Formation and Evolution of Structure: Growth of Inhomogenieties & the Linear Power Spectrum

last time:
how fluctuations are generated
and
how the smooth Universe grows

main goal: understand how fluctuations grow

- Inflation gives the "primoridal" power spectrum (describes density fluctuations in Fourier space at the end of inflation)
  - $P(k) \sim k^n$,
  - $n \sim 1$
  - Gaussian fluctuations
- We want the power spectrum (distribution of density fluctuations) at later times

define the density contrast

$$\delta = (\rho-\rho_0)/\rho_0$$

- inflation can produce initial density fluctuations
- density perturbations may be modified by
  - amplification due to gravitational instability
  - pressure
  - dissipation

growth of inhomogenieties

- $\delta << 1$: linear theory
- $\delta \sim 1$: need specific assumptions (ie, spherical symmetry)
- $\delta >> 1$ non-linear regime. solve numerically (or higher order perturbation theory)

- in general, Universe is lumpy on small scales and smoother on large scales -- consider inhomogenieties as a perturbation to the homogeneous solution

schematically

- $\ddot{\delta}$+ [Pressure - Gravity] + δ= 0
- in the absense of an expanding universe
  - if pressure is low, δ grows exponentially
  - if pressure is large , δ oscillates with time.
  - the relevant scale here is called the Jeans length (fluctuations grow if perturbations are longer than λ_J = sound speed/ dynamical time)
- now solve in an expanding universe

Fundamental equations of fluid motion

in Lagrangian, comoving coordinates

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\mathbf{u}] = 0$$

energy equation

$$\frac{\partial \mathbf{u}}{\delta t} + \frac{\dot{a}}{a}\mathbf{u} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{a}\nabla \phi$$

Euler equation

Poisson equation

$$\nabla^2 \phi = 4\pi G \rho_b a^2 \delta$$

consider small perturbations

$u = u_0 + \delta u$ velocity

$\rho = \rho_0 + \delta\rho$ density

$p = p_0 + \delta p$ pressure

$\phi = \phi_0 + \delta\phi$ potential

keep only terms linear in δ

linearized equations

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\delta t} + \frac{\dot{a}}{a} \mathbf{u} + \frac{1}{a} \nabla \phi = 0$$

$$\nabla^2 \phi = 4\pi G \rho_b a^2 \delta$$

for cold dark matter the pressure is zero.

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}\delta$$

this is identical to the equation of motion
for a mini FRW Universe!
can be solved parametrically in the same way
as we solved the Friedmann eqn.

general solution can be written

$$\delta(\mathbf{x}) = f_1(\mathbf{x})D_1(t) + f_2(\mathbf{x})D_2(t)$$

growing mode

decaying mode

$$\delta(t) = \delta(t_0)\frac{D_1(t)}{D_1(t_0)}$$

in a matter dominated Universe $$H = \frac{2}{3t}$$

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

$$\delta = A(x)t^{2/3} + B(x)t^{-1}$$

linear growth

$$\delta = \delta_0(x)a$$

expansion of the universe slows growth from exponential to linear in a lambda dominated Universe

