Chain Recurrence in Graph Determined Hybrid Systems

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Abstract

This paper examines a continuous time dynamical system that is an extension of a discrete time dynamical system previously examined, and considers this system together in a product space with a compact subset of Euclidean space. Together, the two systems give a skew product flow. We first examine limit behavior and recurrence in our continuous time extension. We then consider analogous limit and recurrence concepts for a skew product flow, and the behavior on the Euclidean space that results.

1 Introduction

In dynamical systems, a major subject of examination is what happens to systems and particular orbits as time goes to infinity. Some orbits are very clearly headed in a particular direction - heteroclinic orbits are an example of this. However, other orbits tend to revisit themselves. The concept of "revisiting" is formalized in the notion of recurrence. There are many different types of recurrence: a few examples are Poincaré recurrence ([8]), nonwandering points, and central points (see [2] for more information). In [6], Conley introduced a new type of recurrence known as "chain recurrence." Chain recurrence is a more general type of recurrence that exhibits many nice properties that we would like to observe in an ideal "recurrent" set: invariance under flow, closure, and topological invariance, among others (again, see [2] for a more thorough discussion). When studying recurrence, one objective is to partition the phase space into a set of recurrent points, and a set of points which are not recurrent, each of these sets being invariant with respect to the flow. In [4], Birkhoff introduced the idea of minimal sets. Topological transitivity gives another way of partitioning the phase space. Chain recurrence and chain equivalence also lead to more general type of dynamical partitioning of the space known as chain transitivity. Chain transitive sets end up being the building blocks of chain recurrence, as the maximal chain transitive sets are the connected components of the set of chain recurrent points. Chain transitive sets also have a relationship with topologically transitive sets, in that every topologically transitive set is also a chain transitive set. However, in order for chain transitivity to be a well defined notion, we require a "flow" on our metric space that corresponds to the dynamics. Without a function to correspond to a single flow, there is no equivalent concept for chain transitivity.

In this paper, we investigate the concept of a finite number of *different* flows on the same compact metric space, M. M is often taken to be a subset of $\mathbb{R}^n$ endowed with the usual Euclidean metric, but this is not a necessary requirement - indeed, any compact metric space will do. We examine the behavior of the system if we switch between the different flows at regular intervals over time. Additionally, we require that the "switching" between systems follow rules given by a (not necessarily symmetric) adjacency matrix, or equivalently, a directed graph. Because the different flows on M are not necessarily related in any way, the limit behavior of this system is not obvious, and indeed, there is no single flow. We attempt to reconcile this problem of "combining" limit behavior - what happens when a space has multiple different dynamical systems, each with distinct limit sets, acting on it? How can limit behavior and recurrence - and in particular, chain transitivity - be studied in this context? We begin in Section 2 by demonstrating that dynamics on M paired with a function space denoted by Δ together form a skew-product flow, allowing for the examination of certain limiting behavior and recurrence concepts.

In [1], we studied a discrete dynamical system on a space Ω which consists of all bi-infinite paths on a directed graph G, endowed with a metric. This space paired with a flow φ given by the left shift-mapping (see [7], pp.48) form a discrete dynamical system. It can then be shown that there exists a finest Morse Decomposition that is nicely correlated with G's structure, and that the Morse sets of this finest decomposition are either a single periodic orbit or chaotic sets. This system is a generalization of the behavior seen in Smale's horseshoe (see [9], pp. 275-280). It is this system (Ω, φ) that we extend to a continuous dynamical system below to line up with the behavior in M to form a skew-product flow.

2 Deterministic Hybrid Systems

In this paper, when we refer to a dynamical system, we mean a pair (M, Φ) satisfying the following definition:

Definition 1. *A dynamical system is a pair (M, Φ) where M is a metric space, and $\Phi : \mathbb{R} \times M \to M$ such that*

1. *$\Phi(0, x) = x$ for all $x \in M$, and*
2. *$\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \in \mathbb{R}$ and $x \in M$.*

If a pair (M, Φ) satisfies this definition, we also say that Φ is a *flow* on M.

Now consider an directed graph $G = (V, E)$ on n vertices such that each vertex has positive in and out degree and loops are allowed (such a graph will heretofore be referred to as an N-graph). Take a collection of n dynamical systems $\{\phi_1, ..., \phi_n\}$ on a compact space $M \subset \mathbb{R}^d$, where each vertex of G corresponds to one dynamical system ϕ_i. We would like to study the dynamics that arise when we switch between these dynamical systems on M at regular time intervals in a manner allowed by the directed graph. (That is, at the moment of the switch, we can only go from ϕ_i to ϕ_j if there is an edge from the vertex corresponding to ϕ_i to the vertex corresponding to ϕ_j, even if $i = j$.) There are somewhat simple examples of this where the limit behavior of the individual systems does not contain the limit behavior of this new system where switching is allowed. Consider the following example:

Example 2. *Consider two systems on the unit interval, systems A and B. In both systems, 0 and 1 are fixed points, but in system A 1 is an attracting fixed point and 0 a repelling fixed point, while in system B, 0 is an attracting fixed point, while 1 is a repelling fixed point. System A is given by the following ordinary differential equation:*

$$\dot{x} = x(1 - x) \tag{1}$$

while system B is given by the following ODE:

$$\dot{x} = -x(1 - x). \tag{2}$$

Let G, the associated graph, be a cycle on two vertices (corresponding to systems A and B). Equation 1 - that is, the ODE associated with system A - is solved by the following function:

$$\phi_A(t,x) = \frac{-x}{-x - e^t + xe^t}$$

and Equation 2 is solved by the function

$$\phi_B(t,x) = \frac{xe^t}{1 - x + xe^t}$$

Note then that $\phi_A(t,x) = \phi_B(-t,x)$ for all t. Thus, for all $x \in [0,1]$,

$$\varphi_B(1, \varphi_A(1,x)) = x. \tag{3}$$

Now imagine we toggle between these two systems at regular time intervals; that is, let $f : \mathbb{R} \to \{A, B\}$ be defined as follows:

$$f(t) = \begin{cases} B & \textit{if } n < x < n+1 \textit{ for } n \textit{ odd} \\ A & \textit{else} \end{cases}$$

Then, $f(t)$ gives the direction of the flow at time t. Note that f is a periodic function with period 2, and with "jump discontinuities" at integer valued inputs. What f then dictates is the system is guided by system B from time 0 to 1, and after one unit of time has passed, it switches to system A, for another unit of time. At time 2, the system switches back to A, and so on. Thus, from Equation 3, this dynamic is periodic with period 2 for all $x \in [0,1]$. Figure 1 shows an example trajectory in $[0,1]$ of $\phi(t, f, 0.75)$ for $t \in [0,5]$.

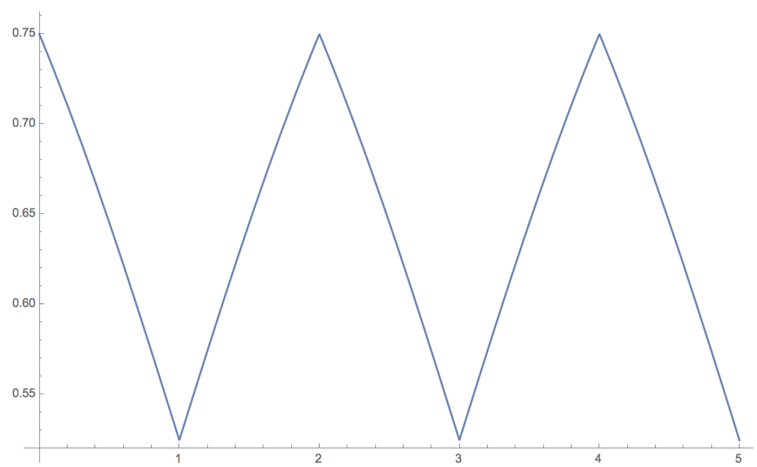

Figure 1: The trajectory of 0.75 in the unit interval under f

By construction, this trajectory is periodic in $[0,1]$ with period 2, and via these dynamics dictated by f, every starting value in $[0,1]$ will similarly have a periodic orbit. Thus, this pair $x \in [0,1]$ along with the function f that dictates when system A is "in control" and when system B is "in control" gives rise to periodicity, which is not seen in the two systems individually. We see then, when combining the systems, new behavior appears that may not exist in any of the systems when considered individually.

In order to study the dynamics here, it is important to note that the dynamics on M alone do not constitute a dynamical system as given in Definition 2. This is because there is no function $\Phi : \mathbb{R}\times M \to M$ such that we can consider the pair (M, Φ) as a dynamical system that give rise to the behavior we wish to study (the behavior after the switch is dependent on which ϕ_j we have switched to). However, we are able to determine dynamics when given a function $f : \mathbb{R} \to \{1, 2, \ldots, n\}$ that specifies which system is in control at which time, as seen in Example 2. Thus, in order to study the behavior, we turn to the concept of a *skew-product flow*, adapted from the definition of a linear skew-product flow defined in [5].

Definition 3. *A skew product flow is a dynamical system Φ with state space $X = B \times M$ of the form $\Phi = (\psi, \varphi) : \mathbb{R} \times B \times M \to B \times M$, where*

1. *$\psi : \mathbb{R} \times B \to B$ satisfies $\psi(0, b) = b$ for all $b \in B$ and $\psi(t+s, b) = \psi(t, \psi(s, b))$ for all $t, s \in \mathbb{R}$ and $b \in B$, and*

2. *$\varphi : \mathbb{R} \times B \times M \to M$ satisfies $\varphi(0, b, m) = m$ for all $b \in B$, $m \in M$ and $\varphi(t + s, b, m) = \varphi(t, \psi(s, b), \varphi(s, b, m))$ for all $t, s \in \mathbb{R}$, $b \in B$, $m \in M$.*

Note then that because

$$\Phi(0, b, m) = (\theta(0, b), \psi(0, b, m)) = (b, m)$$

and

$$\begin{aligned}\Phi(t + s, b, m) &= (\psi(t + s, b), \varphi(t + s, b, m)) \\ &= (\psi(t, \psi(s, b)), \varphi(t, \psi(s, b), \varphi(s, b, m))) \\ &= \Phi(t, \psi(s, b), \varphi(s, b, m)) \\ &= \Phi(t, \Phi(s, b, m))\end{aligned}$$

that $(B \times M, \Phi)$ comprises a dynamical system as given in Definition 2. Thus, we can discuss limit sets and recurrence concepts in this concepts, and in particular, we can study chain recurrent sets of $B \times M$.

We plan to study these switching hybrid systems by configuring them into skew-product flows. In [3], we introduced a space Δ that is a subset of the functions mapping $\mathbb{R}$ into the set $\{1, \ldots, n\}$ that are piecewise constant on intervals of length 1. We further required that the bi-infinite sequence

$$\{f(i)\}_{i\in\mathbb{Z}}$$

be a bi-infinite path on G. We then introduced a metric on this set, and demonstrated that this space is compact with respect to the topology induced by the metric. We further analyzed the dynamics on this space given by the left shift mapping:

$$\psi : \mathbb{R} \times \Delta \to \Delta,\ f(\cdot) \mapsto f(\cdot + t)$$

We can now consider a skew product flow on the space $\Delta \times M$. For $f \in \Delta$, let $0 \leq a_f < 1$ be such that f only has jump discontinuities at values congruent to $a_f \mod 1$. Then let $\varphi(t, f, x) : \mathbb{R} \times \Delta \times M \to M$ be defined by

$$\varphi(t, f, x) = \varphi_t(f, x) = \phi_{f(t)}((t - a_f) \mod 1, (\phi_{f(t-1)}(1, ..., \phi_{f(0)}(a_f, x))). \tag{4}$$

Since we assume that every ϕ_i is invertible, this function is defined backwards in time as well. Additionally, since φ, is a composition of continuous functions, it itself is continuous with respect to x.

While the function defined in Equation 4 is messy to look at, what it attempts to do is define a system where $\varphi(t, f, x)$ is given by the flow along the dynamical system ϕ_i during the period of time for which $f = i$. We then claim that $(\Delta \times M, \Phi)$, where $\Phi : \mathbb{R} \times \Delta \times M \to \Delta \times M$ is given by

$$\Phi(t, f, x) = \begin{pmatrix} \psi(t, f) \\ \varphi(t, f, x) \end{pmatrix}$$

is a dynamical system on $\Delta \times M$, and is in fact a skew-product flow.

Proposition 4. *$(\Delta \times M, \Phi)$ is a skew product flow.*

Proof. The left shift mapping $\psi : \mathbb{R} \times \Delta \to \Delta$ clearly satisfies condition (1) as given in Definition 3. It remains to show that $\varphi : \mathbb{R} \times \Delta \times M \to M$ satisfies condition (2). Let $(f, x) \in \Delta \times M$. Then $\varphi(0, f, m) = \phi_{f(0)}(0, x) = x$ because $\phi_{f(0)}$ is a dynamical system on M. Now let $t, s \in \mathbb{R}$. Then

$$\begin{aligned} \varphi(t, \psi(s, f), \varphi(s, f, x)) &= \phi_{\psi(s,f)(t)}((t - a_{\psi(s,f)}) \mod 1, (\phi_{\psi(s,f)(t-1)}(1, ..., \phi_{\psi(s,f)(0)}(a_{\psi(s,f)}, \varphi(s, f, x)))) \\ &= \phi_{f(s+t)}((t - a_{\psi(s,f)}) \mod 1, \phi_{f(s+t-1)}(1, ..., \phi_{f(s)}(a_{\psi(s,f)}, \varphi(s, f, x)))) \end{aligned}$$

since $\psi(s, f(\cdot)) = f(s+\cdot)$. Note further that if f has jump discontinuities at values congruent to a_f mod 1, then $f(s+\cdot)$ has jump discontinuities at values congruent to $(a_f - s) \mod 1$, and therefore,

$$a_{\psi(s,f)} = (a_f - s) \mod 1.$$

Thus,

$$\begin{aligned} &\phi_{f(s+t)}((t - a_{\psi(s,f)}) \mod 1, \phi_{f(s+t-1)}(1, ..., \phi_{f(s)}(a_{\psi(s,f)}, \varphi(s, f, x)))) \\ &= \phi_{f(s+t)}((t - (a_f - s) \mod 1) \mod 1, \phi_{f(s+t-1)}(1, ..., \phi_{f(s)}((a_f - s) \mod 1, \varphi(s, f, x)))) \\ &= \phi_{f(s+t)}((t + s - a_f) \mod 1, \phi_{f(s+t-1)}(1, ..., \phi_{f(s)}((a_f - s) \mod 1, \varphi(s, f, x)))) \\ &= \phi_{f(s+t)}((t + s - a_f) \mod 1, \phi_{f(s+t-1)}(1, ..., \phi_{f(s)}((a_f - s) \mod 1, \phi_{f(t)}((s - a_f) \mod 1, (\phi_{f(s-1)}(1, ..., \\ &\phi_{f(0)}(a_f, x)))))) \\ &= \phi_{f(t+s)}((t + s - a_f) \mod 1, (\phi_{f(t+s-1)}(1, ..., \phi_{f(0)}(a_f, x))) \\ &= \varphi(t + s, f, x) \end{aligned}$$

Thus, $(\Delta \times M, \Phi)$ is a skew-product flow. $\square$

This definition of φ, however, is rather unintuitive, bulky, and notationally annoying. It is easier to consider a less rigorous description of the system. Consider $(f, x) \in \Delta \times M$. We first consider the dynamical system dictated by $f(0)$; that is, as time moves forward, the orbit of x is given by that dictated by the dynamical system corresponding to $\psi(f, t)(0)$. Recall that ψ simply shifts the function f to the right. If the function changes values, there is an instantaneous switch in which dynamical system is dictating the orbit of x. Because of the instantaneous change, this could then result in a non-smooth orbit. This continues, with a possible change in dynamical system on M occurring after a time interval of length 1. This can be further understood via the following commuting diagram (where π_1 and π_2 are the usual projection mappings):

$$\begin{tikzcd} \mathbb{R} \times \Delta \times M \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & \Delta \times M \arrow[d, "\pi_2"] \\ \mathbb{R} \times \Delta \arrow[r, "\psi"] & \Delta \end{tikzcd}$$

For the set $\Delta \times M$, we use the metric induced by the L_1 norm; that is, the metric in $\Delta \times M$ is given by the sum of the metrics used in M (usually given by the standard Euclidean norm in $\mathbb{R}^n$), and the metric used in Δ. Note then that by Tychonoff's Theorem that $\Delta \times M$ is compact, and that Φ is continuous (as it is continuous in each of its components separately).

