Introduction to Smooth Manifolds - Some solutions

Keshav Balwant Deoskar

January 25, 2024

Abstract

This is a collection of my personal solutions to some exercises from the book 'Introduction to Smooth Manifolds' (2nd edition) by John M. Lee.

These solutions are being written up and I work through the book, however I do not write up every exercise/problem and leave many to be updated later.

This has been written solely to deepen my own understanding of the material, but please feel free to contact me with corrections, concerns, or comments at kdeoskar@berkeley.edu.

Contents

1 Chapter 1: Smooth Manifolds

1.1 Topological Manifolds

Exercise 1.6: Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof:

To show Hausdorffness, consider two distinct points $x, y \in \mathbb{RP}^n, x \neq y$. Then,

To show second-countability, we use the following lemma:

For a topological space X with countable basis $\mathcal{B}$

(a) Any of its subsets $Y \subseteq X$ has a countable basis as well ($\mathcal{B}_Y = \{U : U = B \cap Y, B \in \mathcal{B}\}$)

(b) If $Z = {}^{X}/_{\sim}$ is a quotient space of X by an equivalence relation $\sim$, then

$$\{[x]_\sim : x \in B \in \mathcal{B}\}$$

will be a countable base for the topology induced on Z

Proof of Lemma: Do later.

Applying (a) tells us that $\mathbb{R}^{n+1} \setminus \{0\}$ as a subspace of $\mathbb{R}^n$ has a countable basis. Then, applying (b) with $Z = \mathbb{RP}^n = {}^{\mathbb{R}^{n+1} \setminus \{0\}}/_{\sim}$ where $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$, tells us that $\mathbb{RP}^n$ too has a countable basis.

Exercise 1.7: Show that $\mathbb{RP}^n$ is compact.

Proof:

Exercise 1.:

Proof:
