Physics 137A: Quantum Mechanics	**Fall 2023**

PSET 09, Due November 15

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Problem 1:

We have a Quantum Harmonic Oscillator in the state

$$\mid \Psi(t)\rangle = c_1 e^{-iE_1 t/\hbar} \mid 1\rangle + c_2 e^{-iE_2 t/\hbar} \mid 2\rangle$$

and we know that the expectation value of energy is

$$\left\langle \Psi(t) \middle| \hat{H} | \Psi(t) \right\rangle = 2\hbar\omega$$

1. First, we want to find the coefficients c_1 and c_2. To do this, let's think about $\left\langle \Psi(t) \middle| \hat{H} | \Psi(t) \right\rangle = 2\hbar\omega$.

   This quantity is just the expectation value of the energy of the state, which can be expressed as

   $$\begin{aligned} \left\langle \Psi(t) \middle| \hat{H} | \Psi(t) \right\rangle &= \mathbb{P}(E_1)E_1 + \mathbb{P}(E_2)E_2 \\ &= |c_1|^2 E_1 + |c_2|^2 E_2 \end{aligned}$$

   The energy of a state $\mid n\rangle$ in a QHO is given by

   $$E_n = \hbar\omega(n + \frac{1}{2})$$

   So,

   $$E_1 = \frac{3}{2}\hbar\omega \text{ and } E_2 = \frac{5}{2}\hbar\omega$$

   Therefore,

   $$\begin{aligned} \left\langle \Psi(t) \middle| \hat{H} | \Psi(t) \right\rangle &= \mathbb{P}(E_1)E_1 + \mathbb{P}(E_2)E_2 \\ \implies 2\hbar\omega &= |c_1|^2 \frac{3}{2}\hbar\omega + |c_2|^2 \frac{5}{2}\hbar\omega \\ \implies 4 &= 3|c_1|^2 + 5|c_2|^2 \end{aligned}$$

   But, we also know at

   $$\left\langle \Psi(t) \middle| \Psi(t) \right\rangle = 1$$
   $$\implies |c_1|^2 + |c_2|^2 = 1$$

So, we have a system of linear equations for $|c_1|^2$ and $|c_2|^2$. Solving the system of linear equations, we find

$$|c_1|^2 = \frac{1}{2} \text{ and } |c_2|^2 = \frac{1}{2}$$
$$\implies c_1 = \frac{1}{\sqrt{2}} \text{ and } c_2 = \frac{1}{\sqrt{2}}$$

where, by convention, we assume $c_1, c_2 \in \mathbb{R}$.

Therefore, the state is

$$\boxed{\mid \Psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{3}{2}\omega t} \mid 1\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{5}{2}\omega t} \mid 2\rangle}$$

2. Now, for $\left\langle \hat{X} \right\rangle = \left\langle \Psi(t) \middle| \hat{X} | \Psi(t) \right\rangle$, we want to show that

$$\frac{d}{dt} \left\langle \hat{X} \right\rangle = -\omega^2 \langle X \rangle$$

Let's start by getting a more computationally useful expression for $\left\langle \hat{X} \right\rangle$.

In a QHO, we have

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators respectively.

To find $\left\langle \hat{X} \right\rangle = \left\langle \Psi(t) \middle| \hat{X} \middle| \Psi(t) \right\rangle$, we must first find $\langle X(0) \rangle = \left\langle \Psi(0) \middle| \hat{X} \middle| \Psi(0) \right\rangle$

This is given by

$$\begin{aligned}
\langle X(0) \rangle &= \frac{1}{2} \left(\langle 1 \mid + \langle 2 \mid \right) |\hat{X}| \left(\mid 1\rangle + \mid 2\rangle \right) \\
&= \frac{1}{2} \left(\langle 1 \mid + \langle 2 \mid \right) |\sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^\dagger + \hat{a})| \left(\mid 1\rangle + \mid 2\rangle \right) \\
&= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left(\langle 1 \mid + \langle 2 \mid \right) |(\hat{a}^\dagger + \hat{a})| \left(\mid 1\rangle + \mid 2\rangle \right) \\
&= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left(\langle 1 \mid + \langle 2 \mid \right) |(\hat{a}^\dagger + \hat{a})| \left(\mid 1\rangle + \mid 2\rangle \right)
\end{aligned}$$

Raising $\mid 1\rangle$ to $\mid 2\rangle$ but taking its inner product with $\mid 1\rangle$ will just return zero, and vice versa. So, writing only the non-zero terms, we have

$$\begin{aligned}
\langle X(0) \rangle &= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left(\left\langle 1 \middle| \hat{a} | 2 \right\rangle + \left\langle 2 \middle| \hat{a}^\dagger | 1 \right\rangle \right) \\
&= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left(2 \cdot \left\langle 1 \middle| 1 \right\rangle + 2 \cdot \left\langle 2 \middle| 2 \right\rangle \right) \\
&= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} (2 + 2)
\end{aligned}$$

So,

$$\boxed{\langle X(0)\rangle = 2\sqrt{\frac{\hbar}{2m\omega}}}$$

Now,

$$\langle X(t)\rangle = 2\sqrt{\frac{\hbar}{2m\omega}} \cdot \left(\langle 1 \mid e^{i\frac{3}{2}\omega t} + \langle 2 \mid e^{i\frac{5}{2}\omega t}\right) |(\hat{a}+\hat{a}^\dagger)| \left(e^{-i\frac{3}{2}\omega t} \mid 1\rangle + e^{-i\frac{5}{2}\omega t} \mid 2\rangle\right)$$

Again, the only terms which are non-zero are those where the ket is raised/lowered to match the bra. So,

$$\begin{aligned}\langle X(t)\rangle &= 2\sqrt{\frac{\hbar}{2m\omega}} \left(\langle 1 e^{i\frac{3}{2}\omega t} \mid \hat{a} \mid e^{-i\frac{5}{2}\omega t} 2\rangle + \langle 2 e^{i\frac{5}{2}\omega t} \mid \hat{a}^\dagger \mid e^{-i\frac{3}{2}\omega t} 1\rangle\right)\\ &= 2\sqrt{\frac{\hbar}{2m\omega}} \cdot 2\left(e^{-i\omega t} + e^{i\omega t}\right)\\ &= 8\sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)\end{aligned}$$

Now that we've found $\langle X(t)\rangle$ we can simply differentiate twice and verify that

$$\boxed{\frac{d}{dt}\left\langle \hat{X}\right\rangle = -\omega^2 \langle X\rangle}$$

<u>Problem 2:</u>

In this problem, we consider several properties of the quantum harmonic oscillator:

1. The annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$ are defined as

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} \hat{P}$$

and

$$\hat{a}^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} - i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} \hat{P}$$

So, we can express the position and momentum operators in terms of the annihilation and creation operators as

$$\hat{X} = \frac{1}{2}\left(\frac{m\omega}{2\hbar}\right)^{-1/2}\left(\hat{a}+\hat{a}^\dagger\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

and

$$\hat{P} = \frac{1}{2i}\left(\frac{1}{2m\omega\hbar}\right)^{-1/2}\left(\hat{a}-\hat{a}^\dagger\right) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$$

2. Now we want to find the expectation values $\left\langle\hat{X}\right\rangle$, $\left\langle\hat{X^2}\right\rangle$, and $\left\langle\hat{V}\right\rangle$ for the n^{th} energy eigenstate $|\,n\rangle$, where $V(\hat{X}) = \frac{1}{2}m\omega^2\hat{X}^2$ is the potential energy.

$$\begin{aligned}
\langle X\rangle = \left\langle n\middle|\hat{X}|n\right\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left\langle n\middle|(\hat{a}+\hat{a}^\dagger)|n\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\left\langle n\middle|\hat{a}|n\right\rangle + \left\langle n\middle|\hat{a}^\dagger|n\right\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\left\langle n\middle|n-1\right\rangle + \sqrt{n+1}\left\langle n\middle|n+1\right\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\delta_{n,n-1} + \sqrt{n+1}\delta_{n,n+1}\right] \\
&= 0
\end{aligned}$$

So,

$$\boxed{\left\langle\hat{X}\right\rangle = 0}$$

For $\left\langle\hat{X^2}\right\rangle$, we have

$$\begin{aligned}
\hat{X^2} &= \hat{X}\cdot\hat{X} \\
&= \frac{\hbar}{2m\omega}\left[\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right]
\end{aligned}$$

