Knot 176: Introduction to Low-Dimensional Topology	**Fall 2023**
Homework 2	
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Compactness

Q2.1.1: Suppose (X, τ_X) and (Y, τ_Y) are two compact topological spaces. Show that $X \times Y$ equipped with the product topology is also compact.

Proof: Given two topological spaces (X, τ_X) and (Y, τ_Y), their product $X \times Y$ is a topological space when endowed with the product topology, which has a basis $\mathcal{B}$ given as

$$\mathcal{B} = \{U_i \times V_j \ | \ U_i \subset \tau_X, V_j \subset \tau_Y\}$$

In order to show that $X \times Y$ inherits compactness from X and Y, it would be useful to study how $X \times Y$ can be "decomposed" in terms of the orginial spaces, so let's think about projections.

Projection Maps

Consider the projection map $\Pi_X : X \times Y \to X$ given by $(x, y) \mapsto y$.

Then, if Z is some open set in $X \times Y$,

$$\Pi_X(Z) = U$$

where U is some open set in X.

Why is this? Because any open set (such as) $Z \in X \times Y$ can be written as a (arbitrary) union or (finite) intersection of the basis sets, and the basis sets of the product topology for $X \times Y$ look like $U_i \times V_j$ where $U_i \in \tau_X$ and $V_i \in \tau_Y$.

For any sets U_i open in X and V_j open in Y, we have $\Pi_X(U_i \times V_j) = U_i$, and clearly, the map respects unions and intersection in that

$$\begin{aligned}
\Pi_X\left(\bigcup_{i \in I} U_i \times V_i\right) &= \Pi_X(\{(x \times y) : \text{x and y are in some } U_i \text{ and } V_i\}) \\
&= \{x : x \in U_i \text{ for some } i \in I\} \\
&= \bigcup_{i \in I} U_i
\end{aligned}$$

and

$$\Pi_X\left(\bigcap_{i\in I} U_i \times V_i\right) = \Pi_X(\{(x \times y) : \text{x and y are in all } U_i \text{ and } V_i, i \in I\})$$
$$= \{x : x \in U_i \text{ for all } i \in I\}$$
$$= \bigcap_{i\in I} U_i$$

The open set Z in $X \times Y$ can be expressed as some union or finite intersection of the basis sets $U_i \times V_j$, so its projection $\Pi_X(Z)$ simply extracts the sets U_i from the basis sets $U_i \times V_j$ which are used to "build" Z via intersections and unions. After extracting these sets, it unions / finitely intersects them together, giving us some open set in X.

So, the projection map $\Pi_X((x, y)) = x$ is an open map.

Similarly, the map $\Pi_Y((x, y)) = y$ is also an open map.

Showing $X \times Y$ is compact

Now, suppose $\{Z_\alpha\}$ is an open cover of $X \times Y$. Pick some $x_0 \in X$ and let

$$A_{x_0} = \{\alpha \mid (x_0, y) \in Z_\alpha\}$$

Since $\{Z_\alpha\}$ covers $X \times Y$, we must have $\Pi_Y(\{Z_\alpha\})_{\alpha\in\mathcal{A}_{x_0}}$ covering Y. So, there is some finite subcover $\Pi_Y(\{Z_\alpha\})_{\alpha\in I_{x_0}}$ where I_{x_0} is finite. Then, there is some $W_{x_0} \in X$ such that $W_{x_0} \times Y$ is covered by $\{Z_\alpha\}_{\alpha\in I_{x_0}}$.

The sets $\{W_{x_0}\}_{x_0\in X}$ form an open cover of X, hence there is a finite subcover of $\{W_{x_0}\}_{x_0\in F}$ where $F \subset X$ is finite.

So, the collection $\{Z_\alpha\}_{\alpha\in\mathcal{A}_{x_0}, x_0\in F}$ is finite and covers the entire space $X \times Y$. Thus, $X \times Y$ equipped with the product topology is compact!

Q2.1.1: The interval $I = [0, 1]$ is a compact space.

Proof:
