Physics 137B Homework 56

Keshav Balwant Deoskar

March 13, 2024

Question 1: Fun with Tensor Products

(a) Write down the corresponding (4 dimensional) vectors for $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

(b) Write down $\hat{S}_x, \hat{S}_y$ as matrices.

(c) Using $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$, write down the 4×4 matrix defining $\hat{S}^2$ for the two spin-1/2 particles.

(d) Find the eigenvalues of $\hat{S}^2$.

(e) Find the eigenvectors of $\hat{S}^2$.

(f) Explain how the eigenvalues and eigenvectors you got are consistent with the ones we obtained in class.

Solution:

(a) We have

$$|\uparrow\downarrow\rangle = |\frac{1}{2}\frac{1}{2}\rangle \otimes |\frac{1}{2}\frac{-1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix}0\\1\end{pmatrix} \\ 0 \cdot \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|\downarrow\uparrow\rangle = |\frac{1}{2}\frac{-1}{2}\rangle \otimes |\frac{1}{2}\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 0 \cdot \begin{pmatrix}1\\0\end{pmatrix} \\ 1 \cdot \begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|\downarrow\downarrow\rangle = |\frac{1}{2}\frac{-1}{2}\rangle \otimes |\frac{1}{2}\frac{-1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 0 \cdot \begin{pmatrix}0\\1\end{pmatrix} \\ 1 \cdot \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

(b) Recall that the matrix representations of the spin operators are

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Now, let's tackle $\hat{S}_x = \hat{S}_x^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_x^{(2)}$. We have

$$
\begin{aligned}
\hat{S}_x &= \hat{S}_x^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_x^{(2)} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 \cdot \mathbb{1} & 1 \cdot \mathbb{1} \\ 1 \cdot \mathbb{1} & 0 \cdot \mathbb{1} \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 \cdot \hat{S}_x^{(2)} & 0 \cdot \hat{S}_x^{(2)} \\ 0 \cdot \hat{S}_x^{(2)} & 1 \cdot \hat{S}_x^{(2)} \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{S}_y &= \hat{S}_y^{(2)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_y^{(2)} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 \cdot \mathbb{1} & -i \cdot \mathbb{1} \\ i \cdot \mathbb{1} & 0 \cdot \mathbb{1} \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 \cdot \hat{S}_y^{(2)} & 0 \cdot \hat{S}_y^{(2)} \\ 0 \cdot \hat{S}_y^{(2)} & 1 \cdot \hat{S}_y^{(2)} \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}
\end{aligned}
$$

(c) Now that we

Question 2: Isospin symmetry for nucleons

(a) What are the allowed isospin states for two nucleons in terms of the proton/neutron basis?

(b) At low energies, most two-nucleon observables are dominated by the $l = 0$ angular momentum. If we fix $l = 0$, what are the allowed spin and isospin two-nucleon states?

Solution:

(a) Suppose

$$|N_1\rangle = |l_1 m_{l_1}\rangle \otimes |S_1 m_{S_1}\rangle \otimes |I_2 m_{I_1}\rangle$$

describes the first nucleon and

$$|N_2\rangle = |l_2 m_{l_2}\rangle \otimes |S_2 m_{S_2}\rangle \otimes |I_2 m_{I_2}\rangle$$

describes the second nucleon. Then the two-nucleon state can be written as

$$|N_1 N_2; l m_l, S m_S, I m_I\rangle = |l m_l\rangle \otimes |S m_S\rangle \otimes |I m_I\rangle$$

where $\vec{l} = \vec{l}_1 + \vec{l}_1$, $\vec{S} = \vec{S}_1 + \vec{S}_1$, and $\vec{I} = \vec{I}_1 + \vec{I}_1$

l is arbitrary, so to obtain the allowed isospin states in terms of the proton neutron basis, we want to write the coupled state $|l m_l\rangle \otimes |S m_S\rangle \otimes |I m_I\rangle$ as a tensor product of the uncoupled basis states.

Although Angular momentum and Isospin are different symmetries and the transformations corresponding to them act on different spaces, they can be studied in nearly identical manner because the rotation group in 3 dimensions $R(3)$ is isomorphic to $SU(2)$ which is the group that describes Isospin Symmetry. (Source: Groups and Symmetries in Nuclear Physics, Jitendra C. Parikh).

Thus, in terms of the proton/neutron basis

$$|I = \frac{1}{2}, I_z = \frac{1}{2}\rangle = |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle = |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the allowed states are of the form

$$|p\rangle \otimes |p\rangle, |p\rangle \otimes |n\rangle, |n\rangle \otimes |p\rangle, |n\rangle \otimes |n\rangle$$

which we can write as

(b) If we fix $l = 0$,

Question 3: Isospin symmetry for two pion states:

(a) Using the tools we have learned about adding spin and angular momentum, what are the allowed isospin states for two pion systems?

(b) The pions are spinless Bosons, so the states of two pions must be symmetric. Given this fact, what are the allowed isospin states for two pion systems for **even** angular momentum l?

(c) What are the allowed isospin states for two pion systems for **odd** angular momentum l?

Solution:

Question 4: States and degeneracy for 2 particle states in a box

Obtain the energy and degeneracy of for the **ground state** and **first excited state** with **zero total momentum**

(a) For two **distinguishable** spinless particles.

(b) For two **identical spinless bosons**.

(c) For two **identical spin-1/2 bosons**.

Solution:

Question 5: Fermi gas model for nuclear matter

Consider a simple model for heavy nuclei that have a large numbero f approximately equal protons and neutrons. If we assume that protons and neutrons are exactly degenerate, and we assume the same exact number of protons and neutrons, what is the Fermi energy of such nuclei? What is the average energy?

Solution:
