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Competition in Multi-characteristics Spaces: Hotelling Was Almost Right*

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Lancasterian models of product differentiation typically assume a one-dimensional characteristics space. We show that standard results on prices and locations no longer hold when firms compete in a multi-characteristics space. In the location game with n characteristics, firms choose to maximize differentiation in the dominant characteristic and to minimize differentiation in the others when the salience coefficient of the former is sufficiently large. Thus, the principle of minimum differentiation holds for all but one characteristic. Furthermore, prices do not necessarily fall when products get closer in the characteristics space because price competition is relaxed when products are differentiated enough in the dominant characteristic. *Journal of Economic Literature* Classification Numbers: L1, M3, R3.

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1. INTRODUCTION

It was Hotelling's [16] belief that firms competing in prices and products tend to supply identical products. This idea became known among economists as the "Principle of Minimum Differentiation". However, as shown 50 years later, Hotelling's analysis was flawed in that no price equilibrium in pure strategies exists when firms are not located

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sufficiently far apart (see d'Aspremont *et al.* [7]). Using mixed strategies to restore existence of a price equilibrium for any location pair, Osborne and Pitchick [22] suggest that firms do not choose the same product specification in the location game, thus invalidating Hotelling's finding. Furthermore, existence and uniqueness of a price equilibrium is guaranteed when transportation costs are quadratic instead of linear. In this case, d'Aspremont *et al.* show that firms want to maximize product differentiation in order to relax price competition. Though this particular product configuration depends on the specifics of the model, a general tendency emerges from most models where firms choose their product strategically, i.e., *firms seek differentiation to avoid unbridled price competition* (see, e.g., Tirole [28], Chap. 7). Intuitively, both equilibrium prices fall when firms get close to each other, and this overcomes the positive impact that a unilateral move towards the market center has on demand. The strategic effect had been neglected by Hotelling and successors.

Most analyses of strategic product positioning assume a one–dimensional characteristics space. Clearly, this assumption is made for mathematical convenience. However, it is implicitly supposed that the general tendencies derived with one dimension carry over to the case of several dimensions. Some preliminary results by Neven and Thisse [21] and Tabuchi [25] cast doubt on this belief for two-dimensional characteristics spaces. Indeed, these authors show that maximum product differentiation is never an equilibrium of the product game. Instead, they identify equilibria where the two firms choose to minimize differentiation in one characteristic while differentiation is maximized along the other. This opens the door to new and interesting questions. If it is clear that maximum product differentiation is not robust against an increase in the number of dimensions, how then do firms choose to differentiate their products and how many characteristics are involved in the differentiation process? Specifically, in the case of n characteristics, do we observe maximum or minimum differentiation along all but one characteristics? In the former case, the market outcome would be close to maximum differentiation. In the latter, Hotelling would be "almost right" in the sense that firms would homogenize their products in $n-1$ characteristics.

The purpose of this paper is to tackle such questions in the context of a simple model *à la* Hotelling with quadratic transportation costs and with several dimensions. As suggested by the marketing literature, we also allow each characteristic to be weighted differently from the others. In the case of n dimensions, we show the following main results. First, when all weights are equal there are n local equilibria in which firms choose maximum differentiation along one characteristic and minimum differentiation along the remaining ones. In this way, duopolists offer similar products but are still able to relax price competition. Second, when there is a *dominant* characteristic, the Nash equilibrium involves maximum differentiation along the dominant characteristic only. In other words, *differentiation in a single dimension is sufficient to relax price competition and to permit firms to enjoy the advantages of a central location in all other characteristics*. The intuition behind this result has nicely been expressed by a referee. In a symmetric equilibrium with fixed market size profits are highest when prices are highest, that is, when the elasticity of demand is lowest. The lower the density of marginal consumers, the lower is the elasticity. Accordingly, as the consumer distribution is uniform, demand has minimal elasticity when the corresponding hyperplane has minimal surface area. Since the strategy space is a hypercube this hyperplane is the one that is parallel to one of the facets, which in turn implies that the products differ only in one dimension. This confirms Hotelling's intuition that duopolists offer similar products. However in another way Hotelling is still wrong since it is known that firms never want to minimize differentiation [8].

Hence, in equilibrium, duopolists offer similar products but relax price competition. This result can only arise in multi-dimensional spaces and gives much more robustness to the preliminary analyses undertaken for the case of two dimensions: *differentiation in one characteristic seems to be enough*. It has been shown that maximum differentiation is never an equilibrium in the 2-dimensional case. We show that this result holds for the case where $n=3$. We also find that maximum differentiation in two characteristics is never an equilibrium either. For the n-dimensional case we show that maximum differentiation in more than one and minimum differentiation in the other dimensions can never be an equilibrium either if the salience coefficients associated with each dimension are identical.

It is worth mentioning that the results above are obtained with quadratic transportation costs, an assumption that seems to favor tough price competition. Indeed, recall that maximum product differentiation arises with one dimension. In addition, Martinez-Giralt and Neven [18] have shown that firms choose each to supply a single product even when they have the opportunity to segment the market by providing two products. As demonstrated by these authors, by selecting a single product firms increase the distance between them and, therefore, soften price competition.

In practice, it is often hard to measure the relative importance consumers attach to the various characteristics and, therefore, to asses the empirical relevance of our main result. However, it seems to fit reasonably well the U.S. weekly magazine market dominated by *Newsweek* and *Time*. Both are supplied in the same stores, close in content and layout, and equally priced. Yet, they tend to differentiate themselves with respect to the cover story which might well be the dominant characteristic from the consumers' point of view. Michael Elliott, editor of *Newsweek International*, confirms this observation in an article published in *The Guardian*, February 19, 1996, when he writes “Time and Newsweek are engaged in a perpetual scrap for market share. Covers are a key battleground in this contest.”[1]

The remainder of the paper is organized as follows. The model is presented in Section 2. We determine equilibrium prices and locations for a n-dimensional characteristics space in Section 3. The proof of our main result is given in the appendix. Section 4 concludes the paper.

2. THE MODEL AND SOME PRELIMINARY RESULTS

The model we study is a variant of Hotelling’s [16] model of spatial competition. Hotelling considers competition along “Main Street” represented by the unit segment, so that products are differentiated in a single characteristic only. Our model extends Hotelling’s analysis to a characteristics space of n dimensions. The product variants are then given by the firms’ location in $\mathfrak{R}^n$. Firm A’s location is described by a vector $\mathbf{a}=(a_1, ..., a_n)$ whereas firm B’s location is given by $\mathbf{b}=(b_1, ..., b_n)$.

There is a continuum of consumers distributed over the unit hypercube $C=[0, 1]^n$ according to a nonnegative continuous density function $g(\mathbf{z})$, where $\mathbf{z}=(z_1, ..., z_n)$ is a consumer’s address, so that $\int_{\mathfrak{R}^n} g(\mathbf{z})\, d\mathbf{z}=N$ is the total population. We assume throughout that $g(\mathbf{z})$ is uniform and, without loss of generality, we normalize the total population to 1. Consumers have a conditional indirect utility function $V_i(\mathbf{z})$, $i=A, B$. A consumer buying at A enjoys a utility equal to (a similar expression holds for a consumer purchasing from firm B)

$$V_A(\mathbf{z})=S-p_A-\sum_{k=1}^{n} t_k(z_k-a_k)^2 \tag{2.1}$$

where S denotes the gross surplus a consumer at $\mathbf{z}$ enjoys from consuming either variant and p_A the price of variant A. The last term in the RHS of this expression is the square of the weighted Euclidian distance between the consumer’s ideal point and the location of variant A; t_k stands for the salience coefficient of characteristic k. Thus we allow characteristics to be weighted differently, which is probably more realistic; however, for simplicity, the weights are assumed to be the same across consumers.[2] Though the specification retained for the utility loss is particular, it emphasizes the impact of competition on the strategic choice of products because all consumers view each characteristic as separated from the others. In other

[1] Other examples are mentioned in Vandenbosch and Weinberg [29].

