Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 January 30 - Manifolds with boundaries, Smooth maps

Recap

- So far, we've seen ***Smooth Manifolds***, which are pairs of Topologcal manifolds with a maximal smooth atlas $(M^n, \mathcal{A})$
- Then, we saw a slight generalization by developing ***Smooth Manifolds with Boundary***.
- Today, we'll build up some more on manifolds with boundaries, and then start talking about smooth maps on/between manifolds.

1.1 Smooth Manifolds with Boundary

Theorem 1.46 (LeeSM, Smooth Invariance of Boundary): If $p \in M$ where M is a smooth manifold with boundary and there are two charts $(U, \phi), (V, \psi)$ covering p then

$$\phi(p) \in \partial\mathbb{H}^n \iff \psi(p) \in \partial\mathbb{H}^n$$
$$\phi(p) \in \mathrm{Int}\mathbb{H}^n \iff \psi(p) \in \mathrm{Int}\mathbb{H}^n$$

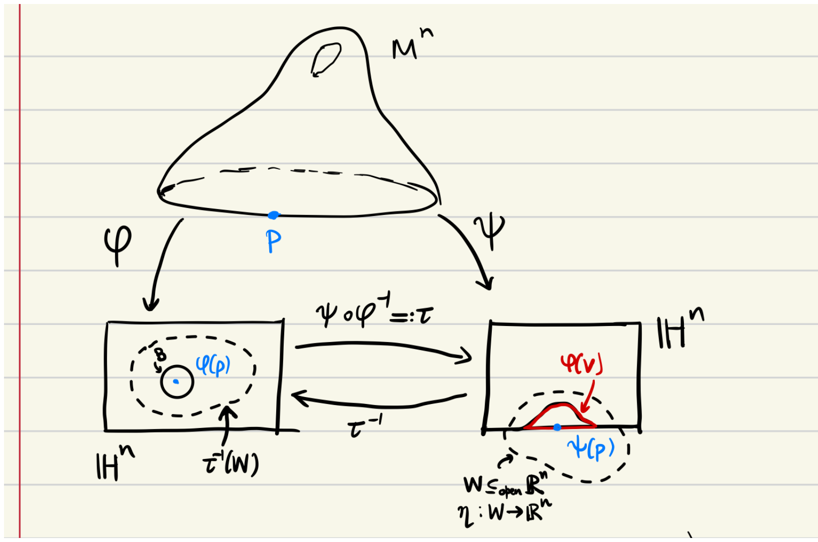
and

$$\psi \circ \phi^{-1}\big|_{\partial\mathbb{H}^n \cap \phi(U \cap V)} : \partial\mathbb{H}^n \cap \phi(U \cap V) \to \partial\mathbb{H}^n \cap \psi(U \cap V)$$

is a smooth transition map from $\phi\big|_{U \cap V \cap \phi^{-1}(\partial\mathbb{H}^n)}$ to $\psi\big|_{U \cap V \cap \psi^{-1}(\partial\mathbb{H}^n)}$

”Proof”: (Sketch; Only first part of Thm.)

Suppose, to the contrary, there exists $p \in M$ which is covered by interior (smooth) chart (U, ϕ) and boundary (smooth) chart (V, ψ) i.e. $\psi(p) \in \partial\mathbb{H}^n$.



Then, denote the transition map $\psi \circ \phi^{-1}$ as τ. By the compatibility of smooth charts, tau and tau^{-1} are smooth, in the sense that they can be extended to smooth functions if necessary.

Then, there exists some open neighborhood W around $\psi(p)$ and continuous map $\eta : W \to \mathbb{R}^n$ such that $\eta\big|_{W\cap\psi(U\cap V)} = \tau^{-1}$.

On the other hand, we assumed ϕ to be an interior map, meaning there exists an open ball B centered around $\phi(p)$ and contained within $\phi(U \cap V)$, so τ is continuous on B in the ordinary sense. After shrinking, if necessary, we can assume $B \subset \tau^{-1}(W)$.

Then, $\eta \circ \tau\big|_B = \tau^{-1} \circ \tau\big|_B = id_B$. So, it follows from the chain rule that $D\eta(\tau(x)) \circ D\tau(x)$ is the identity map for each $x \in B$.

The derivative map is a linear transformation, so it is a square matrix. This means it is non-singular. Corollary C.36 from LeeSM then tells us that $\tau(x)$ is an open map (maps open sets to open sets). But this contradicts our assumption that (V, ψ) is a boundary chart! According to that assumption, $\psi(V)$ is not open.

So it must be the case that p is an interior point or boundary point in *both* charts.

Corollary: Given a smooth manifolds M, we can split it up into two disjoint parts

$$M = \mathrm{Int}M \coprod \partial M$$

and Int$M$ us a smooth $n-$manifold without boundary, while ∂M is a smooth $(n-1)$-manifold without boundary.

Note: There's a *further* generalization called ***Manifolds with corners*** on which calculus can be extended as well (requires much more machinery than extending calculus to manifolds with boundary though).

Remark: It may be the case that

$$\text{Int}_{mfd}M \neq \text{Int}_{top}M$$

and

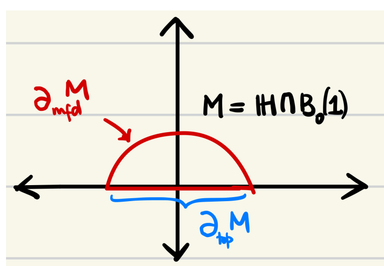
$$\partial_{mfd}M \neq \partial_{top}M$$

Example:

- $M = \{x^n > 0\} \subset \mathbb{R}^n$. As a manifold it has no boundary i.e. $\partial_{mfd}M\emptyset$. However, as a topological space, its boundary is just the $x^n = 0$ plane $\partial_{top}M = \{x^n = 0\}$.
- Consider any manifold with nonempty boundary, M. Then viewing M as a subspace of itself tells us

$$\partial_{top}M = \emptyset$$
$$\text{Int}_{top}M = \emptyset$$

- Consider $M = S^n \subset \mathbb{R}^{n+1}$. Then, $\partial_{mfd}M = \emptyset$ but $\partial_{top}M = \mathbb{S}^n$.
- Consider $M = \mathbb{H}^n \cap B_1(0)$. Then, $\partial_{mfd}M$ is just the "diameter" of the hemi-sphere but $\partial_{top}M$ is the entire "circumference."



1.2 Smooth maps (between mfds w/ or w/out bdy)

1.2.1 Smooth maps from a Manifold to $\mathbb{R}^n$

We say a function $f : M^n \to \mathbb{R}^m$ is ***smooth*** if for any point $p \in M$ there exists a smooth chart (U, ϕ) such that $p \in M$ and

$$f \circ \phi^{-1}\big|_{\phi(U)} : \phi(U) \to \mathbb{R}^m$$

is smooth.

Notation: We defnote the set of all smooth functions $f : M \to \mathbb{R}$ as C^∞.

Lemma: If $f : M^n \to \mathbb{R}^m$ is smooth, the for *any* smooth chart (V, ψ), the coordinate representation $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}^m$ is smooth.

Proof: Consider the chart (V, ψ) and a point $p \in V$. Now, around p, there is another chart (U, ϕ) such that $p \in U$.

Now, by the definition of smoothness, $f \circ \phi^{-1}$ is smooth. Using the compatibility of charts, the map $\phi \circ \psi^{-1}$ is also smooth. Thus, $f \circ \psi^{-1}$ is also smooth because

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

Example: Charts give smooth maps to $\mathbb{R}^n$.

1.2.2 Smooth maps between Manifolds

Consider $m-$ and $n-$ dimensional smooth manifolds, M^m and N^n.

