Chapter 10

The Ramsey model

As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician and economist Frank Ramsey (1903-1930). Ramsey's contribution was mathematically demanding and did not experience much response at the time. Three decades had to pass until his contribution was taken up seriously (Samuelson and Solow, 1956). The model was fusioned with Solow's simpler growth model (Solow 1956) and became a cornerstone in neoclassical growth theory from the mid 1960s. The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the *Ramsey-Cass-Koopmans model*.

The model is one of the basic workhorse models in macroeconomics. It can be seen as placed at one end of a line segment. At the other end appears another workhorse model, namely Diamond's overlapping generations model. In the Diamond model there is an *infinite* number of agents (since in every new period a new generation enters the economy) and these have a *finite* time horizon. In the Ramsey model there is a *finite* number of agents with an *infinite* time horizon; further, these agents are completely alike. The Ramsey model is thus a *representative agent* model, whereas the Diamond model has heterogeneous agents (young and old) interacting in every period. There are important economic questions where these differences in the model setup lead to salient differences in the answers.

Along the line segment where these two frameworks are polar cases, less abstract models are scattered, some being closer to the one pole and others closer to the other. A given model may open up for different regimes, one close to Ramsey's pole, another close to Diamond's. An example is Robert Barro's model with parental altruism presented in Chapter 7. When the bequest motive is operative, the Barro model coincides with a Ramsey model (in discrete time). But when the bequest motive is not operative, the Barro model coincides with a Diamond OLG model. The Blanchard (1985) OLG model in continuous time to be considered in the chapters 12 and 13 also belongs to the interior of this line segment, but closer to the Diamond pole than the Ramsey pole.

This chapter concentrates on the continuous time version of the Ramsey model as did Ramsey's own original contribution. We first study the Ramsey framework under the conditions of a perfectly competitive market economy. In this context we will see, for example, that the Solow growth model comes out as a special case of the Ramsey model. Next we consider the Ramsey framework in a setting with an "all-knowing and all-powerful" social planner.

10.1 Preliminaries

We consider a closed economy. Time is continuous. We assume that the households own the capital goods and hire them out to firms at a market rental rate $\hat{r}$. This is just to have something concrete in mind. If instead the capital goods were owned by the firms using them in production and the capital investment by these firms were financed by issuing shares and bonds, the conclusions would remain unaltered as long as we ignore uncertainty.

The variables in the model are considered as (piecewise) continuous and differentiable functions of time, t. Yet, to save notation, we shall write them as w_t, $\hat{r}_t$, etc. instead of $w(t)$, $\hat{r}(t)$, etc. as in the previous chapter. In every short time interval $(t, t+\Delta t)$, the individual firm employs labor at the market wage w_t and rents capital goods at the rental rate $\hat{r}_t$. The combination of labor and capital produces the homogeneous output good. This good can be used for consumption as well as investment. So in every short time interval there are at least three active markets, one for the "all-purpose" output good, one for labor, and one for capital services (the rental market for capital goods). For the sake of intuition it may be useful to imagine that there is also a market for loans, which we shall name the *credit market* and which has a variable short-term interest rate, r_t. All households are alike, however, and so this market will not be active in general equilibrium. There is perfect competition in all markets, that is, prices are exogenous to the individual households and firms. Any need for means of payment − money − is abstracted away. Prices are measured in current output units.

There are no stochastic elements in the model. We assume that households understand precisely how the economy works and can predict the future path of wages and interest rates. That is, we assume "rational expectations" which in our non-stochastic setting amounts to perfect foresight. (It turns out that the Ramsey model always generates unique equilibrium paths, so that the strong assumption of perfect foresight is slightly less problematic than in the Diamond OLG model, where under certain circumstances multiple equilibria paths could arise, cf. Chapter 3.)

As uncertainty is assumed absent, rates of return on alternative assets must in equilibrium be the same. So if otherwise the credit market were active, the interest rate would equal the rate of return on holding capital goods, i.e.,

$$r_t = \hat{r}_t - \delta, \tag{10.1}$$

where δ (≥ 0) is a constant rate of capital depreciation. This no-arbitrage condition indicates how the rental rate of capital is related to the short-term interest rate.

Below we present, first, the households' behavior and next the firms' behavior. After this, the interaction between households and firms in general equilibrium and the resulting dynamics will be analyzed.

10.2 The agents

10.2.1 Households

There is a fixed number of identical households with an infinite time horizon. This feature makes aggregation very simple: we just have to multiply the behavior of a single household with the number of households. We may interpret the infinite horizon of the household as reflecting the presence of an altruistic bequest motive. That is, the household is seen as an infinitely-lived family, a dynasty, whose current members act in unity and are concerned about the utility from own consumption as well as the utility of the future generations within the family.[1] Every family has L_t members and L_t changes over time at a constant rate, n :

$$L_t = L_0 e^{nt}, \qquad L_0 > 0. \tag{10.2}$$

Indivisibility problems are ignored.

Each family member supplies inelastically one unit of labor per time unit. Equation (10.2) therefore describes the growth of the population as well as the labor force. Since there is only one consumption good, the only decision problem is how to distribute current income between consumption and saving.

[1] The Barro model of Chapter 7 exemplifies such a structure in discrete time. In that chapter we also discussed the shortcomings of this dynasty interpretation.

Intertemporal utility function

From now we consider a single household/family/dynasty. Its preferences can be represented by an additive intertemporal utility function with a constant rate of time preference, ρ. Seen from time 0, the function is

$$U_0 = \int_0^\infty u(c_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member. The instantaneous utility function, $u(c)$, has $u'(c) > 0$ and $u''(c) < 0$, i.e., positive but diminishing marginal utility of consumption. The utility contribution from consumption per family member is weighed by the number of family members, L_t. So it is the sum of the family members' utility that counts. Such a utility function is sometimes referred to as a *classical-utilitarian* utility function (with discounting).

In passing we note that in the Ramsey setup births (into adult life) do not reflect the emergence of *new* economic agents with independent interests. Births and population growth are seen as just an expansion of the size of already existing infinitely-lived households. In contrast, in the Diamond OLG model births imply entrance of new economic decision makers whose preferences no-one has cared about in advance.

Because of (10.2), U_0 can be written as

$$U_0 = \int_0^\infty u(c_t) e^{-(\rho - n)t} dt, \tag{10.3}$$

where the unimportant positive factor L_0 has been eliminated. Here $\bar{\rho} \equiv \rho - n$ is known as the *effective* rate of time preference while ρ is the *pure* rate of time preference. We later introduce a restriction on $\rho - n$ to ensure upward boundedness of the utility integral in general equilibrium.

The household chooses a consumption-saving plan which maximizes U_0 subject to its budget constraint. Let $A_t \equiv a_t L_t$ be the household's (net) financial wealth in real terms at time t. It is of no consequence whether we imagine the components of this wealth are capital goods or loans to other agents in the economy. We have

$$\dot{A}_t = r_t A_t + w_t L_t - c_t L_t, \qquad A_0 \text{ given.} \tag{10.4}$$

This equation is a book-keeping relation telling how financial wealth or debt $(-A)$ is evolving depending on how consumption relates to current income. The equation merely says that the increase in financial wealth per time unit equals saving which equals income minus consumption. Income is the sum of the net return on financial wealth, r_tA_t, and labor income, w_tL_t, where w_t is the real wage.[2] Saving can be negative. In that case the household "dissaves" and does so simply by selling a part of its stock of capital goods or by taking loans in the credit market.[3]

When the dynamic budget identity (10.4) is combined with a requirement of solvency, we have a budget *constraint*. The relevant solvency requirement is the No-Ponzi-Game condition (NPG for short):

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{10.5}$$

This condition says that financial wealth far out in the future cannot have a negative present value. That is, in the long run, debt is at most allowed to rise at a rate *less* than the real interest rate r. The NPG condition thus precludes permanent financing of the interest payments by new loans.[4]

The decision problem is: choose a plan $(c_t)_{t=0}^{\infty}$ so as to achieve a maximum of U_0 subject to non-negativity of the control variable, c, and the constraints (10.4) and (10.5). The problem is a slight generalization of the problem studied in Section 9.4 of the previous chapter.

To solve the problem we shall apply the Maximum Principle. This method can be applied directly to the problem as stated above or to an equivalent problem with constraints expressed in per capita terms. Let us follow the latter approach. From the definition $a_t \equiv A_t/L_t$ we get by differentiation w.r.t. t

$$\dot{a}_t = \frac{L_t\dot{A}_t - A_t\dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Substitution of (10.4) gives the dynamic budget identity in per capita terms:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given.} \tag{10.6}$$

By inserting $A_t \equiv a_tL_t = a_tL_0e^{nt}$, the No-Ponzi-Game condition (10.5) can be rewritten as

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{10.7}$$

where the unimportant factor L_0 has been eliminated.

[2]Since the technology exhibits constant returns to scale, in competitive equilibrium the firms make no (pure) profit to pay out to their owners (presumably the households).

[3]The market prices, w_t and r_t, faced by the household are assumed to be piecewise continuous functions of time.

[4]In the previous chapter we saw that the NPG condition, in combination with (10.4), is equivalent to an ordinary *intertemporal* budget constraint which says that the present value of the planned consumption path cannot exceed initial total wealth, i.e., the sum of the initial financial wealth and the present value of expected future labor income.

We see that in both (10.6) and (10.7) a kind of corrected interest rate appears, namely the interest rate, r, minus the family size growth rate, n. Although deferring consumption gives a real interest rate of r, this return is diluted on a per head basis because it will have to be shared with more members of the family when $n > 0$. In the form (10.7) the NPG condition requires that debt, if any, in the long run rises at most at a rate *less* than $r - n$.

Solving the consumption/saving problem

The decision problem is now: choose $(c_t)_{t=\infty}^{0}$ so as to a maximize U_0 subject to the constraints: $c_t \geq 0$ for all $t \geq 0$, (10.6), and (10.7). The solution procedure is similar to that in the simpler problem of the previous chapter:

1) We set up the current-value Hamiltonian

$$H(a, c, \lambda, t) = u(c) + \lambda \left[(r - n) a + w - c \right],$$

where λ is the *adjoint variable* associated with the dynamic constraint (10.6).

2) We differentiate H partially w.r.t. the control variable and put the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0,$$

that is,

$$u'(c) = \lambda. \tag{10.8}$$

3) We differentiate H partially w.r.t. the state variable, a, and put the result equal to the effective discount rate (appearing in the integrand of the criterion function) multiplied by λ minus the time derivative of the adjoint variable λ:

$$\frac{\partial H}{\partial a} = \lambda(r - n) = (\rho - n)\lambda - \dot{\lambda},$$

that is,

$$\dot{\lambda} = -(r - \rho)\lambda. \tag{10.9}$$

4) Finally, we apply the Maximum Principle: an interior optimal path $(a_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda(t)$ such that for all $t \geq 0$, (10.8) and (10.9) hold along the path and the transversality condition,

$$\lim_{t \to \infty} a_t \lambda_t e^{-(\rho - n)t} = 0, \tag{10.10}$$

is satisfied.

