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Math 215A: Algebraic Topology

Homework 10

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Question 1: Prove that the cap-product between $H_{m+n}(X, A \cup B; R)$ and $H^m(X, B; R)$ is well-defined to take values in $H_n(X, A; R)$ and the cup-product between $H^m(X, A; R)$ and $H^n(X, B; R)$ is well-defined to take values in $H^{m+n}(X, A \cup B; R)$ *provided that* A and B are deformational retracts of their neighborhoods U_A and U_B inside $A \cup B$.

Solution: (I read through Allen Hatcher's Algebraic Topology Sections 2.2, 3.3 for this question)

We consider spaces A, B and $U_A, U_B \subseteq A \cup B$ so that U_A, U_B deformation retract onto A, B respectively. There are two things we want to prove:

(a) The Cap-Product $H_{m+n}(X, A \cup B; R) \smile H^m(X, B; \mathbb{R})$ is well-defined to take values in $H_n(X, A; R)$, and

(b) The Cup-Product $H^m(X, A; R) \frown H^n(X, B; R)$ is well-defined to take values in $H^{m+n}(X, A \cup B; R)$

I'm going to swap the order.

(a) **Cup-product**:

Let's first recall the definitions of Relative Cohomology groups. To define $H^n(X, A; G)$ for a pair (X, A) we take the exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

and dualize it by applying $\mathrm{Hom}(-, G)$ to get

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0$$

where $C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G)$. This sequence is exact for the following reasons:

The map i^* restricts cochains on X to cochains on A. So for a function from the singular n-simplices on X to G, the image of the function under i^* is exactly the same function - just with its domain restricted to A rather than X. Any functions from singular $n-$simplices on A to G can be extended to a function from the singular $n-$simplices on X to G - eg. just by assigning value 0 to those simplices not in A. Thus, i^* is surjective. So the composition of i^* with the zero map sending $C^n(A;G) \mapsto 0$ is exact.

Now, the kernel of i^* is all those cochains on X which take value 0 on singular $n-$simplices in A. Thus, such cochains can be thought of as homomorphisms

$$C_n(X, A) = C_n(X)/C_n(A) \to G$$

so the kernel is exactly $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G)$, giving us exactness of the entire sequence.

The main thing to remember from this discuss is that **$C^n(X, A; G)$ can be viewed as the functions from singular $n-$simplices on X which vanish on simplices in A.**

Let's also quickly recall the absolute versions:

The <u>absolute</u> cup-product on cochains $\varphi \in C^m(X;G)$ and $\psi \in C^n(X;G)$ is defined to be the cochain $\varphi \smile \psi \in C^{m+n}(X;G)$ which acts on $\sigma : \Delta^{k+l} \to X$ as

$$(\varphi \smile \psi)(\sigma) = \varphi([v_0, \cdots, v_k])\, \psi([v_{k+1}, \cdots, v_{k+l}])$$

The coboundary map δ acts on the cup-product as

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi)$$

From this formula, we can tell that

- the cup-product of two cocycles is again a cocycle.
- the cup-product a cocycle and a coboundary (in either order) is again a coboundary.

Thus, there is an induced Cup-Product on the Cohomology groups

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R)$$

The relative cup product is obtained by noticing that the absolute cup-product on the cochains

$$C^m(X;R) \times C^n(X;R) \xrightarrow{\smile} C^{m+n}(X;R)$$

restricts to a cup-product

$$C^m(X, A;R) \times C^n(X, B;R) \xrightarrow{\smile} C^{m+n}(X, A+B;R)$$

where $C^{m+n}(X, A+B; R)$ is the subgroup of $C^{m+n}(X; R)$ consisting of cochains vanishing on sums of chains in A and B. The inclusions $C^n(X, A\cup B; R) \hookrightarrow C^n(X, A+B; R)$ induce isomorphisms on the cohomology via the Five-lemma and the fact that the restriction maps $C^n(A\cup B; R) \hookrightarrow C^n(A+B; R)$ induce isomorphismss on Cohomology via excision.

Therefore the cup-product

$$C^m(X, A; R) \times C^n(X, B; R) \xrightarrow{\smile} C^{m+n}(X, A+B; R)$$

induces the relative cup-product of the cohomology groups

$$H^m(X, A; R) \times H^n(X, B; R) \xrightarrow{\smile} H^{m+n}(X, A\cup B; R)$$

(b) **Cap-Product**: Again, let's recall that the cap-product for absolute chains and cochains is defined as follows: For Arbitrary space X and coefficient ring R, we have

$$\frown \; : C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$$

where $k \geq l$ defined as

$$\sigma \frown \varphi = \varphi\left(\sigma\big|_{[v_0,\cdots,v_l]}\right)\sigma\big|_{[v_l,\cdots,v_k]}$$

form $\sigma : \Delta^k \to X$ and $\varphi \in C^l(X; R)$

This induces a cap-product in Homology and Cohomolgy due to the formula

$$\partial\left(\sigma \frown \varphi\right) = (-1)^l\left(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi\right)$$

in the sense that this formula tells us

- The cap-product of a cycle σ and a cocycle φ is a cycle.
- The cap-product of a cycle and a coboundary is a coboundary.
- The cap-product of a boundary and a cocycle is a boundary.

