Math 214 Homework 9

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Q9-2. Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold, and V is a smooth vector field on M that is tangent to S.

(a) Show that for any integral curve γ of V such that $\gamma(t_0) \in S$, there exists $\epsilon > 0$ such that $\gamma\left((t_0 - \epsilon, t_0 + \epsilon)\right) \subseteq S$.

(b) Now assume S is properly embedded. Show that every integral curve that intersects S is contained in S.

(c) Give a counterexample to (b) if S is not closed.

Proof:

(a) Let $\gamma : J \to M$ be an integral curve such that $q := \gamma(t_0) \in S$. Since S is a smooth (immersed) submanifold of X, the restriction $V\big|_S$ is a smooth vector field on S which is $\iota-$related to V where $\iota : S \to M$ is the inclusion map.

By Proposition 9.2, there exists $\epsilon > 0$ and a smooth curve $\gamma_S : (-\epsilon, \epsilon) \to S$ such that γ_S is an integral curve starting at q.

Then, by Proposition 9.6 (Naturality of Integral curves), V and $V\big|_S$ being $\iota-$related means $\iota\left(\gamma_S\right) = \gamma_S$ is an integral curve in M. But due to the uniqueness of integral curves, it must be the case that $\gamma_S(t) = \gamma(t)$ for $t \in (-\epsilon, \epsilon) \cap J$. Therefore, for $t \in (-\epsilon, \epsilon) \cap J(?)$, we have $\gamma(t) \in S$ i.e. $\gamma\left((t_0 - \epsilon, t_0 + \epsilon)\right) \subseteq S$.

(b) Not sure yet.

(c) A embedded submanifold is properly embedded if and only if it is closed. So, for my counterexample, I'm thinking of the embedding $\mathbb{R}^n \to \mathbb{S}^n$ whose image is $\mathbb{S} \setminus \{N\}$ where N is the north pole. Then, any integral curve passing through the north pole intersects $\mathbb{R}^n$ but is not contained in $\mathbb{R}^n$.

Q9-3. Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

(a)

$$V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

(b)

$$W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

(c)

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

(d)

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Solution:

(a) An integral curve $\gamma(t) = (x(t), y(t))$ of this vector field satsifies the condition $\gamma'(t) = V_{\gamma(t)}$ which translates to

$$\begin{aligned} x'(t)\partial_x\big|_{\gamma(t)} + y'(t)\partial_y\big|_{\gamma(t)} &= x(t)\partial_x\big|_{\gamma(t)} + 1\cdot\partial_y\big|_{\gamma(t)} \\ \implies &\begin{cases} x'(t) = x(t) \\ y'(t) = 1 \end{cases} \\ \implies &\begin{cases} x(t) = ae^t \\ y(t) = bt \end{cases} \end{aligned}$$

So, the flow of the vector field is

$$\boxed{\tau_t(x,y) = \left(xe^t, yt\right)}$$

(b) An integral curve $\gamma(t) = (x(t), y(t))$ of this vector field is characterized by

$$\begin{aligned} &\begin{cases} x'(t) = x(t) \\ y'(t) = 2y(t) \end{cases} \\ \implies &\begin{cases} x(t) = ae^t \\ y(t) = b\left(e^{2t}\right) \end{cases} \end{aligned}$$

So, the flow of the vector field is

$$\boxed{\tau_t(x,y) = \left(xe^t, ye^{2t}\right)}$$

(c) Integral curves of this vector field are characterized by

$$\begin{cases} x'(t) = x(t) \implies x(t) = ae^t \\ y'(t) = -y(t) \implies y(t) = be^{-t} \end{cases}$$

So, the flow of the vector field is

$$\boxed{\tau_t(x,y) = \left(xe^t, ye^{-t}\right)}$$

(d) An integral curve of this vector field satisfies

$$\begin{aligned} &\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases} \\ \implies &\begin{cases} x''(t) = -x(t) \\ y''(t) = -y(t) \end{cases} \\ \implies &\begin{cases} x(t) = a\cos(t) - b\sin(t) \\ y(t) = a\sin(t) + b\sin(t) \end{cases} \end{aligned}$$

So the flow associated with this vector field is

$$\boxed{\tau_t(x,y) = (x\cos(t) - y\sin(t), x\cos(t) + y\sin(t))}$$

Q9-6. Prove Lemma 9.19 (the escape lemma).