The interpretation of these optimality conditions is as follows. The condition (10.8) can be considered a $MC = MB$ condition (in utility terms). It illustrates together with (10.9) that the adjoint variable λ can be seen as the shadow price, measured in current utility, of per head financial wealth along the optimal path. Rearranging (10.9) gives, $r_t = \rho - \dot{\lambda}_t/\lambda_t$; the left-hand-side of this equation is the market rate of return on saving while the right-hand-side is the *required* rate of return (as in the previous chapter, by subtracting the shadow price "inflation rate" from the required utility rate of return, ρ, we get the required real rate of return). The household is willing to save the marginal unit only if the actual return equals the required return.

The transversality condition (10.10) says that for $t \to \infty$ the present shadow value of per head financial wealth should go to zero. Combined with (10.8), the condition is that

$$\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} = 0 \tag{10.11}$$

must hold along the optimal path. This requirement is not surprising if we compare with the case where instead $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} > 0$. In this case there would be over-saving; U_0 could be increased by ultimately consuming more and saving less, that is, reducing the "ultimate" a_t. The opposite case, $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} < 0$, will not even satisfy the NPG condition in view of Proposition 2 of the previous chapter. In fact, from that proposition we know that the transversality condition (10.11) is equivalent with the NPG condition (10.7) being satisfied with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s-n)ds} = 0. \tag{10.12}$$

Recall that the Maximum Principle gives only *necessary* conditions for an optimal plan. But since the Hamiltonian is jointly concave in (a, c) for every t, the necessary conditions are also *sufficient*, by Mangasarian's sufficiency theorem.

The first-order conditions (10.8) and (10.9) give the Keynes-Ramsey rule:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(r_t - \rho), \tag{10.13}$$

where $\theta(c_t)$ is the (absolute) elasticity of marginal utility,

$$\theta(c_t) \equiv -\frac{c_t}{u'(c_t)}u''(c_t) > 0. \tag{10.14}$$

As we know from previous chapters, this elasticity indicates the consumer's wish to smooth consumption over time. The inverse of $\theta(c_t)$ is the elasticity of intertemporal substitution in consumption. It indicates the willingness to incur variation in consumption over time in response to a change in the interest rate.

Interestingly, the population growth rate, n, does not appear in the Keynes-Ramsey rule. Going from $n = 0$ to $n > 0$ implies that r_t is replaced by $r_t - n$ in the dynamic budget identity and ρ is replaced by $\rho - n$ in the criterion function. This implies that n cancels out in the Keynes-Ramsey rule. Yet n appears in the transversality condition and thereby also in the *level* of consumption for given wealth, cf. (10.18) below.

CRRA utility

In order that the model can accommodate Kaldor's stylized facts, it should be able to generate a balanced growth path. When the population grows at the same constant rate as the labor force, here n, by definition balanced growth requires that per capita output, per capita capital, and per capita consumption grow at constant rates. This will generally require that the real interest rate is constant in the process. But (10.13) shows that having a constant per capita consumption growth rate at the same time as r is constant, is only possible if the elasticity of marginal utility does *not* vary with c. Hence, we will from now on assume that the right-hand-side of (10.14) is a positive constant, θ. This amounts to assuming that the instantaneous utility function is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0. \tag{10.15}$$

Recall, that the right-hand side can be interpreted as $\ln c$ when $\theta = 1$. So our Keynes-Ramsey rule simplifies to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{10.16}$$

The Keynes-Ramsey rule characterizes the optimal *rate of change* of consumption. The optimal initial *level* of consumption, c_0, will be the highest feasible c_0 which is compatible with both the Keynes-Ramsey rule and the NPG condition. And for this reason the choice will exactly comply with the transversality condition (10.12). Although an explicit determination of c_0 is not actually necessary to pin down the equilibrium path of the economy, we note in passing that c_0 can be found by the method described at the end of Chapter 9. Indeed, given the book-keeping relation (10.6), we have by Proposition 1 of Chapter 9 that the transversality condition (10.12) is equivalent with satisfying the following intertemporal budget constraint (with strict equality):

$$\int_0^\infty c_t e^{-\int_0^t (r_\tau - n)d\tau} dt = a_0 + h_0. \tag{10.17}$$

Solving the differential equation (11.39) we get $c_t = c_0 e^{\frac{1}{\theta}\int_0^t (r_\tau - \rho)d\tau}$, which we substitute for c_t in (10.17). Isolating c_0 now gives[5]

$$\begin{aligned} c_0 &= \beta_0(a_0 + h_0), \quad \text{where} \qquad (10.18)\\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_\tau - \rho}{\theta} + n)d\tau} dt}, \quad \text{and} \\ h_0 &= \int_0^\infty w_t e^{-\int_0^t (r_\tau - n)d\tau} dt. \end{aligned}$$

Thus, the entire expected future evolution of wages and interest rates determines c_0. The marginal propensity to consume out of wealth, β_0, is less, the greater is the population growth rate n.[6] The explanation is that the effective utility discount rate, $\rho - n$, is less, the greater is n. The propensity to save is greater the more mouths to feed in the future. The initial saving level will be $r_0 a_0 + w_0 - c_0 = r_0 a_0 + w_0 - \beta_0(a_0 + h_0)$.

In the Solow growth model the saving-income ratio is parametrically given and constant over time. The Ramsey model endogenizes the saving-income ratio. Solow's parametric saving rate is replaced by two "deeper" parameters, the rate of impatience, ρ, and the desire for consumption smoothing, θ. As we shall see the resulting saving-income ratio will not generally be constant outside the steady state of the dynamic system implied by the Ramsey model.

10.2.2 Firms

There is a large number of firms which maximize profits under perfect competition. All firms have the same neoclassical production function with CRS,

$$Y_t = F(K_t^d, T_t L_t^d) \tag{10.19}$$

where Y_t is supply of output, K_t^d is capital input, and L_t^d is labor input, all measured per time unit, at time t. The superscript d on the two inputs indicates that these inputs are seen from the demand side. The factor T_t

[5] These formulas can also be derived directly from Example 1 of Section 9.4.4 of Chapter 9 by replacing $r(\tau)$ and ρ by $r(\tau) - n$ and $\rho - n$, respectively.

[6] This also holds if $\theta = 1$, since in that case $\beta_0 = \rho - n$.

represents the economy-wide level of technology as of time t and is exogenous. We assume there is technological progress at a constant rate g (≥ 0) :

$$T_t = T_0 e^{gt}, \qquad T_0 > 0. \tag{10.20}$$

Thus the economy features Harrod-neutral technological progress, as is needed for compliance with Kaldor's stylized facts.

Necessary and sufficient conditions for the factor combination (K_t^d, L_t^d), where $K_t^d > 0$ and $L_t^d > 0$, to maximize profits are that

$$F_1(K_t^d, T_t L_t^d) = \hat{r}_t, \tag{10.21}$$
$$F_2(K_t^d, T_t L_t^d)T_t = w_t. \tag{10.22}$$

10.3 General equilibrium

We now consider the economy as a whole and thereby the interaction between households and firms in the various markets. For simplicity, we assume that the number of households is the same as the number of firms. We normalize this number to one so that $F(\cdot, \cdot)$ can from now on be interpreted as the aggregate production function and C_t as aggregate consumption.

Factor markets

In the short term, that is, for fixed t, the available quantities of labor, $L_t = L_0 e^{nt}$, and capital, K_t, are predetermined. The factor markets clear at all points in time, that is,

$$K_t^d = K_t, \quad \text{and} \quad L_t^d = L_t, \tag{10.23}$$

for all $t \geq 0$. It is the rental rate, $\hat{r}_t$, and the wage rate, w_t, which adjust (immediately) so that this is achieved. Aggregate output can be written

$$Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0, \tag{10.24}$$

where $\tilde{k}_t \equiv K_t/(T_t L_t)$. Substituting (10.23) into (10.21) and (10.22), we find the equilibrium interest rate and wage rate:

$$r_t = \hat{r}_t - \delta = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{10.25}$$
$$w_t = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial L_t} T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \tag{10.26}$$

where $\tilde{k}_t$ is at any point in time predetermined and where in (10.25) we have used the no-arbitrage condition (10.1).

Capital accumulation

From now we leave out the explicit dating of the variables when not needed for clarity. By national product accounting we have

$$\dot{K} = Y - C - \delta K. \tag{10.27}$$

Let us check whether we get the same result from the wealth accumulation equation of the household. Because physical capital is the only asset in the economy, aggregate financial wealth, A, at time t equals the total quantity of capital, K, at time t.[7] From (10.4) we thus have

$$\begin{aligned}
\dot{K} &= rK + wL - cL \\
&= (f'(\tilde{k}) - \delta)K + (f(\tilde{k}) - \tilde{k}f'(\tilde{k}))TL - cL \qquad \text{(from (10.25) and (10.26))} \\
&= f(\tilde{k})TL - \delta K - cL \qquad \text{(by rearranging and use of } K \equiv \tilde{k}TL) \\
&= F(K, TL) - \delta K - C = Y - C - \delta K \qquad \text{(by } C \equiv cL).
\end{aligned}$$

Hence, the book-keeping is in order (the national income account is consistent with the national product account).

We now face a fundamental difference as compared with models where households have a finite horizon, such as the Diamond OLG model. Current consumption cannot be determined independently of the expected *long-term* evolution of the economy. This is because consumption and saving, as we saw in Section 10.2, depend on the expectations of the entire future evolution of wages and interest rates. And given the presumption of perfect foresight, the households' expectations are identical to the prediction that can be calculated from the model. In this way there is interdependence between expectations and the level and evolution of consumption. We can determine the level of consumption only in the context of the overall dynamic analysis. In fact, the economic agents are in some sense in the same situation as the outside analyst. They, too, have to think through the entire dynamics of the economy in order to form their rational expectations.

The dynamic system

We get a concise picture of the dynamics by reducing the model to the minimum number of coupled differential equations. This minimum number is two. The key endogenous variables are $\tilde{k} \equiv K/(TL)$ and $\tilde{c} \equiv C/(TL) \equiv c/T$.

[7] Whatever financial claims on each other the households might have, they net out for the household sector as a whole.

Using the rule for the growth rate of a fraction, we get

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n) \qquad \text{(from (10.2) and (10.20))} \\
&= \frac{F(K, TL) - C - \delta K}{K} - (g+n) \qquad \text{(from (10.27))} \\
&= \frac{f(\tilde{k}) - \tilde{c}}{\tilde{k}} - (\delta + g + n) \qquad \text{(from (10.24))}.
\end{aligned}$$

The associated differential equation for $\tilde{c}$ is obtained in a similar way:

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{1}{\theta}(r_t - \rho) - g \quad \text{(from the Keynes-Ramsey rule)} \\
&= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right] \qquad \text{(from (10.25))}.
\end{aligned}$$

We thus end up with the dynamic system

$$\begin{aligned}
\dot{\tilde{k}} &= f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \quad \text{given}, \qquad (10.28) \\
\dot{\tilde{c}} &= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \qquad (10.29)
\end{aligned}$$

The lower panel of Fig. 10.1 shows the *phase diagram* of the system. The curve OEB represents the points where $\dot{\tilde{k}} = 0$; from (10.28) we see that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv \tilde{c}(\tilde{k}). \qquad (10.30)$$

The upper panel of Fig. 10.1 displays the value of $\tilde{c}(\tilde{k})$ as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = (\delta + g + n)\tilde{k}$ (to save space the proportions are distorted).[8] The maximum value of $\tilde{c}(\tilde{k})$, if it exists, is reached at the point where the tangent to the OEB curve in the lower panel is horizontal, i.e., where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - (\delta + g + n) = 0$ or $f'(\tilde{k}) - \delta = g + n$. The value of $\tilde{k}$ which satisfies this is the golden rule capital intensity, $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = g + n. \qquad (10.31)$$

From (10.28) we see that for points above the $\dot{\tilde{k}} = 0$ locus we have $\dot{\tilde{k}} < 0$, whereas for points below the $\dot{\tilde{k}} = 0$ locus, $\dot{\tilde{k}} > 0$. The horizontal arrows in the figure indicate these directions of movement.