[2] The form of the utility loss considered in (2.1) has been widely used in address models of product differentiation: see, e.g., [7, 10, 11, 20, 25]. In the marketing literature, conjoint analysis models also often use such a specification; see [14].

words, when a firm chooses a particular value for one characteristic, it has no direct impact on the choice of the others. This implies that the equilibrium product placement reflects the pure effect of competition.

Consumers have unit demands. Furthermore, it is assumed that S is large enough for all consumers to buy at the price equilibrium corresponding to any location pair. The demand for variant A is then defined by the mass of consumers for whom variant A is weakly preferred to B, i.e.,

$$D_A = \int_{\{\mathbf{z};\ V_A(\mathbf{z}) \geqslant V_B(\mathbf{z})\}} g(\mathbf{z})\, d\mathbf{z}.$$

Finally, any variant can be produced at the same constant marginal cost, which is normalized to zero without loss of generality.

The next section deals with the question of how firms place their products in the characteristics space, when there is no threat of entry. As usual, we assume a two-stage game where firms determine first the location of their product and then compete in prices; we seek a subgame perfect Nash equilibrium. Since the decision of where to place one's product in the characteristics space is made first, firms must be able to correctly predict the impact of their location on profits. A sufficient condition for this to be possible is the existence and uniqueness of the price equilibrium for any location pair.

Caplin and Nalebuff [6] have identified conditions under which both existence and uniqueness of the price equilibrium hold.[3] The existence of a price equilibrium depends on the functional form and the distribution of consumer preferences. Since the utility function of (2.1) is a special case of the utility considered by Caplin and Nalebuff (Assumption 1, p. 29) and since the uniform distribution complies with ρ-concavity (Assumption 2, p. 30), there exists a pure strategy price equilibrium for any location pair (Theorem 2, p. 39). In addition, the demand functions are twice differentiable (see the next section) and the uniform distribution is log-concave, thus implying the uniqueness of the price equilibrium (Proposition 6, p. 42) for each location pair.

Thus, firms are able to evaluate their profit at the price equilibrium and to compete in location. Since each characteristic induces a new dimension of the choice space, it is reasonable to assume that the strategy space of each firm is given by the unit hypercube C. In other words, any characteristic must be chosen in the unit interval. Note, however, that this assumption is much less restrictive than it seems at first sight. Indeed, our

[3] See also Dierker, E. [9].

analysis can easily be extended to the case of a strategy space given by a box whose sides have arbitrary lengths since the unit of each characteristic may be chosen for the length of each side to be one.

3. PRODUCT POSITIONING WITH n CHARACTERISTICS

In this section, we study firms' location choice in a n-dimensional characteristics space. First, the demands of both firms are introduced in 3.1 and the equilibrium prices are analyzed in 3.2 for the most relevant case. Then, in 3.3 we propose a candidate location equilibrium where *product differentiation is maximized in the characteristic with the largest salience coefficient and minimized in the other characteristics.* Hence, maximum product differentiation arises in one characteristic while minimum differentiation occurs in all others. It is proven that this product configuration is always a local equilibrium. More importantly, an intuitive condition is found under which this local equilibrium is a global one. At the other extreme, when all salience coefficients are equal, we show that there exist n local equilibria where products are differentiated along one characteristic and standardized along the others. Finally, it is demonstrated that some "natural" candidate configurations can be disregarded as a location equilibrium. Specifically, for the case of identical salience coefficients we show that maximum product differentiation is never an equilibrium. Similarly, maximum product differentiation with respect to at least two characteristics and minimum product differentiation in the remaining characteristics is never an equilibrium either. We also show that these results hold when salience coefficients differ and $n=3$.

3.1. *Demand*

Given the assumptions set out in Section 2, the characteristics space is a hypercube whose side length equals one. Consumers indifferent between purchasing product A or B are now located on an $(n-1)$-dimensional hyperplane defined by

$$p_A + \sum_{k=1}^{n} t_k(z_k - a_k)^2 = p_B + \sum_{k=1}^{n} t_k(z_k - b_k)^2.$$

It is readily verified that

$$\hat{z}_n(z_1, z_2, ..., z_{n-1}) = \frac{p_B - p_A + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2)}{2t_n(b_n - a_n)} - \sum_{k=1}^{n-1} \frac{t_k(b_k - a_k)}{t_n(b_n - a_n)} z_k.$$

Assuming that $\mathbf{b} \geqslant \mathbf{a}$, the slope of this hyperplane is negative along each dimension. An illustration for $n=3$ is provided in Fig. 3.1.

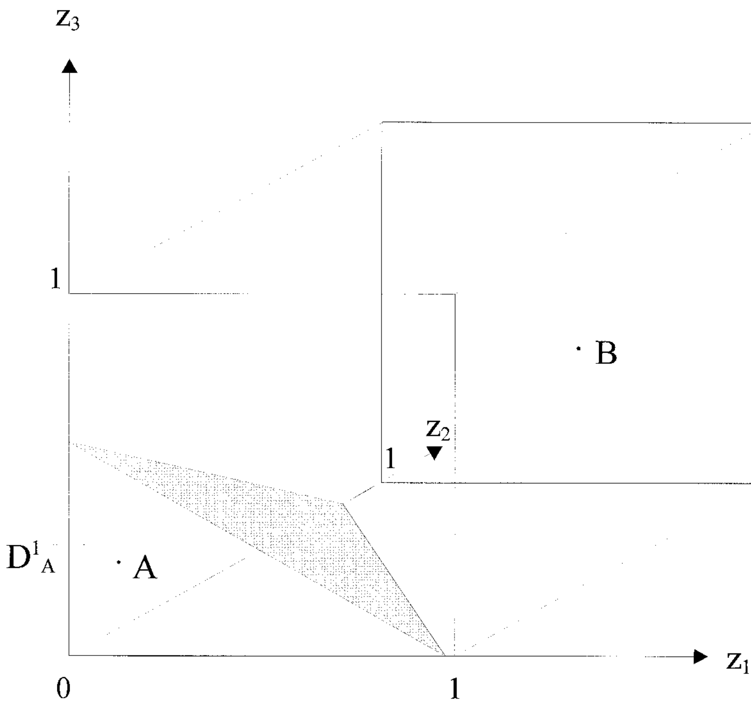

Fig. 3.1. Marginal consumer plane when $n=3$.

Since the location space is a hypercube, each firm's demand function is composed of at most 2^n-1 pieces, which can be shown to define a twice continuously differentiable function because the slope of the hyperplane is constant along each dimension. In the case of three characteristics the geometrical intuition is that the size of a firm's market space is identical to the sum of the volume of pyramids. The boundaries of the corresponding price intervals are determined by the price differentials $(p_B - p_A)$ for which the marginal consumer hyperplane given above hits a corner of the hypercube. Which corner is hit first, second... depends on the slope of the marginal consumer hyperplane. Without loss of generality, we assume from now on that the n characteristics can be ordered such that $t_n(b_n - a_n) \geqslant t_{n-1}(b_{n-1} - a_{n-1}) \geqslant, ..., \geqslant t_1(b_1 - a_1)$. In this case, we say that the nth characteristic is *dominant*, while all others are said to be *dominated*. Two subcases may then arise:

1. We say that the nth characteristic *weakly dominates* when

$$\sum_{k=1}^{n-1} t_k(b_k - a_k) \geqslant t_n(b_n - a_n). \tag{3.1}$$

2. The nth characteristic *strongly dominates* when

$$t_n(b_n - a_n) > \sum_{k=1}^{n-1} t_k(b_k - a_k). \tag{3.2}$$

Intuitively, the latter condition means that, for a given pair of variants A and B, the difference between the marginal utility losses along one characteristic is larger than the sum of these differences along the others. Observe that dominance depends on the salience coefficients as well as on the position of the variants. Also, both conditions can still hold in the special case of equally weighted characteristics, that is, $t_k = t$ for all $k = 1, ..., n$. In interpreting conditions (3.1) and (3.2), it should be kept in mind that consumers typically concentrate on a relatively small number of characteristics because of the difficulty they have to structure their preferences when this number is large. For example, Miller [19] argues from experiments that many individuals cannot deal with more than seven dimensions because of their limited ability to process information. Similarly, Simon has stressed in various publications that comparing a large number of alternatives (here characteristics) is a highly problematic task for an individual (see, e.g. Simon [23]). Hence, (3.2) is in fact less restrictive than it might appear at first sight. As shown below, this condition is *always* satisfied at the local Nash equilibrium in locations, even if all salience coefficients are identical.