$$H^2 = \frac{\Lambda c^2 \pi G}{3}$$

$$\ddot{\delta} + 2H\dot{\delta} = 0$$

$$\delta = A(x) + B(x)e^{-2Ht}$$

frozen fluctuations

- solution for the matter dominated case

$$\delta = A(x)t^{2/3} + B(x)t^{-1}$$

$$\delta = \delta_0(x)a$$

linear growth

- solution for the lambda dominated case

$$\delta = A(x) + B(x)e^{-2Ht}$$

frozen fluctuations

- general case

$$\delta = \delta_0(x)ag(a, \Omega_{m0})$$

g is constant at early times and scales as 1/a at late times

for our cosmology, the action ended around z=0.5

Linear Growth Function

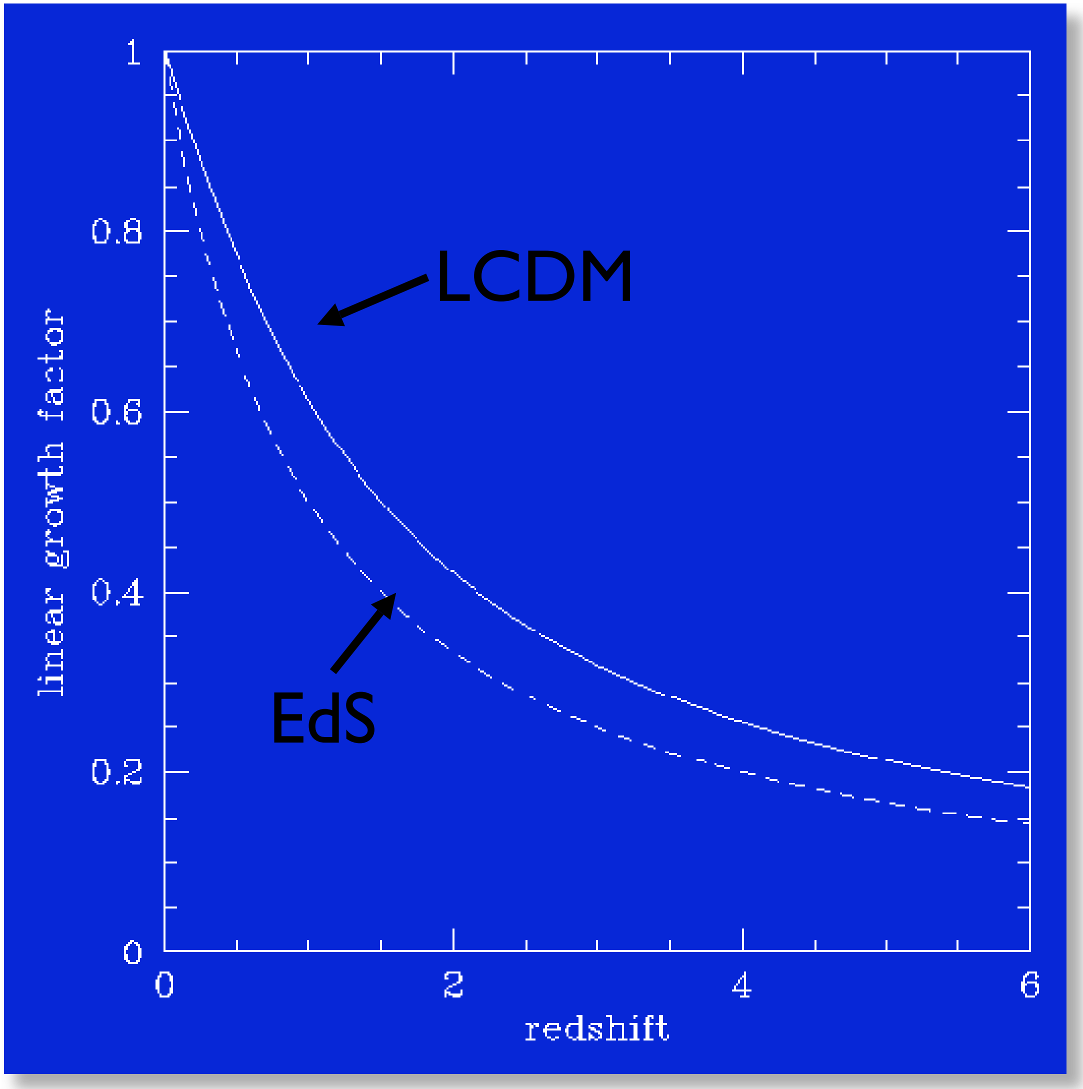
$$D_1(t) \propto a(t) = [1 + z(t)]^{-1}$$

Einstein- de Sitter

$$D_1(t) \propto \frac{(\Omega_\Lambda a^3 + \Omega_R a + \Omega_m)^{1/2}}{a^{3/2}} \int^a \frac{da\, a^{3/2}}{(\Omega_\Lambda a^3 + \Omega_R a + \Omega_m)^{3/2}}$$

Linear Growth of fluctuations

fluctuations grow more slowly in low Ω_m Universe



- CMB indicates that fluctuations in the baryon distribution at recombination had an amplitude less than 10^{-4}
- The existence of non-linear structures today implies that the growth of fluctuations must have been driven by non-baryonic dark matter which was not relativistic at recombination.
- After recombination, baryons decouple and fall into dm potential wells.

- after $\delta > 1$, non-linear regime
- matter accumulates in dense regions
- hard to approximate without numerical simulations
- random motions of particles halts the growth
- non-linear collapse on small scales doesn't change the linear evolution of the large-scale perturbations (they don't care whether the small scale power is lumpy -- Gauss's law!)

Linear Power Spectrum =
Primordial Power Spectrum
* Transfer Function
* Growth Fucntion

primordial power spectrum

$$P_k \sim k^n$$

n < 1 : ‘blue tilt’, less power on small scales
n=1 : ‘scale-free’, Harrison-Zeldovich-Peebles
n>1: ‘red tilt’, more power on small scales
n(k): running scale index

current constraints from WMAP+LSS: n = 0.95

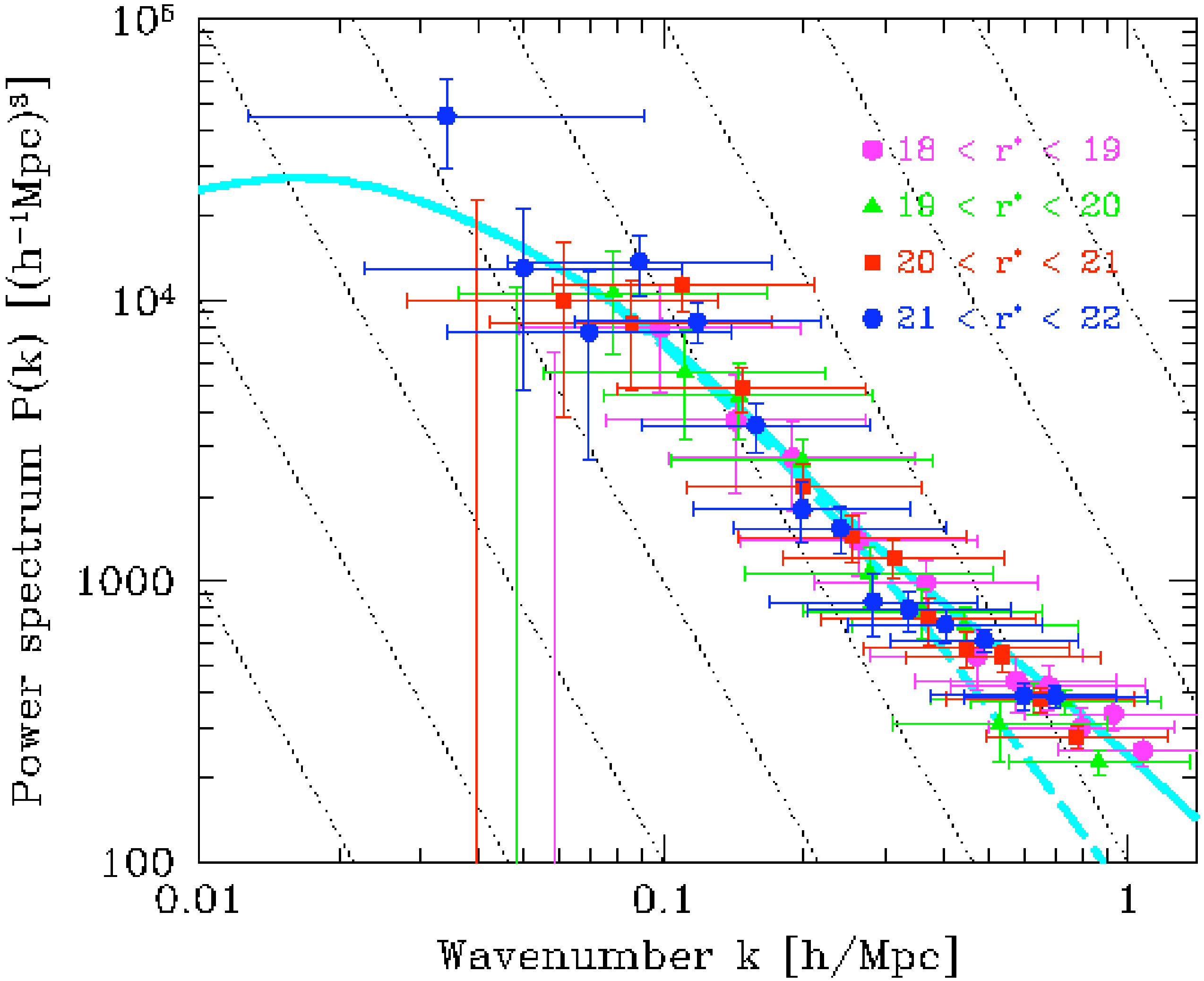
the transfer function

- the primordial power spectrum is 'processed' during the early Universe
  - this is called the transfer function T(k) (e.g. Bardeen et al. 1986; (BBKS); Eisenstein & Hu 1999)

