Math 214 Homework 3

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Q2-1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Proof:
The way we've defined smoothness implies continuity. Since $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$$

is clearly not continuous at $x = 0$, it cannot be smooth (according to our definition).

However, let's show there still exist smooth coordinate charts (U, ϕ) and (V, ψ) such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from $\phi(U \cap f^{-1}(V))$ to $\psi(V)$. Away from zero f is smooth so this is definitely true. To deal with the origin, let $\epsilon > 0$ and $U = (-\epsilon, \epsilon)$ and let $V = (\frac{1}{2}, \frac{3}{4})$.

Then, U contains x, V contains $f(0) = 1$, and (U, id), (V, id) are charts on $\mathbb{R}$, and $\psi \circ f \circ \phi^{-1} = \mathrm{id} \circ f \circ \mathrm{id}^{-1}$ is just the contant map on $\phi(U \cap f^{-1}(V))$ to $\psi(V)$.

Q2-3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the n^{th} ***power map*** for $n \in \mathbb{Z}$, given in complex notation as $p_n(z) = z^n$.

(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the ***antipodal map*** $\alpha(x) = -x$.

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$$

where we think of $\mathbb{S}^3$ as the subset $\{(w, z) : |w|^2 + |w|^2 = 1\}$ of $\mathbb{C}^2$.

Proof:

(a) For (a), let's recall,

From Problem 1-8 we know that if $U \subseteq \mathbb{S}^n$ is an open subset such that $U \neq \mathbb{S}^n$ there exists a continuous (angle) function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ and $\theta^{-1}(x) = e^{i\theta(\theta^{-1}(x))} = e^{ix}$.

We have that $e^{i\theta(z)} = z$, so $\theta(z) = -i\log(z)$ where we can make a branch cut on the line passing through the origin and any $p \in \mathbb{S}^{-1} \setminus U$.

If $z = e^{it}$, then

$$\begin{aligned}\theta(e^{it}) &= -i\log(e^{it}) && (1)\\ &= -i(it) + k(e^{it}) \cdot 2\pi && (2)\\ &= t + k(e^{it}) \cdot 2\pi && (3)\end{aligned}$$

for some $k(e^{it})$ which is integer valued. But, θ is continuous, so k must also be continuous, so it must attain constant values on each connected component of U.
So,

$$\theta(e^{it}) = t + k \cdot 2\pi$$

Let $z \in \mathbb{S}^1$, and let (U, θ) be a chart containing z where θ is an angle function, and let (V, ϕ) be a chart containing z^n where ϕ is an angle function as well. Then, the coordinate representation is $\phi \circ p_n \theta^{-1}(x) = \phi \circ p_n(e^{ix}) = \phi(e^{inx}) = nx + k \cdot 2\pi$ for some k which must be constant on each connected component of $U \cap p^{-1}(V)$. Note that $U \cap p^{-1}(V)$ is open since p_n is continuous, meaning $p^{-1}(V)$ is open.

We've shown the coordinate representation is smooth on any chart of $\mathbb{S}^1$, so the map p_n is smooth.

(b) WLOG suppose $x \in \mathbb{S}^n$ such that x is contained in the chart $(\mathbb{S} \setminus \{N\}, \sigma)$ where σ denotes the usual stereographic projection. Then, the chart $(\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})$ contains the antipodal point $\alpha(x)$. We can calculate the coordinate representation to be

$$\begin{aligned}(\tilde{\sigma} \circ \alpha \circ \sigma^{-1})(u^1, \ldots, u^n) &= (\tilde{\sigma} \circ \alpha)\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right)\\ &= \tilde{\sigma}\left(\frac{(-2u^1, \ldots, -2u^n, -(|u|^2 - 1))}{|u|^2 + 1}\right)\\ &= -\sigma\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right) \quad (\text{Since } \tilde{\sigma} = -\sigma(-x)\ )\\ &= -u\end{aligned}$$

Since the coordinate representation is smooth, so too is the map α.