We now present the middle piece of firm A's demand for the case of strong dominance. Since $D_B = 1 - D_A$, firm B's middle demand piece is thus implicitly determined. Observe that all the magnitudes associated with strong dominance are denoted with a tilde.

The middle piece of firm A's demand under strong dominance arises when the number of vertices of the hypercube is the same on either side of the marginal consumer hyperplane. It is given by

$$\tilde{D}_A^{2^{n-1}} = \int_0^1 \int_0^1 \cdots \int_0^{\hat{z}_n} dz_n \, dz_{n-1} \cdots dz_1,$$

which yields

$$\tilde{D}_A^{2^{n-1}} = \frac{\tilde{p}_B - \tilde{p}_A + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2) - \sum_{k=1}^{n-1} t_k(b_k - a_k)}{2t_n(b_n - a_n)}. \tag{3.3}$$

This expression is linear in prices and represents firm A's demand over the interval

$$\tilde{p}_A \in \left[\tilde{p}_B + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2) - 2t_n(b_n - a_n), \right.$$

$$\left. \tilde{p}_B + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2) - 2\sum_{k=1}^{n-1} t_k(b_k - a_k) \right]. \tag{3.4}$$

The two limits of this interval are obtained when the marginal consumer hyperplane just hits one vertex from above or from below, that is, when $\hat{z}_n(1, 1, ..., 1) = 0$ and $\hat{z}_n(0, 0, ..., 0) = 1$.

Given that specification of demand, we can proceed with the analysis of the corresponding price subgames.

3.2. *Price Equilibrium under Strong Dominance*

As mentioned in Section 2 we know that a unique price equilibrium exists for all demand pieces. Explicit computations of the equilibrium prices, however, turn out to be very tedious (first order conditions can be of order n) but, as will be seen, this will not impede us from solving for a locational equilibrium which belongs to the domain of strong dominance. This requires to study the price equilibrium for the middle demand piece when there is strong dominance.

Using the demand function of equation (3.3), it is readily verified that the first order conditions of profit maximization have a unique solution

$$\tilde{p}_A^* = \frac{2t_n(b_n - a_n) - \sum_{k=1}^{n-1} t_k(b_k - a_k) + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2)}{3} \tag{3.5}$$

$$\tilde{p}_B^* = \frac{4t_n(b_n - a_n) + \sum_{k=1}^{n-1} t_k(b_k - a_k) - \sum_{k=1}^{n} t_k(b_k^2 - a_k^2)}{3}. \tag{3.6}$$

These prices define an equilibrium as long as both of them belong to the p_A-interval given by (3.4) for which the middle piece of the demand functions arises. For that, $\tilde{p}_A^*$ must be smaller than or equal to the upper bound of this price interval, which holds iff

$$4\sum_{k=1}^{n-1} t_k(b_k - a_k) - 2t_n(b_n - a_n) \leqslant \sum_{k=1}^{n} t_k(b_k^2 - a_k^2). \tag{3.7}$$

Similarly, $\tilde{p}_A^*$ must be larger than or equal to the lower bound, which holds iff

$$4t_n(b_n - a_n) - 2\sum_{k=1}^{n-1} t_k(b_k - a_k) \geqslant \sum_{k=1}^{n} t_k(b_k^2 - a_k^2). \tag{3.8}$$

The same conditions are obtained for p_B. Hence, $(\tilde{p}_A^*, \tilde{p}_B^*)$ is the unique equilibrium as long as the product locations belong to the domain determined by conditions (3.7) and (3.8). Clearly, this domain is nondegenerate iff (3.2) holds.

The above equilibrium prices have interesting comparative static properties:

$$\frac{d\tilde{p}_A^*}{da_i} > 0 \qquad \text{for} \quad a_i < \frac{1}{2} \qquad i = 1, ..., n-1$$

$$\frac{d\tilde{p}_B^*}{db_i} < 0 \qquad \text{for} \quad b_i > \frac{1}{2} \qquad i = 1, ..., n-1.$$

Hence, unlike what we observe in the one dimensional model, *both equilibrium prices rise as the products become more similar in the dominated characteristics*, which run against conventional wisdom. On the other hand, as in the one-dimensional model, *equilibrium prices fall when products become more similar in the dominant characteristic*. The more striking result is undoubtedly the first one. It can be understood as follows. When firms choose their product specification, two effects are at work. First, the price competition effect is relaxed when firms locate far apart. Second, as intuitively perceived by Hotelling, firms benefit from a central location in each characteristics segment because a higher demand arises there. In the one-dimensional model, the former always dominates the latter, so that maximum differentiation emerges in equilibrium. With at least two dimensions, firms may relax price competition by differentiating their products in some characteristics and, simultaneously, benefit from more central locations in the other ones. The question is then to know the number of characteristics along which products need to be differentiated for the price competition effect to be dominated by the demand effect.

The sensitivity of equilibrium prices with respect to location parameters suggests that minimum differentiation in the first $n-1$ characteristics may occur in equilibrium provided that price competition is relaxed enough by maximum differentiation in the nth characteristic. Thus, it makes sense to study the stability of such a product configuration. Indeed, under strong dominance, differentiation in the corresponding dimension turns out to be sufficient because its salience coefficient is large enough compared to the others in the sense of (3.2), thus leading firms to raise their prices when they are better located in *all* dominated dimensions.

3.3. *The Location Equilibrium*

In what follows, we first show that the candidate configuration described above is a local Nash equilibrium. It is our belief that this concept is empirically relevant because of the difficulty that many decision makers face in accounting for all possible actions.[4] We then identify a condition under which this candidate equilibrium is indeed a Nash equilibrium.

In the location game, firm A's profit function is given by

$$\pi_A^* = p_A^*(\mathbf{a}, \mathbf{b})\, D_A(\mathbf{a}, \mathbf{b}, p_A^*(\mathbf{a}, \mathbf{b}), p_B^*(\mathbf{a}, \mathbf{b})).$$

Introducing the equilibrium prices of (3.5) and (3.6) into A's demand as given by (3.3) yields the following profit

$$\tilde{\pi}_A^{2^{n-1}} = \frac{[2t_n(b_n - a_n) - \sum_{k=1}^{n-1} t_k(b_k - a_k) + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2)]^2}{18t_n(b_n - a_n)}.$$

Deriving with respect to the location of each dominated characteristic a_i, $i = 1, ..., (n-1)$, we find that, for any configuration $\mathbf{b}$, $a_1^* = a_2^* = \cdots = a_{n-1}^* = 1/2$ is firm A's best reply. Hence, minimum differentiation in all dominated characteristics must occur in equilibrium. This also means that the demand effect, which is positive, dominates the strategic effect, which is negative, provided that $a_i < 1/2$.