So, the expected value is calculated as

$$\left\langle\hat{X^2}\right\rangle = \frac{\hbar}{2m\omega}\left[\left\langle n\middle|\hat{a}\hat{a}|n\right\rangle + \left\langle n\middle|\hat{a}\hat{a}^\dagger|n\right\rangle + \left\langle n\middle|\hat{a}^\dagger\hat{a}|n\right\rangle + \left\langle n\middle|\hat{a}^\dagger\hat{a}^\dagger|n\right\rangle\right]$$

But we notice that applying $\hat{a}\hat{a}$ on $|\,n\rangle$ will give us $\sqrt{n\cdot(n-1)}\,|\,n-2\rangle$ and due to the orthogonality of the different energy states, we have $\left\langle n\middle|n-2\right\rangle = 0$ so the entire term is zero.

The same argument applies for $\hat{a}^\dagger\hat{a}^\dagger$, since that gives us $\sqrt{(n+1)(n+2)}\,|\,n+2\rangle$.
So, the only non-zero terms are the cross terms.
Now,

$$\left\langle n\middle|\hat{a}\hat{a}^\dagger|n\right\rangle = \left\langle\hat{a}^\dagger n\middle|\hat{a}^\dagger n\right\rangle = (\sqrt{n})^*(\sqrt{n})\cdot\left\langle n-1\middle|n-1\right\rangle = n$$

and similarly,

$$\left\langle n\middle|\hat{a}^\dagger\hat{a}|n\right\rangle = \left\langle\hat{a}n\middle|\hat{a}n\right\rangle = (\sqrt{n+1})^*(\sqrt{n+1})\cdot\left\langle n+1\middle|n+1\right\rangle = n+1$$

So, plugging these in,

$$\begin{aligned}
\left\langle\hat{X^2}\right\rangle &= \frac{\hbar}{2m\omega}\left[n+n+1\right] \\
&= \frac{\hbar}{2m\omega}\cdot(2n+1)
\end{aligned}$$

Thus,

$$\boxed{\left\langle \hat{X}^2 \right\rangle = \frac{\hbar}{m\omega} \cdot \left(n + \frac{1}{2}\right)}$$

Lastly, we want to find the expectation value $\langle V \rangle$ where $V(\hat{X}) = \frac{1}{2}m\omega^2\hat{X}^2$. Since we are just multiplying $\hat{X}^2$ by a constant, we can immediately find the mean value to be

$$\boxed{\langle V \rangle = \frac{\hbar\omega}{2} \cdot \left(n + \frac{1}{2}\right)}$$

3. In this part, we want to find the expectation values of $\left\langle \hat{P} \right\rangle$, $\left\langle \hat{P}^2 \right\rangle$, and $\langle T \rangle$ where $T = \frac{\hat{P}^2}{2m}$ is the kinetic energy, for the n^{th} energy eigenstates.

$$\begin{aligned}
\left\langle \hat{P} \right\rangle = \left\langle n \middle| \hat{P} | n \right\rangle &= i\sqrt{\frac{m\omega\hbar}{2}} \left\langle n \middle| \hat{a}^\dagger - \hat{a} | n \right\rangle \\
&= i\sqrt{\frac{m\omega\hbar}{2}} \left[\sqrt{n+1}\delta_{n,n+1} - \sqrt{n}\delta_{n,n-1}\right] \\
&= 0
\end{aligned}$$

So,

$$\boxed{\left\langle \hat{P} \right\rangle = 0}$$

We can express $\hat{P}^2$ as

$$\begin{aligned}
\hat{P}^2 &= \hat{P} \cdot \hat{P} \\
&= \left(i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\right) \cdot \left(i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\right) \\
&= (-1) \cdot (\frac{m\omega\hbar}{2}) \left[\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right]
\end{aligned}$$