[8] As the graph is drawn, $f(0) = 0$, i.e., capital is assumed essential. But none of the conclusions we are going to consider depends on this.

From (10.29) we see that

$$\dot{\tilde{c}} = 0 \text{ for } f'(\tilde{k}) = \delta + \rho + \theta g \qquad \text{or} \qquad \tilde{c} = 0. \tag{10.32}$$

Let $\tilde{k}^* > 0$ satisfy the equation $f'(\tilde{k}^*) - \delta = \rho + \theta g$. Then the vertical line $\tilde{k} = \tilde{k}^*$ represents points where $\dot{\tilde{c}} = 0$ (and so does of course the horizontal half-line $\tilde{c} = 0, \tilde{k} \geq 0$). For points to the left of the $\tilde{k} = \tilde{k}^*$ line we have, according to (10.29), $\dot{\tilde{c}} > 0$ and for points to the right of the $\tilde{k} = \tilde{k}^*$ line we have $\dot{\tilde{c}} < 0$. The vertical arrows indicate these directions of movement.

Steady state

The point E has coordinates $(\tilde{k}^*, \tilde{c}^*)$ and represents the unique steady state.[9] From (10.32) and (10.30) follows that

$$f'(\tilde{k}^*) = \delta + \rho + \theta g, \qquad \text{and} \tag{10.33}$$

$$\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*. \tag{10.34}$$

From (10.33) it can be seen that the real interest rate in steady state is

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g. \tag{10.35}$$

The capital intensity satisfying this equation is known as the *modified golden rule* capital intensity, $\tilde{k}_{MGR}$. The modified golden rule is the rule saying that for a representative agent economy to be in steady state, the capital intensity must be such that the net marginal product of capital equals the required rate of return, taking into account the pure rate of time preference, ρ, and the desire for consumption smoothing, measured by θ.[10]

We show below that the steady state is, in a specific sense, asymptotically stable. First we have to make sure, however, that the steady state exists and

[9] As (10.32) shows, if $\tilde{c}_t = 0$, then $\dot{\tilde{c}} = 0$. Therefore, mathematically, point B (if it exists) in Fig. 10.1 is also a stationary point of the dynamic system. And if $f(0) = 0$, then, according to (10.29) and (10.30), also the point $(0, 0)$ in the figure is a stationary point. But these stationary points have zero consumption forever and are therefore not steady states of any *economic* system. From an economic point of view they are "trivial" steady states.

[10] Note that the ρ of the Ramsey model corresponds to the intergenerational discount rate R of the Barro dynasty model in Chapter 7. Indeed, in the discrete time Barro model we have $1 + r^* = (1 + R)(1 + g)^\theta$, which, by taking logs on both sides and using first-order Taylor approximations around 1 gives $r^* \approx \ln(1 + r^*) = \ln(1 + R) + \theta \ln(1 + g) \approx R + \theta g$. Recall, however, that in view of the considerable period length (about 25-30 years) of the Barro model, this approximation will not be very good.

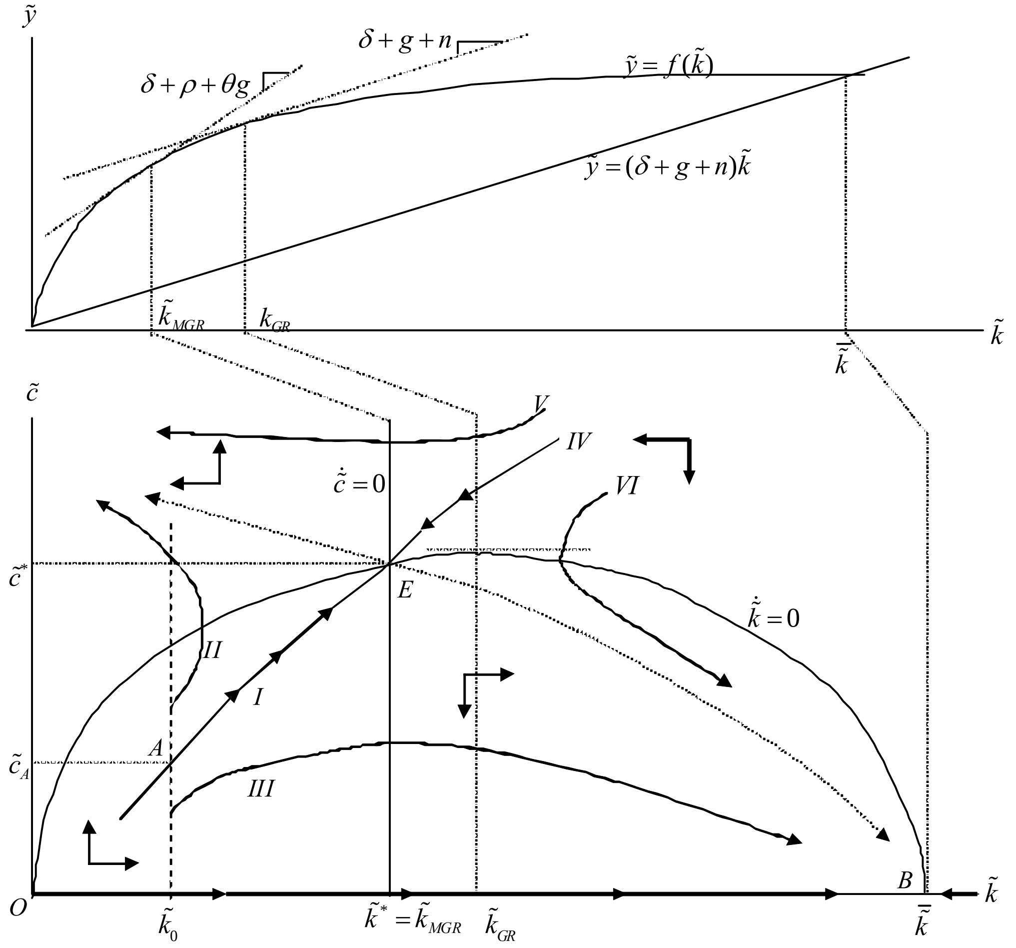

Figure 10.1: Phase portrait of the Ramsey model.

is consistent with general equilibrium. This consistency requires that the household's transversality condition (10.12) holds in the point E. Using $a_t = K_t/L_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$ and $r_t = f'(\tilde{k}_t) - \delta$, we get

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s)-\delta-g-n\right)ds} = 0. \tag{10.36}$$

In the point E $\tilde{k}_t = \tilde{k}^*$ and $f'(\tilde{k}_t) - \delta = \rho + \theta g$ for all t. So the condition (10.36) becomes

$$\lim_{t\to\infty} \tilde{k}^* e^{-(\rho+\theta g-g-n)t} = 0.$$

This is fulfilled if and only if $\rho + \theta g > g + n$, that is,

$$\rho - n > (1-\theta)g. \tag{A1}$$

This condition ensures that the improper integral U_0 is bounded, at least when the system is in steady state (see Appendix B). If $\theta \geq 1$, (A1) is fulfilled as soon as the effective utility discount rate, $\rho - n$, is positive; (A1) may even hold for a negative $\rho - n$ if not "too" negative). If $\theta < 1$, (A1) requires $\rho - n$ to be "sufficiently positive".

Since the parameter restriction (A1) can be written $\rho + \theta g > g + n$, it implies that the steady-state interest rate, r^*, given in (10.35), is higher than the "natural" growth rate, $g + n$. If this did not hold, the transversality condition (10.12) would fail in the steady state. Indeed, along the steady state path we have

$$a_t e^{-r^* t} = a_0 e^{(g+n)t} e^{-r^* t} = k_0 e^{(g+n-r^*)t},$$

which would take the value $k_0 > 0$ for all $t \geq 0$ if $r^* = g + n$ and would go to ∞ for $t \to \infty$ if $r^* < g + n$. The individual households would be over-saving. Each household would in this situation alter its behavior and the steady state could thus not be an equilibrium path.

Another way of seeing that $r^* \leq g + n$ can never be an equilibrium in a Ramsey model is to recognize that this condition would make the household's human wealth infinite because wage income, wL, would grow at a rate, $g+n$, at least as high as the real interest rate, r^*. This would motivate an immediate increase in consumption and so the considered steady-state path would again not be an equilibrium.

To have a model of interest, from now on we assume that the parameters satisfy the inequality (A1). As an implication, the capital intensity in steady state, $\tilde{k}^*$, is less than the golden rule value $\tilde{k}_{GR}$. Indeed, $f'(\tilde{k}^*) - \delta = \rho + \theta g > g + n = f'(\tilde{k}_{GR}) - \delta$, so that $\tilde{k}^* < \tilde{k}_{GR}$, in view of $f'' < 0$.

So far we have only ensured that if the steady state, E, exists, it is consistent with general equilibrium. Existence of both a steady state and a golden rule capital intensity requires that the marginal productivity of capital is sufficiently sensitive to variation in the capital intensity. We therefore assume that f has the properties

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g \quad \text{and} \quad \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta < g + n. \tag{A2}$$

Together with (A1) this implies $\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g > g + n > \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta$. By continuity of f', these inequalities ensure the existence of both $\tilde{k}^* > 0$ and $\tilde{k}_{GR} > 0$.[11] Moreover, the inequalities ensure the existence

[11] The often presumed Inada conditions, $\lim_{\tilde{k} \to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k} \to \infty} f'(\tilde{k}) = 0$, are stricter than (A2) and not necessary.

of a $\bar{\bar{k}} > 0$ with the property that $f(\bar{\bar{k}}) - (\delta + g + n)\tilde{k} = 0$, as in Fig. 10.1.[12] Because $f'(\tilde{k}) > 0$ for all $\tilde{k} > 0$, it is implicit in the technology assumption (A2) that $\delta + g + n > 0$. Even without deciding on the sign of n (a decreasing workforce should not be excluded in our days), this seems like a plausible presumption.