3 Chain Transitivity, Chain Recurrence, and Chain Sets

Now that we have an understanding of the behavior of our system $(\Delta \times M, \Phi)$, we turn to examining some recurrence concepts having to do with $(\varepsilon, T)-$chains. The traditional definition of an $(\varepsilon, T)-$chain, as used in the context of flows, is as follows:

Definition 5. *Let ϕ^t be a flow on a metric space (X, d). Given $\varepsilon > 0$, $T > 0$, and $x, y \in X$, an $(\varepsilon, T)-$chain from x to y with respect to ϕ^t and d is a pair of finite sequences $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ in X and $t_0, \dots, t_{n-1}$ in $[T, \infty)$, denoted together by $(x_0, \dots, x_n; t_0, \dots, t_{n-1})$ such that*

$$d(\phi^{t_i}(x_i), x_{i+1}) < \varepsilon$$

for $i = 0, 1, 2, \dots, n-1$. [2]

In the case where we have a *flow* (see Definition 2) on a metric space, we can use Definition 5 to discuss a type of recurrence called chain recurrence.

Definition 6. *Let ϕ^t be a flow on a metric space (X, d). The forward chain limit set of $x \in X$ with respect to ϕ^t and d is the set*

$$\Omega^+(x) = \bigcap_{\varepsilon, T > 0} \{y \in X | \exists \text{ an } (\varepsilon, T)\text{-chain from } x \text{ to } y \text{ with respect to } \phi^t\}.$$

Definition 7. *Let ϕ^t be a flow on a metric space (X, d). Two points $x, y \in X$ are chain equivalent with respect to ϕ^t and d if $y \in \Omega^+(x)$ and $x \in \Omega^+(y)$. [2]*

Definition 8. *Let ϕ^t be a flow on a metric space (X, d). A point $x \in X$ is called chain recurrent with respect to ϕ^t and d if x is chain equivalent to itself. [2]*

Definition 9. *Let ϕ^t be a flow on a metric space (X, d). A set $A \subseteq X$ is chain transitive with respect to ϕ^t if A is a nonempty, closed, invariant set with respect to ϕ^t such that for each $x, y \in A$, the pair x, y is chain equivalent. [2]*

By making the choice $y = x$ in Definition 9, we can see that every element of a chain transitive set is also chain recurrent. In fact,

Theorem 10. *The maximal chain transitive sets of a flow on a compact metric space are the connected components of the chain recurrent set of the flow*

A proof of Theorem 10 can be found in [2], pp. 116-118.

Note that the above definitions all require a *flow* on a metric space; however, our dynamics on M do not constitute a flow, and therefore we can not apply these definitions. Our goal, then, is to adapt the above definition for this case, and examine the consequences. We begin by adapting our definition of chain transitive sets, to new idea, known as chain sets:

Definition 11. *A set $E \subset M$ is called a chain set of a system if*

1. *for all $x \in E$ there exists $f \in \Delta$ such that $\varphi(t, f, x) \in E$ for all $t \in \mathbb{R}$, and*

2. *for all $x, y \in E$ and for all $\varepsilon, T > 0$ there exist $n \in \mathbb{N}$, $x_0, \dots, x_n \in M$, $f_0, \dots, f_{n-1} \in \Delta$ and $t_0, \dots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and*

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \dots, n-1.$$

Such a sequence is called an (ε, T)-chain from x to y. We also require that a chain set E be maximal with respect to these conditions.

It is important to note that chain sets as defined are distinct from the usual chain transitive sets as defined in Definition 9. This is because the behavior on M is not a flow, and thus the concept of chain transitivity is not applicable here. However, in Lemmas 12, 13, and 14, we do demonstrate that chain sets, when considered as maximal components, exhibit nice properties that we would expect of sets that demonstrate some type of recurrence.

Lemma 12. *Chain sets are pairwise disjoint.*

Proof. Let E_1 and E_2 be two chain sets, and suppose, by way of contradiction, there exists $x \in E_1 \cap E_2$ (that is, E_1 and E_2 are not disjoint). Then let $y \in E_1$ and $z \in E_2$. Then, given $\varepsilon, T > 0$, by definition of a chain set there exist $n \in \mathbb{N}$, $x_0, \dots, x_n \in M$, $f_0, \dots, f_{n-1} \in \Delta$ and $t_0, \dots, t_{n-1} \geq T$ with $x_0 = y$, $x_n = x$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \dots, n-1.$$

Similarly, for all $\varepsilon, T > 0$ there exist $n \in \mathbb{N}$, $x_0, \dots, x_n \in M$, $f_0, \dots, f_{n-1} \in \Delta$ and $t_0, \dots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = z$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \dots, n-1.$$