These facts induce the cap-product

$$H_k(X; R) \times H^l(X; R) \xrightarrow{\frown} H_l(X; R)$$

which is $R-$linear in each variable.

The Relative Cap-Product

$$H_k(X, A\cup B; R) \times H^l(X, B; R) \xrightarrow{\frown} H_l(X, A; R)$$

is defined when A, B are open subsets of X and using the fact that $H_k(X, A\cup B; R)$ can be computed using the chain groups

$$C_n(X, A+B; R) = C_n(X; R)/C_n(A+B; R)$$

Our question is slightly different, however. We have A and B not necessarily being open subsets, but we have open neighborhoods $U_A, U_B \subseteq U$ which deformation retract onto A, B

respectively.

What we've described above works perfectly well for U_A, U_B. So, we need to show isomorphism between the relative homology groups using U_A, U_B and the relative homology groups using A, B.

The quotient map
$$C_\bullet(X)/[C_\bullet(A+B)] \to C_\bullet(X)/C_\bullet(A \cup B)$$
induces isomorphism between the homology groups.

Question 2: Call a degree-n integer homology class $[M]$ of a closed oriented $n-$dimensional manifold M the ***fundamental class*** if for every point $x \in m$ the projection of this class from $H_n(M)$ to $H_n(M, M - \{x\}) = H_n(\mathbb{S}^n) = \mathbb{Z}$ equals 1. Embed M into an oriented sphere $\mathbb{S}^N$ and let U be a tubular neighborhood of M in $\mathbb{S}^N$ considered as a disk bundle over M. Show that the composition of the natural map from $H_N(\mathbb{S}^N) = \mathbb{Z}$ to $H_N(\mathbb{S}^N, \mathbb{S}^N - U)$ $(= H_N(U, \partial U)$ by excision) while the Thom Isomorphism between $H_N(U, \partial U)$ and $H_n(M)$ maps $[\mathbb{S}^N]$ to $[M]$ (thus proving the existence of the latter).

<u>Solution:</u>

text

Question 3: Use Morse Theory to show that a Morse function on $\mathbb{RP}^n$ has at least $n+1$ critical points. Give an example of a Morse function on $\mathbb{RP}^n$ with exactly $n+1$ critical points, and find critical values and Morse indices of the critical points in your example.

Solution: (Inspired by these Morse Theory notes ; Missed lecture this week but I heard the Weak Morse Inequality was covered in class)

Recall that for a smooth manifold M, a smooth function $f \in C^\infty(M)$, $f : M \to \mathbb{R}$ is said to be a *Morse Function* if it has no degenerate critical points on M.

Theorem 0.1. ***(Weak Morse Inequalities:)*** *Let f be a Morse function on a manifold M. Let N_k denote the number of index k critical points of f. Then,*

$$N_k \geq b_k(M)$$

where $b_k(M)$ is the k^{th} Betti Number.

Theorem 0.2. ***(Corollary of the Weak Morse Inequality):*** *Let f be a Morse function on M. Then, f has at least as many critical points as the sum of the ranks of the homology groups on M (i.e. the betti numbers).*

Recall that the homology groups of projective spaces are slightly different for n odd and n even:

n odd:

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, \ i = 0, n \\ \mathbb{Z}/2\mathbb{Z}, \ i \text{ odd}, 1 \leq i \leq n-1 \\ 0, \ \text{otherwise} \end{cases}$$

n even:

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, \ i = 0 \\ \mathbb{Z}/2\mathbb{Z}, \ i \text{ odd}, 1 \leq i \leq n-1 \\ 0, \ \text{otherwise} \end{cases}$$

Namely, the top-homology group vanishes when n is even, so the sum of the ranks of the homology groups for n is less that that of n odd.

Now, for n odd, we have the betti numbers of the homology groups

$$b_i = \begin{cases} 1, & 0 \leq 1 \leq n \\ 0, & \text{otherwise} \end{cases}$$

Thus, any Morse function on $\mathbb{RP}^n$ has at least

$$\sum_i b_i = n + 1$$

critical points.

A Morse function $f : \mathbb{RP}^n \to \mathbb{R}$ with exactly $n + 1$ critical points is

$$f(x) = \sum_{i=1}^{n+1} i \cdot |x|^2$$