Lemma 9.19: Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.

Proof:

We have the maximal integral curve $\gamma : J \to M$ where J has the form $(a, b), a \in [-\infty, \infty), b < \infty$. Suppose, for contradiction, there exists some $t_0 \in J$ such that $\gamma([t_0, b))$ is completely contained in some compact subset $K \subseteq M$. Then as $t_0 < t \to b$, the point $\gamma(t)$ must be approaching some limit point $\gamma(b)$.

But we know from Proposition 9.2 that given a smooth vector field V on M, for any point $p \in M$ there exists some $\epsilon > 0$ and smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of V starting at p.

Thus, for the point $\gamma(b)$, there exists some $\epsilon_b > 0$ such that $\Gamma : (-\epsilon_b, \epsilon_b) \to M$ is an integral curve starting at $\gamma(b)$ i.e. $\Gamma(0) = \gamma(b)$.

By the uniqueness of integral curves, Γ must agree with γ on the overlap of their domains. However, this means γ can be extended beyond $t = b$ by defining $\gamma(t) = \Gamma(t - b)$ for $t \in [b, \epsilon_b)$. This contradicts the assumption that γ is the maximal curve passing through $\gamma(b)$. Therefore, such a t_0 cannot exist.

Q9-10. For each vector field in Problem 9-3, find smooth coordinates in a neighborhood of $(1, 0)$ for which the given vector field is a coordinate vector field.

Solution:

For each vector field, we want to find coordinates $\left(s^i\right)$ around the point $(1, 0)$ such that the vector field is a coordinate vector field.

We can do this by finding a smooth curve passing through $(1, 0)$ which is not tangent to the vector field near the point. For each point on the curve, we can apply the flow of the vector field for time $t, t \in (-\epsilon, \epsilon)$. Doing this will generate a small neighborhood around $(1, 0)$.

(a) Take, say, the line $x = 1$ which we can parametrize using the $y-$coordinate. Then, applying the flow $\tau_t(x, y) = (xe^t, yt)$ we get the coordinate transformation $(x, y) \leftrightarrow (t, v)$ as $(x, y) = (e^t, vt)$.

(b) The smooth curve can be the vertical line $x = 1$ which we can parametrize using the y coordinate. Then, the new coordinate system (t, v) is obtained by starting at $(1, v)$ and flowing for time t using $\tau_t(x, y) = (xe^t), ye^{2t}$. This gives the coordinate tranformation $(x, y) = (e^t, ve^{2t})$.

(c) In this case, we get $(x, y) = (e^t, ve^{-t})$

(d) Here, we get $(x, y) = (\cos(t) - v\sin(t), \cos(t) + v\sin(t))$

Q9-19. Let M be $\mathbb{R}^3$ wth the $z-$axis removed. Define $V, W \in \mathfrak{X}(M)$ by

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2 + y^2}\frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial y} + \frac{x}{x^2 + y^2}\frac{\partial}{\partial z}$$

and let θ and ψ be the flows of V and W, respectively. Prove that V and W commute, but there exist $p \in M$ and $s, t \in \mathbb{R}$ such that $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are both defined but are not equal.

Proof:

Two vector fields commute if and only if their Lie Bracket is zero. Recall that, by Proposition 8.26, the Lie Bracket of two vector fields can be calulated as

$$[V,W] = \left(VW^j - WV^j\right)\frac{\partial}{\partial x^j}$$

In our case,

$$\begin{aligned}
[V,W] &= [V(0) - W(1)]\frac{\partial}{\partial x} + [V(1) - W(0)]\frac{\partial}{\partial y} + \left[V\left(\frac{x}{x^2+y^2}\right) - W\left(\frac{-y}{x^2+y^2}\right)\right]\frac{\partial}{\partial z} \\
&= 0 + 0 + \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\right]\frac{\partial}{\partial z} \\
&= 0
\end{aligned}$$

So the fields certainly commute. However, if we let $p = (1,0,0)$ and $s = t = 1$ and compute $\theta_t \circ \psi_s(p)$, $\psi_s \circ \theta_t(p)$ we see that they are NOT equal.

An integral curve $\gamma(t) = (x(t), y(t), z(t))$ of V through p satisfies the system of differential equations

$$\begin{aligned}
x'(t) &= 1 \\
y'(t) &= 0 \\
z'(t) &= -\frac{y}{x^2+y^2}
\end{aligned}$$

with the initial condition $(x(0), y(0), z(0)) = (1,0,0)$. The solution is given by

$$\begin{aligned}
x(t) &= t+1 \\
y(t) &= 0 \\
z(t) &= 0
\end{aligned}$$

Thus, $\theta_1(1,0,0) = (2,0,0)$. Next, let's conmpute $\psi_1(p)$. An integral curve $\beta(t) = (u(t), v(t), w(t))$ satisfies the system of differential equations

$$\begin{aligned}
u'(t) &= 0 \\
v'(t) &= 1 \\
w'(t) &= \frac{x}{x^2+y^2}
\end{aligned}$$

giving us the solution

$$\begin{aligned}
u(t) &= 1 \\
v(t) &= t \\
w(t) &= \arctan(t)
\end{aligned}$$

Thus, $\psi_1(p) = (1,1,\arctan(1)) = (1,1,\pi/4)$. Let's now find $\psi_1 \circ \theta_1(p)$ i.e. the same system of equations as above but instead with the initial condition $(u(0), v(0), w(0)) = (2,0,0)$. The solution is

$$\begin{aligned}
u(t) &= 2 \\
v(t) &= t+1 \\
w(t) &= \arctan\left(\frac{t+1}{2}\right) - \arctan\left(\frac{1}{2}\right)
\end{aligned}$$

Thus, $\psi_1 \circ \phi_1(p) = (2, 2, \arctan(1) - \arctan(1/2))$. Lastly, we calculate $\theta_1 \circ \psi_1(p)$. To do this, we solve the first system of equations but with the initial condition $(x(0), y(0), z(0)) = (1,1,\pi/4)$. Solving the system of equations gives us

$$\begin{aligned} x(t) &= t + 1 \\ y(t) &= 1 \\ z(t) &= -\arctan(t+1) + \frac{\pi}{2} \end{aligned}$$

Thus, $\theta_1 \circ \psi_1(p) = (2, 1, -\arctan(1) + \pi/2)$.
So, we find that $\theta_1 \circ \psi_1(p) \neq \psi_1 \circ \theta_1(p)$.

Q9-21. Let M be a smooth manifold. A ***smooth isotopy of*** M is a smooth map $H : M \times J \to M$, where $J \subseteq \mathbb{R}$ is an interval, such that for each $t \in J$, the map $H_t : M \to M$ defined by $H_t(p) = H(p,t)$ is a diffeomorphism.

(a) Suppose $J \subseteq \mathbb{R}$ is an open interval and $H : M \times J \to M$ is a smooth isotopy. Show that the map $V : J \times M \to TM$ defined by

$$V(t,p) = \frac{\partial}{\partial s}\bigg|_{s=t} H_s(H_t^{-1}(p))$$

is a smooth time-dependent vector field on M, whose time-dependent flow is given by $\psi(t, t_0, p) = H_t \circ H_{t_0}^{-1}(p)$ with domain $J \times J \times M$.

(b) Conversely, suppose J is an open interval and $V : J \times M \to M$ is a smooth time-dependent vector field on M whose time-dependent flow is defined on $J \times J \times M$. For any $t_0 \in J$, show that the map $H : M \times J \to M$ defined by $H(t,p) = \psi(t, t_0, p)$ is a smooth isotopy of M.