So, the expectation value is

$$\left\langle \hat{P}^2 \right\rangle = \frac{-m\omega\hbar}{2} \left[\left\langle n \middle| \hat{a}^\dagger\hat{a}^\dagger | n \right\rangle - \left\langle n \middle| \hat{a}^\dagger\hat{a} | n \right\rangle - \left\langle n \middle| \hat{a}\hat{a}^\dagger | n \right\rangle + \left\langle n \middle| \hat{a}\hat{a} | n \right\rangle\right]$$

Once again, the only contributing terms are the cross terms, so we find

$$\begin{aligned}
\left\langle \hat{P}^2 \right\rangle &= \frac{-m\omega\hbar}{2} \cdot (-\left[(n+1) + n\right]) \\
&= \frac{m\omega\hbar}{2} \cdot (2n+1)
\end{aligned}$$

So,

$$\boxed{\left\langle \hat{P}^2 \right\rangle = m\omega\hbar \cdot \left(n + \frac{1}{2}\right)}$$

And to obtain the kinetic Energy, we just divide by $2m$, so again, we can directly find $\langle K \rangle$ to be

$$\boxed{\langle K \rangle = \frac{\hbar\omega}{2} \cdot \left(n + \frac{1}{2}\right)}$$

So, the relation between the expected Kinetic Energy and Potental is

$$\boxed{\left\langle \hat{K} \right\rangle = \left\langle \hat{V} \right\rangle}$$

The expected values for Kinetic and Potential Energy are the same!

4. Our state is a generic combination of the 0^{th} and 1^{st} states:

$$\mid \psi \rangle = a \mid 0 \rangle + be^{i\phi} \mid 1 \rangle$$

where a, b, ϕ are real and $a^2 + b^2 = 1$.
Using the results from parts (b) and (c) of this question, the expected values of kinetic and potential energy are

$$\begin{aligned}
\langle K \rangle &= |c_0|^2 K_0 + |c_1|^2 K_1 \\
&= a^2 \cdot \frac{\hbar\omega}{2}\left(0 + \frac{1}{2}\right) + |b \cdot e^{-i\phi}|^2 \frac{\hbar\omega}{2}\left(1 + \frac{1}{2}\right) \\
&= \frac{\hbar\omega}{2}\left(\frac{a^2}{2} + \frac{3b^2}{2}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle V \rangle = \langle K \rangle &= \frac{\hbar}{2}\left(\frac{a^2}{2} + \frac{3b^2}{2}\right) \\
\implies \langle V \rangle = \langle K \rangle &= \frac{\hbar}{4}(a^2 + 3b^2)
\end{aligned}$$

No, <u>the result will not change</u> if we consider the time evolution of the state because the coefficients $c_0^{'} = c_0 e^{-iEt/\hbar}$ and $c_1^{'} = c_1 e^{-iEt/\hbar}$ will still have the same squared magnitudes

$$|c_0^{'}|^2 = |c_0|^2 \text{ and } |c_1^{'}|^2 = |c_1|^2$$

because the complex exponential $e^{-iEt/\hbar}$ has modulus one.

Problem 3:

In this problem, we find the relation between $\langle V \rangle$ and $\langle T \rangle$ using another method:

1. Recalling that the hamiltonian for a quantum harmonic oscillator reads as

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2$$

Let us first calculate the commutator $\left[\hat{H}, \hat{P}\hat{X}\right]$.

We can express each of these operators in terms of the annihilation and creation operators:

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right)$$

$$\hat{P} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Now, the product $\hat{P}\hat{X}$ is

$$\begin{aligned}
\hat{P}\hat{X} &= i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right) \cdot \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \\
&= \frac{i\hbar}{2}\left(\hat{a}^\dagger - \hat{a}\right)\left(\hat{a}^\dagger + \hat{a}\right) \\
&= \frac{i\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)
\end{aligned}$$

But recall that $[\hat{a}^\dagger, \hat{a}] = \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger = 1$, So

$$\hat{P}\hat{X} = \frac{i\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 1\right)$$

We can now calucate the commutator as

$$\left[\hat{H}, \hat{P}\hat{X}\right] = \hat{H}\left(\hat{P}\hat{X}\right) - \left(\hat{P}\hat{X}\right)\hat{H}$$

Another way we can calculate the commutator is

$$\left[\hat{H}, \hat{P}\hat{X}\right] = \left[\hat{H}, \hat{P}\right]\hat{X} + \hat{P}\left[\hat{H}, \hat{X}\right]$$

