Bound Notes

VCE Specialist Mathematics, MAT1085

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2008

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Last update	Version 1.43, 29th September 2008

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Algebra

Linear Algebra (Systems and Matrices, Vectors), Algebra of Functions, Complex Numbers

1.1Linear Algebra

1.1.1 Systems and Matrices

A $m \times n$ matrix is a rectangular array of m rows and n columns, denoted by $A_{m\times n}$. Its entries at i row and j column is denoted as $a_{i\,j}$. The matrix can also be expressed as $A = \left(a_{i\,j}\right)_{m\times n}$

Addition/subtraction are only legal where the order (dimension) of the matrices are the same.

$$A_{m\times n} + B_{m\times n} = \left(a_{i\,j} + b_{i\,j}\right)_{m\times n}$$

Two matrices are considered equal if their order and all their entries are the same.

Properties of addition:

- Commutative, (A+B)=(B+A)
- Associative, A+(B+C)=(A+B)+C
- Addition of zero matrix has no effect, A+0=A
- There exist a negative matrix D of A, where each and all its entries are negative that of A, such that, D+A=A+D=0

Scalar multiple of a matrix is obtained by multiplying each entry by the scalar.

Properties of scalar multiplication:

- Scalar “1” has no effect, 1A=A
- Collective, kA+nA=(k+n)A
- Distributive, k(A+B)=kA+kB
- Associative and commutative, k(nA)=n(kA)=(nk)A
- Zero scalar nulls the matrix, 0A=0

1.1.1.1 Matrix Multiplication

Matrix multiplication between A and B is only legal if the number of columns is the same as the number of rows. The product inherit the number of rows of A, and the number of columns of B

$$A_{m\times p} \times B_{p\times n} = C_{m\times n}$$

$$c_{i\,j} = \sum_{t=1}^{p} (a_{i\,t} \cdot b_{t\,j})$$

For example

$$\begin{bmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} \\ a_{2\,1} & a_{2\,2} & a_{2\,3} \end{bmatrix} \cdot \begin{bmatrix} b_{1\,1} & b_{1\,2} \\ b_{2\,1} & b_{2\,2} \\ b_{3\,1} & b_{3\,2} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Where the matrices have the right dimensions to multiply each other, they are said to be “conformable”

Matrix division involves the matrix inverse, which doesn’t always exist. This will be explored later on in the notes.

Properties of multiplication:

- Associative, A(BC)=(AB)C
- Distributive, A(B+C)=AB+AC
- NOT commutative

1.1.1.2 Transpose

A transpose matrix is a matrix with rows and columns switched, or inverted about its primary axis [a_{11}, a_{22}, a_{33}…]

$$\left((a_{i\,j})_{m\times n}\right)^T = (a_{j\,i})_{n\times m}$$

For example

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Properties of transpose:

- The transpose of a transpose is itself, $(A^T)^T=A$
- Transpose of a sum is the sum of transposes, $(A+B)^T=A^T+B^T$
- Transpose of a scalar multiple is the scalar multiple of the transpose, $(kA)^T=k(A^T)$
- Transpose of a matrix product is the product of the transposes in the REVERSE ORDER, $(AB)^T=B^TA^T$

1.1.1.3 Special Types of Matrices

1.1.1.3.1 Zero

Zero matrices are matrices where all entries are 0:

$$0_{m\times n} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

1.1.1.3.2 Square

Square matrices are matrices which has the same horizontal and vertical dimension, such as $A_{m\times m}$

Only a square matrix can have determinants, inverses and powers.

1.1.1.3.3 Symmetrical

The transpose of a symmetrical matrix is equal to itself. I.e. the matrix is equivalent on either side of its primary axis.

$\begin{bmatrix} \boldsymbol{a} & x & z \\ x & \boldsymbol{b} & y \\ z & y & \boldsymbol{c} \end{bmatrix}$ is square.

1.1.1.3.4 Diagonal

A diagonal matrix is a symmetrical matrix where all entries except those on the primary axis are zero.

$\begin{bmatrix} \boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c} \end{bmatrix}$ is a diagonal matrix.

The non-zero entries of the inverse of the diagonal matrix are the reciprocal of the non-zero entries of the diagonal matrix. $\begin{bmatrix} \boldsymbol{1/a} & 0 & 0 \\ 0 & \boldsymbol{1/b} & 0 \\ 0 & 0 & \boldsymbol{1/c} \end{bmatrix}$

1.1.1.3.5 Identity

An identity matrix is a diagonal matrix where all the non-zero entries are 1.

$$I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A matrix multiplied by a conformable identity matrix is itself. IA=AI=A.

The inverse of the identity matrix is itself.

1.1.1.3.6 Orthogonal

Orthogonal matrices are square matrices where their transpose is equal to their inverse.

$$A \times A^T = A^T \times A = I$$

One of such case is the rotational matrix, $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

1.1.1.4 Linear Transformation

For an n-dimensional space, the position vector can be transformed linearly by an $n \times n$ transformation matrix T, such that:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = T_{n\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

1.1.1.4.1 Homogenous Linear Transformation

Homogenous linear transformation are transformation in the R^2 space by a 2x2 matrix.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Generally, where $x_2=mx_1+c$ (a straight line), given that $a_{11}a_{22} - a_{12}a_{21} \neq 0$:

$$y_2 = m'y_1 + c'$$

$$m' = \frac{a_{21} + ma_{22}}{a_{11} + ma_{12}}, \quad c' = \frac{c(a_{11}a_{22} - a_{12}a_{21})}{a_{11} + ma_{12}}$$

1.1.1.4.1.1 Special Cases

Where $a_{11}a_{22} - a_{12}a_{21} = 0$,

$$y_2 = \frac{a_{22}}{a_{12}} y_1, \quad a_{12} \neq 0$$

Or $y_1 = a_{22}$ where a_{22} is not 0. Or $y_2 = \frac{a_{21}}{a_{11}} y_1$ where both a_{12} and a_{22} are 0.

1.1.1.4.2 Dilation

$\begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}$, where h is the dilation factor from the y axis (parallel to x), and k is the dilation factor from x axis (parallel to y)

1.1.1.4.3 Reflection

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects about the y axis.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects about the x axis.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects about the y=x line, or takes the inverse of the relationship.

1.1.1.4.4 Rotation

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ rotates by θ in the anticlockwise direction.

1.1.1.4.5 Shearing

$\begin{bmatrix} 1 & h \\ k & 1 \end{bmatrix}$, where h is the amount of sheer in the x direction, and k is the amount of sheer in the y direction.

1.1.1.5 Row Operations

There are three elementary row operations:

- Row swap – swap any row with another row
- Multiply by a scalar (non-zero) – multiply any row by a number (not zero)
- Add a multiple of another row

By using row operations, a matrix can be made into ref or rref.

1.1.1.5.1 Row Echelon Form

Entries below any leading entries (the first non-zero entry in a row) are zero. This also implies that all entries to the left of any leading entries are zero. Leading entries are preferred to be, but not necessarily, "1".

$$\begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

1.1.1.5.2 Reduced Row Echelon Form

Entries below and above any leading entries (the first non-zero entry in a row) are zero. This also implies that all entries to the left of any leading entries are zero. All leading entries are "1".

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \end{bmatrix}$$

1.1.1.6 Determinants

The determinant is a numerical valued function of a square matrix that determines whether it is invertible. This allows the calculation of the matrix inverse, and hence allows "division".

Determinants are denoted by $\det A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$.

It is granted that $\det\left(\left(a_{ij}\right)_{1\times 1}\right) = a_{11}$.

For a 2 by 2 matrix, we can express a linear system as

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

By the process of elimination, $(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2$

For x_1 to have a unique solution, its coefficient must not be 0. This coefficient is the determinant.

$$\det\left(\left(a_{ij}\right)_{2\times 2}\right) = a_{11} = a_{11}a_{22} - a_{12}a_{21}$$

1.1.1.6.1 Minor

The minor (i j) of matrix A is the matrix with the i^th row and j^th column struck out:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

1.1.1.6.2 Cofactor

The cofactor is the determinant of the minor multiplied by -1 to the power of the row plus column:

$$A_{i\,j} = (-1)^{i+j}\left|M_{i\,j}\right|$$

For the above case, $A_{(11)} = (-1)^{1+1} \times (a_{22}a_{23} - a_{23}a_{32}) = a_{22}a_{23} - a_{23}a_{32}$

The -1 term will mean that the cofactors of different entries will have differing signs. They follow this general pattern:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

1.1.1.6.3 Cofactor Expansion

Cofactor expansion can take any row and any column. The determinant is the sum of the product of each entry and its cofactor in a particular row or column. For matrix $A_{m\times m}$

By column:

$$\det A = \sum_{i=1}^{m} a_{i\,k} \cdot A_{i\,k} = \sum_{i=1}^{m} (-1)^{i+k} \cdot a_{i\,k} \cdot |M_{i\,k}|, \qquad \text{where } k \text{ is a constant}$$

By row:

$$\det A = \sum_{j=1}^{m} a_{k\,j} \cdot A_{k\,j} = \sum_{j=1}^{m} (-1)^{k+j} \cdot a_{k\,j} \cdot \left|M_{k\,j}\right|, \qquad \text{where } k \text{ is a constant}$$

For example:

$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, to find det A, expanding by first row:

$$\begin{aligned} \det A &= (-1)^{1+1} \times 1 \times \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + (-1)^{1+2} \times 0 \times \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + (-1)^{1+3} \times 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ &= (2-6) + 0 + (2-6) = -8 \end{aligned}$$

1.1.1.6.4 Properties of Determinants

Some properties of determinants are:

- Det(A)=Det(A^T)
- If there is any row or column that is entirely consisted of zeros, the determinant is zero
- If any row or columns are identical, the determinant is zero
- If any row or column are multiples of another row/column, then the determinant is zero
- Scalar multiple of a single row or column gives the scalar multiple of the determinant,

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Decomposition of rows/columns,

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Multiples of another row or column,

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Row swap,

$$\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Det(AB)=Det(A)Det(B)

1.1.1.7 Inverse

A matrix inverse is one such that A(A^{-1})=(A^{-1})A=I

1.1.1.7.1 By Cofactor

The matrix inverse is the transpose of the cofactor matrix divided by its determinant. Hence a matrix with a 0 determinant has no inverse.

$$A^{-1} = \frac{1}{\det(A)}\left(A_{j\,i}\right)_{m\times m}$$

The transpose of the cofactor matrix is called the adjoint matrix, denoted by adj(A).

$$A^{-1} = \frac{\operatorname{adj} A}{\det A}$$

For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, A_{i\,j} = \begin{bmatrix} -4 & 8 & -4 \\ 2 & -2 & -2 \\ -2 & -2 & 2 \end{bmatrix}, \det(A) = -8$

$$A^{-1} = \frac{1}{-8}\begin{bmatrix} -4 & 0 & -4 \\ 8 & -2 & -2 \\ -2 & -2 & 2 \end{bmatrix}^T = -\frac{1}{8}\begin{bmatrix} -4 & 2 & -2 \\ 8 & -2 & -2 \\ -4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ -1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

1.1.1.7.2 By Row Operations

By augmenting the square matrix with the identity of equal dimension on the right, using elementary row operations to make the left-hand matrix into identity, the right hand matrix will be the inverse.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Where an entire row or column of the left hand side becomes entirely 0, the identity matrix can never be obtained, i.e. no matrix inverse.

1.1.1.7.3 Properties of Inverse

- A matrix has an inverse if and only if it has a non-zero determinant.
  - If a matrix has no inverse, it is "singular"
  - If a matrix has an inverse, it is "non-singular" or "invertible"
- $\det(A)\det(A^{-1})=1$
- $(A^T)^{-1}=(A^{-1})^T$
- $(AB)^{-1}=B^{-1}A^{-1}$

1.1.1.8 Elementary Matrices

Elementary matrices are square transformation matrices that perform a single elementary row operation. These matrices are invertible.

An $m \times m$ matrix is considered an elementary matrix if it differs from the identity matrix by a single row operation.

1.1.1.8.1 Types and Properties

Type I – Interchange two rows

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Type II – Multiply a row by a non-zero number

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Type III – add a multiple of another row

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we let E be an $m \times m$ elementary matrix, and A be an $m \times n$. The matrix product EA would be the same as applying that row operation to A.

1.1.1.8.2 Inverse Elementary Matrices

Type I – Interchange two rows

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(This type of elementary matrix is its self-inverse.)

Type II – Multiply by the reciprocal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Type III – Subtract multiple of another row

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrices fundamentally characterises matrix inverses:

If A is the product of elementary matrices $E_k \ldots E_3E_2E_1$, then:

$$A = E_k \ldots E_3 E_2 E_1 I$$

$$E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_k^{-1} A = E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_k^{-1} E_k \ldots E_3 E_2 E_1 I = I$$

$$\therefore A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \ldots E_k^{-1}$$

A matrix has an inverse if and only if it is the product of elementary matrices.

1.1.1.9 Systems of Linear Equations

A system of linear equation can be generalised to be:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 \cdots + a_{kn}x_n = b_k$$

For n-variable n-equations system, it can be expressed as the matrix equation:

$$(a_{ij})_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = B$$

1.1.1.9.1 Solving

1.1.1.9.1.1 Using Inverses

In the case that A is a non-singular matrix,

$$Ax = B \Longrightarrow x = A^{-1}B$$

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

1.1.1.9.1.2 Cramer's Rule

Cramer's rule is considered to be easier than using matrix inverses.

Since $A^{-1} = \frac{adj(A)}{\det A}$, the solution of the system can be expressed as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} A_{11}b_1 + A_{21}b_2 + \cdots + A_{n1}b_n \\ A_{12}b_1 + A_{22}b_2 + \cdots + A_{n2}b_n \\ \vdots \\ A_{1n}b_1 + A_{2n}b_2 + \cdots + A_{nn}b_n \end{bmatrix}$$

It is evident that the rightmost matrix entries is a cofactor expansion of a column, where the entry $a_{i\,j} = b_i$ for the jth column, and the rest of the entries are identical to that of A.

A$^{(j)}$ is used to denote the matrix obtained from A by replacing the jth column of A by the column vector B.

For example, $A^{(2)} = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}$

In general,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} \det A^{(1)} \\ \det A^{(2)} \\ \vdots \\ \det A^{(3)} \end{bmatrix}$$

In particular, $x_n = \frac{\det A^{(n)}}{\det A}$

1.1.1.9.1.3 Gaussian Elimination

Gaussian elimination uses elementary row operations of the augmented matrix [A|B], carrying it to its row echelon or reduced row echelon form.

For example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

Using row echelon forms

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$\boldsymbol{x_3 = \frac{1}{2}}$**;** $x_2 + 2x_3 = \frac{1}{2} \Longrightarrow \boldsymbol{x_2 = -\frac{1}{2}}$**;** $x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{1}{3} \Longrightarrow \boldsymbol{x_1 = \frac{1}{2}}$

Using reduced row echelon forms

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$\Longrightarrow x_1 = \frac{1}{2}, x_2 = -\frac{1}{2}, x_3 = \frac{1}{2}$$

Gaussian elimination can be used to solve not only n by n systems, but any system of linear equations.

1.1.1.9.2 Consistency

In a system of linear equations, exactly one of the following occurs.

1.1.1.9.2.1 No Solutions

A system of linear equations is said to be inconsistent if there are no solutions.

When there are no solutions, the determinant of the coefficient matrix A is zero, and the matrix inverse does not exist (The converse is not necessarily true). Using matrix inverse or Cramer's rule to solve a system would yield an indeterminate result (divide by zero).

Using Gaussian eliminations, there are no solutions when there are row of the type $[0 \quad 0 \quad \cdots \quad 0 \quad | \quad *]$, where * is a non-zero number.

1.1.1.9.2.2 Unique Solution

For a system with a set of unique solutions,

- The number of equations must be equal or more than the number of variables.
- The determinant of the coefficient matrix must not be 0
- The coefficient matrix must be invertible
- The reduced row echelon form of the coefficient matrix must resemble the identity matrix.
- The reduced row echelon form of the coefficient matrix must not have a column entirely consisted of zeros.

$$\text{unique solution } \Leftrightarrow A^{-1} \text{ exists} \Leftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A_{n\times n}^{-1} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow \det(A) \neq 0$$

1.1.1.9.2.3 Infinite Solutions

A system may have infinite solutions, such as two planes intersecting on a line or two equations coincide. Generally, if there are more variables than equations, there are infinite solutions to the system.

If the coefficient matrix of a system with infinite solutions is a square matrix, its determinant will be zero (the converse is not necessarily true). Using matrix inverse and Cramer's rule will yield an indeterminate result (divide by zero).

Using Gaussian elimination, in ref or rref, the columns of the coefficient which do not contain any leading entries are unbound, and its corresponding coefficient becomes a parameter.

For example, $\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 3 \\ -1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$

The augmented matrix will hence be $\begin{bmatrix} 1 & -3 & 1 & 1 \\ 2 & -6 & 3 & 4 \\ -1 & 3 & 0 & 1 \end{bmatrix}$, and its rref is $\begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Ignoring the row of zero entries, we can see that $\boldsymbol{z = 2}, \boldsymbol{x = 3y - 1}$, and y is unbound.

Let $\boldsymbol{y = t}, \boldsymbol{x = 3t - 1}, \boldsymbol{z = 2}$. This describes a line in R^3 space.

1.1.2 Vectors

1.1.2.1 *Definition*

A vector is a quantity with both magnitude and direction. It is also possible to define the sense of a vector, i.e. one of the two ways the vector can be pointing towards.

A vector is represented by a straight line segment with an arrow.

1.1.2.1.1 Equality of Vectors

Two vectors are equal if and only if they have the same magnitude, same direction and same sense.

Vectors are "free", where the starting point of a vector is irrelevant. The same line segments pointing in the same direction always represent the same vector, regardless of their starting point.

Displacement vectors are free vectors without a bound starting point. Position vectors are free vectors with the starting point bound at origin.

1.1.2.1.2 Special Vectors

1.1.2.1.2.1 Unit Vectors

Unit vectors have a magnitude of 1.

1.1.2.1.2.2 Spatial Dimensions

The defined dimensions are a way to coordinate n-dimensional space.

In particular, these dimensional vectors are unit vectors, and are perpendicular to each other.**i** is the first dimension, **j** is the second dimension perpendicular to **i**,and **k** is the third dimension perpendicular to both **i** and **j**.

A vector is often resolved into components in the direction of spatial dimensions.

1.1.2.1.2.3 Zero Vectors

Zero vector is a 0-dimensional vector, with zero magnitude, unspecified direction and sense. It is a single point.

1.1.2.1.3 Magnitude

The magnitude of a vector can be calculated by Pythagoras' theorem when it is expressed as perpendicular components.

$$|\vec{\boldsymbol{a}}| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots}$$

1.1.2.1.4 Angle with Axis

The cosine of the angle a vector makes with an axis is its component in that direction divided by its magnitude (ratio of cosine).

If α is the angle a vector makes with the x axis (**i** direction), then $\alpha = \cos^{-1}\left(\frac{a_1}{|\vec{a}|}\right)$

For a vector **u**, where α is the angle between it and the x axis, β to the y axis and γ to the z axis,

$$\cos\alpha = \frac{u_1}{|u|}, \cos\beta = \frac{u_2}{|u|}, \cos\gamma = \frac{u_3}{|u|}$$

And, since $\frac{1}{|u|}(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})$ would be a unit vector $\frac{u}{|u|}$, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

For example, $\tilde{s} = 40\tilde{\imath} + 60\tilde{\jmath} - 49\tilde{k}$, find the acute angle this vector makes with the horizontal.

Let γ be the angle $\tilde{s}$ makes with the z axis.

$$\cos\gamma = \frac{-49}{\sqrt{40^2 + 60^2 + 49^2}} = -0.562$$

$$\gamma \approx 2.168^c \approx 124.2^o$$

Hence the acute angle it makes with the horizontal is $124.2^o - 90^o = 34.2^o$

1.1.2.1.5 Sums, Differences and Scalar Multiples (Parallel Vectors)

Sum of vectors are calculated by joining each vector head to tail. The resultant vector is the vector which joins the tail of the first vector to the head of the last vector.

When vectors are expressed in their components, the sum of the vectors in a particular direction is the sum of the components in that direction. The components of the resultant vector are the sum of the components.

Vector subtraction are addition of negative vectors (i.e. reversed vector).

Scalar multiple of a vector changes the magnitude by the factor of the scalar, direction and sense are not changed.

Vectors addition and scalar multiplication are

- Commutative: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
- Associative: $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
- Addition of zero vector has no effect: $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
- Scalar multiplication is associative: $n(k\mathbf{u})=(nk)\mathbf{u}$
- Collective: $n\mathbf{u}+k\mathbf{u}=(n+k)\mathbf{u}$
- Distributive over addition: $n(\mathbf{u}+\mathbf{v})=n\mathbf{u}+n\mathbf{v}$
- Multiplication by 1: $1\mathbf{u}=\mathbf{u}$
- Multiplication by 0: $0\mathbf{u}=\mathbf{0}$

Two vectors are considered parallel if they are scalar multiples of each other.

1.1.2.1.6 Linear Dependence

Vectors are linearly dependant if the sum of multiples of vectors is the zero vector.

$$k_1\boldsymbol{a} + k_2\boldsymbol{b} + k_3\boldsymbol{c} + \cdots = \boldsymbol{0}$$

If this relationship is true for a set of coefficients that are not *all* zero.

This implies that any set of vectors are automatically linearly dependant if one of them is a zero vector.

Linear independency can be described as when a quantity cannot be described in terms of multiples of other quantities in a set. This applies not only to vectors, but algebraic expressions also.

1.1.2.2 Scalar Dot Product

1.1.2.2.1 Definition and Interpretation

The scalar dot product is defined as followed:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\theta$$

Where θ is the angle between the vectors.

The implication of this is that unit vectors which point in the same direction have a dot product of 1, and perpendicular vectors have a dot product of 0.

In three dimensional space (in terms of **i**, **j**, and **k**), the dot product of two vectors is

$$(a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) = a_1b_1 + a_2b_2 + a_3b_3$$

$$\theta = \cos^{-1}\frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}| \cdot |\boldsymbol{b}|}$$

Angle between vectors is never more than 180° (π).

Properties of the dot product:

- Commutative, **a.b=b.a**
- Distributive over addition, **a.(b+c)=a.b+a.c**
- Distributive over scalar multiplication, **a**.(k**b**)=k(**a**.**b**)=(k**a**).**b**
- The dot product of any vector with a zero vector is 0.
- The dot product of any vector with itself is its magnitude squared, **a.a=|a|**^{2}
- Two non-zero vectors are orthogonal if and only if their dot product is zero.

1.1.2.2.2 Orthogonal Vectors

Orthogonal vectors are vectors which point in perpendicular directions.

For non-zero orthogonal vectors, their dot product is always 0.

1.1.2.2.3 Resolute

The vector resolute of a vector in the direction of another vector:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \cdot \boldsymbol{b} = \left(\boldsymbol{a} \cdot \widehat{\boldsymbol{b}}\right) \cdot \widehat{\boldsymbol{b}}$$

The scalar resolute of a vector in the direction of another vector is $\boldsymbol{a} \cdot \widehat{\boldsymbol{b}}$, simply the magnitude of the vector resolute.

The perpendicular resolute is $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u}$.

1.1.2.3 Vector Cross Product

1.1.2.3.1 Definition and Interpretation

The vector cross product is defined as $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\theta \cdot \boldsymbol{u}$

Where **u** is a unit vector perpendicular to both **a** and **b**.

The cross product of two vectors in the three dimensional space can be computed by a determinant:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The magnitude of the vector cross product can be interpreted as the area of parallelogram formed by the two vectors.