Trajectories in the phase diagram

A first condition for a path $(\tilde{k}_t, \tilde{c}_t)$, with $\tilde{k}_t > 0$ and $\tilde{c}_t > 0$ for all $t \geq 0$, to be a solution to the model is that it satisfies the system of differential equations (10.28)-(10.29). Indeed, it must satisfy (10.28) to be technically feasible and it must satisfy (10.29) to comply with the Keynes-Ramsey rule. Technical feasibility of the path also requires that its initial value for $\tilde{k}$ equals the historically given (pre-determined) value $\tilde{k}_0 \equiv K_0/(T_0 L_0)$. In contrast, for $\tilde{c}$ we have no exogenously given initial value. This is because $\tilde{c}_0$ is a so-called *jump variable* or *forward-looking variable.* By this is meant an endogenous variable which can immediately shift to another value if new information arrives so as to alter expectations about the future. We shall see that the terminal condition (10.36), reflecting the transversality condition of the households, makes up for this lack of an initial condition for c.

In Fig. 10.1 we have drawn some possible paths that could be solutions as t increases. We are especially interested in the paths which start at the historically given $\tilde{k}_0$, that is, start at some point on the stippled vertical line in the figure. If the economy starts out with a high value of $\tilde{c}$, it will follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time (see Appendix C). If the economy starts out with a low level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the capital intensity converges towards $\bar{\bar{k}}$ in the figure.

All in all this suggests the existence of an initial level of consumption somewhere in between, which gives a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value.[13] We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$. The path which emerges from this, namely the

---

[12] We claim that $\bar{\bar{k}} > \tilde{k}_{GR}$ *must* hold. Indeed, this inequality follows from $f'(\tilde{k}_{GR}) = \delta + n + g \equiv f(\bar{\bar{k}})/\bar{\bar{k}} > f'(\bar{\bar{k}})$, the latter inequality being due to $f'' < 0$ and $f(0) \geq 0$.

[13] As an implication of the uniqueness theorem for differential equations, two solution paths in the phase plan cannot intersect.

path I, starting at the point A, is special in that it converges towards the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Paths starting above A do not, as we just saw. It is similar for a path starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ and then it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it has $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every $\tilde{c}$-value that path III has in common with path I, path III has a higher $\dot{\tilde{k}}$ and a lower $\dot{\tilde{c}}$ than path I (use (10.28) and (10.29)). Hence, path III diverges from point E.

Equivalently, had we considered a value of $\tilde{k}_0 > \tilde{k}^*$, there would also be a unique value of $\tilde{c}_0$ such that the path starting from $(\tilde{k}_0,\ \tilde{c}_0)$ would converge to E (see path IV in Fig. 10.1).

The point E is a *saddle point*. By this is meant a steady-state point with the following property: there exists exactly two paths, one from each side of $\tilde{k}^*$, that converge towards the steady-state point; all other paths (at least starting in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths, the dotted North-West and South-East curves in Fig. 10.1. The two converging paths together make up what is known as the *stable arm*; on their own they are referred to as *saddle paths*.[14] The dotted diverging paths in Fig. 10.1 together make up the *unstable arm*.

The equilibrium path

A solution to the model is a path which is technically feasible and in addition satisfies a set of equilibrium conditions. In analogy with the definition in discrete time (see Chapter 3) a path $(\tilde{k}_t,\ \tilde{c}_t)_{t=0}^{\infty}$ is called a *technically feasible path* if (i) the path has $\tilde{k}_t \geq 0$ and $\tilde{c}_t \geq 0$ for all $t \geq 0$; (ii) it satisfies the accounting equation (10.28); and (iii) it starts out, at $t = 0$, with the historically given initial capital intensity. An *equilibrium path* with perfect foresight is then a technically feasible path $(\tilde{k}_t,\ \tilde{c}_t)_{t=0}^{\infty}$ with the properties that the path (a) is consistent with the households' optimization given their expectations; (b) is consistent with market clearing for all $t \geq 0$; and (c) has the property that the evolution over time of the pair $(w_t,\ r_t)$, where $w_t = \tilde{w}(\tilde{k}_t)T_t$ and $r_t = f'(\tilde{k}_t) - \delta$, is as expected by the households. The condition (a) in this definition requires the transformed Keynes-Ramsey rule (10.29) and the transversality condition (10.36) to hold for all $t \geq 0$.

[14] An algebraic definition of a saddle point, in terms of eigenvalues, is given in Appendix A. There it is also shown that if $\lim_{\tilde{k}\to 0} f(\tilde{k}) = 0$, then the saddle path on the left side of the steady state in Fig. 10.1 will start out infinitely close to the origin.

Consider the case where $0 < \tilde{k}_0 < \tilde{k}^*$, as illustrated in Fig. 10.1. Then, the path starting at point A and following the saddle path towards the steady state is an equilibrium path because, by construction, it is technically feasible and in addition has the required properties, (a), (b), and (c). More intuitively: if the households expect an evolution of w_t and r_t corresponding to this path (that is, expect a corresponding underlying movement of $\tilde{k}_t$, which we know unambiguously determines r_t and w_t), then these expectations will induce a behavior the aggregate result of which is an actual path for $(\tilde{k}_t, \tilde{c}_t)$ that confirms the expectations. And along this path the households find no reason to correct their behavior because the path allows both the Keynes-Ramsey rule and the transversality condition to be satisfied.

No other path than the saddle path can be an equilibrium. This is because no other technically feasible path is compatible with the households' individual utility maximization under perfect foresight. An initial point above point A can be excluded in that the implied path, II, does not satisfy the household's NPG condition (and, consequently, not at all the transversality condition).[15] So, if the individual household expected an evolution of r_t and w_t corresponding to path II, then the household would immediately choose a *lower* level of consumption, that is, the household would *deviate* in order not to suffer the same fate as Charles Ponzi. In fact *all* the households would react in this way. Thus path II would not be realized and the expectation that it would, can not be a rational expectation.

Likewise, an initial point below point A can be ruled out because the implied path, III, does not satisfy the household's transversality condition but implies over-saving. Indeed, at some point in the future, say at time t_1, the economy's capital intensity would pass the golden rule value so that for all $t > t_1$, $r_t < g + n$. But with a rate of interest permanently below the growth rate of wage income of the household, the present value of human wealth is *infinite*. This motivates a *higher* consumption level than that along the path. Thus, if the household expects an evolution of r_t and w_t corresponding to path III, then the household will immediately *deviate* and choose a higher initial level of consumption. But so will *all* the households react and the expectation that the economy will follow path III can not be rational.

We have presumed $0 < \tilde{k}_0 < \tilde{k}^*$. If instead $\tilde{k}_0 > \tilde{k}^*$, the economy would move along the saddle path *from above*. Paths like V and VI in Fig. 10.1 can be ruled out because they violate the NPG condition and the transversality condition, respectively. With this we have shown:

PROPOSITION 1 Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then the Ramsey model exhibits a unique equilibrium path, characterized

[15] This is shown in Appendix C.

by $(\tilde{k}_t,\ \tilde{c}_t)$ converging, for $t \to \infty$, towards a unique steady state with a capital intensity $\tilde{k}^*$ satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. In the steady state the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, the per capita consumption path is $c_t^* = \tilde{c}^* T_0 e^{gt}$, where $\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*$, and the real wage path is $w_t^* = \tilde{w}(\tilde{k}^*) T_0 e^{gt}$.

A numerical example based on one year as the time unit: $\theta = 2$, $g = 0.02$, $n = 0.01$ and $\rho = 0.01$. Then, $r^* = 0.05 > 0.03 = g + n$.

So output per capita, $y_t \equiv Y_t/L_t \equiv \tilde{y}_t T_t$, tends to grow at the rate of technological progress, g :

$$\frac{\dot{y}_t}{y_t} \equiv \frac{\dot{\tilde{y}}_t}{\tilde{y}_t} + \frac{\dot{T}_t}{T_t} = \frac{f'(\tilde{k}_t)\dot{\tilde{k}}_t}{f(\tilde{k}_t)} + g \to g \quad \text{for} \quad t \to \infty,$$

in view of $\dot{\tilde{k}}_t \to 0$. This is also true for the growth rate of consumption per capita and the real wage, since $c_t \equiv \tilde{c}_t T_t$ and $w_t = \tilde{w}(\tilde{k}_t) T_t$.

The intuition behind the convergence lies in the neoclassical principle that starting from a low capital intensity and therefore high marginal and average product of capital, the resulting high aggregate saving[16] will be more than enough to maintain the capital intensity which therefore increases. But when this happens, the marginal and average product of capital decreases and the resulting saving, as a proportion of the capital stock, declines until eventually it is only sufficient to replace worn-out machines and equip new "effective" workers with enough machines to just maintain the capital intensity. If instead we start from a high capital intensity a similar story can be told in reverse. Thus in the Ramsey model the long-run state is attained when the marginal saving and investment yields a return as great as the representative household's willingness to postpone the marginal unit of consumption.

The equilibrium path generated by the Ramsey model is necessarily dynamically efficient and satisfies the modified golden rule in the long run. Why this contrast to Diamonds OLG model where equilibrium paths *may* be dynamically inefficient? The reason lies in the fact that only a "single infinity", not a "double infinity", is involved in the Ramsey model. The time horizon of the economy is infinite but the number of decision makers is finite. Births (into adult life) do not reflect the emergence of new economic agents with separate interests. It is otherwise in the Diamond OLG model where births imply the entrance of new economic decision makers whose preferences no-one has cared about in advance. In that model neither is there any final

[16] Saving will be high because the substitution and wealth effects on current consumption of the high interest rate dominate the income effect.

date, nor any final decision maker. It is this difference that lies behind that the two models in some respects give different results. A type of equilibria, namely dynamically inefficient ones, can be realized in the Diamond model but not so in the Ramsey model. A rate of time preference low enough to generate a *tendency* to a long-run interest rate below the income growth rate is inconsistent with the conditions needed for general equilibrium in the Ramsey model. And such a low rate of time preference is in fact ruled out in the Ramsey model by the parameter restriction (A1).

The concept of saddle-point stability

The steady state of the model is globally asymptotically stable for arbitrary initial values of the capital intensity (the phase diagram only verifies local asymptotic stability, but the extension to global asymptotic stability is verified in Appendix A). If $\tilde{k}$ is hit by a shock at time 0 (say by a discrete jump in the technology level T_0), the economy will converge toward the same unique steady state as before. At first glance this might seem peculiar considering that the steady state is a saddle point. Such a steady state is unstable for arbitrary initial values of *both* variables, $\tilde{k}$ and $\tilde{c}$. But the crux of the matter is that it is only the initial $\tilde{k}$ that *is* arbitrary. The model assumes that the decision variable c_0, and therefore the value of $\tilde{c}_0 \equiv c_0/T_0$, immediately adjusts to the given circumstances and information about the future. That is, the model assumes that $\tilde{c}_0$ always takes the value needed for the household's transversality condition under perfect foresight to be satisfied. This ensures that the economy is initially on the saddle path, cf. the point A in Fig. 10.1. In the language of differential equations *conditional* asymptotic stability is present. The condition that ensures the stability in our case is the transversality condition.

We shall follow the common terminology in macroeconomics and call a steady state of a two-dimensional dynamic system (locally) *saddle-point stable* if:

1. the steady state is a saddle point;
2. there is one predetermined variable and one jump variable; and
3. the saddle path is not parallel to the jump variable axis.