Thus, the concatenation of these two ε, T-chains results in a ε, T-chain from y to z, and thus y and z are in the same chain set, and thus, since the choice of y and z was arbitrary, $E_1 = E_2$. □

Lemma 13. *Chain sets are compact.*

Proof. Let E be a chain set, and let $x \in M$ be a limit point of E. Then there exists a sequence in E, $\{x_i\}_{i=1}^{\infty}$ such that $x_i \to x$. Let $y \in E$, and let $\varepsilon, T > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $d(x_N, x) < \varepsilon$. By definition of a chain set, there exist $n \in \mathbb{N}$, $x_0, \dots, x_n \in M$, $f_0, \dots, f_n \in \Delta$ and $t_0, \dots, t_n \geq T$ with $x_0 = y$, $\varphi(t_n, x_n, f_n) = x_N$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \dots, n-1.$$

Thus, since $d(\varphi(t_n, x_n, f_n), x) < \varepsilon$, setting $x_{n+1} = x$ gives an ε, T-chain from y to x, and similarly from x to y, and thus $x \in E$, and thus E is closed. Since E is then a closed subset of the compact set M, E is thus compact. □

Lemma 14. *Chain sets are connected.*

Proof. Let E be a chain set and let A, B be open sets such that $E \subset A \cup B$ and $A \cap B = \emptyset$. If $\inf\{d(a,b)|a \in A, b \in B\} > 0$, then there exists $\varepsilon < \inf\{d(a,b)|a \in A, b \in B\}$ and thus there exists no $\varepsilon, T-$chain from any $a \in A$ to any $b \in B$, and thus one of A or B must be empty. If $\inf\{d(a,b)|a \in A, b \in B\} = 0$, then there exists some x such that $\inf\{d(a,x)|a \in A\} = 0$ and $\inf\{d(b,x)|b \in B\} = 0$. Since by Lemma 13, E is closed, this implies that $x \in E$, and thus $x \in A$ or $x \in B$. Without loss of generality, let $x \in A$. Since $\inf\{d(b,x)|b \in B\} = 0$, this implies that if B is nonempty, for every neighborhood of N of X, $N \cap B \neq \emptyset$. However, since A is open, there is a neighborhood N of x such that $N \subset A$. This then implies that $A \cap B \neq \emptyset$, a contradiction. Thus B is empty, and E is connected.

□

Note that the above lemma does not demonstrate that chain sets are necessarily path connected; indeed, the following is an example of a chain set which is not path connected.

Example 15. *Let G, the graph governing Δ, be the complete graph on 2 vertices, and let $M = \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\} \cup \{0\} \times [-1, 1]$, otherwise known as the Topologist's Sine Curve. Let the two systems defined on M be given by the following differential equations:*

$$A : \dot{x} = -x(1/2\pi - x) \quad (5)$$

$$B : \dot{x} = x(1/2\pi - x) \quad (6)$$

(Note that it is sufficient to describe the dynamics of the systems with just the behavior in the x-coordinate alone as, except where $x = 0$ - which consists entirely of fixed points - there is exactly one y-value for each x-value.) Thus, in both systems, the set $\{0\} \times [-1, 1]$ is entirely made up of fixed points, and in System A, along the set $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ the system moves from right to left (with a fixed point at $x = 1/2pi$), the speed converging to zero as x approaches zero. Similarly, in system B, along $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, the system moves left to right, with a fixed point at $x = 1/2\pi$. We then claim that the entirety of M is forms one chain set. Let $x, y \in M$. It should be clear that, if $x, y \in \{0\} \times [-1, 1]$, or if $x, y \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, then for all $\varepsilon, T > 0$, there exists an $\varepsilon, T -$ chain from x to y. If $x \in \{0\} \times [-1, 1]$ and $y \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, then a chain can be formed by staying at x for a time of at least T, and then jumping by ε onto a point $z \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ (since M is the closure of $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, x is a limit point of $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, and thus there exists a point z within ε of x). Thus, since we have established that there is a chain from z to y, there is a chain from x to y. Similarly, if $x \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ and $y \in \{0\} \times [-1, 1]$, then, if we let z be a point on $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ within ε of y, then there exists a chain from x to z, and then jumping to y gives a chain from x to y. Thus, M is a chain set.

It is well known that M, the Topologist's Sine Curve, is a set that is connected but not path connected. A proof can be found in [10], pages 137-138.

Thus, while chain sets are always connected, they may not exhibit path connectivity.

Since a chain set E is a subset of M, it helps to have an extension of it to a set contained in $\Delta \times M$.

Definition 16. *Given $E \subset M$, the lift of E to $\Delta \times M$ is given by*

$$\ell(E) = \{(f, x) \in \Delta \times M, \Phi(t, f, x) \in E \text{ for all } t \in \mathbb{R}\}.$$

Definition 17. *A set $A \subset M$ is said to be invariant if for all $x \in A$, $\varphi(t, f, x) \in A$ for all $t \in \mathbb{R}$ and $f \in \Delta$. A set $A \subset M$ is said to be forward invariant if for all $x \in A$, $\varphi(t, f, x,) \in A$ for all $t \in \mathbb{R}^+$ and $f \in \Delta$. Similarly, a set $A \subset M$ is said to be backward invariant if for all $x \in A$, $\varphi(t, f, x,) \in A$ for all $t \in \mathbb{R}^-$ and $f \in \Delta$.*

Remark 18. *Notice that if E is invariant, $\ell(E) = \Delta \times E$.*

It is important to remember that chain sets are not the usual chain transitive sets. This is because we can not consider the behavior on M alone as a flow; it is dependent on orbits in Δ. The following example further demonstrates why the concepts of a chain set and a chain transitive set are not the same.

