SEARCH FOR INVARIANT SETS OF THE GENERALIZED TENT MAP

Abstract. The article uses the predictive control method to search for periodic orbits of the generalized tent map. The main problem is that the invariant set containing periodic orbits as a subset is a repeller and set of the Cantor type. Therefore, to find a periodic orbit, a simple local stabilization of the orbit may not be enough due to the small measure of the basin of attraction. It is shown that for certain values of the control parameter, not only the local behavior of solutions changes in the controlled system, but also the global one, namely, an interval or the entire real axis becomes an invariant set. The computational particularities of using the control system are considered, the necessary conditions for the found orbit to be periodic are given. The question about local asymptotic stability of subcycles of the controlled system's stable cycles is fully investigated. Some statistical properties of the subset of the classical Cantor set, which is determined by the periodic points of the generalized tent map, are studied.

Keywords. Generalized tent map, predictive control, periodic orbits, local stabilization, invariant sets.

Introduction

Condense this introduction to one paragraph

In the seminal paper [1] *R. Lozi* noted that numerical computations using computers play a central role in analyzing solutions of nonlinear dynamical systems, that computer-aided proofs are complex and necessarily require additional special validation of the results. Nevertheless, numerous researches in the fields related to chaotic dynamical systems are confident in the numerical solutions that they found using popular software, with no careful check on the reliability of these results, though the researches have to be very cautious to back up theory with numerical computations.

Computationally, computers store numbers in registers and memory cells with a limited number of digits. Thus, the system of real numbers represented in the machine is discrete and finite - irrational numbers, and rational numbers with infinite decimal expansions are rounded to decimal expansions that terminate. This can lead to problems when attempting to find unstable periodic orbits of discrete chaotic dynamical systems.

As an example, in this paper, we consider the *generalized tent map*:

$$x_{n+1} = f(x_n),\ n = 1,\, 2, \dots, \tag{1}$$

where

$$f(x) = H\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right) = \begin{cases} Hx,\ x \le \frac{1}{2}, \\ H(1-x),\ x > \frac{1}{2}, \end{cases} \tag{2}$$

$x \in (-\infty; +\infty)$, $H \ge 2$. Note that, despite the relative simplicity of function (2), equation (1) is of great theoretical importance and can also be used to solve some applied problems [24, 25]. Consider the classical tent map, given by $H = 2$. Note that $x_0 = \frac{2}{3}$ is a fixed point: $f(\frac{2}{3}) = \frac{2}{3}$. Since $|f'(\frac{2}{3})| = 2 > 1$, $\frac{2}{3}$ is an unstable fixed point. Thus, any amount of rounding error will cause an orbit to eventually diverge from this fixed point, and we can see this computationally.

In the present article, it will be shown how we can correct the computational procedure in the problem of finding periodic orbits of a nonlinear discrete system on the example of the tent map.

The dynamics of even the simplest nonlinear discrete systems can be quite complex [3–5]. Such systems are often characterized by extremely unstable motions in phase space, which are defined as chaotic [3]. In dissipative systems, these motions define invariant sets, which can be strange attractors or repellers. Trajectories on such invariant sets have positive Lyapunov exponents; therefore, these trajectories are exponentially sensitive to the initial conditions. Invariant sets can be studied by considering unstable periodic orbits, which constitute the framework of these invariant sets. Periodic orbits have a hierarchical structure determined by their length, which makes it possible to calculate various characteristics of invariant sets and their subsets, for example, topological dimension and entropy [6]. However, applying numerical methods of searching for periodic orbits (periodic points that determine the orbit) encounters a number of fundamental problems. Due to the sensitivity to initial conditions and rounding errors, after several calculation steps, the results can vary greatly depending on the chosen calculation accuracy, the so-called "butterfly effect" occurs. Even with a real possibility to choose a very high accuracy of calculations, we will never be able to say with certainty what we actually found: a long cycle, a pseudo-cycle or a strange attractor [1].

There exist various methods of searching for periodic orbits of a non-linear discrete system, which can be divided into two groups: methods that do not use the correction of the original discrete dynamical system, for example, the method of interval arithmetic analysis [7–9], the method connected with the construction of special Hamiltonian systems [6]; and methods based on local stabilization of an unknown periodic orbit of a given length [10–15]. The second group of methods is more preferable in the sense that with the growth of calculations, their accuracy increases due to the correction of the original dynamical system. If with the help of the control action we can locally stabilize the orbit, then the trajectories of the system will remain in the neighborhood of the orbit and will be attracted to it, i.e. the periodic orbit will be found. By choosing different initial points, different periodic orbits can be found. To solve the problems of stabilization and search of periodic orbits, there were proposed various control schemes that use information about the states of the controlled system at previous points in time [16–19] (delayed control) or about the states of the initial system at future points in time [20–23] (predictive control).

The purpose of this article is to illustrate the effectiveness of the predictive control method using the example of the generalized tent map. The invariant sets of the generalized tent map are repellers that have the structure of the Cantor set. It is shown that in a controlled system, along with a change in the local behavior of solutions, the global behavior also changes. The invariant sets of the controlled system have been found, this is either an interval or the union of intervals. Locally asymptotically stable periodic orbits are subsets of the invariant set. The basins of attraction of these orbits are estimated.

The paper is organized as follows. In Section 1, we present the main Theorem of the work [23], which substantiates the generalized predictive control scheme. Section 2 poses the problem of finding a cycle of length T, and Section 3 provides necessary and sufficient conditions for the local asymptotic stability of this cycle. Section 4 is the main one in the present work, it provides conditions under which all solutions of the controlled system are bounded (Theorem 2), or solutions with initial values from a given interval are bounded (Theorem 3). It was also noted that the cycles do not lose the property of local asymptotic stability. In the fifth section, some computational particularities of using the control system for finding cycles are considered, the necessary conditions for the found periodic sequence to be a cycle are given. In the sixth section, using the example of stabilization of cycles of small lengths, we consider the relationship between the geometry of the original map and the map determined by the controlled system. In Section 7, the question about local asymptotic stability of subcycles of the controlled system's stable cycles is fully investigated (Theorem 4). Section 8 studies subsets of the classical Cantor set. The Cantor set is characterized by points of the first and second type. The points that are end-points of the adjacent to the Cantor set intervals are called points of the first type. All the other points of the set are called points of the second type. Among the points of the second type, we can select a subset consisting of all cyclic points of the generalized tent map. For specific values of the period, a set of periodic points and a graph of the corresponding distribution density function are constructed. The graph is compared with the graph of the density function for the distribution of the Cantor set's points of the first type.

1. Preliminary results

Background section starts here.

We consider the nonlinear discrete system

$$x_{n+1} = f(x_n),\ x_n \in R^m,\ n = 1,\, 2, \dots, \tag{3}$$

where $f(x)$ is a differentiable vector function of the corresponding dimension. It is assumed that this system has one or several unstable T-cycles $(\eta_1, \dots, \eta_T)$, where all the vectors $\eta_1, \dots, \eta_T$ are different, i.e. $\eta_{j+1} = f(\eta_j),\ j = 1, \dots, T-1,\ \eta_1 = f(\eta_T)$. Cycle vectors are called cyclic points and constitute a periodic orbit. The multipliers of the considered unstable cycles are defined as the eigenvalues of the products of Jacobi matrices $\prod_{j=1}^{T} f'(\eta_{T-j+1})$ of dimension $m \times m$ at the points of the cycle. The matrix $\prod_{j=1}^{T} f'(\eta_{T-j+1})$ is called the Jacobi matrix of the cycle $(\eta_1, \dots, \eta_T)$. The collection of all multipliers $\{\mu_1, \dots, \mu_m\}$ is called the spectrum of the Jacobi matrix. As a rule, the cycles $(\eta_1, \dots, \eta_T)$

of system (3) are not known a priori. Consequently, the spectrum is not known either.

Consider the control system

$$x_{n+1} = F\left(\vartheta_1 x_n + \sum_{j=2}^{N} \vartheta_j f^{((j-1)T)}(x_n)\right). \tag{4}$$

The numbers $\vartheta_1, \dots, \vartheta_N$ are real. It is easy to verify that when $\sum\limits_{j=1}^{N} \vartheta_j = 1$, system (4) also has a cycle $\{\eta_1, \dots, \eta_T\}$. The problem is to choose a parameter N and coefficients $\vartheta_1, \dots, \vartheta_N$ so that the cycle $\{\eta_1, \dots, \eta_T\}$ of system (4) is locally asymptotically stable.

Theorem 1 ([23])**.** *Suppose $f \in C^1$ and that system (3) has an unstable T-cycle with multipliers $\{\mu_1, \dots, \mu_m\}$. Then this cycle will be a locally asymptotically stable cycle of system (4) if*

$$\mu_j \left[r(\mu_j)\right]^T \in D, \ j = 1, \dots, m,$$

where $D = \{z \in C : \ |z| < 1\}$ is an open central unit circle on the complex plane, $r(\mu) = \sum\limits_{j=1}^{N} \vartheta_j \mu^{j-1}$.

2. Problem statement

A set U is called invariant for equation (1) if for any $x_0 \in U$ it follows that $f^{(k)}(x_0) \in U$, $k = 1, 2, \dots$. It is shown in [26] that at $H = 3$ the invariant set of equation (1) is the classical Cantor set. Analogously, it can be shown that when $H > 2$, the equation (1) invariant set is a set of the Cantor type, that is: closed, with cardinality of the continuum, with zero Lebesgue measure. Note that each point of the invariant set can be represented in the form $\sum\limits_{j=1}^{\infty} \frac{\alpha_j}{H^j}$, where $\alpha_j \in \{0, H-1\}$. This set includes a countable subset of all periodic points of map (2). If x_0 does not belong to the invariant set, then the corresponding sequence $\left\{f^{(k)}(x_0)\right\}_{k=1}^{\infty}$ tends to $-\infty$. Such invariant sets are called repellers of map (2). The set $\{\eta_1, \dots, \eta_T\}$ is called a T-cycle of map (2), if the numbers $\eta_1, \dots, \eta_T$ are different, and $\eta_{j+1} = f(\eta_j)$, $j = 1, \dots, T-1$, $\eta_1 = f(\eta_T)$, and in this case, each point of the T-cycle is called a T-periodic point. The problem is: for given T, to find T-periodic points of map (2).

3. Control system and local behavior of the control system trajectories

To solve the stated problem, we will use the predictive control method. Along with equation (1), consider the equation

$$x_{n+1} = F(x_n),\ n = 1, 2, \dots, \tag{5}$$

where $F(x) = f\left(\vartheta x + (1-\vartheta) f^{(T)}(x)\right)$, ϑ is some real number, called the control parameter and to be determined later. Equation (5) we will call the control system for equation (1).

Let $\{\eta_1, \dots, \eta_T\}$ be the equation (1) cycle. Since $\vartheta\eta_k + (1-\vartheta) f^{(T)}(\eta_k) = \eta_k$, then $F(\eta_k) = f(\eta_k)$, which means that the cycle of the equation (1) will also be the cycle of equation (5). Note that the converse statement is generally not true.

The multiplier of the equation (1) cycle is defined by the formula

$$\mu = f'(\eta_T) \cdot \ldots \cdot f'(\eta_1).$$

Since $|f'(\eta_j)| = H$, then $|\mu| = H^T > 1$, that is, any cycle of equation (1) is unstable. Let us find the value of the multiplier λ of the same cycle $\{\eta_1, \dots, \eta_T\}$, but for equation (5). From *Theorem 1* we get that

$$\lambda = \mu\left(\vartheta + (1-\vartheta)\mu\right)^T.$$

In what follows, we will separate two cases: $\mu > 0$ and $\mu < 0$. Let $\mu = H^T$. Then the condition for local asymptotic stability of the T-cycle of equation (5) is: $|\lambda| = \left|H^T\left(\vartheta + (1-\vartheta)H^T\right)^T\right| < 1$, from which it follows that

$$1 < \frac{H^T - \frac{1}{H}}{H^T - 1} < \vartheta < \frac{H^T + \frac{1}{H}}{H^T - 1} < \frac{5}{2}. \tag{6}$$

If $\mu = -H^T$, then the condition for local asymptotic stability of the equation (5) cycle is: $|\lambda| = \left|H^T\left(\vartheta - (1-\vartheta)H^T\right)^T\right| < 1$, from which

$$\frac{1}{2} < \frac{H^T - \frac{1}{H}}{H^T + 1} < \vartheta < \frac{H^T + \frac{1}{H}}{H^T + 1} < 1. \tag{7}$$

This isn't really a lemma, it is a statement of sufficient conditions on θ for the stability of the cycles.

Lemma 1. *Given a T-cycle of equation (1) with the multiplier μ. This cycle will be a locally asymptotically stable cycle of equation (5) if inequalities (6), in the case $\mu > 0$, or inequalities (7), when $\mu < 0$, are satisfied.*

4. Global behavior of the control system trajectories

If there are locally asymptotically stable cycles in equation (5), then the invariant set of this equation also includes the basins of attraction of these cycles. Since the basin of attraction of a cycle is an open set, its measure is positive. However, there remains the question about how large this measure is. This question is important for choosing the initial point for which the corresponding orbit should eventually get into the basin of attraction the desired cycle. Thus, as we passing from equation (1) to equation (5), there is ensured a change in the nature of the periodic orbit (the set of points of the cycle) which turns from a repeller into an attractor, but it is also important to additionally provide a sufficiently large measure of the basin of attraction.

Below we will establish the properties of invariant sets of equation (5) and the global behavior of its solutions. The properties are formulated as lemmas. From these properties, two main theorems, allowing us to find all cycles of arbitrary lengths with any given accuracy, will follow.

Here is the result of the paper.

Theorem 2. *If inequalities (6) are satisfied, then any solution of equation (5) is bounded. Moreover, any T-cycle $\{\eta_1, \dots, \eta_T\}$ of this equation, for which the quantity $\mu = f'(\eta_T) \cdot \ldots \cdot f'(\eta_1)$ is positive, is locally asymptotically stable.*

Theorem 3. *If the inequalities $\frac{H^T - \frac{1}{H}}{H^T+1} < \vartheta \leq \frac{H^T}{H^T+1}$ are satisfied, the set $\left[0, \frac{H}{2}\right]$ is an invariant set of equation (5). Moreover, any T-cycle $\{\eta_1, \dots, \eta_T\}$ of this equation, for which the quantity $\mu = f'(\eta_T) \cdot \ldots \cdot f'(\eta_1)$ is negative, is locally asymptotically stable.*

We could illustrate these two cases, and maybe the subcases, with some figures. Relegate the algebra in the following lemmas to an appendix??

We proceed to formulate and prove the main lemmas.

Let inequalities (6) be satisfied. Divide conditions (6) into three cases: $\vartheta = \frac{H^T}{H^T-1}$, $\frac{H^T-\frac{1}{H}}{H^T-1} < \vartheta < \frac{H^T}{H^T-1}$, $1 < \frac{H^T}{H^T-1} < \vartheta < \frac{H^T+\frac{1}{H}}{H^T-1}$.

Lemma 2. *Let $\vartheta = \frac{H^T}{H^T-1}$. Then, if $x_0 \le 0$, it follows that $F(x_0) = 0$, hence, $F^{(k)}(x_0) = 0$ when $k = 2, 3, \ldots$; if $x_0 \ge 1$, then $F(x_0) < 0$, hence, $F^{(k)}(x_0) = 0$ when $k = 2, 3, \ldots$; if $x_0 \in (0, 1)$, then $\left\{F^{(k)}(x_0)\right\}_{k=2}^{\infty} \in [0, 1]$.*

Proof. Let $x_0 \le 0$, then $f(x_0) = Hx_0 \le 0$, $f^{(2)}(x_0) = H^2 x_0 \le 0$, $\ldots$, $f^{(T)}(x_0) = H^T x_0$. Find

$$\zeta = \vartheta x_0 + (1-\vartheta) f^{(T)}(x_0) = \frac{1}{H^T-1}\left(H^T x_0 - H^T x_0\right) = 0.$$

Hence, $F(x_0) = f(\zeta) = 0$.

Let $x_0 \ge 1$, then $f(x_0) = H(1-x_0) \le 0$, $f^{(2)}(x_0) = H^2(1-x_0) \le 0$, $\ldots$, $f^{(T)}(x_0) = H^T(1-x_0)$, $\zeta = \vartheta x_0 + (1-\vartheta) f^{(T)}(x_0) = \frac{1}{H^T-1}\left(H^T x_0 - H^T(1-x_0)\right) = \frac{H^T}{H^T-1}(2x_0-1) > 1$, $F(x_0) = f(\zeta) = H(1-\zeta) < 0$. Hence, $F^{(2)}(x_0) = 0$.

The last case remains: at $x_0 \in (0, 1)$, either starting from some number $k > k_0$, $F^{(k)}(x_0) = 0$, or $\left\{F^{(k)}(x_0)\right\}_{k=2}^{\infty} \in (0, 1)$. The lemma is proved. □

The case $\frac{H^T - \frac{1}{H}}{H^T-1} < \vartheta < \frac{H^T}{H^T-1}$ is considered in a similar way.