Thus, to establish saddle-point stability all three properties must be verified. If for instance point 1 and 2 hold but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$

In the present case, we have already verified point 1 and 2. And as the phase diagram indicates, the saddle path is not vertical. So also point 3 holds. Thus, the Ramsey model is saddle-point stable. In Appendix A it is shown that the positively-sloped saddle path in Fig. 10.1 ranges over *all* $\tilde{k} > 0$ (there is nowhere a vertical asymptote to the saddle path). Hence, the steady state is *globally* saddle point stable. All in all, these characteristics of the Ramsey model are analogue to those of Barro's dynasty model in discrete time when the bequest motive is operative.

10.4 Comparative statics and dynamics

10.4.1 The role of key parameters

The conclusion that in the long run the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, tells us that only three parameters matter: the rate of time preference, the elasticity of marginal utility, and the rate of technological progress. A higher ρ, i.e., more impatience and thereby less willingness to defer consumption, implies less capital accumulation and thus smaller capital intensity and in the long run a higher interest rate and lower consumption than otherwise. The long-run growth rate is unaffected. A higher desire for consumption smoothing, θ, will have the same effect in that it implies that a larger part of the greater consumption opportunities in the future, as brought about by technological progress, will be consumed immediately. The long-run interest rate depends positively on the growth rate of labor productivity, g, because the higher this is, the greater is the expected future wage income and the associated consumption possibilities even without any current saving. This discourages current saving and we end up with lower capital accumulation and lower effective capital intensity in the long run, hence higher interest rate. It is also true that the higher is g, the higher is the rate of return needed to induce the saving required for maintaining a steady state and to resist and overcome the desire for consumption smoothing. The long-run interest rate is independent of the particular form of the aggregate production function, f. This function matters for *what* effective capital intensity and *what* consumption level per unit of effective labor are compatible with the long-run interest rate. This kind of results are specific to representative agent models. This is because only in these models will the Keynes-Ramsey rule hold not only for the individual household, but also at the aggregate level.

Unlike the Solow growth model, the Ramsey model provides a *theory* of the evolution and long-run level of the saving rate. The endogenous gross saving rate of the economy is

$$\begin{aligned} s_t &\equiv \frac{Y_t - C_t}{Y_t} = \frac{\dot{K}_t + \delta K_t}{Y_t} = \frac{\dot{K}_t / K_t + \delta}{Y_t / K_t} = \frac{\dot{\tilde{k}}_t / \tilde{k}_t + g + n + \delta}{f(\tilde{k}_t)/\tilde{k}_t} \\ &\to \frac{g + n + \delta}{f(\tilde{k}^*)/\tilde{k}^*} \equiv s^* \quad \text{for} \quad t \to \infty. \end{aligned} \tag{10.37}$$

By determining the path of $\tilde{k}_t$, the Ramsey model determines how s_t moves over time and adjusts to its constant long-run level. Indeed, for any given $\tilde{k} > 0$, the equilibrium value of $\tilde{c}_t$ is uniquely determined by the requirement that the economy must be on the saddle path. Since this defines $\tilde{c}_t$ as a function, $\tilde{c}(\tilde{k}_t)$, of $\tilde{k}_t$, there is a corresponding function for the saving rate in that $s_t = 1 - \tilde{c}(\tilde{k}_t)/f(\tilde{k}_t) \equiv s(\tilde{k}_t)$; so $s(\tilde{k}^*) = s^*$.

To see how the long-run saving rate may depend on basic parameters, let us consider the case where the production function is Cobb-Douglas:

$$\tilde{y} = f(\tilde{k}) = A\tilde{k}^{\alpha}, \qquad A > 0, 0 < \alpha < 1. \tag{10.38}$$

Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1} = \alpha f(\tilde{k})/\tilde{k}$. In steady state we get, by use of the steady-state result (10.33),

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{1}{\alpha} f'(\tilde{k}^*) = \frac{\delta + \rho + \theta g}{\alpha}.$$

Substitution in (10.37) gives

$$s^* = \alpha \frac{\delta + g + n}{\delta + \rho + \theta g} < \alpha, \tag{10.39}$$

where the inequality follows from our parameter restriction (A1). Indeed, (A1) implies $\rho + \theta g > g + n$.

We note that the long-run saving rate is a decreasing function of the rate of impatience, ρ, and the desire of consumption smoothing, θ; it is an increasing function of the capital depreciation rate, δ, the rate of population growth, n, and the elasticity of production w.r.t. to capital, α.[17] It can be shown (see Appendix D) that if, by coincidence, $\theta = 1/s^*$, then $s'(\tilde{k}) = 0$,

[17] Partial differentiation w.r.t. g yields $\partial s^*/\partial g = \alpha\left[\rho - \theta n - (\theta - 1)\delta\right]/(\delta + \rho + \theta g)^2$, the sign of which cannot be determined in general.

that is, the saving rate s_t is also outside of steady state equal to s^*. In view of (10.39), the condition $\theta = 1/s^*$ is equivalent to the "knife-edge" condition $\theta = (\delta+\rho)/\left[\alpha(\delta + g + n) - g\right] \equiv \bar{\theta}$. More generally, assuming $\alpha(\delta+g+n) > g$ (which seems likely empirically), we have that if $\theta \lesseqgtr 1/s^*$ (i.e., $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively (and if instead $\alpha(\delta + g + n) \leq g$, then $s'(\tilde{k}) < 0$, unconditionally).[18] Data presented in Barro and Sala-i-Martin (2004, p. 15) indicate no trend for the US saving rate, but a positive trend for several other developed countries since 1870. One interpretation is that whereas the US has been close to its steady state, the other countries are still in the adjustment process toward the steady state. As an example, consider the parameter values $\delta = 0.05$, $\rho = 0.02$, $g = 0.02$ and $n = 0.01$. In this case we get $\bar{\theta} = 10$ if $\alpha = 0.33$; given $\theta < 10$, these other countries should then have $s'(\tilde{k}) < 0$ which, according to the model, is compatible with a rising saving rate over time only if these countries are approaching their steady state from *above* (i.e., they should have $\tilde{k}_0 > \tilde{k}^*$). It may be argued that α should also reflect the role of education and R&D in production and thus be higher; with $\alpha = 0.75$ we get $\bar{\theta} = 1.75$. Then, if $\theta > 1.75$, these countries would have $s'(\tilde{k}) > 0$ and thus approach their steady state from *below* (i.e., $\tilde{k}_0 < \tilde{k}^*$).

10.4.2 Solow's growth model as a special case

The above results give a hint that Solow's growth model, with a given constant saving rate $s \in (0, 1)$ and given δ, g, and n (with $\delta + g + n > 0$), can, under certain circumstances, be interpreted as a special case of the Ramsey model. The Solow model is given by

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t.$$

The constant saving rate implies proportionality between consumption and income which, in growth-corrected terms, per capita consumption is

$$\tilde{c}_t = (1 - s)f(\tilde{k}_t).$$

For the Ramsey model to yield this, the production function must be like in (10.38) (i.e., Cobb-Douglas) with $\alpha > s$. And the elasticity of marginal utility, θ, must satisfy $\theta = 1/s$. Finally, the rate of time preference, ρ, must be such that (10.39) holds with s^* replaced by s, which implies $\rho = \alpha(\delta+g+n)/s - \delta - \theta g$. It remains to show that this ρ satisfies the inequality, $\rho - n > (1 - \theta)g$, which is necessary for existence of an equilibrium in the Ramsey model. Since

[18] See Appendix D.

$\alpha/s > 1$, the chosen ρ satisfies $\rho > \delta+g+n - \delta - \theta g = n+(1-\theta)g$, which was to be proved. Thus, in this case the Ramsey model generates an equilibrium which implies a time path identical to that generated by the Solow model with $s = 1/\theta$.[19]

With this foundation of the Solow model, it will always hold that $s = s^* < s_{GR}$, where s_{GR} is the golden rule saving rate. Indeed, from (10.37) and (10.31), respectively,

$$s_{GR} = \frac{(\delta+g+n)\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \frac{f'(\tilde{k}_{GR})\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \alpha > s^*,$$

from the Cobb-Douglas specification and (10.39), respectively.

A point of the Ramsey model vis-a-vis the Solow model is to replace a mechanical saving rule by maximization of discounted utility and thereby, on the one hand, open up for a wider range of possible evolutions and on the other hand at the same time narrow the range in certain respects.

, the model opens up for studying welfare consequences of alternative economic policies.

10.5 A social planner's problem

Another implication of the Ramsey setup is that the decentralized market equilibrium (within the idealized presumptions of the model) brings about the same allocation of resources as would a social planner with the same criterion function as the representative household. As in Chapter 8, by a social planner we mean a hypothetical central authority who is "all-knowing and all-powerful". The social planner is not constrained by other limitations than those arising from technology and initial resources and can thus fully decide on the resource allocation within these confines.

Let the economy be closed and let the social welfare function be time separable with constant elasticity, $\hat{\theta}$, of marginal utility and a pure rate of

[19] A more elaborate account of the Solow model as a special case of the Ramsey model is given in Appendix D.

time preference $\hat{\rho}$.[20] Then the social planner's optimization problem is

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} W_0 &= \int_0^{\infty} \frac{c_t^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-(\hat{\rho}-n)t} dt \qquad \text{s.t.} && (10.40)\\
c_t &\geq 0, && (10.41)\\
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, && (10.42)\\
\tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0. && (10.43)
\end{aligned}$$

We assume $\hat{\theta} > 0$ and $\hat{\rho} - n > (1 - \hat{\theta})g$. In case $\hat{\theta} = 1$, the expression $c_t^{1-\hat{\theta}}/(1-\hat{\theta})$ should be interpreted as $\ln c_t$. The dynamic constraint (10.42) reflects the national product account. Because the economy is closed, the social planner does not have the opportunity of borrowing or lending from abroad and hence there is no solvency requirement. Instead we just impose the definitional constraint (10.43) of non-negativity of the state variable $\tilde{k}$. The problem is the continuous time analogue of the social planner's problem in discrete time in Chapter 8. Note, however, a minor conceptual difference, namely that in continuous time there is in the short run no *upper* bound on the *flow* variable c_t, that is, no bound like $c_t \leq T_t \left[f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t \right]$. A consumption intensity c_t which is higher than the right-hand side of this inequality, will just be reflected in a negative value of the flow variable $\dot{\tilde{k}}_t$.[21]

To solve the problem we apply the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = \frac{c^{1-\hat{\theta}}}{1-\hat{\theta}} + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k} \right],$$

where λ is the adjoint variable associated with the dynamic constraint (10.42). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda(t)$ such that, for all $t \geq 0$,

$$\begin{aligned}
\frac{\partial H}{\partial c} &= c^{-\hat{\theta}} - \frac{\lambda}{T} = 0, \text{ i.e., } c^{-\hat{\theta}} = \frac{\lambda}{T}, \quad \text{and} && (10.44)\\
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - \delta - g - n) = (\hat{\rho} - n)\lambda - \dot{\lambda} && (10.45)
\end{aligned}$$

[20] Reasons for allowing these two preference parameters to deviate from the corresponding parameters in the private sector are given Chapter8.

