Physics 198 Notes

Keshav Balwant Deoskar

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These are notes taken from lectures on Topology, Geometry, and Algebra delivered by EVibhu Ravindran, Pablo Castano, and Michelle Dong for UC Berekley's Physics 198 class (DeCal) in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 January 23 - Classical Mechanics Review, Linear Algebra

Classical mechanics is the field of physics everyone first encounters, and usually this only extends to **Newtonian Mechanics**.

Newtonian Mechanics

$F = ma$

$E = K + V$

Newtonian Mechanics is powerful, but it is a local theory. One has to use the second law at each point in time to study the behavior of a system. For some problems, it's easier to work with the equivalent, but **global** formulations of classical mechanics, which are **Lagrangian** and **Hamiltonian** mechanics.

Lagrangian

$L = K - V$

S.H.O. $= \dfrac{1}{2}m\dot{x}^2 - \dfrac{1}{2}kx^2$

Action: $S = \displaystyle\int dtL = \int dtd^3x\mathcal{L}$

Euler Lagrange Equations: $\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial(\partial_t x)}\right) = 0$

$\vdots$

*** Show equivalence of Euler-Lagrange Equations and Netwon's 2nd Law.

Hamiltonian Mechanics has the same global structure as Lagrangian Mechanics, but is useful because it makes it easier to spot symmetries and conserved quantities. The Hamiltonian H is obtained from the Lagrangian L by a **Legendre Transformation**.

Hamiltonian

Write equations later.

*** Be sure to explain conjugate momentum, phase/configuration space, Hamiltons Equations.

1.1 Linear Algebra

Linearly Algebra is a remarkably powerful branch of math because of how many different systems can be modelled using vector spaces. Familiar examples of vector spaces are $\mathbb{R}$, $\mathbb{C}$, and in fact $\mathbb{R}^n$ and $\mathbb{C}^n$, but we'll see some wild examples later.

1.1.1 Vector Spaces

Vector Space: A set V with some sense of addition and multiplication is said to be a vector space if its operations satify

- **Commutativity:** $u + v = v + u, \forall u, v \in V$
- **Associativity:** $(u + v) + w = u + (v + w), \forall u, v, w \in V$
- **Additive Identity:**
- **Additive Inverse:**
- write the rest of the vector space properties.

1.1.2 Subspaces

A subset U of V is called a **subspace** of V if U is itself a vector space.

1.1.3 Linear Map

Given Definition

Some spaces associated with a Linear Map are the **Kernal** and **Image**.

For a linear map $T : V \to W$,

- The **Kernal of T** is defined as

$$\ker(T) = \{v \in V : Tv = 0\}$$

- The **Image of T** is defined as

$$\mathrm{Im}(T) = \{w \in W : w = Tv \text{ for some } v \in V\}$$

[Include diagram or something to make this not dry]

1.1.4 Injectivity, Surjectivity, etc.

Write about when a linear map is injective, surjective etc.

1.1.5 Linear Functionals, Inner Products

Talk about linear functionals (aka dual vectors aka covectors) and inner Products

Exercise: Find an example of an infinite dimensional vector space which is not isomorphic to its dual.

1.1.6 Spectral Theorem for Self-Adjoint Operators

1.1.7 Tensors

An $(r, s)-$Tensor T is a multi-linear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to \mathbb{F}$$

2 January 25 - Algebra, Group Theory

2.1 Revisiting vector spaces, duals, and tensors

Last time, we defined the set of linear maps between vector spaces V, W and denoted it as $\mathcal{L}(V, W)$.

The set of linear maps $\mathcal{L}(V, W)$ is itself a vector space with addition and scalar multiplication defined as :

$$\begin{aligned} Addition &: (T + S)(v) := T(v) + S(v) \\ Multiplication &: (\lambda T)(v) := \lambda \cdot (T(v)) \end{aligned}$$

where $T, S \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}, v \in V$.

Note: If V is a vector space over the field $\mathbb{F}$, another name for $\mathcal{L}(V, \mathbb{F})$ is the **Dual Space, $\mathbf{V}^\star$**. Also, last time we saw the formal definition of a tensor, but we didn't see a concete example.

Example: For an example of a physical tensor, we can consider the Lagrangian of a free (classical) particle:

$$\begin{aligned} L &= T - V \\ &= \sum_i \frac{1}{2} m \dot{x}^2 \\ &= \sum_i \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j \end{aligned}$$

and this describes the motion in flat space. However, in curved space (eg. space-time) we can take the influence of the curvature into account by replacing δ_{ij} with g_{ij} where g_{ij} depends only on tim i.e.

$$\boxed{L = \sum_i \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j}$$

Solving the Euler-Lagrange equations, we have

$$\begin{aligned} \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) &= 0 \\ \implies \frac{1}{2} m \frac{\partial g_{ik}}{\partial x^i} x^j x^k - \frac{d}{dt}\left(m g_{ij} x^i x^j\right) &= 0 \\ \implies \text{Complete this example later} \end{aligned}$$

2.2 Group Theory

Now that we've covered that stuff, we move onto some **group theory**. This branch of math is essentially the study of **symmetries** i.e. transformations that leave a system unchanged or **invariant**.

Formal definition of a Group

A group is a pair $(G, \star)$, where G is a set and $\star : G \to G$ is a *bilinear operation*, satifying three properties. Namely,

(a) Associativity: $a \star (b \star c) = (a \star b) \star c$

(b) Identity: There exsits an element $e \in G$ such that, for all $g \in G$,

$$g \star e = e \star g = g$$

(c) Inverses: For each $a \in G$, there must exist an *inverse* element denoted a^{-1} such that

$$a \star a^{-1} = a^{-1} \star a$$

For example,

- Any vector space V with vector addition being the bilinear product $\star$ is a group.
- The set of permutations of three objects, called the **Symmetric group of order 3**, denoted S_3 is a group.

Note that in a vector space, $v \star w = w \star v$. This is a very special property called **commutativity** i.e. the order in which we operate group elements does not matter. Such a group is said to be **commutative** or **abelian**. In constrast to this, the symmetric group of order 3, S_3 is **non-abelian** (in fact, it's the smallest such group!)