Lemma 3. *Let $\frac{H^T - \frac{1}{H}}{H^T-1} < \vartheta < \frac{H^T}{H^T-1}$. Then, if $x_0 \le 0$ or $x_0 \ge 1$, $F^{(k)}(x_0) \xrightarrow[k\to\infty]{} 0$; when $x_0 \in (0, 1)$, either $F^{(k)}(x_0) \xrightarrow[k\to\infty]{} 0$ or $\left\{F^{(k)}(x_0)\right\}_{k=1}^{\infty} \in (0, 1)$.*

Proof. Let $x_0 \le 0$, then $f(x_0) = Hx_0 \le 0$, $\ldots$, $f^{(T)}(x_0) = H^T x_0$. Find

$$\zeta = \vartheta x_0 + (1-\vartheta) f^{(T)}(x_0) = \left(H^T - \vartheta(H^T-1)\right) x_0.$$

It follows from the inequality $\frac{H^T - \frac{1}{H}}{H^T-1} < \vartheta < \frac{H^T}{H^T-1}$ that, first, $\zeta \le 0$, and $F(x_0) = f(\zeta) = H\zeta = H\left(H^T - \vartheta(H^T-1)\right) x_0 \le 0$, and second, $0 < \alpha = H\left(H^T - \vartheta(H^T-1)\right) < 1$, from which $F(x_0) = -\alpha|x_0| > -|x_0|$. Then $\left|F^{(k)}(x_0)\right| = \alpha^k |x_0| \xrightarrow[k\to\infty]{} 0$.

Let $x_0 \ge 1$, then $f(x_0) = H(1-x_0) \le 0$, $\ldots$, $f^{(T)}(x_0) = H^T(1-x_0)$,

$$\zeta = \vartheta x_0 + (1-\vartheta) f^{(T)}(x_0) = x_0 + (\vartheta - 1)\left(x_0 + H^T(x_0-1)\right) > x_0 > 1.$$

Then $F(x_0) = f(\zeta) = H(1-\zeta) < 0$. Hence, $F^{(2)}(x_0) = -\alpha\left|F(x_0)\right| > -\left|F(x_0)\right|$, $\left|F^{(k)}(x_0)\right| = \alpha^{k-1}\left|F(x_0)\right| \xrightarrow[k\to\infty]{} 0$.

Let $x_0 \in (0, 1)$. If for some k_0 $F^{(k_0)}(x_0) \le 0$ or $F^{(k_0)}(x_0) \ge 1$, then $F^{(k)}(x_0) \xrightarrow[k\to\infty]{} 0$. Otherwise, $\left\{F^{(k)}(x_0)\right\}_{k=1}^{\infty} \in (0, 1)$. The lemma is proved. □

The case $\frac{H^T}{H^T-1} < \vartheta < \frac{H^T+\frac{1}{H}}{H^T-1}$ remains.

Lemma 4. *Let $\frac{H^T}{H^T-1} < \vartheta < \frac{H^T+\frac{1}{H}}{H^T-1}$. In this case, if $-H^2+\frac{H}{2} \le x_0 \le \frac{H}{2}$, then*

$$-H^2 + \frac{H}{2} \le F^{(k)}(x_0) \le \frac{H}{2}, \ k = 1, 2, \dots . \tag{8}$$

If $x_0 > \frac{H}{2}$ or $x_0 < -H^2 + \frac{H}{2}$, there exists a number $k_0 \le 0$ such that inequalities (8) are satisfied for all k greater than k_0.

Proof. Let $x_0 \le 0$, then $f(x_0) = Hx_0 \le 0, \dots, f^{(T)}(x_0) = H^T x_0$,

$$\zeta = \left(H^T - \vartheta(H^T - 1)\right) x_0.$$

The inequality $\frac{H^T}{H^T-1} < \vartheta < \frac{H^T+\frac{1}{H}}{H^T-1}$ is equivalent to $-\frac{1}{H} < H^T - \vartheta(H^T - 1) < 0$, from which it follows that $0 \le \zeta \le \frac{|x_0|}{H}$. If $-H \le x_0 \le 0$, then $0 \le \zeta \le 1$ and $0 \le F(x_0) \le \frac{H}{2}$.

If $x_0 < -H$ and $\zeta \ge 1$, then $0 \ge F(x_0) = H(1-\zeta) > H - |x_0|$. Moreover, if $F^{(k)}(x_0) < -H$ and $\left(H^T - \vartheta(H^T-1)\right) F^{(k)}(x_0) \ge 1$, then $0 \ge F^{(k+1)}(x_0) > H - \left|F^{(k)}(x_0)\right| > (k+1)H - |x_0|$. It means that there exists such k_0 that $-H \le F^{(k_0)}(x_0) \le 0$. And hence, $0 \le F^{(k_0+1)}(x_0) \le \frac{H}{2}$.

Let $0 \le x_0 \le \frac{H}{2}$, then $H^T\left(1 - \frac{H}{2}\right) \le f^{(T)}(x_0) \le \frac{H}{2}$. Since $\vartheta > 1$, then

$$\begin{aligned}
-\frac{3}{4}H < \vartheta x_0 - (\vartheta - 1)\frac{H}{2} \le \zeta \le \vartheta x_0 + (\vartheta-1)H^T\left(\frac{H}{2} - 1\right) < \\
< \vartheta\frac{H}{2} + (\vartheta - 1)H^T\left(\frac{H}{2} - 1\right) < \\
< \frac{1}{H^T-1}\left(\left(H^T + \frac{1}{H}\right)\frac{H}{2} + \left(1 + \frac{1}{H}\right)H^T\left(\frac{H}{2} - 1\right)\right) = \\
= \frac{1}{H^T-1}\left(H^{T+1} - \frac{H^T}{2} - H^{T-1} + \frac{1}{2}\right).
\end{aligned}$$

Note that $1 < H-1 < \frac{1}{H^T-1}\left(H^{T+1} - \frac{H^T}{2} - H^{T-1} + \frac{1}{2}\right) < H + \frac{1}{2}$. Then

$$F(x_0) > H\left(1 - \frac{1}{H^T-1}\left(H^{T+1} - \frac{H^T}{2} - H^{T-1} + \frac{1}{2}\right)\right) > -H\left(H - \frac{1}{2}\right).$$

Therefore, if $0 \le x_0 \le \frac{H}{2}$, then $-H^2 + \frac{H}{2} \le F(x_0) \le \frac{H}{2}$.

Let $x_0 > \frac{H}{2}$, then $f^{(T)}(x_0) = H^T(1-x_0)$, $\zeta = \vartheta x_0 + (\vartheta - 1)H^T(x_0 - 1) > 1$, and $F(x_0) < 0$. It means that for some k_0 the inequality $0 \le F^{(k_0)}(x_0) \le \frac{H}{2}$ is satisfied.

Summing up the above four cases, we get that when $-H^2+\frac{H}{2}\le x_0\le\frac{H}{2}$, inequalities (8) hold. If $x_0>\frac{H}{2}$ or $x_0<-H^2+\frac{H}{2}$, then inequalities (8) will be satisfied starting from some number. The lemma is proved. $\square$

From Lemmas 1, 2, 3, 4, the assertion of Theorem 2 follows.

Let us now pass to the study of the global behavior of the T-cycles $\{\eta_1,\dots,\eta_T\}$ of equation (5) for which the quantity $\mu=f'(\eta_T)\cdot\ldots\cdot f'(\eta_1)$ is negative, assuming that conditions (7) are satisfied.

Lemma 5. *Let conditions (7) be satisfied and $x_0<0$. Then $F^{(k)}(x_0)\xrightarrow[k\to\infty]{}-\infty$.*

Proof. Since $x_0<0$, then $f(x_0)=Hx_0\le 0,\ \dots,\ f^{(T)}(x_0)=H^Tx_0$, and

$$\zeta=\vartheta x_0+(1-\vartheta)f^{(T)}(x_0)=\left(\vartheta+(1-\vartheta)H^T\right)x_0.$$

Since $\frac{1}{2}<\vartheta<1$, then $\zeta<x_0$, $F(x_0)=H\zeta<Hx_0$, $F^{(k)}(x_0)<H^kx_0$ when $k=1,2,\dots$, from which the conclusion of the lemma follows. $\square$

Lemma 6. *Let the inequalities $\frac{H^T-\frac{1}{H}}{H^T+1}<\vartheta\le\frac{H^T}{H^T+1}$ be satisfied, and $0\le x_0\le\frac{H}{2}$. Then $0\le F(x_0)\le\frac{H}{2}$.*

Proof. Let $1\le x_0\le\frac{H}{2}$, then $f(x_0)=H(1-x_0)\le 0,\ \dots,\ f^{(T)}(x_0)=H^T(1-x_0)$,

$$\zeta=\vartheta x_0+(1-\vartheta)f^{(T)}(x_0)=\frac{1}{H^T+1}\left((H^T+\alpha)x_0+(1-\alpha)H^T(1-x_0)\right),$$

where $\vartheta=\frac{H^T+\alpha}{H^T+1}$, $-\frac{1}{H}<\alpha\le 0$. Since $0\le x_0\le\frac{H}{2}$, then

$$\begin{aligned}\zeta&=\frac{1}{H^T+1}\left(H^T+\alpha(x_0+x_0H^T-H^T)\right)>\\&>\frac{1}{H^T+1}\left(H^T-\frac{1}{H}\left(\frac{H}{2}+\frac{H}{2}H^T-H^T\right)\right)=\\&=\frac{1}{H^T+1}\left(\frac{1}{2}H^T-\frac{1}{2}+H^{T-1}\right)>\frac{1}{2}.\end{aligned}$$

On the other side, $\zeta\le\frac{H^T}{H^T+1}<1$. Hence, $0<F(x_0)\le\frac{H}{2}$.

Let $0\le x_0<1$, $p\in\{1,\dots,T\}$ is the smallest number at which $f^{(p-1)}(x_0)<1$, but $f^{(p)}(x_0)>1$ (here we mean that $f^{(0)}(x)=x$). It is clear that $f^{(p)}(x_0)\le\frac{H}{2}$.

Note that

$$\frac{1}{H}f^{(p)}(x_0) \le f^{(p-1)}(x_0) \le 1 - \frac{1}{H}f^{(p)}(x_0),$$

$$\frac{1}{H^2}f^{(p)}(x_0) \le \frac{1}{H}f^{(p-1)}(x_0) \le f^{(p-2)}(x_0) \le 1 - \frac{1}{H}f^{(p-1)}(x_0) \le 1 - \frac{1}{H^2}f^{(p)}(x_0),$$

$$\dots,$$

$$\frac{1}{H^p}f^{(p)}(x_0) \le x_0 \le 1 - \frac{1}{H^p}f^{(p)}(x_0).$$

If $p = T$, then $\zeta > 0$ and, since $x_0 - f^{(T)}(x_0) < 0$,

$$\zeta = \vartheta x_0 + (1-\vartheta)f^{(T)}(x_0) < \frac{1}{H^T+1}\left(\left(H^T - \frac{1}{H}\right)x_0 + \left(1 + \frac{1}{H}\right)f^{(T)}(x_0)\right),$$

and because of (7),

$$\left(H^T - \frac{1}{H}\right)x_0 + \left(1+\frac{1}{H}\right)f^{(T)}(x_0) \le \left(H^T - \frac{1}{H}\right)\left(1 - H^{-T}f^{(T)}(x_0)\right) +$$

$$+\left(1+\frac{1}{H}\right)f^{(T)}(x_0) = H^T - H^{-1} - f^{(T)}(x_0) + H^{-T-1}f^{(T)}(x_0) + f^{(T)}(x_0) +$$

$$+H^{-1}f^{(T)}(x_0) < H^T + \frac{1}{2}H^{-T} + \frac{1}{2} < H^T + 1.$$

This implies $\zeta < 1$.

Let $p < T$. Then $\zeta < 1$. Check the inequality $\zeta > 0$. Since $f^{(T)}(x_0) = H^{T-p}\left(1 - f^{(p)}(x_0)\right) \le 0$, then $x_0 - f^{(T)}(x_0) > 0$ and

$$\zeta = \vartheta x_0 + (1-\vartheta)f^{(T)}(x_0) > \frac{1}{H^T+1}\left(\left(H^T - \frac{1}{H}\right)x_0 + \left(1 + \frac{1}{H}\right)f^{(T)}(x_0)\right).$$

Because of (7),

$$\left(H^T - \frac{1}{H}\right)x_0 + \left(1+\frac{1}{H}\right)f^{(T)}(x_0) \ge \left(H^T - \frac{1}{H}\right)H^{-p}f^{(p)}(x_0) +$$

$$+\left(1+\frac{1}{H}\right)H^{T-p}\left(1 - f^{(p)}(x_0)\right) = H^{T-p}f^{(p)}(x_0) - H^{T-p-1}f^{(p)}(x_0) +$$

$$+\left(1+\frac{1}{H}\right)H^{T-p} - \left(H^{T-p} + H^{T-p-1}\right)f^{(p)}(x_0) = \left(1+\frac{1}{H}\right)H^{T-p} -$$

$$-2H^{T-p-1}f^{(p)}(x_0) \ge \left(1+\frac{1}{H}\right)H^{T-p} - H^{T-p} > 0.$$

Thus, when $0 \le x_0 < 1$, the inequalities $0 < \zeta < 1$ hold and, therefore, $0 < F(x_0) \le \frac{H}{2}$. The lemma is proved. □

The result of Lemmas 5, 6 implies the assertion of Theorem 3.

Note that when the inequalities $\frac{H^T}{H^T+1} < \vartheta < \frac{H^T+\frac{1}{H}}{H^T+1}$ are satisfied, the invariant set of equation (5) will no longer be a segment, but will be the union of a finite or countable number of intervals, and the measure of this set may be small.

5. Computational particularities of using the control system

This section seems to be about basins of attraction: how to find particular cycles.

Let us consider the computational particularities of the iterative scheme (5) for finding the cycles of equation (1). For cycles with positive multipliers, it is theoretically possible to take any number from the interval $\left(\frac{H^T-\frac{1}{H}}{H^T-1}, \frac{H^T+\frac{1}{H}}{H^T-1}\right)$ as the control parameter; however, if this parameter belongs to the half-interval $\left(\frac{H^T-\frac{1}{H}}{H^T-1}, \frac{H^T}{H^T-1}\right]$, then the trivial equilibrium will have a sufficiently large basin of attraction. Geometrically, this fact will be illustrated below. Therefore, it is reasonable to choose the control parameter closer to $\frac{H^T+\frac{1}{H}}{H^T-1}$. For cycles with negative multipliers, the control parameter can be taken from the interval $\left(\frac{H^T-\frac{1}{H}}{H^T+1}, \frac{H^T}{H^T+1}\right)$. Moreover, in order to find the largest possible number of cycles, the initial value x_0 should belong to the nodes of a sufficiently dense grid of the interval $(0, 1)$.

Note that for large values of T, the control parameter ϑ is close to one, and the value $1-\vartheta$ is close to zero. In this case, the lengths of the intervals of possible changes in the control parameter are equal to $\frac{1}{H(H^T-1)}$ or $\frac{1}{H(H^T+1)}$, i.e. as T grows, they tend to zero exponentially.

Therefore, although the suggested above method for determining cycles, implemented on a computer, theoretically allows us to solve the stated problem, there still remain practical questions: when can we rely on numerical solutions? How to control numerical results? What calculation accuracy should be chosen?

In practice, intermediate calculations should be introduced to control the results. Let the sequence $\{x_n\}_{n=1}^{\infty}$ be a solution of equation (5), and let the quantity $1-\vartheta$ have the order of magnitude 10^{-p}, where p is large enough. Need to figure out exactly what is meant by "point of control". Then the *first point of control* will be the estimate of the residual $U_n = \left\| f\left(\vartheta x_n + (1-\vartheta) f^{(T)}(x_n)\right) - f(x_n) \right\|$. If the sequence $\{x_n\}$ tends to the solution of system (3), then the sequence $\{U_n\}$ tends to zero. However, if the sequence $\{x_n\}$ does not tend to the solution of system (3), then the residual can have the order of magnitude $1-\vartheta \sim$

10^{-p}, i.e. be close to zero. To understand that the residual tends to zero, we have to choose the calculation accuracy $\delta = 10^{-p_1}$, where p_1 should be significantly greater than p. Then the first point of control will be the condition $U_n \sim 10^{-p_1}$, $n \geq n_1$.

The *second point of control* is the check of periodicity of the obtained numerical solution: $\|x_{n+T} - x_n\| \sim 10^{-p_1}$, $n \geq n_1$. Of course, it is additionally necessary to check that T is a cycle proper, but not a subcycle (i.e., a cycle of shorter length) of equation (1).

The fulfilled conditions of the points of control are only necessary for the found sequence $\{x_n\}$ to be a T-cycle of equation (1). The effectiveness of these necessary conditions is due to the fact that they are quite simple to check.