Properties of vector product:

- Distributive over scalar multiples, **a**x(k**b**)=k(**a**x**b**)=(k**a**)x**b**
- NOT commutative. By the property of determinants, reversing the order swaps two rows, making the determinant negative of what it was. **b**x**a**=-(**a**x**b**)
- Distributive over addition, **a**x**(b+c)=a**x**b+a**x**c**
- Vector product with itself is the zero vector
- Vector product with a zero vector is the zero vector

1.1.2.3.2 Scalar Triple Product and Co-Planarity

The scalar triple product, or box product [a,b,c], is defined as

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By the properties of determinants,
A row swap makes the determinant negative, hence, -[a,b,c]=[b,a,c]=[a,c,b]=[c,b,a]
Swapping two rows makes the determinant positive hence, [a,b,c]=[c,a,b]=[b,c,a]

An interpretation of the value of the box product (scalar triple product) is the volume of the parallelepiped of the three vectors.

If the scalar triple product is zero, the vectors are coplanar (they exist on the same plane).

1.1.2.4 *Vector Geometry*

1.1.2.4.1 Line

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}t$$

$$\boldsymbol{r}(t) = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k} + t(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})$$

$$\begin{gathered} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{gathered}$$

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

1.1.2.4.2 Plane

For a plane in 3D, a normal vector **n** is perpendicular to the plane at all points.

1.1.2.4.2.1 Equation for a Plane

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

$$\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{n} \cdot \boldsymbol{r}_0$$

Where **n**=<a,b,c>

$$ax + by + cz = d = (a, b, c) \cdot (x_0, y_0, z_0)$$

1.1.2.4.2.2 Perpendicular Distance From Origin

(x_0,y_0,z_0) denotes the point where the plane is closest to the origin, i.e. its perpendicular distance from origin.

At that point, the position vector $\boldsymbol{r}_0$ is a multiple of the normal vector, $\boldsymbol{r}_0 = (ka, kb, kc)$

Also that $ax + by + cz = d$

$$\therefore k = \frac{d}{a^2 + b^2 + c^2} = \frac{d}{|\boldsymbol{n}|^2}$$

$$|\boldsymbol{r}_0| = k \cdot |\boldsymbol{n}| = \frac{d}{|\boldsymbol{n}|}$$

1.1.2.4.2.3 Angles Between Planes

The angle between planes are simply the angle between the normal vectors.

$$\theta = \cos^{-1}\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2|}$$

1.1.2.4.3 Parameterisation and Cartesian Equivalence

1.1.2.4.3.1 Ellipses

$$x = a \cdot \cos t + h, y = b \cdot \sin t + k$$

or

$$x = a \cdot \sin t + h, y = b \cdot \cos t + k$$

$$\Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Ellipses with going to +/- a in the x direction and +/- b in the y direction, centred at (h,k)

1.1.2.4.3.2 Hyperbola

Hyperbola on the left and right.

$$x = a \cdot \sec t + h, y = b \cdot \tan t + k$$

$$\Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Hyperbola on the top and bottom.

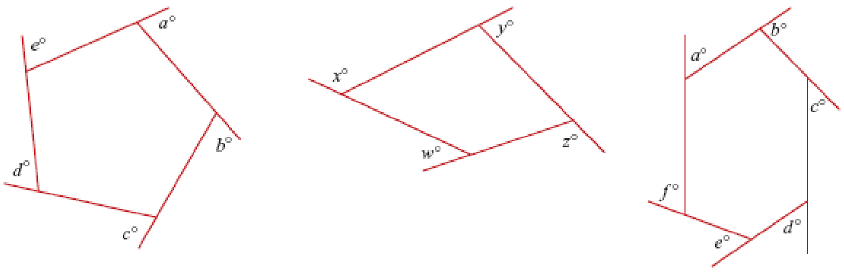
$$x = a \cdot \tan t + h, y = b \cdot \sec t + k$$

$$\Rightarrow \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

1.1.2.4.4 Vector Proofs

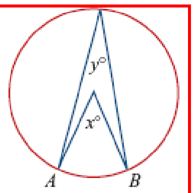
1.1.2.4.4.1 Geometry Prerequisites

The sum of the exterior angles of a convex polygon is 360°.

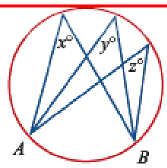


The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference.

$x = 2y$

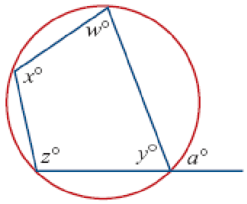


Angles in the same segment of a circle are equal.



$x = y = z$

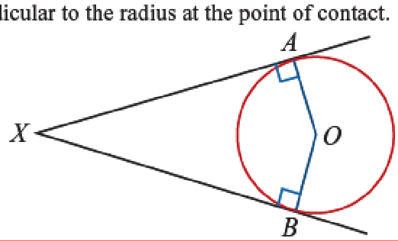
The sum of the opposite angles of a cyclic quadrilateral is 180°.



$x + y = 180$
$z + w = 180$

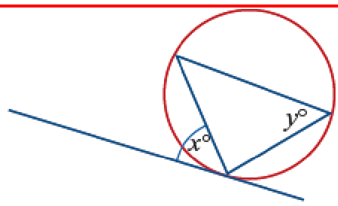
$a = x$

A trangent to a circle is perpendicular to the radius at the point of contact.
The two tangents to a circle from an exterior point are equal in length.



$XA = XB$

An angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.



$x = y$

For example

Prove the cosine rule for any angle.

Let ABC be a triangle

$$\overrightarrow{AB} = \boldsymbol{a}$$

$$\overrightarrow{AC} = \boldsymbol{b}$$

$$\overrightarrow{BC} = -\boldsymbol{a} + \boldsymbol{b}$$

$$\overrightarrow{BC} \cdot \overrightarrow{BC} = (-\boldsymbol{a} + \boldsymbol{b})(-\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a} - 2 \cdot \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$$

$$\left|\overrightarrow{\mathrm{BC}}\right|^{\mathbf{2}} = \left|\overrightarrow{AB}\right|^{\mathbf{2}} + \left|\overrightarrow{AC}\right|^{\mathbf{2}} - 2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| cos\,(\theta)$$

$$c^2 = a^2 + b^2 - 2ab \cdot cos\,(\theta)$$

QED

Prove the mid-point of the hypotenuse of a right-angled triangle is equidistant from all vertices.

Let ABC be a right angled triangle

$$\overrightarrow{AB} \perp \overrightarrow{BC}$$

$$\overrightarrow{AB} = \boldsymbol{a}$$

$$\overrightarrow{BC} = \boldsymbol{b}$$

$$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$$

Let M be the midpoint of $\overrightarrow{AC}$

$$\overrightarrow{AM} = \overrightarrow{MC} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$\left|\overrightarrow{AM}\right| = \left|\overrightarrow{MC}\right| = \frac{1}{2}\sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2}$$

$$\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB} = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

Since $\boldsymbol{a} \perp \boldsymbol{b}$

$$\left|\overrightarrow{MB}\right| = \frac{1}{2}\sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2} = \left|\overrightarrow{AM}\right| = \left|\overrightarrow{MC}\right|$$

$\therefore$ Point M is equidistant from A, B and C

QED

Three points P(-1,2), Q(-1,-2) and M(4,0) lie on the circumference of a circle, which also has another x intercept D(d,0), where d is negative. Find d.

$$\overrightarrow{PD} = \tilde{d} - \tilde{p} = -d\tilde{\imath} + \tilde{\imath} - 2\tilde{\jmath} = (1-d)\tilde{\imath} - 2\tilde{\jmath}$$

$$\overrightarrow{PM} = \tilde{m} - \tilde{p} = 4\tilde{\imath} + \tilde{\imath} - 2\tilde{\jmath} = 5\tilde{\imath} - 2\tilde{\jmath}$$

Since MD is a diameter of the circle, and P lies on the circumference, $\angle DPM = 90^o$

$$\overrightarrow{PD} \cdot \overrightarrow{PM} = 0$$

$$-5d + 5 + 4 = 0$$

$$d = -\frac{5}{9}$$

1.2 Algebra of Functions

1.2.1 Circular Functions

1.2.1.1 Symmetrical Identities

	$\theta+\frac{\pi}{2}$	$\theta-\frac{\pi}{2}$	$\theta+\pi$	$\theta-\pi$	$-\theta$	$\frac{\pi}{2}-\theta$	$-\frac{\pi}{2}-\theta$	$\pi-\theta$	$-\pi-\theta$
sin	cos	-cos	-sin	-sin	-sin	cos	-cos	sin	sin
cos	-sin	sin	-cos	-cos	cos	sin	-sin	-cos	-cos
tan	-cot	-cot	tan	tan	-tan	cot	cot	-tan	-tan
sec	-cosec	cosec	-sec	-sec	Sec	cosec	-cosec	-sec	-sec
cosec	sec	-sec	-cosec	-cosec	-cosec	sec	-sec	cosec	cosec
cot	-tan	-tan	cot	cot	-cot	tan	tan	-cot	-cot

1.2.1.2 Cartesian Identities

$$\sin^2\theta+\cos^2\theta=1$$
$$\tan^2\theta+1=\sec^2\theta$$
$$1+\cot^2\theta=\csc^2\theta$$

1.2.1.3 Compound Angle Formulae

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta$$
$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$
$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$$
$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$$

For example,

$$\frac{1-\cos\theta}{\sin\theta}=\tan\left(\frac{\theta}{2}\right)$$

$$\text{LHS}=\frac{1-2\cos^2\left(\frac{\theta}{2}\right)+1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}=\frac{2\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}=\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}=\tan\left(\frac{\theta}{2}\right)=\text{RHS}$$

$$\tan\left(\frac{\pi}{8}\right)=\frac{1-\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}=\frac{2-\sqrt{2}}{\sqrt{2}}=\sqrt{2}-1$$

$$\sec^2\left(\frac{\pi}{8}\right)=1+\tan^2\left(\frac{\pi}{8}\right)=4-2\sqrt{2}$$

$$\sec\left(\frac{\pi}{8}\right) = \sqrt{4 - 2\sqrt{2}}$$

1.2.1.3.1 Multiple Angle Formulae

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

1.2.1.4 Sine and Cosine Rule

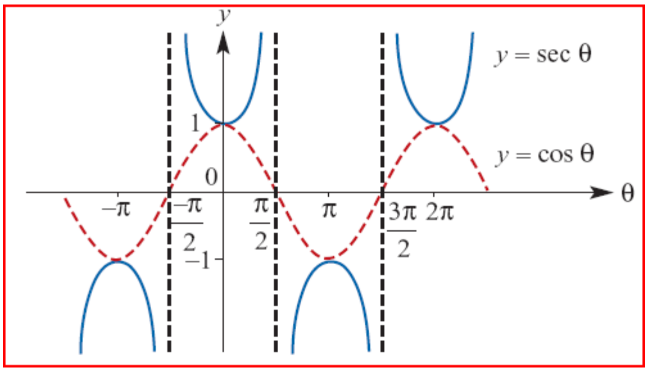
In a triangle with sides a, b and c and the angle opposite them A, B and C,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

1.2.1.5 Reciprocal Functions

Secant



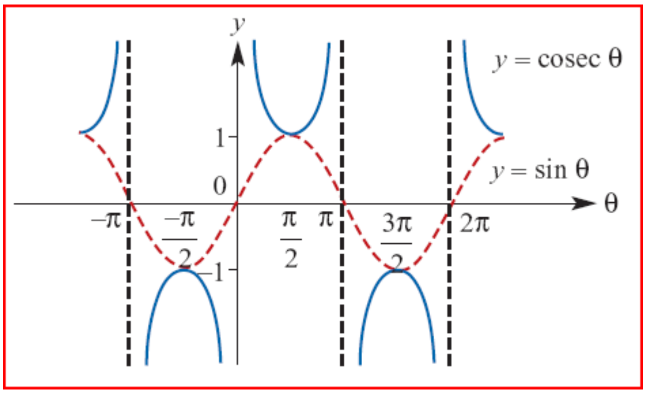
$$\sec\theta = \frac{1}{\cos\theta}$$

Vertical asymptotes every π from π/2.

Domain: $\left\{x : x \in \mathbb{R} \backslash \left\{\frac{\pi}{2}(2n+1)\right\}\right\}$

Range: $\{y : y < -1 \cup y > 1\}$

Cosecant



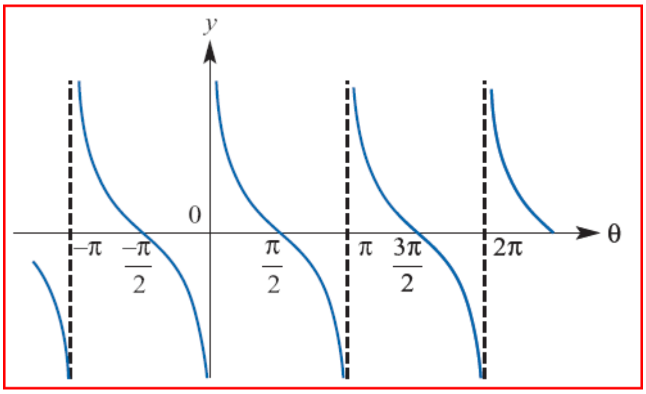
$$\csc\theta = \frac{1}{\sin\theta}$$

Vertical asymptotes every π from 0.

Domain: $\{x : x \in \mathbb{R} \backslash \{n\pi\}\}$

Range: $\{y : y < -1 \cup y > 1\}$

Cotangent



$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Vertical asymptotes every π from 0.

Domain: $\{x : x \in \mathbb{R} \backslash \{n\pi\}\}$

Range: $\{y : y \in \mathbb{R}\}$

Graphing combinations of these must take careful consideration in terms of the asymptotic behaviours. This is the case with $f(x) = a \cdot \csc x + b \cdot \cot x = \frac{a + b \cdot \cos x}{\sin x}$.

When *a*>*b*, the numerator is always positive and the sine determines the sign of the function, which behaves similarly to the cosecant. When *b*>*a*, the numerator varies similarly to cosine, and the function behaves similarly to cotangent.

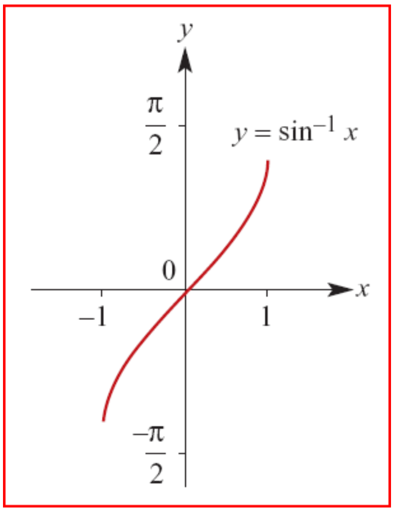
When *a*=*b*, the function is convergent at $x = (2k-1)\pi$, where y converges to 0. The function is equivalent to the cotangent dilated by a factor of 2 from the y axis. $\frac{1+\cos x}{\sin x} = \frac{1 + 2\cos^2\frac{x}{2} - 1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\frac{x}{2}$

1.2.1.6 Restricted Functions and Inverses

$\text{Sin}\,\theta$ is defined for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$y = \text{Sin}^{-1}\,x$ is defined for $-1 \le x \le 1$

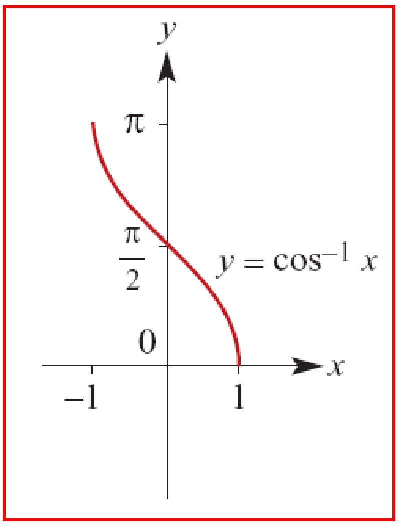
Range: $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$



$\text{Cos}\,\theta$ is defined for $0 \le \theta \le \pi$

$y = \text{Cos}^{-1}\,x$ is defined for $-1 \le x \le 1$

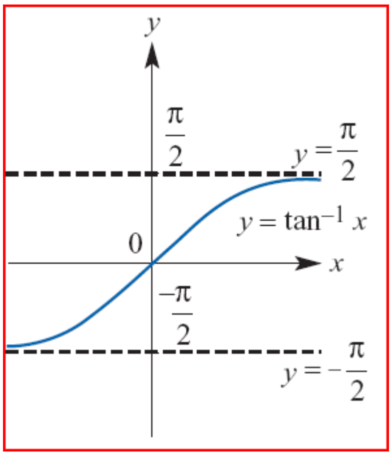
Range: $0 \le y \le \pi$



$\text{Tan}\,\theta$ is defined for $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

$y = \text{Tan}^{-1}\,x$ is defined for $x \in \mathbb{R}$

Range: $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$



1.2.1.6.1 Further Identities

$$\sec^{-1} x = \cos^{-1}\frac{1}{x}$$

$$\csc^{-1} x = \sin^{-1}\frac{1}{x}$$

$$\cot^{-1} x = \tan^{-1}\frac{1}{x}$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2}$$

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}$$

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\cos^{-1} x) = \frac{\sqrt{1-x^2}}{x}$$

1.2.1.7 Circular Arcs and Chords

14 A curve on a light rail track is an arc of a circle of length 300 m and the straight line joining the two ends of the curve is 270 m long.

a Show that, if the arc subtends an angle of $2\theta°$ at the centre of the circle, θ is a solution of the equation $\sin\theta° = \frac{\pi}{200}\theta°$.

b Solve, correct to two decimal places, the equation for θ.

a) If we bisect the angle, the bisector ray would perpendicularly bisect the straight line, cutting it in half to 135m.

$$\sin\theta = \frac{135}{r} \Longrightarrow r = \frac{135}{\sin\theta}$$

Also, the arc length would be halved to 150m

$$\frac{150}{2\pi r} = \frac{\theta}{2\pi} \Longrightarrow r = \frac{150}{\theta}$$

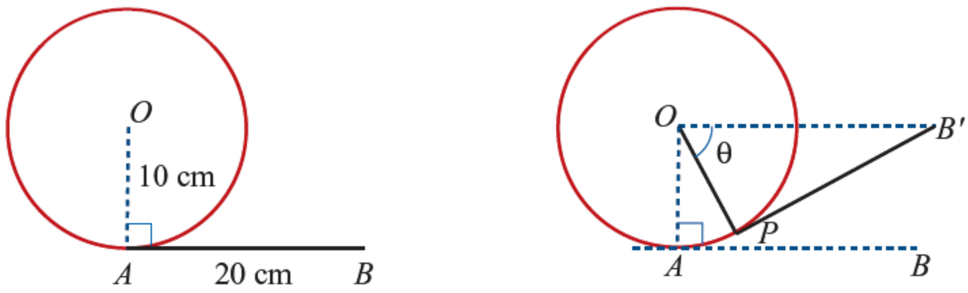
Equating,

$$\frac{135}{150}\theta = \sin\theta$$
$$\frac{9}{10}\theta = \sin\theta$$

Converting to degrees

$$\frac{9}{10}\theta^o \cdot \frac{\pi}{180} = \sin\theta^o$$
$$\sin\theta^o = \frac{\pi}{200}\theta^o, \text{as required.}$$

19 A string is wound around a disc and a horizontal length of the string AB is 20 cm long. The radius of the disc is 10 cm. The string is then moved so that the end of the string, B', is moved to a point at the same level as O, the centre of the circle. $B'P$ is a tangent to the circle.



a Show that θ satisfies the equation $\frac{\pi}{2} - \theta + \tan\theta = 2$.

b Find the value of θ, correct to two decimal places, which satisfies this equation.

a) The new length can be broken into two sections, arc AP and the line segment PB'.

$$|AP| = \frac{\left(\frac{\pi}{2} - \theta\right)}{2\pi} \cdot 2\pi r = 10\left(\frac{\pi}{2} - \theta\right)$$

$$|PB'| = r \cdot \tan\theta = 10\tan\theta$$

$$|AP| + |PB'| = 20$$
$$\frac{\pi}{2} - \theta + \tan\theta = 2, \qquad \text{as required}$$

1.2.2 Hyperbolic Functions

1.2.2.1 Definitions and Interpretations

The hyperbolic functions are odd and even parts of the natural exponential.

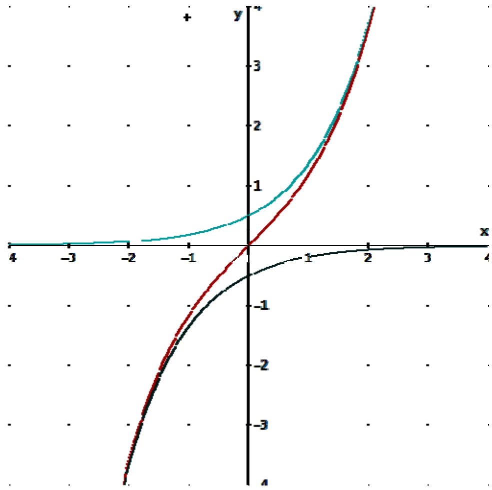
$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$e^x = \cosh(x) + \sinh(x)$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Domain: $x \in \mathbb{R}$

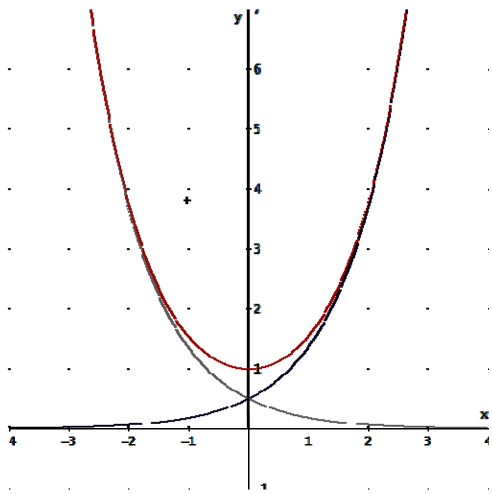
Range: $y \in \mathbb{R}$



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Domain: $x \in \mathbb{R}$

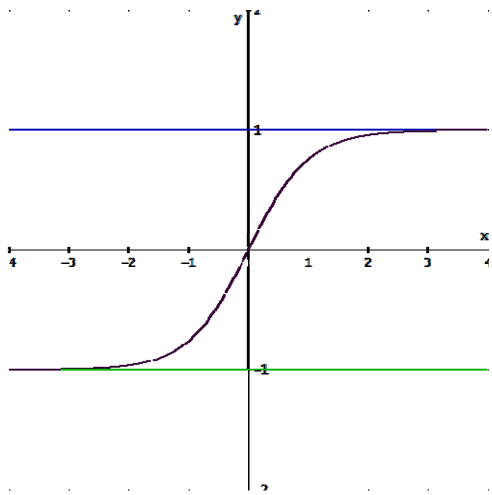
Range: $y \geq 1$



$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Domain: $x \in \mathbb{R}$

Range: $-1 > y > 1$



$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\text{Domain: } x \in \mathbb{R}\backslash\{0\}$$

$$\text{Range: } y \in \mathbb{R}\backslash\{0\}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

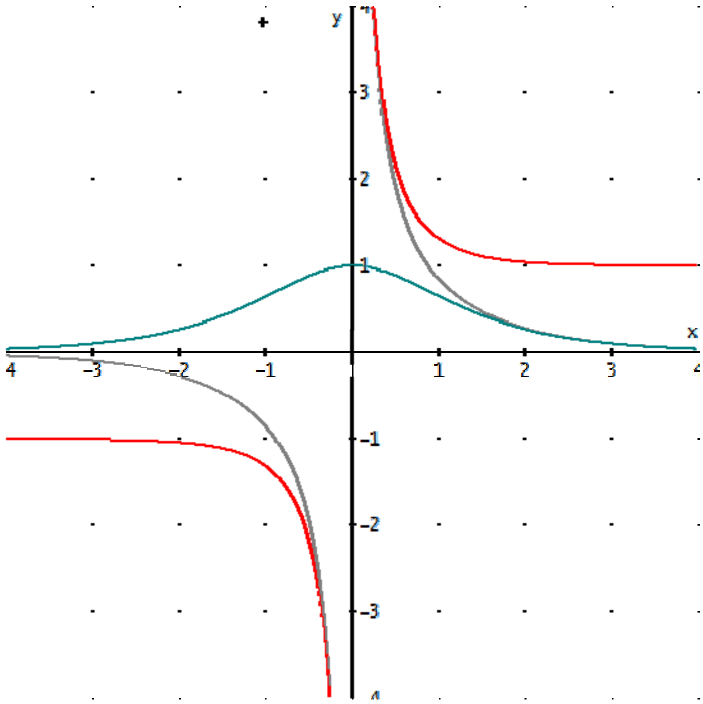
$$\text{Domain: } x \in \mathbb{R}$$

$$\text{Range: } 0 < y \leq 1$$

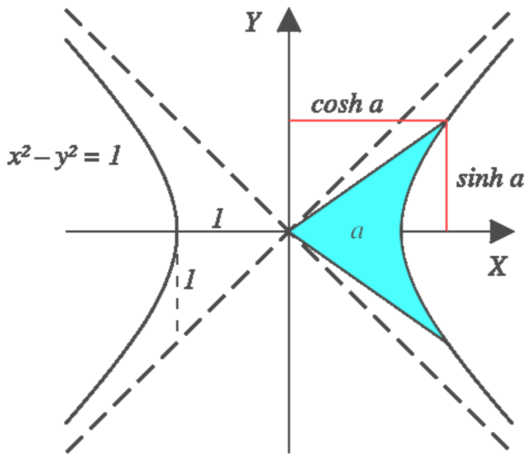
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\text{Domain: } x \in \mathbb{R}\backslash\{0\}$$

$$\text{Range: } y < -1 \cup y > 1$$



The hyperbolic functions are very similar to the circular functions. Where the circular functions are functions of the area of the sector, hyperbolic functions are functions of the area enclosed by the unit hyperbola x^2-y^2=1, a straight line from the origin to the hyperbola and its vertical reflection.