(depends on Ω_m, Ω_Λ, H_0, Ω_b, type of dark matter particle...)

- in pure CDM:
  - in the matter dominated era all scales grow equally
  - in radiation-dominated era, pressure is important.
  - scales smaller than the horizon: growth is stalled by the presence of radiation pressure. growth slows as a^2, and small scale modes are damped as k^{-2} --> power spectrum suppressed by k^4.
  - Universe is expanding too quickly for the dark matter to collapse.
  - scales above the horizon continue to collapse
  - scales that enter horizon during the radiation-dominated era grow more slowly than those that enter during matter domination.

- P~ k on large scales P ~ k^{-3} on small scales
- power below the Jeans length is suppressed
- the Jeans scale of the total system grows to the size of the horizon at matter-radiation equality and then shrinks to zero -- the transition scale marks the scale of the horizon at matter-radiation equality



power spectrum for SDSS early data release

fluctuation damping

- perturbations on very small scales can be erased entirely.
- scales that have entered the horizon while dark matter particles are relativistic get erased by "free streaming" (fast random particle velocities disperse the fluctuations)
- for CDM, this is way before zeq (this is why it's "cold")
- for HDM, this happens at zeq, so only large scale perturbations survive.
- for baryons, "Silk damping" (mean free path of photons due to scattering by the plasma is non-zero.

effect of massive neutrinos

- neutrino dark matter slows the growth of structure
- neutrino species with cosmological abundance contribute to matter as:

$$\Omega_\nu h^2 = \Sigma m_\nu / 94eV$$

- current data from SDSS + CMB give good constraints on neutrino masses

$$\Sigma m_\nu < 1eV$$

baryon wiggles

- baryons are caught up in the acoustic oscillations of the CMB, and impart acoustic wiggles to the Transfer function.
- detected in the SDSS LRG survey (and in SDSS clusters)
- an excellent standard ruler for the angular diameter distance (no evolution in redshift)
- radial extent of the wiggles gives a measure of H(z) (goal of next generation redshift surveys

Transfer Function

- Can calculate using a Boltzman code
- most popular code is CMBFAST
- Or, use a fitting function. For CDM:

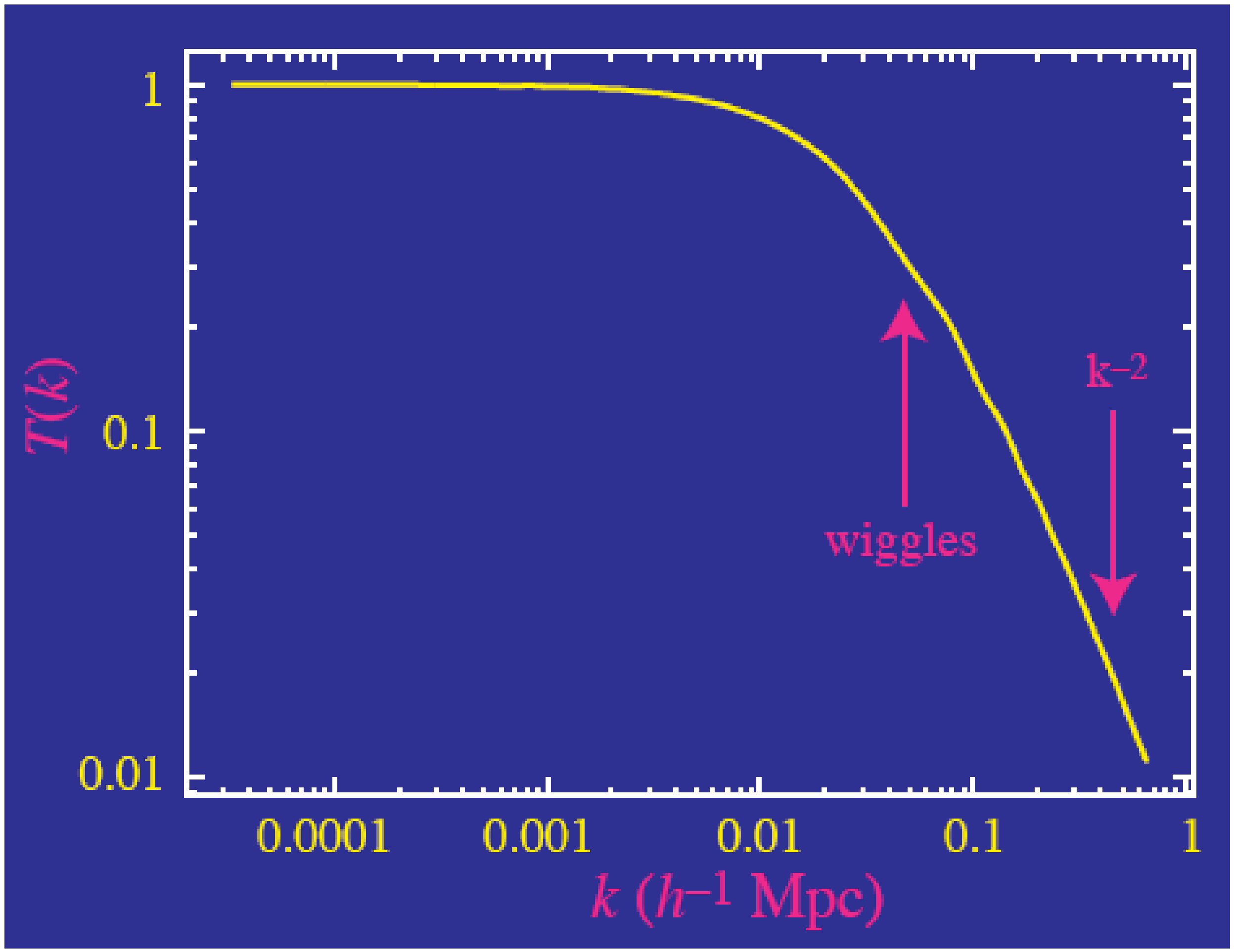
$$T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2}$$

$$L(q) = \ln(e + 1.84q)$$

$$C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}}$$

$$q = k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\mathrm{CMB}}/2.7K)^2$$

Transfer Function



normalization of the power spectrum

- CMB normalization first extracted from COBE at l ~ 10 (k ~ H0). At low l normalization suffers from cosmic variance -- only 2l+1 samples of a given l mode.
- WMAP measures the first peak, at l ~ 200. This is the current best normalization scale (k ~ 0.02 Mpc^{-1})
- WMAP chose k = 0.05 Mpc^{-1} to account for future improvements.

normalization of the power spectrum

- present (matter dominated) power spectrum compared to inflationary initial conditions (normalized by the CMB): constrains the growth function
- present fluctuations are often characterized by
- s8, RMS of density field filtered by a tophat of 8 h^{-1}Mpc

what can we learn from measurements of the power spectrum

- initial primordial power spectrum (tilt and running)
- the nature of dark matter -- affects the transfer function (ratio of large to small scale power)
- dark energy and curvature -- affects the growth function (normalization of CMB power spectrum compared to normalization today)

ways of characterizing lumpyness

- correlation function

  $\xi(r) = dP/(4\pi n r^2\, dr) - 1$

- power spectrum P_k
- variance $\sigma(M)$
- scales: $R \sim M^{1/3} \sim 1/k$

- rms density flucuations

$$\sigma^2(R) = \int \frac{dk}{k} \frac{k^3 P(k)}{2\pi^2} W^2(kR)$$

- rms fluctuations on a scale of 8 Mpc/h is roughly 1 today -- this is roughly the non-linear mass scale (2 x 10^14 Msun/h)