A map $F : M^m \to N^n$ is ***smooth*** if for all points $p \in M$ there are charts (U, ϕ) on M and (V, ψ) on N such that

- $p \in U$
- $F(U) \subset V$ (OR $F^{-1}(V) \cap U$ is open in M)
- $\psi \circ F \circ \phi^{-1}$ is smooth as a map between euclidean spaces.

Note: The function $\psi \circ F \circ \phi^{-1}$ is called the "coordinate representation of F with respect to $(U, \phi), (V, \psi)$".

The above definition gives us the implication that smooth maps are continuous, which is important. There are other ways we could have generalized the notion of smoothness but many of them have pathological cases of non-continuous functions satisfying their conditions.

(See also Proposition 2.4 in LeeSM)

Lemma: If $F : M \to N$ is smooth, $(U, \phi), (V, \psi)$ are charts on M, N respectively, then:

- If $F(U) \subset V$, then the coordinate representation $\psi \circ F \circ \phi^{-1}$ is smooth.

Proof: Exercise.

Remark: (Relating coordinate representations by diffeomorphisms)

If we have a function $F : M \to N$, charts $(U, \phi), (V, \psi)$ on M, N respectively such that $F(U) \subset V$, and another set of charts $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi})$ on M, N respectively such that $F(\tilde{U}) \subset \tilde{V}$, then we can pass between these dfferent coordinate representations by the transition maps $\phi \circ \psi{-1}$, $\psi \circ \psi^{-1}$, etc.

1.2.3 Examples

- Scalar functions $f : M \to \mathbb{R}$, such as morse functions (say, from a torus to the reals).
- Paths on a manifold, $f : [0, 1] \to M$.

1.3 Partitions of Unity

- These give us nice ways of combining things to endow some global properties on our manifold.

- In particular, they give us the paracompactness of manifolds.

Motivating example:

Suppose we're given two smooth functions on $\mathbb{R}$, $f_-, f_+ \in C^\infty(\mathbb{R})$ and we want to paste the two functions together to get a new smooth function which coincides with f_- below $x = 0$ and with f_+ above $x = 0$.

[Fill in missing stuff from picture and book]

Observe that
$$\begin{cases} \psi_+ + \psi_- = 1 \\ \operatorname{supp}(\psi_-) \subset (-\infty, 1) \\ \operatorname{supp}(\psi_+) \subset (-1, \infty) \end{cases}$$
We say that $\{\psi_+, \psi_-\}$ is a ***partition of unity*** subordinate to the open cover $\{(-infty, 1), (-1, \infty)\}$

Let $\mathfrak{X} = \{X_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X. Then, a ***partition of unity subordinate to*** $\mathfrak{X}$ is a family of continuous maps $\{\psi_\alpha : X \to \mathbb{R}\}$ such that

1. $0 \leq \psi_\alpha \leq 1$
2. $\operatorname{supp}(\psi_\alpha) \subset X_\alpha$
3. $\{\operatorname{supp}(\psi_\alpha)\}_\alpha$ is locally finite
4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in X$.

Theorem: (Existence of Partition of Cover)
For any open cover $\mathfrak{X}$ of a smooth manifold with boundary M, there exists a smooth partition of unity subordinate to $\mathfrak{X}$.

Proof: We will construct smooth $(\tilde{\psi}_\alpha)_{\alpha \in A}$ on M such that

1. $\tilde{\psi}_\alpha \geq 0$
2. $\operatorname{supp}(\tilde{\psi}_\alpha) \subset X_\alpha$
3. $\operatorname{supp}(\tilde{\psi}_\alpha)$ locally finite
4. $\bigcup_{\alpha \in A} \underbrace{\{\tilde{\psi}_\alpha > 0\}}_{=\{x \in M : \tilde{\psi}_\alpha(x) > 0\}} = M$

(We'll construct the collection $\{\tilde{\psi}_\alpha\}$ next time, but once we have them we can define

$$\psi_\beta =$$

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