Math H185 Lecture 7

Keshav Balwant Deoskar

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

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1 January 31 - Cauchy's Theorem

Review

- Recall that the boundary of a set $\Omega \subset \mathbb{C}$ is
$$\partial\Omega = \underbrace{\overline{\Omega}}_{\text{closure}} \setminus \underbrace{\Omega^\circ}_{\text{interior}}$$
- For example, the boundary of the (closed or open) ball of radius r centered at z_0 i.e. $\Omega = \partial B_r(z_0)$ is $\partial B_r(z_0) = \partial \overline{B_r(z_0)}$ is the circle of radius r around z_0.
- When integrating over a curve, we can do a sort of u-substitution
$$\int_\gamma f(z)dz = \int_{u\circ\gamma} f(z)u'(z)dz$$
as long as u is holomorphic in an open neighborhood of γ.

1.1 Last time

Last time, we calculated the following integral:

$$\int_{\partial B_r(0)} z^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$$

Doing the substitution $u(z) = z - z_0$. The integral on the circle around some other point z_0 also gives us the same answer i.e.
$$\int_{\partial B_r(z_0)} (z-z_0)^n dz = \int_{\partial B_r(0)} z^n dz$$

1.2 Cauchy's Theorem

Theorem (Cauchy): If $U \subset_{open} \mathbb{C}$ has piece-wise smooth boundary and $f(z)$ is holomorphic around a neighborhood of $\overline{U}$ then
$$\int_{\partial U} f(z) = 0$$

As you may have read in many books, this is ***the most important result in Complex Analysis!*** So many other results fall out as a result of this, but it'll take us some time to really appreciate its importance.

Example: If we want to calculate the integral

$$\int_{\partial B_r(0)} z^n dz$$

we know that z^n is holomorphic on the ball for all $n \geq 0$, so Cauchy's Theorem immediately tells us the integral is zero.

For $n < 0$ though, the function is not holomorphic on the ball because it is not holomorphic at the origin. So, Cauchy's theorem doesn't immediately apply. We'll need to be bit more clever.

Consider the annulus formed the the crcles around $z = 0$ of radius r and r', $(r' < r)$. The boundary of this region consists of the two circles, but with the inner circle having the reversed orientation i.e. $\partial U = \partial B_r(0) \cup \partial B_{r'}(0)^-$.

[Include picture]

The origin is excluded, so z^n is holomorphic and Cauchy's theorem tells us the integral is zero, for all n.

So,

$$\begin{aligned} &\int_{\partial B_r(0)} z^n dz + \int_{\partial B_{r'}(0)^-} z^n dz = 0 \\ \implies &\int_{\partial B_r(0)} z^n dz = -\int_{\partial B_{r'}(0)^-} z^n dz \\ \implies &\boxed{\int_{\partial B_r(0)} z^n dz = \int_{\partial B_{r'}(0)} z^n dz} \end{aligned}$$

This verifies our observation from last lecture that the integral of z^n along a circle centered at the origin is ***independent of*** r.

Upshots

- We will see that it's useful to apply Cauchy's Theorem to creative choices of U.
- Cauchy's Theorem allows us to do contour manipulation i.e. we can deform the curve γ in the domain where f (the function we're integrating) is holomorphic. [Include squiggly wiggly nasty curve diagram and show it's equivalent to just a ball]

1.3 Cauchy's Formula

Suppose $f : \Omega_{open} \to \mathbb{C}$ be holomorphic on Ω and let $z_0 \in \Omega, r > 0$ such that $B_r(z_0) \in \Omega$. Then, for all $z \in B_r(z_0)$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z} dw$$

This, too, is an incredible statement. Cauchy's formula allows us to calculate the value of the function at *any* point inside $B_r(z_0)$, only using the values of the function on the ball $B_r(z_0)$ or any path homologous to it.

<u>Proof:</u> Define a $g(w) = \frac{f(w)}{w-z}$. This function is holomorphic on $B_r(z_0) \setminus \{z\}$. Let $\delta > 0$ such that $B_\delta(z_0) \subset B_r(z_0)$ and define $U = B_r(z_0) \setminus B_\delta(z)$.

Then, the boundary of U is $\partial U = \partial B_r(z_0) \cup \partial B_\delta(z)^-$.

By Cauchy's Theorem,

$$\int_{\partial B_r(z)} g(w)dw = \int_{\partial_\delta(z)} g(w)dw$$

Then, since ff is holomorphic at z, we have $f(w) = f(z) + \epsilon(w)$ where ϵ
[Complete later, when recording is out]

<u>Summary:</u> Cauchy's formula says

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{w-z} dw$$

Explain how but the fact that a holomorphic functions is automtically infinitely differentiable is a result of Cauchy's formula.