[21] As usual, we presume that capital can be "eaten". That is, we consider the capital good to be instantaneously convertible to a consumption good. Otherwise there *would* be at any time an upper bound on c, namely $c \leq Tf(\tilde{k})$, saying that the per capita consumption flow cannot exceed the per capita output flow. The role of such constraints is discussed in Feichtinger and Hartl (1986).

hold along the path and the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\hat{\rho}-n)t} = 0, \tag{10.46}$$

is satisfied.[22]

The condition (10.44) can be seen as a $MC = MB$ condition and illustrates that λ_t is the social planner's shadow price, measured in terms of current utility, of $\tilde{k}_t$ along the optimal path.[23] The differential equation (10.45) tells us how this shadow price evolves over time. The transversality condition, (11.49), together with (10.44), entails the condition

$$\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{gt} e^{-(\hat{\rho}-n)t} = 0,$$

where the unimportant factor T_0 has been eliminated. Imagine the opposite were true, namely that $\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{[g-(\hat{\rho}-n)]t} > 0$. Then, intuitively U_0 could be increased by reducing the long-run value of $\tilde{k}_t$, i.e., consume more and save less.

By taking logs in (10.44) and differentiating w.r.t. t, we get $-\hat{\theta}\dot{c}/c = \dot{\lambda}/\lambda - g$. Inserting (10.45) and rearranging gives the condition

$$\frac{\dot{c}}{c} = \frac{1}{\hat{\theta}}(g - \frac{\dot{\lambda}}{\lambda}) = \frac{1}{\hat{\theta}}(f'(\tilde{k}) - \delta - \hat{\rho}). \tag{10.47}$$

This is the social planner's Keynes-Ramsey rule. If the rate of time preference, $\hat{\rho}$, is lower than the net marginal product of capital, $f'(\tilde{k}) - \delta$, the social planner will let per capita consumption be relatively low in the beginning in order to attain greater per capita consumption later. The lower the impatience relative to the return on capital, the more favorable it becomes to defer consumption.

Because $\tilde{c} \equiv c/T$, we get from (11.50) qualitatively the same differential equation for $\tilde{c}$ as we obtained in the decentralized market economy. And the dynamic resource constraint (10.42) is of course identical to that of the decentralized market economy. Thus, the dynamics are in principle unaltered and the phase diagram in Fig. 10.1 is still valid. The solution of the social planner implies that the economy will move along the saddle path towards the steady state. This trajectory, path I in the diagram, satisfies both the first-order conditions and the transversality condition. However, paths such as III

[22] The infinite-horizon Maximum Principle itself does not guarantee validity of such a straightforward extension of a necessary transversality condition from a finite horizon to an infinite horizon. Yet, this extension *is* valid for the present problem, cf. Appendix E.

[23] Decreasing c_t by one unit, increases $\tilde{k}_t$ by $1/T_t$ units, each of which are worth λ_t utility units to the social planner.

in the figure do not satisfy the transversality condition of the social planner but imply permanent over-saving. And paths such as II in the figure will experience a sudden end when all the capital has been used up. Intuitively, they cannot be optimal. A rigorous argument is given in Appendix E, based on the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. Thence, not only is the saddle path an optimal solution, it is the *only* optimal solution.

Comparing with the market solution of the previous section, we have established:

PROPOSITION 2 *(equivalence theorem)* Assume (A1) and (A2) with θ and ρ replaced by $\hat{\theta}$ and $\hat{\rho}$, respectively. Let there be a given $\tilde{k}_0 > 0$. Then the perfectly competitive market economy, without externalities, brings about the same resource allocation as that brought about by a social planner with the same criterion function as the representative household, i.e., with $\hat{\theta} = \theta$ and $\hat{\rho} = \rho$.

This is a continuous time analogue to the discrete time equivalence theorem of Chapter 8.

The capital intensity $\tilde{k}$ in the social planner's solution will not converge towards the golden rule level, $\tilde{k}_{GR}$, but towards a level whose distance to the golden rule level depends on how much $\hat{\rho} + \hat{\theta}g$ exceeds the natural growth rate, $g + n$. Even if society would be able to consume more in the long term if it aimed for the golden rule level, this would not compensate for the reduction in current consumption which would be necessary to achieve it. This consumption is relatively more valuable, the greater is the social planner's effective rate of time preference, $\hat{\rho} - n$. In line with the market economy, the social planner's solution ends up in a *modified golden rule.* In the long term, net marginal productivity of capital is determined by preference parameters and productivity growth and equals $\hat{\rho} + \hat{\theta}g > g + n$. Hereafter, given the net marginal productivity of capital, the capital intensity and the level of the consumption path is determined by the production function.

Average utilitarianism

In the above analysis the social planner maximizes the sum of discounted per capita utilities *weighed* by generation size. This is known as discounted *classical utilitarianism.* As an implication, the *effective* utility discount rate, $\rho - n$, varies negatively (one to one) with the population growth rate. Since this corresponds to how the per capita rate of return on saving, $r - n$, is "diluted" by population growth, the net marginal product of capital in steady state becomes independent of n, namely equal to $\hat{\rho} + \hat{\theta}g$.

An alternative to discounted classical utilitarianism is to maximize discounted *per capita* utility. This accords with the principle of discounted *average utilitarianism.* Here the social planner maximizes the sum of discounted per capita utilities without weighing by generation size. Then the effective utility discount rate is independent of the population growth rate, n. With $\hat{\rho}$ still denoting the pure rate of time preference, the criterion function becomes

$$W_0 = \int_0^{\infty} \frac{{c_t}^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-\hat{\rho}t} dt.$$

The social planner's solution then converges towards a steady state with the net marginal product of capital

$$f'(\tilde{k}^*) - \delta = \hat{\rho} + n + \hat{\theta} g. \tag{10.48}$$

Here, an increase in n will imply higher long-run net marginal product of capital and lower capital intensity, everything else equal.

The representative household in the Ramsey model may of course also have a criterion function in line with discounted average utilitarianism, that is, $U_0 = \int_0^{\infty} u(c_t) e^{-\rho t} dt$. Then, the interest rate in the economy will in the long run be $r^* = \rho + n + \theta g$ and so an increase in n will increase r^* and decrease $\tilde{k}^*$.

Ramsey's original zero discount rate and the overtaking criterion

It was mostly the perspective of a social planner, rather than the market mechanism, which was at the center of Ramsey's original analysis. Ramsey maintained that the social planner should "not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey 1928). Ramsey also assumed $g = n = 0$. Given the instantaneous utility function u, with $u' > 0, u'' < 0$, and given $\rho = 0$, Ramsey's original problem was: choose $(c_t)_{t=0}^{\infty}$ so as to optimize (in some sense, see below)

$$\begin{aligned} W_0 &= \int_0^{\infty} u(c_t) dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= f(k_t) - c_t - \delta k_t, \\ k_t &\geq 0 \qquad \text{for all } t \geq 0. \end{aligned}$$

Since the improper integral W_0 will generally not be bounded in this case, Ramsey could not use maximization of W_0 as an optimality criterion.

Instead he used a criterion akin to the overtaking criterion we considered in a discrete time context in the previous chapter. We only have to reformulate this criterion for a continuous time setting.

Let $(c_t)_{t=0}^{\infty}$ be the consumption path associated with an arbitrary technically feasible path and let $(\hat{c}_t)$ be the consumption path associated with our candidate as an optimal path, that is, the path we wish to test for optimality. Define

$$D_T \equiv \int_0^T u(\hat{c}_t)dt - \int_0^T u(c_t)dt. \tag{10.49}$$

Then the feasible path $(\hat{c}_t)_{t=0}^{\infty}$ is *overtaking optimal*, if for any alternative feasible path, $(c_t)_{t=0}^{\infty}$, there exists a number $T' \geq 0$ such that $D_T \geq 0$ for all $T \geq T'$. That is, if from some date on, cumulative utility of the candidate path up to *all* later dates is greater than that of any alternative feasible path, then the candidate path is overtaking optimal. We say it is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred*.[24]

Optimal control theory is also applicable with this criterion. The Hamiltonian is

$$H(k, c, \lambda, t) = u(c) + \lambda \left[f(k) - c - \delta k \right].$$

The Maximum Principle states that an interior overtaking-optimal path will satisfy that there exists an adjoint variable λ such that for all $t \geq 0$ it holds along this path that

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0, \text{ and} \tag{10.50}$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - \delta) = -\dot{\lambda}. \tag{10.51}$$

Since $\rho = 0$, the Keynes-Ramsey rule reduces to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta), \qquad \text{where } \theta(c) \equiv -\frac{c}{u'(c)}u''(c).$$

One might conjecture that the transversality condition,

$$\lim_{t \to \infty} k_t \lambda_t = 0, \tag{10.52}$$

is also necessary for optimality but, as we will see below, this turns out to be wrong in this case with no discounting.

[24] A more generally applicable optimality criterion is the *catching-up* criterion. The meaning of this criterion in continuous time is analogue to its meaning in discrete time, cf. Chapter 8.

Our assumption (A2) here reduces to $\lim_{k\to 0} f'(k) > \delta > \lim_{k\to\infty} f'(k)$ (which requires $\delta > 0$). Apart from this, the phase diagram is fully analogue to that in Fig. 10.1, except that the steady state, E, is now at the top of the $\dot{k} = 0$ curve. This is because in steady state, $f'(k^*) - \delta = 0$, and this equation also defines k_{GR} in this case. It can be shown that the saddle path is again the unique solution to the optimization problem (by essentially the same method as in the discrete time case of Chapter 8).

A noteworthy feature is that in this case the Ramsey model constitutes a counterexample to the widespread presumption that an optimal plan with infinite horizon *must* satisfy a transversality condition like (10.52). Indeed, by (10.50), $\lambda_t = u'(c_t) \to u'(c^*)$ for $t \to \infty$ along the overtaking-optimal path (the saddle path). Thus, instead of (10.52), we get

$$\lim_{t\to\infty} k_t\lambda_t = u'(c^*)k^* > 0.$$

With CRRA utility it is straightforward to generalize these results to the case $g \geq 0, n \geq 0$ and $\hat{\rho} - n = (1-\hat{\theta})g$. The social planner's overtaking-optimal solution is still the saddle path approaching the golden rule steady state; and this solution violates the seemingly "natural" transversality condition. What we learn from this is that an infinite horizon and the golden rule are sometimes associated with remarkably distinct results.

Note also that with zero effective utility discounting, there can not be equilibrium in the *market* economy version of this story. The real interest rate would in the long run be zero and thus the human wealth of the infinitely-lived household would be infinite. But then the demand for consumption goods would be unbounded and equilibrium thus be impossible.

10.6 Concluding remarks

The Ramsey model has played an important role as a way of structuring economists' thoughts about many macrodynamic phenomena including economic growth. The model should not be considered directly descriptive but rather as an examination of a benchmark case. As just noted this case is in some sense the opposite of the Diamond OLG model. Both models build on very idealized assumptions. Whereas the Diamond model ignores any bequest motive and emphasizes life-cycle behavior and at least some heterogeneity in the population, the Ramsey model implicitly assumes an altruistic bequest motive which is always operative and which turns households into homogeneous, infinitely-lived agents. In this way the Ramsey model ends up as an easy-to-apply framework, implying, among other things, a clear-cut theory of the level of the real interest rate in the long run. The model's usefulness lies in allowing general equilibrium analysis of an array of problems in a "vacuum".

