Math H185 Homework 10

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Question 1

Show that if a Mobius transformation $g \in \mathrm{PSL}_2(\mathbb{C})$ sends $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, then g is an element of $\mathrm{PGL}_2(\mathbb{R})$. Exhibit a Mobius transformation which sends $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$ but is not given by an element of $\mathrm{PSL}_2(\mathbb{R})$.

Proof:

Consider a Mobius transformation $g \in \mathrm{PSL}_2(\mathbb{C})$ which has corresponding matrices $M, -M \in \mathrm{SL}_2(\mathbb{R})$ described by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Assume M has determinant 1 (so $-M$ has det $= -1$). This gives us the constraint $ad - bc = 1$. We also know that g sends $\hat{\mathbb{R}}$ to itself, so a complex number $z = x + 0i$ for any $x \in \hat{\mathbb{R}}$ gets sent to another complex number $g \cdot z = y + 0i$, $y \in \hat{\mathbb{R}}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$
$$\implies \begin{cases} ax = y \\ cx = 0 \end{cases}$$

So, we have an additional constraint: $c = 0$. Which means $ad = 1 \implies a = 1/d$. We still have no constraints on

[Come back to this one]

Question 2

Let U, V be open subsets of $\mathbb{C}$. Suppose $f : U \to V$ is holomorphic and injective. Show that $f'(z) \neq 0$ for all $z \in U$.

Proof:

Suppose for contradiction there exists $z_0 \in U$ such that $f'(z_0) = 0$. Then, WLOG, we can assume the power series expansion of f about z_0 has the form

$$\begin{aligned} f(z) &= a_k (z - z_0)^k + a_{k+1}(z - z_0)^{k+1} \cdots \\ &= (z - z_0)^k \left(a_k + a_{k+1}(z - z_0) + \cdots\right) \end{aligned}$$

wth $a_k = f^{(k)}(z_0), k \geq 2$.

Let $g(z) = a_k + a_{k+1}(z - z_0) + \cdots$. Note that $g(z_0) \neq 0$. Hence, there exists $\delta > 0$ so that in $D_\delta(z_0)$ we have a branch $g^{1/k}(z)$ of the $k-$th-root of $g(z)$. Therefore, $f(z) = (h(z))^k$, where h is a holomorphic function defined by

$$h(z) = (z - z_0)g^{1/k}z$$

and hence $h(z_0) = 0, h;(z_0) = g^{1/k}(z_0) \neq 0$. Since, on a small disk, h is injective and z^k is also one-to-one, f is $k-$to-one in a neighborhood of z_0. This contradicts the assumption that $f(z)$ is injective.

Question 3

Find a biholomorphism between the upper half-disk $\mathbb{D} \cap \mathbb{H}$ and the unit disk $\mathbb{D}$.

__Solution:__

First, the map

$$f_1 : z \mapsto \frac{1+z}{1-z}$$

takes the upper half of the unit disk to the first quadrant.

Then,

$$f_2 : z \mapsto z^2$$

takes the first quadrant to the upper half plane.

Finally,

$$f_3 : z \mapsto \frac{z-i}{z+i}$$

maps the upper half-plane to the unit disk.

Altogether, the map we need is

$$\begin{aligned} f(z) = (f_3 \circ f_2 \circ f_1)(z) &= \frac{\left(\frac{1+z}{1-z}\right)^2 - i}{\left(\frac{1+z}{1-z}\right)^2 + i} \\ &= -i \cdot \frac{z^2 + 2iz + 1}{z^2 - 2iz + 1} \end{aligned}$$

Question 4

Find a biholomorphism between the half-strip

$$U = \{z = x + iy \in \mathbb{C} \ : \ 0 < y < \pi, x > 0\}$$

and the strip

$$V = \{z = x + iy \in \mathbb{C} \ : \ 0 < y < 2\pi\}$$

__Solution:__

The map

Question 5

Find a biholomorphism between the half-strip

$$\{z = x + iy \in \mathbb{C} \ : \ 0 < y < 2\pi\}$$

and the upper half-plane.

Solution:

The map $z \mapsto 2\log(z)$ with the negative imaginary axis deleted takes the upper half-plane to the half-strip. The inverse map is $\omega \mapsto e^{2\omega}$.

Question 6

Does there exist a holomorphic surjection from the unit disk to $\mathbb{C}$? Either write down an explicit formula for one, or prove that none exists.

Solution:

Yes, we can construct a holomorphic surjection from the unit disk to $\mathbb{C}$ via a composition of multiple maps.

We have the following chain of maps:

$$\mathbb{D} \xrightarrow{F} \mathbb{H} \xrightarrow{g} (\mathbb{H} - i) \xrightarrow{h} \mathbb{C}$$

- First, we have the conformal map $F : \mathbb{D} \to \mathbb{H}$ given by
$$F(z) = \frac{i - z}{i + z}$$
which was discussed in class (and proven to be a conformal map).
- Then, we can shift the entire half-plane down by one unit via the map $g : z \mapsto z - i$. Denote the image of this map as
$$\mathbb{H} - i = \{z \in \mathbb{C} : \mathrm{Im}(z) > -1\}$$
- Finally we have the square-map $h : z \mapsto z^2$. This map doubles the argument of z i.e. it sends $re^{i\theta} \mapsto r^2 e^{i(2\theta)}$ so the image is the entire complex plane.

