1 Proof of Equation (4)

et us define a country by a closed, connected set $\Theta \in \mathbb{R}^2$, which is broken up into a countable collection of open sets $\theta_i,\ i = 1, ..., n$, such that for any $i \neq j,\ \theta_i \cap \theta_j = \emptyset$, and letting the closure of θ_i be denoted, $\bar{\theta}_i$, $\cup_{i=1}^{k} \bar{\theta}_i = C$.[1]

Moreover, two sets θ_i, θ_j are denoted *adjacent* if $\bar{\theta}_i \cap \bar{\theta}_j \neq 0$, for $i \neq j$, and that the cardinality, $\#(\bar{\theta}_i \cap \bar{\theta}_j) = \infty$.[2] Then for two adjacent counties, i, j, we denote their border by $B_{ij} = \bar{A}_i \cap A_j$. Further, they are *cornered* if $\bar{A}_i \cap \bar{A}_j \neq 0,\ i \neq j$, and $\#(\bar{\theta}_i \cap \bar{\theta}_j) < \infty$.

We now make the following definitions.[3]

$$B = \{b \in \bar{\theta}_i \cap \theta_j, \forall i, j,\ i \neq j\} \tag{1}$$

$$B_{ij} = \{b \in \bar{\theta}_i \cap \bar{\theta}_j,\ i \neq j\} \tag{2}$$

$$H(\epsilon) = \{\cup_{b \in B} \left((\mathcal{B}_\epsilon(b) \cap \theta_i) \cup (\mathcal{B}_\epsilon(b) \cap \theta_j) \right),\ \forall i, j,\ i \neq j\} \tag{3}$$

Where, $\mathcal{B}_\epsilon(b)$ denotes the epsilon ball around a point b.

Now, let us have a set of K covariates, where each one is in $\mathbb{R}$ that we are interested in over T time periods, such that we have the random matrix $\{\Omega_\alpha, \epsilon_\alpha : \alpha \in \Theta\}$ be a set of random matrices such that Ω_α is a $T \times K$ matrix, and ϵ is a $T \times 1$ vector, and we believe that firm entry rate for each location is determined by,

$$y_\alpha = \Omega_\alpha \Gamma_\alpha + \epsilon_\alpha$$

Where, let us split Ω_α into a partitioned matrix, $\Omega_\alpha = [Z_\alpha\ X_\alpha]$, and $\Gamma_\alpha = [\gamma_\alpha \beta_\alpha]$. Then,

$$y_\alpha = Z_\alpha \gamma_\alpha + X_\alpha \beta_\alpha + e_\alpha$$

Then, for any $\alpha \in \theta_i,\ \alpha \in \theta_j$ such that $\bar{\theta}_i \cap \bar{\theta}_j \neq \emptyset$,

$$y_\alpha - y_{\alpha'} = Z_\alpha \gamma_\alpha - Z_{\alpha'} \gamma_{\alpha'} + X_\alpha \beta_\alpha - X_{\alpha'} \beta_{\alpha'} + e_\alpha - e_{\alpha'}$$

Now, we need that, $E[Z_\alpha \gamma_\alpha - Z_{\alpha'} \gamma_{\alpha'}] \to 0$ as $\alpha, \alpha' \to b = \{\lambda \alpha + (1 - \lambda) \alpha' : \forall \lambda \in (0, 1)\} \cap \{\theta_i \cap \theta_j\}$. Then,

$$Z_\alpha \gamma_\alpha - Z_{\alpha'} \gamma_{\alpha'} = (Z_\alpha - Z_{\alpha'})(\gamma_\alpha - \gamma_{\alpha'}) + (Z_\alpha - Z_{\alpha'}) \gamma_{\alpha'} + (\gamma_\alpha - \gamma_{\alpha'}) \beta_{\alpha'}$$

[1] The point of this is to think of Θ as a country, and θ as a collection of states. This process will be iterated into local counties as well. I state that they are all open under the perception that the border is not in any particular object, but exists in the limit. Finally, $\mathcal{F}$ defines the data generating process on the set of points around each border.

[2] i.e. state pairs have to be connected at more than a finite number of points

[3] The B take all the border points into a set and a state-pair respectively. The H, take all the points in each state that fit a fixed bandwidth for each point along the borders.

But, as $\alpha, \alpha' \to b$, $||\alpha, \alpha'||_2 < \delta \implies (Z_\alpha - Z_{\alpha'}) < \epsilon$, and $((\gamma_\alpha - \gamma_{\alpha'}) < \epsilon$. Thus,

$$Z_\alpha \gamma_\alpha - Z_{\alpha'} \gamma_{\alpha'} = \epsilon^2 + 2\epsilon\gamma_b$$

Since ϵ is arbitrary, $Z_\alpha \gamma_\alpha - Z_{\alpha'} \gamma_{\alpha'} \to 0$, and

$$\lim_{\alpha, \alpha' \to b} y_\alpha - y_{\alpha'} = X_\alpha - X_{\alpha'} \beta_b + e_\alpha - e_{\alpha'}$$