mathrm{d}\theta) k(\theta, y) \ : \ \mu \in \mathsf{M} \right\}$$

and propose an update formula for μ that ensures a systematic decrease in the α-divergence at each step

$\rightarrow$ Finite Mixture Models : $\mu = \sum_{j=1}^{J} \lambda_j \delta_{\theta_j}$

NB: The mapping $\mu \mapsto \Psi_\alpha(\mu k; p)$ is convex!

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Optimisation problem

$$\inf_{\mu \in \mathsf{M}} \Psi_\alpha(\mu k; p) \quad \text{with} \quad \Psi_\alpha(\mu k; p) := \int_{\mathsf{Y}} f_\alpha\left(\frac{\mu k(y)}{p(y)}\right) p(y) \nu(\mathrm{d}y)$$

- p is a nonnegative measurable function defined on $(\mathsf{Y}, \mathcal{Y})$
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Let $\mu_1 \in \mathrm{M}_1(\mathsf{T})$ be such that $\Psi_\alpha(\mu_1 k) < \infty$. The sequence of probability measures $(\mu_n)_{n \geqslant 1}$ is defined iteratively by

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Conditions for a monotonic decrease

(A1) For all $(\theta, y) \in \mathsf{T} \times \mathsf{Y}$, $k(\theta, y) > 0$, $p(y) \geqslant 0$ and $\int_{\mathsf{Y}} p(y)\nu(\mathrm{d}y) < \infty$.

(A2) The function $\Gamma : \mathrm{Dom}_\alpha \to \mathbb{R}_{>0}$ is decreasing, continuously differentiable and satisfies the inequality

$$[(\alpha - 1)(v - \kappa) + 1] (\log \Gamma)'(v) + 1 \geqslant 0 \, .$$

Theorem

Assume (A1) and (A2). Let $\mu \in \mathrm{M}_1(\mathsf{T})$ be such that $\Psi_\alpha(\mu k) < \infty$ and $\mu(\Gamma(b_{\mu,\alpha} + \kappa)) < \infty$. Then,

1. $\Psi_\alpha(\mathcal{I}_\alpha(\mu)k) \leqslant \Psi_\alpha(\mu k)$
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$$\Gamma(v) = e^{-\eta v}$$

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Assume (A1) and that $|b|_{\infty,\alpha} = \sup_{\theta \in \mathsf{T}, \mu \in \mathrm{M}_1(\mathsf{T})} |b_{\mu,\alpha}(\theta)| < \infty$

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$$\mu_{n+1}(\mathrm{d}\theta) = \frac{\mu_n(\mathrm{d}\theta) \cdot \Gamma(b_{\mu_n,\alpha}(\theta) + \kappa)}{\mu_n(\Gamma(b_{\mu_n,\alpha} + \kappa))}, \quad n \geqslant 1$$

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$\to$ $O(1/N)$ convergence rates when Γ is L-smooth and $-\log \Gamma$ is concave increasing

- Entropic Mirror Descent : $\Gamma(v) = e^{-\eta v}$ with $\eta \in (0, \frac{1}{|\alpha-1||b|_{\infty,\alpha}+1})$, any α, κ
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Let $\Theta = (\theta_1, \ldots, \theta_J) \in \mathsf{T}^J$, $\boldsymbol{\lambda}_1 = (\lambda_{1,1}, \ldots, \lambda_{J,1}) \in \mathcal{S}_J$ and denote

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Algorithm

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with $Y_{1,n}, ..., Y_{M,n} \overset{\text{i.i.d}}{\sim} \mu_n k$.

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Outline

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- Gaussian kernel with density k_h and bandwidth h, $\mathsf{T} = \mathbb{R}^d$

$$\left\{ y \mapsto \mu_{\boldsymbol{\lambda},\Theta} k_h(y) = \sum_{j=1}^J \lambda_j k_h(y - \theta_j) \ : \ \boldsymbol{\lambda} \in \mathcal{S}_J, \Theta \in \mathsf{T}^J \right\} .$$

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1. Exploitation step : optimise $\boldsymbol{\lambda}$ using the (α, Γ)-descent.
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$p(y) = Z \times [0.5\mathcal{N}(\boldsymbol{y}; -2\boldsymbol{u_d}, \boldsymbol{I_d}) + 0.5\mathcal{N}(\boldsymbol{y}; 2\boldsymbol{u_d}, \boldsymbol{I_d})]$, $Z = 2$

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Algorithm

1. Exploitation step : optimise $\boldsymbol{\lambda}$ using the (α, Γ)-descent.
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- Toy example
$p(y) = Z \times [0.5\mathcal{N}(\boldsymbol{y}; -2\boldsymbol{u_d}, \boldsymbol{I_d}) + 0.5\mathcal{N}(\boldsymbol{y}; 2\boldsymbol{u_d}, \boldsymbol{I_d})]$, $Z = 2$