(c) Denote the stereographic projection (from the north) on the $\mathbb{S}^3$, identified with a subset of $\mathbb{R}^4$ as σ_3 (respectively σ_2 for the 2-sphere, which has one less coordinate in the chart, and denote projection from the south as $\tilde{\sigma}_i$), which we know are given by

$$\sigma_3 : (x^1, \ldots, x^4) \mapsto \left(\frac{x^1}{1 - x^4}, \ldots, \frac{x^3}{1 - x^4}\right) \text{ defined on } \mathbb{S}^3 \setminus \{(0,0,0,1)\}$$

$$\sigma_{-1}^3 : (x^1, x^2, x^3) \mapsto \left(\frac{2x^1}{(x^1)^2 + (x^2)^2 + (x^3)^2 + 1}, \frac{2x^2}{(x^1)^2 + (x^2)^2 + (x^3)^2 + 1}, \frac{2x^3}{(x^1)^2 + (x^2)^2 + (x^3)^2 + 1}\right)$$

Then, define f as the real counterpart to F, given by:

$$\begin{aligned} f(x^1, x^2, x^3, x^4) &= F(x^1 + ix^2, x^3 + ix^4) \\ &= (2x^1x^3 + 2x^2x^4, 2x^2x^3 - 2x^1x^4, (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2) \end{aligned}$$

Writing out all the coordinate representations of $f : \mathbb{S}^3 \to \mathbb{S}^2$ for the required charts gives us

$$\begin{aligned} \sigma^2 \circ f \circ \sigma_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2 - 1)^2 + (x^3)^2}, x^2 \right) \\ \sigma^2 \circ f \circ \tilde{\sigma}_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2 - 1)^2 + (x^3)^2}, \frac{x^2}{(x^1)^2 + (x^2)^2 + (x^3)^2} \right) \\ \tilde{\sigma}^2 \circ f \circ \tilde{\sigma}_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2 + 1)^2 + (x^3)^2}, x^2 \right) \\ \tilde{\sigma}^2 \circ f \circ \sigma_3^{-1}(x^1, x^2, x^3) &= \left(\frac{2x^1}{(x^1)^2 + (x^2 + 1)^2 + (x^3)^2}, \frac{x^2}{(x^1)^2 + (x^2)^2 + (x^3)^2} \right) \end{aligned}$$

Which are all smooth as rational functions with no singularities in the domains. Then, the smoothness of f implies the smoothness of F.

Q2-4. Show that the inclusion map $\overline{B}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{B}^n$ is regarded as a smooth manifold with boundary.

Proof:

Consider the collection of charts $\{(U_i^\pm, \phi_i^\pm)\}$ where

$$\begin{aligned} U_i^+ &= \{x = (x^1, \dots, x^n) \ : \ |x|^2 \leq 1, x^i > 0\} \\ U_i^- &= \{x = (x^1, \dots, x^n) \ : \ |x|^2 \leq 1, x^i < 0\} \end{aligned}$$

and $\phi_i^\pm : U_i^\pm \to B_i^{n,\pm}$ is defined by the mapping

$$(x^1, \dots, x^n) \mapsto \left(x^1, \dots, x^i \mp \sqrt{1 - (x^1)^2 - \dots - \widehat{(x^i)^2} - \cdots (x^n)^2}, \cdots, (x^n)^2 \right)$$

These define a smooth structure on $\overline{\mathbb{B}^n}$. Now, let $x \in \overline{\mathbb{B}^n}$ be contained in some chart $(U_i^\pm, \phi_i^\pm)$. The image of x under inclusion is simply $\iota(x) = x$ and is certainly contained in the chart $(\mathbb{R}, \mathrm{id})$. Then, $\mathrm{id} \circ \iota \circ (\phi_i^\pm)^{-1} = (\phi_i^\pm)^{-1}$ is smooth.