Example. Consider the problem of finding 5-cycles of equation (1) when $H = 4$. Choose two initial conditions and for each of them find one 5-cycle with a positive and negative multiplier respectively. Set $x_0 \in \{0.25, 0.85\}$, $\vartheta \in \left\{\frac{H^T - \frac{0.4}{H}}{H^T+1}, \frac{H^T + \frac{0.4}{H}}{H^T-1}\right\}$. Thus, we got four iterative schemes. For each scheme, when visualizing, choose its color: at $x_0 = 0.25$, $\vartheta = \frac{H^T - \frac{0.4}{H}}{H^T+1} = 0.9989...$ it will be red; at $x_0 = 0.85$, $\vartheta = \frac{H^T - \frac{0.4}{H}}{H^T+1}$ – blue; at $x_0 = 0.25$, $\vartheta = \frac{H^T + \frac{0.4}{H}}{H^T-1} = 1.0010...$ – green; at $x_0 = 0.85$, $\vartheta = \frac{H^T + \frac{0.4}{H}}{H^T-1}$ – black. The chosen calculation accuracy is 10^{-15}. The corresponding cyclic points are shown in Fig. 1, 2. Fig. 3 shows the graphs of the residual for different variation intervals of n. The periodicity condition $\|x_{n+5} - x_n\| = 0$ hold for all four cases, starting from $n = 43$.

Similarly, all cycles of any length can be found. The limitation is the calculation accuracy, which should be chosen approximately $H^{1.05T}$. Fig. 4 represents the cyclic points of four cycles of length 100. The accuracy of calculations was taken 10^{65}. For negative multipliers, the control parameter is chosen $\vartheta = \frac{H^T - \frac{0.4}{H}}{H^T+1}$; for positive ones, $\vartheta = \frac{H^T + \frac{0.4}{H}}{H^T-1} = 1 + 0.68... \cdot 10^{-62}$. Note that the lengths of the intervals of possible changes in the control parameter have the order of magnitude $H^{-(T+1)}$, that is, they are approximately equal to $1.5... \cdot 10^{-61}$. The necessary smallness of the residual value and the condition of periodicity are attained at the 250th step (Fig. 5).

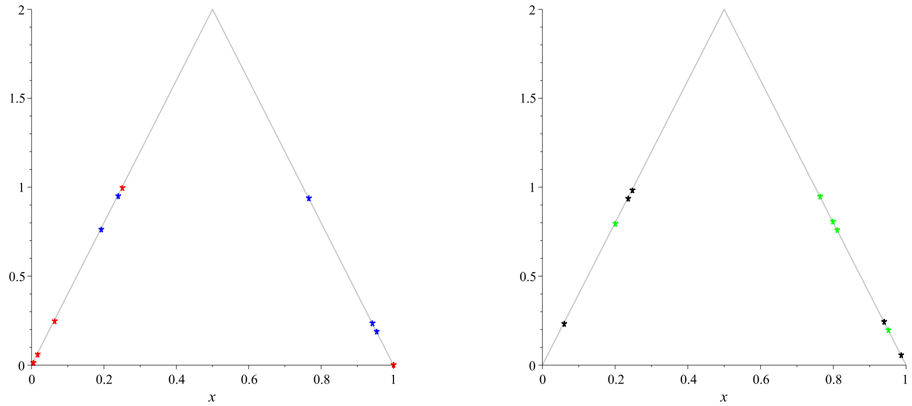


FIGURE 1. Cyclic points of 5-cycles of the tent map in the case of positive (red and blue) and negative (green and black) multipliers

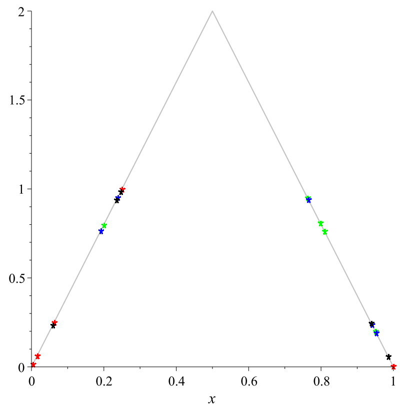


FIGURE 2. Four sets of cyclic points of the tent map 5-cycles

6. Graphical study of the control system map

This section is about how the basins of attraction for the various cycles are distributed. Some examples are visualized.

Equation (1) has two equilibria $\eta = 0$ and $\eta = \frac{H}{H+1}$, moreover, both are unstable; their multipliers are H and $-H$ respectively. Consider the equation (5)

$$x_{n+1} = f\left(\vartheta x_n + (1-\vartheta) f(x_n)\right)$$

for different values of $\vartheta \in \left[0, \frac{H^T + \frac{1}{H}}{H^T - 1}\right]$. For definiteness, set $H = 3$. Then $\left[0, \frac{H^T + \frac{1}{H}}{H^T - 1}\right] = \left[0, \frac{5}{3}\right]$. To stabilize the equilibrium $\eta = 0$, we need to choose the control parameter from the interval $\left(\frac{4}{3}, \frac{5}{3}\right)$, and with an

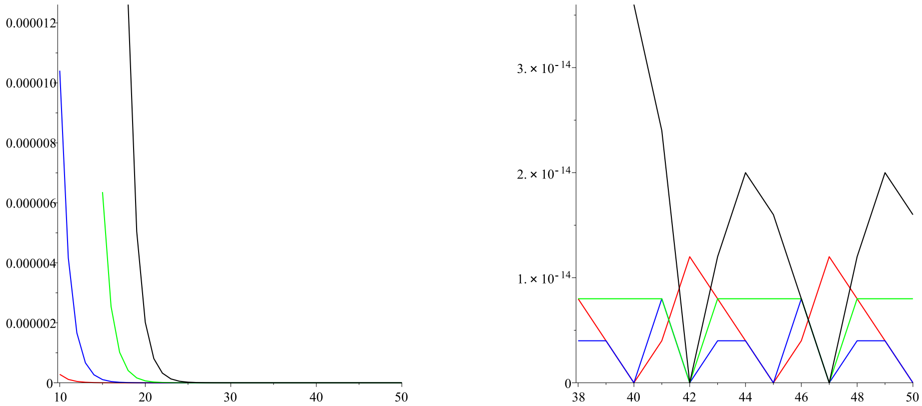


FIGURE 3. Residual for the periodic points of 5-cycles at $n = 10..50$ and $n = 38..50$

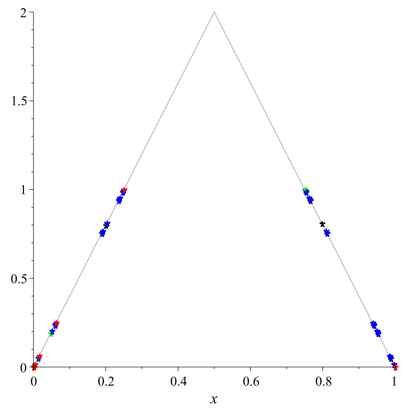


FIGURE 4. Four sets of cyclic points of the tent map 100-cycles

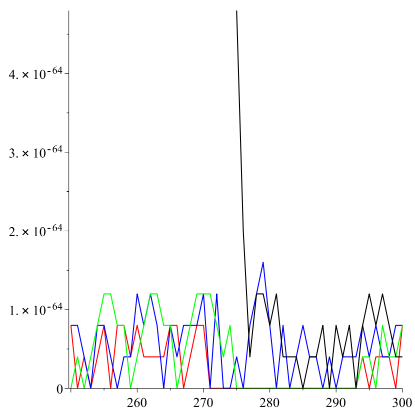


FIGURE 5. Residual for the periodic points of 100-cycles at $n = 250..300$

increase in the control parameter from $\frac{4}{3}$ to $\frac{5}{3}$, the multiplier of this equilibrium decreases from 1 to -1 (when $\vartheta = \frac{3}{2}$, the multiplier equals zero). The graphs of the function $y = f\left(\vartheta x + (1-\vartheta)f(x)\right)$ for different values of ϑ are shown in Fig. 6.

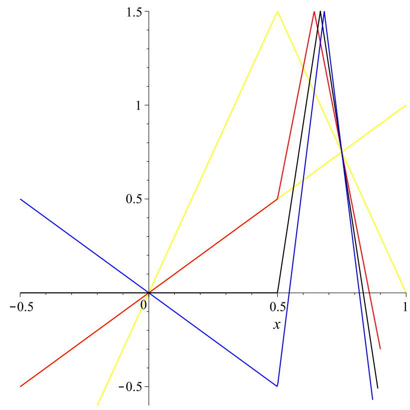


FIGURE 6. Graphs of the function $y = f\left(\vartheta x + (1-\vartheta)f(x)\right)$ at $\vartheta = \frac{4}{3}$ (red); at $\vartheta = \frac{3}{2}$ (black); at $\vartheta = \frac{5}{3}$ (blue); graphs of the functions $y = x$ and $y = f(x)$ are marked in yellow

The equilibrium $\eta = \frac{3}{4}$ will be a locally asymptotically stable equilibrium of equation (5) at $\vartheta \in \left(\frac{2}{3}, \frac{5}{6}\right)$. As the control parameter increases, the multiplier of this equilibrium decreases from 1 to -1. When $\vartheta = \frac{3}{4}$, the multiplier equals zero. The graphs of the function $y = f\left(\vartheta x + (1-\vartheta)f(x)\right)$ for different values of ϑ are depicted in Fig. 7.

