Math H185 Lecture 4

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

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Whereas in the reals, we can simply look at the graph of a function and tell whether it's differentiable or not, functions on $\mathbb{C}$ are different.

Our intuition that a function $\mathbb{R}^d \to \mathbb{R}^d$ is differentiable is to see whether it is "smooth-ish".

The goal for today is to develop some sort of similar intuition for Holomorphic functions on $\mathbb{C}$. (Note: A characteristic equivalent to Holomorphicity s "conformality". We'll explore this today as well.)

1.1 Review of Differentiation

For a function $f : \mathbb{R}^d \to \mathbb{R}^d$, the derivative is supposed to be the *best linear approximation* to f.

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$$

for $x \approx x_0$. (In 1D, $f'(x_0)$ is just a number, but in higher dimensions it's generally a *matrix*).

Now, the same idea holds for functions on the complex plane. That is, for $f : \Omega \subset_{open} \mathbb{C} \to \mathbb{C}$,

$$\boxed{f(z_0) \approx f(z_0) + \underbrace{f'(z_0)}_{\in\mathbb{C}}(z - z_0), z \approx z_0}$$

Contrast this with $f : \Omega \subset_{open} \mathbb{R}^2 \to \mathbb{R}^2$ where

$$f(x, y) \approx f(x_0, y_0) + (x - x_0, y - y_0)f'(x_0, y_0)$$

where

$$f'(x_0, y_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

[Listen to lecture recording and write about comparison between derivatives on $\mathbb{C}$ and $\mathbb{R}^2$ later]

1.2 Cauchy-Riemann Equations

Consider a complex function $f : \Omega \subset_{open} \to \mathbb{C} = \mathbb{R}^2$ and denote

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y) = f(x, y)$$

Lemma: If f is holomorphic at $z_0 = x_0 + iy_0$, then

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$$
$$\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

Proof: Let f be holomorphic at z_0. Then,

$$\lim_{x\in\mathbb{R}\to 0} \frac{f(z_0 + x) - f(z_0)}{x} = f'(z_0) = \lim_{y\in\mathbb{R}\to 0} \frac{f(z_0 + iy) - f(z_0)}{y}$$

but the Left Hand Expresson above is, by definition, $\partial f / \partial x(z_0)$, or in other words,

$$\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$$

and similarly the Right Hand Expression is $\frac{1}{i} \cdot \partial f / \partial x(z_0)$, or in other words,

$$\frac{1}{i}\left(\frac{\partial u}{\partial xy}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right)$$

Then, since both of them are equal to $f'(z_0)$, we should have

$$\boxed{\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = \frac{1}{i}\left(\frac{\partial u}{\partial xy}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right)}$$

The Real and Imaginary parts of the above must then be equal, so we get

$$\begin{aligned} \frac{\partial u}{\partial x}(z_0) &= \frac{\partial v}{\partial y}(z_0) \\ \frac{\partial u}{\partial y}(z_0) &= -\frac{\partial v}{\partial x}(z_0) \end{aligned}$$

These are the *Cauchy-Riemann Equations.*

Remark: There is a converse which is harder to prove.

If f is C^1 and the Cauchy-Riemann Equations hold at z_0, then f is holomorphic at z_0.

This has important applications in PDEs.

Summary: The partial derivative matrix of a holomorphic function has the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a, b \in R$. Then, using polar coordinates $(a, b) \to (r, \theta)$ wherein $a = r\cos(\theta), b = r\sin(\theta)$ then the matrix is

$$r\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

So, the derivative $f'(z_0)$ of a complex function f is a linear map of the form "scaling + rotaton". There are *conformal mappings* i.e. they infinitessimally preserve angles or scale to zero.

Example: Consder $f(z) = \lambda z$, $\lambda - re^{i\theta} \in \mathbb{C}$. Angles are certainly preserved by this map: [insert figure].

Example: In contrast to the last example, $f(z) = \overline{z}$ does *not* preserve angles so it's not Holomorphic. [insert figure]

Example: $f(z) = z^2$: This one's a bit tricky. One may think this map is *not* conformal because the real axis stays fixed while the positive imaginary axis becomes aligned with the negative real axis, changing the angle between them from 90 degrees to 180 degrees.

However, recall that conformal maps can also *scale to zero.* In fact, that's essentially what happens to numbers in a very small neighborhood around the origin.
[insert image and write some more explanation]