Note: The number of elements in a group G is called its *order* and is denoted by $|G|$.

2.2.1 Subgroup

Subgroup

- For a group G, a subset $H \subset G$ is called a subgroup if
  (a) $e \in H$
  (b) $a, b \in H \implies ab \in H$ (this is called *closure*)
  (c) $a \in H \implies a^{-1} \in H$
- If H is a subgroup of G, we write $H \leq G$.

2.2.2 Coset

Coset

- For a group G and subgroup $H \subset G$, we can consider an element $g \in G, g \notin H$ and define the *left coset of* H to be
$$gH := \{gh : h \in H\}$$
- There is a one to one correspondence between a subgroup H and any coset of H.

2.2.3 Lagrange's Theorem

Theorem: If G is a finite group and $H \subset G$ is a subgroup, then H divides G.

Proof Sketch: Let's define the equvalence relation $g_1 \sim g_2$ iff $g_1^{-1}g_2 \in H$. Suppose
$$g_2H = g_1H$$
Then
$$g_1^{-1}g_2H = H$$
Any two distinct cosets are disjoint, thus all cosets are of equal order since they are disjoint equivalence classes. So,
$$\begin{aligned} |G| &= |g_1H| + |g_2H| + c \ldots && (1) \\ &= \underbrace{k}_{\#of cosets} \cdot |H| && (2) \end{aligned}$$
Thus, $|H|$ divides $|G|$.

2.2.4 Normal subgroups

Some subgroups have the very special propert that their *left and right cosets are equal.* This is unusual since group element multiplication is not generally guaranteed to be commutative.

Normal Subgroup

A subset of a group G is said to be a **normal subgroup** if

- It is a subgroup
- $gN = Ng$

A subgroup is denoted as $N \trianglelefteq G$.

Normal subgroups allow us to produce a very important class of groups called *Quotient Groups.*

2.2.5 Quotient Groups

Quotient Group

Given a group G and normal subgroup $N \trianglelefteq G$, we can define the **quotient group** as

$$G/N := \{[a] = aN : a \in G\}$$

<u>Exercise:</u> Verify that the bilinear product on the quotient group

$$[a] \star [b] = [ab]$$

is **well defined** i.e. does not depend on the specific elements a, b chosen to represent the equivalence classes $[a], [b]$.

2.3 Group Homomorphisms

So far we've spoken about a group and its subgroups, all stuff that was relatively self-contained. However, in math, we often want to study *maps between objects.* The "natural" map between groups is called a *homomorphism.*

Now, the numbers 1 to 10 in english are "One", "Two", ...,"Ten" whereas in say Spanish they are "Unos", "Dos", ..., "Diez". Furthermore, to convey the idea of adding numbers in English, we can say "One **plus** Two" whereas the same idea in Spanish would be "Uno más Dos".

Clearly the numbers 1 to 10 are the same regardless of the names we use to describe them in different languages, and similarly for the process of adding them (which is a *really* just a bilinear product). So, there is a sort of mapping between the two.

Drawing from this (imperfect) analogy, a homomorphism between two groups is a like a dictionary, which maps the words between the two languages and allows us to translate.

Group Homomorphisms and Isomorphisms

- For groups $(G, \cdot)$ and $(H, \star)$, a map $\phi : G \to H$ is a homeomorphism if

$$\phi(a \cdot b) = \phi(a) \star \phi(b)$$

- Further, if the map if *bijective* then it is called an **Isomorphism.** We then say G and H are isomorphic, denoted as $G \cong H$.

- For example, a group G is isomorphic to *itself*. The map $G \to G$ is then called an **automorphism**
- An **inner automorphism** is of the form

$$\phi_h(g) = h^{-1}gh$$

i.e. it conjugates each element $g \in G$ with respect to some particular $h \in G$.

Group Homomorphisms are *structure preserving*.

2.4 Kernel, Image, and the First Isomorphism Theorem:

Similar to vector spaces, we define the *Kernel* and *image* of a map $\phi : G \to H$ to be

$$\begin{aligned} \mathrm{Ker}(\phi) &:= \{g \in G : \phi(g) = e\} \\ \mathrm{Im}(\phi) &:= \{h \in H : h = \phi(g), g \in G\} \end{aligned}$$

Then, the first isomorphism theorem is as follows:

First Isomorphism Theorem: For groups G, H and map $\phi : G \to H$

(a) $\mathrm{Ker}(\phi)$ is a normal subgroup of G i.e. $\mathrm{Ker}(\phi) \trianglelefteq G$.

(b) $\mathrm{Im}(\phi)$ is a subgroup of H i.e. $\mathrm{Im}(\phi) \leq H$.

(c) The quotient of G by $\ker(\phi)$ is isomorphic to $\mathrm{Im}(\phi)$ i.e.

$$G/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$$

2.5 Fundamental Group

Consider a donut shaped surface called a *Torus*, denoted T, and some point $p \in T$ [Complete this later].

3 January 30 - Point set Topology

3.1 What is a Topological Space?

Topologcal Space

- A topology is a pair $(M, \mathcal{O})$ where M is some arbitrary set and $\mathcal{O}$ is a collection of subsets of M i.e. $\mathcal{O} \subset \mathcal{P}(M)$ such that
  - $\emptyset, M \in \mathcal{O}$
  - Finite intersections of sets already in $\mathcal{O}$ are also in $\mathcal{O}$: $\{U, V\} \subset \mathcal{O} \implies \bigcap\{U, V\} \subset \mathcal{O}$
  - Arbitrary unions of sets already in $\mathcal{O}$ are also in $\mathcal{O}$:$C \subseteq \mathcal{O} \implies \bigcup C \subseteq \mathcal{O}$

Examples:

- Let M be a set. Then, $\mathcal{O} = \{\emptyset, M\}$ is called the *chaotic topology* on M.
- Let $\mathcal{O} = \mathcal{P}(M)$ is called the *discrete topology* on M.

More goofy ahh adjectives for topologies:

Let M be a set with two topologies τ_1 and τ_2.