If we represent the Lamerey diagram Not sure what this means ... on the graphs of Fig. 6, then we can observe that for any initial value x_0 and $\vartheta \in \left(\frac{4}{3}, \frac{5}{3}\right)$, the corresponding solution of equation (5) will tend to zero. Analogously, Fig. 7 can illustrate that the segment $\left[0, \frac{3}{2}\right]$, under the mapping $y = f\left(\vartheta x + (1-\vartheta)f(x)\right)$ (and at $\vartheta \in \left(\frac{2}{3}, \frac{5}{6}\right)$) goes into itself. Moreover, for any $x_0 \in \left(0, \frac{3}{2}\right)$, the solution tends to the equilibrium $\eta = \frac{3}{4}$. These facts are not difficult to obtain analytically as well.

Let $T = 2$. Equation (1) has the only 2-cycle, and its multiplier is negative. The corresponding equation (5) takes the form $x_{n+1} = f\left(\vartheta x_n + (1-\vartheta)f^{(2)}({}_n x)\right)$. Consider the graphs of the functions $y = F(x)$, $y = F^{(2)}(x)$, where $F(x) = f\left(\vartheta x + (1-\vartheta)f^{(2)}(x)\right)$ at $\vartheta = \frac{H^T}{H^T-1} = \frac{9}{8}$ and $\vartheta = \frac{H^T}{H^T+1} = \frac{9}{10}$ (Fig. 8, Fig. 9 – a), b)).

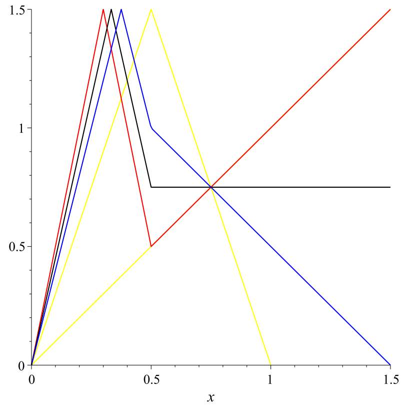


FIGURE 7. Graphs of the function $y = f\left(\vartheta x + (1-\vartheta) f(x)\right)$ at $\vartheta = \frac{2}{3}$ (red); at $\vartheta = \frac{3}{4}$ (black); at $\vartheta = \frac{5}{6}$ (blue); graphs of the functions $y = x$ and $y = f(x)$ are marked in yellow

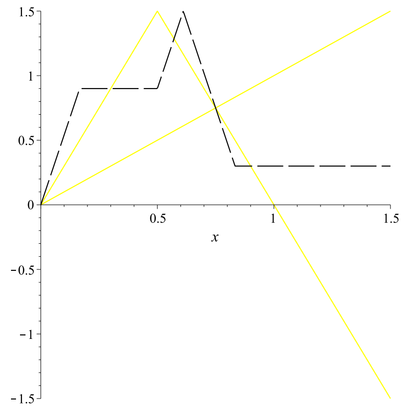
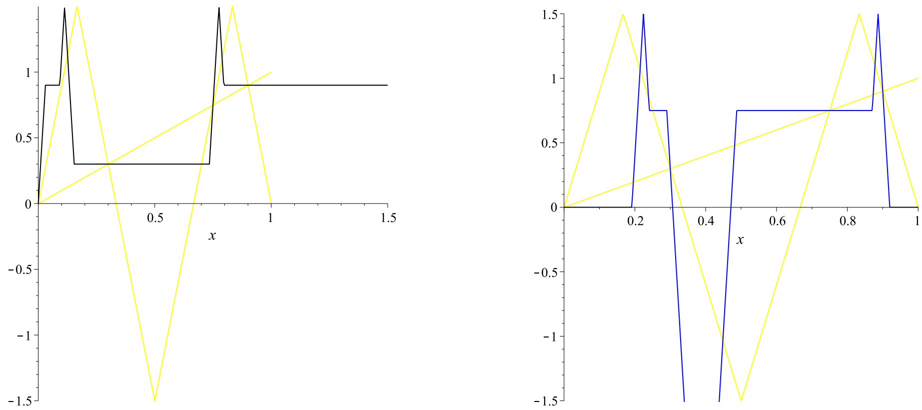


FIGURE 8. Graphs of the function $y = F(x)$ at $\vartheta = \frac{9}{10}$ (black dashed line); $y = x$ and $y = f(x)$ (yellow)

From Fig. 8, we can see that the set $\left[0,\, \frac{3}{2}\right]$ is invariant under the mapping $y = F(x)$, in Fig. 9-a) – that the 2-cycle of equation (5) becomes locally asymptotically stable with the multiplier equal to zero. From Fig. 9-b), it is seen that the 2-cycle of equation (5) is unstable, but both equilibria are locally asymptotically stable (with zero multipliers). An explanation of this fact will be given in the next section.

Consider the global behavior of solutions of equation (5) at $\vartheta = \frac{H^T}{H^T+1}$. The function

$$\zeta(x) = \frac{H^T}{H^T+1} x + \frac{1}{H^T+1} f^{(T)}(x)$$



a) b)

FIGURE 9. Graphs of the function $y = F^{(2)}(x)$ at $\vartheta = \frac{9}{10}$ (black); at $\vartheta = \frac{9}{8}$ (blue); graphs of the functions $y = x$ and $y = f^{(2)}(x)$ are marked in yellow

does not decrease on $(-\infty;\infty)$. Indeed, $\zeta'(x) = \frac{H^T \pm H^T}{H^T+1} = \begin{cases} 0, & x \in \Sigma, \\ \frac{2H^T}{H^T+1}, & x \notin \Sigma, \end{cases}$ where Σ is the set on which the function $f^{(T)}(x)$ decreases. Note that the sets of increasing of the functions $f^{(T)}(x)$ and $\zeta(x)$ coincide. In particular, the function $\zeta(x)$ increases when $x \in \left(\frac{1}{2}, 1 - \frac{1}{H} + \frac{1}{2H^{T-1}}\right)$ from $-\frac{1}{2}H^{T-1}(H-1)$ to $\frac{H}{2}$. The point $\left(\frac{1}{2}, -\frac{1}{2}H^{T-1}(H-1)\right)$ is the only point of the global minimum of the function $f^{(T)}(x)$. Fig. 10 presents the graphs of the functions $f^{(T)}(x)$ and $\zeta(x)$ at $H = 3$, $T = 3$.

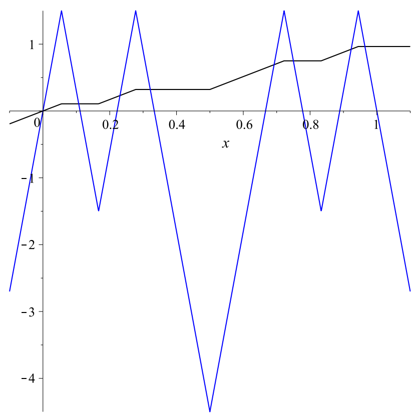
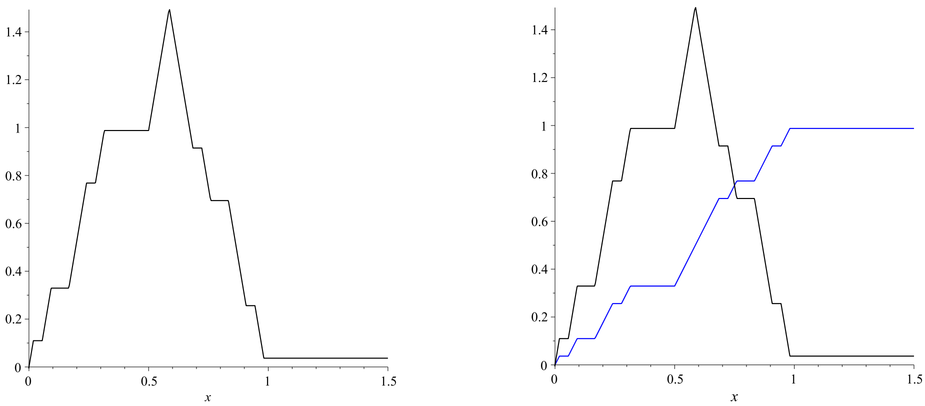


FIGURE 10. Graphs of the functions $f^{(T)}(x)$ (blue) and $\zeta(x)$ (black) at $H = 3$, $T = 3$.

Let $x \in \left(\frac{1}{2}, 1 - \frac{1}{H} + \frac{1}{2H^{T-1}}\right)$, $T > 1$. Denote $x = 1 - \frac{1}{H} + \frac{\alpha}{H^T}$. Then $\alpha \in \left(-\frac{1}{2}H^{T-1}(H-2), \frac{H}{2}\right)$. Compute $f^{(T)}(x) : f(x) = 1 -$

$\frac{\alpha}{H^{T-1}} > \frac{1}{2}$, $f^{(2)}(x) = \frac{\alpha}{H^{T-2}} < \frac{1}{2}$, $\ldots$, $f^{(T)}(x) = \alpha$. This implies that the equation $\frac{H^T}{H^T+1}\left(1 - \frac{1}{H} + \frac{\alpha}{H^T}\right) + \frac{1}{H^T+1}\alpha = \frac{1}{2}$ has only one root on $\left(-\frac{1}{2}H^{T-1}(H-2), \frac{H}{2}\right) : \widehat{\alpha} = \frac{1}{4} - \frac{1}{4}H^T + \frac{1}{2}H^{T-1}$. Since $\zeta\left(\frac{1}{2}\right) = \frac{H^{T-1}}{H^T+1} < \frac{1}{2}$, $\zeta\left(1 - \frac{1}{H} + \frac{1}{2H^{T-1}}\right) = \frac{H^T - H^{T-1} + H}{H^T+1} > \frac{1}{2}$, then in the interval $x \in \left(\frac{1}{2}, 1 - \frac{1}{H} + \frac{1}{2H^{T-1}}\right)$ there is exactly one root of the equation $\zeta(x) = \frac{1}{2} : \hat{x} = 1 - \frac{1}{H} + \frac{\widehat{\alpha}}{H^T} = \frac{3}{4} - \frac{1}{2H} + \frac{1}{4H^T}$. This means that the function $F(x) = f(\zeta(x))$ does not decrease on the interval $[0, \hat{x}]$, does not increase on $[\hat{x}, 1]$, $F(\hat{x}) = \frac{H}{2}$ and $F(x) = F\left(1 - \frac{1}{2H^{T-1}}\right) = \frac{H^T}{H^T+1}$ when $\hat{x} \geq 1 - \frac{1}{2H^{T-1}}$. The graph of the function $F(x)$ at $H = 3$, $T = 4$ is given in Fig. 11-a). Fig. 11-b) shows the corresponding graphs of the functions $F(x)$ and $\zeta(x)$.



a) b)

FIGURE 11. Graphs of the functions $F(x)$ (black) and $\zeta(x)$ (blue) at $H = 3$, $T = 4$.

On the interval $[0, 1]$, there exist 2^{T-1} disjoint intervals, where the function $F(x)$ is constant. This means that there exist 2^{T-1} periodic points, where $F'(x) = 0$. These points are T-periodic points of the map $f(x)$ with negative multipliers. Generally speaking, the proper period of these points may be less than T, if T is not a prime number. The question of stability of the so-called subcycles is considered in the next section. Equality to zero of the derivative means that the iterative process defined by equation (5) has ultrafast convergence to cyclic points. In particular, if $x_0 \geq \hat{x}$, then $x_1 = F(x_0) = \frac{H^T}{H^T+1} > \frac{1}{2}$, $x_2 = F(x_1) = f(x_1) = \frac{H}{H^T+1} < \frac{1}{2}$, $\ldots$, $x_T = F(x_{T-1}) = f(x_{T-1}) = \frac{H^{T-1}}{H^T+1} < \frac{1}{2}$, $x_{T+1} = F(x_T) = f(x_T) = \frac{H^T}{H^T+1} = x_1$, i.e., $\{x_1, \ldots, x_T\}$ is a cycle of equations (1) and (5).

7. Subcycles

This section is about which cycles are stabilized together.

In equation (1), for any natural $T \geq 1$ there exist T-cycles. Moreover, it is not difficult to estimate their number. The graph of the function $y = f^{(T)}(x)$ intersects with the graph of the function $y = x$ when $x \in [0, 1]$ exactly 2^T times. Two points of intersection correspond to equilibria $x = 0$, $x = 1 - \frac{1}{H+1}$. This means that there are no more than $\frac{2^T-2}{T}$ cycles of the length T (excluding equilibria). If T is a prime number, then there are exactly $\frac{2^T-2}{T}$ cycles. The well-known result is easily obtained from this: for a prime number $T > 2$, the number $2^{T-1}-1$ is divisible by T. If $T = \prod\limits_{j=1}^{s} \tau_j^{\rho_j}$, where $\tau_1, \ldots, \tau_s$ are prime numbers, then there are exactly $\frac{2^T - \sum\limits_{j=1}^{s} 2^{\frac{T}{\tau_j}} + \sum\limits_{\substack{i,j=1\\ i<j}}^{s} 2^{\frac{T}{\tau_i\tau_j}} + \ldots + (-1)^s 2^{\frac{T}{\tau_1\cdot\ldots\cdot\tau_s}}}{T}$ cycles of the length T. The fact that the given fraction is an integer is a special case of Gauss's theorem.

Let τ be a factor of the number T (if T is a prime number, then we assume $\tau = 1$). And let the T-cycle of equation (5) be locally asymptotically stable. *The task* is: to find out in what cases τ-cycles of the same equation will be locally asymptotically stable?

Theorem 4. *Let condition (6) be satisfied, i.e. T-cycles of equation (1), for which the multipliers are positive, are locally asymptotically stable cycles of equation (5). Then all the τ-cycles of equation (1), for which the multipliers are positive, and all the τ-cycles of equation (1), for which the multipliers are negative and the number $\frac{T}{\tau}$ is even, are locally asymptotically stable cycles of equation (5).*

Let condition (7) be satisfied, i.e. T-cycles of equation (1), for which the multipliers are negative, are locally asymptotically stable cycles of equation (5). Then all the τ-cycles of equation (1), for which the multipliers are negative and the number $\frac{T}{\tau}$ is odd, are locally asymptotically stable cycles of equation (5).

Proof. Let $\{\eta_1, \ldots, \eta_T\}$ be a T-cycle and $\{\widehat{\eta}_1, \ldots, \widehat{\eta}_\tau\}$ be a τ-cycle of equation (1). Let $\mu_T = f'(\eta_T) \cdot \ldots \cdot f'(\eta_1)$, $\mu_\tau = f'(\widehat{\eta}_\tau) \cdot \ldots \cdot f'(\widehat{\eta}_1)$ be the corresponding multipliers of these cycles. These cycles will also be the cycles of equation (5). We assume that the condition for local asymptotic stability of the T-cycle of equation (5)

$$\left|\mu_T\left(\vartheta+(1-\vartheta)\mu_T\right)^T\right|<1 \tag{9}$$

is satisfied. Let us find the multiplier of the τ-cycle of equation (5). Let, as before, $F(x)=f\left(\vartheta x+(1-\vartheta)f^{(T)}(x)\right)$, and $p=\frac{T}{\tau}$. Calculate

$$F'(\widehat{\eta}_j)=f'(\widehat{\eta}_j)\left(\vartheta+(1-\vartheta)\left(f'(\widehat{\eta}_\tau)\cdot\ldots\cdot f'(\widehat{\eta}_1)\right)^p\right)=$$
$$=f'(\widehat{\eta}_j)\left(\vartheta+(1-\vartheta)(\mu_\tau)^p\right),\ j=1,\ldots,\tau.$$

Then the multiplier of the τ-cycle of equation (5) equals $F'(\widehat{\eta}_\tau)\cdot\ldots\cdot F'(\widehat{\eta}_1)=\mu_\tau\left(\vartheta+(1-\vartheta)(\mu_\tau)^p\right)^\tau$, and, accordingly, the condition for local asymptotic stability of the τ-cycle of equation (5) is:

$$\left|\mu_\tau\left(\vartheta+(1-\vartheta)(\mu_\tau)^p\right)^\tau\right|<1 \tag{10}$$

Since $\mu_T=\pm H^T$, $\mu_\tau=\pm H^\tau$, the following cases are possible:

a) $\mu_T<0,\ \mu_\tau<0$, then $\begin{cases}\mu_T=(\mu_\tau)^p,\ p \text{ is odd},\\ \mu_T=-(\mu_\tau)^p,\ p \text{ is even},\end{cases}$

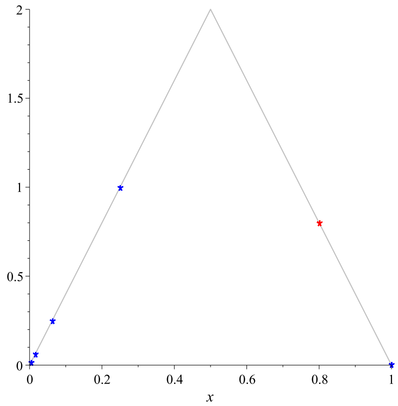
b) $\mu_T<0,\ \mu_\tau>0$, then $\mu_T=-(\mu_\tau)^p$ for any p,

c) $\mu_T>0,\ \mu_\tau<0$, then $\begin{cases}\mu_T=-(\mu_\tau)^p,\ p \text{ is odd},\\ \mu_T=(\mu_\tau)^p,\ p \text{ is even},\end{cases}$

d) $\mu_T>0,\ \mu_\tau>0$, then $\mu_T=(\mu_\tau)^p$ for any p.

