Physics 137A: Quantum Mechanics **Fall 2023**

PSET 10, Due November 21

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Problem 1:

We consider two non-interacting particles in a harmonic oscillator potential with one particle being in state $|\psi\rangle = |\psi_0\rangle$ and the other is in the state $|\phi\rangle = \frac{1}{\sqrt{2}}(|\psi_0\rangle + |\psi_1\rangle)$ where $|\psi_i\rangle$ represents the i^{th} energy eigenstate of the Harmonic Oscillator.

1. First, we want to construct the symmetrized and anti-symmetrized wavefunctions of the two-particle system, corresponding to Bosons and Fermions respectively.

   The (un-normalized) symmetrized wavefunction would be

$$\begin{aligned}|\Psi, S\rangle &= |\psi\rangle + |\phi\rangle \\ &= |\psi_0\rangle + \frac{1}{\sqrt{2}}(|\psi_0\rangle + |\psi_1\rangle) \\ &= \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)|\psi_0\rangle + \frac{1}{\sqrt{2}}|\psi_1\rangle\end{aligned}$$

   Whereas, the (unnormalized) anti-symmetrized wavefunction is

$$\begin{aligned}|\Psi, A\rangle &= |\psi\rangle - |\phi\rangle \\ &= \left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)|\psi_0\rangle - \frac{1}{\sqrt{2}}|\psi_1\rangle\end{aligned}$$

   The states can then be appropriately normalized.

2. Perhaps we could do an experiment like the Stern-Gerlach, which separates particles into different classes of spin (bosons and fermions have different spins.)

Problem 2:

1. **Write down the totally symmetric three-particle state with proper normalization:**

   A totally symmetric three-particle state is one where switching any two indices does not change the state. i.e.

$$|\omega_1\omega_2\omega_3, S\rangle = |\omega_2\omega_1\omega_3, S\rangle$$

   A general symmetric 3-particle state is then a linear combination of the six states obtained by swapping two indices at a time.

$$|\omega_1\omega_2\omega_3, S\rangle = A\left[|\omega_1\omega_2\omega_3\rangle + |\omega_1\omega_3\omega_2\rangle + |\omega_2\omega_3\omega_1\rangle + |\omega_2\omega_1\omega_3\rangle + |\omega_3\omega_2\omega_1\rangle + |\omega_3\omega_1\omega_2\rangle\right]$$

where A is the overall normalization factor.

We can find A using the normalization condition

$$\left\langle \omega_1\omega_2\omega_3, S \middle| \omega_1\omega_2\omega_3, S \right\rangle = 1$$

That is,

$$1 = |A^2| \, (\text{A number of inner products})$$

Writing out each inner product would be a pain. We can skip that by recalling that only inner products of the form

$$\left\langle \omega_i\omega_j\omega_k \middle| \omega_i\omega_j\omega_k \right\rangle = \delta ii \delta_{jj} \delta kk = 1$$

will contribute, while the other inner products will vanish.

Thus, we have

$$6 \cdot |A|^2 = 1 \tag{10.1}$$

Thus,

$$\boxed{A = \frac{1}{\sqrt{6}}}$$

2. Similarly, we can find the totally antisymmetric state as a linear combination of the states obtained by swapping two indices at a time (introducing a minus sign each time we swap!):

$$|\omega_1\omega_2\omega_3, A\rangle = A\left[|\omega_1\omega_2\omega_3\rangle - |\omega_1\omega_3\omega_2\rangle + |\omega_2\omega_3\omega_1\rangle - |\omega_2\omega_1\omega_3\rangle + |\omega_3\omega_1\omega_2\rangle - |\omega_3\omega_2\omega_1\rangle\right]$$

Once again, we use the normalization condition,

$$\left\langle \omega_1\omega_2\omega_3, A \middle| \omega_1\omega_2\omega_3, A \right\rangle = 1$$

Again, the only inner products that survive are those in which the ordering is the exact same in the first and second argument. Additionally, since any state which has a (-1) takes an inner product with itself, also having a factor of (-1), the negative factors cancel each other and once again we arrive at

$$6 \cdot |A|^2 = 1$$

or

$$\boxed{A = \frac{1}{\sqrt{6}}}$$

3. The totally anti-symmetric state is a linear combination of states such that swapping any two indices introduces a negative sign. So, for example,

$$|\omega_1\omega_2\omega_3, A\rangle = -|\omega_2\omega_1\omega_3, A\rangle$$

Or, more generally, we have

$$|\omega_i\omega_j\omega_k, A\rangle = -|\omega_j\omega_i\omega_k, A\rangle$$

and so on.

But this is exactly the kind of behavior which is encapsulated by the **Levi-Civita symbol** defined as

$$\epsilon_{ijk} = \begin{cases} +1, & \text{symmetric permutation of ijk} \\ 0, & \text{otherwise} \\ -1, & \text{anti-symmetric permutation of ijk} \end{cases}$$

Thus, the totally anti-symmetric three-particle state is

$$\boxed{|\omega_i\omega_j\omega_k\rangle = \sum_{i\neq j\neq k} \frac{1}{\sqrt{6}}\epsilon_{ijk}|\omega_i\omega_j\omega_k, A\rangle}$$

Problem 3:

1. **Using the representations of the position and momentum operators, we want to verify the canonical commutation relations:**

The position and momentum operators are represented as

$$X_i \to x_i, \quad P_i \to -i\hbar\frac{\partial}{\partial x_i}$$

where $i, j = 1, 2, 3$ correspond to the x,y,z directions.

