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Qualitative Theory of Dynamical Systems

A Behavioral Characterization of Discrete Time Dynamical Systems over Directed Graphs

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Abstract Using a directed graph, a Markov chain can be treated as a dynamical system over a compact space of bi-infinite sequences, with a flow given by the left shift of a sequence. In this paper, we show that the Morse sets of the finest Morse decomposition on this space can be related to communicating classes of the directed graph by considering lifting the communicating classes to the shift space. Finally, we prove that the flow restricted to these Morse sets is chaotic.

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1 Introduction

Much research has gone into both the continuous and discrete time of dynamical systems. Many books, such as [4] consider symbolic dynamical systems as dictated by a finite, complete graph, or as a graph with an irreducible associated transition matrix, such as in [5]. This paper makes generalizations to graphs for which all vertices have in-coming and out-going edges. Initially, focus is given to communicating classes in these graphs for the applications they have to symbolic dynamics. We apply the ideas of Morse sets and chaotic sets, concepts previously reserved for continuous time dynamics, to these discrete time symbolic dynamical system, and examines how the three are related in symbolic dynamics.

Recently effort has been made to amalgamate the cases and create hybrid systems, combining the two areas. Generally, however, studies have chosen to asses these systems from one of two perspectives: time or space. Motivation for this paper comes from the desire to create and examine a system that is hybridized in both time and space. To do so it is necessary to understand completely the behavior of the discrete time symbolic dynamical system presented here, and also to have a means of infusing it into a continuous case, which, though it is not presented in this paper, is an area of ongoing research.

The paper begins by introducing key terms used in association with directed graphs. We then introduce a symbolic dynamical system based on a directed graph, and discuss the behavior of trajectories within this system. The paper concludes by defining a Morse decomposition for the system. Ideas of chain recurrence and chaos are also discussed. For related results in the case of continuous time dynamical systems, see [3].

2 Directed Graphs

In this section, we introduce basic terms and structures relevant to the directed graphs associated with Markov chains that will be used in the remainder of the paper. It is important to have a thorough understanding of these structures and concepts, as they provide the foundation for the rest of the work presented.

A finite directed graph $G = (V, E)$ is a pair of sets $V = \{1, \dots, n\}$ called vertices, and $E \subseteq V \times V$, called edges. For any graph $G = (V, E)$ an admissible path γ is given by $(x_1, \dots, x_k)$ with $(x_i, x_{i+1}) \in E$ for all $i \in \{1, \dots, k-1\}$, and $k-1 \in \mathbb{N}$ is called the length of γ. The set of all admissible paths is denoted by $\mathcal{P}$. For any $\gamma \in \mathcal{P}$: we define γ_1 and γ_F as the initial and final vertices of γ, respectively. For any $\gamma \in \mathcal{P}$ and $\alpha \in V$: γ is said to contain α if there is some $\gamma_i = \alpha$.

A minimal path between two distinct vertices $\gamma_1, \gamma_F \in V$ is any $\gamma \in \mathcal{P}$ with length l such that there exists no other $\gamma^* \in \mathcal{P}$ with $\gamma_1^* = \gamma_1$ and $\gamma_f^* = \gamma_F$ of length $k < l$. It is to see that a minimal path between distinct vertices $\gamma_1, \gamma_F \in V$ contains no loops (paths $\gamma \in \mathcal{P}$ with length ≥ 1 for which $\gamma_1 = \gamma_F$).

For any $\alpha \in V$, we define the set $O^+(\alpha) := \{\beta \in V \mid \text{ there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 = \alpha, \gamma_F = \beta\}$. This definition may be extended to subsets $A \subseteq V$ by defining the set $O^+(A) := \{\beta \in V \mid \text{ there exists } \gamma \in \mathcal{P} \text{ with } \gamma_1 \in A, \gamma_F = \beta\}$. Similarly, for any $\alpha \in V$, we define the set $O^-(\alpha) := \{\beta \in V \mid \text{ there exists } \gamma \in \mathcal{P}$ with

$\gamma_1 = \beta, \gamma_F = \alpha\}$, and for any subset $A \subseteq V$, we define the set $O^-(A) := \{\beta \in V \mid$ there exists $\gamma \in \mathcal{P}$ with $\gamma_1 = \beta, \gamma_F \in A\}$.

The out-degree of any $\alpha \in V$, denoted $o(\alpha)$, is the number of $\gamma \in \mathcal{P}$ with length 1 and $\gamma_1 = \alpha$. The in-degree of any $\alpha \in V$, denoted $i(\alpha)$, is the number of $\gamma \in \mathcal{P}$ with length 1 and $\gamma_F = \alpha$. Any finite directed graph $G = (V, E)$ with $o(\alpha) \geq 1$; for all $\alpha \in V$ is called an M-graph, while any finite directed graph $G = (V, E)$ with $o(\alpha), i(\alpha) \geq 1$; for all $\alpha \in V$ is called an N-graph. Note that all N-graphs are M-graphs, but the converse does not hold. It is trivially true, but still worth noticing, that given any M-graph, $\alpha \in O^+(\alpha)$ if and only if $\alpha \in O^-(\alpha)$. Furthermore, if $\alpha \in O^+(\alpha)$ or if $\alpha \in O^-(\alpha)$, then there exists a loop through α.

Given any M-graph with n vertices and $\gamma \in \mathcal{P}$ from α to $\beta, \alpha \neq \beta$, there exists a $\gamma' \in \mathcal{P}$ of length at most $n - 1$ such that $\gamma'_1 = \alpha$ and $\gamma'_F = \beta$. Any path that is longer than this must hit some vertex more than once and so must contain at least one loop; all such loops may be removed from that path to produce a minimal path between α and β whose length is less than n.

Given a path δ in G of length n another path γ in G of length m such that $\delta_F = \gamma_1$, the concatenation $\delta \circ \gamma$ of δ and γ is defined as the path of length $m + n - 1$ where

$$\delta \circ \gamma = (\delta_1, \delta_2, \ldots, \delta_F = \gamma_1, \gamma_2, \ldots, \gamma_F).$$

A vertex $\alpha \in V$ is variant if there exists no $\gamma \in \mathcal{P}$ containing α which has $\gamma_1 = \gamma_F$. A vertex $\beta \in V$ is invariant if every $\gamma \in \mathcal{P}$ containing β has $\gamma_F = \beta$. Note that 'not invariant' is not the same as 'variant.'

We now come to discuss the particularly useful sets called communicating classes. These communicating classes will provide important information on the behavior of a Markov chain after successive iterations as well as on the direction of flow within the dynamical system spaces. These are arguably the most useful structures we will deal with in this paper.

Definition 1 A communicating class in G is a (nonempty) subset $C \subseteq V$ for which two things are true:

1. For all $\alpha, \beta \in C$ there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha$ and $\gamma_F = \beta$.
2. There exists no $C' \supset C$ where for all $\alpha', \beta' \in C'$ there exists $\gamma' \in \mathcal{P}$ such that $\gamma'_1 = \alpha'$ and $\gamma'_F = \beta'$. This condition is called maximality.

Clearly, maximality implies that given two communicating classes $A, B \subset V$ in G, either $A = B$ or $A \cap B = \emptyset$. Communicating classes can be classified further in two ways:

1. A communicating class, C, is variant if there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in C$ and $\gamma_F \notin C$.
2. A communicating class, C, is invariant if for all $\gamma \in \mathcal{P}$ with $\gamma_1 \in C, \gamma_F \in C$.

Lemma 2 *If $O^+(\alpha) \cap O^-(\alpha) \neq \emptyset$, then $O^+(\alpha) \cap O^-(\alpha)$ is a communicating class.*

Proof (1) Let $\beta, \kappa \in A = O^+(\alpha) \cap O^-(\alpha)$. Then there exists γ such that $\gamma_1 = \beta$ and $\gamma_F = \alpha$, and γ' such that $\gamma'_1 = \alpha$ and $\gamma'_F = \kappa$. Since $\gamma_F = \gamma'_1$ we can construct $\Gamma = (\gamma_1, \ldots, \gamma_F, \ldots, \gamma'_F)$ to be an admissible path in P from β to κ. Since β and κ are arbitrary there exists a path between any two vertices in $O^+(\alpha) \cap O^-(\alpha)$.

