Feynman Diagrams & QFT Notes

Keshav Balwant Deoskar

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These are notes taken from lectures on Feynman Diagrams and QFT delivered by Ivan Burbano. Any errors that may have crept in are solely my fault.

Contents

1 January 26 - First meeting! Setting goals.

1.1 The goal

This course wll be a very first-principles, barebones experience. Our goal for the next month will be to develop the tools to solve the integral

$$\int_{-\infty}^{\infty} dx\ e^{-S(x)}O(x)$$

where $S(x) = \frac{1}{2}ax^2 + \frac{1}{3!}gx^3 + \frac{1}{4!}\lambda x^2 + \cdots$, the constants a, g, λ etc. must be positive reals and are called *coupling constants* and $O(x)$ is a polynomial in x.

What is the physical motvation for this integral?

There are *two* sides to the physics related to this integral: Quantum (this is what we want!) and Statistical (this is what we do!).

[Insert picture]

1.2 The Quantum regime

In the Quantum regime, *fields* become *fuzzy!* We can't quite pin down what the configuration of the field is, rather we can tell what the *probability amplitude* of any given field configuration will be at a point in (space)time.

So, the fundamental question in QM is:

> If at time t_i we prepare a field ϕ_i, what is the probability amplitude that, at t_f, I measure ϕ_f?

We can say that

$$\phi_f = \langle \phi_f | U(t_f, t_i) | \phi_i \rangle$$

where $U(t_f, t_i)$ is called the *propagator* or the *time-evolution operator.*

According to feynman, the answer can be found by integrating the *action* over the space of all field configurations.

i.e.

$$= \int \mathcal{D}\Phi e^{iS(\Phi)}$$

where we are integrating over all field configurations $\Phi(x)$ such that $\Phi = \phi_i$ at t_i and $\Phi = \phi_f$ at t_f.

In more formal notation, the set over which we are integrating is

$$\{\phi \in C^{\infty}([t_i, t_f] \times \Sigma) \quad : \quad \phi\big|_{\Sigma_i} = \phi_i ; \phi\big|_{\Sigma_f} = \phi_f\}$$

where Σ denoted the space we're working on.

1.3 The Statistical Side of Things

This is what we actually do!

Suppose we have some region, say a table, T populated by some *field* $\phi(x)$. For instance, it could be some spin distrbiution i.e. the field assigns each point on the table T with some spin.

Boltzmann showed that the probability (*not* probability amplitude) for the field to be in configuration ϕ is proportional to

$$e^{-S(\phi)}$$

where $S(\phi)$ is the energy of the field configuration.

In particular, the probablity is

$$\frac{e^{-S(\phi)}}{\mathcal{Z}}$$

where

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)}$$

is the integral over the space (of all field configurations) of probabilities. It's called the **Partition Function**.

Quick aside, what about dimensions?

The partition function is dimensionless, whereas the feynman path integral we covered in the Quantum Regime *is* dimension-ful. This is our first hint that something is amiss with the path-integral. (Has to do with renomalization).

Note: The ket $|x\rangle$ *does have units.* In particular, the completeness relation tells us

$$\int dx\, |x\rangle\langle x| = 1$$

So, $|x\rangle$ has units of $\frac{1}{\sqrt{\text{Length}}}$

1.4 An example from Stat. Mech: Kinetic Theory of gasses

Now, let's actually solve an integral! Let's compute

$$\int_{-\infty}^{\infty} d\phi e^{-S(\phi)} O(\phi)$$

where

$$S(\phi) = \frac{1}{2}\phi m^2 \phi - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$$

where $m^2 > 0$.

1. Partition Function : $O = 1$ For a free theory, we have $g = \lambda = \cdots = 0$

   So, our integral turns into

$$\boxed{\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2}}$$

   This integral is intimately connected to the Kinetic Theory of Gasses.
   Sol:

$$
\begin{aligned}
\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2} &= \sqrt{\left(\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2}\right)^2} \\
&= \sqrt{\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2} \int_{-\infty}^{\infty} d\psi e^{-\frac{1}{2}m^2\phi^2}} \\
&= \sqrt{\int_{\mathbb{R}^2} d\phi d\psi \; e^{-\frac{1}{2}m^2(\phi^2+\psi^2)}}
\end{aligned}
$$

Now, we convert to polar coordinates with $r^2 = \psi^2 + \phi^2$, $u = \frac{1}{2}r^2m^2$, $du = drr$

$$
\begin{aligned}
&= \sqrt{\int_{\mathbb{R}^2} \underbrace{d\phi d\psi}_{=rdrd\theta} \; e^{-\frac{1}{2}m^2(\phi^2+\psi^2)}} \\
&= \sqrt{2\pi \int_{0}^{\infty} dr \; r \; e^{-\frac{1}{2}r^2m^2}} \\
&= \sqrt{\frac{2\pi}{m^2}} \int_{0}^{\infty} du \; e^{-u} \\
&= \sqrt{\frac{2\pi}{m^2}}
\end{aligned}
$$

Exercises!

1.
$$\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2+J\phi}$$

2.
$$\int_{-\infty}^{\infty} d\phi_1 d\phi_2 e^{-\frac{1}{2}\vec{\phi^T}\cdot A\vec{\phi}}$$

where A is any symmetric 2×2 matrix (can generalize to $n \times n$ matrices!).

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2.1 Last time

- We computed the integral ($m^2 > 0$)

$$\mathcal{Z} = \int_{-\infty}^{\infty} d\phi e^{\phi m^2 \phi} = \sqrt{\frac{2\pi}{m^2}}$$

- We called this integral the ***partition function of our free theory*** (the "free" tells us that the action $S(\phi) = \frac{1}{2}\phi m^2 \phi$ is quadratic).