Explicitly, the formula for the map is

$$z \mapsto (F(z) - 1)^2$$

Question 7

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function satisfying $|f(z)| \leq 1$ and $f(i) = 0$. Prove that

$$|f(z)| \leq \left|\frac{z - i}{z + i}\right|$$

for all $z \in \mathbb{H}$.

Proof:
Since $|f(z)| \leq 1$ we can essentially think of f as a function between $\mathbb{H}$ and $\overline{\mathbb{D}}$.
Let $T : \mathbb{D} \to \mathbb{H}$ be the map defined by

$$T(z) = i\frac{1+z}{1-z}$$

Since $z \in \mathbb{D}$ this map is holomorphic. Then, we see that $f \circ T : \mathbb{D} \to \overline{\mathbb{D}}$ is holomorphic with $f(T(0)) = f(i) = 0$. By the Schwartz Lemma, we have $|f(T(z))| \leq |z|$.
Also, we have

$$T^{-1}(z) = i\frac{z-1}{z+1}$$

Using this fact, we see that

$$\begin{aligned} |f(z)| = |\left(f \circ T \circ T^{-1}\right)(z)| &= \left|(f \circ T)\left(i\frac{z-1}{z+1}\right)\right| \\ &\leq \left|\left(i\frac{z-1}{z+1}\right)\right| \\ &= \left|\frac{z-1}{z+1}\right| \end{aligned}$$

Thus,

$$\boxed{|f(z)| \leq \left|\frac{z-1}{z+1}\right|}$$

Question 8

A complex number $\omega \in \mathbb{D}$ is a *fixed point* of a map $f : \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity map.

(b) Must every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ have a fixed point?

Proof:

(a) The function $f : \mathbb{D} \to \mathbb{D}$ is analytic (and thus also holomorphic) and there exist two fixed points $w_1, w_2 \in \mathbb{D}$ of f.

We know by the Schwartz Lemma that since $f(z)$ has a fixed point, it must be a rotation i.e. $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. But recall that there are *two* distinct fixed points i.e. $w_1 \neq w_2$. i.e. at least one of these fixed points must be non-trival (both can't be the origin else we'd have $w_1 = w_2$). Let w_1 be the (guaranteed) non-trivial fixed point. Then, $f(z_1) = e^{i\theta}z_1$ for non-zero z_1 means we must have $\theta = 0$.

(b) No. For a counterexample, consider the function

$$f(z) = \frac{z+1}{2}$$

This function essentially moves every point along the real axis a little bit, but the factor of $1/2$ ensures that $f(z) \in \mathbb{D}$ given $z \in \mathbb{D}$.

Question 9

Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ take the form

$$z \mapsto e^{i\theta}\frac{z-\beta}{z+\beta}, \quad \theta \in \mathbb{R}, \beta \in \mathbb{H}$$

Proof:

Let $f : \mathbb{H} \to \mathbb{D}$ be a generic conformal map from the upper half plane to the unit disk. Let $G : \mathbb{D} \to \mathbb{H}$ be the conformal map defined as

$$G(w) = i\frac{1-w}{1+w}$$

This map has inverse

$$G^{-1}(z) = \frac{i-z}{i+z}$$

Then, $(f \circ G) : \mathbb{D} \to \mathbb{D}$ is an automorphism on the unit disk. We know the automorphisms on the unit disk are of the form

$$F(w) = e^{i\theta}\frac{\alpha - z}{1-\overline{\alpha}z}$$

where $\alpha \in \mathbb{D}$, $\theta \in \mathbb{R}$.

Thus, for $w \in \mathbb{D}$ and some appropriate value α, we have

$$\begin{aligned}
&(f \circ G)(w) = F(w) \\
\implies &f\left(i\frac{1-w}{1+w}\right) = e^{i\theta}\frac{\alpha - w}{1-\overline{\alpha}w}
\end{aligned}$$

So, solving for $f(z)$ we get

$$\begin{aligned}
&f(z) = e^{i\theta}\frac{\alpha - G^{-1}(z)}{1-\overline{\alpha}G^{-1}(z)} \\
\implies &f(z) = e^{i\theta}\frac{\alpha - \left(\frac{i-z}{i+z}\right)}{1-\overline{\alpha}\left(\frac{i-z}{i+z}\right)} \\
\implies &f(z) = e^{i\theta}\underbrace{\left(\frac{1-\alpha}{1+\overline{\alpha}}\right)}_{\text{modulus}=1}\frac{z-\beta}{z+\beta} \\
\implies &f(z) = e^{i\theta'}\frac{z-\beta}{z+\beta}
\end{aligned}$$

where $\beta = i\frac{1-\alpha}{1+\alpha} \in \mathbb{H}$.