(2) Suppose there exists some $A' \supset A$ for which for all $\kappa', \beta' \in A'$ there exists $\gamma' \in \mathcal{P}$ such that $\gamma'_1 = \kappa'$ and $\gamma'_F = \beta'$ and $A' \neq O^+(\alpha) \cap O^-(\alpha)$. Since $\alpha \in A'$, for all $\beta' \in A'$ there exists a path in both directions between α and β'. Then it follows that $\beta' \in A$, proving that $A = A'$, hence A is maximal. □

We want to define an order on the set of all communicating classes in G. This order is not only useful for future lemmas and theorems on directed graphs, but also has strong connections to behavior of the flow on our dynamical systems in the coming sections. First, we must establish a few minor results.

Lemma 3 *For all $\alpha \in V$: α is in some communicating class if and only if α is in some loop.*

Proof ($\Rightarrow$) Pick $\alpha \in C \subset V$ such that C is a communicating class of V. By Definition 1 there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha$ and $\gamma_F = \alpha$, which means that γ is a loop from α to itself.

($\Leftarrow$) Suppose α is contained in some loop $\gamma \in \mathcal{P}$ of length k. Then $\alpha = \gamma_i$ for some $i \in \mathbb{N}$. So we can construct $\gamma' = (\gamma_i, \ldots, \gamma_F, \gamma_2, \ldots, \gamma_i)$ to be an admissible path in $\mathcal{P}$ with $\gamma'_1 = \gamma'_F = \alpha$. So $\alpha \in O^+(\alpha)$, which implies $\alpha \in O^-(\alpha)$. Thus $\alpha \in O^+(\alpha) \cap O^-(\alpha)$ which by Lemma 2 is a communicating class. □

Thus, the existence of a loop within an M-graph automatically implies the existence of a communicating class. We will usually prove the existence of a loop in order to show existence of a communicating class in the following theorems which will allow us to generalize characteristics from our definition of an M-graph.

Theorem 4 *Every M-graph has a path of arbitrary finite length and contains a communicating class.*

Proof (by induction) Let G be an M-graph. Since $o(\alpha) \geq 1$ for all $\alpha \in V$, any $\alpha \in V$ must have associated at least one $e = (\alpha, \eta) \in E$ which constitutes a path of length 1 in G. Assume there exists a path $\gamma \in \mathcal{P}$ of length k from $\gamma_1 = \alpha \in V$ to $\gamma_{k+1} = \beta \in V$. Since $o(\gamma_{k+1} = \beta) \geq 1$, there exists a $\alpha' \in V$ with $e' = (\gamma_{k+1}, \alpha') \in E$. Therefore, there exists a path of length k+1 from γ_1 to α'. Therefore, G contains some path of arbitrarily large finite length.

Next, consider an M-graph G with n vertices. From above, we know there exists a path γ length n. Hence, this path contains $n + 1$ vertices. So for at least one pair $1 \leq i < j \leq n + 1$, we have $\gamma_i = \gamma_j$. Therefore, there exists a loop from γ_i to γ_j, which by Lemma 3 implies the existence of a communicating class. □

In the next lemmas we show that every positive orbit in an M-graph is also an M-graph. These two lemmas will be very useful when we begin to consider point trajectories in our symbolic dynamical system space in the coming sections.

Lemma 5 *For all $\alpha \in V$: $O^+(\alpha)$, along with its associated edges*

$$E_\alpha = \{(x_i, x_{i=1}) \in E | x_i, x_{i+1} \in O^+(\alpha)\}$$

is an M-graph.

Proof Consider the graph $(O^+(\alpha), E_\alpha)$. For all $\beta \in O^+(\alpha)$, β is an element of the M-graph G, so by definition there exists $\eta \in V$ such that $e = (\beta, \eta) \in E$. Since $\beta \in O^+(\alpha)$, for each $\gamma \in \mathcal{P}$ with $\gamma_1 = \beta$ and length n, $\gamma_i \in O^+(\alpha)$ for all $i \leq n$. Thus for all η such that there exists some $e = (\beta, \eta) \in E$, $\eta \in O^+(\alpha)$, which in addition implies $e \in E_\alpha$. So each element of $O^+(\alpha)$ has out-degree of at least one. So $O^+(\alpha)$ with E_α is an M-graph. □

Lemma 6 *Any communicating class C, along with its associated edges*

$$E_C = \{(x_i, x_{i+1}) \in E | x_i, x_{i+1} \in C\}$$

is an M-graph.

Proof Consider a communicating class C with $\alpha \in C$. Since α is in a communicating class, there exists $\beta \in C$, (where β may equal α) such that $(\alpha, \beta) \in E$. Thus, $o(\alpha) \geq 1$, and thus any communicating class is an M-graph. □

Definition 7 Let $S = \{\ C \mid C$ is a communicating class in $G\ \}$. We define a partial order, $\prec$, on S by saying, for $C_i, C_j \in S$, $C_i \prec C_j$ when $C_j \subset O^+(C_i)$.

We must show that $\prec$ is asymmetric and transitive. For asymmetry, suppose there were $C_i, C_j \in$S such that $C_i \prec C_j$ and $C_j \prec C_i$ and $C_i \neq C_j$. Then $C_j \subset O^+(C_i)$ and $C_i \subset O^+(C_j)$. Let $C_T = C_i \bigcup C_j$. Notice then that for all $\alpha', \beta' \in C_T$ there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha'$ and $\gamma_F = \beta'$, making C_T a communicating class that strictly contains both C_i and C_j, contradicting their maximality.

For transitivity, take communicating classes $C_i, C_j, C_k \subset G$ such that $C_i \prec C_j$ and $C_j \prec C_k$. Then $C_j \subset O^+(C_i)$ and $C_k \subset O^+(C_j)$. Hence, there exists a $\alpha \in C_i$ and $\beta \in C_j$ such that there is a path γ^1 from α to β. There also exist $\varepsilon \in C_j$ and $\sigma \in C_k$ such that there is a path γ^2 from ε to σ. Since β and ε are both in C_j, there exists a path γ^3 from β to ε. Thus, the path $(\gamma_1^1, \ldots, \gamma_F^1 = \gamma_1^2, \ldots, \gamma_F^2 = \gamma_1^3, \ldots, \gamma_F^3)$ is a path from α to σ. Additionally, since there exist paths from any vertex in C_i to α and from any vertex in C_k to σ, there exists paths from any element of C_i to any element of C_k, and thus $C_i \prec C_k$.

Note that this is in fact a partial order on S, as there does not necessarily exist a relationship between two given communicating classes.

Lemma 8 *Invariant communicating classes are exactly the maximal elements of S.*

Proof ($\Rightarrow$) Let $C \in S$ be an invariant communicating class of an M-graph G containing an invariant communicating class. Suppose there exists another $C' \in S$, $C \neq C'$, such that $C \prec C'$. Then $C' \subset O^+(C)$, so there exists $\gamma \in$P with $\gamma_1 \in C$ and $\gamma_F \in C'$. But by the definition of invariant communicating class, for all $\gamma \in$P with $\gamma_1 = \alpha \in C$, $\gamma_F = \beta \in C$. Thus C is not invariant, which is a contradiction. ($\Leftarrow$) Choose $C \in S$ such that there exists no $C' \in S$ where $C \prec C'$ and $C \neq C'$. Then, since C is a communicating class, $C \subset O^+(C)$. Suppose there exists $\alpha \in O^+(C)$ such that $\alpha \notin C$. By Lemma 5 and Theorem 4, there exists $C' \in S$ where $C' \subset O^+(\alpha)$. But then $C \prec C'$, implying that $C = C'$, contradicting the assumption that $C \neq C'$. □

We will now use this order on the set of all communicating classes in an M-graph to show that along with a general communicating class, every M-graph must also contain an invariant communicating class:

Theorem 9 *Every M-graph contains an invariant communicating class.*

Proof By Theorem 4, every M-graph contains a communicating class C. If C is not invariant, it is not maximal. Hence for $\alpha \in C$ there is $\beta \in O^+(\alpha)\backslash C$. By Lemma 5, $(O^+(\beta), E_\beta)$ is an M-graph, hence contains a communicating class C' which is clearly different from C. Since the number of communicating classes is finite, there exists a maximal, hence invariant, communicating class.