Q2-6. Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Proof:

$$\begin{tikzcd} \mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "P"] \arrow[d, "\pi^n"'] & \mathbb{R}^{k+1} \setminus \{0\} \arrow[d, "\pi^k"] \\ \mathbb{RP}^n \arrow[r, "\tilde{P}"'] & \mathbb{RP}^k \end{tikzcd}$$

Well-defined:

If we have two points in $n-$Real Projective space $[x], [y] \in \mathbb{RP}^n$ such that $[x] = [y]$, then $x = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. So,

$$P(x) = P(\lambda y) = \lambda^d P(y)$$

Then, under the action of the quotient map $\pi^k : \mathbb{R}^{k+1} \to \mathbb{RP}^k$, defined in the usual way, $P(x)$ and $P(y)$ get mapped to the same point i.e. $\pi^k(P(x)) = \pi^k(P(y)) = [P(x)]$. Therefore, the map $\tilde{P}(x)$ which has the action $[x] \mapsto [P(x)]$ is well defined.

Smoothness:

The quotient maps π^n, π^k are diffeomorphisms, as shown when we proved Real Projective Spaces are smooth manifolds in class meaning $(\pi^n)^{-1}, \pi^k$ are smooth, and P is smooth by hypothesis. Therefore, their composition $\pi^k \circ P \circ (\pi^n)^{-1} = \tilde{P}$ is smooth.

Q2-10. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C^\infty(N) \to C^\infty(M)$ by $F^*(f) := f \circ F$.

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof:

(a) The linearity of F^* follows from the distributivity of function composition, and associativity of scalar multiplication.

- For $x \in M$

$$\begin{aligned} F^*(f+g)(x) &= [(f+g) \circ F](x) \\ &= [(f \circ F) + (g \circ F)](x) \\ &= (f \circ F)(x) + (g \circ F)(x) \\ &= F^*(f)(x) + F^*(g)(x) \end{aligned}$$

$$\implies \boxed{F^*(f+g) = F^*(f) + F^*(g)}$$

- For $x \in M, a \in \mathbb{R}$

$$\begin{aligned} F^*(af)(x) &= ((af) \circ F)(x) \\ &= a(f \circ F)(x) \\ &= aF^*(f)(x) \end{aligned}$$

$$\implies \boxed{F^*(af) = aF^*(f)}$$

Thus, F^* is a linear map.

(b) <u>" $\implies$ " Direction:</u> Consider any $f \in C^\infty(N)$. By hypothesis, F is smooth, therefore their composition $F * (f) := f \circ F$ is also smooth. Thus, $F^*(C^\infty(N)) \subseteq F^*(C^\infty(M))$.

$" \Longleftarrow "$ Direction: Suppose $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $\mathcal{B}_\mathcal{N}$ be the atlas of coordinate balls (V,ψ) covering N^n, and let $\mathcal{A}_\mathcal{M}$ be an atlas of M^m. By the Extension Lemma for smooth functions, each ψ can be extended to a smooth function $\tilde{\psi} = \left(\tilde{\psi}^1(x), \dots, \tilde{\psi}^n(x)\right) : N \to R^n$. Further, notice that each coordinate function is smooth i.e. $\tilde{\psi}^i(x) : N \to \mathbb{R} \in C^\infty(N)$

Now, for any $p \in M$ contained in chart (U,ϕ) such that $F(p)$ is contained in the chart (V,ψ), we want to show the smoothness of the coordinate function $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.

Note that $U \cap F^{-1}(V)$ is open in M since F is continuous. Now, since each $\tilde{\psi}^i$ is continuous, we have that $\tilde{\psi} \circ F = (\tilde{\psi}^1 \circ F, \dots, \tilde{\psi}^n \circ F)$ is smooth as each coordinate function $\tilde{\psi}^i \in F^*(C^\infty(N)) \subseteq C^\infty(M)$. Since $U \cap F^{-1}(V)$, the restriction $\tilde{\psi} \circ F\Big|_{U \cap F^{-1}(V)}$ is also smooth. As a result, the coordinate representaton

$$\psi \circ F \circ \phi^{-1}\big|_{\phi(U \cap F^{-1}(V))} \phi(U \cap F^{-1}(V)) \to \psi(V)$$

is smooth. So, we conclude that F is smooth.

(c) Suppose $F : M \to N$ is a homeomorphism.