The next chapter presents examples of different applications of the Ramsey model. Because of the model's simplicity, one should always be aware of the risk of non-robust conclusions. The assumption of a representative household is a main limitation of the Ramsey model. It is not easy to endow the dynasty portrait of households with plausibility. One of the problems is, as argued by Bernheim and Bagwell (1988), that this portrait does not comply with the fact that families are interconnected in a complex way via marriage of partners coming from different parent families. And the lack of heterogeneity in the model's population of households implies a danger that important interdependencies between different classes of agents are unduly neglected. For some problems these interdependencies may be of only secondary importance, but for others (for instance, issues concerning public debt or interaction between private debtors and creditors) they are crucial.

Another critical limitation of the model comes from its reliance on saddle-point stability with the associated presumption of perfect foresight infinitely far out in the future. There can be good reasons for bearing in mind the following warning (by Solow, 1990, p. 221) against overly reliance on the Ramsey framework in the analysis of a market economy:

> "The problem is not just that perfect foresight into the indefinite future is so implausible away from steady states. The deeper problem is that in practice – if there is any practice – miscalculations about the equilibrium path may not reveal themselves for a long time. The mistaken path gives no signal that it will be "ultimately" infeasible. It is natural to comfort oneself: whenever the error is perceived there will be a jump to a better approximation to the converging arm. But a large jump may be required. In a decentralized economy it will not be clear who knows what, or where the true converging arm is, or, for that matter, exactly where we are now, given that some agents (speculators) will already have perceived the need for a mid-course correction while others have not. This thought makes it hard even to imagine what a long-run path would look like. It strikes me as more or less devastating for the interpretation of quarterly data as the solution of an infinite time optimization problem."

10.7 Bibliographical notes

1. Frank Ramsey died at the age of 26 but he published several important articles. Ramsey discussed economic issues with, among others, John Maynard Keynes. In an obituary published in the Economic Journal (March 1932) some months after Ramsey's death, Keynes described Ramsey's article about the optimal savings as "one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject".

2. The version of the Ramsey model we have considered is in accordance with the general tenet of neoclassical preference theory: saving is motivated only by higher consumption in the future. Other versions assume accumulation of wealth is also motivated by a desire for social prestige and economic and political power rather than consumption. In Kurz (1968b) an extended Ramsey model is studied where wealth is an independent argument in the instantaneous utility function.

3. The equivalence in the Ramsey model between the decentralized market equilibrium and the social planner's solution can be seen as an extension of the first welfare theorem as it is known from elementary textbooks, to the case where the market structure stretches infinitely far out in time, and the finite number of economic agents (families) face an infinite time horizon: in the absence of externalities etc., the allocation of resources under perfect competition will lead to a Pareto optimal allocation. The Ramsey model is indeed a special case in that all households are identical. But the result can be shown in a far more general setup, cf. Debreu (1954). The result, however, does not hold in overlapping generations models where new generations enter and the "interests" of the new households have not been accounted for in advance.

4. Cho and Graham (1996) consider the empirical question whether countries tend to be above or below their steady state. Based on the Penn World Table they find that *on average*, countries with a relatively low income per adult are above their steady state and that countries with a higher income are below.

10.8 Appendix

A. Algebraic analysis of the dynamics around the steady state

To supplement the graphical approach of Section 10.3 with an exact analysis of the adjustment dynamics of the model, we compute the Jacobian matrix for the system of differential equations (10.28) - (10.29):

$$J(\tilde{k},\tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta + g + n) & -1 \\ \frac{1}{\theta}f''(\tilde{k})\tilde{c} & \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho + \theta g) \end{bmatrix}.$$

Evaluated in the steady state this reduces to

$$J(\tilde{k}^*,\tilde{c}^*) = \begin{bmatrix} \rho - n - (1-\theta)g & -1 \\ \frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* & 0 \end{bmatrix}$$

This matrix has the determinant

$$\frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* < 0.$$

Since the product of the eigenvalues of the matrix equals the determinant, the eigenvalues are real and opposite in sign.

In standard math terminology a steady-state point in a two dimensional continuous-time dynamic system is called a *saddle point* if the associated eigenvalues are opposite in sign.[25] For the present case we conclude that the steady state is a saddle point. This mathematical definition of a saddle point is equivalent to that given in the text of Section 10.3. Indeed, with two eigenvalues of opposite sign, there exists, in a small neighborhood of the steady state, a stable arm consisting of two saddle paths which point in opposite directions. From the phase diagram in Fig. 10.1 we know that the stable arm has a positive slope. At least for $\tilde{k}_0$ sufficiently close to $\tilde{k}^*$ it is thus possible to start out on a saddle path. Consequently, there is a (unique) value of $\tilde{c}_0$ such that $(\tilde{k}_t, \tilde{c}_t) \to (\tilde{k}^*, \tilde{c}^*)$ for $t \to \infty$. Finally, the dynamic system has exactly one jump variable, $\tilde{c}$, and one predetermined variable, $\tilde{k}$. It follows that the steady state is (locally) *saddle-point stable.*

We claim that for the present model this can be strengthened to *global* saddle-point stability. Our claim is that for *any* $\tilde{k}_0 > 0$, it is possible to start out on a saddle path. For $0 < \tilde{k}_0 \leq \tilde{k}^*$, this is obvious in that the extension

[25]Note the difference compared to the discrete time system in Chapter 8. In the discrete time system we have next period's $\tilde{k}$ and $\tilde{c}$ on the left-hand side of the dynamic equations, not the increase in $\tilde{k}$ and $\tilde{c}$, respectively. Therefore, the criterion for a saddle point is different in discrete time.

of the saddle path towards the left reaches the y-axis at a non-negative value of $\tilde{c}^*$. That is to say that the extension of the saddle path cannot, according to the uniqueness theorem for differential equations, intersect the $\tilde{k}$-axis for $\tilde{k} > 0$ in that the positive part of the $\tilde{k}$-axis is a solution of (10.28) - (10.29).[26]

For $\tilde{k}_0 > \tilde{k}^*$, our claim can be verified in the following way: suppose, contrary to our claim, that there exists a $\tilde{k}_1 > \tilde{k}^*$ such that the saddle path does not intersect that region of the positive quadrant where $\tilde{k} \geq \tilde{k}_1$. Let $\tilde{k}_1$ be chosen as the smallest possible value with this property. The slope, $d\tilde{c}/d\tilde{k}$, of the saddle path will then have no upper bound when $\tilde{k}$ approaches $\tilde{k}_1$ from the left. Instead $\tilde{c}$ will approach ∞ along the saddle path. But then $\ln \tilde{c}$ will also approach ∞ along the saddle path for $\tilde{k} \to \tilde{k}_1$ ($\tilde{k} < \tilde{k}_1$). It follows that $d \ln \tilde{c}/d\tilde{k} = (d\tilde{c}/d\tilde{k})/\tilde{c}$, computed along the saddle path, will have no upper bound. Nevertheless, we have

$$\frac{d \ln \tilde{c}}{d\tilde{k}} = \frac{d \ln \tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}/\tilde{c}}{\dot{\tilde{k}}} = \frac{\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho - \theta g)}{f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}}.$$

When $\tilde{k} \to \tilde{k}_1$ and $\tilde{c} \to \infty$, the numerator in this expression is bounded, while the denominator will approach $-\infty$. Consequently, $d \ln \tilde{c}/d\tilde{k}$ will approach zero from above, as $\tilde{k} \to \tilde{k}_1$. But this contradicts that $d \ln \tilde{c}/d\tilde{k}$ has no upper bound, when $\tilde{k} \to \tilde{k}_1$. Thus, the assumption that such a $\tilde{k}_1$ exists is false and our original hypothesis holds true.

B. Boundedness of the utility integral

We claimed in Section 10.3 that if the parameter restriction

$$\rho - n > (1 - \theta)g \tag{A1}$$

holds, then the utility integral, $U_0 = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt$, is bounded along the steady-state path, $c_t = \tilde{c}^* T_t$. The proof is as follows. For $\theta \neq 1$,

$$\begin{aligned}(1-\theta)U_0 &= \int_0^\infty c_t^{1-\theta} e^{-(\rho-n)t} dt = \int_0^\infty (c_0 e^{gt})^{1-\theta} e^{-(\rho-n)t} dt \\ &= c_0 \int_0^\infty e^{[(1-\theta)g-(\rho-n)]t} dt \\ &= \frac{c_0}{\rho - n - (1-\theta)g},\end{aligned} \tag{10.53}$$

[26] Because the extension of the saddle path towards the left in Fig. 10.1 can not intersect the $\tilde{c}$-axis at a value of $\hat{c} > f(0)$, it follows that if $f(0) = 0$, the extension of the saddle path ends up in the origin.

by (A1). If $\theta = 1$, we get

$$U_0 = \int_0^\infty (\ln c_0 + gt)e^{-(\rho-n)t}dt,$$

which is also finite, in view of (A1) implying $\rho - n > 0$ in *this* case. It follows that also any path converging to the steady state will entail bounded utility, when (A1) holds.

On the other hand, suppose that (A1) does *not* hold, i.e., $\rho - n \leq (1-\theta)g$. Then by the third equality in (10.53) and $c_0 > 0$ follows that $U_0 = \infty$.

C. The diverging paths

In Section 10.3 we stated that paths of types *II* and *III* in the phase diagram in Fig. 10.1 can not be equilibria with perfect foresight. Given the expectation corresponding to any of these paths, every single household will choose to *deviate* from the expected path (i.e., deviate from the expected "average behavior" in the economy). We will now show this formally.

We first consider a path of type *III.* A path of this type will not be able to *reach* the horizontal axis in Fig. 10.1. It will only *converge* towards the point $(\bar{\bar{\tilde{k}}}, 0)$ for $t \to \infty$. This claim follows from the uniqueness theorem for differential equations with continuously differentiable right-hand sides. The uniqueness implies that two solution curves cannot cross. And we see from (10.29) that the positive part of the x-axis is from a mathematical point of view a solution curve (and the point $(\bar{\bar{\tilde{k}}}, 0)$ is a trivial steady state). This rules out another solution curve hitting the x-axis.

The convergence of $\tilde{k}$ towards $\bar{\bar{\tilde{k}}}$ implies $\lim_{t\to\infty} r_t = f'(\bar{\bar{\tilde{k}}}) - \delta < g + n$, where the inequality follows from $\bar{\bar{\tilde{k}}} > \tilde{k}_{GR}$. So,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (r_s - g - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = \bar{\bar{\tilde{k}}} e^{\infty} > 0. \tag{10.54}$$

Hence the transversality condition of the households is violated. Consequently, the household will choose higher consumption than along this path and can do so without violating the NPG condition.