Corollary 10 $(O^+(\alpha), E_\alpha)$ *contains at least one invariant communicating class for any vertex* α *in an M-graph.*

Proof From Theorem 9, all M-graphs contain at least one invariant communicating class. Since $(O^+(\alpha), E_\alpha)$ is an M-graph by Lemma 5, $(O^+(\alpha), E_\alpha)$ necessarily contains an invariant communicating class. □

3 Dynamical Systems on Graphs

Much of what we have previously discussed and proved culminates in the generation and characterization of a symbolic dynamical system from a directed graph. We begin this section with a brief overview devoted to definitions for general dynamical systems. We will then create our space directly from the N-graphs with which we became familiar in the earlier sections, and we will study the topology, structures, and eventually the full characterization of the dynamical system associated with these N-graphs.

Definition 11 A dynamical system on a metric space S for some (continuous or discrete) time set $\mathbb{T}$ is given by a map $\Phi : \mathbb{T} \times S \to S$ that satisfies $\Phi(0, x) = x$ and $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ for all $x \in S$ and all $t, s \in \mathbb{T}$. Φ can be expressed by two different but equivalent notations for $x,\ x' \in S$ and $t \in \mathbb{T}$:

$$\Phi(t, x) = x' \quad or \quad \Phi_t(x) = x'.$$

We say that a dynamical system is 1-sided when $\mathbb{T} = \mathbb{N} \cup \{0\}$ or $\mathbb{T} = \mathbb{R}^+ \cup \{0\}$, and 2-sided when $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. Any 2-sided dynamical system with mapping Φ_t has an inverse mapping Φ_{-t} where

$$\Phi_t \circ \Phi_{-t}(x) = \Phi_{-t} \circ \Phi_t(x) = Id(x) = x.$$

3.1 Generating Shift Spaces

The space Υ is defined so as to better understand the subspace we will shortly define and primarily work with, Ω. Υ will also play a role later in the paper when we study topological properties of Ω.

Definition 12 Given an N-graph $G = (V, E)$, the bi-infinite product space Υ of the set $V = \{1, \ldots, n\}$ is the set of all bi-infinite sequences $x = (\ldots x_{-1}, x_0, x_1, \ldots)$ where $x_i \in V$ for all $i \in \mathbb{Z}$.

Definition 13 Given an N-graph $G = (V, E)$ with $A \subset V$ and $\alpha \in V$, we define:

- $\Omega = \{ (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \mid (x_i, x_{i+1}) \in E \}$ to be the shift space of G.
- $\Omega_A = \{ x \in \Omega \mid x_i \in A$ for all $i \in \mathbb{Z} \}$ to be the lift of A.
- $\Omega_p = \{ x \in \Omega \mid$ there exists $k \in \mathbb{N}$ such that $x_i = x_{i+k}$ for all $i \in \mathbb{Z} \}$ to be the space of all periodic sequence points in Ω.
- For any $y \in \Omega$, $D_y^N = \{ x | x_j = y_j \; \forall j \in [-N, N]\}$, to be a cylinder set of order N which contains points $x \in \Omega$ that agree with y in all its entries between y_{N+1} and y_{-N-1}.
- $\Omega_\alpha = \{ x \in \Omega \mid x_0 = \alpha \}$ to be the points in Ω centered at α.

Notice that the points of Ω_p are periodic both in the sense that the trajectories through them are periodic orbits, and that the sequence of vertices that appears in their bi-infinite vertex expansion $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ is periodic. Notice also that $\Upsilon = \Omega$ if and only if G is complete.

Lifts are a particularly important part of Ω and a thorough understanding of their definitions will be very useful in the following pages.

Definition 14 Given $x \in \Omega$ and $i \in \mathbb{Z}$, the mapping $\pi_i : \Omega \to V$ is called the projection of x and is given by $\pi_i(x) = x_i \in V$.

Projections will serve as a bridge connecting points in Ω back to the graph generating Ω. Clearly they will be used when we wish to apply the properties of an N-graph to Ω. This next definition is a slight extension, also for this purpose:

Definition 15 For any $x \in \Omega$, the range of x is given by: $range(x) = \bigcup_{i \in \mathbb{Z}} \{\pi_i(x)\}$.

Now we describe our means of "flow" on Ω:

Definition 16 We define as the left shift on Ω the mapping $\Phi : \Omega \to \Omega$ where if $\Phi(x) = y$, then $y_i = x_{i+1}$ for all $i \in \mathbb{Z}$.

Though it is quite simple, we will see the Φ will have many interesting consequences for the system, many of which are not entirely intuitive. Next some lemmas on lifts are given, pointing out some of their properties and how they are related to the communicating classes of a graph.

Lemma 17 *All non-empty lifts are Φ-invariant.*

Proof Let $A \subset V$ and Ω_A be its non-empty lift. Suppose there exists some $x \in \Omega_A$ such that $\Phi(x) = x' \notin \Omega_A$. Since $x' \notin \Omega_A$ then for some $i \in \mathbb{Z}$, $\pi_i(x') \notin A$ or for some $x_i, x_{i+1} \in A$, $(x_i', x_{i+1}') \notin E$. But if $\Phi(x) = x'$ then $\pi_{i-1}(x) = \pi_i x' \notin A$ so $x \notin \Omega_A$, and if $(x_i', x_{i+1}') \notin E$, then $(x_{i-1}, x_i) \notin E$ implying $x \notin \Omega$, contradicting that $x \in \Omega_A$. $\square$

Since the lifts of communicating classes in an N-graph are non-empty, they are invariant.

Lemma 18 *For an N-graph* $G = (V, E)$ *and* $\alpha \in V$, $O^+(\alpha) = \bigcup_{i \in \mathbb{N}} \{\pi_i(x) | x \in \Omega_\alpha\}$

Proof ($\subseteq$) Choose $\beta \in O^+(\alpha)$. Then there exists $\gamma \in P$ such that $\gamma_0 = \alpha$ and $\gamma_i = \beta$ for some $i \in \mathbb{N}$. Since Ω_α is made up of all points in Ω which are admissible sequences that have an $x_0 = \alpha$, there must be some $x \in \Omega_\alpha$ which has $x_i = \beta$ for some $i \in \mathbb{N}$ because γ is an admissible path in G. Thus $\beta \in \bigcup_{j \in \mathbb{N}} \{\pi_j(x) | x \in \Omega_\alpha\}$. ($\supseteq$) Now pick $\beta \in \bigcup_{i \in \mathbb{N}} \{\pi_j(x) | x \in \Omega_\alpha\}$. Then there exists some $x \in \Omega$ which is an admissible sequence and has $x_0 = \alpha$ and $x_i = \beta$ for some $i > 0$. Thus if there is an admissible forward sequence from α to β, by definition it must be true that there is an admissible path in G from α to β so $\beta \in O^+(\alpha)$. □

Lemma 19 *Given* $A \neq B \subset V$ *communicating classes of* G: $\Omega_A \cap \Omega_B = \emptyset$.

Proof Since A and B are distinct communicating classes in G, they are disjoint. Thus by the definition of lifts, $\Omega_A \cap \Omega_B = \emptyset$. □

We now move on to discussing the topology for Ω, including a metric and an overview of the properties of some of its subsets.

3.2 Characterizing the Shift Space

In this section a metric is first defined on the space Ω. Once this metric has been shown to be satisfactory the discussion will move more generally to the topology it induces on Ω. Important topics covered will be those of completeness, closed subsets, and compactness, all of which will be immensely useful in the last two subsections.

Let

$$f(x_i, y_i) = \begin{cases} 0 & \text{when } x_i = y_i \\ 1 & \text{when } x_i \neq y_i. \end{cases}$$

Lemma 20 *The function* $d(x, y) = \sum_{i=-\infty}^{\infty} \dfrac{f(x_i, y_i)}{4^{|i|}}$, *is a metric on* Υ.

Proof 1. (**Non-negativity**) For any $x, y \in \Upsilon$ and all $i \in \mathbb{Z}$, $\frac{f(x_i, y_i)}{4^{|i|}}$ is a quotient of two nonnegative numbers, so it is nonnegative. Nonnegative reals are closed under addition, so $d(x, y) \geq 0$.