- If $\tau_1 \subset \tau_2$ then τ_1 is weaker than τ_2, and equvalently, τ_2 is stronger than τ_1.

3.2 Openness and Closedness

You may have heard of open sets in $\mathbb{R}$, or even $\mathbb{R}^n$. Well, these open sets are actually just the members of the *standard topology on* $\mathbb{R}^n$.

In general, ”open” just means ”member of topology”.

Open and closed sets

Let $(M, \mathcal{O})$ be a topological space.

- A subset $S \subseteq M$ is ***open*** (with respect to $\mathcal{O}$) if $S \in \mathcal{O}$.
- A subset $U \subseteq M$ is ***closed*** (with respect to $\mathcal{O}$) if $M \setminus U$ is open.

Note that ”open” does not = ”closed”! A set can be *both* open and closed. If we find sufficiently weird spaces, we can even find subsets which are *neither* open or closed! [Include some examples later].

Open balls in $\mathbb{R}^d$

For instance, the real numbers with the usual metric

$$d(x, y) = \sqrt{\sum (y_i - x_i)^2}$$

we can define ***open ball centered around*** x of ***radius*** r to be

$$\boxed{B_r(x) := \{y \in \mathbb{R}^d \ : \ d(x,y) < r\}}$$

Standard Topology on $\mathbb{R}^d$

We define the standard topology on $\mathbb{R}^d$, $\mathcal{O}_{std}$ by

$$U \in \mathcal{O}_{std} \iff \forall p \in U, \ \exists r \in \mathbb{R}^+ \ : \ B_r(p) \subset U$$

3.3 Building new topologies out of old ones

Given one or more topological spaces, there are many ways to combine them/restrict them/quotient them to form new topological spaces.

3.3.1 Subspace Topology

Given a topological space $(M, \mathcal{O})$ and some subset $N \subset M$, we can turn N into a topological space by endowing it with the ***subspace topology*** defined as :

$$\mathcal{O}\big|_N = \{V \subset M | V = U \cap N \ \text{ for some open set } U \in \mathcal{O}\}$$

3.3.2 Quotient Topology

Given a topological space $(M, \mathcal{O})$ and an equivalence relation $\sim$ defined on M, we can construct the a new topological space by endowing the ***quotient set*** with the ***quotient topology***:

$$\begin{aligned}&\text{Quotient set: } {}^{M}/_{\sim} = \{[m] \subset \mathcal{P}(M) : m \in M\} \\ &\text{Quotient topology: } \mathcal{O}_{M/\sim} = \{U \subset {}^{M}/_{\sim} : \bigcup U = \bigcup_{[a] \in U} [a] \in \mathcal{O}\}\end{aligned}$$

Examples:

- For example, the standard topology on the unit circle $\mathbb{S}^1$ is [write later]

3.3.3 Product Topology

Given two topological spaces $(A, \mathcal{O}_A)$, $(B, \mathcal{O}_B)$ we can define a topology on $A \times B$ as

$$U \in \mathcal{O}_{A \times B} \iff \forall p \in U : \exists (S,T) \in \mathcal{O}_A \times \mathcal{O}_B : S \times T \subseteq U$$

3.4 Sequences and Convergence

- Let M be a set. A sequence in N is a function $q : \mathbb{N} \to M$.
- Let $(M, \mathcal{O})$ be a topological space. A sequence q in M converges against a limit point $a \in M$ if $\forall U \in \mathcal{O} : a \in U \implies \exists N : \forall n \geq N : q(n) \in U$.
- An open set $U \subseteq M$ such that $a \in U$ is called an ***open neighborhood*** of a : $U(a)$.

Continuous maps

A map $\phi : M \to N$ between topological spaces $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$ is called ***continuous*** if for every $S \in \mathcal{O}_n$, the pre-image is open in M i.e. $\phi^{-1}(S) \in \mathcal{O}_M$.

This definition extends the $\epsilon - \delta$ definition that we are familiar with the spaces that don't have a metric.

Homeomorphism

A map $\phi : M \to N$ is said to be a ***homomorphism*** if ϕ is bijective, and ϕ, ϕ^{-1} are both continuous maps between topological spaces.

If there exists a homeomorphism between two spaces M, N then they are said to be ***homeomorphic***.

3.5 Classification of Topological Spaces

- A topologcial space $(M, \mathcal{O}_M)$ is said to be T_1 if $p, q \in M, p \neq q$ there exists $U(p) \in \mathcal{O}_M$ such that $q \notin U(p)$ (Also called Kolmogorov spaces)
- A space is called T_2 if for any $p, q \in M$ there exists open sets $O_p \ni p$, $O_q \ni q$ such that $O_p \cap O_q = \emptyset$. (Also called Hausdorff spaces).

Note: T_2 implies T_1 [Write proof]

3.6 Covers, Compactness, and more

- Given a topological space $(M, \mathcal{O}_M)$, a set $C \subseteq \mathcal{P}(M)$ is called a ***cover*** of M if

$$\bigcap C = M$$

- Let C be a cover of M. Then, $\tilde{C} \subset C$ such that $\tilde{C}$ is also a cover is called a ***subcover*** of M. If $|\tilde{C}|$ is finite, then it's called a finite subcover.
- A topological space is called ***compact*** if and only if every open cover also has a fintie subcover.
- Given a topological space $(M, \mathcal{O}_M)$ and cover C, a ***refinement*** of X is another cover R such that $\forall U \in R : \exists V \in C : U \subseteq V$.

More about compactness

[Write some brief expo]

Theorem: (Heine-Borel) Let $(\mathbb{R}^d, \mathcal{O}_{std})$ be a topological space. A subset of $\mathbb{R}^d$ is compact if and only if it is **closed and compact**.

Theorem: Given two compact topologcial spaces M and N, the product topological space $M \times N$ is also compact.

Another property that is incredibly useful in many cases, for instance in proving certain properties of manifolds, is ***paracompactness***.

Paracompactness

A topological space $(M, \mathcal{O}_M)$ is said to be paracompact if for any open cover there exists a locally finite, open refinement.

What the hell does locally finite mean?

A collection of sets $\mathcal{U}$ is said to be locally finite if for any point $p \in M$ there exists an open neighborhood which intersects only finitely many sets in $\mathcal{U}$.