Consider now instead a path of type *II.* We shall first show that if the economy follows such a path, then depletion of all capital occurs in finite time. Indeed, in the text it was shown that any path of type *II* will pass the $\dot{\tilde{k}} = 0$ locus in Fig. 10.1. Let t_0 be the point in time where this occurs. If path *II* lies above the $\dot{\tilde{k}} = 0$ locus for all $t \geq 0$, then we set $t_0 = 0$. For

$t > t_0$, we have

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t < 0.$$

By differentiation w.r.t. t we get

$$\ddot{\tilde{k}}_t = f'(\tilde{k}_t)\dot{\tilde{k}}_t - \dot{c}_t - (\delta + g + n)\dot{\tilde{k}}_t = [f'(\tilde{k}_t) - \delta - g - n]\dot{\tilde{k}}_t - \dot{c}_t < 0,$$

where the inequality comes from $\dot{\tilde{k}}_t < 0$ combined with $\tilde{k}_t < \tilde{k}_{GR} \Rightarrow f'(\tilde{k}_t) - \delta > f'(\tilde{k}_{GR}) - \delta = g + n$. Therefore, there exists a $t_1 > t_0 \geq 0$ such that

$$\tilde{k}_{t_1} = \tilde{k}_{t_0} + \int_{t_0}^{t_1} \dot{\tilde{k}}_t dt = 0,$$

as was to be shown. At time t_1, $\tilde{k}$ cannot fall any further and $\tilde{c}_t$ immediately drops to $f(0)$ and stay there hereafter.

Yet, this result does not in itself explain why the individual household will deviate from such a path. The individual household has a negligible impact on the movement of $\tilde{k}_t$ in society and correctly perceives r_t and w_t as essentially independent of its own consumption behavior. Indeed, the economy-wide $\tilde{k}$ is not the household's concern. What the household cares about is its own financial wealth and budget constraint. Nothing prevents the household from planning a negative financial wealth, a, and possibly a continuously declining financial wealth, if only the NPG condition,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0,$$

is satisfied.

But we can show that paths of type *II* will *violate* the NPG condition. The reasoning is as follows. The household plans to follow the Keynes-Ramsey rule. Given an expected evolution of r_t and w_t corresponding to path *II*, this will imply a planned gradual transition from positive financial wealth to debt. The transition to positive net debt, $\tilde{d}_t \equiv -\tilde{a}_t \equiv -a_t/T_t > 0$, takes place at time t_1 defined above.

The continued growth in the debt will meanwhile be so fast that the NPG condition is violated. To see this, note that the NPG condition implies the requirement

$$\lim_{t\to\infty} \tilde{d}_t e^{-\int_0^t (r_s - g - n)ds} \leq 0, \tag{NPG}$$

that is, the productivity-corrected debt, $\tilde{d}_t$, is allowed to grow in the long run only at a rate *less* than the growth-corrected real interest rate. For $t > t_1$ we get from the accounting equation $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ that

$$\dot{\tilde{d}}_t = (r_t - g - n)\tilde{d}_t + \tilde{c}_t - \tilde{w}_t > 0,$$

where $\tilde{d}_t > 0$, $r_t > \rho+\theta g > g+n$, and where $\tilde{c}_t$ grows exponentially according to the Keynes-Ramsey rule, while $\tilde{w}_t$ is non-increasing in that $\tilde{k}_t$ does not grow. This implies

$$\lim_{t\to\infty} \frac{\dot{\tilde{d}}_t}{\tilde{d}_t} \geq \lim_{t\to\infty}(r_t - g - n),$$

which is in conflict with (NPG).

Consequently, the household will choose a lower consumption path and thus *deviate* from the reference path considered. Every household will do this and the evolution of r_t and w_t corresponding to path *II* is thus *not* an equilibrium with perfect foresight.

The conclusion is that all individual households understand that the only evolution which can be expected rationally is the one corresponding to the saddle path.

D. A constant saving rate as a special case

As we noted in Section 10.4 Solow's growth model can be seen as a special case of the Ramsey model. Indeed, a constant saving rate may, under certain conditions, emerge as an endogenous result in the Ramsey model.

Let the rate of saving, $(Y_t - C_t)/Y_t$, be s_t. We have generally

$$\tilde{c}_t = (1 - s_t) f(\tilde{k}_t), \qquad \text{and so} \tag{10.55}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t = s_t f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t. \tag{10.56}$$

In the Solow model the rate of saving is a constant, s, and we then get, by differentiating with respect to t in (10.55) and using (10.56),

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = f'(\tilde{k}_t)[s - \frac{(\delta + g + n)\tilde{k}_t}{f(\tilde{k}_t)}]. \tag{10.57}$$

By maximization of discounted utility in the Ramsey model, given a rate of time preference ρ and an elasticity of marginal utility θ, we get in equilibrium

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g). \tag{10.58}$$

There will not generally exist a constant, s, such that the right-hand sides of (10.57) and (10.58), respectively, are the same for varying $\tilde{k}$ (that is, outside steady state). But Kurz (1968a) showed the following:

CLAIM Let δ, g, n, α, and θ be given. If the elasticity of marginal utility θ is greater than 1 and the production function is $\tilde{y} = A\tilde{k}^{\alpha}$ with $\alpha \in (1/\theta, 1)$, then a Ramsey model with $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g$ will generate a constant saving rate $s = 1/\theta$. Thereby the same resource allocation and transitional dynamics arise as in the corresponding Solow model with $s = 1/\theta$.

Proof. Let $1/\theta < \alpha < 1$ and $f(\tilde{k}) = A\tilde{k}^{\alpha}$. Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1}$. The right-hand-side of the Solow equation, (10.57), becomes

$$A\alpha\tilde{k}^{\alpha-1}[s - \frac{(\delta+g+n)\tilde{k}_t}{A\tilde{k}^{\alpha}}] = sA\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \tag{10.59}$$

The right-hand-side of the Ramsey equation, (10.58), becomes

$$\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\rho+\theta g}{\theta}.$$

By inserting $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g$, this becomes

$$\begin{aligned} & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\theta\alpha(\delta+g+n)-\delta-\theta g+\theta g}{\theta} \\ = \; & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \end{aligned} \tag{10.60}$$

For the chosen ρ we have $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g > n + (1-\theta)g$, because $\theta\alpha > 1$ and $\delta+g+n > 0$. Thus, $\rho - n > (1-\theta)g$ and existence of equilibrium in the Ramsey model with this ρ is ensured. We can now make (10.59) and (10.60) the same by inserting $s = 1/\theta$. This also ensures that the two models require the same $\tilde{k}^*$ to obtain a *constant* $\tilde{c} > 0$. With this $\tilde{k}^*$, the requirement $\dot{\tilde{k}}_t = 0$ gives the same steady-state value of $\tilde{c}$ in both models, in view of (10.56). It follows that $(\tilde{k}_t, \tilde{c}_t)$ is the same in the two models for all $t \geq 0$. $\square$

On the other hand, maintaining $\tilde{y} = A\tilde{k}^{\alpha}$, but allowing $\rho \neq \theta\alpha(\delta+g+n) -\delta - \theta g$, so that $\theta \neq 1/s^*$, then $s'(\tilde{k}) \neq 0$, i.e., the Ramsey model does not generate a constant saving rate except in steady state. Defining s^* as in (10.39) and $\bar{\theta} \equiv (\delta+\rho)/\left[\alpha(\delta+g+n) - g\right]$, we have: When $\alpha(\delta+g+n) > g$ (which seems likely empirically), it holds that if $\theta \lesseqqgtr 1/s^*$ (i.e., if $\theta \lesseqqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqqgtr 0$, respectively; if instead $\alpha(\delta+g+n) \leq g$, then $\theta < 1/s^*$ and $s'(\tilde{k}) < 0$, unconditionally. These results follow by considering the slope of the saddle path in a phase diagram in the $(\tilde{k}, \tilde{c}/f(\tilde{k}))$ plane and using that $s(\tilde{k}) = 1 - \tilde{c}/f(\tilde{k})$, cf. Exercise 10.?? The intuition is that when $\tilde{k}$ is rising over time (i.e., society is becoming wealthier), then, when the desire for consumption smoothing is "high" (θ "high"), the prospect of high consumption in the future is partly taken out as high consumption already today, implying that saving is initially low, but rising over time until it eventually settles down in the steady state. But if the desire for consumption smoothing is "low" (θ "low"), saving will initially be high and then gradually fall in the process towards the steady state. The case where $\tilde{k}$ is falling over time gives symmetric results.

E. The social planner's solution

In the text of Section 10.5 we postponed some of the more technical details. First, by (A2), the existence of the steady state, E, and the saddle path in Fig. 10.1 is ensured. Solving the linear differential equation (10.45) gives $\lambda_t = \lambda_0 e^{-\int_0^t (f'(\tilde{k}_s)-\delta-\hat{\rho}-g)ds}$. Substituting this into the transversality condition (11.49) gives

$$\lim_{t\to\infty} e^{-\int_0^t (f'(\tilde{k}_s)-\delta-g-n)ds}\tilde{k}_t = 0, \tag{10.61}$$

where we have eliminated the unimportant positive factor $\lambda_0 = c_0^{-\hat{\theta}}T_0$.

This condition is essentially the same as the transversality condition (10.36) for the market economy and holds in the steady state, given the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$, which is satisfied in view of (A1). Thus, (10.61) also holds along the saddle path. Since we must have $\tilde{k} \geq 0$ for all $t \geq 0$, (10.61) has the form required by Mangasarian's sufficiency theorem. Thus, if we can show that the Hamiltonian is concave in $(\tilde{k}, c)$ for all $t \geq 0$, then the saddle path *is* a solution to the social planner's problem. And if we can show strict concavity, the saddle path is the *only* solution. We have:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - (\delta+g+n)), \quad \frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T}, \\
\frac{\partial^2 H}{\partial \tilde{k}^2} &= \lambda f''(\tilde{k}) < 0 \quad (\text{by } \lambda = c^{-\hat{\theta}}T > 0), \quad \frac{\partial^2 H}{\partial c^2} = -\hat{\theta}c^{-\hat{\theta}-1} < 0, \\
\frac{\partial^2 H}{\partial \tilde{k}\partial c} &= 0.
\end{aligned}$$

Thus, the leading principal minors of the Hessian matrix of H are

$$D_1 = -\frac{\partial^2 H}{\partial \tilde{k}^2} > 0, \quad D_2 = \frac{\partial^2 H}{\partial \tilde{k}^2}\frac{\partial^2 H}{\partial c^2} - \left(\frac{\partial^2 H}{\partial \tilde{k}\partial c}\right)^2 > 0.$$

Hence, H is strictly concave in $(\tilde{k}, c)$ and the saddle path is the unique optimal solution.

It also follows that the transversality condition (11.49) *is* a *necessary* optimality condition. Note that we had to derive this conclusion in a different way than when solving the household's consumption/saving problem in Section 10.2. There we could appeal to a link between the No-Ponzi-Game condition (with strict equality) and the transversality condition to verify necessity of the transversality condition. But that proposition does not cover the social planner's problem.

As to the diverging paths in Fig. 10.1, note that paths of type II (those paths which, as shown in Appendix C, in finite time deplete all capital) can not be optimal, in spite of the temporarily high consumption level. This follows from the fact that the saddle path is the unique solution. Finally, paths of type III in Fig. 10.1 behave as in (10.54) and thus violate the transversality condition (11.49), as claimed in the text.

10.9 Exercises