Example 19. *Consider the system where Δ is given by the complete graph on two vertices labeled A and B, and $M = [0, 2]$. Let the system corresponding to vertex A be given by the differential equation:*

$$\dot{x} = -x(x-1)(x-2),$$

and the system corresponding to vertex B is given by

$$\dot{x} = -x(x-2).$$

Both of these systems are bounded on either end by fixed points at 0 and 2. System A has a repelling fixed point at 1, while in system B on the interval $(0, 2)$ the flow moves in the positive direction. We claim that in this system, the interval $[0, 1]$ is a chain set for all $T, \varepsilon > 0$. (The fixed point at $x = 2$ is also rather trivially a chain set.) Note that on the open interval $(0, 1)$, the flow moves in different directions in systems A and B. Note further that the lift of $[0, 1]$ is not equal to $\Delta \times [0, 1]$, as $[0, 1]$ is not invariant in system B; if, we consider the function $f \equiv B$, for any $x \in (0, 1)$, there exists $T > 0$ such that $\varphi(T, f, x) > 1$. Now, let $y, z \in [0, 1)$. Given $\varepsilon, T > 0$, we wish to construct an $(\varepsilon, T)-$chain from y to z. Let $a_n = \varphi(-nT, f, z)$, where $f \equiv B$, and consider the sequence $\{a_n\}_{n=0}^{\infty}$. Notice then that

$$\lim_{n\to\infty} a_n = 0.$$

Let N be such that $|0 - a_N| < \varepsilon/2$. Notice that there exists a time $T' > T$ such that

$$|0 - \varphi(T', g, y)| < \varepsilon/2$$

where $g \equiv A$. Thus, by the triangle inequality,

$$|\varphi(T', g, y) - a_N| < \varepsilon.$$

Therefore, the sequence $y, \varphi(T', g, y), a_N, z$ forms an $(\varepsilon, T)-$chain from y to z. Since we know chain sets are closed by Lemma 13, we know now that $[0, 1]$ is a chain set for all $T, \varepsilon > 0$.

Now, in the above system, let the graph G be the cycle on two vertices. Then we claim that $[0, 1]$ is no longer a chain set for all $T > 0$. Note now that the only functions in Δ are shifts of the periodic function that alternates between A and B on intervals of length 1. Without loss of generality, let $z < y$ such that $\varphi(1, f, \varphi(1, g, z)) \neq z$. Such a point exists because for all $\varepsilon > 0$, there exists a point $x \in (1/2, 1)$ such that $\varphi'(t, g, x) < \varepsilon$ as the flow in system A converges to 1. However, since $\phi'(t, f, 1) \neq 0$, the flow is not symmetric, and thus we can not have that $\varphi(1, f, \varphi(1, g, z)) = z$ for all $z \in (0,)$, and thus such a point z exists. We would like to show that there exist ε, T such that there is no longer an (ε, T)-chain from z to y. Let us start at z with system A. Let $T = 2$, and $x_2 = \varphi(1, f, \varphi(1, g, z))$. If $x_2 < z$, then pick $\varepsilon < |z - x_2|$ - notice now that all solutions must be less *than z for all times $t > 1$. Similarly, if $x_2 > z$, we can choose ε such that we can not reach any points less than z with a particular ε. If $z = x_2$, let ε be small enough such that $\varphi(-1, f, 1+\varepsilon) < z$.*

However, note that if we take $T = 1$, since we are allowed to switch functions after each ε-jump, we are essentially in the same case as when the graph G is complete (since we may jump by $\varepsilon = 0$ and let $f(0)$ take either value A or B after the jump, and start with those functions whose jump discontinuities happen at integer values). Thus, in this case $[0, 1]$ is an $(\varepsilon, 1)$-chain set. Note that this example then implies that, $(\varepsilon, 1)$-chain sets and (ε, T)-chain sets for a general T may not be equivalent, and thus Theorem 22 does not apply in this case. This is important because Theorem 22 is applicable to traditional chain transitive sets, and we have thus demonstrated a key difference between these chain transitive sets and our newly introduced chain sets.

The above example illustrates that $(\varepsilon, 1)$-chain sets are like having the complete graph (see Lemma 24); that is, if $\mathcal{E} \subset \Delta \times M$ is a maximal invariant chain transitive set, then $\mathcal{E} = \ell(\pi_M \mathcal{E})$, and $\pi_M \mathcal{E}$ is a chain set for all T (see Theorem 20).

Thus, we see that the relationship between chain sets, subsets of M, and chain transitive sets, subsets of $\Delta \times M$, is rather complicated. As of right now, there is no general theory about the relationship between the two concepts. Above we have explored certain examples of relationships, but future work may entail coming up with a more general result that relates the two. In addition, concepts such as Poincaré recurrence and nonwandering sets could be explored within this context.

Since Φ is a flow on the compact set $\Delta \times M$, $\Delta \times M$ contains chain transitive sets. We would like to make connections between a chain transitive set that is a subset of $\Delta \times M$ and a chain set that is a subset of M. One way to map sets in $\Delta \times M$ to sets in M is via the usual projection map; a way to map sets in M to sets in $\Delta \times M$ is via lifts. The following results relate the ideas of chain transitive sets, chain sets, projections, and lifts, and what properties are retained when projecting onto M or lifting to $\Delta \times M$.

Theorem 20. *Let $\mathcal{E} \subset \Delta \times M$ be a maximal invariant chain transitive set for the flow. Then $\pi_M \mathcal{E}$ is a chain set.*

Proof. Let $\mathcal{E}$ be an invariant, chain transitive set in $\Delta \times M$. For $x \in \pi_M \mathcal{E}$ there exists $f \in \Delta$ such that $\varphi(t, f, x) \in \mathcal{E}$ for all t by definition of invariance. Now let $x, y \in \pi_M \mathcal{E}$ and choose $\varepsilon, T > 0$. Then by chain transitivity of $\mathcal{E}$, there exist x_j, f_j, t_j for $j \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ such that the corresponding trajectories satisfy the required condition. The proof is concluded by noticing that $\pi_M \mathcal{E}$ is maximal if $\mathcal{E}$ is maximal.

□

Lemma 21. *Given a maximal invariant chain transitive set $\mathcal{E} \subset \Delta \times M$, $\mathcal{E} \subset \ell(\pi_M \mathcal{E})$.*

Proof. Let $(f, x) \in \mathcal{E}$. Then $x \in \pi_M \mathcal{E}$. Since $\mathcal{E}$ is invariant, $\Phi(t, f, x) \in \mathcal{E}$ for all $t \in \mathbb{R}$. Thus, $\pi_M \Phi(t, f, x) = \varphi(t, f, x) \in \pi_M \mathcal{E}$ for all $t \in \mathbb{R}$. This then implies that $(f, x) \in \ell(\pi_M \mathcal{E})$, by definition of the lift. □

We then wondered if it was possible to establish a more general theory about chain sets and chain transitive sets, and how they are related via lifts and projections. In order to accomplish this task, we made use of the following theorem, taken from [2], Theorem 2.7.18.

Theorem 22. *If ϕ^t is a flow on a compact metric space (X, d) and $x, y \in X$, then the following statements are equivalent.*

1. *The points x and y are chain equivalent with respect to ϕ^t.*

2. *For every $\varepsilon > 0$ and $T > 0$ there exists an $(\varepsilon, 1)$-chain*

$$(x_0, \ldots, x_n; t_0, \ldots, t_{n-1})$$

from x to y such that

$$t_0 + \cdots + t_{n-1} \geq T,$$

and there exists an $(\varepsilon, 1)$-chain

$$(y_0, \ldots, y_m; s_0, \ldots, s_{m-1})$$

from y to x such that

$$s_0 + \cdots + s_{m-1} \geq T.$$

3. *For every $\varepsilon > 0$ there exists an $(\varepsilon, 1)$-chain from x to y and a $(\varepsilon, 1)$-chain from y to x.*

4. *The points x and y are chain equivalent with respect to ϕ^1.*

Notice then, that by this theorem, for chain sets E such that $E = \pi_M(\mathcal{E})$, where $\mathcal{E}$ is the lift of E, it is sufficient to take Definition 11 and consider all chains where all t_i's take the value 1. However, this may not be true for all chain sets in general, as there exist chain sets E such that $E \neq \pi_M(\mathcal{E})$.

Theorem 23. *If G is a complete graph, then given a chain set $E \subset M$, $\ell(E)$ is chain transitive.*

Proof. Let $E \subset M$ be a chain set, and let $x, y \in E$ By definition of a chain set, there exist $f, g \in \Delta$ such that $\varphi(t, f, x) \in E$ and $\varphi(t, g, y) \in E$ for all $t \in \mathbb{R}$. As defined in [3],

$$d(f,g) = \sum_{i=-\infty}^{\infty} \left(\int_i^{i+1} \delta(f,g,t)dt \right) * 4^{-|i|}$$

where

$$\delta(f,g,t) = \begin{cases} 1 & f(t) \neq g(t) \\ 0 & f(t) = g(t) \end{cases}.$$

There exists $N \in \mathbb{N}$ such that

$$2 \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon/2.$$

E being a chain set means there exists $k \in \mathbb{N}$ and $x_0, \ldots, x_k \in M$, $f_0, \ldots, f_{k-1} \in \Delta$, $t_0, \ldots, t_{k-1} > T$ with $x_0 = \varphi(2T, f, x)$ and $x_k = \varphi(-T, y, g)$ with

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) < \varepsilon.$$

Without loss of generality, let $t > 1$. Then by Theorem 22, we can set

$$t_0 = \cdots = t_{k-1} = 1.$$

Define

$$t_{-2} = N, \;\; x_{-2} = x, \;\; g_{-2} = f$$

$$t_{-1} = t, x_{-1} = \varphi(N, f, x), g_{-1} = \begin{cases} f(t_{-2} + t) & t \leq t_1 \\ f_0(t - t_{-1}) & t > t_1 \end{cases}$$

Let $t_0, \ \ldots, \ t_{k-1}$ and $x_0, \ldots, x_k$ be given as before, and let

$$t_k = N, \ x_{k+1} = y, \ g_{k+1} = g.$$

Now, for $j = 0, \ldots, k-2$ we define

$$g_j(t) = \begin{cases} g_{j-1}(t_{j-1} + t) & t \leq 0 \\ f_j(t) & 0 < t \leq t_j \\ f_{j+1}(t - t_j) & t > t_j \end{cases}$$

$$g_{k-1} = \begin{cases} g_{k-2}(t_{k-2} + t) & t \leq 0 \\ f_{k-1}(t) & 0 < t \leq t_{k-1} \\ g(t - t_{k-1} - N) & t > t_{k-1} \end{cases}$$

$$g_k = \begin{cases} g_{k-1}(t_{k-1} + t) & t \leq 0 \\ g(t - N) & t > 0 \end{cases}$$

We then claim that, by construction, all g_j's are elements of Δ. In [3] we discuss how functions in Δ require that

$$\{f(i)\}_{i \in \mathbb{Z}} \in \Omega$$

for all $f \in \Delta$. Since the graph G is complete, clearly for each f_i the jumps between vertices are allowed by the graph. The "stitching" together of pieces of the functions f_i's is also allowed by the graph G associated Δ because of the completeness of G.
We further require that the functions g_i be piecewise constant on intervals of length 1. By setting $t_j = N$ for all $j \in \{-2, -1, \ldots, k-1\}$ this property is satisfied as well. Thus, $g_j \in \Delta$ for all j.

We further claim that for all $j = -2, -1, \ldots, k$,

$$d(g_j(t_j + \cdot), g_{j+1}) < \varepsilon.$$

By choice of N, on has that for all $d_1, d_2 \in \Delta$

$$\begin{aligned} d(d_1, d_2) &= \sum_{i=-\infty}^{\infty} \left(\int_i^{i+1} \delta(d_1, d_2, t) dt \right) * 4^{-|i|} \\ &\leq \sum_{i=-N}^{N} \left[\left(\int_i^{i+1} \delta(d_1, d_2, t) dt \right) * 4^{-|i|} \right] + \varepsilon/2 \end{aligned}$$

Thus it suffices to show that for the considered functions, the integrands vanish. Notice by definition, for all $i \in \{-2, -1, \ldots, k-1\}$, $g_i(t+N) = g_{i+1}$ for all $-N < t < N$. Thus, $\delta(g_i(t+N), g_{i+1}(t), t) = 0$ for all $-N < t < N$, and therefore,

$$\int_i^{i+1} \delta(f, g, t) dt$$

for all $i \in \{-N, \ldots, N-1\}$. Thus for all $j = -2, -1, \ldots, k$,

$$d(g_j(t_j + \cdot), d_{j+1}) < \varepsilon.$$

$\square$

Thus, given a complete N-graph, if $\mathcal{E} \subset \Delta \times M$ is a chain transitive set, Theorem 20 demonstrates its projection onto M, $\pi_M \mathcal{E}$ is a chain set, and furthermore, the lift of that projection, $\ell(\pi_M \mathcal{E})$, is in fact equal to $\mathcal{E}$ and is therefore a chain transitive set. It then follows that all chain sets in M that are projections of chain transitive sets in $\Delta \times M$ are then also chain transitive sets once lifted up into $\Delta \times M$. However, not all chain sets of M may be projections of chain transitive sets in $\Delta \times M$, and therefore it is important to recognize that this result may not extend to all chain sets.

This theorem lends itself to the proof of the following theorem:

Theorem 24. *If G is a complete graph, then given a maximal invariant chain transitive set $\mathcal{E} \subset \Delta \times M$, $\mathcal{E} = \ell(\pi_M \mathcal{E})$.*

Proof. Because Lemma 21 shows that $\mathcal{E} \subseteq \ell(\pi_M \mathcal{E})$, it remains to show that $\ell(\pi_M \mathcal{E}) \subseteq \mathcal{E}$. Since $\mathcal{E}$ is a chain transitive set, by Theorem 20, $\pi_M \mathcal{E}$ is a chain set. Thus, by Theorem 23, $\ell(\pi_M \mathcal{E})$ is chain transitive. Since $\mathcal{E} \subseteq \ell(\pi_M \mathcal{E})$ and $\ell(\pi_M \mathcal{E})$ is chain transitive, it follows that $\mathcal{E} \subseteq \ell(\pi_M \mathcal{E})$, since $\mathcal{E}$ is maximal with respect to chain transitivity. □

References

[1] J. Ackerman, K. Ayers, E. J. Beltran, J. Bonet, D. Lu, and T. Rudelius, *A behavioral characterization of discrete time dynamical systems over directed graphs*, Qualitative Theory of Dynamical Systems, 13 (2014), pp. 161–180.

[2] J. Alongi and G. Nelson, *Recurrence and Topology*, Graduate studies in mathematics, American Mathematical Society, 2007.

[3] K. Ayers, X. Garcia, J. Kunze, T. Rudelius, A. Sanchez, S. Shao, and E. Speranza, *Limit and morse sets for deterministic hybrid systems*, Qualitative Theory of Dynamical Systems, 12 (2013), pp. 335–360.

[4] G. Birkhoff, *Quelque théorèms sur le movement des systèms dynamiques*, Bulletin de la Société Mathématique de France, 40 (1912), pp. 305–323.

[5] F. Colonius and W. Kliemann, *Dynamical Systems and Linear Algebra:*, Graduate Studies in Mathematics, American Mathematical Society, 2014.

[6] C. Conley, *Isolated Invariant Sets and the Morse Index*, American Mathematical Society, 1978.

[7] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1997.

[8] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste Vols. 1-3*, Gautier-Villars, 1892.

[9] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1999.

[10] L. Steen and J. Seebach, *Counterexamples in Topology*, Dover Books on Mathematics, Dover Publications, 2013.