Proof:

Q10-1. Let E be the total space of the Möbius bundle constructed in Example 10.3.

(a) Show that E has a unique smooth structure such that the quotient map $q : \mathbb{R}^2 \to E$ is a smooth covering map.

(b) Show that $\pi : E \to \mathbb{S}^1$ is a smooth rank-1 vector bundle.

(c) Show that it is not a trivial bundle.

Proof:

(a) By Proposition 4.33, a topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

But Recall that if we have a smooth manifold M and a covering map $p : M \to N$, then N has a unique smooth structure such that p is locally a diffeomorphism.

So, in particular, given a topological covering map $p : \mathbb{R}^2 \to N$ there is a unique smooth structure on N such that p is a smooth covering map.

These two facts combined imply that E has a un ique smooth structure such that $q : \mathbb{R}^2 \to E$ is a smooth covering map.

Q10-10. Suppose M is a compact smooth manifold and $E \to M$ is a smooth vector bundle of rank k. Use transversality to prove that E admits a smooth section σ with the following property: if $k > \dim M$, then σ is nowhere vanishing; while if $k \leq \dim M$, then the set of points where σ vanishes is a smooth compact codimension-k submanifold of M. Use this to show that M admits a smooth vector field with only finitely many singular points.

Proof:

For each point $p \in M$, let U be the coordinate ball centered around p and B an open set whose closure is contained in U.

Since M is compact, we can choose finitely many points $p_1, \cdots, p_n$ such that $B_1, \cdots, B_n$ cover M. Let $\Phi_1, \cdots, \Phi_n$ be the local trivializations of the bundle over $U_1, \cdots, U_n$.

Now, let $\phi_i : M \to \mathbb{R}$ be a smooth function that is 1 on B_i and supported in U_i. Then, if we instead replace ϕ_i with $\phi_i / \sum_{i=1}^n \phi_i$ we can assume that $\sum_{i=1}^n \phi_i = 1$.

Define a map $F : M \times \left(\mathbb{R}^k\right)^n \to E$ as

$$F\left(p, (a_1^1, \cdots, a_k^1), \cdots, (a_1^n, \cdots, a_k^n)\right) = \sum_{i=1}^n \Phi_i^{-1}(pma_1^i, \cdots, a_k^i)\phi_i$$

where $\Phi_i^{-1}(p, a_1^i, \cdots, a_k^i)$ is defined to be 0 if $p \notin U_i$. This functions is smooth because it is smooth on each open set $U_i \times \left(\mathbb{R}^k\right)^n$.

Let's show that F is a submersion. Suppose $p \in B_j \subset U_j$ and $v \in T_pM$. Let $\gamma : (\epsilon, \epsilon) \to M$ be a smooth curve such that starting at p with $\gamma'(0) = (v, 0, \cdots, 0)$. Then, $(\Phi_j \circ F \circ \gamma)'(0) = (v, 0, \cdots, 0)$ because $\sum_{i=1}^n \phi_1 = 1$. Now, if for any $a \in \mathbb{R}$, the smooth curve $\tau : (-\epsilon, \epsilon) \to \left(\mathbb{R}^k\right)^n$ given by $\tau(\epsilon) = (p, a\epsilon, 0, \cdots, 0)$ then it satisfies $\tau(0) = p$ and $\tau'(0) = (0, a, 0, \cdots, 0)$ and $(\Phi_i \circ \tau)'(0) = (0, a, 0, \cdots, 0)$. This can be done for all kn coordinates of $(\mathbb{R}^k)^n$ in $M \times (\mathbb{R}^k)^n$. Since the B_i's cover M, this shows that F is a submersion.

F intersects transversely with M when viewed as an embedded submanifold of E. Hence, by the parametric transversality theorem, there exists $w \in (\mathbb{R}^k)^n$ such that the map from M to E given by $x \mapsto F(x, w)$ intersects transversely with M. This map is a smooth section by construction.

If $k > dim(M)$, then