We now show that firm A wants to maximize differentiation in the dominant characteristic n. Denote $E \equiv [2t_n(b_n - a_n) - \sum_{k=1}^{n-1} t_k(b_k - a_k) + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2)]$. Deriving $\tilde{\pi}_A^{2^{n-1}}$ with respect to a_n yields

$$\frac{d\tilde{\pi}_A^{2^{n-1}}}{da_n} = \frac{E(E - 4t_n(1 + a_n)(b_n - a_n))}{18t_n(b_n - a_n)^2}.$$

Maximum differentiation in this characteristic requires $d\tilde{\pi}_A^{2^{n-1}}/da_n \leqslant 0$ for any a_i, b_i, $i = 1, ..., n$, or

$$\sum_{k=1}^{n-1} t_k(b_k - a_k)(b_k + a_k - 1) \leqslant t_n(b_n - a_n)(2 + 3a_n - b_n).$$

Since $t_n(b_n - a_n) > \sum_{k=1}^{n-1} t_k(b_k - a_k)$, it is sufficient for the above to hold that

$$\sum_{k=1}^{n-1} t_k(b_k - a_k)(b_k + a_k - 1) \leqslant (2 + 3a_n - b_n) \sum_{k=1}^{n-1} t_k(b_k - a_k).$$

[4] The same idea underlies the work of Bonanno [4], Bonanno and Zeeman [5], and Gary-Bobo [13].

Rearranging yields

$$\sum_{k=1}^{n-1} t_k(b_k - a_k)(b_k + a_k + b_n - 3(1 + a_n)) \leqslant 0.$$

The latter is always satisfied since $b_k + a_k + b_n \leqslant 3$. Hence, firm A's best reply to any location of firm B in the admissible domain is $\mathbf{a}^* = (1/2, 1/2, ..., 1/2, 0)$. Similarly, one can show that firm B's best reply to any location of firm A in the admissible domain is given by $\mathbf{b}^* = (1/2, 1/2, ..., 1/2, 1)$. Hence, the configuration exhibiting maximum differentiation in the dominant characteristic and minimum differentiation in the others is an equilibrium in the admissible location domain.

The analysis above is summarized in the following proposition.

PROPOSITION 1. *Assume* $t_n(b_n - a_n) > \sum_{k=1}^{n-1} t_k(b_k - a_k)$. *If deviations are restricted to the location domain defined by* (3.7) *and* (3.8), *then*

$$\mathbf{a}^* = (1/2, 1/2, ..., 1/2, 0), \qquad \mathbf{b}^* = (1/2, 1/2, ..., 1/2, 1)$$

is the only equilibrium of the first stage game.

In other words, there exists a local Nash equilibrium in locations where both firms choose to differentiate their products along the dominant characteristic only, while they select the same central position in each of the dominated characteristics.

Hence, the candidate configuration is always *a local Nash equilibrium of the location game*. Using (3.5) and (3.6), the corresponding equilibrium prices are $p_A^* = p_B^* = t_n$. Hence, somewhat surprisingly, *the price level depends only upon the salience coefficient of the dominant characteristic*. The intuition is that products are homogeneous in terms of all dominated characteristics so that they have no influence on the determination of the equilibrium prices. Both firms equally share the market and equilibrium profits are therefore equal to $t_n/2$. For $n = 3$, the equilibrium constellation is depicted in Fig. 3.2.

In the special case of identical salience coefficients, Proposition 1 implies the following, stronger result.

COROLLARY 1. *Assume* $t_k = t$ *for all* k. *Then, for each* $k = 1, ..., n$ *there exists* $\varepsilon > 0$ *such that* $\mathbf{a}^* = (1/2, ..., 1/2, 0, 1/2, ..., 1/2)$, $\mathbf{b}^* = (1/2, ..., 1/2, 1, 1/2, ..., 1/2)$, *where* 0 *and* 1 *are the k's component of the two vectors, is the only equilibrium of the first stage game if deviations by firm* A (*resp.* B) *are restricted to the location domain defined by* $1/2 - \varepsilon < a_i < 1/2 + \varepsilon$ (*resp.* $1/2 - \varepsilon < b_i < 1/2 + \varepsilon$) *for all* $i \neq k$ *and* $0 \leqslant a_k < \varepsilon$ (*resp.* $1 - \varepsilon < b_k \leqslant 1$).

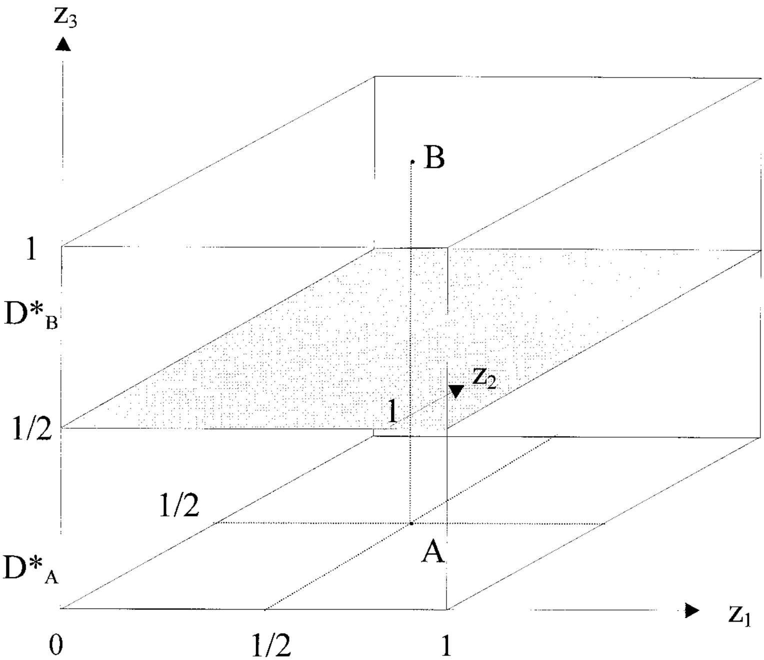

Fig. 3.2. Equilibrium configuration when $n = 3$.

This follows from the fact that both inequalities (3.7) and (3.8) are strict at the equilibrium. Therefore, we have shown that, if all salience coefficients are equal, *there exist n local equilibria in which both products are maximally differentiated along any single characteristic and bunched along all the others at the center of the unit interval.* At each of these local equilibria, prices are equal to t and firms earn the same profits. In a sense, the distinction between characteristics is here inessential in that any of them can support differentiation in equilibrium. Note also that the argument above shows that Corollary 1 still holds if the salience coefficients differ only by a trifle.

A referee has offered the following, nice geometrical intuition for Corollary 1. Starting from any configuration identified there, it is clear that a unilateral move of variant A can be broken down into one move along the line L_{ab} connecting $\mathbf{a}^*$ and $\mathbf{b}^*$ and one move along the hypersphere S_{ab} with a center at (1/2, ..., 1/2) and passing through the points $\mathbf{a}^*$ and $\mathbf{b}^*$. Consider first a move of $\mathbf{a}$ toward $\mathbf{b}^*$ along L_{ab}. Since the distribution of consumers is uniform over the hypercube C, the projection of consumer types on L_{ab} results in a uniform distribution along this line. In other words, the multi-dimensional problem collapses to a one dimensional problem. Hence we can appeal to d'Aspremont *et al.* [7] to check that any such move towards $\mathbf{b}^*$ hurts firm A. Observe, in passing, that this argument shows that *firms always want differentiate their products in at least one dimension.*

We now come to a move along the hypersphere S_{ab}. If prices remain fixed, the structure of preferences implies that any move along S_{ab} leads to a rotation of the marginal consumer hyperplane around the center of gravity of C, thus leaving demand unchanged. The idea is then to show that any marginal move along S_{ab} is never profitable to firm A. If firm A changes $\mathbf{a}$ in a way that holds its demand constant, then the impact on its profits depends only upon how firm B responds with its price. Then, a sufficient condition for the candidate configuration to be a local equilibrium is that firm B does not change its price, at least to the first order. Indeed, any marginal move along S_{ab} would have no direct effect on demands and no first-order effect on equilibrium prices and, consequently, a zero derivative of profit. For this to be true, the first-order condition for firm B shows that $D'_B \equiv \partial D_B / \partial p_B$ must remain constant. This derivative is made up of two components: (i) the change in the mass of consumers on the surface of the marginal consumer hyperplane between $\mathbf{a}^*$ and $\mathbf{b}^*$, and (ii) the shifting effect associated with the move of this hyperplane in response to the change in p_B. The latter is inversely related to the distance between $\mathbf{a}^*$ and $\mathbf{b}^*$.

Since the marginal consumer hyperplane is parallel to one facet of the hypercube and passes through its center of gravity, the mass of consumers along its surface is minimal. Furthermore, the center of gravity of the hyperplane coincides with the one of the hypercube. Therefore, to the first order, a marginal rotation of the hyperplane around its center of gravity does not change the first component of D'_B. This leaves the issue of the impact of the shifting effect on D'_B. To the first order, the move in $\mathbf{a}$ along the hypersphere with center at $\mathbf{b}^*$ and passing through $\mathbf{a}^*$ also lies on S_{ab}. Because the line L_{ab} passes through the center of gravity of the marginal consumer hyperplane, the two hyperspheres are tangent at $\mathbf{a}^*$ so that the distance between $\mathbf{a}$ and $\mathbf{b}^*$ has the same order of magnitude as the distance between $\mathbf{a}^*$ and $\mathbf{b}^*$. Hence, to the first order, D'_B is constant. Consequently, any marginal move away from $\mathbf{a}^*$ along the hypersphere S_{ab} leaves firm A's profit unchanged. Adding the two moves shows that firm A is always worse–off by deviating locally.

A natural question then arises: when is the local equilibrium identified in Proposition 1 or in Corollary 1 a global one ? The intuitive argument that led us to conjecture that $\mathbf{a}^* = (1/2, 1/2, ..., 1/2, 0)$, $\mathbf{b}^* = (1/2, 1/2, ..., 1/2, 1)$ was an equilibrium, suggests that the salience coefficient of the dominant characteristic must be large enough (see, however, our discussion in the last section). This is confirmed in our main result given below. Assume without loss of generality that $t_n \geqslant t_{n-1} \geqslant \cdots \geqslant t_1$ and let

$$\alpha(n) \equiv \frac{\left[\sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^n \right]^2}{\left| \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^{n-1} \right|} (2^n n n!)^{-1}$$

$$\beta(n) \equiv \frac{2^{n-2}(n-1)!}{\left| \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n-2i)^{n-1} \right|}.$$

PROPOSITION 2. *If the inequality*

$$\max\{\alpha(n), \beta(n)\}\ t_{n-1} < t_n \tag{3.9}$$

is satisfied, then $\mathbf{a}^* = (1/2, 1/2, ..., 1/2, 0)$, $\mathbf{b}^* = (1/2, 1/2, ..., 1/2, 1)$ *is a Nash equilibrium of the location game.*

Proof. See Appendix.

Proposition 2 emphasizes the role of the weights consumers assign to the different product characteristics. More precisely, when the intensity of preferences about the dominant characteristic is "large enough" compared to the intensity of preferences concerning the dominated characteristics, firms' equilibrium location exhibits maximum product differentiation in the dominant characteristic whereas minimum differentiation occurs in the remaining $n-1$ characteristics. It is worth mentioning what the intuition developed after Corollary 1 becomes in this case. The only difference lies in the value D'_B which takes on its smallest value with respect to any maximum differentiation along any single characteristic. Hence the first-order condition for firm B implies that both equilibrium prices are higher while the market remains equally split.

In order to evaluate how strongly the nth characteristic has to dominate the others, we have computed the coefficients of t_{n-1} in (3.9) for different values of n. Table 3.1 gives these values for $n = 3, 5, ..., 31$. Thus, the salience coefficient of the dominant characteristic need not be very large compared to those of the dominated characteristics, even when n exceeds any value which is relevant in practice.

TABLE 3.1

Parameter Values Guaranteeing the Existence of the Location Equilibrium

n	3	5	7	9	11	13	15	21	31
$\max\{\alpha(n), \beta(n)\}$	2.77	2.62	2.61	2.64	2.69	2.75	2.83	3.34	4.05

We may wonder what the other configurations identified in Proposition 1 become under the conditions of Proposition 2. They remain local equilibria but Corollary 2 shows that they are never global.

COROLLARY 2. *If* (3.9) *holds, then all configurations* $\mathbf{a} = (1/2, ..., 1/2, 0, 1/2, ..., 1/2)$, $\mathbf{b} = (1/2, ..., 1/2, 1, 1/2, ..., 1/2)$ *such that firms homogenize on the characteristic n are not Nash equilibria.*

Proof. Consider a configuration where $a_k = b_k = 1/2$ for all $k \neq i$ and $a_i = 0$, $b_i = 1$ for any given $i \neq n$. To show the result, it is sufficient to find a unilateral profitable deviation for firm A. For that, assume that $a_n = 0$ while the other components of $\mathbf{a}$ are unchanged. Then, we are in a two-dimensional problem and it is readily verified that firm A's profit evaluated at the deviation is given by $t_n(5 + t_i/t_n)^2/144$.[5] Since the profit at the initial configuration is equal to $t_i/2$, it is a matter of simple calculation to check that the deviation proposed is always strictly profitable when $t_n > 2.464 t_i$. The result then follows from the inequality $\max\{\alpha(n), \beta(n)\} > 2.464$.

Q.E.D.

Hence, when t_n is large enough, only the local equilibrium involving maximum differentiation along the dimension n is global.

We do not know whether the location equilibrium identified above is unique. However, we show below that some "natural" candidate configurations can be disregarded as possible equilibria. Assume for the moment that the salience coefficients are equal across characteristics. First, we show that maximal product differentiation where $\mathbf{a} = (0, ..., 0)$, $\mathbf{b} = (1, ..., 1)$ can never be a location equilibrium. Consider the linear segment L_{ab} connecting $\mathbf{a}$ and $\mathbf{b}$. We can always map all the consumers distributed over the hypercube on L_{ab} so that we fall back on a one–dimensional problem with quadratic transportation costs. Since the distribution over C is uniform, the distribution over L_{ab} is triangular with its maximum at $(1/2, ..., 1/2)$. For $(\mathbf{a}, \mathbf{b})$ to be a location equilibrium, it must be that $(\mathbf{a}, \mathbf{b})$ is also an equilibrium of the game along L_{ab}. Now, we know from Proposition 2 of Tabuchi and Thisse [26] that $(\mathbf{a}, \mathbf{b})$ is not an equilibrium of this game so that maximal product differentiation cannot be a location equilibrium of the global game.

This argument also applies to any configuration of the type $\mathbf{a} = (\underbrace{0, ..., 0}_{k>1}, 1/2, ..., 1/2)$ and $\mathbf{b} = (\underbrace{1, ..., 1}_{k>1}, 1/2, ..., 1/2)$. Such a configuration

[5] Using numerical analysis, Ansari *et al.* [3] claim that this deviation is the most profitable one.

cannot be an equilibrium because the projection of all consumers onto the line connecting the two candidate points remains triangular provided that $k > 1$. On the contrary, when $k = 1$, the distribution over L_{ab} is uniform and Proposition 2 of Tabuchi and Thisse can no longer be used.

Thus, we have the following result.

COROLLARY 3. *Assume $t_k = t$ for all k. Then, among all combinations max/min, the only possible location equilibria are those identified in Proposition* 1.

When the salience coefficients are not the same, the game on the hypercube is equivalent to a game on a box with different sides but equal salience coefficients. The distribution over L_{ab} is then no longer triangular. For $n = 2$ and $n = 3$ we have been able to show that the max/min configurations excluded in Corollary 3 cannot be an equilibrium either (Irmen and Thisse [17]). We expect this to hold for any $n > 1$.