mass variance

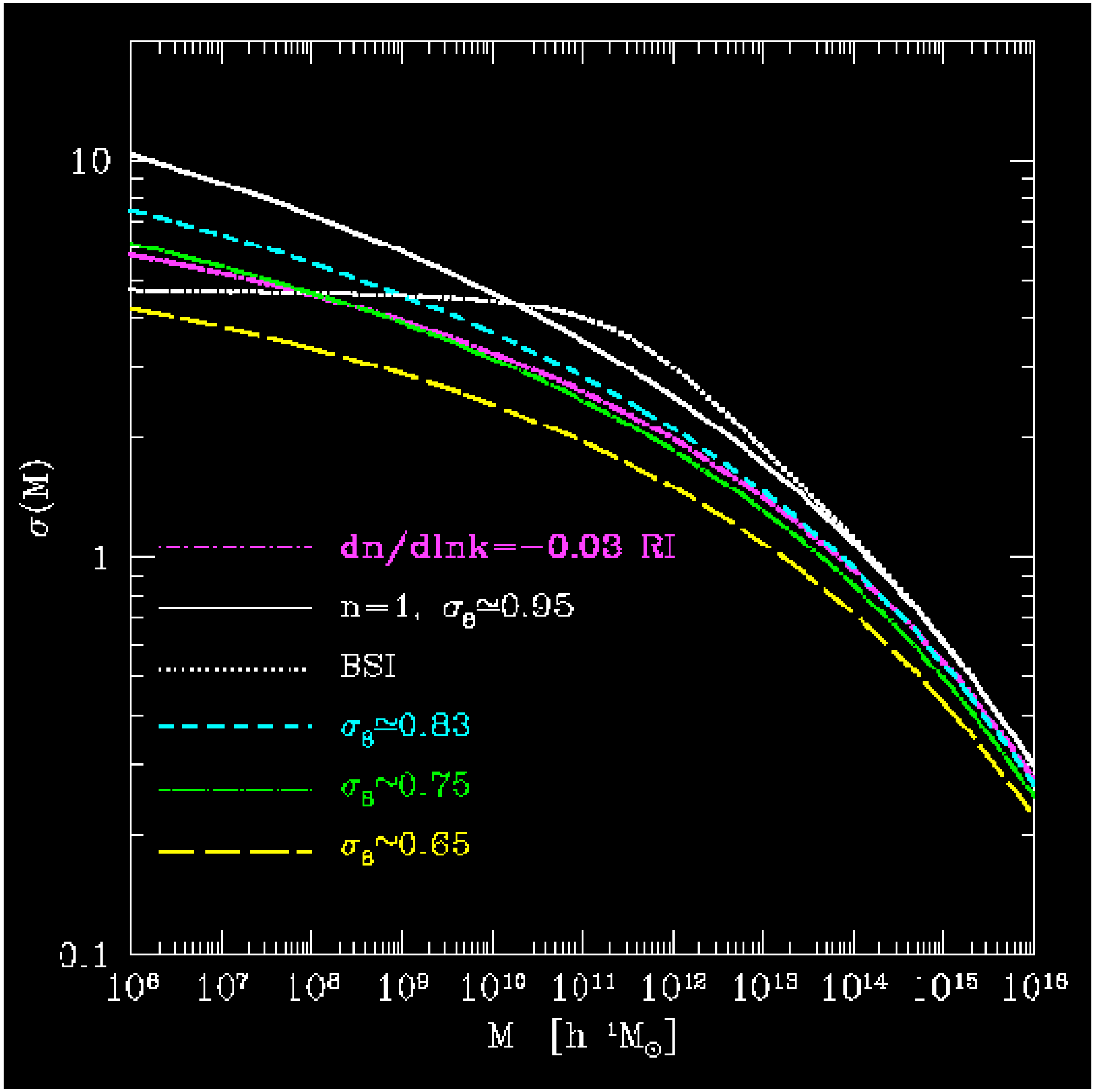
$$\langle M \rangle = \bar{\rho} V = \frac{4\pi}{3} R^3 \, \bar{\rho}$$

$$\begin{aligned}
\sigma^2(M) &= \frac{\langle (M - \langle M \rangle)^2 \rangle}{\langle M \rangle^2} = \frac{\langle \delta M^2 \rangle}{\langle M \rangle^2} \\
&= \frac{1}{2\pi^2} \int P(k) W^2(kR) k^2 dk \\
W(kR) &= \frac{3[\sin(kR) - kR\cos(kR)]}{(kR)^3}
\end{aligned}$$

ere P_k is the linear
ower spectrum,
(kR) is the Fourier
ansform of a real-
space
'tophat'

variance on small scales depends on details of inflation...