We'll not try to evaluate each of the commutators on the RHS, using the following properties:

$$\begin{aligned}
\left[\hat{a}^\dagger, \hat{H}\right] &= \left[\hat{a}^\dagger, \hat{H}\right] \\
&= \left[\hat{a}^\dagger, \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right] \\
&= \left[\hat{a}^\dagger, \hbar\omega\left(\hat{a}^\dagger\hat{a}\right)\right] \\
&= \hbar\omega\left[\hat{a}^\dagger, \hat{a}^\dagger\hat{a}\right] \\
&= \hbar\omega(\left[\hat{a}^\dagger, \hat{a}^\dagger\right]\hat{a} + \hat{a}^\dagger\left[\hat{a}^\dagger, \hat{a}\right]) \\
&= \hbar\omega(0\cdot\hat{a} - \hat{a}^\dagger\left[\hat{a}, \hat{a}^\dagger\right]) \\
&= -\hbar\omega\hat{a}^\dagger
\end{aligned}$$

So, in conclusion,

$$\boxed{\left[\hat{a}^\dagger, \hat{H}\right] = -\hbar\omega\hat{a}^\dagger}$$

Doing a similar calculation for $\left[\hat{a}, \hat{H}\right]$, we find that

$$\boxed{\left[\hat{a}, \hat{H}\right] = \hbar\omega\hat{a}}$$

(a) Okay now let's actually compute the commutators. First, the one with momentum and the Hamiltonian:

$$\begin{aligned}
\left[\hat{H}, \hat{P}\right] &= -\left[\hat{P}, \hat{H}\right] \\
&= -\left[i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right), \hat{H}\right] \\
&= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\left(\hat{a}^\dagger - \hat{a}\right), \hat{H}\right] \\
&= -i\sqrt{\frac{m\omega\hbar}{2}}(\left[\hat{a}^\dagger, \hat{H}\right] - \left[\hat{a}, \hat{H}\right]) \\
&= -i\sqrt{\frac{m\omega\hbar}{2}}\left(-\hbar\omega\hat{a}^\dagger - \hbar\omega\hat{a}\right) \\
&= (i\hbar\omega)\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger + \hat{a}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right)\cdot\sqrt{\frac{2m\omega}{\hbar}}\cdot(i\hbar\omega)\sqrt{\frac{m\omega\hbar}{2}} \\
&= \left(\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right)\right)\cdot i\hbar\omega m \\
&= i\hbar\omega m\hat{X}
\end{aligned}$$

So,

$$\boxed{\left[\hat{H}, \hat{P}\right] = i\hbar m^2\omega^2 \hat{X}}$$

(b) Next, the commutator between the position operator and the Hamiltonian:

$$\begin{aligned}
\left[\hat{H}, \hat{X}\right] &= -\left[\hat{X}, \hat{H}\right] \\
&= -\left[\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right), \hat{H}\right] \\
&= -\sqrt{\frac{\hbar}{2m\omega}}\left[\left(\hat{a}^\dagger + \hat{a}\right), \hat{H}\right] \\
&= -\sqrt{\frac{\hbar}{2m\omega}}\left(\left[\hat{a}^\dagger, \hat{H}\right] + \left[\hat{a}, \hat{H}\right]\right) \\
&= -\sqrt{\frac{\hbar}{2m\omega}}\left(-\hbar\omega\hat{a}^\dagger + \hbar\omega\hat{a}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \cdot \hbar\omega\left(\hat{a}^\dagger - \hat{a}\right) \\
&= \left(\sqrt{\frac{\hbar}{2m\omega}} \cdot \hbar\omega\right) \cdot \left(\frac{1}{i}\sqrt{\frac{2}{m\omega\hbar}}\right) \cdot \left(i\sqrt{\frac{m\omega\hbar}{2}}\right)\left(\hat{a}^\dagger - \hat{a}\right) \\
&= -i\frac{\hbar\omega}{m\omega}\hat{P}
\end{aligned}$$

So,

$$\boxed{\left[\hat{H}, \hat{X}\right] = -i\frac{\hbar}{m}\hat{P}}$$

So, finally, let's tackle the original commutator we were trying to evaluate: $\left[\hat{H}, \hat{P}\hat{X}\right]$