2.2 Today

- If we have a polynomial $\mathcal{O}(\phi)$, we want to calculate the expectation value

$$\langle \mathcal{O} \rangle := \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}\phi m^2 \phi} \mathcal{O}(\phi)$$

2.3 Schwinger-Dyson equation

Consider the following where $S(\phi)$ and $\mathcal{O}(\phi)$ are just some polunomails and $S(\phi) \to \infty$ as $\phi \to \pm\infty$

Theorem (Version 1)

$$\int_{-\infty}^{\infty} d\phi e^{-S(\phi)} \frac{d\mathcal{O}}{d\phi} = \int_{-\infty}^{\infty} d\phi e^{-S(\phi)} \frac{dS}{d\phi} \mathcal{O}$$

We can divide both integrals by $\mathcal{Z}$ to obtain the other form of the theorem:

Theorem (Version 2)

$$\langle \frac{d\mathcal{O}}{d\phi} \rangle = \langle \frac{dS}{d\phi} \mathcal{O} \rangle$$

Proof:

$$\int_{\infty}^{\infty} d\phi \frac{d}{d\phi} \left(e^{-S(\phi)} \mathcal{O}(\phi) \right) = \left[e^{-S(\phi)} \mathcal{O}(\phi) \right]_{-\infty}^{\infty} = 0$$

Now, why is this useful? Our whole goal today was to compute Expectation values but we haven't been calculating a whole lot of them.

Well, notice the following:

- $\langle 1 \rangle = 1$
- $\deg \frac{d\mathcal{O}}{d\phi} = \deg \mathcal{O}$ - 1

- $\deg\left(\frac{dS}{d\phi}\cdot\mathcal{O}\right) = \deg S + \deg\mathcal{O}$ - 1

s By proving the Schwniger-Dyson equation, we've gotten a relation between something of higher of degree and something of lower degree. So, we can go recursively until we reach a polynomial of degree 1, whose expectation value will be piss easy to calculate since $\langle 1\rangle = 1$.

> Also, quick side note, $\frac{dS}{d\phi}$ give us the ***Lagrange Equations***, and so are called the ***Equations of motion!***

Note: We call anything that's a function of ϕ an operator. We'll relate this to the more familiar notion of an operator in Quantum Mechanics later.

2.4 Returning to our Free Theory

Now, returning to our free theory with action $S(\phi) = \frac{1}{2}\phi m^2\phi$ (Equations of motion $\frac{dS}{d\phi} = m^2\phi$). To caluclate the expectation value $\langle\phi^2\rangle$ we express it as

$$\begin{aligned}\langle\phi^2\rangle &= \frac{1}{m^2}\langle\underbrace{m^2\phi}_{dS/d\phi}\underbrace{\phi}_{\mathcal{O}}\rangle \\ &= \frac{1}{m^2}\langle\frac{dS}{d\phi}\mathcal{O}\rangle\end{aligned}$$

Then, applying the Schwinger-Dyson Equation, we get

$$\langle\phi^2\rangle = \underbrace{\frac{1}{m^2}}_{\langle\phi\phi\rangle}\langle\frac{d\phi}{d\phi}\rangle = \frac{1}{m^2}$$

This term $1/m^2$ is "contracted" from the $\langle\phi\phi\rangle$ term, and is called the ***propagator***.

2.5 ϕ^4 Free Theory

Let's now do something similar for ϕ^4.

$$\begin{aligned}\langle\phi^4\rangle &= \frac{1}{m^2}\langle m^2\phi\ \phi^3\rangle \\ &= \frac{1}{m^2}\langle\underbrace{\frac{d}{d\phi}(\phi^3)}_{\frac{d}{d\phi}(\phi\phi\phi)}\rangle \text{ (By Schwinger-Dyson)} \\ &= \frac{1}{m^2}\langle\frac{d\phi}{d\phi}\phi\phi\rangle + \frac{1}{m^2}\langle\phi\frac{d\phi}{d\phi}\phi\rangle + \frac{1}{m^2}\langle\phi\phi\frac{d\phi}{d\phi}\rangle \\ &= \langle\phi\phi\phi\rangle + \langle\phi\phi\phi\rangle + \langle\phi\phi\phi\rangle \\ &= \text{[Draw feynman diagrams]}\end{aligned}$$

[WATCH RECORDING AND ADD THE FEYNMAN DIAGRAM REPRESENTATIONS OF THESE TERMS – IMPORTANT

- Every ϕ is a dot with a line coming out of It

- When we contract two ϕ's we connect their lines

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<u>Exercises:</u>

1. Show that
$$\langle \phi^4 \rangle = \frac{(n-1)!!}{m^n}$$
where the double exclaimation is the double-factorial.

2. Write the diagrams for $\langle \phi^6 \rangle$ in a few dfferent ways:

3.
$$\begin{aligned} \langle \phi^6 \rangle &= \langle \phi\phi\phi\phi\phi\phi \rangle \\ &= \langle \phi^2 \phi^4 \rangle \\ &= \langle \phi^3 \phi^3 \rangle \\ &= \langle (\phi^6) \phi^0 \rangle \end{aligned}$$

4. Compute the partition function
$$\mathcal{Z} = \frac{1}{h} \int_0^L \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p \; e^{-\frac{-\beta p^2}{2m}}$$
explicitly (*Hint: Convert into the same form as we've solved in class using the substitution* $\phi = \frac{Lp}{h}$ *and then figure out what the action* $S(\phi)$ *should be.*)

5. Also Compute the free Energy F, where $e^{-\beta F} = \mathcal{Z}$.
   - Note that $F = E - TS, dF = -SdT - pdV$
   - $p = -\left(\frac{\partial F}{\partial V}\right)_\beta \implies$ Ideal Gas Law

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