$" \Longrightarrow "$ Direction: If F is additionally a diffeomorphism, then F, F^{-1} are smooth homeomorphisms between M and N. Applying the result from part (b) in each direction gives $F^*(C^\infty(N)) \subseteq C^\infty(M)$ and $(F^{-1})^*(C^\infty(M)) \subseteq C^\infty(N)$. Further, F is a homeomorphism so it's bijective meaning F^* must be injective. We conclude that $F^* : C^\infty(N) \to C^\infty(M)$ is a bijection.

From part (a), we know it is a linear map. Furthermore, for functions $f, g \in C^\infty(N)$ we have

$$F^*(fg) := (fg) \circ F = (f \circ F) \cdot (g \circ F) = F^*(f) \cdot F^*(g)$$

So, F^* respects the binary operation of pointwise-multiplication on the algebra.

Therefore, F^* is an algebra isomorphism between $C^\infty(N)$ and $C^\infty(M)$.

$" \Longleftarrow "$ Direction: Suppose $F^*(C^\infty(N)) \cong C^\infty(M)$. Then, $F^*(C^\infty(N)) \subseteq C^\infty(M)$ and $F^*(C^\infty(M)) \subseteq C^\infty(N)$ combined with the result from part (b) show that F and F^{-1} are smooth maps. Therefore, F is a diffeomorphism.

Q2-11. Suppose that V is a real vector space of dimension $n \geq 1$. Define the ***projectivization of*** V, denoted by $\mathbb{P}(V)$, to be the set of 1-D linear subspaces of V with the quotient topology induced by $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ that sends $v \in V \setminus \{0\}$ to its span. Show that $\mathbb{P}(V)$ is an $(n-1)-$topological manifold, and has a unique smooth structure with the property that for each basis $(E_1, \dots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E\left[v^1, \dots, v^n\right] = \left[v^i E_i\right]$ is a diffeomorphism.

Proof: Rather than proving that $\mathbb{P}(V)$ has each of the three properties required of Topological Manifolds, let's instead show that the projectivization $\mathbb{P}(V)$ is homeomorphic to Real Projective space $\mathbb{RP}^n$.

Consider two $n-$dimensional vector spaces V, W with the standard topologies induced by their norms and let $T : V \to W$ be a linear transformation between them. Let $\pi^V : V \setminus \{0\} \to \mathbb{P}(V)$ and $\pi^W : W \setminus \{0\} \to \mathbb{P}(W)$ be the natural projections and let $\tilde{T} : \mathbb{P}(V) \to \mathbb{P}(W)$ be the linear transformation between them defined as

$$[v] \mapsto [T(v)]$$

We show that $\tilde{T}$ is a homeomorphism.

The map is well defined as for $[u], [v] \in \mathbb{P}(V)$ which are equal, we have

$$\begin{aligned} & [u]_{\mathbb{P}(\mathbb{P}(W))} = [v]_{\mathbb{P}(\mathbb{P}(W))} \\ \implies & u = \lambda v, \lambda \in \mathbb{R} \setminus \{0\} \\ \implies & T(u) = T(\lambda v) = \lambda T(v) \\ \implies & \pi^W(T(u)) = \pi^W(\lambda T(v)) \\ \implies & [T(u)]_{\mathbb{P}(W)} = [T(u)]_{\mathbb{P}(V)} \end{aligned}$$

<u>***$\tilde{T}$ is bijective***</u>

<u>Injectivity:</u> Suppose $[u]_{\mathbb{P}(\mathbb{P}(W))} \neq [v]_{\mathbb{P}(\mathbb{P}(W))} \in \mathbb{P}(V)$. Then,

$$\begin{aligned} & [u]_{\mathbb{P}(\mathbb{P}(W))} \neq [v]_{\mathbb{P}(\mathbb{P}(W))} \\ \implies & u \neq \lambda v, \lambda \in \mathbb{R} \setminus \{0\} \\ \implies & T(u) \neq T(\lambda v) = \lambda T(v) \\ \implies & \pi^W(T(u)) \neq \pi^W(\lambda T(v)) \\ \implies & [T(u)]_{\mathbb{P}(W)} \neq [T(u)]_{\mathbb{P}(V)} \end{aligned}$$

<u>Surjectvity:</u> We know that T and the two natural projections are all surjective maps. So for any $[w] \in \mathbb{P}(W)$, there exists an element $[v] = \left(\pi^V \circ T \circ (\pi^W)^{-1}\right)([w])$ such that $\tilde{T}([v] = [w])$.