Inequality (9) implies inequality (10) if and only if one of the conditions holds: $\mu_T<0$, $\mu_\tau<0$, p is odd; $\mu_T>0$, $\mu_\tau<0$, p is even; $\mu_T>0$, $\mu_\tau>0$ for any p. The first condition is possible for the parameters ϑ that satisfy inequalities (6), the second and third conditions are so for the parameters ϑ that satisfy inequalities (7), whence the conclusion of the Theorem follows. □

Consider again the example of stabilization of 5-cycles from Section 4, but take $x_0=0.001$ and $x_0=0.801$ as the initial points. Then, besides locally asymptotically stable 5-cycles, equilibria $\eta=0$ at $\vartheta=\frac{H^T+\frac{0.4}{H}}{H^T-1}$ and $\eta=0.8$ at $\vartheta=\frac{H^T-\frac{0.4}{H}}{H^T+1}$ will also be stable. The selected initial points get into the basins of attraction of the corresponding equilibria. Periodic and fixed points are shown in Fig. 12 (the colors remain the same).



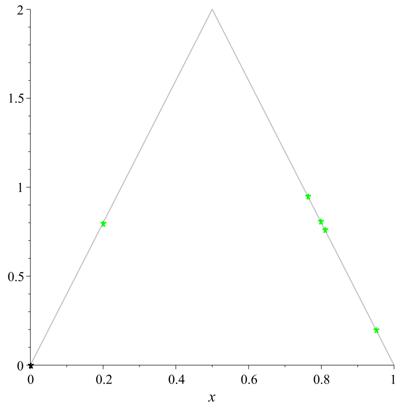


FIGURE 12. Cyclic points of 5-cycles and fixed points of the tent map in the case of positive (red and blue) and negative (green and black) multipliers

8. Distribution of cyclic points and visualization of the Cantor set

This is an example of waht can be done with the method: visualizing fairly accurate Cantor sets.

As noted above, the invariant set of equation (1) at $H = 3$ is the classical Cantor set. However, due to its strong instability, it is impossible to visualize the points of this set using equation (1). The Cantor set is characterized by points of the first and second type. The points that are end-points of the adjacent to the Cantor set intervals are called points of the first type. All the other points of the set are called points of the second type. Points of the first type can be obtained as the union of all roots of the equations $f^{(k)}(x) = 1$, $k = 1, 2, \ldots$. The set of points of the first type is countable. Among the points of the second type, we can select a subset consisting of all cyclic points of map (2). They are obtained as the union of all the roots of the equations $f^{(k)}(x) = x$, $k = 1, 2, \ldots$. The set of cyclic points of map (2) is also countable.

If an infinite number of points of the Cantor set lie between two points of the first type, then there also lie an infinite number of periodic points between them. Consequently, since each point of the Cantor set is a limit point for the set of points of the first type, then each point of the Cantor set will be limit for the set of cyclic points as well.

The following question arises: how are the periodic points of one orbit of a given period T, of several such orbits located in the Cantor set? More precisely, how uniformly do the periodic points of the orbit of a given period T fill the Cantor set? The paper does not consider the analytical solution of this problem, only some examples of visualization of the density functions for the distribution of periodic points are constructed, and they are graphically compared with the analogous function for a random sample of elements from the set of the first-type points of the Cantor set.

Let us take for example the period $T = 1009$, accuracy 10^{-525}, initial value $x_0 = 0.555$, values for the control parameter $\vartheta = \frac{H^T \pm \frac{0.6}{H}}{H^T+1}$. We will get $2T = 2018$ cyclic points. Let us construct the density function for the distribution of the cyclic points set (Fig. 13-a). The graph shows that the found periodic points are not quite evenly distributed in the Cantor set. Let us now find *two hundred* orbits with the period $T = 1009$, i.e., we get 20180 cyclic points. Surprisingly, the graph of the distribution density function for the new set of points (Figure 13-b) does not differ much from the plot in the previous case. Finally, let us plot the density function for the distribution of randomly selected 200000 points of the first type of the Cantor set. For this purpose, represent the points from the set of the first type of the Cantor set in the form $s = \sum_{j=1}^{N} \alpha_j \frac{2}{3^j}$, where the set $\{\alpha_1, \ldots, \alpha_N\}$ consists of zeros and ones. Take $N = 25$. Then the maximum number of possible points is 2^{25}. Let us randomly choose values for α_j: either zero or one, and thus construct 200000 points. Next, we build a graph of the distribution density function of the resulting set (Fig. 14). Comparison of the graphs in Figures 13, 14 shows that the points of the first type, constructed by the above method, are distributed more evenly on the Cantor set.

9. Conclusions

In [23], there was solved the problem of constructing the framework of the nonlinear dynamic systems attractor through a set of the periodic orbits, which were found using generalized predictive control. If the invariant set is a repeller, then the analogous problem turns out to be much more complicated: simple local stability of the orbit is not sufficient, due to the fact that the basin of attraction can have a small measure, and for most of the initial points the trajectory will go to infinity. However, the method of generalized predictive control, used to stabilize the orbit, has an important characteristic: in addition to

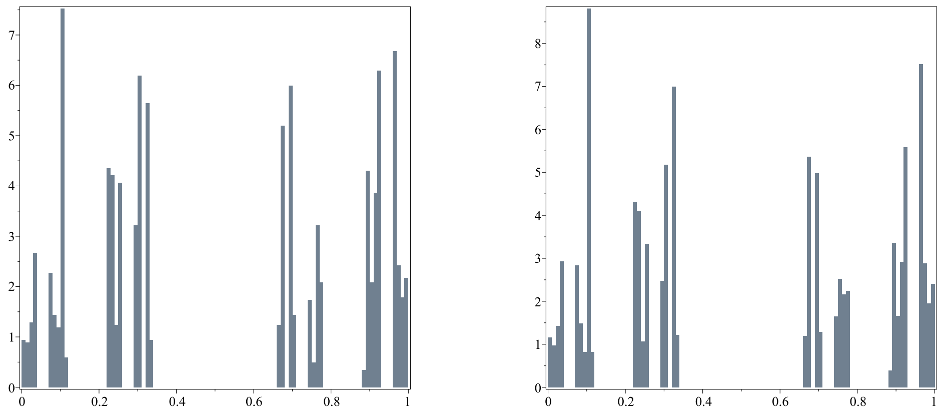


FIGURE 13. The graph of the distribution density function of the set of cyclic points of two and two hundred 1009-periodic orbits of map (2) at $H = 3$

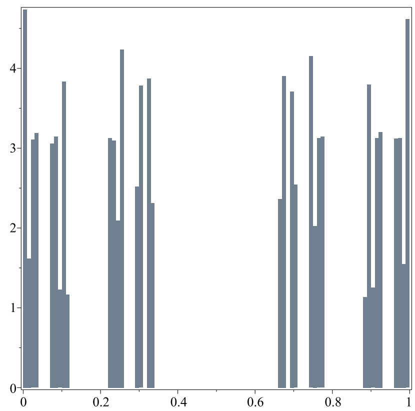


FIGURE 14. Graph of the distribution density function of the set of 200000 random points of the first type of the Cantor set

the local asymptotic stability of the orbits of controlled motion, the rest of the orbits remain bounded for a sufficiently large set of initial values. For the generalized tent map, it is possible to ensure that all the solutions are bounded. The global behavior of solutions for the generalized logistic map, the generalized Lozi and Hnon maps, etc. is somewhat more complicated. For such systems, apparently, it is necessary to choose the control parameter as a function of the current state of the system. Need a reference, maybe even an example? Investigation of the global behavior of the controlled systems solutions for the mentioned maps is the task for future research.

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