Now we get to some meaty stuff. Buckle up mfs because we're about to define a ***manifold*** !

3.7 Manifolds

Topological Manifolds

A Paracompact, Hausdorff topological space $(M, \mathcal{O}_M)$ is said to be an $n-$dimensional manifold if for all points $p \in M$ there exists an open neighborhood $U(p)$ and a homeomorphism $\phi : U(p) \to \tilde{U} \subseteq \mathbb{R}^n$.

4 February 1 - Properties of Topological Spaces

4.1 Path-Connectedness

Path-connected

A topologcial space is path-connected if every pair of points can be joined by a path. i.e. for any two point $x, y \in X$ in the space there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma: If X is path-connected and $f : X \to Y$ is a homeomorphism, then Y is also path-connected.

Fixed Point

A topological space X has a fixed point property if *every* self-map $f : X \to X$ has a fixed point $f(x_0) = x_0$.

Path-connectedness is super useful when working with Homotopy.

Homotopy

[write more later]

Surface

A Surface is a Hausdorff Topological Space such that every point has an open neghborhood homeomorphic to an open set in $\mathbb{R}^2$.

[Show examples of surfaces like a sphere, torus and the map to an open set in $\mathbb{R}^2$]

[Show a non-example like the disjoint union of the 2D plane and an $\mathbb{R}^1$ line]

4.2 Boundary

Write about surfaces with boundary.

4.3 Quotient Topological Spaces

- Write about Torus construction
- Write about Klein bottle construction
- Write about Planar Diagrams
- Write about Projective Space

-Write about Torus $\mathbb{T}^2$ being compact+oriented, inf cylinder C_2 being non-compact+oriented, Klein-bottle compact+unoriented, mobius strip (with bdy) non-compact+unoriented.

4.4 Orientation

Write about orientable/non-orientable surfaces

4.5 Connected Sum

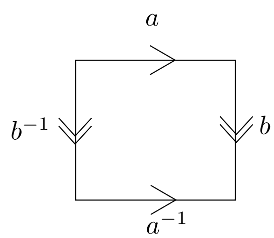
- Operation that connects topological spaces. For example, we can connect a $2-$ genus torus and a $3-$ genus torus to get a $5-$genus torus.
- Denoted by $\#$. For example, $S_1 \# S_2$ denotes the connect sum of spaces S_1 and S_1.
- This is distinct from the disjoint sum (denoted $\coprod$)

5 February 6 -

Recap

Recall from last time that

- An orientable surface of genus g is homeomorphic to the torus of genus g. So, any orientable surface can be written as the Connect sum of g 1-genus torii.
- Further, we can write Torii in terms of Projectvie Planes.
- A planar diagram P is a polyhedron. [write more]
- We can denote as being constructed from a planar diagram by labelling the edges that should be "glued" together. We also spoke about the Klein Bottle. [Insert planar diagram constructions of both.]



[Write more stuff from images]

n-Cells

FOr $n \geq 1$, an n-cell is a topological space which is homeomorphic to an open $n-$dimensional ball. For $n = 0$, a $0-$cell is just a point.

5.1 Cell Complex

A cell complex is a topological space which is a disjonit union of cells of various dimensions.

CW Complex

Write later

Euler Characteristic

Write later

5.2 Next Time

We'll begin studying ***Algebraic Topology***. We'll study different types of complexes.

6 Feburary 8 - Starting off with Algebraic Toplogy!

6.1 Recap of the past

- So far, we've seen Linear Algebra, Group Theory, and Point-set Topology with their main objects of study being Vector Spaces + Linear Transformations, Groups + Group Homomorphisms, and finally Topological Spaces + Homeomorphisms.
- We studied some important properties of Topological Spaces such as **openness and closedness, compactness and continuity.**

6.2 Alg. Top. At. Last.

Preview of the future

Skipping ahead real quick, let's see where the topology we're going to develop shows up in physics and *why* it's useful.

Quantum Hall Effect:

- Explain the effect later.
- It turns out that $\frac{ve^2}{h}$ is a *topological invariant* of the many-body wavefunction.
- In fact, this is a special case of *Topologcal Band Theory*.
- In 2017, it was realized that $\approx 27\%$ of our known materials have Topological bands!

Topological Phases of Matter:

- Could be used for Topological Quantum Computing.
- Their classification is still an open problem.

Quantum Field Theory:

- Any kind of non-perturbative niformation we can gain about our QFT's is valuable, and so Topologcial Invariants are valuable.
- Further, there is the focused branch *Topological Quantum Field Theory* [write more later]

Why do there only exist fermions and bosons?

- We'll cover this next week!

6.3 Topological Invariants

A ***Topological Invariant*** is a property of a space which remains unchanged (or invariant) up to Homeomorphism. That is, it does not change as long as we only continuously deform our space.

A homeomorphism f is a continuous map between topological spaces $f : X \to Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_Y X$

Why are we even interested in Topological Invariants? Well, to show that two spaces X, Y are homeomorphic we have to *find* a homeomorphism between them. But how do we show that two spaces are ***not*** homeomorphic?

We can't sit and verify *every* map between the spaces. So, instead, we find some topologcal nivarant and show that the invarant differs for the two spaces!

The issue is that the challenge of finding *all topological invariants* is essentially intractible. It's simply too hard! So, instead, we weaken the condition. Rather than consider spaces up to Homeomorphism, we consider them *up to homotopy*.

Homotopy between Functions

- We say there exists a Homotopy between continuous functions $f, g : X \to Y$ if there exsts a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.
- We denote this as

$$f \sim g$$

Homotopy Equivalence

- Two spaces are homotopy equivalent if
- Gven example of Circle and Cylinder

6.4 Fundamental Group

This is the first topological invariant we'll study. It gives us a way to classfy all the possible *paths* on a space.

Paths

- A ***Path*** on a space X is a map $\alpha : I \to X$ where $I = [0,1]$. We say the path starts at x_0 and ends at x_1 if $\alpha(0) = x_0$ and $\alpha(1) = x_1$.
- A ***loop*** is a path such that $\alpha(0) = \alpha(1)$.
- A constant path is a path of the form $\alpha(t) = x_0$ for all $t \in I$.