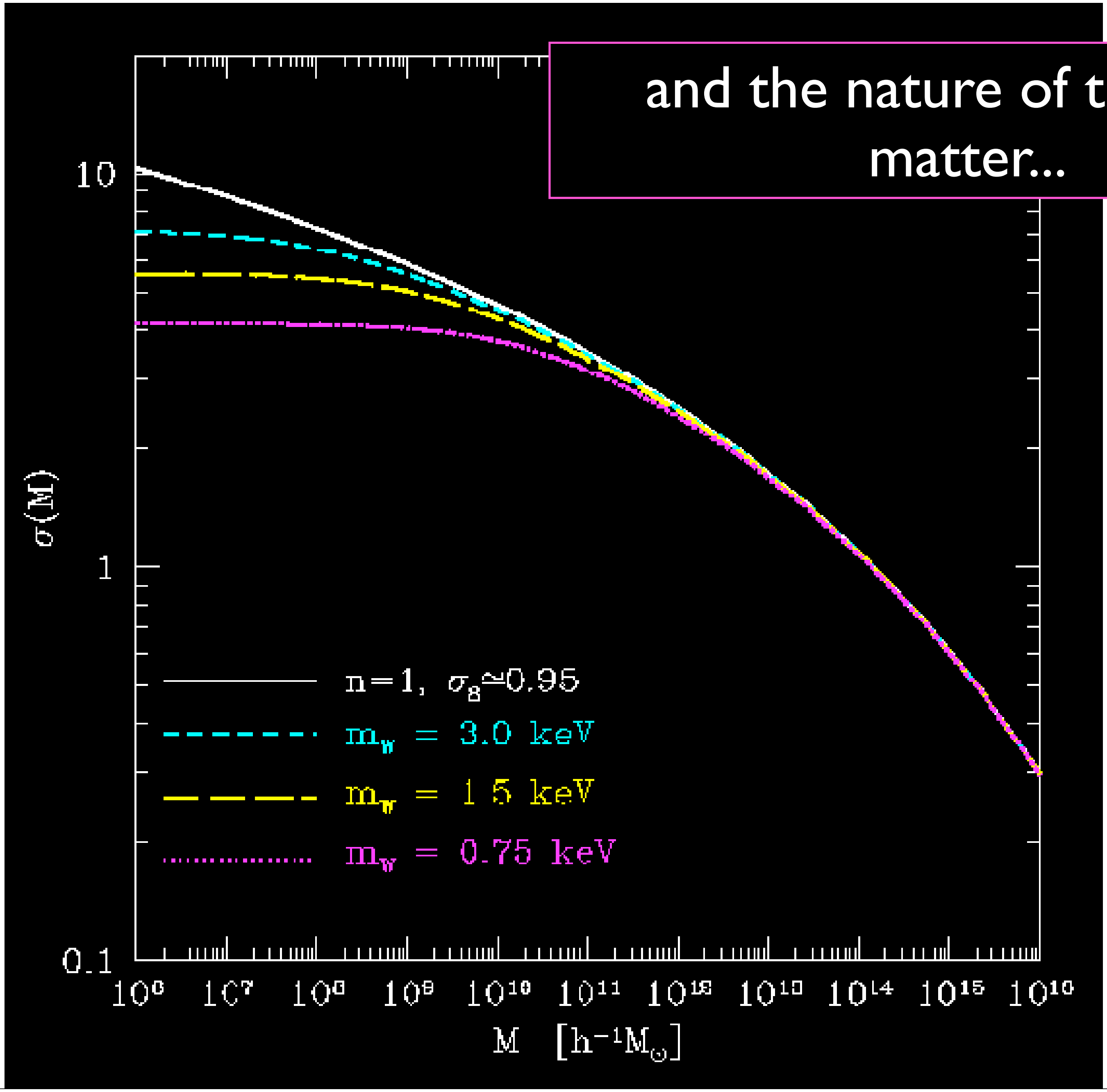
mass variance

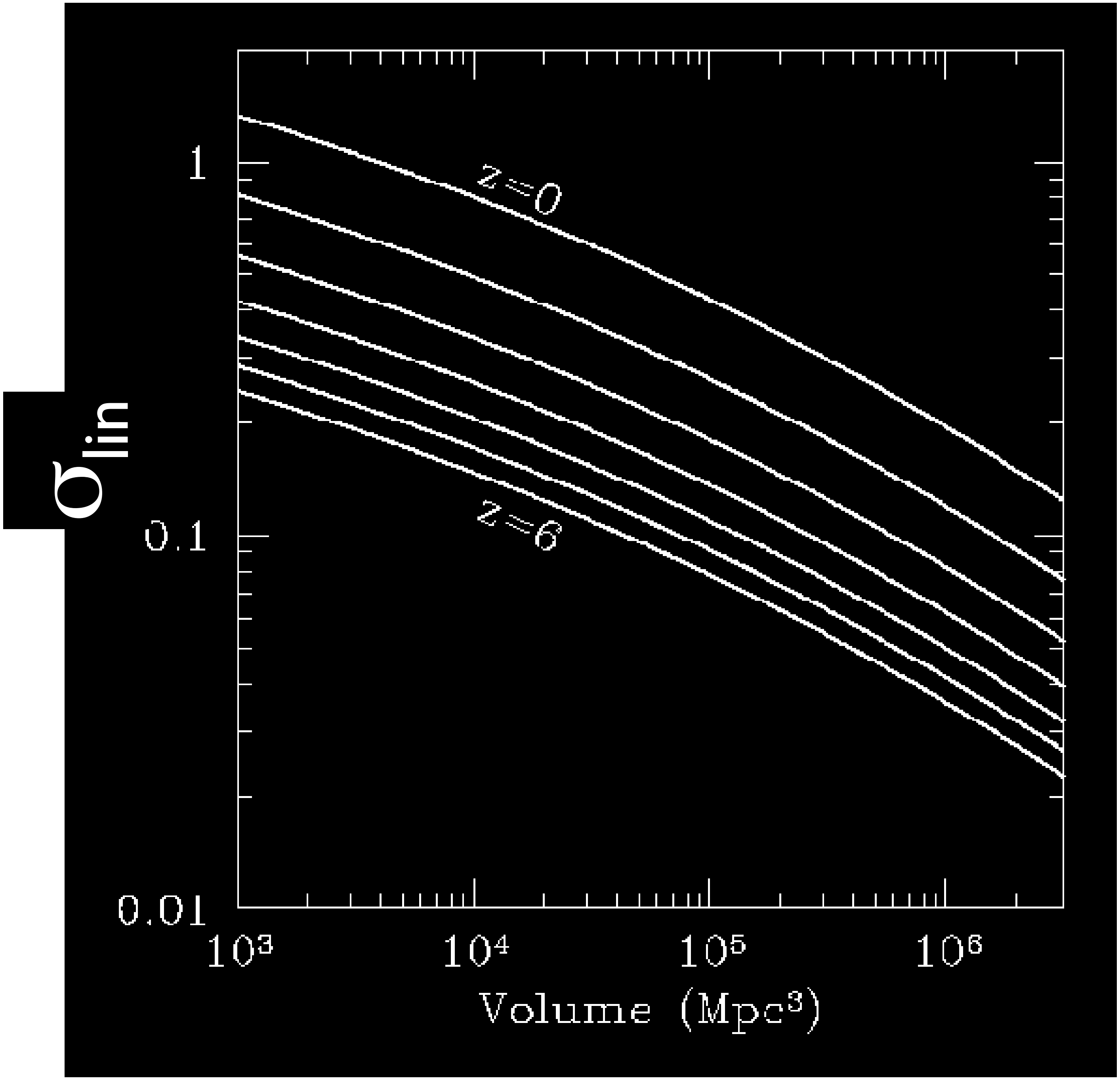


Zentner & Bullock

$\sigma_8 = \sigma(R=8\ h^{-1}\ \mathrm{Mpc})$ is the mass variance w/in a sphere with $R=8\ h^{-1}$ Mpc of the primordial density field projected forwards to $z=0$ using linear theory

it probes scales comparable to rich clusters and the nature of the dark matter...





σ_{lin}
1
0.1
0.01
10^3
10^4
10^5
10^6
Volume (Mpc^3)
z=0
z=6

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