2. (**Identity of Indiscernables**) Suppose $x \neq y$. Then there is some i such that $x_i \neq y_i$, so $f(x_i, y_i) = 1$. Thus, $d(x, y) \geq \frac{1}{4^{|i|}} > 0$. If $x = y$, then $x_i = y_i$ for all i, and $f(x_i, y_i) = 0$ for all i. So then $d(x, y) = \sum_{i=-\infty}^{\infty} 0 = 0$. So $d(x, y)$ is zero if and only if $x = y$.
3. (**Symmetry**) Clearly, $f(a, b) = f(b, a)$ for all $a, b \in V$, so $d(x, y) = \sum_{i=-\infty}^{\infty} \dfrac{f(x_i, y_i)}{4^{|i|}} = \sum_{i=-\infty}^{\infty} \dfrac{f(y_i, x_i)}{4^{|i|}} = d(y, x)$.

4. (**Triangle Inequality**) First, we need to show $f(x_i, y_i) + f(y_i, z_i) \geq f(x_i, z_i)$. If $x_i = z_i$, the case is trivial. If $x_i \neq z_i$, then $f(x_i, z_i) = 1$. Consider two cases- $y_i = x_i$ or $y_i \neq x_i$. If $y_i = x_i$, then $y_i \neq z_i$, and $f(y_i, z_i) = 1$. Then $f(x_i, y_i) + f(y_i, z_i) = 1 \geq f(x_i, z_i)$. If $y_i \neq x_i$, then $f(x_i, y_i) = 1$, so $f(x_i, y_i) + f(y_i, z_i) \geq 1 = f(x_i, z_i)$. Thus, $d(x, y) + d(y, z) = \sum_{i=-\infty}^{\infty} \frac{(f(x_i, y_i) + f(y_i, z_i))}{4^{|i|}} \geq \sum_{i=-\infty}^{\infty} \frac{f(x_i, z_i)}{4^{|i|}} = d(x, z)$. □

The topology induced by using 4 in the denominator is slightly different from those induced by 2 or 3. In the case of 4, two points in Ω that agree in the i^{th} position, but are different before and beyond, will have a distance less than two points that disagree in the i^{th} position and agree before and beyond, which is not necessarily the case with 2 or 3. All integers greater than 4 will induce an equivalent metric to that induced by 4, so the choice to use 4 was made for simplicity. We now examine some topological properties of Ω and Ω_C.

Lemma 21 *All Cauchy sequences in Ω converge to some x in Υ.*

Proof Let $\{x^n\}_{n=1}^{\infty}$ be a Cauchy sequence in Ω. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x^m, x^n) < \varepsilon$ for all $n, m > N$. For all $i \in \mathbb{Z}$, there exists $\varepsilon' > 0$ such that $\frac{1}{4^i} > \varepsilon'$. Then there exists $N' \in \mathbb{Z}$ such that $x_i^m = x_i^n$ for some $i \in \mathbb{Z}$ for all $n, m > N'$. Therefore, $\lim_{n\to\infty} x_i^n$ exists for all $i \in \mathbb{Z}$, and is in V. Let $y = (\ldots, \lim_{n\to\infty} x_{-1}^n, \lim_{n\to\infty} x_0^n, \lim_{n\to\infty} x_1^n, \ldots) \in \Upsilon$. Then for all $\varepsilon > 0$, there exists $j \in \mathbb{N}$ such that $\sum_{i=j}^{\infty} \frac{2}{4^i} < \varepsilon$. There exists $N_{-j}, N_{-j+1}, \ldots, N_j$ such that $y_i = x_i^n$ for all $n > N_i$, $i \in [-j, j]$. Let $N = \max\{N_{-j}, \ldots N_j\}$. So for $n > N$, $x_i^n = y_i$, and so

$$d(x^n, x) < \sum_{i=j}^{\infty} \frac{2}{4^i} < \varepsilon.$$

Thus, $\{x^n\}_{n=1}^{\infty}$ converges to y.

Lemma 22 *The shift operator Φ_t is continuous for all $t \in \mathbb{Z}$.*

Proof If we show Φ_1 is continuous, then Φ_t is continuous by the continuity of continuous compositions. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/4$. Pick any $x, y \in \Omega$ such that

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} < \delta,$$

$$d(\Phi_1(x), \Phi_1(y)) = \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i|}}.$$

We may then form

$$(1/4)d(\Phi_1(x), \Phi_1(y)) = \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i|+1}} \leq \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i+1|}}.$$

From here,

$$\sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i+1|}} = \sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} = \delta.$$

This yields:

$$d(\Phi_1(x), \Phi_1(y)) \leq 4\delta = \varepsilon.$$

□

Lemma 23 *For all $x \in \Omega$, $\varepsilon > 0$, $B(x, \varepsilon)$ contains a cylinder set of $2N + 1$ fixed elements about the origin for some N.*

Proof By definition, $B(x, \varepsilon)$ contains all points y such that $d(x, y) < \varepsilon$. Because $\sum_{i=-\infty}^{\infty} \frac{1}{4^{|i|}}$ is a convergent series, there exists an N such that

$$\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} + \sum_{i=-\infty}^{-N} \frac{1}{4^{|i|}} = 2\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

For any y in the cylinder set with $2N + 1$ fixed points about the origin defined by $D^N = \{x^i | x^i_j = x_j \;\; \forall j \in [-N, N]\}$, $d(x, y) \leq 2\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$. Thus, $D^N \subset B(x, \varepsilon)$. □

This last lemma insures that in any ε-ball of point $p \in \Omega$ it is possible to find other points in the ball which have the same entries as p from entries p_{-N} to p_N for some N. This will be most useful when discussing convergent sequences and especially when it is necessary to construct points with certain properties in ε-balls in Ω. All that will need to be done is form a ε-ball and show that it is possible find a point within that ball which is equal to all other points in the ball for at some fixed $-N$ to N values but contains a constructed sequence before or beyond those entries.

Lemma 24 *Ω is closed in Υ.*

Proof Assume we have a convergent sequence $\{x^n\}_{n=1}^{\infty} \in \Omega$ such that $\lim_{n\to\infty} x^n = y$. Assume that $y \notin \Omega$. Then there exist $i \in \mathbb{Z}$ such that $(y_i, y_{i+1}) \notin E$. Note that there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4^{|i+1|}}$. Thus, there exists $N \in \mathbb{N}$ such that for all $n > N$, $x^n_i = y_i$, and $x^n_{i+1} = y_{i+1}$. Therefore, for all $n > N$, $(x^n_i, x^n_{i+1}) \notin E$, implying for all $n > N$, $x^n \notin \Omega$, contradicting that the sequence $\{x^n\}_{n=1}^{\infty}$ is a sequence in Ω. □

Theorem 25 *Ω is compact.*

Proof We want to prove that Ω is compact by showing that it is complete and totally bounded. We have shown that Ω is closed and that all Cauchy sequences in Ω converge, so all Cauchy sequences in Ω converge to a point in Ω. Hence, Ω is complete.

To prove that Ω is totally bounded, we need to show that given $\varepsilon > 0$, there exist a finite collection of ε-balls covering Ω. Take N such that

$$\sum_{i=-\infty}^{-N} \frac{1}{4^{|i|}} + \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

This ensures that any cylinder set of order N will be contained in a ball of radius ε. If the graph generating Ω has M vertices, then there exist at most M^{2N+1} possible cylinder sets of order N in ε. Take a collection of M^{2N+1} ε-balls, each of which covers a cylinder set of order N. Clearly, this collection of ε-balls is finite. And, since any element of Ω is contained in some cylinder set of order N, the collection of ε-balls covers Ω. Hence, Ω is totally bounded.

So, Ω is complete and totally bounded, and thus is compact. □

Lemma 26 *For any $C \subset V$ a communicating class in G, Ω_C is closed and hence compact.*

Proof Assume we have a convergent sequence of $\{x^n\}_{n=1}^{\infty} \in \Omega_C$ such that $\lim_{n\to\infty} x^n = x$. In order to prove that Ω_C is closed, we have to show $x \in \Omega_C$. Because the sequence is convergent to x, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. Assume $x \notin \Omega_C$. Therefore, there exists an entry of x, x_i, such that $x_i \notin C$. Hence $d(x^n, x) \geq \frac{1}{4^{|i|}}$, contradicting convergence. Thus, $x \in \Omega_C$ and the set is closed. Since Ω_C is a closed subset of the compact space Ω, it follows that Ω_C is compact. □

Finally, we will prove the following lemma on periodic points in Ω. Its significance will become relevant later when the topic of chaos is examined more closely.

