(Instructor: Chien-I Chiang)

Physics 105: Analytical Mechanics notes

Keshav Balwant Deoskar
kdeoskar@berkeley.edu

These are some very terse notes taken from UC Berkeley's Physics 105 during the Summer '24 session, taught by Chien-I Chiang.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 First topic

text

2 July 3, 2024:

2.1 Finishing up discussion from last lecture

- Finish this from lecture recording

Continuing on with out discussion of when $H \neq E$, we can parametrize the position of a particle as $\vec{r} = \vec{r}(q_k, t)$

We have

$$\frac{\partial K}{\partial \dot{q_k}} = \frac{1}{2} m \left[2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_m} + \cdots \right]$$

Then,

$$\begin{cases} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} \right] \\ 2K = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + 2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \right] \end{cases}$$

(The expression for $2K$ is obtained by expandingo out

$$K = \frac{1}{2} m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}$$

in terms of indices – write this out explicitly later)

Which gives us the relation

$$\begin{aligned} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} &= 2K - m \frac{\partial \vec{r}}{\partial t} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial q_k} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \right)}_{= \frac{d\vec{r}}{dt}} \\ &= 2K - \vec{p} \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

The question we were originally considering is **When is $H = E$?**

Now,

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{q_k}} \dot{q} - L \\ &= \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = (K - V) \\ &= 2K - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} - K + V \\ &= K + V - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

So we see that $H = E = K + V$ only when

$$\frac{\partial \vec{r}}{\partial t} = 0$$

i.e. when $\vec{r} = \vec{r}(q_k, t)$ has no time dependence i.e. $\vec{r} = \vec{r}(q_k)$

Earlier, we considered the following setup:

and we showed that

$$H = E - m\omega^2\rho^2$$

So, let's check that

$$m\omega^2\rho^2 = \vec{p} \cdot \frac{\partial \vec{r}}{\partial t}$$

$$\begin{aligned} \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} &= \vec{p} \cdot (-\rho\omega \sin(\omega t)\hat{x} + \rho\omega \sin(\omega t)\hat{y}) \\ &= \vec{p} \cdot \left[\rho\omega\hat{\phi}\right] \\ &= m v_\phi \rho\omega \\ &= m\rho^2\omega^2 \end{aligned}$$

where $v_\phi = \rho\omega$

Since the hamiltonian itself has no time dependence, ***H* is conserved**. However, ***E* is not**. We can check that

$$dH = dE = d(m\omega^2\rho^2)$$

is indeed zero.

[Include figure]

If we break the force on the bead into a normal force (denoted N) and a centripetal(?) force, then

$$\begin{aligned} dW &= \underbrace{\overbrace{N\rho}}_{\text{torque about z-axis}} d\phi \\ &= \frac{dl_z}{dt} d\phi \\ &= d\left(\rho m \rho \omega\right)\omega \\ &= d\left(m\rho^2\omega^2\right) \end{aligned}$$

This is the energy that goes into the system.

By energy conservation, $dW = dE$.

$$\implies 0 = dE - dW = dE - d(m\rho^2\omega^2)$$

i.e. $E - m\rho^2\omega^2 = H$ is a conserved quantity.

So, the **Hamiltonian being conserved** and the **Hamiltonian being equal to Energy** are two different scenarios with two different conditions.

- The Lagrangian is time-independent i.e. $\frac{\partial L}{\partial t} = 0 \implies H$ is conserved.
- The position vector centered in an inertial frame $\vec{r} = \vec{r}(q_k, t)$ is time independent i.e $\frac{\partial \vec{r}}{\partial t} = 0 \implies H = E$

Now we move on to a powerful technique.

2.2 The Method of Lagrange Multipliers

We have a block constrained to move on the xy-plane, and we have gravity. Previously, we would say

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

i.e we would start with an unconstrained lagrangian, and then plug in the constraints $z = 0, \dot{z} = 0$

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &\implies \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \end{cases} \end{aligned}$$

Alternatively, we can implement the constraint $\ddot{z} = 0$ in the following way: We have the original lagrangian

$$L' = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz + \lambda z$$

where λ is the Lagrange multiplier and we can think of z as being the constraint function $f(z)$ and our constraint is $f(z) = 0$.

If we treat λ as an independent degree of freedom, we can write the Euler-Lagrange equation for λ as

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0 \implies z = 0 \text{ (constraint)}$$

On the other hand, if we look at the equation of motion for z, we get

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \implies \lambda - mg - m\ddot{z} = 0 \implies m\ddot{z} = \lambda - mg$$

and using the constraint $z = 0 \implies \ddot{z} = 0$ we get $-mg + \lambda = 0 \implies \lambda = mg$. Okay, but what physical meaning does λ have? It has to do with the **Normal force**. i.e. λ is encoding the **constraint** that the block can only move on the xy-plane due to the Normal force.

So, in general, for N constraints we have Lagrange Multipliers $\lambda_1, \cdots, \lambda_N$.

Why do we call $\boldsymbol{\lambda}$ a Lagrange Multiplier?

Recall from Calc 3 that if we have contours of a function $f(x, y)$ on the xy-plane and we are constrained to move along some other curve $g(x, y) = c$ on the plane, if we ask "What is the extremum of $f(x, y)$ as we move along the curve $g(x, y) = c$?" then visually we can tell that the extremum corresponds to the point where $g(x, y)$ intersects the contour of $f(x, y)$ only once. This is because at such a point, the gradients of the two functions are parallel:

$$\nabla g = \lambda \nabla f$$

This constant multiplier is the **Lagrange Multiplier**

So, in general, if we have a Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

we know that $\delta L = 0$ gives the Equations of Motion. But if we want to do this variation δL under some constraint $C(x, y, z) = 0$ then we need to consider

$$\delta L = \lambda \delta C \implies L' = L - \lambda C$$

Generally, if we have P constraints, $C_l(q_1, \cdots, t) = 0$, $l = 1, \cdots, P$ on the lagrangian L, we can write a new lagrangian

$$L' = L + \sum_{l=1}^{P} \lambda_l C_l$$

The Euler-Lagrange equation for λ_l leads to $C_l = 0$ and the Euler-Lagrange equation for the generalized coordinate q_k is

$$\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k} \right) = 0$$

$$\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} + \underbrace{\sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}}_{\text{generalized force}}$$

On the physical point of view, consider the following system:

[include picture of block and sledge which can both move]

If we consider the system as a whole, the normal forces due to the block and the sledge are equal and opposite, so they cancel each other out - and so does the work that they do(?).

Howeverm if we consider the block only - we do have a normal force. The block is constrained the only move on the surface of the slope, so we can write

$$L' = K - V + \int^{\vec{r}} \vec{F}_C(\vec{r}) \cdot d\vec{r}'$$

(This is a bit handwavy - watch the lecture recording and think about this)

Then, if we conpare this with

$$L' = L - V + \sum_l \lambda_l C_l$$

we have

$$\sum_l \lambda_l C_l = \int^{\vec{r}} \vec{F}_C \cdot d\vec{r}' = \int^{\vec{r}} \vec{F} \cdot \left(\frac{\partial \vec{r}'}{\partial q_k} \cdot \mathrm{d}q_k\right)$$

$$\implies \frac{\partial}{\partial q_k}\left(\sum_l \lambda_l c_l\right) = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial q_k}\right) \equiv \mathcal{F}_k \text{ (generalized force)}$$