Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 1

kdeoskar@berkeley.edu

Question 1: Which of the following spaces are pairwise homeomorphic and which are not?

(a) The Orthogonal Group SO_3

(b) The space T_1S^2 of unit tangent vectors to S^2

(c) The Stiefel Manifold $V(3,2)$

(d) In the complex 3−space $\mathbb{C}^3$ with coordinates z_1, z_2, z_3, the set of unit vectors i.e. $\{(z_1, z_2, z_3) : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ satisfying $z_1^2 + z_2^2 + z_3^2 = 0$. Let's denote this set as X.

(e) The real projective space $\mathbb{RP}^3$

(f) $\mathbb{S}^2 \times \mathbb{S}^1$

Solution:

We have

$$SO_3 \cong \mathbb{RP}^3 \cong SO_3 \cong T_1\mathbb{S}^2 \cong X$$

i.e. all of them except $\mathbb{S}^2 \times \mathbb{S}^1$ are homeomorphic.

- SO_3 is the set of rotations in three dimensions. We can describe any 3D rotation by specifying the axis of rotation and angle of rotation about that axis (in radians). With this perspective, a solid ball of radius π in $\mathbb{R}^3$, $\mathbb{B}^3$, is homeomorphic to SO_3 with some modifications. If we let the line between the origin and a point x describe the axis of rotation and the length of the line be the angle of rotation about that point. However, note that $\pi \cdots \mathbf{x}$ and $\pi \cdot (-\mathbf{x})$ both correspond to rotating by π radians around the axis defined by $\mathbf{x}$. So, to make o ur map from $\mathbb{B}^3$ to SO_3 injective we need to identify antipodal points on the boundary. Thus, $SO_3 \cong \mathbb{RP}^3$.
- Again, we can think of $T_1\mathbb{S}^2$ as SO_3 using the rotation axis and angle perspective. The space of unit tangent vectors at each point $x \in \mathbb{S}^2$ is just $\mathbb{S}^1$. So $T_1\mathbb{S}^2$ is a **circle bundle** over $\mathbb{S}^2$. So, again, a point on $\mathbb{S}^2$ can be thought to define an axis of rotation and the value of the circle bundle at that point gives us the amount of rotation.

- Thinking of the Stiefel Manifold as a homogeneous space of actions of classical groups we know that (since $3 > 2$)

$$V(3,2) = O(3)/O(3-2) = SO(3)/SO(1) = SO(3)$$

Alternatively, we can think of $SO_3, T_1\mathbb{S}^2, V(3,2)$ as collections of pairs $(\vec{x}, \vec{y})$ which are orthogonal and have fixed length. (In the case of SO_3,)

- Writing each complex number as $z_i = x_i + y_i$, the condition $|z_1|^2 + |z_2|^2 + |z_2|^2 = 1$ gives us

$$\sum_i x_i^2 + \sum_i y_i^2 = 1 \tag{1}$$

and the condition $z_1^2 + z_2^2 + z_3^2 = 0$ gives us

$$(x_1^2 - y_1^2 + i \cdot 2x_1y_1) + (x_2^2 - y_2^2 + i \cdot 2x_2y_2) + (x_3^2 - y_3^2 + i \cdot 2x_3y_3) = 0$$

and the requirement that the real and imaginary parts are separately zero gives us

$$\sum_i x_i^2 - \sum_i y_i^2 = 0 \tag{2}$$

$$\sum_i x_iy_i = 0 \tag{3}$$

Equations (1) and (2) give tell us that

$$x_1^2 + x_1^2 + x_1^2 = y_1^2 + y_1^2 + y_1^2 = \frac{1}{2}$$

and Equation (3) tells us that

$$(x_1, x_2, x_2) \cdot (y_1, y_2, y_2) = 0$$

i.e. we have two fixed length vectors $\vec{a} = (x_1, x_2, x_2)$, $\vec{b} = (y_1, y_2, y_2)$ which are ortogonal to each other. So, X is homeomorphic to SO_3.

Why is $SO(3) \neq \mathbb{S}^2 \times \mathbb{S}^1$?

We might naively think that since a rotation in $SO(3)$ can be specified by an axis of rotation + an angle of rotation, which would correspond to an element of $\mathbb{S}^2$ for the rotation and an element of $\mathbb{S}^1$ for the amount of rotation.

However, trying to do this would effectively by to take every point in $\mathbb{S}^2$ and assign to it an element of the tangent space at that point which just happens to have unit length (and is thus an element of $\mathbb{S}^1$ since $T_x\mathbb{S}^2 \cong \mathbb{R}^2$ for any $x \in \mathbb{S}^2$).

But that means we're trying to assign a non-vanishing smooth vector field to $\mathbb{S}^2$, which is impossible due to the **Hairy Ball Theorem**.

If we wanted to describe $SO(3)$ rotations by assigning values of $\mathbb{S}^1$ to points on $\mathbb{S}^2$, we would have to do it via a **circle bundle** which is a fiber bundle over $\mathbb{S}^2$.

We can also tell that $\mathbb{S}^2 \times \mathbb{S}^1 \ncong SO_3$ by looking at their fundamental groups:

$$\pi\left(\mathbb{S}^2 \times \mathbb{S}^1\right) = \mathbb{Z}^3 \neq \pi(SO_3) = \mathbb{Z}/2\mathbb{Z}$$

Question 2: Prove that a compact subset in $\mathbb{R}^\infty := \lim \mathbb{R}^n$ (the union of a nested sequence of Euclidean Spaces) must lie in a finite dimensional subspace $\mathbb{R}^n$.

Solution:

$\mathbb{R}^\infty$ is the space of sequence $(x_1, x_2, x_3, \cdots)$ and the topology on it is described as :

A subset $F \subseteq \mathbb{R}^\infty$ is said to be open/closed if and only if $F \cap \mathbb{R}^n$ is open/closed in $\mathbb{R}^n$ for all $n \in N$.

We'll show the claim by contrapositive. Consider a subset $A \subseteq \mathbb{R}^\infty$ which doesn't lie in a finite dimensional subspace, and denote $A_i = A \cap \mathbb{R}^i$. Let's define $B \subseteq \mathbb{R}^\infty$ such that

$$\pi_j(B_i) \begin{cases} = \varnothing, \ j \neq i \\ \supseteq A_i \text{ and open } j = i \end{cases}$$

where π_j is the projection $\mathbb{R}^\infty \to \mathbb{R}^j$.

Then, B is open in $\mathbb{R}^\infty$ since each $B \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ since it's either an empty set or an open set (by definition). Since each B_i covers each A_i, we can say B is an open cover of A.

But no finite subcover of B we take can possibly cover A since it will miss out on some intersections $A_i = A \cup \mathbb{R}^i$. Thus, A cannot be compact.

Question 3: For a base-point space X, construct a natural (with respect to X) homeomorphism between the suspension and the smash-product of X with $\mathbb{S}^1$.

__**Solution:**__ (Inspired by [1])

For a base point space (X, x_0) we define the **(Reduced) Suspension** as

$$\Sigma X = (X \times I) \Big/ ((X \times \partial I) \cup (\{x_0\} \times I))$$

where I is the unit interval $I = [0, 1]$. The **Smash Product** of X with $\mathbb{S}^1$ is

$$X \# \mathbb{S}^1 = (X \times \mathbb{S}^1) \Big/ (X \vee \mathbb{S}^1)$$

where $\vee$ denotes the Wedge sum

$$X \vee \mathbb{S}^1 = \left(X \coprod \mathbb{S}^1\right) /\sim$$

Also, it doesn't matter which point of $\mathbb{S}^1$ we choose as the base-point, but let's choose the point associated with $[0] = [1]$ in the equivalence relation $I/\partial I$.

We can also think of the Smash Product of (X, x_0) and $(\mathbb{S}^1, [0])$ as

$$X \# \mathbb{S}^1 = X \times \mathbb{S}^1 \Big/ \left(\{x_0\} \times X \cup \mathbb{S}^1 \times \{[0]\}\right)$$

To show that $\Sigma X \cong_h X \# \mathbb{S}^1$, let's show the existence of maps $f : X \times \mathbb{S}^1 \to \Sigma X$ and $g : \Sigma X \to X \times \mathbb{S}^1$, then show that these maps descend to the appropriate quotients and serve as each others' inverses.

Define the function

$$\begin{aligned} f : X \times \mathbb{S}^1 &\to \Sigma X \\ (x, [t]) &\mapsto [(x, t)] \end{aligned}$$

where we can think of $\mathbb{S}^1 \cong I/\partial I$ as a base-point. It doesn't matter which point on $\mathbb{S}^1$ we choose, but let's choose $[0] = [1] \in I/\partial I$ as our base-point.

We need to

1. Verify that f is well-defined as a map $X \times \mathbb{S}^1 \to \Sigma X$
2. Verify that f passes to the quotient i.e. if $\sim$ denotes the equivalence relation that sends $X \times \mathbb{S}^1 \to X \# \mathbb{S}^1$ then for $x, y \in X \times \mathbb{S}^1$ we have $x \sim y$ if and only if $f(x) = f(y)$.

To check that f is well-defined
we just need to verify that it's independent of the representative used for the equivalence class $[0] = [1]$ i.e. $(x, [0])$ and $(x, [1])$ get mapped to the same point in ΣX.

Indeed, under this map we have $f(x, [0]) = [x, 0] = [x, 1] = f(x, [1])$ since we quotient out by

$X \times \partial I$ so both $(x, [0])$ and $(x, [1])$ get sent to the new base point.

To check that f passes to the quotient
Consider $(x_1, [t_1]) \sim (x_2, [t_2]) \in X \times \mathbb{S}^1$ which get sent to the base point in $X \# \mathbb{S}^1$ when we quotient out by $\{x_0\} \times X \cup \mathbb{S}^1 \times \{[0]\}$. So under the map f they both get mapped to the base point in ΣX beause the equivalence relation on $X \times I$ used to define ΣX identifies such points together.

So, f induces a continuous map $\tilde{f} : X \# \mathbb{S}^1 \to \Sigma X$ such that $\tilde{f}([w]) = f(w)$ for $w \in X \times \mathbb{S}^1$.

The inverse is induced by

$$\begin{aligned} g : X \times I &\to X \# \mathbb{S}^1 \\ (x, t) &\mapsto [(x, [t])] \end{aligned}$$

We don't need to check that g is well-defined since the domain doesn't involve an equivalence relation.

To check that g passes to the quotient
Consider any $(x, t) \in X \times \mathbb{S}^1$ that gets sent to the base point of ΣX under the quotient map $q : X \times I \to \Sigma X$ i.e. $(x, t) \in ((X \times \partial I) \cup (\{x_0\} \times I))$. Any such (x, t) gets mapped to the base point of $X \# \mathbb{S}^1$ under the map g because $[(x, [t])]$ is the equivalence class obtained by identifying all elements of $X \times \partial I \cup \{x_0\} \times I$ together as the base point.

So, g induces a continuous map $\tilde{g} : \Sigma X \to X \# \mathbb{S}^1$ such that $\tilde{g}([z]) = g(z)$ for $g \in X \times I$. Almost by definition, $\tilde{f}$ and $\tilde{g}$ are inverses. Thus, $\Sigma X \cong X \# \mathbb{S}^1$.

References

[1] Alex Provost. https://math.stackexchange.com/questions/2463770/how-to-prove-sigma-x-cong-s1-wedge-x, 2017.