Bonferroni Correction/Adjustment

Math 699: Design and Analysis of Experiments

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Multiple Comparisons

- *Consider Fixed Effect Model:* $Y_i = \mu_i + \epsilon_{ij} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,n \end{cases}$; $where\ \mu_i = \mu + \tau_i$
- $H_0: \mu_1 = \mu_2 = \cdots = \mu_a\ versus\ H_a: H_0\ is\ false$
- Say we reject the null, now we want to find out which treatments means are different.
- We begin conducting n different tests; but we are concerned with the familywise error rate.
- Family-wise error rate = $P(reject\ at\ least\ one\ H_0 \mid all\ H_0\ are\ true\ )$
- = $\begin{cases} 1 - [1 - P(reject\ H_0 \mid H_0\ is\ true)\ ]^n, if\ independent \\ \leq\ nP(reject\ H_0 \mid H_0\ is\ true), generally \end{cases}$

Multiple Comparison (Continued)

- In a single test the chances of making an error is α
- Therefore, the probability of making one or more error is approximately $n\alpha$
- If n is large, then the chance of making an error will be nearly 100%
- Therefore, we need to either adjust the p-values for the number of hypotheses or control the Type I error rate

What is the Bonferroni Correction?

- The Bonferroni correction addresses the problem with multiple comparisons

- It was introduced and developed by Carlo Emilio Bonferroni.

- The correction is based on the idea if an experimenter is testing n independent or dependent hypotheses, than one way of maintaining or controlling the familywise error rate is to test each individual hypothesis at a statistical significance level of 1/n times.

- The method simply divides α by n

Bonferroni Inequality

- Consider simple linear regression: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$
- $(1 - \alpha)100\%$ Confidence intervals for β_0 and β_1:
- $b_0 \pm t\left(1 - \frac{\alpha}{2}; n - 2\right) s\{b_0\}$
- $b_1 \pm t\left(1 - \frac{\alpha}{2}; n - 2\right) s\{b_0\}$

- Let A_1 $denote\ the\ event\ that\ the\ first\ confidence\ interval\ does\ not\ cover\ \beta_0$
- Let A_2 $denote\ the\ event\ that\ the\ second\ confidence\ interval\ \ does\ not\ cover\ \beta_1$
- $P(A_1) = \alpha\ and\ P(A_2) = \alpha$

Bonferroni Inequality (Continued)

- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
- $P(both\ intervals\ are\ correct) = P(\overline{A_1} \cap \ \overline{A_2}) = 1\ - P(A_1 \cup A_2)$
- $= 1\ - P(A_1) - P(A_2) + P(A_1 \cap A_2)$

- Therefore, $P(\overline{A_1} \cap \ \overline{A_2}) \geq 1 - P(A_1) - P(A_2); since\ P(A_1 \cap A_2) \geq 0$
- $P(\overline{A_1} \cap \ \overline{A_2}) \geq \ 1\ -\ \alpha - \alpha = 1 - 2\alpha$

- Thus, if β_0 *and* β_1 are separately estimated with 95% Confidence intervals, the Bonferroni inequality guarantees us a family confidence coefficient of at least 90% that both intervals are correct

Bonferroni Inequality (Continued)

- What if we wanted to obtain a family confidence coefficient of at least $(1-\ \alpha)\ for\ estimating\ \beta_o\ and\ \beta_1$?
- We can do this simply by estimating $\beta_o\ and\ \beta_1$ separatly with statement confidence coefficients of $1-\frac{\alpha}{2}\ each$.
- Therefore, the Bonferroni inequality will equal $1-\frac{\alpha}{2}-\frac{\alpha}{2}=1-\alpha$
- The Bonferroni Inequality for n simultaneous confidence intervals:
- $P(\bigcap_{i=1}^{n}\overline{A_i})\geq 1-n\alpha$

Comparison of two formula

Number of Tests	$1-(1-\alpha)^{\frac{1}{n}}$	α/n
1	.05	.05
2	.02532	.025
3	.01695	.0166
5	.01021	.01
10	.00521	.005
20	.00256	.0025
100	.000513	.0005

Holm Bonferroni (HB) Method

- Method is based on ordered p-values and the corresponding hypotheses are rejected one at a time
- It is considered as a stepwise (step down) procedure
- The method starts with the smallest p-value and continues with next smallest p-value
- This method was proposed by Holm (1979)
- Let $p_{(1)} < p_{(2)} < \cdots < p_{(n)}$ *be the order p − values*
- Let $\mathrm{H}_{(1)}, \mathrm{H}_{(2)}, \ldots, \mathrm{H}_{(n)}$ be the corresponding null hypotheses

Steps for HB Method

- Step 1: Look at smallest p-value $p_{(1)}$
  - If it is $\leq \frac{\alpha}{n}$ $then\ reject\ H_{(1)}$ and go on to step 2.
  - If it is $> \frac{\alpha}{n}$ then fail to reject $\mathrm{H}_{(1)}$, $\mathrm{H}_{(2)}, \ldots, \mathrm{H}_{(\mathrm{n})}$ and stop

- Step k: Look at the smallest p-value $p_{(k)}$
  - If it is $\leq \frac{\alpha}{n-k+1}$ $then\ reject\ H_{(k)}$ and go on to step k+1.
  - If it is $> \frac{\alpha}{n-k+1}$ then fail to reject $\mathrm{H}_{(\mathrm{k})}, \mathrm{H}_{(k+1)}, \ldots, \mathrm{H}_{(\mathrm{n})}$ and stop

- Step n: Look at the largest p-value $p_{(n)}$
  - If it is $\leq \alpha\ then\ reject\ H_{(n)}\ and\ stop.$
  - If it is $> \alpha$ then fail to reject $\mathrm{H}_{(\mathrm{n})}$ and stop

References

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