4. SUMMARY AND CONCLUSION

We have shown that competition in a multi-dimensional characteristics space is inherently different from the standard one-dimensional model. Contrary to the prediction of the latter according to which the market would supply excessive diversity our analysis suggests that consumers are confronted with excessive sameness. Hotelling was "almost right" since minimum differentiation can arise in equilibrium along all but one dimension. In our model, competition governs the process of differentiation since characteristics are independent in consumer preferences. Since firms are always better–off when price competition is relaxed, they choose to differentiate their products. However, our main result suggests that firms find it sufficient to relax price competition by differentiating products in a single characteristic so that the Principle of Minimum Differentiation almost holds. One should also mention that a referee has suggested a lower bound on the distance between the two products depending on the consumer density (which here is uniform), thus invalidating once and for all the Principle of Minimum Differentiation.

This result is in accord with previous findings presented in Anderson *et al.* [1, Chap. 9]. These authors revisit the Hotelling model under the logit and show that firms reduce more and more their spatial differentiation as the parameter of product differentiation in the logit rises. However, product differentiation is implicit in the logit and is not modeled explicitly unlike here where we have determined the equilibrium degree of product differentiation. In other words, our model allows for an endogeneous determination of product differentiation in a market environment where *all* characteristics are chosen strategically.

When there is a dominant characteristic in the sense of Proposition 2, maximum differentiation takes place only along this characteristic because it allows for higher equilibrium prices. The dominant characteristic need not be the same over time. For example, before the Industrial Revolution, it is a well-documented fact that the geographical dimension was dominant and firms were often dispersed. As transportation costs start to decrease, the relative importance of the geographical factor declines relative to other dimensions of the characteristics space. Nowadays, product variety is rather determined by facets of consumer preferences. As a result, firms geographically agglomerate and prefer being differentiated along other, intangible characteristic(s). This observation is in accordance with several results obtained in recent developments in economic geography [12].

Previous results obtained by Neven and Thisse [21] who consider both horizontal and vertical characteristics and by Hirlé *et al.* [15] and Vandenbosch and Weinberg [29] who study a model with two vertical characteristics suggest that the min-min-$\cdots$-min-max configuration identified in the foregoing analysis is also an equilibrium in a multi-characteristics space when all characteristics are vertical or when both types of characteristics are combined. In other words, we offer the conjecture that firms will pick the same amount of each dominated characteristic provided that one firm chooses the highest amount of the dominant characteristic while its rival chooses the smallest amount. Besides cost considerations, it seems that the difference between horizontal and vertical characteristics does not matter for our results.

In the 2-dimensional case, as shown by Neven and Thisse [21], Tabuchi [25], and Ansari *et al.* [3] when the salience coefficients are not too different multiple equilibria may arise but each equilibrium involves maximum differentiation along one characteristic and minimum differentiation along the other. Under the same conditions, we suspect that, in the n-dimensional case, the local equilibria found in Corollary 1 are also Nash equilibria but our proof strategy used for Proposition 2 is no longer appropriate. Furthermore, the sufficient condition identified in Proposition 2 is not necessary, implying that a min-min-$\cdots$-min-max configuration is an equilibrium for lower values of the salience coefficients. It is worth noting, however, that with three characteristics the numerical value found by Ansari *et al.* is close to that given in Proposition 2 when only one salience coefficient is large. Finally, we have seen that with n characteristics configurations with maximum differentiation in more than one and minimum differentiation in the remaining characteristics can be excluded as possible location equilibria.

Our analysis gives rise to several new questions. They include: Can "intermediate" locations be chosen by the firms in the n-dimensional model, as suggested by Vandenbosch and Weinberg [29] for the two-dimensional case with vertically differentiated characteristics? Is there a relationship between the set of location equilibria for different values of the number of dimensions? In particular, does this set expand? We also saw in the intuitive argument presented after Corollary 1 that the uniform distribution of consumers was critical for our proof. A recent paper by Anderson *et al.* [2] characterizes the class of distributions in the one-dimensional model for which a pure strategy equilibrium in location exists. A next step on the research agenda would be to integrate the two analyses. In the case of a distribution defined by the product of n i.i.d. univariate log-concave distributions, we offer the conjecture that firms want to minimize differentiation along $n-1$ characteristics.

Another set of questions deal with the specification of products under entry. Some empirical studies suggest that in some markets entry could well be associated with the addition of one new dimension to the existing characteristics space (see, e.g., Swann [24]). This seems to be consistent with our main result. In an entry context, this can be achieved by adding a new characteristic while imitating the incumbents along the existing characteristics. Finally, it remains to investigate in detail the case of vertically differentiated characteristics and the combination of both types of characteristics in the case of multiple characteristics. Clearly, more work remains to be done.

5. APPENDIX: PROOF OF PROPOSITION 2

The following notation is used throughout the appendix:

$$\begin{aligned}
X &\equiv p_B - p_A + \sum_{k=1}^{n} t_k(b_k^2 - a_k^2) \\
K &\equiv 2t_1(b_1 - a_1) \\
L &\equiv 2t_2(b_2 - a_2) \\
M &\equiv 2t_3(b_3 - a_3) \\
&\vdots \;\vdots\; \vdots \\
V &\equiv 2t_{n-1}(b_{n-1} - a_{n-1}) \\
W &\equiv 2t_n(b_n - a_n)
\end{aligned}$$

We proceed as follows. Since we do not have the explicit forms for the profit functions evaluated at the price equilibrium for each demand piece, we determine an upper bound on these profits. As will be seen in Section 5.2, this bound is obtained when all n characteristics are of equal importance, i.e., $K=L=M=\cdots=V=W$. Therefore, we first derive in Section 5.1 the demand functions under this assumption. We then develop in Section 5.2 a sufficient condition under which the profits associated with the candidate equilibrium, i.e., $t_n/2$, exceed this bound on any other region of the hypercube. This condition is sufficient for the local equilibrium to be a Nash equilibrium.

5.1. *The Symmetric Case*

The n-dimensional unit cube has 2^n corners. Assume that $K<L<M<\cdots<V<W$. The slope of the marginal consumer hyperplane is different with respect to each of the $(n-1)$ characteristics. There are 2^{n-1} demand pieces and the sequence of corners hit is fully determined. First, the corner $(1, 0, ..., 0)$ is hit, i.e., the corner associated with the least important characteristic K. Notice that when $K=L$ both corners $(1, 0, ..., 0)$ and $(0, 1, ..., 0)$ are hit at the same time which reduces the number of demand pieces by one. Furthermore, the market space corresponding to the first demand piece contains more consumers when $K=L$. More generally, when the number of corners hit at the same time rises, more consumers belong to the corresponding market space.

Assume now that all characteristics are equally important, i.e., $K=L=M=\cdots=V=W$. All magnitudes associated with this symmetric case are denoted with a bar. Then, the marginal consumer hyperplane becomes

$$\bar{z}_n(z_1, z_2, ..., z_{n-1})=\frac{\bar{X}}{V}-\sum_{k=1}^{n-1} z_k. \tag{5.1}$$

In this case, the number of demand pieces is reduced to n, i.e., it coincides with the number of characteristics. The slope of (5.1) with respect to all characteristics is one. Thus, by shifting upward the hyperplane several corners are hit at the same time. Let $1\leqslant j\leqslant n$ be the index of these demand pieces. The number of corners hit at the upper bound of the jth demand piece equals $\binom{n}{j}$. We derive firm A's first three demand pieces. We then develop the expression of the demand function.