Lemma 27 *Given G and associated Ω: points of Ω_p are dense in the lifts of communicating classes.*

Proof Pick $x \in \Omega_C$ and form an ε-ball $B(x, \varepsilon)$ about x for $\varepsilon > 0$. By Lemma 23, $B(x, \varepsilon)$ contains a cylinder set of 2N+1 fixed elements about the origin for some $N \in \mathbb{N}$. Therefore, a periodic sequence of period 2N+1+k exists such that the repeated portion is the finite subsequence $(x_{-N}, \ldots, x_0, \ldots, x_N, \ldots, x_{N+k})$ where $x_{N+k} \in V$ and $(x_{N+k}, x_{-N}) \in E$. A periodic sequence of this form can always be found because x_{-N} and x_N are in a communicating class together, thus there does exist a path from x_N to x_{-N}. Hence, because for all $\varepsilon > 0$, $B(x, \varepsilon)$ contains an element of Ω_p, points of Ω_p are dense in Ω_C. □

Now that a metric and its induced topology on Ω have been defined and to some degree described, it will be useful to take a more thorough look at how Φ affects points in Ω. To do so we introduce α- and ω-limit sets, sets common to the study of dynamical systems. These sets will allow us to answer the general question of: what goes where?

3.3 α- and ω-Limit Sets

In this section we define and study α- and ω- limit sets for points and subsets of Ω. These sets essentially show all of the spaces a given x could end up after being acted upon by Φ in either positive or negative time. Of course, it may often be quite tedious to find a single point in a limit set, but by using our understanding of N-graphs we will be able to characterize entire sets. In addition it will quickly become evident that these will be particularly useful for a later characterization of behavior on and off of the lifts of communicating classes. For now, we begin with the general definition of α- and ω-limit sets:

Definition 28 For all $x \in \Omega$: $\alpha(x) = \{y \in \Omega |$ there exist some sequence $\{t_n\}_{n=1}^{\infty}$, with $t_{n+1} < t_n < 0$ for all n and $t_n \to -\infty$, such that $\lim_{n\to\infty} \Phi_{t_n}(x) = y\}$. $\omega(x) = \{y \in \Omega |$ there exists some sequence $\{t_n\}_{n=1}^{\infty}$, with $t_{n+1} > t_n > 0$ for all n and $t_n \to \infty$, such that $\lim_{n\to\infty} \Phi_{t_n}(x) = y\}$. $\alpha(x)$ and $\omega(x)$ are called the α- and ω-limit sets of x, respectively.

- For all $M \subset \Omega$: the ω- (α-)limit set of M is defined by $\omega(M)$ $(\alpha(M)) = \{y \in \Omega |$ there exist sequences $\{x^n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$, with $t_n \to \infty(-\infty)$ with $t_{n+1} > t_n > 0$ $(t_{n+1} < t_n < 0)$ and $x^n \in M$ for all n, such that $\lim_{n\to\infty} \Phi_{t_n}(x^n) = y\}$.

Note that these definitions of the limit sets of sets is not equal to the union of the limit sets of the points contained in the set. It is well known that for all $x \in \Omega$, $\omega(x)$ and $\alpha(x)$ are closed sets in Ω.

This next definition given for cycles within Ω is a means of characterizing the action of Φ on points of Ω.

Definition 29 For $\Omega_A, \Omega_B \subset \Omega$, a cycle between Ω_A and Ω_B exists if there exist some $p, q \in \Omega$ such that $\omega(p) \subset \Omega_A$, $\omega(q) \subset \Omega_B$ and $\alpha(p) \subset \Omega_B$, $\alpha(q) \subset \Omega_A$.

Cycles become very important when Morse Decompositions for Ω are discussed in Sect. 3.4. Before discussing where these cycles can and cannot exist in Ω, however, it is important to first understand how α- and ω-limit sets in Ω are related to the communicating classes of the associate graph.

Lemma 30 *For all $x \in \Omega$, if an N-graph G has communicating classes $C_1, \ldots, C_k$, $\alpha(x), \omega(x) \subset \bigcup_{i=1}^{k} \Omega_{C_i}$. In particular, each $\alpha(x)$ and $\omega(x)$ are contained in a single Ω_{C_i}.*

Proof Let $a \in \pi(\omega(x))$. Then there exists a $y \in \omega(x)$ and an $N \in \mathbb{Z}$ such that $y_N = a$. Since $y \in \omega(x)$, there exists a sequence of natural numbers $n_k \to \infty$ such that $\Phi_{n_k}(x) \to y$. Recall that

$$\Phi_{n_k}(x) = \Phi_{n_k}(x_i, i \in \mathbb{Z}) = (x_{i+n_k}, i \in \mathbb{Z}).$$

Note that there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4^{|N|}}$. Thus, there exists K such that $\pi_N(\Phi_{n_k}(x)) = a$, for all $k > K$, that is, $\pi_N(x_{i+n_k}, i \in \mathbb{Z}) = a$ for all $k > K$. So for all i such that $i = N + n_k$ for $k > K$, $x_i = a$. Since $n_k \to \infty$, for all $n \in \mathbb{Z}$, there exists n_k such that $N + n_k > n$.

Thus, if a vertex a is in $\pi(\omega(x))$, there exists a path $(x_{N+n_k}, x_{N+n_k+1}, \dots, x_{N+n_{k+1}})$ in G from a to itself. Thus, there is a loop through a, and so by Lemma 3, a is in a communicating class.

Now let $a, b \in \pi(\omega(x)$. Then there exists n_1 such that $\pi_{n_1}(x) = a$ and $m > n_1$ such that $\pi_m(x) = b$. There also exists $n_2 > m$ such that $\pi_{n_2}(x) = a$. Thus, there is a path $(x_{n_1}, x_{n_1+1}, \dots, x_m)$ a path from a to b in G and a path $(x_m, x_{m+1}, \dots, x_{n_2})$ a path from b to a in G. Thus, a and b must be elements of the same communicating class. □

Lemma 30 shows that the ω- and α-limit sets must each be contained in the lift of only one communicating class, but it does not show that the two communicating classes must be the same. We want to describe necessary conditions for the lift of the communicating class of the ω-limit set to differ from the lift of the communicating class of the α-limit set. For this, the following lemma will prove useful.

Lemma 31 *Given $C \subset V$ a communicating class, its non-empty lift $\Omega_C \subset \Omega$, and some point $p \in \Omega$: $\omega(p) \subset \Omega_C$. Then, there exists some integer i such that $t > i$ implies $\pi_t(p) \in C$.*

Proof Suppose that there is no such i. Then, given any $k \in \mathbb{N}$, there must exist $m > k$ such that $\pi_m(p) \notin \Omega_C$. Furthermore, since $\omega(p) \subset \Omega_C$, given any $k \in \mathbb{N}$, there must exist $n > k$ such that $\pi_n(p) \in \Omega_C$. So, take n_1 such that $\pi_{n_1}(p) \in \Omega_C$, take $m > n_1$ such that $\pi_m(p) \notin \Omega_C$, and take $n_2 > m$ such that $\pi_{n_2}(p) \in \Omega_C$. Then, there is a path from a point $\pi_{n_1}(p)$ in C that hits a point $\pi_m(p)$ not in C and returns to a point $\pi_{n_2}(p)$ in C. This implies that there must be a loop between a point of the communicating class C and a point outside it, which contradicts the assumption that C is a communicating class. Hence, such an i must exist.

This allows us to easily establish the following lemma.