Further, we can define a sort of *product* of two paths. We define the product as

Product of two paths

$$\alpha \star \beta := \begin{cases} \alpha(2s), \ s \leq 1/2 \\ \alpha(2s+1), \ s > 1/2 \end{cases}$$

Intuitively, this means we follow α for half of our time, and then β for the remaining half of our time.

Inverse of a Loop

$$\alpha^{-1}(s) := \alpha(1-s)$$

Also, although gonig around a circle counterclockwise and then clockwise is not the same as just the constant map, we define an equivalence class on the set of paths such that $\alpha \sim \beta$ if the two paths are *homotopic*. Under this relation, the two are the same.

Notice that we have bilinear product, an identity loop, and an inverse for each loop. These are all the properties required of a group! So, we define the ***Fundamental Group*** as the set of equivalence classes of paths

Fundamental Group based at a point

The fundamental group of a space X with basepoint x_0 is defined as

$$\pi_1(X, x_0) = \{[\alpha_{x_0}]\}$$

with the product, inverse, and identity defined as above.

Show later that the product is well defined!

Notice that the fundamental group dependson the ***base point***. It turns out that if X is simply-connected, then it *doesn't matter which point we choose as the base!* The fundamental groups based at each point are Isomorphic.

Theorem: If $f : X \to Y$ is a homotopy equivalence between topological spaces X, Y then the fundamental groups of the spaces are isomorphic

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$$

Prove later!

6.4.1 Fundamental Group of the Circle

Now that we've defined the Fundamental Group, let's actually compute it!

Draw Image and write formal argument later.

Intuitively, the *number of windings* is the topological invariant here. So,

$$\boxed{\pi_1(S^1) = \mathbb{Z}}$$

Quick note: The number of windings is called the ***degree of the map***. Later, we'll see how the degree of a map is physically impact to Field Configurations.

Deformation Retract

Deformation Retract

- A subset $R \subseteq X$ of a toplogical space X is said to be a ***deformation retract of*** X if there exists a homotopy H between R and X such that

$$H(r,t) = H(r,0)$$

for all $r \in R$ and $t \in I$.

- X is contractible if it deformaton retracts to a point $x_0 \in X$.

Note that if X is a contractible space and deformation retracts to $x_0 \in X$, then

$$\pi_1(X) \cong \pi_1(x_0) \cong \mathbf{1}$$

Theorem:

$$\pi_1(x \times Y) \cong \pi_1(X) \oplus \pi_1(Y)$$

where $\oplus$ denotes the direct product in additive notation.

6.5 Fundamental Group of a Torus

Using the above theorem, we have

$$\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$$

We also have the relation $aba^-1b^1 \sim 1$ which basically tells us that it is a commutative group $ab = ba$.

[Insert image]

6.6 Fundamental Group of a Wedge Sum: $\mathbb{S}^1 \vee \mathbb{S}^1$

[write this section later from image]

Note that in this case the fundamental group is not commutative.

$$\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$$

where $*$ denotes the free product.

7 February 13 -

Recap

- Write from picture.
- Write about different perspectives on $\mathbb{RP}^n$.

$$\pi_1\left(\mathbb{RP}^2\right) \cong \mathbb{Z}/2\mathbb{Z}$$

The fundamental group is very simple to define, but it's generally very hard to compute. Further, there are spaces which which are *not* homotopy equivalent but which the fundamental group *cannot* tell apart. For instance, $\mathbb{S}^2$ and a single point both have trivial fundamental group.

So, the fundamental group is simple but has some issues: notably, it kind of *fails* at doing its job as an invariant. We will spend the next few weeks building more advanced invariants which become harder and harder to fool.

7.1 Lens spaces

[write later]

7.2 Induced Homomorphisms

If we have a map between topological spaces $f : X \to Y$ which is continuous at x_0, this map *induces* a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by

$$[\phi] \mapsto [f \circ \phi]$$

Let's verify that this map is well-defined. Consider two homotopic paths on X, $\psi \sim \psi$, for which there exists a homotopy H. Then, their images $f \circ \phi$ and $f \circ \psi$ are also homotopic with the homotopy $f \circ H$.

> Note:
>
> - $(f \circ g)_* = f_* \circ g_*$
> - $\mathbf{1}_* = \mathbf{1}$

Some properties of Induced Homomorphisms

- f is a homeomorphism $\implies$ f_* is an isomorphism.
- f is a homotopy equivalence $\implies$ f_* is an isomorphism.
- f is a retraction $\implies$ f_* is an injective.
- f is a deformation $\implies$ f_* is an isomorphism.

7.3 Covering Spaces and Maps

A covering space of X is $(\tilde{X}, P)$ where $P : \tilde{X} \to X$ is such that

- P is surjective.
- $\forall x \in X$, there exxsts an open U_x such that $p^{-1}(U_x)$ is

$$\coprod U_\alpha \cong_h U_x$$

For example, $R \to \mathbb{S}^1$, $\mathbb{S}^n \to \mathbb{RP}^n$. [Write more about these].

7.4 CW Complexes

- CW Complexes are spaces built recursively by starting off with points, then adding lines, then planes, and so on. . . .
- Most surfaces we'll work with will be CW Complexes.

$n-$cell

An $n-$

[Write later.]

7.5 Limitations of the Fundamental Group

- π_1 is really sensitive to dimesnions $n \leq 2$, but fails to detect higher dimensional holes.

Add missing sections

7.6 Summary of Homotopy

8 February 15 - Some Physics applications

8.1 Outline for today

- What are phases of matter? Why are they interesting?
- Phase transitions, Landau Paradigm/Classification
- Examples of Homotopy defects
- Superfluidity/Mean Field Theory/Superconductivity

8.2 Phases of Matter

- In Condensed matter, we're concerned with systems with particles on the scale of Avogadros constant i.e. we're interested in *macroscopic* behavior. It's impossible to study such systems by just solving Schrodinger's Equation.
- Further, there are some behaviors like Superconductivity which only arise in macroscopic settings.
- Consider a chain of N individual electron spins for example. Insert image. The total hilbert space is the tensor product of the each spin hilbert space.
- $$\mathcal{H} = \otimes_n \mathcal{H}_i$$
  and has dimension 2^N, which is huge!
- Furthermore, Condensed matter is usually only concerned with the states in a small corner of the total hilbert space. For example, only states at low temperatures.
- So, rather than studying the entire hilbert space, we partition it into *phases*.