1.2.2.2 Identities

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x)$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

Compound angle identities:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$$
$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$
$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

1.2.2.3 Inverse Hyperbolic Functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), -1 < x < 1$$

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

$$\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$$

$$\sinh(\tanh^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$$

$$\cosh(\sinh^{-1} x) = \sqrt{x^2 + 1}$$

$$\cosh(\tanh^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$\tanh(\cosh^{-1} x) = \frac{\sqrt{x^2 - 1}}{x}$$

1.2.3 Relationships and their Graphs

1.2.3.1 Ellipses

Follows the general equation

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Which is an ellipse centred at (h,k) spanning a units to the left and right, and b units to the top and bottom.

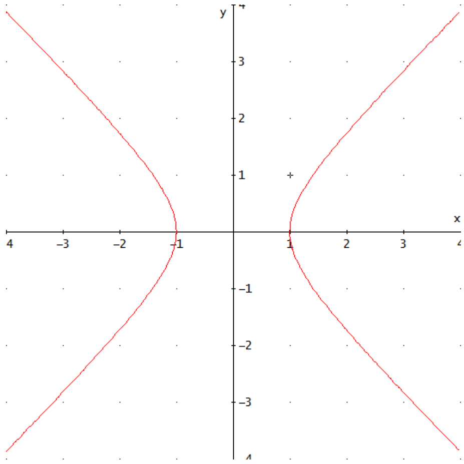
Ellipse have the domain $[h - a, h + a]$ and the range $[k - b, k + b]$.

An ellipse can also be described by the equation

$$ax^2 + by^2 + cx + dy = k$$

1.2.3.2 Hyperbolas

A hyperbola can take two forms:



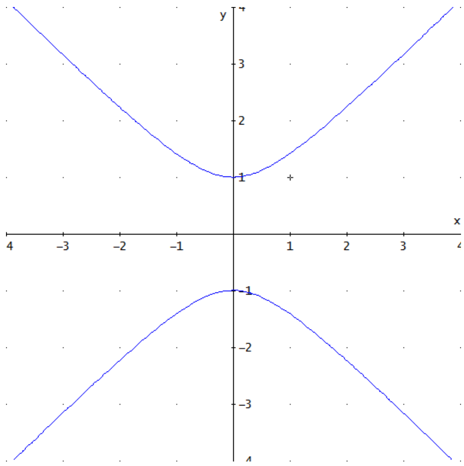
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Is a left-right hyperbola, centred at (h,k). The two branches are centred at (h-a,k) and (h+a,k).

The domain is $\mathbb{R}\backslash(h-a,h+a)$, and the range is $\mathbb{R}$.

This type of hyperbola can also be described by the equation

$$ax^2-by^2+cx+dy=k$$



$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

Is an up-down hyperbola, centred at (h,k). The two branches are centred at (h,k-b) and (h,k+b).

The domain is $\mathbb{R}$, and the range is $\mathbb{R}\backslash(k-b,k+b)$.

This type of hyperbola can also be described by the equation

$$by^2-ax^2+cx+dy=k$$

In both cases, the equations of the tangents can be obtained as followed:

$\frac{(y-k)^2}{b^2}=\frac{(x-h)^2}{a^2}\pm 1$,as x and y get large, the 1 can be ignored.

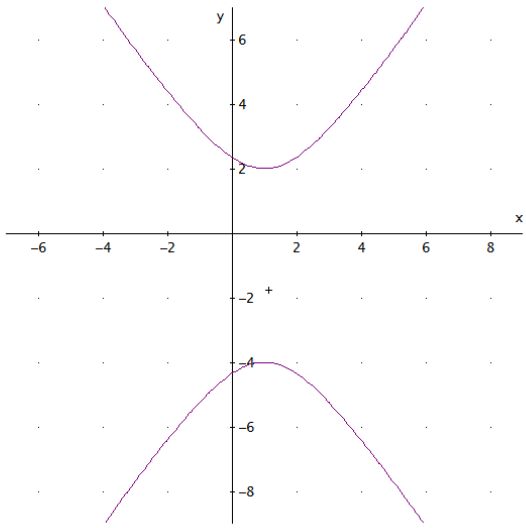
$$\frac{(y-k)^2}{b^2}\simeq\frac{(x-h)^2}{a^2}$$

$$y\simeq\pm\frac{b}{a}(x-h)+k$$

For example, consider the relationship $y^2 - 9x^2 + 8y + 18x = 41$

$$\begin{aligned} 4y^2 - 9x^2 + 8y + 18x &= 41 \\ 4(y^2 + 2y + 1) - 4 - 9(x^2 - 2x + 1) + 9 &= 41 \\ 4(y+1)^2 - 9(x-1)^2 &= 36 \\ \frac{(y+1)^2}{9} - \frac{(x-1)^2}{4} &= 1 \end{aligned}$$

This is an up-down hyperbola, hence the range is $(-\infty, -4] \cup [2, \infty)$



The equation of the asymptotes

$$y + 1 = \pm\frac{3}{2}(x - 1)$$

$$y = \frac{3}{2}x - \frac{5}{2}, \qquad y = -\frac{3}{2}x + \frac{1}{2}$$

1.2.4 Transformation

Transformation	Rule
Dilation	
By a factor of *a* from the y axis (parallel to x axis)	$f(x) \to f\left(\frac{1}{a}x\right)$
By a factor of 1/*a* from the y axis (parallel to x axis)	$f(x) \to f(ax)$
By a factor of *a* about *x=h*	$f(x-h) \to f\left(\frac{x-h}{a}\right)$
In general, dilation in the horizontal direction from h can be represented by substituting x with $\frac{x-h}{a}$.	
By a factor of *a* from the x axis (parallel to y axis)	$g(y) \to g\left(\frac{1}{a}y\right) \Leftrightarrow f(x) \to af(x)$
By a factor of 1/*a* from the x axis (parallel to y axis)	$g(y) \to g(ay) \Leftrightarrow f(x) \to 2f(x)$
By a factor of *a* about *y=k*	$g(y-k) \to g\left(\frac{y-k}{b}\right)$ $\Leftrightarrow f(x)+k \to af(x)+k$
In general, dilation in the vertical direction from k can be represented by substituting y with $\frac{y-k}{a}$	
Reflection	
Reflection in either axis can be represented by dilating by a factor of -1 from the axis.	

For example, consider the relationship $y^2 - 9x^2 + 8y + 18x = 41$, what are the equations of the asymptotes after a dilation by a factor of ½ from the y axis then a translation of -1 units parallel to the x axis?

Before transformation,

$$y = \frac{3}{2}x - \frac{5}{2}, \qquad y = -\frac{3}{2}x + \frac{1}{2}$$

Dilation by factor of ½ from y, $x \to 2x$

$$y = 3x - \frac{5}{2}, \qquad y = -3x + \frac{1}{2}$$

Translation of -1 units parallel to x, $x \to x + 1$

$$y = 3x + \frac{1}{2}, \qquad y = -3x - \frac{5}{2}$$

1.3 Complex Numbers

A complex number has two parts:

A real part, consisting of any real number,

And an imaginary part, consisting of any real multiples of the imaginary number i, where $i = \sqrt{-1}$.

Some properties of the imaginary number:

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$$

1.3.1 Argand Diagram

The argand diagram can be used to graphically represent complex numbers.

Its x axis is the real part, Re(z). Its y axis is the imaginary part, Im(z).

The Cartesian form (rectangular coordinates) of a complex number is

$$z = Re(z) + Im(z)i$$

1.3.2 Operations

1.3.2.1 Addition

$$z_1 + z_2 = \big(Re(z_1) + Re(z_2)\big) + \big(Im(z_1) + Im(z_2)\big)i$$

1.3.2.2 Multiplication

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

1.3.2.3 Conjugates

The conjugate of a complex number is denoted by a bar,

$$\text{If } z = a + bi, \text{then } \bar{z} = a - bi$$

Characteristically, $z \cdot \bar{z} = \big(Re(z)\big)^2 + \big(Im(z)\big)^2$

1.3.2.4 Division

Evaluation of a complex fraction is achieved when the fraction is multiplied by the conjugate of the denominator on top and bottom.

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 - (di)^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

More specifically, the denominator of the final expression is the square of the magnitude of the complex number, or the square of its modulus. Also, the reciprocal of an imaginary number:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

1.3.3 Polar Form

A complex number can also be expressed in polar form, i.e. direction and magnitude.

1.3.3.1 Modulus

The modulus is the magnitude of a complex number, i.e. its distance from the origin.

Using Pythagoras' theorem with the rectangular coordinates,

$$|z| = \sqrt{\left(Re(z)\right)^2 + \left(Im(z)\right)^2} = \sqrt{z \cdot \bar{z}}$$

1.3.3.2 Argument

The Argument is the angle the complex number makes with the positive real axis. This gives us three relationships using the rectangular coordinates,

$$\sin\left(Arg(z)\right) = \frac{Im(z)}{|z|}$$
$$\cos\left(Arg(z)\right) = \frac{Re(z)}{|z|}$$
$$\tan\left(Arg(z)\right) = \frac{Im(z)}{Re(z)}$$

The Argument is restricted to $(-\pi, \pi]$. It is possible to find the Argument with any of the above three relationships in conjunction with the knowledge of quadrants.

A complex number can hence be expressed as:

$$z = Re(z) + Im(z)i$$

$$= |z|\left(\frac{Re(z)}{|z|} + \frac{Im(z)}{|z|}i\right)$$

$$= |z|\left[\cos\left(Arg(z)\right) + i\sin\left(Arg(z)\right)\right]$$

The $\cos\left(Arg(z)\right) + i\sin\left(Arg(z)\right)$ component is abbreviated to $\text{cis}\left(Arg(z)\right)$, but it is more formally known as $\text{Exp}[i \cdot Arg(z)]$

$$z = Re(z) + Im(z)i = |z|\,\text{cis}\left(Arg(z)\right) = |z|\,\text{Exp}[i \cdot Arg(z)] = |z| \cdot e^{i \cdot Arg(z)}$$

Note that for cis, since it is the sum of cos and i*sin, $\text{cis}(-\theta) \neq \text{cis}(\theta)\,, \text{cis}(-\theta) \neq -\text{cis}(\theta)$

The only symmetrical properties it has are $\text{cis}(\theta + \pi) = -\text{cis}(\theta)\,$, and $\cos(\theta + 2n\pi) = \cos(\theta)\,$, where $n \in \mathbb{Z}$.

More conventionally, cis is usually expressed as:

$$\text{cis}(\theta) = e^{i\theta}$$

Where θ is the argument of z.

Some special numbers:

- 0+0i cannot be expressed in the polar form, its argument is undefined.
- All positive real numbers, a+0i, have an Argument of 0.
- All negative real numbers, -a+0i, have an Argument of π.
- All positive imaginary numbers, 0+ai, have an Argument of π/2.
- All negative imaginary numbers, 0-ai, have an Argument of -π/2.

A complex number can have its argument expressed in terms of the arctangent:

- 1st quadrant, a+bi
    - $Arg(z) = \tan^{-1}\frac{b}{a}$
- 2nd quadrant, -a+bi
    - $Arg(z) = -\tan^{-1}\frac{b}{a} + \pi$
- 3rd quadrant, -a-bi
    - $Arg(z) = \tan^{-1}\frac{b}{a} - \pi$
- 4th quadrant, a-bi
    - $Arg(z) = -\tan^{-1}\frac{b}{a}$

A complex number's conjugate can be derived as followed:

$$\begin{aligned}\bar{z} &= Re(z) - Im(z)i \\ &= |z|\left[\cos\left(Arg(z)\right) - i\sin\left(Arg(z)\right)\right] \\ &= |z|\left[\cos\left(-Arg(z)\right) + i\sin\left(-Arg(z)\right)\right] \\ &= |z|\operatorname{cis}\left(-Arg(z)\right)\end{aligned}$$

For example, solve for z such that $|z-3i|=3$ and $\arg(z-3i)=\frac{3\pi}{4}$

$$x^2+(y-3)^2=9, \qquad [1]$$

$$\frac{y-3}{x}=\tan\frac{3\pi}{4}=-1, \qquad [2]$$

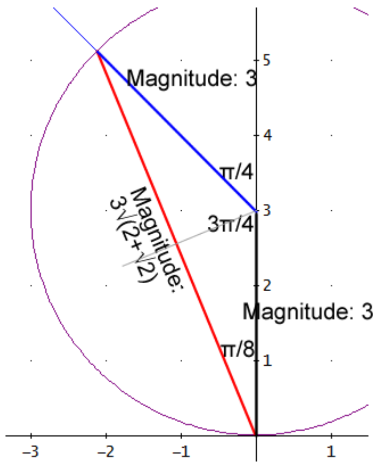
$$x=3-y$$

$$\Longrightarrow 2(y-3)^2=9$$

$$z=-\frac{3\sqrt{2}}{2}+\left(3+\frac{3\sqrt{2}}{2}\right)i$$

$$|z|=\sqrt{\frac{9}{2}+9+\frac{9}{2}+9\sqrt{2}}=3\sqrt{2+\sqrt{2}}$$

Since z is the intersection of [1] and [2], it is the point of intersection of the ray and the circle. It is easy to construct a triangle to find cosine of $\frac{\pi}{8}$



$$\cos\frac{\pi}{8}=\frac{\frac{3}{2}\sqrt{2+\sqrt{2}}}{3}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$Arg(z)=\frac{\pi}{8}+\frac{\pi}{2}=\frac{5\pi}{8}$$

Hence, the polar form of z is

$$z=6\cos\frac{\pi}{8}\operatorname{cis}\frac{5\pi}{8}$$

1.3.3.3 Multiplication and Division

Multiplication and division using polar forms is a lot easier than using the Cartesian form.

For two complex numbers, $z_1 = |z_1| \operatorname{cis}(\theta)$ and $z_2 = |z_2| \operatorname{cis}(\phi)$

$$z_1 \cdot z_2 = |z_1||z_2| \operatorname{cis}(\theta + \phi)$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \operatorname{cis}(\theta - \phi)$$

The reciprocal can also be worked out fairly easily:

$$\frac{1}{z_1} = \frac{|1|}{|z_1|} \operatorname{cis}(0 - \theta) = \frac{1}{|z_1|} \operatorname{cis}(-\theta) = \frac{\overline{z_1}}{|z_1|^2}$$

With polar coordinates, the idea of rotation is introduced. When a complex number z_1 is multiplied by another complex number z_2 such that $|z_2|=1$, then z_1 is rotated anticlockwise by the angle z_2 makes with the positive real axis:

$$z_1 \cdot z_2 = |z_1||z_2| \operatorname{cis}(\theta + \phi) = |z_1| \operatorname{cis}(\theta + \phi)$$

In particular, multiplying by the complex number i rotates the complex number 90° anticlockwise.

1.3.3.4 De Moivre's Theorem

$$z^n = |z|^n \operatorname{cis}(n\theta)$$

For example, $z_1 = 3 - i$, $z_2 = -1 - i$, find $z_3 = (z_1)^2 z_2$

$$z_3 = -14 - 2i$$

$$Arg(z_3) = Arg((z_1)^2 z_2) = 2Arg(z_1) + Arg(z_2)$$

$$\left(\tan^{-1}\frac{1}{7} - \pi\right) = 2\tan^{-1} -\frac{1}{3} + (\tan^{-1} 1 - \pi)$$

$$\therefore \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

1.3.4 Complex Roots

1.3.4.1 Polynomials

Fundamental Theorem of Algebra

Any polynomials *p(z)* of degree n has n roots in the complex plane.

Any polynomials *p(z)* of degree n can be factorised into n complex linear factors (some of which may be repeated).

1.3.4.1.1 Algebra

$$[f(x)]^2 + k^2 = [f(x)]^2 - (ki)^2 = (f(x) + ki)(f(x) - ki)$$

$$\begin{aligned} z^4 + 1 &= z^4 + 2z^2 + 1 - 2z^2 = (z^2+1)^2 - \left(\sqrt{2}z\right)^2 = \left(z^2 + \sqrt{2}z + 1\right)\left(z^2 - \sqrt{2}z + 1\right) \\ &= (z^2 + i)(z^2 - i) = \left(z + \sqrt{-i}\right)\left(z - \sqrt{-i}\right)\left(z + \sqrt{i}\right)\left(z - \sqrt{i}\,\right) \end{aligned}$$

It is also helpful to remember that

$$z^n - a^n = (z - a)(z^{n-1} + az^{n-2} + a^2z^{n-3} \cdots + a^{n-1})$$

$$z^n + a^n = (z + a)(z^{n-1} - az^{n-2} + (-a)^2z^{n-3} \cdots + (-a)^{n-1})$$

1.3.4.1.2 Conjugate pair theorem

For a polynomial $p(z)$ with real coefficients, if z_1 is a root of *p*, then $\overline{z_1}$ is also a root of *p*.

1.3.4.2 Roots of Numbers

If $z^n = c$, where $c \in \mathbb{C}$, then

$$z = |c|^{(1/n)} \operatorname{cis}\left(\frac{Arg(c) + 2k\pi}{n}\right), \text{where } k \in \{0,1,2, \dots, n-1\}$$

On an Argand diagram, all roots are evenly spaced.

1.3.5 Relationships in the Complex Plane

A locus in the complex plane is a set of points on the Argand diagram.

For example, $\{s: |z| = 1, z \in \mathbb{C}\}$

1.3.5.1 Line

$$\{z: |z - z_0| = |z - z_1|, z \in \mathbb{C}\}$$

This describes a straight line on the Argand diagram. It is the perpendicular bisector of the line z_1z_2.

Particular cases include:

$$\{z: Re(z) = c, z \in \mathbb{C}\} \equiv \{z: z + \bar{z} = 2c\}, \text{where } c \text{ is a constant}$$

This describes a vertical line, x=c

$$\{z: Im(z) = c, z \in \mathbb{C}\} \equiv \{z: z - \bar{z} = 2c\}, \text{where } c \text{ is a constant}$$

This describes a horizontal line, y=c

$$\{z: z = i\bar{z}, z \in \mathbb{C}\}$$

This describes the set of points that makes an angle of 90^o with their reflection in the horizontal axis, i.e. the line y=x

$$\{z: |1 - iz| = |z + 1|\}$$

This describes a line also: $|1 - iz| = |(-i)(i) - i(z)| = |-i(z + i)| = |z + i|$

Inequalities involving a line can be found using a simple point substitution, usually the origin.

1.3.5.2 Ray

$$\{z: Arg(z - z_0) = \theta, z \in \mathbb{C}\}, \text{where } \theta \text{ is a constant}$$

This describes a ray from (but not including) z_0 at an angle of θ from the horizontal.

Inequalities can be found using a simple point substitution. It should also be noted that

- For $\{z: Arg(z - z_0) > \theta, z \in \mathbb{C}\}$ or $\{z: Arg(z - z_0) \geq \theta, z \in \mathbb{C}\}$, the horizontal to the left $\{z: Arg(z - z_0) = \pi, z \in \mathbb{C}\}$ is the end of the region, and is included.
- For $\{z: Arg(z - z_0) < \theta, z \in \mathbb{C}\}$ or $\{z: Arg(z - z_0) \leq \theta, z \in \mathbb{C}\}$, the horizontal to the left $\{z: Arg(z - z_0) = -\pi, z \in \mathbb{C}\}$ is the end of the region, and is **not** included.
- z_0 is not included.

1.3.5.3 Hyperbola

By the geometric definition of hyperbola,

$$\{z: |z - z_0| - |z - z_1| = c, z \in \mathbb{C}\}$$

This describes a hyperbola on the side of z_1. The centre is at $\frac{z_0+z_1}{2}$.

The centre of the branch is located $\frac{c}{2}$ units away from the centre in the direction of z_1.

$$\frac{z_0 + z_1}{2} + \frac{c}{2} \times \frac{z_1 - z_0}{|z_1 - z_0|}$$

1.3.5.4 Circles

$$\{c: |z - z_0| = c, z \in \mathbb{C}\}, \text{where } c \text{ is a constant}$$

This describes a circle centred at z_0 with a radius of c.

If we let $w = z - z_0$, then $\overline{w} = \bar{z} - \overline{z_0}, \Longrightarrow |z - z_0|^2 = w \cdot \overline{w} = (z - z_0)(\bar{z} - \overline{z_0}) = c^2$

$$\{c: (z - z_0)(\bar{z} - \overline{z_0}) = c^2, z \in \mathbb{C}\}, \text{where } c \text{ is a constant}$$

This also describes a circle centred at z_0 with a radius of c.

1.3.5.4.1 Ellipses

By the geometric definition of an ellipses,

$$\{c: |z - z_0| + |z - z_1| = c, z \in \mathbb{C}\}, \text{where } c \text{ is a constant}$$

This describes the ellipses with its two foci at z_0 and z_1.

Axis along the two foci: $\pm\frac{c}{2}$

Axis perpendicular to the foci:

$$\sqrt{\left(\frac{c}{2}\right)^2 - \left(\frac{|z_1 - z_0|}{2}\right)^2} = \frac{1}{2}\sqrt{c^2 - |z_1 - z_0|^2}$$

1.3.5.4.2 Arcs

By the geometric theorem that angles subtended by a chord/arc at all points on the circumference are equal:

$$\arg(z+z_0)-\arg(z-z_i)=\theta$$

This describes an arc such that the angle made between two segments, $\overline{zz_1}$ and $\overline{zz_2}$ is always θ, i.e. this exists as an arc. Note that this is only **one** side of the arc, the other side of the arc is described by $\arg(z+z_0)-\arg(z-z_i)=\pi+\theta$.

It is easier to convert this to Cartesian form trying to draw it.