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Toy example : Entropic Mirror Descent vs Power Descent

Comparison between

- 0.5-Mirror descent : $\Gamma(v) = e^{-\eta v}$ and $\alpha = 0.5$,
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$J = M = 100$, initial mixture weights : $[1/J, ..., 1/J]$, $N = 10$, $T = 20$
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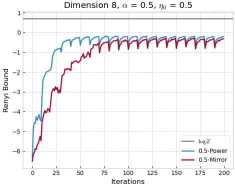

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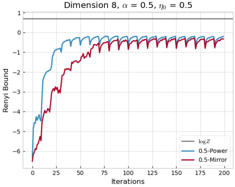

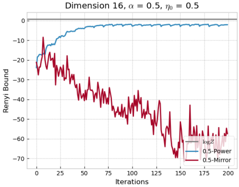

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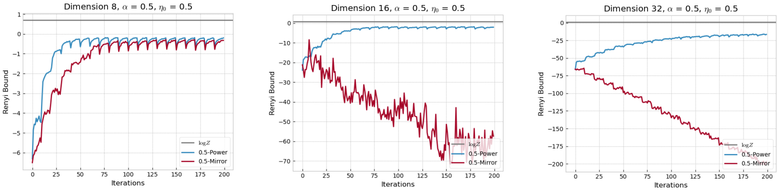

Toy example : the case $\alpha = 1$

Comparison between:

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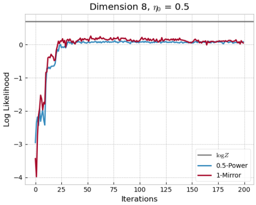

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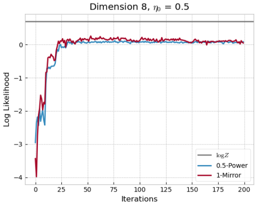

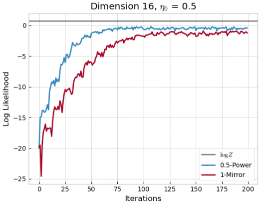

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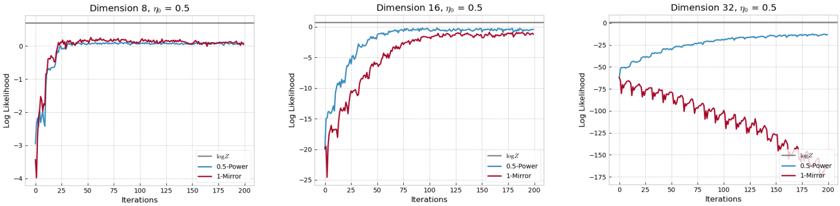

Bayesian Logistic Regression

→ $\mathscr{D} = \{\boldsymbol{c}, \boldsymbol{x}\}$: I binary class labels, $c_i \in \{-1, 1\}$, L covariates for each datapoint, $\boldsymbol{x}_i \in \mathbb{R}^L$

→ Model : L regression coefficients $w_l \in \mathbb{R}$, precision parameter $\beta \in \mathbb{R}^+$

$$p_0(\beta) = \mathrm{Gamma}(\beta; a, b) \ ,$$
$$p_0(w_l|\beta) = \mathcal{N}(w_l; 0, \beta^{-1}) \ , \quad 1 \leqslant l \leqslant L$$
$$p(c_i = 1|\boldsymbol{x}_i, \boldsymbol{w}) = \frac{1}{1 + e^{-\boldsymbol{w}^T \boldsymbol{x}_i}} \ , \quad 1 \leqslant i \leqslant I$$

where $a = 1$ and $b = 0.01$

Nonparametric variational inference S. Gershman, M. Hoffman, and D. Blei (2012). ICML

→ Quantity of interest : $p(y|\mathscr{D})$ with $y = [\boldsymbol{w}, \log \beta]$

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- Typical AIS

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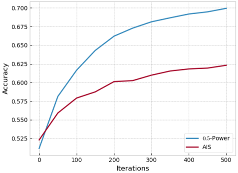

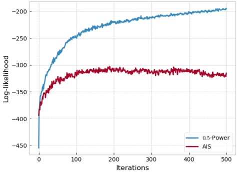

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Outline

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General framework for infinite-dimensional α-divergence minimisation over

$$\mathcal{Q} = \left\{ q : y \mapsto \int_{\mathsf{T}} \mu(\mathrm{d}\theta) k(\theta, y) \; : \; \mu \in \mathsf{M} \right\}$$

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- conditions for a systematic decrease + convergence results
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Thank you for your attention!

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