First demand piece. Firm A's first demand piece is characterized by the fact that no corner of the hypercube is hit. The price p_A for which A's demand is zero can be determined from $\bar{z}_n(0, ..., 0)=0$. Hence, $\bar{D}_A^1>0$ iff $\bar{X}>0$. The lower bound of the corresponding price interval, i.e., the lowest price for which $\bar{D}_A^1$ is firm A's demand, is reached when the marginal consumer hyperplane hits the first n corners of the hypercube C. It is given by $\bar{z}_n(0, ..., 0) = 1 : \bar{X} = V$. Hence, $\bar{D}_A^1$ is defined over the $\bar{X}$-interval

$$\bar{X} \in [0, V[.$$

Observe that the marginal consumer hyperplane intersects the z_1-axis at the point $\bar{z}_1(0, ..., 0)$, the z_2-axis in a line $\bar{z}_2(z_1, 0, ..., 0)$, and finally the z_{n-1}-axis in the $(n-2)$-dimensional hyperplane $\bar{z}_{n-1}(z_1, z_2, ..., z_{n-2}, 0)$. Thus, the analytical expression for $\bar{D}_A^1$ is given by

$$\bar{D}_A^1 = \int_0^{\bar{z}_1(0, ..., 0)} \int_0^{\bar{z}_2(z_1, 0, ..., 0)} \cdots \int_0^{\bar{z}_{n-1}(z_1, z_2, ..., z_{n-2}, 0)} \bar{z}_n(z_1, z_2, ..., z_{n-1}) \times dz_{n-1}\, dz_{n-2} \cdots dz_1. \tag{5.2}$$

This is equivalent to

$$\bar{D}_A^1 = \frac{\bar{X}^n}{n!\, V^n}.$$

Second demand piece. The second piece of A's demand arises for $\bar{X} \geqslant V$. The lower bound of the interval associated with $\bar{D}_A^2$ is given by the value of $\bar{X}$ for which the next $\binom{n}{2}$ corners are hit. The corresponding price level is obtained from $\bar{z}_n(0, 0, ..., 0) = 2 : \bar{X} = 2V$. Hence, $\bar{D}_A^2$ is defined for

$$\bar{X} \in [V, 2V[.$$

Using (5.2) we have[6]

$$\bar{D}_A^2 = \bar{D}_A^1 - n \int_0^{\bar{z}_1 - 1} \int_0^{\bar{z}_2 - 1} \cdots \int_0^{\bar{z}_{n-1} - 1} \bar{z}_n(z_1, z_2, ..., z_{n-1}) \times dz_{n-1}\, dz_{n-2} \cdots dz_1 \tag{5.3}$$

which gives

$$\bar{D}_A^2 = \frac{X^n - n(X - V)^n}{n!\, V^n}.$$

Third demand piece. As p_A falls further, the marginal consumer hyperplane shifts outward until the next $\binom{n}{3}$ corners are hit, i.e., where

[6] Observe that we will omit the arguments of the upper bounds of the integrals to save space. It should however be understood that $\bar{z}_1 = \bar{z}_1(0, ..., 0)$; $\bar{z}_2 = \bar{z}_2(z_1, 0, ..., 0)$; ...; $\bar{z}_{n-1} = \bar{z}_{n-1}(z_1, z_2, ..., z_{n-2}, 0)$.

$\bar{z}_n(0, 0, ..., 0) = 3 : \bar{X} = 3V$. Therewith, the third demand piece is defined over

$$\bar{X} \in [2V, 3V[.$$

Using (5.3), $\bar{D}_A^3$ is given by

$$\bar{D}_A^3 = \bar{D}_A^2 + \binom{n}{2} \int_0^{\bar{z}_1 - 2} \int_0^{\bar{z}_2 - 2} \cdots \int_0^{\bar{z}_{n-1} - 2} \bar{z}_n(z_1, z_2, ..., z_{n-1}) \times dz_{n-1}\, dz_{n-2} \cdots dz_1$$

which results in

$$\bar{D}_A^3 = \frac{\bar{X}^n - n(\bar{X} - V)^n + \binom{n}{2} (\bar{X} - 2V)^n}{n!\, V^n}.$$

The jth demand piece. Following the same constructive approach, the expression for the jth demand piece can be derived. Specifically, the analytical expression for $\bar{D}_A^j$ and $n \geqslant 1$ is given by

$$\bar{D}_A^j = \sum_{i=0}^{j-1} (-1)^i \binom{n}{i} \int_0^{\bar{z}_1 - i} \int_0^{\bar{z}_2 - i} \cdots \int_0^{\bar{z}_{n-1} - i} \bar{z}_n(z_1, z_2, ..., z_{n-1}) \times dz_{n-1}\, dz_{n-2} \cdots dz_1 \tag{5.4}$$

which reduces to

$$\bar{D}_A^j = \frac{\sum_{i=0}^{j-1} (-1)^i \binom{n}{i} (\bar{X} - iV)^n}{n!\, V^n}. \tag{5.5}$$

The $\bar{X}$-interval over which $\bar{D}_A^j$ is defined is

$$\bar{X} \in [(j-1)\, V, jV[.$$

Based on this expression for demand, we can proceed with the computations of the profit suprema.

5.2. *Profit Bounding*

In this section, we derive the upper bounds on firms' A and B profits. We limit ourselves to the first $2^{n-1} - 1$ demand pieces under weak and strong dominance, and to the 2^{n-1}th (middle) demand piece under weak dominance, for firms A and B. This is sufficient to show that $\mathbf{a}^*$ is firm A's best reply, because by symmetry the bounds on firm A's profit in the

$2^{n-1}+1, 2^{n-1}+2, ..., 2^n-1$ pieces are identical to those of firm B in the first $2^{n-1}-1$ pieces.

Furthermore, we restrict our attention to the case where n is odd. Indeed, the sufficient condition for n is also sufficient for $n-1$, since the optimal deviation in lower dimensional characteristics spaces can always be replicated in higher dimensional ones, so that the case where n is even is covered.[7]

Firm A. Here we determine an upper bound on firm A's profit for the first 2^{n-1} demand pieces. We show that this bound is associated with the 2^{n-1}th demand piece under weak dominance. We then demonstrate that the bound for this demand piece is obtained when all characteristics are of equal importance. Finally, we determine the upper bound on A's profit for this demand piece. This leads to the sufficient condition under which any unilateral deviation from $\mathbf{a}^*=(1/2, 1/2, ..., 1/2, 0)$ is unprofitable for firm A when firm B is located at $\mathbf{b}^*=(1/2, 1/2, ..., 1/2, 1)$.

The upper bound on A's demand is determined as follows. Recall first that all demand pieces can be expressed as a function $D_A(X)$. By definition of X, we have $\partial D_A(X)/\partial X=\partial D_A(X)/\partial p_B \geqslant 0$, i.e., $D_A(X)$ increases in X. Consequently, the highest possible value of firm A's demand over the first 2^{n-1} demand pieces is given for $X=\tilde{X}=K+L+M+\cdots+V$. Hence, the demand pieces corresponding to lower prices can be eliminated. Under weak dominance, this highest demand corresponds to the upper bound of the 2^{n-1}th demand piece, i.e., $D_A^{2^{n-1}}|_{X=K+L+M+\cdots+V}$, while it corresponds to the upper bound of the $(2^{n-1}-1)$th demand piece under strong dominance, i.e., $\tilde{D}_A^{2^{n-1}-1}|_{\tilde{X}=K+L+M+\cdots+V}$. In order to show that the bound on profits is associated with the 2^{n-1}th demand piece under weak dominance, it is then sufficient to prove that a deviation in the $(2^{n-1}-1)$th piece under strong dominance is never profitable. First, $\tilde{D}_A^{2^{n-1}-1}|_{\tilde{X}=K+L+M+\cdots+V} \leqslant 1/2$. Second, due to the continuity of the best reply function we have $\tilde{p}_A^{2^{n-1}-1} \leqslant t_n$.