For example, $\arg(z+i)-\arg(z-i)=\frac{\pi}{4}$

$$\tan^{-1}\frac{y+1}{x}-\tan^{-1}\frac{y-1}{x}=\frac{\pi}{4}, \qquad x>0$$

$$\frac{\frac{y+1}{x}-\frac{y-1}{x}}{1+\frac{y+1}{x}\cdot\frac{y-1}{x}}=1$$

$$y+1-y+1=x+\frac{y^2-1}{x}$$

$$(x-1)^2+y^2=2, \qquad x>0$$

Axis intercepts: $y=\pm1, x=\sqrt{2}$+1

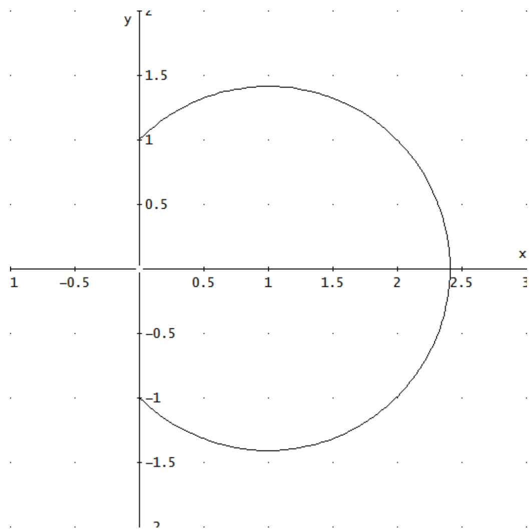
$$\tan^{-1}\frac{y+1}{x}-\pi-\tan^{-1}\frac{y-1}{x}+\pi=\frac{\pi}{4}, \qquad x<0, y<-1$$

$$(x-1)^2+y^2=2, \qquad x<0, y<-1, \qquad \text{FALSE}$$

$$\tan^{-1}\frac{y+1}{x}+\pi-\tan^{-1}\frac{y-1}{x}-\pi=\frac{\pi}{4}, \qquad x<0, y>1$$

$$(x-1)^2+y^2=2, \qquad x<0, y>1, \qquad \text{FALSE}$$

For $x<0, -1< \ y<1$, $z+i$ is in the second quadrant, and its argument lies in $\left(\frac{\pi}{2},\pi\right)$. $z-i$ is in the third quadrant, and its argument lies in $\left(-\pi,-\frac{\pi}{2}\right)$. Hence, $\arg(z+i)-\arg(z-i)$ has the maximum 2π, and the minimum π. Therefore, $\arg(z+i)-\arg(z-i)=\frac{\pi}{4}$ not true for $x<0, -1<$ $y<1$.



Calculus

Single Variable Calculus, Vector Calculus, Multivariable Calculus

2.1Single Variable Calculus

2.1.1 Limits

A limit of a function at a is the function's value as the variable approaches a.

2.1.1.1 Definition and Interpretation

The limit has two parts, the left hand limit $\lim_{x \to a^-} f(x)$, which approaches the value from the left hand negative side, and the right hand limit $\lim_{x \to a^+} f(x)$, which approaches the value from the right hand positive side.

A limit exists if and only if:

- The left hand limit $\lim_{x \to a^-} f(x)$ exists
- The right hand limit $\lim_{x \to a^+} f(x)$ exists
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

The function must be defined on some open interval that contains (with the possible exception of) a.

Under formal definition,

$\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

The two side limits can be similarly defined with the bounds on x: $-\delta < x - a < 0$ for left hand limits, and $0 < x - a < \delta$ for right hand limits.

Some limits may evaluate to infinite. That is, $\lim_{x \to a} f(x) = \infty$ for every positive number M, there is a positive δ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta$$

Similarly, for limits that evaluate to the negative infinity: $\lim_{x \to a} f(x) = -\infty$ for every negative number N number, there is a positive δ such that

$$f(x) < N \text{ whenever } 0 < |x - a| < \delta$$

For limits at the infinity, a limit can be either divergent, divergent to infinity, or convergent.

A limit that converges to a number at the positive infinity, under formal definition,

$\lim_{x \to \infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a positive number M such that

$$|f(x) - L| < \varepsilon \text{ whenever } x > M$$

A similar definition may be formed for convergency at the negative infinity. Convergency at either infinity indicates a horizontal asymptote.

A limit that diverges to infinity gets infinitely large, (or small), under formal definition,

$\lim_{x \to \infty} f(x) = \infty$ if for every positive number M there is a positive number C such that

$$f(x) > M \text{ whenever } x > C$$

A similar definition may be formed for the other three possibilities (negative infinity to the right, and positive/negative infinity to the left).

A limit can also diverge without getting to infinity. This type of functions usually oscillates. An example is the sine ratio.

2.1.1.2 *Limit Laws*

Evaluation of a limit can be as simple as substituting the number. However, these are often not enough.

Some limit laws include:

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[c \cdot f(x)] = c \lim_{x \to a} f(x)$$

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \lim_{x \to a} f(x) \lim_{x \to a}[g(x)]^{-1}$$

2.1.1.3 L'Hospital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

This is only applicable where an indeterminate form is reached.

2.1.1.3.1 Indeterminate Form

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

2.1.1.3.2 Indeterminate Product

When an indeterminate product $0\cdot\infty$ is reached, there is a simple manipulation that can be done:

$$f(a)\cdot g(a)=0\cdot\infty\Leftrightarrow\frac{f(a)}{1/g(a)}=\frac{0}{0}\text{ or }\frac{\infty}{\infty}$$

Similar rearrangements can sometime also be used to evaluate the indeterminate difference $\infty-\infty$

2.1.1.3.3 Indeterminate Power

An indeterminate power is $0^0,\infty^0$ or 1^∞. In each of these cases, the logarithm may be taken.

$$\lim_{x\to a}[f(x)]^{g(x)}=\lim_{x\to a}y$$

$$g(x)\ln[f(x)]=\ln y$$

$$\lim_{x\to a}g(x)\ln[f(x)]=\lim_{x\to a}\ln y$$

$$\therefore\lim_{x\to a}[f(x)]^{g(x)}=\operatorname{Exp}\left(\lim_{x\to a}g(x)\ln[f(x)]\right)$$

2.1.1.4 Squeeze Theorem

The squeeze theorem can be used for evaluation of oscillating functions.

If, on an open interval including (with the possible exception of) a,

$$f(x)\le g(x)\le h(x)$$

Then

$$\lim_{x\to a}f(x)<\lim_{x\to a}g(x)<\lim_{x\to a}h(x)$$

Hence if

$$\lim_{x\to a}f(x)=\lim_{x\to a}h(x)$$

$$\therefore\lim_{x\to a}g(x)=\lim_{x\to a}f(x)=\lim_{x\to a}h(x)$$

2.1.1.5 Continuity

A function is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

If a point is continuous, then

- $\lim_{x \to a^+} f(x)$ exists
- $\lim_{x \to a^-} f(x)$ exists
- $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

The left or right hand limit can be used to define continuity on one side, where a is the end point of an open interval.

2.1.1.6 Differentiability

A function is differentiable at a if

- $f(a)$ exists
- f is continuous at a
- There is no abrupt change of direction at a (i.e. the derivative is continuous at a)

2.1.2 Methods of Differentiation

First principle

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.1.2.1 Differentiation Rules

Addition rule:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product rule:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$$

And hence, the constant rule:

$$\frac{d}{dx}\big(c \cdot f(x)\big) = c \cdot \frac{df}{dx}, \text{where } c \text{ is a real constant}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

Some particular cases:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2.1.2.2 Implicit Differentiation

By the chain rule, if y is a function of x:

$$\frac{d}{dx}\big(f(y)\big) = \frac{df}{dy} \cdot \frac{dy}{dx}$$

Or,

$$\frac{d}{dx}\big(f(x,y)\big) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

2.1.2.3 Circular Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

2.1.2.4 Inverse Circular Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sin^{-1} \sqrt{x}) = \frac{1}{2\sqrt{x-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} \sqrt{x}) = -\frac{1}{2\sqrt{x-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} \sqrt{x}) = \frac{1}{2\sqrt{x\cdot(1+x)^2}}$$

$$\frac{d}{dx}\left(\sin^{-1} \frac{1}{x}\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} \frac{1}{x}\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} \frac{1}{x}\right) = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

2.1.2.5 Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

2.1.2.5.1 Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

2.1.2.6 Exponential and Logarithms

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a \cdot x}$$

$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

2.1.2.7 Logarithmic Differentiation

$$\text{If } y = [f(x)]^{g(x)}$$

$$\ln(y) = g(x)\ln\big(f(x)\big)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\Big(g(x)\ln\big(f(x)\big)\Big)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{dg}{dx}\cdot\ln\big(f(x)\big) + \frac{df}{dx}\cdot\frac{g(x)}{f(x)}$$

$$\frac{dy}{dx} = y\left(\frac{dg}{dx}\cdot\ln\big(f(x)\big) + \frac{df}{dx}\cdot\frac{g(x)}{f(x)}\right)$$

2.1.2.8 Second Derivatives

$$\frac{d^2}{dx^2}f(x) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$$

2.1.2.8.1 Concavity

For $\frac{d^2y}{dx^2}$ at $x = a$, if $\frac{d^2y}{dx^2}$ is

- Positive, the concavity is upwards
  - If it is also a stationary point, it is a local minimum
- Negative, the concavity is downwards
  - If it is also a stationary point, it is a local maximum
- Zero,
  - If the third derivative is also zero, concavity test is inconclusive
  - Otherwise it is a point of inflection
    - If the third derivative is positive (sign change from negative to positive), it is the minimum gradient
    - If the third derivative is negative (sign change from positive to negative), it is the maximum gradient
    - If it is also a stationary point, it is a stationary point of inflection

2.1.3 Applications of Differential Calculus

2.1.3.1 Graphing

A – Domain

B – Find x and y intercepts

C – Look for symmetry: if f(-x)=f(x), it is evenly symmetrical; if f(-x)=-f(x), it is oddly symmetrical

D – Asymptotes, vertical, horizontal, slant

E – Intervals which the graph is increasing/decreasing

F – Stationary points

G – Points of inflection

2.1.3.2 Addition of Ordinates

When graphing a hybrid function, the method of addition of ordinates may be used.

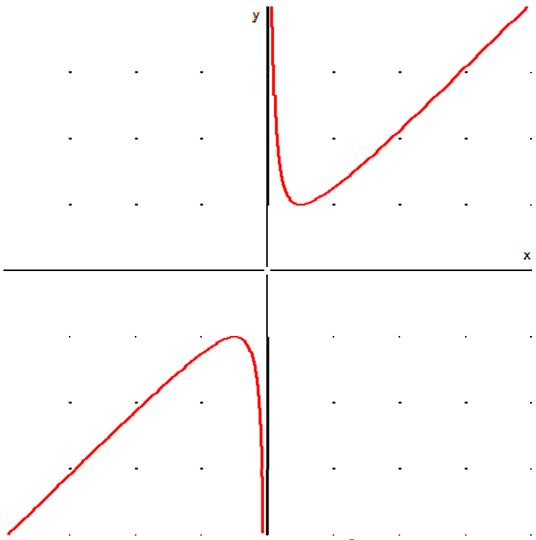
$$f(x) = g(x) + h(x)$$

Then for $y = f(x)$ at $x = a, y = g(a) + h(a)$

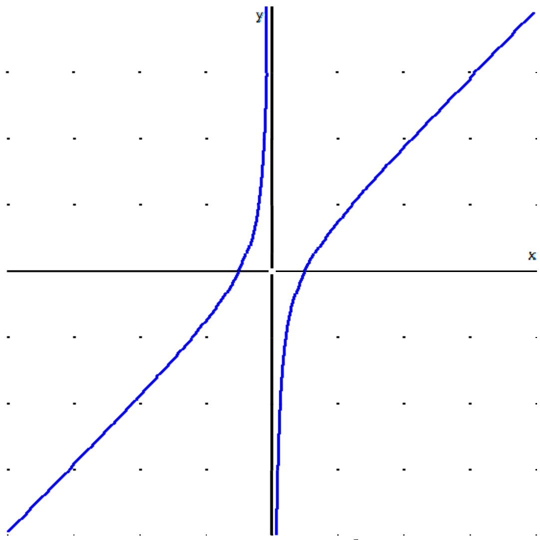
2.1.3.2.1 Some Rational Functions

$$f(x) = ax + \frac{b}{x} = \frac{ax^2 + b}{x}$$

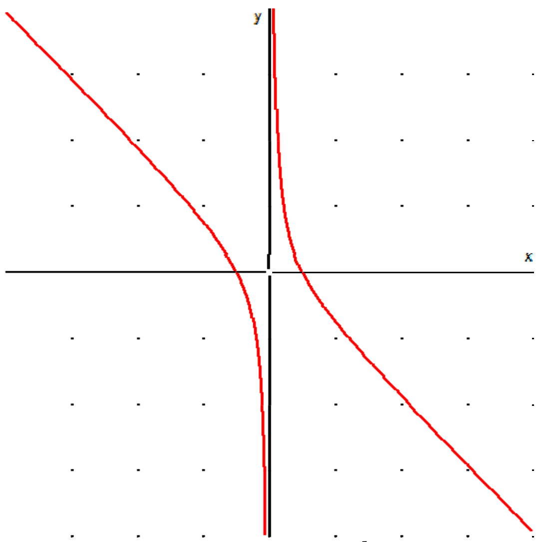
- Vertical asymptote at x=0
- Slant asymptote y=ax



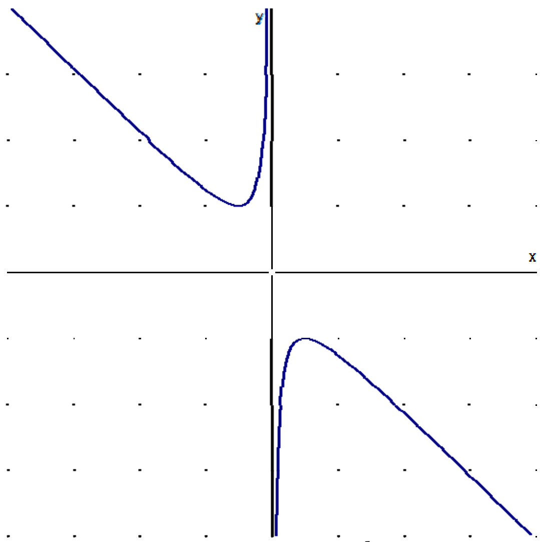
$$y = ax + \frac{b}{x}$$



$$y = ax - \frac{b}{x}$$



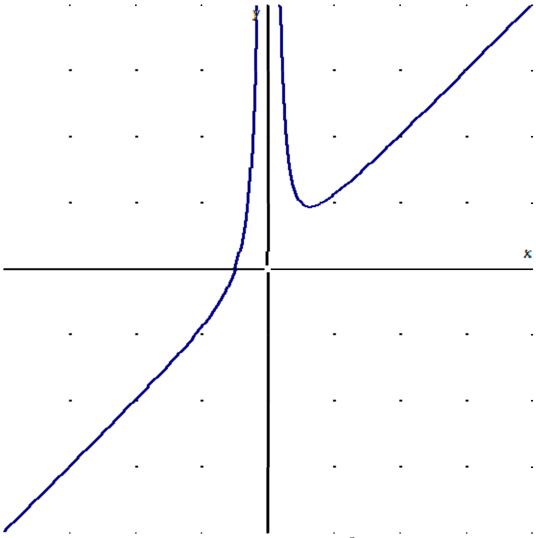
$$y = -ax + \frac{b}{x}$$



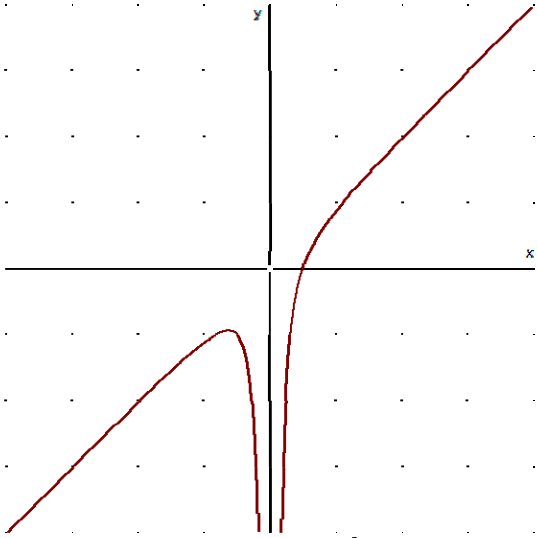
$$y = -ax - \frac{b}{x}$$

$$f(x) = ax + \frac{b}{x^2} = \frac{ax^3 + b}{x^2}$$

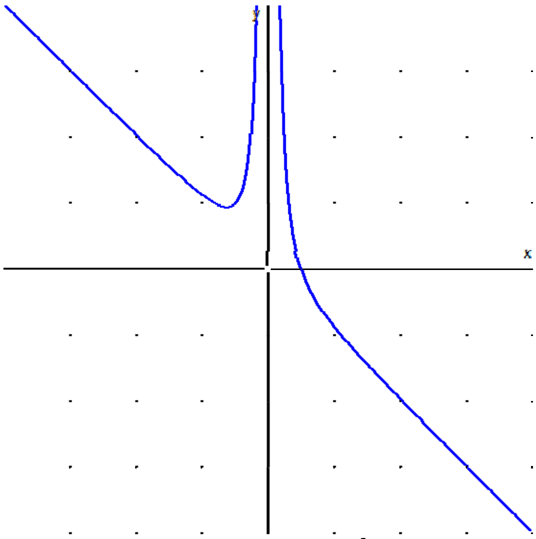
- Vertical asymptote at x=0
- Slant asymptote y=ax



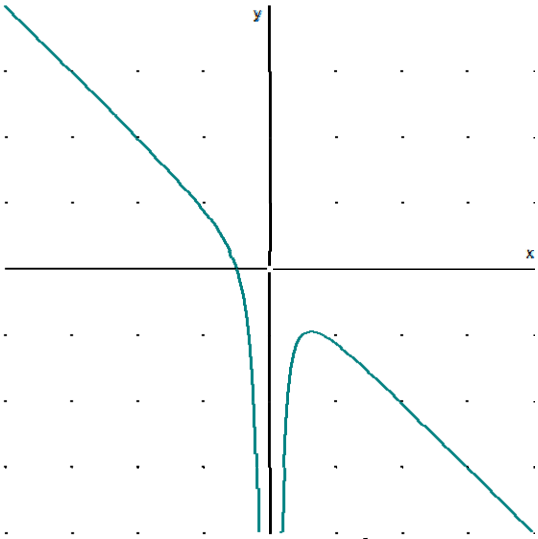
$$y = ax + \frac{b}{x^2}$$



$$y = ax - \frac{b}{x^2}$$



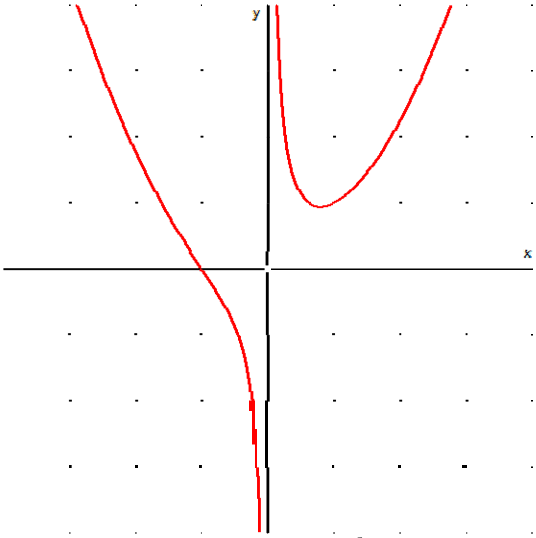
$$y = -ax + \frac{b}{x^2}$$



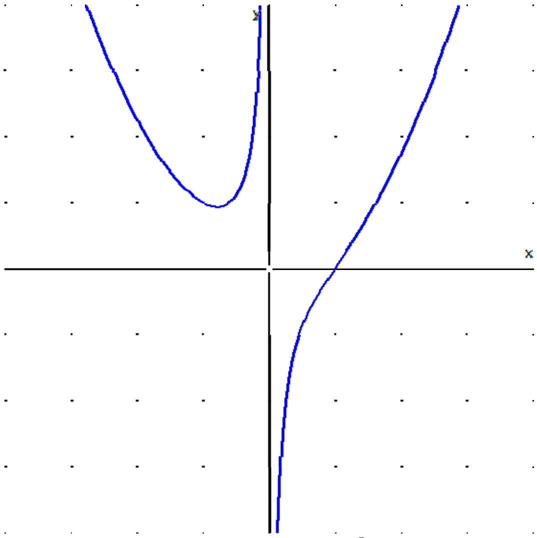
$$y = -ax - \frac{b}{x^2}$$

$$f(x) = ax^2 + \frac{b}{x} = \frac{ax^3 + b}{x}$$

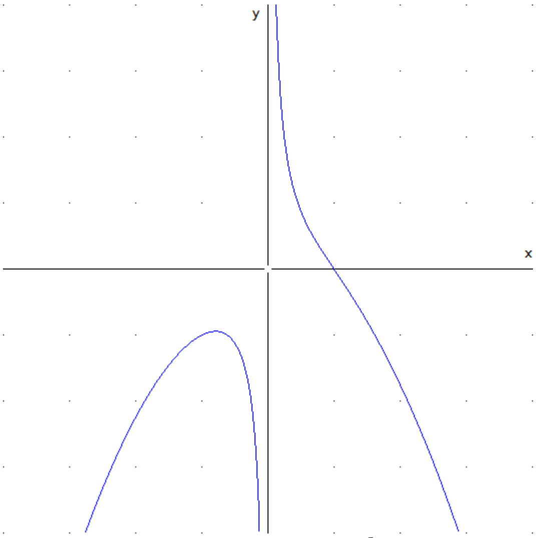
- Vertical asymptote at x=0
- Parabolic asymptote y=ax²



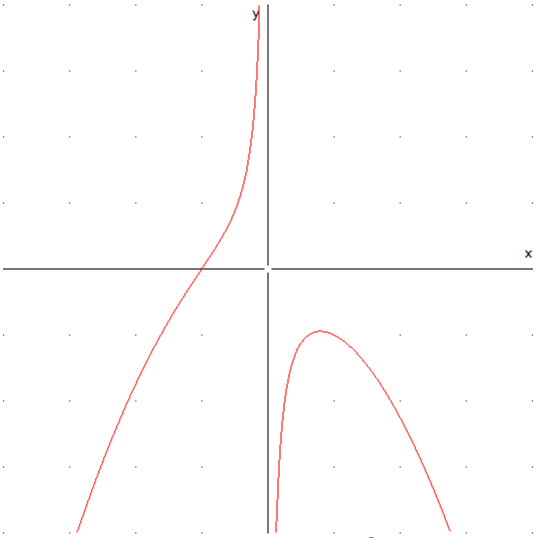
$$y = ax^2 + \frac{b}{x}$$



$$y = ax^2 - \frac{b}{x}$$



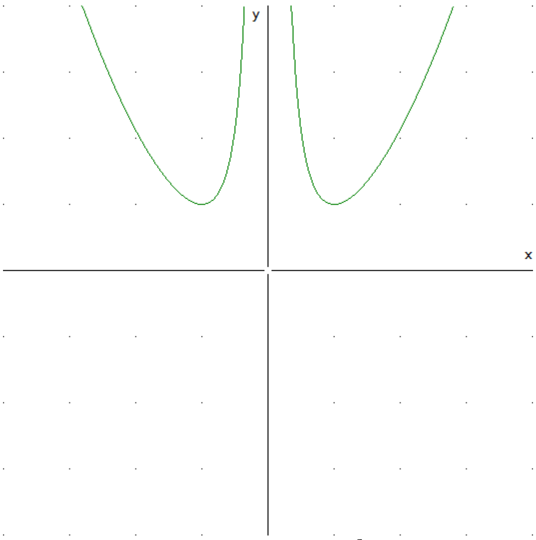
$$y = -ax^2 + \frac{b}{x}$$



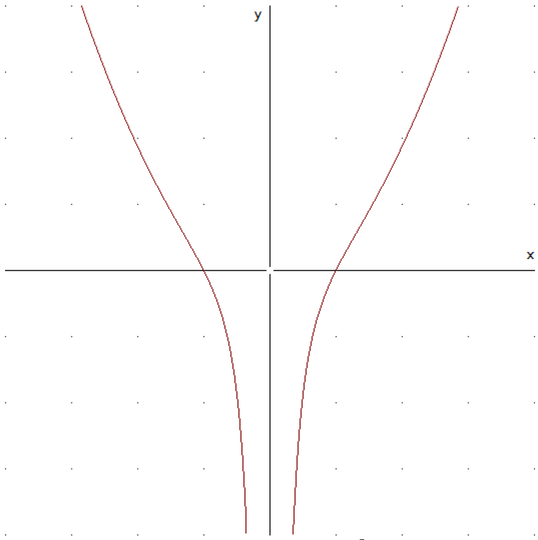
$$y = -ax^2 - \frac{b}{x}$$

$$f(x) = ax^2 + \frac{b}{x^2} = \frac{ax^4 + b}{x^2}$$

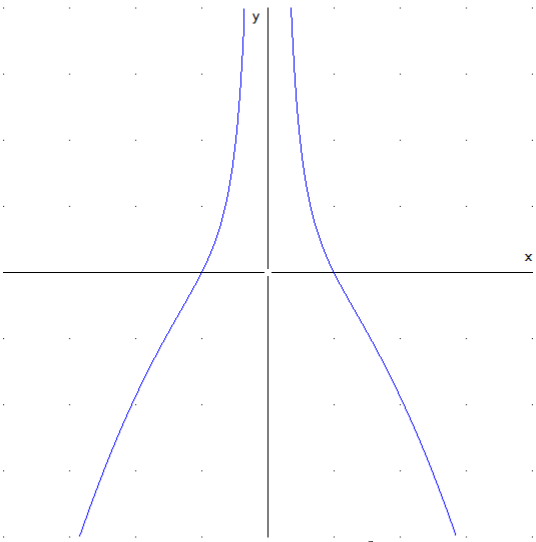
- Vertical asymptote at x=0
- Parabolic asymptote y=ax^2



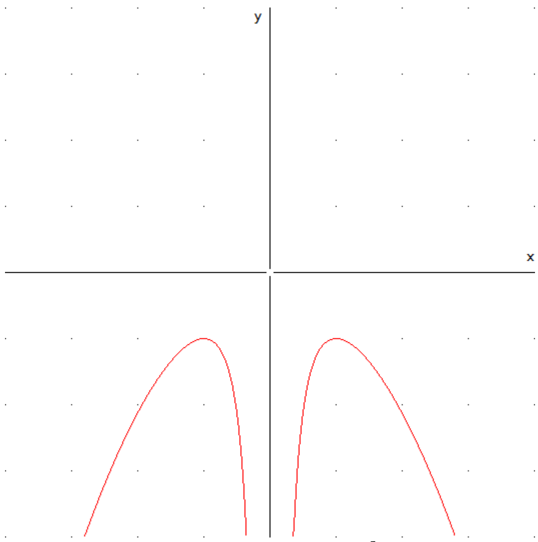
$$y = ax^2 + \frac{b}{x^2}$$



$$y = ax^2 - \frac{b}{x^2}$$



$$y = -ax^2 + \frac{b}{x^2}$$



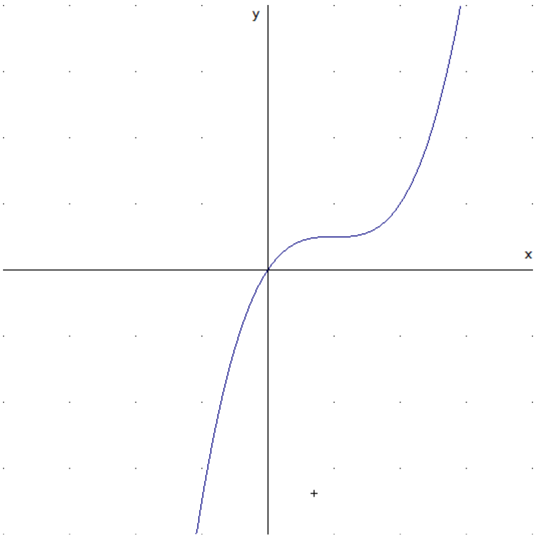
$$y = -ax^2 - \frac{b}{x^2}$$

2.1.3.3 Reciprocal Functions

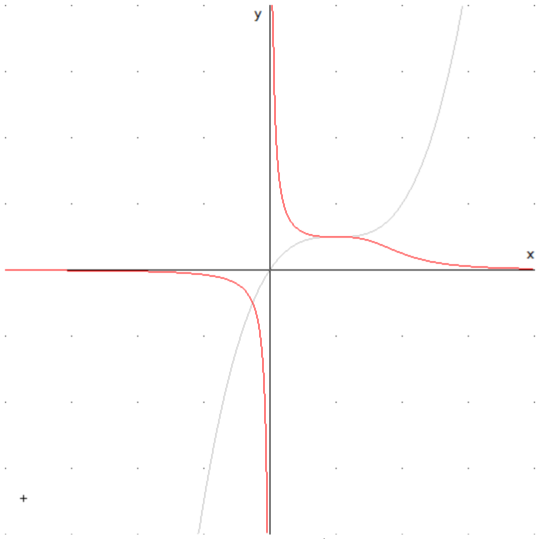
A reciprocal function can be graphed by taking the reciprocal of the function's value.

When the function

- Crosses the x axis from positive to negative
  - The reciprocal function goes to positive infinity on the left, and to negative infinity on the right (asymmetric vertical asymptote)
- Crosses the x axis from negative to positive
  - The reciprocal function goes to negative infinity on the left, and to positive infinity on the right (asymmetric vertical asymptote)
- Touches the x axis
  - Depending on whether the function touches the axis on the positive side or the negative side, the reciprocal function goes to positive/negative infinity on both sides (symmetric vertical asymptote)
- Has a local minimum
  - The reciprocal function has a local maximum
- Has a local maximum
  - The reciprocal function has a local minimum
- Has a stationary point of inflection (not when y=0)
  - The reciprocal function has a stationary point of inflection
- Has a point of inflection (not when y=0)
  - The reciprocal function has a point of inflection



$y = (x-1)^3 + 1$



$$y = \frac{1}{(x-1)^3 + 1}$$

2.1.4 Methods of Antidifferentiation

Indefinite integrals:

Some rules of the integral:

- $\int[f(x)+g(x)]\,dx = \int f(x)dx + \int g(x)dx$
- $\int c\cdot f(x)\,dx = c\int f(x)\;dx$, where c is a real constant
- $\int f(x)dx = F(x) + C$, where C is a real constant

And for **definite integrals**:

Fundamental theorem of calculus

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Fundamental theorem of calculus (II)

If $F(x)$ is the antiderivative of $f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Some rules:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^a f(x)dx$
- $\int_a^a f(x)dx = 0$

The most obvious method of antidifferentiation is via recognition.

$$\int e^x\,dx = e^x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

2.1.4.1 Substitution

Method of substitution is the reverse of the chain rule of differentiation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y = \int \left(\frac{dy}{du} \cdot \frac{du}{dx}\right) dx = \int \frac{dy}{du}\, du$$

By making the appropriate u substitution, an integral can be solved.

Usually, the integral is present as

$$\int \Big(f\big(g(x)\big) \cdot g'(x)\Big)\, dx = F\big(g(x)\big) + C$$

However, this needs not to be the case,

$$\int f\big(g(x)\big)\, dx = \int \frac{f(g)}{g'(x)} dg$$

In this case, if $g'(x)$ can be rearranged in terms of g, the integral may be evaluated.

Note that for definite integrals, when a substitution is made, the terminals must also change correspondingly

$$\int_a^b \Big(f\big(g(x)\big) \cdot g'(x)\Big)\, dx = \int_{g(a)}^{g(b)} f(g)\, dg$$

2.1.4.1.1 Linear Substitution

Where the integral consists of linear factors that cannot be expanded (such as linear factors in the denominator, or square root of linear factors) multiplied by other expandable factors, a simple substitution of that linear factor can be made, and other factors can be arranged in terms of this linear factor.

2.1.4.2 Inverse Circular Functions

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^1 x + C$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^1 x + C = -\sin^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{a}{a^2+x^2} dx = \tan^{-1}\frac{x}{a} + C$$

$$\int \frac{a}{\sqrt{1-a^2x^2}} dx = \sin^{-1} ax + C$$

$$\int \frac{a}{1+a^2x^2} dx = \tan^{-1} ax + C$$

2.1.4.3 Integration by Recognition

By finding the differentiating a function, which derivative contains all or part of the integrand, can allow integration by recognition.

Generally, if g(x) has an antiderivative G(x), and

$$\frac{d}{dx}F(x) = f(x) + g(x)$$

Then

$$\int f(x)\,dx = F(x) - G(x) + c$$

Some algebra and identities may be used in this process.

For example, differentiate $x \cdot \sin^{-1} x$, hence antidifferentiate $y = \cos^{-1} x$

$$\frac{d}{dx}(x \cdot \sin^{-1} x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$$

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

$$\frac{d}{dx}(x \cdot \sin^{-1} x) = \frac{\pi}{2} - \cos^{-1} x + \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \int \frac{\pi}{2} - \cos^{-1} x + \frac{x}{\sqrt{1-x^2}}\,dx = x \cdot \sin^{-1} x + C$$

$$\frac{\pi}{2}x - \int \cos^{-1} x\;dx - \sqrt{1-x^2} = x \cdot \left(\frac{\pi}{2} - \cos^{-1} x\right) + C$$

$$\int \cos^{-1} x\;dx = x \cdot \cos^{-1} x - \sqrt{1-x^2} + C$$

2.1.4.4 Integration by Parts

Reverse product rule.

$$\frac{d}{dx}(u \cdot v) = \frac{du}{dx}v + \frac{dv}{dx}u$$

$$\int \left(\frac{d}{dx}(u \cdot v)\right) dx = \int \left(\frac{du}{dx}v\right) dx + \int \left(\frac{dv}{dx}u\right) dx$$

$$u \cdot v = \int v \cdot du + \int u \cdot dv$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Or

$$\int f(x) \cdot g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$$

Generally, *f(x)* is preferred in the following order:

- Logarithm
- Inverse trig
- Algebraic
- Trigonometric
- Exponential

$g'(x)$ sometimes may not be apparent, but can be as simple as "1".

For example,

$$\int \log_e x \;\, dx = \int \log_e x \cdot \frac{d}{dx}(x) \cdot dx = x \cdot \log_e x - \int \frac{1}{x} \cdot x\, dx = x \cdot \log_e x - x + C$$

$$\begin{aligned}\int x^2 e^x\, dx &= \int x^2 \cdot \frac{d}{dx}(e^x)\, dx \\ &= x^2 \cdot e^x - 2\int x \cdot e^x\, dx \\ &= x^2 e^x - 2\int x \cdot \frac{d}{dx}(e^x) \cdot dx \\ &= x^2 e^x - 2\left(x \cdot e^x - \int e^x\, dx\right) \\ &= x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C \\ &= (x^2 - 2x + 2) \cdot e^x + C\end{aligned}$$

2.1.4.5 Trigonometric Integration

$$\int \sin x\,dx = -\cos x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

For others, the connection may not be so apparent. The use of the double angle identities and the Pythagorean identity and the substitution method is extensive.

In general,

- For $\int \sin^m x \cos^n x\,dx$
  - If n is odd, use the Pythagorean identity to factor out all cosine but one. Then make the substitution u=sin x

$$\begin{aligned}\int \sin^m x \cos^n x\,dx &= \int \sin^m x \cos^{2k} x \cos x\,dx\\ &= \int \sin^m x\,(1-\sin^2 x)^k \cos x\,dx\\ &= \int u^m (1-u^2)^k dx\end{aligned}$$

  - If m is odd, use the Pythagorean identity to factor out all sine but one, then make the substitution u=cos x

$$\begin{aligned}\int \sin^m x \cos^n x\,dx &= \int \sin^{2k} x \cos^n x \sin x\,dx\\ &= \int \cos^n x\,(1-\cos^2 x)^k \sin x\,dx\\ &= -\int u^n (1-u^2)^n dx\end{aligned}$$

  - If both m and n are even
    - $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
    - $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
    - $\sin x \cos x = \frac{1}{2}\sin 2x$

- For $\int \tan^m x \sec^n x \, dx$
  - If n is even, use the Pythagorean identity to factor out all secants but one square, then make the substitution u=tan x

$$\begin{aligned}\int \tan^m x \sec^n x \, dx &= \int \tan^m x \sec^{2k} x \sec^2 x \, dx \\ &= \int \tan^m x \,(1+\tan^2 x)^k \sec^2 x \, dx \\ &= \int u^m (1-u^2)^k dx\end{aligned}$$

  - If m is odd, use the Pythagorean identity to factor out all but one tan, and separate out one factor of sec x tan x, then make the substitution u=sec x

$$\begin{aligned}\int \tan^m x \sec^n x \, dx &= \int \tan^{2k} x \sec^n x \tan x \, dx \\ &= \int \sec^{n-1} x \,(\sec^2 x - 1)^k \tan x \sec x \, dx \\ &= \int u^{m-1}(u^2-1)^k dx\end{aligned}$$

  - Otherwise, the distinction is not so clear-cut. The integral of tan and sec would be employed.

Some selected integrals

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} dx \text{, let } u = \cos x \\ &= -\int \frac{du}{u} \\ &= -\ln|\cos x| + C \\ &= \ln|sec\, x| + C\end{aligned}$$

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} dx \\ &= \int \frac{(1-\cos^2 x)\sin x}{\cos^3 x} dx \text{, let } u = \cos x \\ &= -\int \frac{1-u^2}{u^3} du \\ &= \int u^{-1} - u^{-3} \, du \\ &= \ln|\cos x| + \frac{\sec^2 x}{2} + C\end{aligned}$$

Alternatively

$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x \,(\sec^2 x - 1) dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\ &= \frac{\tan^2 x}{2} - \ln|\sec x| + C\end{aligned}$$

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}dx\,, \text{let } u = \sec x + \tan x \\ &= \int \frac{du}{u} \\ &= \ln|sec\,x + tan\,x| + C\end{aligned}$$

Or alternatively,

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\cos x}{\cos^2 x}dx \\ &= \int \frac{\cos x}{1-\sin^2 x}dx, \text{let } u = \sin x \\ &= \int \frac{du}{1-u^2} \\ &= \tanh^{-1} u + C, \text{since} - 1 \le \sin x \le 1 \\ &= \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C\end{aligned}$$

$$\int \sec^3 x\,dx$$

Using integration by parts, $u = \sec x\,, du = \sec x \tan x\,dx, dv = \sec^2 x\,, v = \tan x$

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ 2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\ \int \sec^3 x\,dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C\end{aligned}$$

For integrals involving multiple angles:

$\int \sin mx \cos nx\,dx$ or $\int \sin mx \sin nx\,dx$ or $\int \cos mx \cos nx\,dx$,

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

2.1.4.6 Trigonometric Substitution

A type of inverse substitution, the independent variable is substituted with a one-to-one function.

In general, we let $x = f(\theta)$, and substitute $dx = f'(\theta)d\theta$

For the radical

- $\sqrt{a^2 - x^2}$, let $x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 + x^2}$, let $x = a\tan\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{x^2 - a^2}$, let $x = a\sec\theta, 0 \leq \theta \leq \frac{\pi}{2}$ or $\pi \leq \theta \leq \frac{3\pi}{2}$

For example,

$$\int \frac{\sqrt{9 - x^2}}{x^2} dx$$

$$\text{Let } x = 3\sin\theta, dx = 3\cos\theta\, d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2} dx &= \int \frac{3\cos\theta}{(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta \\
&= \int \cot^2\theta\, d\theta \\
&= \int \csc^2\theta - 1\, d\theta \\
&= -\cot\theta - \theta + C \\
&= -\frac{1}{\tan\left(sin^{-1}\frac{x}{3}\right)} - \sin^{-1}\frac{x}{3} + C \\
&= -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\frac{x}{3} + C
\end{aligned}$$

2.1.4.7 Hyperbolic Substitution

Hyperbolic substitution is almost identical to trigonometric substitution, and is sometimes preferred over using the substitution $x = a\tan\theta$ or $x = a\sec\theta$.

For the radical

- $\sqrt{a^2 + x^2}$, let $x = a\sinh u, u \in \mathbb{R}$
- $\sqrt{x^2 - a^2}$, let $x = a\cosh\theta, u \geq 0$ or $u \leq 0$

2.1.4.8 Partial Fractions

Partial fractions can be used to simplify a rational function so that it can be integrated.

A rational function $f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where all of these functions are polynomials, and the degree of R is less than the degree of Q.

Part I

The denominator $Q(x)$ is the product of **distinct** linear factors.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots$$

Part II

The denominator $Q(x)$is a product of linear factors, some of which are repeated

$$\frac{R(x)}{Q(x)} = \frac{A_{1\,1}}{a_1x + b_1} + \frac{A_{1\,2}}{(a_1x + b_1)^2} + \cdots + \frac{A_{1\,r}}{(a_1x + b_1)^r} + \cdots$$

Part III

The denominator $Q(x)$contains **distinct** irreducible quadratic factors

$$\frac{R(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \cdots$$

Part IV

The denominator $Q(x)$contains irreducible quadratic factors, some of which are repeated

$$\frac{R(x)}{Q(x)} = \frac{A_{1\,1}x + B_{1\,1}}{a_1x^2 + b_1x + c_1} + \frac{A_{1\,2}x + B_{1\,2}}{(a_1x^2 + b_1x + c_1)^2} + \cdots + \frac{A_{1\,r}x + B_{1\,r}}{(a_1x^2 + b_1x + c_1)^r} + \cdots$$

2.1.4.8.1 Quartics

For quartics or higher degree polynomials without real roots, it is possible to factorise these using "complete the square" to irreducible quadratics.

$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - \left(\sqrt{2}x\right)^2 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$$

$$\begin{aligned} x^6 + 1 &= (x^2 + 1)(x^4 - x^2 + 1) = (x^2 + 1)(x^4 + 2x^2 + 1 - 3x^2) \\ &= (x^2 + 1)\left(x^2 - \sqrt{3}x + 1\right)\left(x^2 + \sqrt{3}x + 1\right) \end{aligned}$$

2.1.4.9 Improper Integrals

For definite integrals at infinity, or where the function is discontinuous in that interval, the integral cannot be evaluated via the normal means of substitution, and limits must be employed.

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)dx = \lim_{t\to\infty} F(t) - F(a)$$

The case for negative infinity is very similar.

For discontinuity within the interval [a,b] at c

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx \\ &= \lim_{t\to c^-} F(t) - \lim_{t\to c^+} F(t) + F(b) - F(a)\end{aligned}$$

That is, if the limit exists at c, then the integral behaves "normally".

However, if both limits evaluate to infinity (or negative infinity), there will be a $\infty - \infty$ term, which will be undefined. If the limits evaluate to opposite infinities, the value is "divergent".

Particular cases of integral at discontinuities are when one (or both) of the end-points is discontinuous.

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx = \lim_{t\to b^-} F(t) - F(a)$$

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx = F(b) - \lim_{s\to a^+} F(s)$$

$$\int_a^b f(x)dx = \lim_{s\to a^+}\left[\lim_{t\to b^-}\int_s^t f(x)dx\right] = \lim_{t\to b^-} F(t) - \lim_{s\to a^+} F(s)$$

2.1.5 Applications of Integral Calculus

2.1.5.1 Area

2.1.5.1.1 Approximation

Area approximation works uses various shapes with defined area formulae to approximate the area under a graph.

2.1.5.1.1.1 Methods

Left and right end point rectangles

Left:

$$\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$$

Right:

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

Midpoint Rectangles

$$\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right) \cdot \Delta x = \frac{b-a}{n} \sum_{i=1}^{n} f(x^*)$$

Trapeziums

$$\sum_{i=1}^{n} \frac{f(x_i) + f(x_{i-1})}{2} \cdot \Delta x = \frac{b-a}{2n} \sum_{i=1}^{n} f(x_i) + f(x_{i-1}) = \frac{b-a}{n}\left(\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)$$

2.1.5.1.1.2 Bounds

- For an increasing function
  - The left rectangles method gives an underestimation
  - The right rectangles method gives an overestimation
- For a decreasing function
  - The left rectangles method gives an overestimation
  - The right rectangles method gives an underestimation
- For a concave up function
  - The mid-point rectangles method gives an underestimation
  - The trapezium method gives an overestimation
- For a concave down function
  - The mid-point rectangles method gives an overestimation
  - The trapezium method gives an underestimation

2.1.5.1.1.3 Error

For a function f(x) on a closed interval [a,b]: there is a number K such that $|f''(x)| \leq K$, then the error bound for the trapezoidal and midpoint rules are

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Hence, the mid-point rule is about twice as accurate as the trapezoidal method.

2.1.5.1.1.4 The Integral as A Sum

Taking the mid-point method, as the number of rectangles is increased, the approximation gets more and more accurate.

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x^*)\Delta x$$

As n goes to infinity, $\Delta x \to dx$, $x^* \to x$

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x^*)\Delta x = \int_a^b f(x)dx$$

2.1.5.1.2 Signed Area

Y values under the x axis are negative, hence the area calculated by an integral would also be negative.

If *f* is negative between in the interval (a,b) [usually a and b would be two x intercepts], then

$$A = -\int_a^b f(x)dx = \int_b^a f(x)dx = \left|\int_a^b f(x)dx\right| = \int_a^b |f(x)|dx$$

When finding area of a function that crosses the x axis several times, the signed area must be taken into account.

2.1.5.1.3 Between Curves

If *f(x)>g(x)* for an interval (a,b) [usually a and b are points of intersection], then

$$\int_a^b [f(x)-g(x)]dx \neq \int_a^b [g(x)-f(x)]dx$$

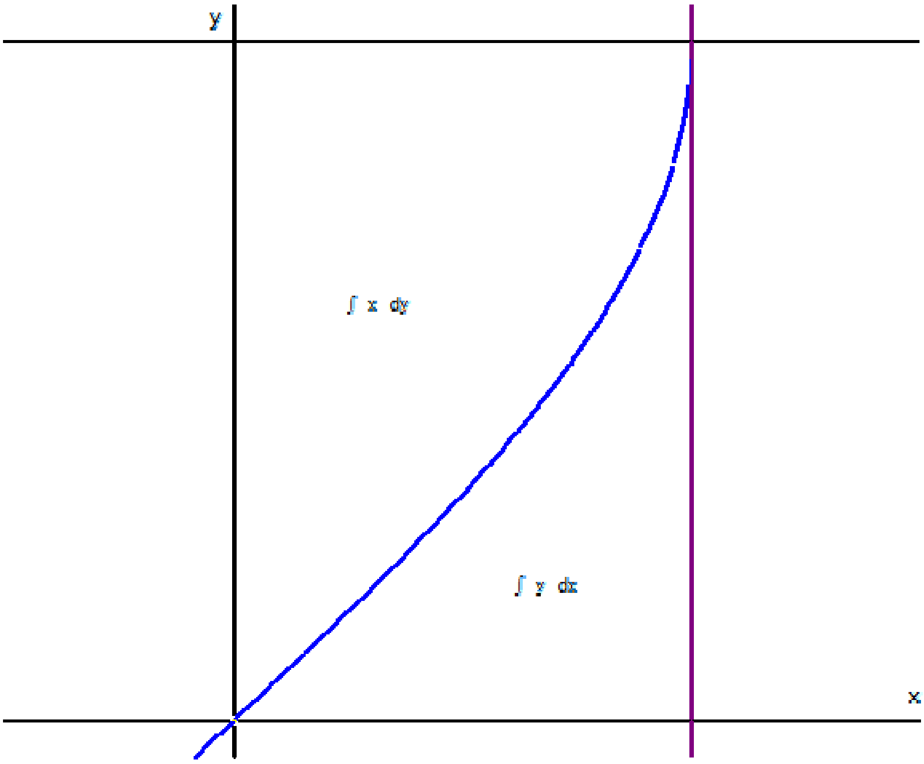
When finding the area between two curves that cross each other several times, the signed difference must be taken into account.