First off, the commutator between X_i and P_j is

$$[X_i, P_j] = X_iP_j - P_jX_i$$

So, applying the commutator to some ket $|\psi\rangle$ gives us

$$[X_i, P_j]|\psi\rangle = (X_iP_j - P_jX_i)|\psi\rangle$$

and projecting this equation onto position space, we have

$$\begin{aligned}[X_i, P_j]\psi(x) &= (X_iP_j - P_jX_i)\psi(x) \\ &= x_i \cdot \left(-i\hbar\frac{\partial\psi(x)}{\partial x_j}\right) - \left(-i\hbar\frac{\partial}{\partial x_j}\right)(x_i\psi(x))\end{aligned}$$

We apply the product rule on the second term, giving us

$$\begin{aligned}[X_i, P_j]\psi(x) &= (X_iP_j - P_jX_i)\psi(x) \\ &= -i\hbar x\frac{\partial\psi(x)}{\partial x_j} - \left(-i\hbar\frac{\partial x_i}{\partial x_j}\psi(x) - i\hbar x_i\frac{\partial\psi(x)}{\partial x_j}\right)\end{aligned}$$

The first and third terms cancel and we are left with the basis independent result

$$\boxed{[X_i, P_j] = i\hbar\delta_{ij}}$$

The commutators of the different position operators amongst themselves, and the momentum operators amongst themselves, are simpler.

Following the same idea, but writing it without reference to the basis in the interest of time, we have

$$\begin{aligned}[X_i, X_j] &= X_i X_j - X_j X_i \\ &= x_i x_j - x_j x_i \\ &= 0\end{aligned}$$

and

$$\begin{aligned}[P_i, P_j] &= P_i P_j - P_j P_i \\ &= \left(-i\hbar\frac{\partial}{\partial x_i}\right)\left(-i\hbar\frac{\partial}{\partial x_j}\right) - \left(-i\hbar\frac{\partial}{\partial x_j}\right)\left(-i\hbar\frac{\partial}{\partial x_i}\right) \\ &= \left(-i\hbar\frac{\partial}{\partial x_i x_j}\right) - \left(-i\hbar\frac{\partial}{\partial x_j x_i}\right) \\ &= 0 \text{ For appropriate wavefunctions, according to Clairaut's Thm.}\end{aligned}$$

2. The Angular Momentum operator is defined to be

$$L_i = \sum_{j,k} \epsilon_{ijk} X_j P_k$$

Now, the commutators of angular momentum with position and momentum are

$$\begin{aligned}[L_i, X_l] &= [\epsilon_{ijk} X_j P_k,\ X_l] \\ &= -[X_, \epsilon_{ijk} X_j P_k]\end{aligned}$$

where we are using **einstein summation notation** and suppressing the sum.

Then, we can use the following commutator identity:

$$[A, BC] = [A, B]C + B[A, C]$$

Thus, we get

$$[X_l, \epsilon_{ijk} X_j P_k] = \epsilon_{ijk}[X_l, X_j]P_k + \epsilon_{ijk} X_j [X_l, P_k]$$

Now, since $[X_l, X_j] = 0$ and $[X_l, P_k] = i\hbar\delta_{lk}$, we have

$$\begin{aligned}[X_l, \epsilon_{ijk} X_j P_k] &= \epsilon_{ijk} i\hbar X_j \delta_{lk} \\ &= i\hbar\epsilon_{ijl} X_j\end{aligned}$$

Therefore,

$$\boxed{[L_i, X_l] = -i\hbar\epsilon_{ijl}X_j}$$

or, being explicit with the summation,

$$\boxed{[L_i, X_l] = -i\hbar\sum_j \epsilon_{ijl}X_j}$$

Carrying out the same procedure but with $[L_i, P_l]$, we find that

$$\boxed{[L_i, X_l] = -i\hbar\sum_j \epsilon_{ijl}P_j}$$

3. Using the above results, we want to find the commutators amongst the different angular momentum operators:

   We have (using different indices less prone to being mistaken for one another)

$$\begin{aligned}[L_i, L_j] &= L_iL_j - L_jL_i \\ &= \epsilon iab\epsilon_{jcd}[X_aP_b, X_C, P_d] \\ &= \epsilon_{iab}\epsilon_{jcd}\left(X_a[P_b, X_c]P_d + X_c[X_a, P_d]P_b\right)\end{aligned}$$

   where the last equality follows from commutator identities.

   Then, using the commutator rules between the position and momentum operators, we can simplify this to be

$$[L_i, L_j] = i\hbar\left(\epsilon_{iab}\epsilon_{bjd} + \epsilon_{dib}\epsilon_{bja}\right)X_aP_d$$

   where we are summing over the indices other than i, j, so

$$\begin{aligned}[L_i, L_j] &= i\hbar\left(\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj} + \delta_{dj}\delta_{ia} - \delta_{da}\delta_{ij}\right)X_aP_d \\ &= i\hbar(X_iP_j - X_jP_i) \\ &= i\hbar\epsilon_{ijk}L_k\end{aligned}$$

4. The commutators $[L_z, r^2]$ and $[L_z, p^2]$ where $r^2 = X^2 + Y^2 + Z^2$ and $p^2 = P_x^2 + P_y^2 + P_z^2$ are each zero.

5.