<u>***$\tilde{T}$ is continuous***</u> For any open set $Y \subseteq \mathbb{P}(W)$, the preimage $\tilde{Y}$ is continuous if and only if $(\pi^W)^{-1}(\tilde{T}^{-1}(Y)) = \left(\tilde{T} \circ \pi^W\right)^{-1}$ is continuous (By the characteristic property). But $\tilde{T} \circ \pi^W = \pi^V \circ T$, which is continuous. Therefore, the map $\tilde{T}$ is continuous. Exactly the same argument works for $\tilde{T}^{-1}$ because T^{-1} is also continuous.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi^V"'] & W \arrow[d, "\pi^W"] \\ \mathbb{P}(V) \arrow[r, "\tilde{T}"'] & \mathbb{P}(W) \end{tikzcd}$$

This shows that $\tilde{T}$ is a homeomorphism. Since P is homeomorphic to $\mathbb{R}^n$, taking via any linear map which sends the basis of V to the standard basis of $\mathbb{R}^n$, setting $W = \mathbb{R}^n$ gives us that

$$\mathbb{P}(V) \cong_h \mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$$

Since, $\mathbb{RP}^n$ is an $(n-1)$-dimension topological manifold, so is $\mathbb{P}(V)$.

Now, let $(E_1, \ldots, E_n)$ be a basis for V. To show the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by

$$E(\left([v^1, \ldots, v^n]\right)) = [v^i E_i]$$

is a diffeomorphism, we must show it is a smooth homeomorphism with smooth inverse.

If we define a linear transformation $T : \mathbb{R}^n \to V$ as

$$(v^1, \ldots, v^n) \mapsto v^i E_i$$

Then, the map $\tilde{T} : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$, $[v] \mapsto [T(v)]$ is exactly our map E. So, it follows from the work we did earlier that E is a homeomorphism.

Now, the coordinate representation of E is

$$\left(\phi_i \circ \tilde{T}^{-1}\right) \circ \phi_j^{-1}(x^1, \ldots, x^{n-1}) = \phi_i \left[T^{-1} \circ F\left(x^1, \ldots, x^{j-1}, 1, x^{i+1}, \ldots, x^{n-1}\right)\right]$$

where $F : \mathbb{R}^n \to V$ is defined by $F(v^1, \ldots, c^n) = v^i E_i$ Now, $T^{-1} \circ F$ is an invertible linear map from $\mathbb{R}^n \to \mathbb{R}^n$ so it is a diffeomorphism. Now, ϕ_i and its inverse π_i are also smooth. Therefore, the map is smooth.

The coordinate representation of E^{-1} is

$$\phi_i \circ E^{-1 \circ \left(\phi_j \circ \tilde{T}^{-1}\right)^{-1}(x^1, \ldots, x^{n-1}) = \phi_i \left[FT^{-1}(x^1, \ldots, x^{j-1}), 1, x^{j+1}, \ldots, x^{n-1}\right]}$$

So, E^{-1} is also smooth.

Q2-14. Suppose that A and B are two disjoint closed subsets of a smooth manifold M. Show that there exists a smooth function $f \in C^\infty(M)$ such that $f^{-1}(0) = A$ and $f-1(1) = B$.

Proof: This follows from the following theorem:

> Level sets of Smooth Functions: (Theorem 2.29 in LeeSM) Let M be a smooth manifold. If K is any closed subset of M, then there is a smooth non-negative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

So, there exist smooth functions $f, g : M \to [0, \infty)$ such that $f^{-1}(0) = A$ and $g^{-1}(0) = B$. Then, consider the function

$$F := \frac{f}{f+g}$$

Then, $F = 0$ if and only if $f = 0$, so $F^{-1}(0) = f^{-1}(0) = A$, and $F = 1$ if and only if $f = f + g$, which occurs for $g = 0$, so $F^{-1}(1) = g^{-1}(0) = B$.