If *g(x)>f(x)* for another interval (b,c), then

$$A = -\int_b^c [f(x)-g(x)]dx = \int_c^b [f(x)-g(x)]dx = \int_b^c [g(x)-f(x)]dx = \int_b^c |f(x)-g(x)|dx$$

2.1.5.1.4 Along the Y Axis

When integrating inverse functions (which are rather difficult), it is often easier to find the area along the y axis, and then subtract that from a rectangle.



For example

$$\int_0^1 \sin^{-1} x \, dx$$

When x=0, y=0. When x=1, y=π/2.

$y = \sin^{-1} x \Longrightarrow x = \sin y$

$$\int_0^1 \sin^{-1} x \, dx = 1 \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin y \, dy = \frac{\pi}{2} + [\cos y]_0^{\pi/2} = \frac{\pi}{2} - 1$$

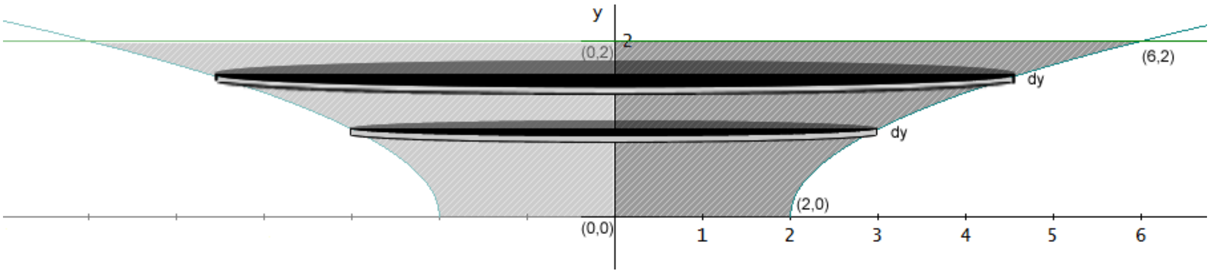
2.1.5.2 Solids of Revolution

When an area is spun around an axis, a solid of revolution is formed.

2.1.5.2.1 Slab Method

The slab method takes the solid of revolution as infinitely thin circular disks (cylinders). The slab method is applicable only when the area being spun is bound by the axis which it is being spun around.

$$V = \sum \pi r^2 \cdot dh$$



In general,

About the x axis:

$$V = \pi \int_a^b [f(x)]^2 dx$$

About the y axis:

$$V = \pi \int_c^d x^2 dy = \pi \int_c^d [f^{-1}(x)]^2 dx = \pi \int_a^b x^2 \cdot \frac{dy}{dx}\, dx$$

2.1.5.2.1.1 Washer Method

The washer method, aka the donut method, is when an area **not** bound by the axis of rotation is spun. The cross-section of such a volume resembles a washer.

This type of area is the area between curves, i.e. $[f(x) - g(x)]$

The volume is calculated by the summation of infinitely thin washers.

$$V = \sum (\pi R^2 - \pi r^2) \cdot dh = \sum \pi (R^2 - r^2) \cdot dh$$

In general, a solid of revolution about the x axis:

$$V = \pi \int_a^b [f(x)]^2 - [g(x)]^2\, dx$$

About the y axis:

$$V = \pi \int_c^d (x_2)^2 - (x_1)^2\, dx$$

2.1.5.2.2 Shell Method

The shell method calculates the volume by cylindrical shells of infinitesimal width.

The volume of a cylindrical shell:

$$V = \pi(R^2 - r^2) \cdot h = \pi(R - r)(R + r)h = 2\pi \frac{(R + r)}{2}(R - r)h$$

Where $\frac{R+r}{2}$ is the average distance from the axis of rotation, and $R - r$ is the thickness of the shell.

As the shell gets small, $\frac{R+r}{2} \to r \approx R, R - r \to dr$

$$V = \sum (2\pi r^*) \cdot dr \cdot h$$

The height of the cylindrical shell is generally the difference of two curves,

In general,

About the y axis,

$$V = 2\pi \int_a^b x \cdot [f(x) - g(x)]\, dx$$

About the x axis,

$$V = 2\pi \int_c^d y \cdot [x_2 - x_1] dy$$

It is easier to remember this form:

$$V = [circumference] \cdot [height] \cdot [thickness]$$

2.1.5.3 Line Integral

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

2.1.5.4 Work

$$W = \boldsymbol{F} \cdot d = m \cdot \boldsymbol{a} \cdot d$$

When a force is applied on an object from a to b, and the force on the object at point x is f(x), then

$$W = \int_a^b f(x)dx$$

Examples

A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done in stretching the spring from 15cm to 18cm?

By Hooke's law, $F = kx, \Longrightarrow 40N = k \cdot 0.05 \Longrightarrow k = 800 \text{ kg s}^{-2}$

$$\int_{0.0.5}^{0.08} 800x\, dx = 1.56\text{J}$$

A 20kg cable of 10m long is hanged down from the top of a building. How much work is required to lift the cable to the top of the building?

Let the top of the building be 0. The density of the cable is 2 kg m^{-1}.

Each small sections of cable, dx, must travel up to the top of the building (x metres above)

Therefore, the work on each small section of cable: 2*dx*g*x=2gx*dx

$$\int_0^{10} 2gx\, dx = 100g\text{J}$$

An inverted conical tank with height of 10m and base radius of 4 is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank.

Let the top of the tank be 0. At some x metres below the top, there is a layer of water with radius of r and thickness of dx. $\frac{r}{10-x} = \frac{4}{10} \Longrightarrow r = \frac{2}{5}(10 - x)$

The mass of that layer of water is hence $m = p \cdot V = p \cdot \pi r^2 dx = \frac{4p\pi}{25}(10 - x)^2 dx$

The work on that layer of water is hence $m \cdot g \cdot x = \frac{4p\pi g}{25} x \cdot (10 - x)^2 dx$

$$\int_2^{10} \frac{4p\pi g}{25} x \cdot (10 - x)^2 dx = \frac{4000\pi g}{25} \int_2^{10} x(10 - x)^2 dx \approx 3.4 \times 10^6\text{J}$$

2.1.5.5 Moments and Centre of Mass

$\text{Moment} = m_n \cdot x_n$, where x_n is distance from the origin.

The centre of mass is where the sum of moments from it equal to zero:

$$\sum m_i(\bar{x} - x_i) = 0$$
$$\sum m_i\,\bar{x} = \sum m_i x_i$$
$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

If M is the total mass of the system, then the centre of mass would be:

$$\bar{x} = \frac{1}{M}\sum m_i x_i$$

In a two dimensional plane, the centre of mass of about the x and the y axis (i.e. distance from the axis) are then:

$$\bar{x} = \frac{1}{M}\sum m_i y_i = \frac{M_y}{M} \qquad \bar{y} = \frac{1}{M}\sum m_i x_i = \frac{M_x}{M}$$

For a shape with uniform density and thickness, represented by an area, the moment of the shape can be considered to be the sum of moments of each infinitesimal rectangle in the x direction:

For each of these infinitesimal rectangle, they have a thickness of dx, a height of $f(x)$, their centre of mass is the centre of the rectangle, $\left(x, \frac{1}{2}f(x)\right)$, and their mass is $p \cdot A = p \cdot f(x) \cdot dx$, where p is the density.

Hence, the moment about the y axis: $M_y = m_i x_i = p \cdot xf(x) \cdot dx$, the sum of moment is hence $p\int_a^b xf(x)dx$

The moment about the x axis: $M_x = m_i y_i = \frac{p}{2}[f(x)]^2 dx$, the sum of moment is hence $p\int_a^b [f(x)]^2 dx$

The mass M of the system is $p \cdot A = p \cdot \int_a^b f(x)\,dx$

Hence, the centre of mass in the x and y direction (the **centroid**) are,

$$\bar{x} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)\,dx}, \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}[f(x)]^2 dx}{\int_a^b f(x)\,dx}$$

For a shape defined by $f(x) - g(x)$

$$\bar{x} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]\,dx}, \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)\,dx}{\int_a^b [f(x) - g(x)]\,dx}$$

Theorem of Pappus

If a shape *R* is rotated about *l*, then the volume of the resulting solid is the product of the area of *R* and the distance the centroid of *R* has travelled.

I.e. if the distance from the centroid perpendicularly to the axis *l* is *r*, then

$$V = 2\pi\bar{r} \cdot A$$

2.1.6 Differential Equations

2.1.6.1 Separable Equations

$$\frac{dy}{dx} = f(x) \Longrightarrow y = \int f(x)\,dx$$

$$\frac{dy}{dx} = f(y) \Longrightarrow x = \int \frac{dy}{f(y)}$$

In general

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \Longrightarrow \int g(y)dy = \int f(x)dx$$

For second case, it should be noted that the antiderivative would most probably be a logarithmic function of an absolute function, which can be positive or negative.

Problems giving an initial state are called "initial value problems", and the initial value determines the position at which the relationship would map to, as well as which branch the function will take.

For example,

$$\frac{dy}{dx} = 2y + 1, y(0) = -1$$

$$\Longrightarrow x = \frac{1}{2}\ln|2y + 1| + c$$

$$\Longrightarrow c = -\frac{1}{2}\ln|-1| = 0$$

$$e^{2x} = |2y + 1|$$

$$2y + 1 = \pm e^{2x}$$

Since $y(0) = -1$, the left hand side is evaluates to a negative number, implying the right hand side must be negative.

$$2y + 1 = -e^{2x}$$

$$y = -\frac{1}{2}(e^{2x} + 1)$$

2.1.6.2 Modelling and Applications

2.1.6.2.1 Variation

Direct proportion: $a \propto b \Leftrightarrow a = kb$

Inverse variation: $a \propto \frac{1}{b} \Leftrightarrow a = \frac{k}{b}$

Note that in these cases, "b" can be "the square", "the square root", "cube", "logarithm", or others as specified by the question.

In particular, for a variation where the rate is proportional to the value, usually modelling population:

$$\frac{dy}{dx} = ky$$

$$x = \frac{1}{k}\ln|y| + c$$

Given (x_0, y_0), $c = x_0 - \frac{1}{k}\ln|y_0|$

$$y = \pm y_0 \cdot e^{k(x-x_0)}$$

(Note the +/- sign. In reality, however, mathematical modelling rarely encounters negative values outside of rates of change)

2.1.6.2.2 Newton's Law of Cooling

A body cools/heats at a rate proportional to the difference between its temperature and its surroundings.

This translates to the differential equation

$$\frac{dT}{dt} = k(S - T), \text{where } S \text{ is the temperature of the surroundings}$$

$$t = -\frac{1}{k}\ln|S - T| + C$$

If at t=0, $T=T_0$, $C = \frac{1}{k}\ln|S - T_0|$

$$e^{-kt} = \left|\frac{S - T}{S - T_0}\right|$$

Since the temperature function is monotonic (it does not "overshoot"), the numerator expression is always the same sign with the denominator expression, and the modulus is not required.

$$T = S - (S - T_0)e^{-kt}$$

Since there are three constants in this expression, the question would need to give at least three conditions to work out these constants.

In this form, solving for constants will be very difficult. The following [non-explicit] form would be more appropriate:

$$t = \frac{1}{k}\ln\left(\frac{S - T_0}{S - T}\right)$$

For example, a thermometer is taken from a house at 21 degrees to the outside. One minute later it reads 27 degrees, another minute later it reads 30 degrees. Find temperature outside house.

$$t = \frac{1}{k}\ln\left(\frac{S - T_0}{S - T}\right)$$

$$T_0 = 21$$

$$1 = \frac{1}{k}\ln\left(\frac{S - 21}{S - 27}\right) \cdots [1]$$

$$2 = \frac{1}{k}\ln\left(\frac{S - 21}{S - 30}\right) \cdots [2]$$

$$2 \times [1]_{RHS} = [2]_{RHS} \Rightarrow \frac{(S - 21)^2}{(S - 27)^2} = \frac{S - 21}{S - 30}$$

$$(S - 30)(S - 21) = (S - 27)^2$$

$$-51S + 620 = -54S + 729$$

$$3S = 99$$

$$S = 33^o$$

2.1.6.2.3 Difference of Rates

For a volume of solution, with an inflow, and whilst it is kept evenly mixed, an outflow, a differential equation for this situation can be modelled by:

$$\frac{dQ}{dt} = \frac{dV_{in}}{dt} \cdot C_{in} - \frac{dV_{out}}{dt} \cdot \frac{Q}{V_0 + \left(\frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}\right) t}$$

Where Q is the amount of solute, $\frac{dV_{in}}{dt}$ is the rate of inflow, C_{in} is the concentration of inflow, and $\frac{dV_{out}}{dt}$ is the rate of outflow.

The differential equation can be thought of as:

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow} = [\text{inflow}] \cdot [\text{concentration of inflow}] - [\text{outflow}] \cdot \frac{\text{Amount}}{\text{Volume}}$$

For systems where the rate of inflow is equal to the rate of outflow:

$$\frac{dQ}{dt} = \frac{V_0 \cdot \left(\frac{dV_{in}}{dt}\right) \cdot C_{in} - \left(\frac{dV_{out}}{dt}\right) \cdot Q}{V_0}$$

Hence,

$$t = \frac{V_0}{\left(\frac{dV_{out}}{dt}\right)} \ln\left(\frac{V_0 \cdot \left(\frac{dV_{in}}{dt}\right) \cdot C_{in} - \left(\frac{dV_{out}}{dt}\right) \cdot Q_0}{V_0 \cdot \left(\frac{dV_{in}}{dt}\right) \cdot C_{in} - \left(\frac{dV_{out}}{dt}\right) \cdot Q}\right)$$

Solving the differential equation where the rate of inflow does not equal to the rate of outflow involves the method of integrating factor, which is not in the Specialist Mathematics course. An example of this will be covered in the relevant section.

2.1.6.2.4 Finite Integral

Differential equations can translate into a finite integral, which can be solved numerically using principles of approximation (or machine approximations). This can be useful for functions without an antiderivative, or a rather complex antiderivative.

Let there be a function f such that its antiderivative is F

$$\frac{dy}{dx} = f(x)$$

$$y = F(x) + C$$

Given the initial condition when $x=x_0$, $y=y_0$, $C = y_0 - F(x_0)$

$$y = F(x) - F(x_0) + y_0$$

By the Fundamental Theorem of Calculus (II)

$$y - y_0 = \int_{x_0}^{x} f(x)dx$$

$$\Delta y = \int_{x_0}^{x} f(x)dx$$

Hence, given the initial value, the differential equation can be solved for any x within a continuous closed set in its domain.

For

$$\frac{dy}{dx} = f(y)$$

$$\Delta x = \int_{y_0}^{y} f(y)dy$$

2.1.6.2.5 Euler's Method

The formula for linear approximation is

$$f(x+h) \approx hf'(x) + f(x)$$

Euler's method uses the gradient function f'(x) to approximate f(x) with small h, numerically solving a differential equation.

A program for Euler's Method on the TI-89 series calculator:

```
:eulert()
:Prgm
:DelVar x,y,dx,dy,t,yp
:ClrIO
:Input "dy/dx",yp
:Input "x initial",x
:Input "y initial",y
:Input "step size",dx
:1→t
:Define list1={}
:Define list2={}
:Input "x final",s
:(s-x)/dx→s
:While t≤s
:Define list1[t]=x
:Define list2[t]=y
:yp*dx→dy
:x+dx→x
:y+dy→y
:t+1→t
:EndWhile
:Define list1[t]=x
:Define list2[t]=y
:Disp y
:Disp approx(y)
:DelVar t,x,y
:DelVar yp,dx,dy
:EndPrgm
```

For example, $\frac{dy}{dx} = \sqrt{2x+1}, y(0) = -1,$ find $y(1)$ using a step size of 0.1

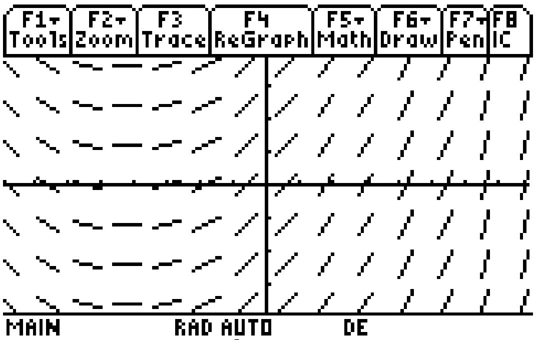
x	y_i	$\frac{dy}{dx}$	dy
0.0	-1	1	0.1
0.1	-0.9	1.0954	0.1095
0.2	-0.7905	1.1832	0.1183
0.3	-0.6721	1.2649	0.1265
0.4	-0.5456	1.3416	0.1342
0.5	-0.4115	1.4142	0.1414
0.6	-0.2701	1.4832	0.1483
0.7	-0.1217	1.5492	0.1559
0.8	0.0332	1.6125	0.1613
0.9	0.1944	1.6733	0.1673
1.0	0.3618		

2.1.6.2.6 Slope Field

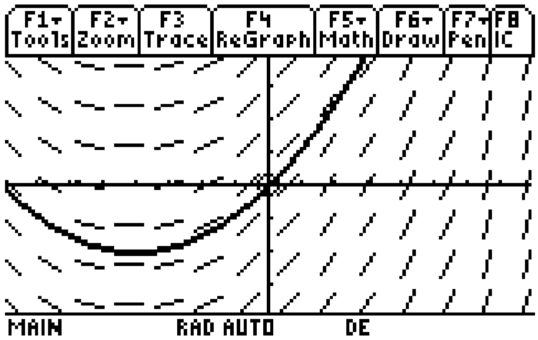
A slope field of a differential equation assigns the value of the gradient to each point on a plane P(x,y).

The gradient is usually given by a short straight line in the direction of the slope at regular intervals in the x and y directions.

With a slope field, any differential equation can be solved for numerically given an initial condition at any point on P(x,y)



$$\frac{dy}{dx} = \frac{x}{4} + 1$$



Initial Condition: x=0, y=0

2.1.6.2.7 Orthogonal Trajectories

Orthogonal trajectories are curves that are always perpendicular to each other at the point of intersection. These satisfy the differential equation

$$\frac{dy_1}{dx} \cdot \frac{dy_2}{dx} = -1$$

In general, a pair of relationships that are orthogonal:

$$y = ax^n, a \in \mathbb{R}, n \in \mathbb{R}\backslash\{0\} \qquad x^2 + ny^2 = b, b \in \mathbb{R}, n \in \mathbb{R}\backslash\{0\}$$

In particular, when n=-1,

$$xy = c, c \in \mathbb{R} \qquad x^2 - y^2 = k, k \in \mathbb{R}$$

2.1.6.3 First Order Linear Differential Equations (Integrating Factors)

The ordinary differential equation

$$y' + P(x)y = Q(x)$$

Is a non-separable first order linear differential equation, and can be solved by multiplying both sides by an “integrating factor”.

Using the integrating factor recognises that:

$$(f(x)y)' = f(x)y' + f'(x)y$$

Hence, multiplying both sides of the DE by the integrating factor *I(x)*

$$I(x)y' + I(x)P(x)y = I(x)Q(x)$$

It can be seen that $I'(x) = I(x)P(x) \Longrightarrow \frac{I'(x)}{I(x)} = P(x) \Longrightarrow I(x) = e^{\int P(x)dx}$

$$(I(x)y)' = I(x)Q(x)$$

$$y = \frac{\int I(x)Q(x)dx + C}{I(x)}$$

For example:

A 20L tank of salt solution initially has 2kg of dissolved salt. Salt is poured into the solution at 0.1kg/min, and the solution is flowing out at a constant rate of 1L/min whilst the solution is kept evenly mixed.

$$\frac{dQ}{dt} = 0.1 - \frac{Q}{20 - t}$$

$$\frac{dQ}{dt} + \left(\frac{1}{20 - t}\right)Q = 0.1$$

$$I(x) = e^{\int \frac{1}{20-t}dt} = e^{-\ln|20-t|} = (|t - 20|)^{-1}$$

The implied domain is t<20 (at that point the tank is empty),

$$I(x) = (20 - t)^{-1}$$

$$\left(\frac{Q}{20 - t}\right)' = 0.1$$

$$\frac{Q}{20 - t} = 0.1t + C$$

Given that Q=2 when t=0, C=0.1

$$\frac{10Q}{20 - t} = t + 1$$

$$Q = -\frac{1}{10}(t - 20)(t + 1)$$

2.1.6.4 Second order Differential Equations

An ordinary second-order linear equation is in the form of:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

2.1.6.4.1 Homogenous

A homogenous second order DE is in the form of

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

Then, if y_1 and y_2 are solutions to this differential equation,

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

Where y(x) is the general solution to the differential equation, and c_1 and c_2 are arbitrary constants. This implies that if two solutions are known, then all solutions are known. This also implies that y_1 and y_2 are linearly independent.

Not all differential equations are solvable, but it is always possible to solve it when P, Q and R are constant functions.

$$ay'' + by' + cy = 0$$

To solve this linear differential equation, let $y=e^{rx}$.

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$ar^2 + br + c = 0$$

The above equation is called the auxiliary equation (or characteristic equation).

Solving for r can have three different outcomes:

- Two solutions
  - $y = c_1e^{r_1x} + c_2e^{r_2x}$
- One solution
  - $y = c_1e^{rx} + c_2xe^{rx}$
- No real solution
  - $r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta$
  - $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$
  - Where c_1 and c_2 can be complex numbers. This gives solution in the real and complex plane.

2.1.6.4.2 Non-Homogenous

For second order non-homogenous linear differential equations with constant coefficients:

$$ay'' + by' + cy = G(x)$$

Then the general solution takes the form

$$y(x) = y_p(x) + y_c(x)$$

Where y_p is a particular solution, and y_c is the general solution to the complementary equation:

$$ay'' + by' + cy = 0$$

Whilst y_c can be found with reasonable ease, finding y_p is more involved, and two of the methods are explained below.

2.1.6.4.2.1 Method of Undetermined Coefficients

$$ay'' + by' + cy = G(x)$$

2.1.6.4.2.1.1 Polynomial Function

Where G(x) is a polynomial.

The particular solution will be of the same degree of G(x), and will take the form:

$$y_p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

Substituting this and its derivatives into the differential equation will give a system of linear equations by equating the coefficients.

2.1.6.4.2.1.2 Exponential Function

If $G(x) = C \cdot e^{kx}$ then,

$$y_p(x) = Ae^{kx}$$

Substituting this and its derivatives into the differential equation will give a system of linear equations by equating the coefficients of the exponential terms.

2.1.6.4.2.1.3 Trigonometric Functions

If $G(x) = C \cdot \cos kx$ or $C \cdot \sin kx$ then,

$$y_p(x) = A \cos kx + B \sin kx$$

Substituting this and its derivatives into the differential equation will give a system of linear equations by equating the coefficients of the trigonometric terms.

2.1.6.4.2.1.4 Product of Functions

If $G(x)$ is a product of the previous types of functions, then a trial solution would be a product of the particular solutions.

If $G(x) = P_1(x) \cdot e^{kx}$, then $y_p(x) = P_2(x) \cdot e^{kx}$

If $G(x) = P_1(x) \cdot \sin kx$ or $P_1(x) \cdot \cos kx$, then $y_p(x) = P_2(x) \cdot \cos kx + P_3(x) \cdot \sin kx$

If $G(x) = Ce^{k_1 x} \cdot \sin k_2 x$ or $Ce^{k_1 x} \cdot \cos k_2 x$, then $y_p(x) = Ae^{k_1 x} \cdot \cos k_2 x + Be^{k_1 x} \cdot \sin k_2 x$

2.1.6.4.2.1.5 Sum of Functions

If $G(x) = g_1(x) + g_2(x)$, then the particular solutions will be the sum of particular solutions to $ay'' + by' + cy = g_1(x)$ and $ay'' + by' + cy = g_2(x)$.

2.1.6.4.2.2 Method of Variation of Parameters

If the complementary equation has already been solved and is expressed with arbitrary constants, the method of variation of parameters then let the arbitrary constants be arbitrary functions and tries to find a particular solution.

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Hence, differentiating gives:

$$y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$$

Since u_1 and u_2 are arbitrary functions, conditions may be imposed on them, such that $u_1'y_1 + u_2'y_2 = 0$.

$$y_p'' = u_1'y_1' + u_2y_2' + u_1y_1'' + u_2y_2''$$

Hence,

$$a(u_1'y_1' + u_2y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G$$

Since *y_1* and *y_2* are particular solutions to the complementary equation, this simplifies to

$$a(u_1'y_1' + u_2'y_2') = G$$

Also since $u_1'y_1 + u_2'y_2 = 0 \Longrightarrow u_1' = -\frac{u_2'y_2}{y_1}$, solving these simultaneously can give expressions for u_1' and u_2', which can be antidifferentiated and hence the particular solution is found.

For example, solve $y'' + y = \tan x, 0 < x < \pi/2$

The complementary equation is $y'' + y = 0$, which gives an auxiliary equation

$$r^2 + 1 = 0 \Longrightarrow r = \pm i$$

$$y_c = e^0 \cdot (c_1 \cos(x) + c_2 \sin(x)) = c_1 \cos x + c_2 \sin x$$

$$y_p = u_1 \cos x + u_2 \sin x$$

Hence, $u_1' y_1 + u_2' y_2 = 0 \Longrightarrow u_1' \cos x + u_2' \sin x = 0$

Also, $u_1'(-\sin x) + u_2'(\cos x) = \tan x$

$$\frac{u_2' \sin^2 x}{\cos x} + u_2' \cos x = \frac{\sin x}{\cos x}$$

$$u_2'(\sin^2 x + \cos^2 x) = \sin x$$

$$u_2 = -\cos x$$

$$u_1' = -\frac{\sin^2 x}{\cos x}$$

$$u_1 = \int -\frac{\sin^2 x}{\cos x} dx = \sin x - \ln(\sec x + \tan x)$$

$$y_p = (\sin x - \ln(\sec x + \tan x)) \cdot \cos x - \cos x \cdot \sin x = -\cos x \cdot \ln(\sec x + \tan x)$$

2.1.6.4.3 Initial Value Problems and Boundary Value Problems

Initial value problems for second order differential equations will provide initial y value as well as the initial gradient. Solve these just as initial value problems in first-order differential equations.

A boundary value problem gives two y values for two x values, and may not always have a solution. Substitute the x and y values into the general solution and solve simultaneously for the arbitrary constants.

2.1.7 Physical Applications

2.1.7.1 Kinematics

2.1.7.1.1 SUVAT and V-t Graphs

For motion with constant acceleration, the following formulae can be used, where *u* is the initial velocity, *v* is the final velocity, *a* is the acceleration, *t* is the time, and *s* is the distance travelled.

$$v = u + at$$

$$a = \frac{v - u}{t}$$

$$s = \frac{u + v}{2}t$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 - u^2 = 2ad$$

For a *v*-*t* graph, the gradient is the acceleration, and the area under the graph is the displacement.

For particular problems where an object has a maximum rate of acceleration *a*, a maximum rate of deceleration *n* times *a*, a maximum velocity *v* at which it reaches and travels at during the journey, and a set distance to travel *s*, the time *t* can be solved by the following:

$$\frac{v^2}{2a} + \frac{v^2}{2na} + vt_p = s$$

$$t = t_p + \frac{v}{a} + \frac{v}{na} = \frac{3}{2} \cdot \left(\frac{nv + v}{na}\right) + \frac{s}{v}$$

2.1.7.1.2 Acceleration

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

For example, find x in terms of t if $a = -4x + 8$ and when $t = 0$, $x = \frac{1}{2}$, $v = -3\sqrt{3}$

$$v \cdot \frac{dv}{dx} = -4x + 8$$

$$\int v\,dv = \int -4x + 8\,dx$$

$$\frac{v^2}{2} = -2x^2 + 8x + C$$

$x = \frac{1}{2}$, $v = -3\sqrt{3}$

$$\frac{27}{2} = -\frac{2}{4} + 4 + C$$

$$C = 10$$

$$v^2 = -4(x^2 - 4x - 5)$$

$$\frac{dx}{dt} = -2\sqrt{5 - (x^2 - 4x + 4) + 4} \text{ since } v \text{ is negative}$$

$$\frac{dt}{dx} = -\frac{1}{2\sqrt{3^2 - (x-2)^2}}$$

$$t = -\frac{1}{2}\cos^{-1}\left(\frac{x-2}{3}\right) + D$$

$t = 0, x = \frac{1}{2}$

$$D = \frac{1}{2}\cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\cos\left(-2\left(t - \frac{\pi}{3}\right)\right) = \frac{x-2}{3}$$

$$x = 3\cos\left(2\left(t - \frac{\pi}{3}\right)\right) + 2$$

2.1.7.2 *Statics and Dynamics*

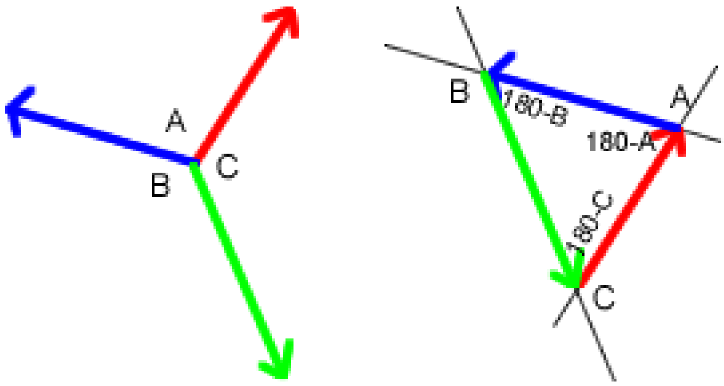
2.1.7.2.1 Force

Force is a vector quantity, with a direction and magnitude. It can be resolved into components (in 2D or 3D).

2.1.7.2.1.1 *Equilibrium*

In equilibrium, the resultant of the forces is a zero. The sum of components of forces in any direction is also zero.

For three forces acting on a particle at equilibrium:



If the magnitudes of the three forces are known, then the cosine rule can be applied.

Let the vector opposite A be **a**, opposite B be **b**, and opposite C be **c**.

$$|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 - 2|\boldsymbol{b}||\boldsymbol{c}|\cos A$$

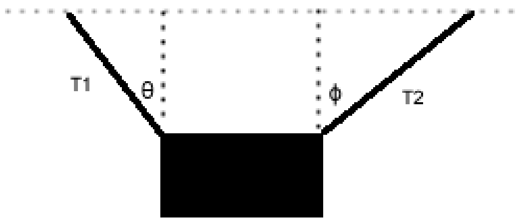
2.1.7.2.1.1.1 *Lami's Theorem*

When an angle is known, the sine rule can be applied to find the magnitude/angle of the other forces.

$$\frac{|\boldsymbol{a}|}{\sin A} = \frac{|\boldsymbol{b}|}{\sin B} = \frac{|\boldsymbol{c}|}{\sin C}$$

2.1.7.2.1.1.2 Hanging Mass

Since the hanging mass is in equilibrium, we can make the following generalisations:



$$\sin\theta\,T_1 = \sin\phi\,T_2$$

$$\cos\theta\,T_1 + \cos\phi\,T_2 = mg$$

Alternatively, where the complement of the angles θ and ϕ are known:

$$\cos\theta'\,T_1 = \cos\phi'\,T_2$$

$$\sin\theta'\,T_1 + \sin\phi'\,T_2 = mg$$

Solving these simultaneously when the angle is given would give the magnitude of the tension forces along the strings. Alternatively, Lami's Theorem can be applied.

Generally, the lengths of the strings are given, and the angle can be worked out henceforth.

Where the angle is required and the forces are known, the cosine rule (pictured as before) can be applied.

2.1.7.2.1.2 Newton's Laws of Motion

Newton's first law of motion

A particle at rest or in constant motion will remain at rest or constant motion unless acted on by an unbalanced force.

For a particle/system in equilibrium, the forces acting on it must balance.

Newton's second law of motion

The force is proportionate to the rate of change of the object's momentum.

$$\boldsymbol{F} = km\frac{d\boldsymbol{v}}{dt} = km\frac{d^2\boldsymbol{s}}{dt^2} = km\boldsymbol{a}$$

The value of one Newton is chosen such that *k* is 1 when a is in ms^{-2} and m is in kg.

Newton's third law of motion

Every force has an equal and opposite force

Weight

The weight is specific to particular gravitational fields. One kilogram in a particular field weighs 1 kg wt. In other words, 1 kg wt = 1g N.

2.1.7.2.1.3 Friction

The friction force always opposes the direction of motion.

The maximum friction force between two particular surfaces is proportional to the normal force (opposing the weight force).

$$F = \mu N$$

Where μ is the coefficient of friction. The coefficients of friction for two static surfaces and two surfaces sliding relatively to each other are different.

2.1.7.2.1.3.1 Static

$$F_{max} = \mu N$$

If an external force acts on a static object on a surface, friction opposes this force parallel to the plane of the surface. The friction force opposes the external force as much as possible up to its maximum limit. Until the external force is greater than the magnitude of maximum force, there is no motion, and the object/system is said to be in equilibrium.

For an object/system in equilibrium, the minimum coefficient of friction possible is when the object/system is on the point of sliding. I.e. the frictional force is at maximum. (If the frictional force is not at maximum, then the coefficient of friction would need to be greater.)

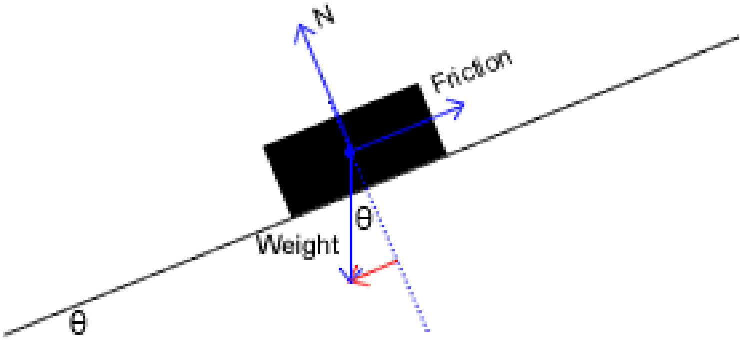
2.1.7.2.1.3.2 Sliding

Sliding friction opposes the direction of motion, and has the magnitude

$$F_{fric} = \mu N$$

2.1.7.2.2 Single Object

2.1.7.2.2.1 Inclined Plane



For example

An object is projected with speed *U* up a rough plane with coefficient of friction μ and inclination of θ degrees to the horizontal.

The distance it travels up the plane (down the plane is negative):

$$F_{net} = ma_{net} = -mg\sin\theta - \mu mg\cos\theta$$

$$a_{net} = -g(\sin\theta + \mu\cos\theta)$$

$$\frac{dv}{dx} = -\frac{g(\sin\theta + \mu\cos\theta)}{v}$$

$$x = \int_U^0 -\frac{v}{g(\sin\theta + \mu\cos\theta)}dv = \frac{1}{g(\sin\theta + \mu\cos\theta)} \cdot \left[\frac{v^2}{2}\right]_0^U = \frac{U^2}{2g(\sin\theta + \mu\cos\theta)}$$

The speed which it returns to ground:

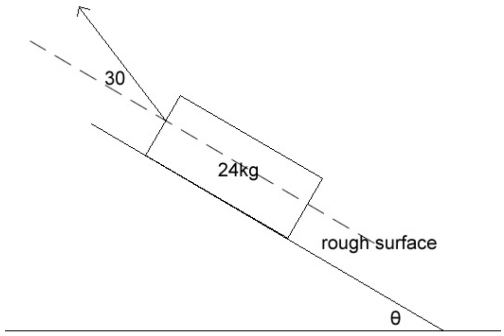
$$a_{net} = -g(\sin\theta - \mu\cos\theta)$$

$$x = \int_0^V -\frac{v}{2g(\sin\theta - \mu\cos\theta)}dv = -\frac{V^2}{2g(\sin\theta - \mu\cos\theta)}$$

$$\frac{V^2}{2g(\sin\theta - \mu\cos\theta)} = \frac{U^2}{2g(\sin\theta + \mu\cos\theta)}$$

$$V = U\sqrt{\frac{\sin\theta - \mu\cos\theta}{\sin\theta + \mu\cos\theta}}$$

Another example



An object of mass 24 kg is on the point of sliding down a rough inclined plane when pulled by a force of 10 kg wt at an angle of 30^o to the inclined plane. When the size of the force is increased to 12 kg wt, the object is on the point of sliding up.

Down the plane is negative. At point of sliding down, the frictional force opposes gravity and points up the plane. At point of sliding up, the frictional force opposes the pulling force and points down the plane. In both instances, the frictional force is maximum, i.e. $F_{fric} = \mu R$

Point of sliding down:

Perpendicular to the plane: $10\sin 30^o + R_1 = 24\cos\theta \Longrightarrow R_1 = 24\cos\theta - 5$

Parallel to the plane: $10\cos 30^o + \mu R_1 = 24\sin\theta \Longrightarrow 5\sqrt{3} + \mu(24\cos\theta - 5) = 24\sin\theta$

Point of sliding up:

Perpendicular to the plane: $12\sin 30^o + R_2 = 24\cos\theta \Longrightarrow R_2 = 24\cos\theta - 6$

Parallel to the plane: $12\cos 30^o = 24\sin\theta + \mu R_2 \Longrightarrow 6\sqrt{3} = 24\sin\theta + \mu(24\cos\theta - 6)$

$$11\sqrt{3} + 24\mu\cos\theta - 5\mu = 48\sin\theta + 24\mu\cos\theta - 6\mu$$

$$\mu = 48\sin\theta - 11\sqrt{3}$$

$$5\sqrt{3} + \left(48\sin\theta - 11\sqrt{3}\right)(24\cos\theta - 5) = 24\sin\theta$$

$$nSolve\left(5\sqrt{3} + \left(48\sin\theta - 11\sqrt{3}\right)(24\cos\theta - 5) = 24\sin\theta\,, \theta\right) | 0 \le \theta \le \frac{\pi}{2}$$

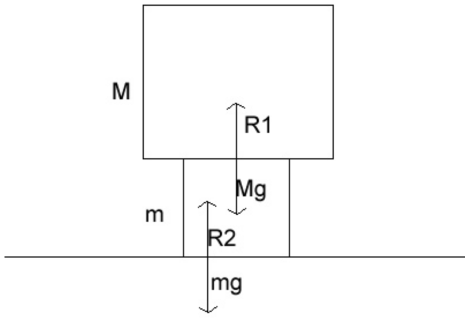
$$\theta \approx 23^o 27' \text{or } 76^o 41'$$

$$\mu \approx 0.05243 \; or \; 27.6552$$

2.1.7.2.3 Multiple Objects

2.1.7.2.3.1 Stacked objects

Boxes

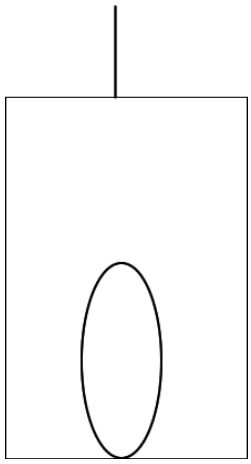


Forces acting on the upper object: Mg downwards, R_1 upwards.

Forces acting on the lower object: mg downwards, R_2 upwards, R_1 downwards (as R_1 is exerted on the upper mass by the lower mass, hence the lower mass must experience the equal and opposite force).

If the object is in equilibrium, then $R_2 = mg + R_1$. Hence, if there is an object below the lower mass, it will experience its own gravitational attraction and reaction force from ground, as well as the reaction force from above.

Lift



For a lift accelerating upwards:

$$T - mg = ma$$

For a lift accelerating downwards:

$$mg - T = ma$$

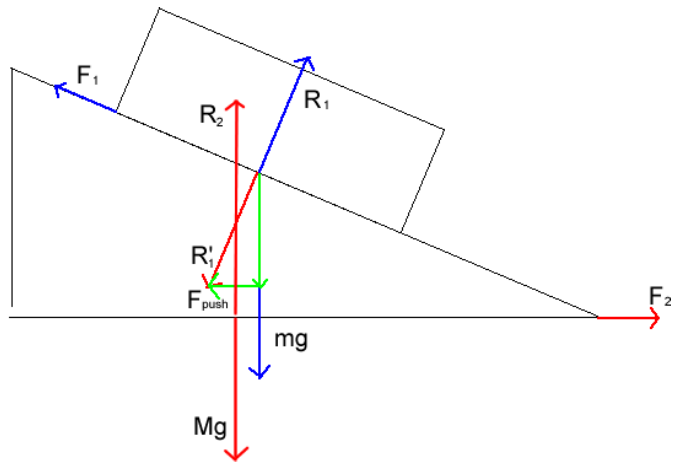
For a person inside the lift, $R - mg = ma$ or $mg - R = ma$, where N is the normal reaction force.

The weight force measured by a weight is the normal reaction force. I.e. When the lift is accelerating upwards, the weight increases, when the lift is accelerating downwards, the weight decreases.

For example, a lift is lowering. During deceleration, the person inside would experience minimum R, and the resultant force on the person would be downwards. During constant speed, R is equal and opposite to the gravitational attraction. During deceleration, the person would experience maximum R, and the resultant force would be upwards.

Moveable Wedge

For a moveable wedge on a surface and a block sitting on top, the block exerts a force perpendicular to the slant face, causing the wedge to move away. The block has a net acceleration towards the slant face as well as parallel to it.



In the diagram on the right, blue forces acts on the block, and red forces acts on the wedge, and the green forces are components of R_1', which are particularly important.

The block is accelerating downwards perpendicular to the slant face (downwards direction is negative).

$$F_{block-net} = ma_\perp = R_1 - mg\cos\theta$$

The block exerts a force on the wedge, R_1', which is equal and opposite to R_1.

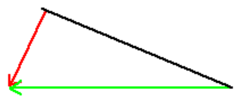
$$R_1' = -R_1 = -(F_{block-net} + mg\cos\theta)$$

Hence, the acceleration of the wedge to the left would be (left is negative)

$$F_{push} = R_1'\sin\theta$$

$$a_{push} = \frac{R_1'}{M}\sin\theta$$

Also, as the block accelerates perpendicularly to the slant face, it does so that it "keeps up" with the wedge which is moving away. i.e. the acceleration of the block is the component of the net acceleration of the wedge perpendicular to the slant face.



$$a_\perp = a_{push}\sin\theta$$

The normal reaction force exerted by the ground on the wedge, R_2, is

$$R_2 = Mg + R_1\cos\theta$$

For example, a 2kg smooth wedge is placed on a smooth table, and a smooth 1kg block is placed on the slant face.

$$a_{\perp} = R_1 - g\cos\theta$$

$$R_1' = -(a_{\perp} + g\cos\theta)$$

$$\begin{aligned} a_{push} &= -\frac{(a_{\perp} + g\cos\theta)}{2}\sin\theta \\ a_{push} &= -\frac{\left(a_{push}\sin\theta + g\cos\theta\right)}{2}\sin\theta \\ a_{push} + \frac{\sin^2\theta}{2}a_{push} &= -\frac{g\cos\theta\sin\theta}{2} \\ (2+\sin^2\theta)a_{push} &= -\frac{g}{2}\sin 2\theta \\ \left(\frac{5}{2} + \sin^2\theta - \frac{1}{2}\right)a_{push} &= -\frac{g}{2}\sin 2\theta \\ \left(\frac{5}{2} - \frac{1}{2}\cos 2\theta\right)a_{push} &= -\frac{g}{2}\sin 2\theta \\ a_{push} &= -\frac{g\sin 2\theta}{5-\cos 2\theta} \end{aligned}$$

Now the frictionless table is replaced by a rough surface. The minimum coefficient of friction would be if the system is now on the point of sliding.

$$R_2 = 2g + R_1\cos\theta$$

Since the wedge is not moving, the block has no net acceleration perpendicular to the slant face.

$$R_1 = g\cos\theta$$

$$\mu R_2 = R_1'\sin\theta$$

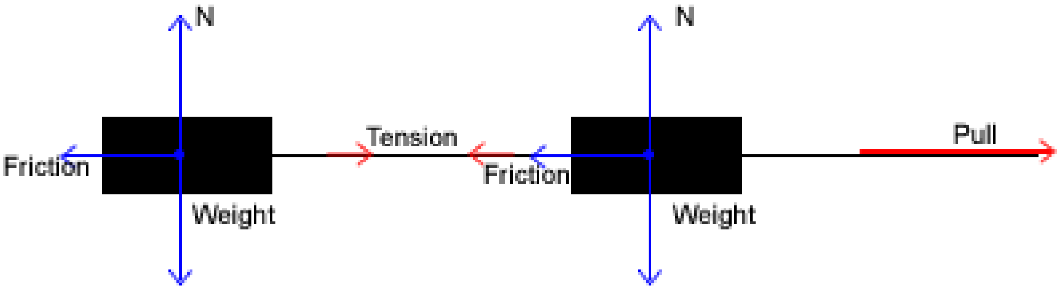
$$\mu\left(g(2+\cos^2\theta)\right) = g\cos\theta\sin\theta$$

$$\mu = \frac{\frac{1}{2}\sin 2\theta}{\frac{1}{2}\cos 2\theta + \frac{5}{2}} = \frac{\sin 2\theta}{\cos 2\theta + 5}$$

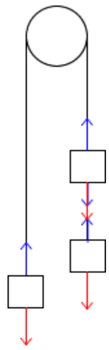
2.1.7.2.3.2 Connected Particles

In connected particles, the rope (inextensible) exerts an equal tension force on both objects connected to it.

Horizontal plane



Pulley



The pulley system moves towards the heavier side.

On the heavier side:

$$m_t a = m_t g - T_1$$

On the lighter side:

$$m_t a = T_1 - m_t g$$

If an object has two strings attached:

$$ma = T_1 - T_2 - mg \text{ or } ma = mg + T_2 - T_1$$

2.1.7.2.4 Vector Force

If a force is given as a vector, the force acts in the direction of the vector, and the magnitude of the force is the magnitude of the vector.

If several forces are involved, the resultant vector can be found.

2.1.7.2.5 Variable Force

If force is variable, then acceleration is not constant.

$$\text{If } F = f(x), a = \frac{1}{m}\big(f(x)\big)$$

For example, an object mass 3kg is projected vertically upwards with initial speed U m/s, and returns to its starting point with speed V m/s. Assume that air resistance is $\frac{gv^2}{20}$, where v is the speed of the object.

Upwards motion is positive.

Maximum height:

$$F_{net} = -mg - \frac{gv^2}{20} = -\frac{g(60+v^2)}{20}$$

$$a_{net} = -\frac{g(60+v^2)}{60}$$

$$v \cdot \frac{dv}{dx} = -\frac{g(60+v^2)}{60}$$

$$\frac{dx}{dv} = -\frac{60v}{g(60+v^2)}$$

$$x = \int_u^v -\frac{60v}{g(60+v^2)}\,dv = \frac{30}{g}\int_0^U \frac{2v}{60+v^2} = \frac{30}{g}\cdot\left[\ln|60+v^2|\right]_0^U = \frac{30}{g}\ln\left(\frac{60+U^2}{60}\right)$$

The time taken for the object to return to starting point from maximum height (friction is upwards [positive] this time)

$$F_{net} = \frac{gv^2}{20} - mg = \frac{g(v^2 - 60)}{20}$$

$$a_{net} = \frac{g(v^2 - 60)}{60}$$

$$\frac{dt}{dv} = \frac{60}{g(v^2 - 60)}$$

$$t = \int \frac{60}{g(v^2 - 60)} dv$$

$$= \frac{\sqrt{15}}{g} \int \frac{2\sqrt{60}}{\left(v + \sqrt{60}\right)\left(v - \sqrt{60}\right)} dv$$

$$= \frac{\sqrt{15}}{g} \int \frac{1}{v + \sqrt{60}} - \frac{1}{v - \sqrt{60}} dv$$

$$= \frac{\sqrt{15}}{g} \ln \left| \frac{v + \sqrt{60}}{v - \sqrt{60}} \right| + C$$

Given when t=0 (maximum height), v=0

$$C = -\frac{\sqrt{15}}{g} \ln 1 = 0$$

$$t = \frac{\sqrt{15}}{g} \ln \left| \frac{v + \sqrt{60}}{v - \sqrt{60}} \right|$$

When v=-V (when it reaches the ground)

$$t = \frac{\sqrt{15}}{g} \ln \left| \frac{-V + \sqrt{60}}{-V - \sqrt{60}} \right| = \frac{\sqrt{15}}{g} \ln \left| \frac{\sqrt{60} - V}{\sqrt{60} + V} \right|$$

Also,

$$a_{net} = \frac{g(v^2 - 60)}{60}$$

$$\frac{dx}{dv} = \frac{60v}{g(v^2 - 60)}$$

$$x = \int_0^{-V} \frac{60v}{g(v^2 - 60)} dv = \frac{30}{g} \int_0^{-V} \frac{2v}{v^2 - 60} dv = \frac{30}{g} \cdot [\ln|v^2 - 60|]_0^{-V} = \frac{30}{g} \ln\left(\frac{V^2 - 60}{60}\right)$$

$$x = -\frac{30}{g} \ln\left(\frac{60}{V^2 - 60}\right)$$

Note that the magnitude of the distance travelled is $\frac{30}{g} \ln\left(\frac{60}{V^2 - 60}\right)$, as the distance travelled is in the downwards direction, i.e. negative.

Since the distance travelled on the upwards journey is the same as the distance travelled on the downwards journey,

$$\frac{30}{g} \ln\left(\frac{60 + U^2}{60}\right) = \frac{30}{g} \ln\left(\frac{60}{V^2 - 60}\right)$$

$$\therefore \frac{60 + U^2}{60} = \frac{60}{V^2 - 60}$$

2.1.8 Sequences and Series

2.1.8.1 Sequence

A sequence is a list of numbers written in a definite order, usually obeying a particular rule.

$$\{a_n\} = \{a_1, a_2, a_3, \cdots\} = \{a_n\}_{n=1}^{\infty}$$

For alternating sequences, $(-1)^n$ is usually incorporated in its rule.

2.1.8.1.1 Limits of Sequences

A sequence may be defined as a function of natural numbers, and this function is a subset of the function over R.

The limit laws applies for limits of sequences, which may be evaluated simply.

Several key notes are:

For a sequence that does not converge at infinity, it is called "divergent" (usually an oscillating function). For a sequence that gets infinitely large or small, it is called "divergent to infinity".

A particular useful identity,

$$\lim_{n\to\infty} |a_n| = 0, \text{then } \lim_{n\to\infty} a_n = 0$$

Where an indeterminate form is encountered when evaluating a limit, L'Hopital's rule cannot be applied directly, but a change of variables may:

$$\text{If } f(n) = a_n \text{ where } n \text{ is an integer, and } \lim_{x\to\infty} f(x) = L \text{ for } x \in \mathbb{R}, \text{then } \lim_{n\to\infty} f(n) = L \text{ also.}$$

L'Hopital's rule can be applied for the limit over reals, but not over integers.

Also, for the series $a_n = r^n$,

$$\lim_{n\to\infty} r^n = f(x) = \begin{cases} 0, & -1 < r < 1 \\ 1, & r = 1 \\ \text{divergent}, & \text{elsewhere} \end{cases}$$

A sequence can be said to be increasing or decreasing. If a sequence is always increasing or decreasing, it is called "monotonic".

For a sequence that have an upper bound or lower bound, it is said to be "bounded above" or "bounded below", and in the case of both, "bounded".

Every bounded monotonic sequence is convergent.

2.1.8.2 Series

A series is the sum of infinite terms of a sequence.

$$S = \sum_{n=1}^{\infty} a_n$$

However, it is not always meaningful to discuss series as they may diverge. Hence partial sums are used to determine whether a series is convergent or not.

$$S_n = \sum_{i=1}^{n} a_i$$

If a series is convergent, then $\lim_{n\to\infty} S_n = S$ exists and is a finite number.

The geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

Is convergent if $|r| < 1, S = \frac{a}{1-r}$, and is otherwise divergent.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

In general, if a series is convergent, then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n > 0$, then its series is divergent.

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

2.1.8.2.1 Tests of Convergence

2.1.8.2.1.1 Integral Test

If f is a continuous, positive and decreasing function on $[1, \infty)$, and $a_n = f(n)$, then

If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is also divergent.

If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is also convergent.

Both of these can be shown graphically by constructing left/right rectangles (overestimation to show it is divergent, underestimation to show it is convergent).

Note that the lower bound is not necessarily 1, if the series is defined from n=k, then the integral would be computed from k.

When the first n terms are used to estimate the series (i.e. using a partial sum), the error made (called the "remainder", $s = s_n + R_n$) is bounded such that

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)\,dx$$

Also, $s_n + \int_{n+1}^\infty f(x)dx \le s \le s_n + \int_n^\infty f(x)\,dx$.This gives a better estimation than partial sums do. In this case, the error is half-way between the upper and lower bounds.

2.1.8.2.1.1.1 p-Series

The *p*-series $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and is divergent otherwise.

2.1.8.2.1.2 Comparison Test

The comparison test compares a given series with a series that is known to be convergent or divergent.

For two series $\sum a_n$ and $\sum b_n$

If $a_n \ge b_n$ for all n, and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

If $a_n \le b_n$ for all n, and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

If both sequences are positive terms and $\lim_{n\to\infty} a_n/b_n = c$, then either both series converges or both diverges.

This rest on the fact that there exists two numbers m and M such that $m < c < M$, hence $mb_n < a_n < Mb_n$. If $\sum b_n$ was convergent, the upper bound is finite hence $\sum a_n$ is also convergent. If $\sum b_n$ was divergent, the lower bound is infinite and hence $\sum a_n$ is also divergent.

If $\sum a_n$ was found to be divergent by comparison with $\sum b_n$, then $\sum a_n$ can be estimated by partial series and the error could be found by comparing remainders. Note that this only works if $a_n \le b_n$, and the remainder of $\sum a_n$ is less than the remainder of $\sum b_n$.

2.1.8.2.1.3 Alternating Series

For an alternating series $\sum \boldsymbol{a_n} = (-\mathbf{1})^{\boldsymbol{n}} \cdot \boldsymbol{b_n}$ or $(-\mathbf{1})^{\boldsymbol{n+1}} \cdot \boldsymbol{b_n}$, where b_n is a monotonic decreasing sequence that converges to zero, $\sum \boldsymbol{a_n}$ converges.

Or in other words, if $b_{n+1} \leq b_n$ for all n, and $\lim_{n\to\infty} b_n = 0$, then $\sum a_n$ converges.

Also, where the above conditions are met, the error (remainder) is bounded such that $|R_n| \leq b_{n+1}$.

2.1.8.2.1.4 Absolute Convergence and Ratio Test

For a series $\sum \boldsymbol{a_n}$, its absolute series is $\sum |\boldsymbol{a_n}|$. If $\sum |\boldsymbol{a_n}|$ is convergent, then $\sum \boldsymbol{a_n}$ is absolutely convergent.

All absolutely convergent series are convergent. A convergent series which is not absolutely convergent is called “conditionally convergent”.

The ratio test uses the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

- **If L<1, then the series $\sum \boldsymbol{a_n}$ is absolutely convergent.**
- **If L>1, then the series $\sum \boldsymbol{a_n}$ is divergent**
- **If L=1, then the ratio test is inconclusive**

However, it should be noted that this limit evaluates to 1 for all *p*-series, and hence all rational/algebraic functions of *n*.

Similarly, there is also a root test for exponential series, using $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

- **If L<1, then the series $\sum \boldsymbol{a_n}$ is absolutely convergent.**
- **If L>1, then the series $\sum \boldsymbol{a_n}$ is divergent**
- **If L=1, then the ratio test is inconclusive**

2.1.8.2.1.5 Mixed

- Evaluate the limit of the sequence first, it is not zero, it is divergent.
- *p*-series and geometric series are easily identifiable.
- For series similar to the *p*-series or geometric series, use the comparison test.
- For alternating series, use the alternating series test.
- If the series involves factorials or other products (or exponentials), try the ratio test or the root test.
- If the integral can be easily computed, then the integral test is effective.

2.1.8.2.2 Power Series

A power series is defined as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \cdots$$

Where c_n is a coefficient. The power series is similar to a polynomial, except that it has infinite number of terms. Where a=0 and c_n is a constant, the power series becomes a geometric series with x=r.

The power series can be convergent/divergent depending on the value(s) of x. In general, the three possibilities are:

- The series converges when x=a
- The series converges for all x
- The series converges for a range of value such that $|x-a| < R$

For the last case, *R* is called the "radius of convergence", and is equal to zero/infinity for the other two cases. The interval of convergence is the interval on x at which the power series converges.

Normally, the ratio test is used to find $|x-a| < R$, and other methods are used to find if the series is convergent when $|x-a| = R$.

Some rational function can be expressed as a power series using the formula for the geometric sequence.

For example,

$$\frac{x^3}{2+x^2} = x^3 \cdot \frac{1}{2} \cdot \left(\frac{1}{1-\left(-\frac{1}{2}x^2\right)}\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}x^2\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{2^{n+1}}$$

This series converges when $\left|-\frac{1}{2}x^2\right| < 1 \Rightarrow |x| < \sqrt{2}$

If the power series has a radius of convergence R>0, then the function f defined by the power series is differentiable (and continuous) on the interval of convergence.

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$

$$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int [c_n(x-a)^n]\, dx$$

For example,

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

Substituting when x=0, $\tan^{-1}(x)$=0,C=0

The radius of convergence of this series is the same as the radius of convergence of the power series for $\frac{1}{1+x^2}$, which is 1.

2.1.8.2.3 Maclaurin and Taylor Polynomials and Series

A Taylor series at a is the power series expansion of a function *f* at a. Assuming that *f* has derivatives of all orders,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{, where } f^{(n)} \text{ is the n}^{\text{th}}\text{derivative}$$

In particular, the Maclaurin series where *a*=0.

A Taylor polynomial T_n is the n^{th} partial sum of the power series, and R_n is the remainder. In general, if *f(x)* is the sum of a Taylor series, then

$$\lim_{n\to\infty} R_n(x) = 0$$

Also, if $\left|f^{(n+1)}(x)\right| \leq M$ for the interval $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

This is called Taylor's Inequality, and is often used to prove the above to show that the Taylor series is equal to the function. In doing so, the following fact is also helpful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Hence $\lim_{n\to\infty} R_n(x) = 0$ is true if there exists an upper-bound *M* such that $\left|f^{(n+1)}(x)\right| \leq M$.

For example, find the Maclaurin series for $\sin x$, and prove that it represents $\sin x$ for all *x*.

$$f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1, f''(0) = -\sin 0 = 0, f'''(0) = -\cos 0 = -1$$

$$f^{(4)}(0) = \sin 0 = 0 \cdots$$

The pattern repeats itself in a cycle, and its derivatives are all sine and cosine functions, hence, its n-th derivative is bounded by 1

$$|R_n(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

Hence R_n is converges to zero, and $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, its Maclaurin series.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Some important Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Multiplication and division of Taylor series can be done using long multiplication and long division.

2.2Vector Calculus

2.2.1 Space Curve and Continuity

A vector space curve is a curve defined in 3D space by a vector equation.

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

A vector is continuous if and only if:

$$\boldsymbol{r}(a) = \lim_{t \to a} r(t)$$

This implies that for the vector space curve to be continuous, all three functions x, y and z must be continuous.

2.2.1.1 Vector Function and Paths

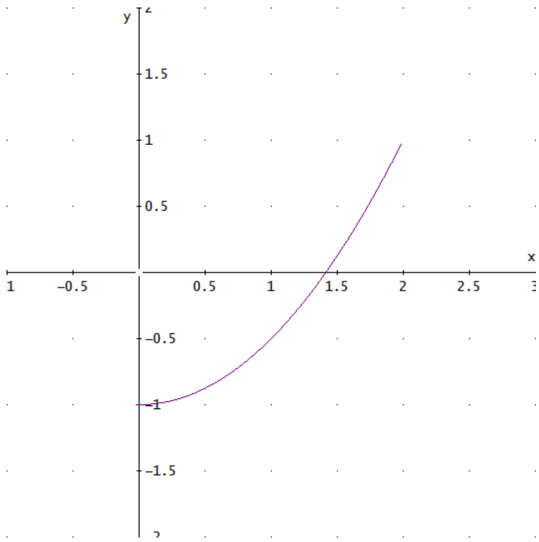
Using various identities or substitutions to equate the x and y terms, a Cartesian equation can be found from the vector function. If the vector function is defined in terms of another variable, then x and y are bounded if that variable is bounded.

For example, $\check{r}(t) = 2\sin\left(\frac{\pi t}{2}\right)\tilde{\imath} - \cos(\pi t)\,\tilde{\jmath}$ for $0 \le t \le \frac{3}{2}$

$$-y = \cos \pi t = 1 - 2\sin^2\left(\frac{\pi t}{2}\right) = 1 - \frac{x^2}{2}$$

$$y = \frac{x^2}{2} - 1$$

$$0 \le t \le \frac{3}{2} \Longrightarrow 0 \le x \le 2, -1 \le y \le 1$$



2.2.2 Derivative

$$\frac{d\boldsymbol{r}}{dt} = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(h)}{h} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

A smooth function's derivative must not be a zero vector at any point.

2.2.3 Vector Tangent

The vector tangent is defined as the unit vector in the direction of the space curve.

$$\overline{\boldsymbol{T}} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

Properties of the vector tangent:

$$|\overline{\boldsymbol{T}}| = 1$$

$$\overline{\boldsymbol{T}} \cdot \frac{d\overline{\boldsymbol{T}}}{dt} = 0$$

$$\left|\overline{\boldsymbol{T}} \times \frac{d\overline{\boldsymbol{T}}}{dt}\right| = \left|\frac{d\overline{\boldsymbol{T}}}{dt}\right|$$

$$\overline{\boldsymbol{T}} \times \overline{\boldsymbol{T}} = 0$$

2.2.4 Curvature

$$\kappa = \left|\frac{d\overline{\boldsymbol{T}}}{ds}\right| = \frac{|\bar{r}'(t) \times \bar{r}''(t)|}{|\bar{r}'(t)|^3}$$

$$\rho = \frac{1}{\kappa}$$

κ is a measure of curvature (the higher it is, the more curved the space curve is).
ρ is the radius of curvature.

2.2.5 Normal and Bi-Normal

$$\overline{\boldsymbol{N}} = \frac{\frac{d\overline{\boldsymbol{T}}}{dt}}{\left|\frac{d\overline{\boldsymbol{T}}}{dt}\right|}$$

$$\overline{\boldsymbol{B}} = \overline{\boldsymbol{T}} \times \overline{\boldsymbol{N}}$$

2.3 Multivariable Calculus

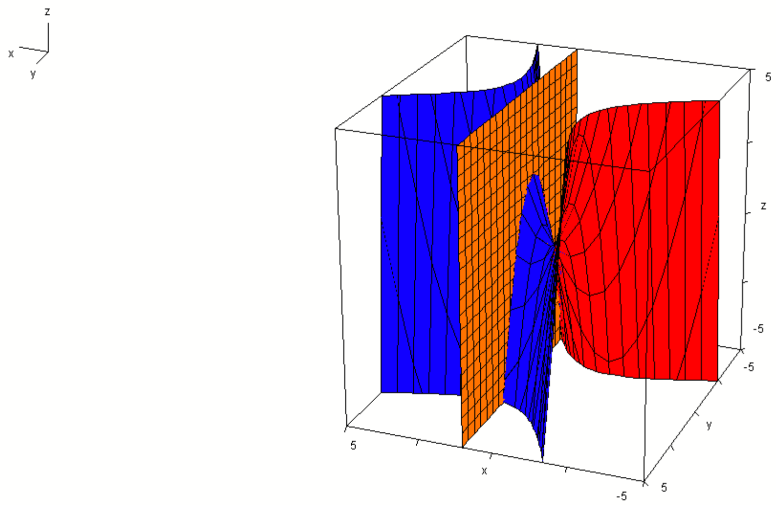
2.3.1 Continuity

A function f of two variables is continuous at (a.b) if:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

2.3.2 Partial Derivatives

When finding a partial derivative with respect to one variable, the other variable is treated as a constant.



$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = D_x f(x,y)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = D_y f(x,y)$$

2.3.2.1 *Tabular Data Interpretation*

Primarily an approximation:

$$f_x(x,y) \approx \frac{\Delta f}{\Delta x} \approx \frac{f(x+a,y) - f(x,y)}{a}$$

$$f_y(x,y) \approx \frac{\Delta f}{\Delta y} \approx \frac{f(x,y+a) - f(x,y)}{a}$$

2.3.2.2 Implicit Derivation of Partial Derivatives

Where $z = f(x, y)$ is stated implicitly as $g(x, y, z) = c$, the partial derivatives can be found implicitly.

To find $\frac{\partial z}{\partial x}$, holding y constant, z can be expressed as $z = f(x)$:

$$\frac{\partial}{\partial x} g(x, y, z) = \frac{\partial}{\partial x} g(x, y, z) + \frac{\partial}{\partial z} g(x, y, z) \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial z}\right)}$$

This is the technique of implicit differentiation (discussed later).

For example:

$$x^3 z - xy\sqrt{z} = 5$$

$$\frac{\partial}{\partial x}\left(x^3 z - xy\sqrt{z}\right) = \frac{\partial}{\partial x}(5)$$

Remembering that z is a function of x, the product rule must be used for both terms:

$$2x^3 z + x^3 \frac{\partial z}{\partial x} - \left(y\sqrt{z} + \frac{xy}{2\sqrt{z}}\frac{\partial z}{\partial x}\right) = 0$$

$$\left(x^3 + \frac{xy}{2\sqrt{z}}\right)\frac{\partial z}{\partial x} = y\sqrt{z} - 2x^3 z$$

$$\frac{\partial z}{\partial x} = \frac{2\sqrt{z}\left(y\sqrt{z} - 2x^3 z\right)}{2x^3\sqrt{z} + xy}$$

2.3.2.3 Higher Partial Derivatives

The first derivative may be partially differentiated further to find higher partial derivatives. The implication is that the first partial derivative with respect to x can be differentiated again with respect to either x **or** y, hence for a function of 2 variables, there are 4 partial derivatives, $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

2.3.2.3.1 Clairaut's Theorem

If f is a function for two variables and is continuous on a disk D containing (a,b), then f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

2.3.2.4 Partial Differential Equations

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

One solution is $u(x, y) = e^x \sin y$.

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

One such solution is $u(x, t) = \sin(x - at)$

2.3.3 Tangent planes

A tangent plane to z=f(x,y) at P(x_0,y_0,z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

2.3.3.1 Linear Approximations

For (x,y) close to (a,b):

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

2.3.3.2 Differentials

For (x,y) close to (a,b):

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy$$

2.3.4 Chain rule

For a function of several variables where each variable is a function of other variable(s):

Let z=f(x,y) and x=g(s,t), y=h(s,t)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

For functions with more variables, the chain rule can be generalised:

Let f(x,y,z,…) be a function, where x=g(s,t), y=h(s,t), z=j(s,t), …

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} \cdots$$

2.3.4.1 Implicit Differentiation Using Partial Derivatives

Let F(x,y)=0 define y=f(x) implicitly

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

2.3.5 Directional Derivatives

For partial derivatives in the direction of a particular unit vector $\hat{\boldsymbol{u}}=\langle a,b\rangle$

$$D_u f(x,y)=\lim_{h\to 0}\frac{f(x+ha,y+hb)-f(x,y)}{h}=f_x(x,y)a+f_y(x,y)b$$

2.3.5.1 Gradient Vector

$$\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle$$

Hence the directional derivative can be expressed as:

$$D_u f(x,y)=\nabla f(x,y)\cdot\hat{\boldsymbol{u}}$$

2.3.5.1.1 Direction

$$D_u f(x,y)=|\nabla f(x,y)|\cdot 1\cdot\cos\theta$$

Hence, the directional derivative is at maximum when the angle between the two vectors is zero.

Hence, the directional derivative points in the direction (on the x-y plane) of maximum rate of change.

For level curves $f(x,y)=c$, the gradient vector points perpendicularly to the level curves in the uphill direction.

2.3.5.1.2 Magnitude

The maximum directional derivative is the magnitude of the gradient vector.

2.3.5.2 Tangent Plane to Level Surface

For a level surface $F(x,y,z)=k$, the equation of the tangent plane is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

Or, for the case $z=f(x,y)$, it can be re-expressed as $F(x,y,z)=f(x,y)-z=0$, where $F_x=f_x,\ F_y=f_y,\ F_z=-1$

2.3.5.2.1 Normal Line

The normal line through origin perpendicular to the tangent plane has the symmetric equation:

$$t=\frac{x-x_0}{F_x}=\frac{y-y_0}{F_y}=\frac{z-z_0}{F_z}$$

2.3.6 Critical Points of a Surface

A critical point of a surface $z = f(x, y)$ occurs when both partial derivatives f_x and f_y are zero.

2.3.6.1 Second Derivative Determinant

The nature of critical points can be found by finding the following determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

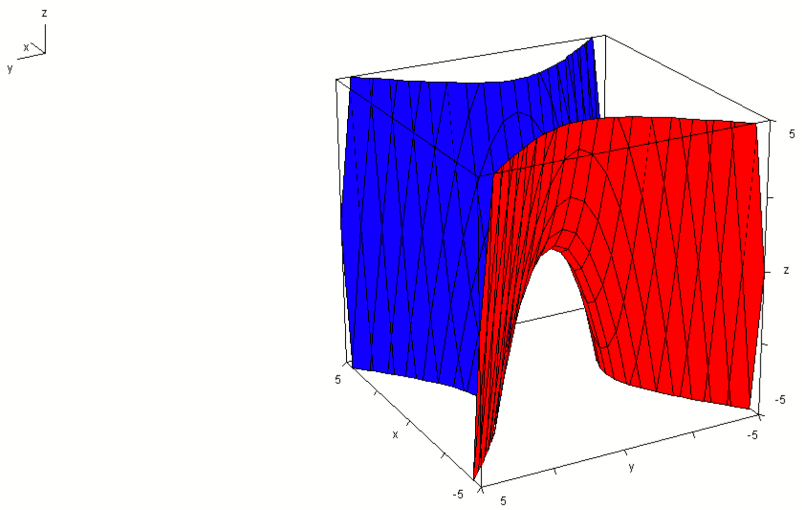
If D is positive and f_{xx} is positive, the critical point is a local minimum.

If D is positive and f_{xx} is negative, the critical point is a local maximum.

If D is negative, the critical point is a saddle point (not a maximum nor minimum).

If D is zero, the test is inconclusive.

Note that for positive D values, where f is defined in the real plane, f_{xx} cannot be zero, as that will imply $(f_{xy})^2$ is negative, hence f_{xy} is a complex number.



2.3.6.2 Absolute Extremum

For a function $f(x, y)$ defined on a closed set D (in two dimensions), the absolute extrema is either at a critical point within D, or on the edge of D. It is necessary to find the relationship between x and y on the boundary of D to find the absolute maximum and the absolute minimum.

I.e. if D is a rectangle (0,0)→(5,0)→(5,3)→(0,3), then it is necessary to find critical points of *f*, the maximum and minimum of *f* when $x = 0, x = 3, y = 0, y = 5$ and by comparison, find the absolute maximum and minimum.