We have

$$\begin{aligned}
\left[\hat{H}, \hat{P}\hat{X}\right] &= \left[\hat{H}, \hat{P}\right]\hat{X} + \hat{P}\left[\hat{H}, \hat{X}\right] \\
&= \left(i\hbar\omega m\hat{X}\right)\hat{X} + \hat{P}\left(\frac{-i\hbar}{m}\hat{P}\right) \\
&= i\hbar\omega m\hat{X}^2 - \frac{i\hbar}{m}\hat{P}^2 \\
&= \frac{2i\hbar}{\omega} \cdot \left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2m}\hat{P}^2\right)
\end{aligned}$$

$$\boxed{\left[\hat{H}, \hat{P}\hat{X}\right] = \frac{2i\hbar}{\omega} \cdot \left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2m}\hat{P}^2\right)}$$

2. We found one expression for the commutator $\left[\hat{H}, \hat{P}\hat{X}\right]$ in the previous section, but another way to find it is as

$$\left[\hat{H}, \hat{P}\hat{X}\right] = \hat{H}\left(\hat{P}\hat{X}\right) - \left(\hat{P}\hat{X}\right)\hat{H}$$

and we found earlier that $\hat{P}\hat{X} = \frac{i\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 1\right)$
So,

$$\begin{aligned}
\left[\hat{H}, \hat{P}\hat{X}\right] &= \left[\hat{H}, \frac{i\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 1\right)\right] \\
&= \frac{i\hbar}{2}\left[\hat{H}, \left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 1\right)\right] \\
&= \frac{i\hbar}{2}\left[\hat{H}, \left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right)\right] \\
&= \frac{i\hbar}{2}\left[\hat{H}, \hat{a}^\dagger\hat{a}^\dagger\right] + \frac{i\hbar}{2}\left[\hat{H}, \hat{a}\hat{a}\right]
\end{aligned}$$

Let's look at each of these commutators separately:

(a) For the first one,

$$\begin{aligned}
\left[\hat{H}, \hat{a}^\dagger\hat{a}^\dagger\right] &= \left[\hat{H}, \hat{a}^\dagger\right]\hat{a}^\dagger + \hat{a}^\dagger\left[\hat{H}, \hat{a}^\dagger\right] \\
&= -\left[\hat{a}^\dagger, \hat{H}\right]\hat{a}^\dagger - \hat{a}^\dagger\left[\hat{a}^\dagger, \hat{H}\right] \\
&= (\hbar\omega\hat{a}^\dagger)\hat{a}^\dagger - \left(\hat{a}^\dagger(\hbar\omega\hat{a}^\dagger)\right) \\
&= \hbar\omega\hat{a}^\dagger\hat{a}^\dagger - \hbar\omega\hat{a}^\dagger\hat{a}^\dagger \\
&= 0
\end{aligned}$$

(b) For the second one,

$$\begin{aligned}
\left[\hat{H}, \hat{a}\hat{a}\right] &= \left[\hat{H}, \hat{a}\right]\hat{a} + \hat{a}\left[\hat{H}, \hat{a}\right] \\
&= -\left[\hat{a}, \hat{H}\right]\hat{a} - \hat{a}\left[\hat{a}, \hat{H}\right] \\
&= -(\hbar\omega\hat{a})\hat{a} - (-\hat{a}(\hbar\omega\hat{a})) \\
&= -\hbar\omega\hat{a}\hat{a} + \hbar\omega\hat{a}\hat{a} \\
&= 0
\end{aligned}$$

So, we find that $\left[\hat{H}, \hat{P}\hat{X}\right] = 0$!

This combined with the result from the first part of the question tells us that, for a quantum harmonic oscillator, we have

$$\frac{2i\hbar}{\omega} \cdot \left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2m}\hat{P}^2\right) = 0$$

Which means,

$$\left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2m}\hat{P}^2\right) = 0$$

So,

$$\boxed{\frac{1}{2m}\hat{P}^2 = \frac{1}{2}m\omega^2\hat{X}^2}$$

Thus, we expect the Kinetic and Potential Energies to be the same!