Lemma 32 *Given $A, B \subset V$ communicating classes, their non-empty lifts $\Omega_A, \Omega_B \subset \Omega$, there exists some point $p \in \Omega$: $\alpha(p) \subset \Omega_A$ such that $\omega(p) \subset \Omega_B$ if and only if there exists some admissible path $\gamma \in \mathcal{P}$ with $\gamma_1 \in A$ and $\gamma_F \in B$.*

Proof
- ($\Rightarrow$). Assume $\alpha(p) \subset \Omega_A$ and $\omega(p) \subset \Omega_B$. Then, by Lemma 31, there exist $i, j \in \mathbb{Z}$, with $i < j$ such that for all $t < i$, $\pi_t(p) \in A$ and for all $t > j$, $\pi_t(p) \in B$. Since $p \in \Omega$, p is an admissible bi-infinite path of G, so the subsequence $(p_{i-1}, \dots, p_{j+1})$ of p is also an admissible path of G. Thus, if we let $\gamma = (p_{i-1}, \dots, p_{j+1})$, then $\gamma_1 \in A$ and $\gamma_F \in B$.
- ($\Leftarrow$). Assume there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in A$ and $\gamma_F \in B$ and let γ have length $k + 1$. Since Ω_A and Ω_B are non-empty, and since all non-empty lifts are Φ-invariant we can find some $x \in \Omega_A$ and $y \in \Omega_B$ such that $\pi_0(x) = \gamma_1 \in A$ and $\pi_0(y) = \gamma_F \in B$. Then it is possible to construct $p = (\dots, \gamma_1, \dots, \gamma_F, \dots)$ with $\pi_i(p) = \pi_i(x)$ for all $i \le 0$, $\pi_{k+i}(p) = \pi_i(y)$ for all $i \ge 0$, and $\pi_{i-1}(p) = \gamma_i$ for all $1 \le i \le k+1$. Then clearly $\alpha(p) \subset A$, $\omega(p) \subset B$, and $p \in \Omega$ since p is made up entirely of admissible paths in $\mathcal{P}$. □

Notice now that the behavior of trajectories in Ω is very closely related to the general flow of paths through the directed graph G. Thus, we use the next lemma to discuss the relationship between behavior in the directed graph and behavior, recalling some characteristics of communicating classes that exist within the context of the graph.

Lemma 33 *Given an N-graph with at least two communicating classes, there are no cycles between the lifts of the communicating classes.*

Proof Consider two distinct communicating classes $A, B \subset V$ and construct their lifts as Ω_A and Ω_B. Suppose their exists $p, q \in \Omega$ such that $\omega(p) \subset \Omega_A, \omega(q) \subset \Omega_B$ and $\alpha(p) \subset \Omega_B, \alpha(q) \subset \Omega_A$. Then this would constitute a cycle between Ω_A and Ω_B. Since the point p is an admissible sequence representing an an admissible path in G, we know there must exist a path from vertices in B to vertices in A because entries in the left tail of p must be contained in A and entries in the right tail of p must be contained in B. In addition since the point q is also an admissible sequence representing an admissible path in G, we know there must also exist a path from vertices in A to vertices in B. The existence of these paths makes the set $A \cup B \subseteq V$ a communicating class in V, which contradicts the maximality of A and B. Thus there can be no cycle between Ω_A and Ω_B. $\square$

The following theorem, definition, and lemma have importance which will become clearer in the next section. For this section, however, it should suffice to see that, using limit sets, we can characterize the action of the shift, Φ, on the lifts of communicating classes as being topologically transitive.

Theorem 34 *Let $X = \{x \in \Omega \mid \omega(x) \subset \Omega_C$ where $C \subset V$ is some invariant communicating class of $G\}$. Then X is open and dense in Ω.*

Proof First, we will show that X is dense in Ω. Consider any $y \in \Omega$, and let $\varepsilon > 0$. Then there exists an N such that $\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$. There is an invariant communicating class in the positive orbit of every vertex in an M-graph by Corollary 10, so there exists a path $\gamma = \gamma_1, \ldots, \gamma_k$ with $\gamma_1 = y_N$ and $\gamma_k = z$ where z is an element of some invariant communicating class. Thus, there is some point x such that $x = (\ldots, y_{N-1}, y_N, \gamma_2, \ldots, \gamma_k, \gamma_1', \ldots)$ where γ' is some infinite path in the invariant communicating class containing z. Then $d(y, x) \leq \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$, so $x \in B(y, \varepsilon)$. Also, $x \in X$, so because y and ε were arbitrary, X is dense in Ω.

Now consider any $x \in X$. There must exist some N such that $x_N \in C$ an invariant communicating class of G. Let $\varepsilon = \frac{1}{4^{N+1}}$. Then for any $y \in B(x, \varepsilon)$, $y_i = x_i$ for all $i = -N, -N+1, \ldots, N$, so y_N is in C. Thus, $y_M \in C$ for all $M > N$, and so $\omega(y) \subset \Omega_C$. So $y \in X$. Thus, $B_\varepsilon(x) \subset X$, and so X is open in Ω. $\square$

Recall that a dynamical system on a metric space X is topologically transitive if there exists some $x \in X$ such that $\omega(x) = X$ [2].

Lemma 35 *Φ is topologically transitive on the lifts of communicating classes; that is, for each communicating class C_i there exists a point x^* such that $\omega(x^*) = \Omega_{C_i}$.*

Proof Consider a communicating class C_i, and consider all loops in C_i of length n, denoted by $A_n^1, \ldots, A_n^{k_n}$. Let $x \in \Omega$ be constructed by the concatenation of all loops, connected by paths linking the last vertex of each loop to the first of the next (which we know to exist, because C_i is a communicating class), starting with those of length 1. That is,

$$x = (\ldots \circ A_n^{k_n} \circ \ldots \circ A_n^1 \circ \ldots \circ A_1^2 \circ p_i \circ A_1^1 \circ p_j \circ A_1^2 \circ \ldots \circ A_n^1 \circ \ldots \circ A_n^{k_n} \circ \ldots).$$

We claim that $\omega(x)$ contains every periodic sequence in Ω_{C_i}. It suffices to show that every periodic sequence is a subsequence of x. Let p be a periodic sequence in Ω_{C_i} with period m. Thus, there exists a loop A in C_i of length m such that $p = (\ldots \circ A \circ A \circ A \circ A \circ \ldots)$. Notice also that the concatenation of A twice is a loop of length $2m$ in C_i, and similarly for all $l \in \mathbb{N}$. Thus, this loop of length m appears infinitely throughout x in both directions, implying that p is a bi-infinite subsequence of x, and thus $p \in \omega(x)$.

Because periodic points are dense in Ω_{C_i}, and $\omega(x)$ is closed, $\omega(x) \subseteq \Omega_{C_i}$, but since $\omega(x)$ is clearly contained in Ω_{C_i}, $\omega(x) = \Omega_{C_i}$. This also ensures that $\alpha(x^*) = \Omega_C$. □

Because this point occurs in the lifts of communicating classes we can use the fact that paths exist between all vertices in any sequence and a similar technique to that used in Theorem 34 to show an extension of the same concept, namely that points for which $\omega(x) = \Omega_C$ are dense in Ω_c.

Lemma 36 *Given an N-graph, G, with communicating class C, lift Ω_C and constructed point $x^* \in \Omega_C$ for which $\omega(x^*) = \Omega_C$, we let $Y = \{ x \in \Omega_C \mid \omega(x) = \Omega_C \}$. Y is dense in Ω_C, as is $Z == \{ x \in \Omega_C \mid \alpha(x) = \Omega_C \}$ and $Y \cap Z$.*

Proof Pick any $p \in \Omega_C$ and form $B(p, \varepsilon)$. Then we know there exists some $N+1 \in \mathbb{N}$ such that $2\sum_{N+1}^{\infty} \frac{1}{4^i} < \varepsilon$, so all points $q \in \Omega_C$ for which $\sum_{-N}^{N} \frac{f(p_i, q_i)}{4^{|i|}} = 0$ are in $B(p, \varepsilon)$. Then, since $\pi_N(p) \in C$ and since for all $i \in \mathbb{Z}$, $\pi_i(x^*) \in C$ from the constructed x^*, we know there exists $\gamma \in \mathcal{P}$ of some length k for which $\gamma_1 = \pi_N(q)$ and $\gamma_F = \pi_0(x^*)$. Then it is possible to form $q^* \in B(p, \varepsilon)$ such that $q_i^* = p_i$ for all $-N \leq i \leq N$, $q_i^* = x_i^*$ for all $N + k \leq i$, and $q_i^* = \gamma_i$ for all $N \leq i \leq k$. Then, with this construction $\omega(q^*) = \omega(x^*) = \Omega_C$ and $q^* \in B(p, \varepsilon)$. Clearly, this proof applies to the case of α limit sets by flowing backward in time. □

Definition 37 A flow on a metric space X has sensitive dependence on initial conditions (is chaotic) if there is $\delta > 0$ such that for every $x \in X$ and every neighborhood B of x there are $y \in B$ and $t > 0$ such that $d(\Phi_t(y), \Phi_t(x)) > \delta$ [2].

Theorem 38 *For an N-graph, G, and associated shift space Ω: The flow, Φ, has sensitive dependence on initial conditions on the lifts of communicating classes of G.*

Proof It was shown in [2] that a flow φ on a space X that is topologically transitive, has a dense subset of periodic points, and is not a single periodic orbit, then φ has sensitive dependence on initial conditions. (This was specifically shown for a continuous time dynamical system, but the results generalize easily to the discrete case.) From

Lemma 35, Φ is topologically transitive on the lifts of communicating classes in G, and from Lemma 27 periodic points are dense in the lifts of communicating classes. Thus, the lifts of communicating classes in G have sensitive dependence on initial conditions.

3.4 Communicating Classes: Chain Recurrence, Morse Decompositions, and Chaos

In the previous sections we discussed the flow of individual points and characteristics of certain types of subsets of our space Ω. In a sense, this all serves the purpose of attaining a global understanding of the nature of our dynamical system and the entirety of its flow. However, doing so requires more than the previously discussed ω- and α-limit sets. Thus, we introduce the concept of chain recurrence (discussed further in [1]):

Definition 39 Let Φ_t be a flow on a metric space Ω. Given $\varepsilon > 0$, $T > 0$ and $x, y \in \Omega$, an (ε, T)-chain from x to y with respect to Φ_t is a pair of finite sequences $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ in Ω and $t_0, \dots, t_{n-1}$ in $[T, \infty)$, denoted together by $(x_0, \dots, x_n; t_0, \dots, t_{n-1})$, such that

$$d(\Phi_{t_i}(x_i), x_{i+1}) < \varepsilon$$

for $i = 0, 1, 2, \dots, n-1$.

Definition 40 Let Φ_t be a flow on a metric space Ω. The forward chain limit set of $x \in \Omega$ with respect to Φ_t is the set $\Psi^+(x) = \bigcap_{\varepsilon, T>0}\{y \in \Omega |$ there exists an (ε, T)-chain from x to y with respect to $\Phi_t\}$. The backward chain limit set of $x \in \Omega$ with respect to Φ_t and d is the set $\Psi^-(x) = \bigcap_{\varepsilon, T>0}\{y \in \Omega |$ there exists an (ε, T)-chain from x to y with respect to $\Phi_{-t}\}$.

Definition 41 Let Φ_t be a flow on a metric space (Ω, d). Two points $x, y \in \Omega$ are chain equivalent with respect to Φ_t and d if $y \in \Psi^+(x)$ and $x \in \Psi^+(y)$.

Definition 42 Let Φ_t be a flow on a metric space (Ω, d). A point $x \in \Omega$ is called chain recurrent with respect to Φ_t and d if x is chain equivalent to itself. The set of all chain recurrent points of Φ_t is the chain recurrent set of Φ_t.

The chain limit sets and chain recurrent points are two more steps towards understanding the flow of the entire system. An educated guess can be made that if all ω- and α-limit sets are in lifts of communicating classes then these lifts must have some significance. In the next theorem we show how all chain recurrent points are contained in those lifts and define more of their properties that will be of importance for the Morse Decomposition.

Theorem 43 *$x \in \Omega$ is a chain recurrent point if and only if $x \in \Omega_C$ for some $C \subset V$ a communicating class of G.*

Proof ($\Rightarrow$). Let $x \notin \Omega_C$ for any C, a communicating class of G. Then there exists some combination of $i, j, k \in \mathbb{Z}$ with either $\pi_i(x)$ a variant vertex or $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$, communicating classes of G, where $C_1 \neq C_2$ and $j < k$. We must show that for some ε, T combination there does not exists an (ε, T)-chain from x to itself.

For the case where there is some $\pi_i(x) \notin C$ for any communicating class, fix $T > i$ and let $\Phi_T(x) = x'$. Then for all $m \geq i$, $\pi_m(x') \in O^+(\pi_i(x))$ and $x'_{i-T} = x_i$. Then for all $\varepsilon < \frac{1}{4^{|m|}}$, $q \in B(x', \varepsilon)$ has $\pi_m(q) \in O^+(\pi_i(x))$ for all $m \geq i$ and $q_{i-T} = x_i$. Then all points, $q' \in \Psi^+(q)$, reachable by sequences of T's and ε-jumps from q, will have $\pi_i(q') \in O^+(\pi_i(x))$. This will not allow a point $p \in \Omega$ with $p_i = x_i$ because $\pi_i(x)$ is variant, i.e. there is no loop between vertices of $O^+(\pi_i(x))$ and $\pi_i(x)$. Thus $x \notin \Psi^+(x)$.

For the case where there is some $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$ where $C_1 \neq C_2$ pick the initial T such that $k - T = j$ and form $\Phi_T(x) = x'$. Then for all $m \geq j$, $\pi_m(x') \in O^+(\pi_k(x))$, $x'_{j-T} = x_j$, and $x'_{k-T} = x'_j = x_k$. So for all $\varepsilon < \frac{1}{4^{|j|}}$, $q \in B(x', \varepsilon)$, has $q_j = x_k$ for all $m \geq j$. Then, similar to before, all points $q' \in \Psi^+(q)$, reachable by sequences of T's and ε-jumps from q will have $\pi_j(q') \in O^+(\pi_k(x)$. So, again, no point $p \in \Omega$ with $p_j = x_j$ will be reached because by the definition of communicating class there will be no loops between elements of C_1 and $O^+(C_2)$. Thus, as before, $x \notin \Psi^+(x)$.

($\Leftarrow$). Let $a = (\ldots a_{-1}, a_0, a_1, \ldots)$ and $c = (\ldots c_{-1}, c_0, c_1, \ldots)$ with $a, c \in \Omega_C$. Given $\varepsilon, T > 0$ there must exist some N such that $\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \frac{\varepsilon}{10}$. Let $M = max(N, T)$. Then let $a' = \Phi_M(a)$. By the definition of communicating class, there is a path $\gamma \in P$ with $\gamma_1 = a'_{M+1}$ and $\gamma_F = c_{-M-1}$. Let k be the length of γ. Then we can construct $e \in C$ such that $e = (\ldots a'_{-1}, a'_0, \ldots a'_M, \gamma_1, \ldots, \gamma_F, c_{-M}, c_{-M+1}, \ldots)$ with a'_0 at the origin to be an admissible sequence. The distance then, between a and e is $d(a', e) \leq \sum_{i=M+1}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$. So we can construct an (ε, T)-chain from a to e. Similarly we can construct $e' \in C$ such that $e' = \Phi_{2M+k+1}(e) = (\ldots a'_{-1}, a'_0, \ldots a'_M, \gamma_1 \ldots, \gamma_F, c_{-M}, c_{-M+1}, \ldots c_0, \ldots)$ with c_0 at the origin to be an admissible sequence. This gives us $d(c, e') \leq \sum_{i=-\infty}^{-M-1} \frac{1}{4^{|i|}} < \varepsilon$. So we can make an (ε, T)-chain from e to c, and thus we can make an (ε, T)-chain from a to c. Since a and c are arbitrary, it follows that all points in the lift of a communicating class are chain recurrent.

We arrive finally at the concept of a Morse decomposition. This concept is discussed further in [2].

Definition 44 An invariant subset $R \subset \Omega$ is isolated if there exists a neighborhood N of R such that for all $x \in \Omega$, $\omega(x) \subset N$ and $\alpha(x) \subset N$ implies $x \in R$.

Definition 45 A Morse Decomposition of a flow on a compact metric space is a finite collection $\{\mathcal{M}_i, i = 1, \ldots, n\}$ of nonvoid, pairwise disjoint, and compact isolated invariant sets such that:

- (i) For all $x \in \Omega$ one has $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$.
- (ii) Suppose there are $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_l$ and $x_1, \ldots, x_l \in \Omega \backslash \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{i-1}$ and $\omega(x_i) \subset \mathcal{M}_i$, for $i = 1, \ldots, l$; $\mathcal{M}_0 \neq \mathcal{M}_l$.

A Morse Decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_{n'}\}$ is called *finer* than a Morse Decomposition $\{\mathcal{M}'_1, \ldots, \mathcal{M}'_n\}$, if for all $j \in \{1, \ldots, n\}$ there is $i \in \{1, \ldots, n'\}$ with $\mathcal{M}_i \subset \mathcal{M}'_j$, and the containment is strict for at least one j. The elements of a Morse Decomposition are called Morse Sets.

From what we have seen so far, it is natural to suspect that the lifts of communicating classes are Morse sets. We will soon see that this is indeed the case. In the next couple of lemmas, we prove that these lifts are isolated after first examining an order defined using ω- and α-limit sets between the lifts of communicating classes.

Definition 46 Let $\mho_S = \{\Omega_C | C$ is a communicating class of $A\}$. We define $\prec$ on $\mho_S$ as: for $\Omega_1, \Omega_2 \in \Omega_S$, $\Omega_1 \prec \Omega_2$ when there exists $p \notin \Omega_1, \Omega_2$ such that $\omega(p) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1$.

Lemma 47 *$\prec$ is an order on $\mho_S$.*

Proof
- $\prec$ is asymmetric: suppose there were $\Omega_1, \Omega_2 \in \mho_S$ such that $\Omega_1 \prec \Omega_2$ and $\Omega_2 \prec \Omega_1$. Then there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_1, \omega(q) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1, \alpha(q) \subset \Omega_2$. But this constitutes a cycle between Ω_1 and Ω_2, which is a contradiction.
- $\prec$ is transitive: assume we have $\Omega_A, \Omega_B, \Omega_C \in \mho_S$ with $\Omega_A \prec \Omega_B$ and $\Omega_B \prec \Omega_C$. So there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ and $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$. Since $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ there exists some $N_1, N_2 \in \mathbb{Z}$ such that $\pi_{N_2+k}(p) \in B, \pi_{N_2-k}(p) \notin B$ and $\pi_{N_1-k}(p) \in A, \pi_{N_1+k}(p) \notin A$ for all $k \in \mathbb{N} \cup \{0\}$. Similarly since $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$ there exists some $N_3, N_4 \in \mathbb{Z}$ such that $\pi_{N_4+k}(p) \in C, \pi_{N_4-k}(p) \notin C$ and $\pi_{N_3-k}(p) \in B, \pi_{N_3+k}(p) \notin B$ for all $k \in \mathbb{N} \cup \{0\}$. Now construct the point p^* to have $\pi_{-k-i}(p^*) = \pi_{N_2-i}$ for all $i \in \mathbb{Z}^-$, and $\pi_{k+i}(p^*) = \pi_{N_3+i}$ for all $i \in \mathbb{Z}^+$. We leave the values $\pi_i(p^*)$ for $-k \leq i \leq k$ to be filled in by the minimal path between $\pi_{N_2}(p) \in B$ and $\pi_{N_3}(q) \in B$ which we know exists because they are in the same communicating class. So from our construction we can see that $\alpha(p^*) \subset \Omega_A$ and $\omega(p^*) \subset \Omega_C$. Thus $\Omega_A \prec \Omega_C$. □

Lemma 48 *Given a communicating class $C \subset V$ and its lift, Ω_C: Ω_C is isolated.*

Proof Let B be the neighborhood of Ω_C formed by taking the union of ε-balls at every point of C for some very small ε (4^{-2} will work, for instance). Note that for any $p \in B$, there exists some $q \in \Omega_C$ such that $d(p, q) < \varepsilon$, by construction of B. Let $x \in \Omega$ be such that $\omega(x), \alpha(x) \subset B$. Since $\alpha, \omega(x) \subset B$, for any $N > 0$ there exist $i, j > N$ such that $d(\Phi_i(x), \Omega_C) < 2\varepsilon$ and $d(\Phi_{-j}(x), \Omega_C) < 2\varepsilon$. This implies, by smallness of ε, that $\pi_i(x) \in C$ and $\pi_{-j}(x) \in C$. Since an admissible path cannot re-enter a communicating class once it has left, this means that $\pi_i(x)$ must be an element of C for all $i \in \mathbb{Z}$, so $x \in \Omega_C$. □

The next lemma is used to show that the lifts of communicating classes can not be divided into subsets that serve as a finer Morse Decomposition.

Lemma 49 *Given nonempty and invariant sets $U_1, U_2 \subset \Omega_C$, where $U_1 \cap U_2 = \emptyset$ and C is any communicating class in the N-graph G, there exists a cycle between U_1, U_2.*

Proof By Lemma 23, for any $x \in U_1$ and $\varepsilon > 0$, there exists a natural number N such that $B(x, \varepsilon)$ contains the set $\{q \in \Omega \mid q_i = x_i$ for $-N \leq i \leq N$ and some $N \in \mathbb{N}\}$. Therefore we can construct a point $r \in B(x, \varepsilon)$ such that $r_i = x_i$ for all $i \leq -N$ and $r_i = z_i$ with $z \in U_2$, for all $i \geq N + k$ where $k \in \mathbb{N}$ is the length of the minimal path between $\pi_N(x)$ and $\pi_{k+1}(z)$. We know this minimal path exists because for all $i \in \mathbb{Z}$, $\pi_i(x), \pi_i(z) \in C$ a communicating class. Then since sequences of shifts of r with $-t$ will converge to shifts of $x \in U_1$, $\alpha(r) \subset U_1$ because U_1 is invariant. Similarly $\omega(r) \subset U_2$ because U_2 is invariant.

With a similar argument a y can be found in the neighborhood of U_2 such that $\alpha(y) \subset U_2$ and $\omega(y) \subset U_1$. Therefore a cycle exists between the two subsets of Ω_C. □

Theorem 50 *Given an N-graph, G, and associated shift space $\Omega \subset \Upsilon$, the lifts of the communicating classes in G, represented as elements of $\mho_S$, are Morse Sets in Ω which form the finest Morse Decomposition for Ω.*

Proof Clearly the set of communicating classes of G is finite because G is finite, so $\mho_S$ has finitely many elements. Also, by the definition of communicating classes, elements of $\mho_S$ are nonvoid. Then, from Lemmas 17, 19, 26, and 48 that they are also disjoint, isolated, compact, and invariant. In addition they fulfill the requirements of (i) and (ii) by Lemmas 30 and 33 respectively. Thus the lifts of communicating classes of an N-graph, G, form a Morse Decomposition for Ω associated with G.

To show that $\mho_S$ is the finest Morse Decomposition we suppose there exists some finer Morse Decomposition $\mathcal{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_{n'}\}$. Then by definition for all $j \in \{1, \ldots, n\}$, where n is the number of communicating classes in G, there is $i \in \{1, \ldots, n'\}$ with $\mathcal{M}_i \subset \Omega_{C_j}$. If $n = n'$, then for each Ω_{C_j} there exists exactly one $\mathcal{M}_i$ such that $\mathcal{M}_i \subset \Omega_{C_j}$. If $\mathcal{M}_i = \Omega_{C_j}$ for all i, j then $\mho_S = \mathcal{M}$, so assume there are some i, j where the strict subset applies. Then pick $x' \in \Omega_{C_j}$ such that $x' \notin \mathcal{M}_i$. By Lemma 35 there exists $x^j \in \Omega_{C_j}$ such that for all $x \in \Omega_{C_j}$, $x \in \omega(x^j)$. Since $x' \notin \mathcal{M}_i$ and $x' \in \omega(x^j)$, $\omega(x^j) \not\subset \mathcal{M}_i$, and because the lifts of communicating classes are disjoint $\omega(x^j) \not\subset \bigcup_{i=1}^{n'} \mathcal{M}_i$. Thus $\mathcal{M}$ is not a Morse Decomposition because it does not follow requirement (i). If $n < n'$ then for some Ω_{C_j} there exist $\mathcal{M}_i, \mathcal{M}_k \subset \Omega_{C_j}$. Since $\mathcal{M}_i, \mathcal{M}_k$ must be invariant to be Morse Sets, from Lemma 49 we know that there are cycles between $\mathcal{M}_i, \mathcal{M}_k$, so $\mathcal{M}$ does not follow requirement (ii). Thus $\mathcal{M}$ is not a Morse Decomposition. Therefore, because there exists no finer Morse Decomposition, $\mho_S$ is the finest Morse Decomposition. □

4 Conclusion

We have found that using an N-graph to generate a symbolic dynamical system implies that the associated shift space will be intricately related to the communicating classes of the graph. In particular, the lifts of these communicating classes in the shift space will define a finest Morse Decomposition for the system. Furthermore, the flow on these lifts are found to have sensitive dependence on initial conditions. All of these characterizations follow essentially from the definitions of communicating class. The follow-up work of [3] examinines how these discrete time systems can be 'injected' and interact with continuous time dynamical systems in the form of hybrid systems.

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