Consequently, the bound on demand is associated with the 2^{n-1}th demand piece under weak dominance. The highest price p_A for which $D_A^{2^{n-1}}$ arises is reached when X is largest, i.e., $X=(n-1)W$. This follows from $X=(n-1)\ W \geqslant (n-1)\ V \geqslant K+L+M+\cdots+V$ and implies that $K=L=M=\cdots=V=W$. Thus, the bound on demand is attained when all characteristics are equally important. Using (5.5), this bound is given by $\bar{D}_A^j|_{\bar{X}=jV}$, where $j=(n+1)/2$. This leads to

$$D_A^{2^{n-1}\,\sup}=\bar{D}_A^{(n+1)/2}|_{\bar{X}=(n+1)\ V/2}=\frac{\sum_{i=0}^{(n-1)/2}(-1)^i\binom{n}{i}(n+1-2i)^n}{2^n n!}. \tag{5.6}$$

[7] Notice that the above does not imply that the profit supremum for higher dimensional product spaces exceeds those found for lower dimensional spaces.

Regarding the upper bound on p^i_A, $i=1, 2, ..., 2^{n-1}$, we choose the highest price on firm A's best reply function. It is given by

$$p^i_A = \frac{D_1}{\left|\dfrac{\partial D_A}{\partial p_A}\right|}. \tag{5.7}$$

The latter defines a function $p_A(X)$. By definition we have $dp_A/dX = dp_A/dp_B$. Since prices are strategic complements we obtain $dp_A/dX \geqslant 0$. Hence, the highest p_A on A's best reply function is obtained for the highest admissible value of X, i.e., $X=(n-1)\,W$ where $K=L=M=\cdots=V=W$. The price bound is then obtained using the demand (5.5) where $j=(n+1)/2$. Hence this bound is given by

$$\begin{aligned} p_A^{2^{n-1}\,\mathrm{sup}} &= \bar{p}_A^{(n+1)/2}\big|_{\bar{X}=(n+1)\,V/2} \\ &= \frac{\sum\limits_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^n}{\left|\sum\limits_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^{n-1}\right|}\,(2n)^{-1}\,V. \end{aligned} \tag{5.8}$$

Accordingly, the profit upper bound

$$\pi_A^{2^{n-1}\,\mathrm{sup}} = \bar{\pi}_A^{(n+1)/2} = \bar{p}_A^{(n+1)/2}\big|_{\bar{X}=(n+1)\,V/2}\bar{D}_A^{(n+1)/2}\big|_{\bar{X}=(n+1)\,V/2}$$

is obtained as

$$\begin{aligned} \pi_A^{2^{n-1}\,\mathrm{sup}} = \frac{\left[\sum\limits_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^n\right]^2}{\left|\sum\limits_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n+1-2i)^{n-1}\right|} \\ \times (2^n n n!)^{-1}\, t_{n-1}(1/2 - a_{n-1}). \end{aligned} \tag{5.9}$$

Denote the RHS by $\alpha(n)\,t_{n-1}(1/2 - a_{n-1})$. The highest attainable profit is reached at $a_{n-1}=0$. Hence, deviating from $\mathbf{a}^*$ into the first 2^{n-1} demand pieces is unprofitable if

$$\alpha(n)\,t_{n-1} < t_n. \tag{5.10}$$

Table 5.1 contains numerical results for $n=3, 5, ..., 31$.

TABLE 5.1

Firm A: The Coefficient of t_n for a Nash Equilibrium

n	3	5	7	9	11	13	15	21	31
$\alpha(n)$	2.77	2.62	2.61	2.64	2.69	2.75	2.81	2.99	3.28

Firm B. The argument used here is different. At any price equilibrium corresponding to the first $2^{n-1}-1$ pieces for B, we have $p_A^* \geqslant p_B^*$. This follows from $D_B^i \geqslant 1/2$, $i=1, 2, ..., 2^{n-1}-1$. Hence, the highest possible value for firm B's equilibrium price coincides with the highest admissible price p_A on A's best reply function, i.e., $p_A^{i\,\sup}$, $i=1, 2, ..., 2^{n-1}-1$. Observe also that $D_B^{1\,\sup}=1$ for the first $2^{n-1}-1$ pieces of firm B's demand.

The profit bound for the first $2^{n-1}-1$ demand pieces for firm B is given by $\pi_B^{i\,\sup} = p_B^{(2^{n-1}-1)\,\sup} = p_A^{(2^{n-1}-1)\,\sup}$, $i=1, 2, ..., (2^{n-1}-1)$.[8] It follows from the continuity of the best reply function that the price bound lies on the boundary of the domains of weak and strong dominance, i.e., when $K+L+M+\cdots+V=W$. Accordingly, we have

$$\begin{aligned}\tilde{p}_B^{(2^{n-1}-1)\,\sup} &= \tilde{p}_A^{(2^{n-1}-1)\,\sup} = p_B^{(2^{n-1}-1)\,\sup} = p_A^{(2^{n-1}-1)\,\sup} \\ &= \tilde{p}_A^{(2^{n-1}-1)}\big|_{X=K+L+M+\cdots+V} = p_A^{(2^{n-1}-1)}\big|_{X=W} \\ &= \frac{K+L+M+\cdots+V}{2} = \frac{W}{2}.\end{aligned}$$

Note that for $K+L+M+...+V=W$ the middle demand piece vanishes and $D_i=1/2$, $i=A, B$. However, when both firms equally share the market one can show that the price level under $K=L=M=\cdots=V=W$ exceeds $W/2$ and is, therefore, an upper bound on firm B's profits. Thus, we evaluate (5.7) by using (5.5) for $j=(n+1)/2$ and $\bar{X}=nV/2$ and obtain

$$p_A^{(2^{n-1}-1)\,\sup} = \bar{p}_A^{(n+1)/2}\big|_{\bar{X}=nV/2} = \frac{2^{n-2}(n-1)!}{\left|\sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{i} (n-2i)^{n-1}\right|} V. \tag{5.11}$$

Denote the RHS by $\beta(n)\, V/2 = \beta(n)\, t_{n-1}(b_{n-1}-1/2)$. The most profitable deviation arises for $b_{n-1}=1$. Hence,

$$\pi_B^{(2^{n-1}-1)\,\sup} = \bar{p}_A^{(n+1)/2}\big|_{X=nV/2} = \frac{\beta(n)}{2}\, t_{n-1}. \tag{5.12}$$

[8] Since $p_A^{2^{n-1}\,\sup} > p_A^{(2^{n-1}-1)\,\sup}$ and $D_B^{1\,\sup}=1$, an upper bound on profit for the first $2^{n-1}-1$ demand pieces of firm B is given by $p_A^{2^{n-1}\,\sup}$ of equation (5.8). However, the bound on profit developed below is lower.

TABLE 5.2

Firm B: The Coefficient of t_n for a Nash Equilibrium

n	3	5	7	9	11	13	15	21	31
$\beta(n)$	1.33	1.67	1.96	2.21	2.43	2.64	2.83	3.34	4.05

Deviation into the first $2^{n-1}-1$ demand pieces is then unprofitable for firm B if

$$\beta(n)\, t_{n-1} < t_n. \tag{5.13}$$

Table 5.2 contains some numerical results for $n = 3, 5, ..., 31$.

By symmetry firm B's $2^{n-1}-1$ demand piece under weak dominance yields the same bound as that of firm A. Hence, we choose $D_A^{(2^{n-1}-1)\,\text{sup}} = D_B^{(2^{n-1}-1)\,\text{sup}}$ as in (5.6) and $p_B^{(2^{n-1}-1)\,\text{sup}} = p_A^{(2^{n-1}-1)\,\text{sup}}$ from (5.8). Thus the profit bound of firm B coincides with $\pi_A^{(2^{n-1}-1)\,\text{sup}}$ of (5.9) and the condition for deviation from $\mathbf{b}^*$ to be unprofitable is identical to (5.10).

To sum-up, we can say that the more restrictive of the conditions (5.10) and (5.13), given by the maximum of $\alpha(n)$ and $\beta(n)$, is sufficient for the local equilibrium to be a Nash equilibrium. In particular, we observe from Tables 5.1 and 5.2 that

$$\alpha(n) > \beta(n) \qquad \text{for} \quad 3 \leqslant n \leqslant 13$$

$$\alpha(n) < \beta(n) \qquad \text{for} \quad n > 13.$$

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