Transverse Ising Field

Consdier the Hamiltonian

$$H = -\sum_{i=1}^{N} (\mathcal{Z}_i \otimes \mathcal{Z}_{i+1} + gX_i)$$

where

$$\mathcal{Z}_i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad X_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and g is a *tuning parameter* that we can control.

- It turns out there are different phases associated with different ranges of values for the parameter g.

What do these two terms in the hamiltonian do? The first term is basically a penalty for misaligned spins, and the second term just points in the x_i direction.

So, the frst guy wants spnis to be aligned, all pointing up or pointing down, whereas the second guy wants them to be pointing in some particular direction which might conflict with the spin up/down preferences of the first term.

$\underline{g = 0}$***:*** In this case, the second guy is not present so our ground states are of the form

$$\begin{aligned} |\psi\rangle &= |\uparrow\uparrow \cdots \uparrow\rangle \text{ or} \\ &= |\downarrow\downarrow \cdots \downarrow\rangle \end{aligned}$$

$\underline{g = \infty}$***:*** The solutions are of the form

$$|\psi\rangle = |+\rangle \otimes |+\rangle \cdots |+\rangle$$

where $|+\rangle = \frac{1}{\sqrt{2}} \cdot (|\uparrow\rangle + |\downarrow\rangle)$

But notice that in the $g = 0$ case we have ***2-dimensional ground state space*** whereas in the $g = \infty$ limit the groun state is a ***1-dimensional space***.

This is actually a consequence of the simplest symmetry group $\mathbb{Z}/2\mathbb{Z}$!

In physics, we call any operator which commutes with the hamiltonian as a *Symmetry operator*. Such an operators has a common eigenbasis with the Hamiltonian so we can label the solution states using the symmetry group instead.

The Transverse Field Ising Hamiltonian commutes with the operator

$$S = \Pi_i^{NX_i}$$

and $S^1 = 1$ so S has the same action as $\mathbb{Z}/2\mathbb{Z}$

If we act S on the ground states of the $g = \infty$ case, they remain the same so we call these the *symmetric* states. If we apply it to the states in the $g = 0$ space, then they flip so these are called the *symmetrized* states.

There is something called the ***Order Parameter*** which tells us on average what the states look like.

8.3 Heisenberg Hamiltonian and Symmetry Breaking

$$H = -J \sim_{i,j} S_i \cdot S_j = g \cdot \sum_i S_i$$

In this case, we can have ***symmetry breaking*** from $SO(3) \to SO(2)$

9 February 20 - Homology and other fun stuff!

Recall the Euler Characteristic, which was a topological invariant we came across a while ago, defined as

$$\chi(M) = \#V - \#E + \#F$$

It turns out that we can generalize this formula, for any CW-complex M, as

$$\boxed{\chi(M) = \sum_n (-1)^n \, (\#n\text{-cell})}$$

But we want to do better than this. So, we're going to ***Categorify*** the Euler Characteristic!.

Roughly, categorification entails replacing our invariant with a better invariant which has more information, and then go back to our original space using the ***Forgetful functor***.

$$\text{Invariant} \longrightarrow \text{Better Invariant}$$

$$\text{Losing additional structure} \longleftarrow \text{More Information}$$

In this class, we will only discuss *Simplicial Homology*, and Singular Homology if time permits. We're going to build Homology to be the *better invariant* corresponding to the Euler Characteristic.

Abelian Group

As a reminder, an ***Abelian Group*** is a group G whose elements commute: $gh = hg$ for all $g, h \in G$.

- For example, the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ are all abelian groups.
- The rotations of a square form an abelian group.

An important *type of element* of a group is one that can produce the rest of the group. For example, we can generate all of $\mathbb{Z}$ by repeated addition or subtraction of 1. So, we say $\mathbb{Z}$ is generated by 1.

Fundamental Theorem of Fintiely Generated Abelian Groups:

$$G = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\text{rank of } G} \;\oplus\; \mathbb{Z}/n\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n\mathbb{Z} \;\; \oplus \cdots$$

9.1 Simplicial Homology

Some definitions are in order

To talk about Simplicial Homology, let's first define what a *simplex* is:

$n-$simplex

- The $n-$simplex is defined to be the ***oriented set***

$$\Delta^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

- These are basically generalizations of triangles to arbitrary dimensions.

[Include some figures.]

Also, $[v_0, \ldots, v_n]$ denotes the n-simplex Δ^n which has vertices $v_1, \ldots, v_n$ and $[v_0, \hat{v}_i, \ldots, v_n]$ denotes the simplex where the i^{th} vertex is deleted from $[v_0, v^i, \ldots, v_n]$. The symbol $\partial\Delta^n$ denotes the union of all faces i.e. the ***boundary*** of the simplex.

[Insert fgure]

Δ structure on X

We can endow a space X withΔ-complex structure by defining a set of maps $\sigma_\alpha : \Delta^n \to X$

-

For example, the Torus is a Δ-complex structure formed by [write later]

Note that the triangle with cyclic orientation is *not* a Δ-structure, but we can turn it into one as follows:

[Insert figure]

We *define* a Homology on a Δ^n structure using an $n-$chain.

n-chains

An $n-$chain on a space X with delta structure Δ^n is denoted as $\Delta^n(X)$ and is the ***Free Abelian Group*** generated by $\{e^n_\alpha\}$. Its elements look like arbitrary linear combinations

$$\sum n_\alpha e^n_\alpha$$

Boundary Map ∂

[Write more crap after class]

Lemma: $\partial_{n-1} \circ \partial_n = 0$ in the sequence

$$\Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \cdots$$

Proof: Write up after class.

Intuitively, this makes sense. The Boundary of a boundary is empty.

Another way to express this fact is to say that

$$\boxed{\mathrm{Im}\left(\partial_{n+1}\right) \subseteq \ker\left(\partial_n\right)}$$

Finally, we define the *Simplicial Homology Group,* $H_n^{\Delta}(X)$ as

$$H_n^{\Delta}(X) = \ker \partial_n / \mathrm{Im}\partial_{n+1}$$

The Homology Group counts the number of "holes" in the space X [Explain the intuition for this with pictures later].

Example: Homology group of $\mathbb{S}^1$

[Write later]

[Insert image]

$$\boxed{H_1(\mathbb{S}^1) = \mathbb{Z}}$$

Note:

$$H_0(X)$$

doesn't tell us much for path connected spaces X. [write more about this]

Homology Group of a Torus

Writelater

9.2 Some useful homology calculations

1. $H_n(\mathbb{S}^n) = \mathbb{Z}$
2. $H_1\left(\mathbb{RP}^n\right) = \mathbb{Z}/2\mathbb{Z}$

9.3 Some properties of Homology Groups

Since we want Homology to be a topological invariant, we require it to satisfy some properties. For instance, the Homology Group must not depend on the specific Δ0structure we endow on the space.

Relation between π_1 and H_1

It turns out (there's a theorem proving this) that the First Homology group is the ***abelianization*** of the Fundamental Group. i.e.

$$\pi_1(X)/\{ghg^{-1} = g\} \cong H_1(X)$$

First Homology group of the wedge-sum of two circles

Calculate later.

The first homotopy and homology groups are very simply related as above. The higher groups have more complicated relations, but there do exist maps between the higher groups (for all orders?).

10 February 22 - Applications of Homology

Recap

- Last time, we saw that the Homology groups of the torus are given by

$$H_n\left(T^2\right) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, n = 1 \\ \mathbb{Z}, n = 2, 0 \\ 0, \text{ else} \end{cases}$$

Let's calculate the Homology groups of the sphere now.

10.1 Homology of $\mathbb{S}^n$

We can endow $\mathbb{S}^n$ with the Δ complex consisting of two n-simplices and the simplex group is

$$\Delta_n\left(\mathbb{S}^n\right) = \mathbb{Z}\{U\} \oplus \mathbb{Z}\{L\}$$

With

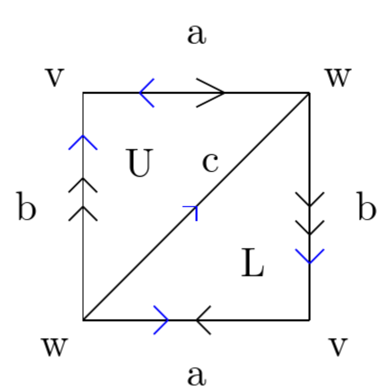
$$\ker$$

[Finish from picture]

Let's do one more calculation.

10.2 Homology of $\mathbb{RP}^2$

The Planar Diagram of $\mathbb{RP}^2$ is as below, where U and L are the 2-simplices, (a, b, c) are the 1-simplices v and w are the 1-simplices. The black arrows denote the gluing, whereas the blue arrows denote the simplex orientations.



[Write more from picture]

10.3 Formal Properties of Homologies

We want to make sure that Homology is indeed a topological *invariant*, and we will do so using *Induced maps* (we came across these when studying homotopy invariance as well).

For homotopy, we defined the induced map f_*. For Homology, we will define $f_\#$.

Rather than defining the map $F_\# : H_n(X) \to H_n(Y)$, we define a map on the level of chain groups

$$f_\# : \Delta_n(X) \to \Delta_n(Y)$$

to have the action

$$\sigma \mapsto f \circ \sigma$$

where $f : X \to Y$ is a homeomorphism.

This then gives us an mnduced map between the homology groups because of the following lemma:

Lemma:

The Kernel and Image of a map ∂_n are preserved under the action of $f_\#$, thus so is their kernel (the homology group).

Note: f is a homotopy equivalence, thus $f_\#$ is an isomorphism.

Degree of a map

In physics, often we are interested in maps from the $n-$Sphere to itself.

Now, for any such map $f : \mathbb{S}^n \to \mathbb{S}^n$, the induced map is $f_\# : \underbrace{H_n(\mathbb{S}^n)}_{\mathbb{Z}} \to \underbrace{H_n(\mathbb{S}^n)}_{\mathbb{Z}}$.

But notice that any such map $\mathbb{Z} \to \mathbb{Z}$ can be characterized as just multiplying each integer by some other integer m. This number m is called the ***degree*** of the map:

Some Properties of the degree

- $deg(\mathbf{1}) = 1$
- $deg(\mathbf{1}) = 1$
- $deg(\mathbf{f}) = 0$ if f is not surjective
- The degree of the antipodal map is $(-1)^{n+1}$

This will allow us to prove the ***Hairy Ball Theorem*** for $\mathbb{S}^n$!

Hairy Ball Theorem: Write statement later.

Proof: Suppose $V(x)$ is a unit tangent vector field i.e. for all $x \in \mathbb{S}^b$ we have $x \perp V(x)$

We can define a homotopy between the maps $\mathbf{x}$ and $\mathbf{v}(\mathbf{x})$. In particular, we can get a homotopy equivalence between $\mathbf{1}$ and $-\mathbf{1}$ as

$$f(tx) =$$

[complete after class]

10.4 Identical Particles

(Based on work by Leinass and Myrrheim) Now we come to a physical application of $\pi_1(X)$!

Let X be the coordinate space or configuration space for a single particle. Then, for n identical particles, we would naively assume the configuration space is $X \times \cdots \times X$, but this does not account for the fact that the n particles are identical!

If we have a configuration $(x_1, x_2, \ldots, x_n)$, swapping any pair of them makes no distance. So, the *actual* configuration space is

$$X/S_n$$

where S_n is the n-th permutation group. Physically, this is fine because X^n/S_n is locally isomorphic to X^n.

Any configuration/path in X^n/S_n can be *lifted* to a number of configurations/paths in X^n – and this is what we do in Classicaly Mechanics! We choose only of the paths in X^n and work in its locality. More formally, X^n/S_n is a covering space for X^n.

However, this doesn't work in Quantum Mechanics.

Dealing with S_n is annoying so let's just consider two particles for now, with $X = \mathbb{R}^n$. Then, the configuration space – when using generalzed radial coordinates – is

$$\left(\underbrace{\mathbb{R}^n}_{C.O.M.} \times \underbrace{\mathbb{R}_+}_{radiallength} \times \underbrace{\mathbb{S}^{n-1}}_{rel.angularposition} \right) / \left(\mathbb{Z}/2\mathbb{Z}\right)$$

And recall that $\mathbb{S}^{n-1}/\left(\mathbb{Z}/2\mathbb{Z}\right) \cong \mathbb{RP}^{n-1}$. This is not a simply connected space, so $\pi_1\left(\mathbb{RP}^{n-1}\right)$ is nontrivial. Now, $\mathbb{S}^{n-1}$ is the universal covering space of $\mathbb{RP}^{n-1}$.

[Fill in some detail]

When we quantize this system, the number of phases that the wavefunction can pick when we switch the two particles is given by

$$\pi_1(\mathbb{RP}^{n-1}) = \begin{cases} \mathbb{Z}, n = 2 \\ \mathbb{Z}/2\mathbb{Z}, n \neq 2 \end{cases}$$

This measn that in 3-dimensions, we can only have two types of particles (bosons and fermions!) but in ***two dimensions***, the phase can be *any* value. Thus, these particles are called ***anyons***!

> *Note:* The correct way of talking about this situation is in terms of *vector bundles* because the wavefunction is actually a vector bundle defined over the configuration space.

[Refine everything above later]

11 February 27 - Starting off with Differential Geometry!

11.1 Differentiable Maninfolds

- Earlier in the course, we saw topological spaces and defined a particular class of topological manifolds called ***Topological Manifolds***.
- Now, we will add more structure on topological manifolds to extend calculus onto these spaces. i.e. we will turn them into ***Differentiable Manifolds.***

M is an ***n-dimensional Differentiable Manifold*** if

1. M is a topological space
2. M has an atlas $\{(U_i, \phi_i)\}$ where each (U_i, ϕ_i) is a chart.
3. The collection $\{U_i\}$ covers M i.e.
$$\bigcup_i U_i = M$$
4. Each ϕ_i is a homeomorphism from U_i onto a set $\tilde{U}_i \subseteq \mathbb{R}^n$
5. For each U_i, U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ from $\psi_j(U_i \cap U_j)$ to $\psi_i(U_i \cap U_j)$ is ***infinitely differentiable.***

It is property number (4) that differentiates differentiable (or smooth) manifolds from topological manifolds.

[insert some pictures]

A function that is n-differentiable is denoted as being a C^n function, so we require the set of transition functions $\{\psi_{ij}\}$ to be C^∞ i.e. infinitely differentiable (aka. smooth).
[Mention theorem which allows us to generalize from n-differentiable to infinitely differentiable.]

11.2 Diffeomorphisms

At this point we've seen a bunch of *morphisms* such as homeomorphism, isomorphism, endomorphism, automorphism etc.

So far, each of these have been maps that allow us to identify different spaces as being "the same". We will define *diffeomorphisms* to be the morphisms for the class (or rather, category) of Smooth Manifolds. But before we get there, we need to define what it means for a map between two smooth manifolds to be smooth.

Smooth maps between manifolds

- We say that a map $f : M \to N$ between smooth manifolds of dimensions m, n is ***smooth*** (i.e. C^∞) if for all charts (U, ϕ_i) on M and (V, ψ) on N, the coordinate representation $\psi \circ F \circ \phi^{-1}$ is C^∞.
- Note that the coordinate representation is always a map between euclidean spaces, so it must be smooth in the normal sense.

[Insert Images]

11.3 Scalar Fields on Smooth Manifolds

- In Physics, often our system is a manifold, and we are interested in the behavior of some quantity at different points on the space, for eg. the spin distribution.
- This is often achieved by using *scalar fields*.

> - A Scalar Field defined on a manifold M is any function in the set of smooth maps
>
> $$C^\infty(M) = \{f : M \to \mathbb{R}\}$$
>
> - In fact, this space forms a commutative ring when equipped with point-wise multiplication [Elaborate on this]

Curves on Manifolds

> A Curve on a manifold is a smooth map $\gamma : J \to M$ where J is a closed (or open) interval in $\mathbb{R}$.

[Write later and Insert Image]

11.4 Tangent and Cotangent Spaces

[Write later]

12 Feb 29 -

Recap

- Last time, we left off while discussing the *pullback*

$$M \xrightarrow{f} N \xrightarrow{\psi} R$$

[Finish this]

Today, we start off by discussing the *pushforward*, which tells us how tangent vectors change when we move between manifolds.

Pushforward

Given smooth manifolds M and N and a smooth map $F : M \to N$, we define the ***pushforward*** at the point $p \in M$ as the map

$$f_* : T_pM \to T_{F(p)}N$$

as

$$f_* \left(\frac{d}{dt}\bigg|_{t=0} \gamma(t) \right) = \frac{d}{dt}\bigg|_{t=0} (f \circ \psi)(t)$$

12.1 Bundles

Often, it is useful to consider not just the tangent space at a particular point $p \in M$, but the collection the tangent spaces at *all* points in M. This is called the *tangent bundle* and it is actually a smooth manifold in itself.

It is also not the only bundle we can define on a manifold. More generally, we define a ***Smooth Bundle*** on a manifold M as the triple (E, π, M) where E is the total space, M is the base space, and $\pi : E \to M$ is a projection map that allows us to from one to the other.

Given a bundle (E, π, M), we define the ***fiber over the point*** $p \in M$ to be the preimage of p under π.

12.2 Sections

The projection map $\pi : E \to M$ takes us one way. To go in the opposite direction, we use the *section* $\sigma : M \to E$, which is defined to be the right inverse of π i.e.

$$(\pi \circ \sigma) = \mathbb{1}_M$$

12.3 Tangent Bundle

Let's now look more closely at the Tangent Bundle.