Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

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Recap

- Last time, we saw a few different types of functions:

immersion dF_p	injective
submersion dF_p	surjective
constant rank $\text{rank}(dF_p)$	constant

1.1 Rank Theorem

The Rank Theorem tells us that if we have a map of constant rank between manifolds, we can essentially think of it as a projection map.

Before tackling the rank theorem, let's recall something from Linear Algebra.

- Recall that if $L : V \to W$ is a linear map between finite dmiensional vector spaces then there exst bases of V and W such that the matrx representation of L is of the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

  where r is the rank of the linear map.

- The above is the ***Canonical Form Theorem*** for linear maps.
- The rank theorem is essentially the non-linear version of this.

Theorem: (Constant Rank Thm) Given smooth manifolds M^m, N^n, smooth map $F : M \to N$ which has constant rank $rank(F) = r$, the for any point $p \in M$ there exsit smooth charts (U, ϕ) on M, (V, ϕ) on N such that $p \in U$, $F(p) \in V$, $F(U) \subseteq V$, and F has the coordinate representation

$$F(x^1, \cdots, x^r, x^{r+1}, \cdots x^m) = (x^1, \cdots, x^n, 0, \cdots, 0)$$

Note: The goal is to find coordinate representations such that the pre-image(s) get "straighted" out.
[Include image]
i.e. we want to find diffeomorphisms

$$\phi \text{ on } \vec{0} \in U \subseteq \mathbb{R}^m$$
$$\psi \text{ on } F(\vec{0}) \in V \subseteq \mathbb{R}^n$$

such that

$$F(U) \subseteq V \psi \circ F \circ \phi^{-1}$$

has the desired form.

Proof: We many assume $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ with $F\mathbb{R}^m \to \mathbb{R}^n$, $\vec{p} = \vec{0}$, $F(\vec{0}) = \vec{0}$

Step 1: Replace F with $A \circ F \circ B$ where $A : \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}^m rightarrow \mathbb{R}^m$ are linear isomorphsms so

$$dF_{\vec{0}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

This "rotates" $\phi(U)$ as needed.

Step 2: Deal with possble stretching.
We replace our old F with $A \circ F \circ B$, and rite

$$\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} \in \mathbb{R}^m \quad \begin{bmatrix} \vec{x'} \\ \vec{y'} \end{bmatrix} \in \mathbb{R}^n$$

where $\vec{x}, \vec{y}, \vec{x'}, \vec{y'}$ has dimensions $r, (m-r), r, (n-r)$ respectively. Also,

$$F\left(\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}\right) = \begin{bmatrix} \vec{Q}(\vec{x}, \vec{y}) \\ \vec{P}(\vec{x}, \vec{y}) \end{bmatrix}$$

Then set

$$\phi\left(\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}\right) = \begin{bmatrix} \vec{Q}(\vec{x}, \vec{y}) \\ \vec{y} \end{bmatrix} \in \mathbb{R}^m$$

We want to set things up to "scale" our domain so that the domain and target have the same scale. This will set us up to use the Inverse Function Theorem.

Note that

$$d\phi_{\vec{0}} = \begin{bmatrix} I_r & 0 \\ 0 & I_{(m-r)} \end{bmatrix} = \mathrm{Id}_{\mathbb{R}^m}$$

So, by the inverse function theorem, ϕ is invertible in the neighborhood of $\vec{0}$.

Step 3: Now that we know ϕ is invertible, let's see if it gives the action we want.

$$\left(F \circ \phi^{-1}\right)\left(\begin{bmatrix} \vec{x^*} \\ \vec{y^*} \end{bmatrix}\right) = \begin{bmatrix} \vec{x^*} \\ \left(\vec{R} \circ \phi^{-1}\right)\left(\vec{x^*}, \vec{x^*}\right) \end{bmatrix}$$

This map is smooth near $\vec{0}$. Computing the differental, we have

$$d\left(F \circ \phi^{-1}\right)_{\begin{bmatrix} \vec{x^*} \\ \vec{y^*} \end{bmatrix}} = \begin{bmatrix} I_r & 0 \\ \frac{\partial(R \circ \phi^{-1})}{\partial \vec{x^*}} & \\ \frac{\partial(R \circ \phi^{-1})}{\partial \vec{y^*}} & \end{bmatrix}$$

But F is a map of constant rank r and ϕ is a diffeomorphism, so it doesn't change the rank. Thus, the differential must also have rank r, which implies we must have

$$\frac{\partial(R \circ \phi^{-1})}{\partial \vec{y^*}} = \vec{0}$$

So, we write $\left(\vec{R} \circ \phi^{-1}\right)\left(\vec{x^*}, \vec{y^*}\right) = \vec{S}(\vec{x^*})$ near $\vec{0}$ since it has not y dependence. i.e.

$$\left(F \circ \phi^{-1}\right)\left(\begin{bmatrix} \vec{x^*} \\ \vec{y^*} \end{bmatrix}\right) = \begin{bmatrix} \vec{x^*} \\ \vec{S}\left(\vec{x^*}\right) \end{bmatrix}$$

Step 4: Define

$$\psi(\begin{bmatrix} \vec{x'} \\ \vec{y'} \end{bmatrix}) = \begin{bmatrix} \vec{x'} \\ S\left(\vec{x'}\right) - \vec{y'} \end{bmatrix}$$

Then,

$$d\psi\big|_{\vec{0}}$$

is invertible, thus it is a local diffeomorphism and

$$\left(\psi \circ F\phi^{-1}\right)\left(\begin{bmatrix}\vec{x^*}\\ \vec{y^*}\end{bmatrix}\right) = \begin{bmatrix}\vec{x^*}\\ \vec{0}\end{bmatrix} \in \mathbb{R}^n$$

near $\vec{0}$.

1.2 Embeddings

- Given smooth manifolds M^m, N^n, the smoooth map $F : M \to N$ is an ***embedding*** if it is an injective immersion and $F : M \to \underbrace{F(M)}_{\text{subspace topology}} \subseteq N$ is a homeomorphism.

Lemma: The map F is an embedding if and only if it is

1. An injective immersion
2. $F(x_i) \xrightarrow{i\to\infty} F(x_\infty) \implies x_i \to x_\infty$

Example:

$$M_1 \times \{p\} \hookrightarrow M_1 \times M_2, p \in M_2$$

is an embedding.

Non-examples of embeddings

- Any curve intersecting itself in non-injectvie, and thus not an embedding.
- The mapping of the open interval into $\mathbb{R}^2$ pictured below is not an embedding since it fails the second condition.
- $F : \mathbb{R}_+ \coprod (-1, 1) \to \mathbb{R}^2$ [complete this example later]
- $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, $F : R \to T^2$ defined as

$$t \to \left(e^{2\pi iat}, e^{2\pi ibt}\right)$$

  - If the ratio $\frac{a}{b} \in \mathbb{Q}$ then the function F is periodic and thus not injective.
  - If $\frac{a}{b} \notin \mathbb{Q}$ then $F(\mathbb{R}) \subseteq T^2$ is dense.

Additional results on Embeddings

Some of the pathalogical examples we saw earlier had to do with the *domain* of the function n ot being nice. If the domain itself is compact, it makes it "easier" for an immersion to be an embedding.

Theorem: If $F : M \to N$ is an injective immersion and M is compact then F is an embedding.

Proof: Given a closed subset $A \subseteq M$ we want to show that $F(A) \subseteq F(M)$ is closed. This follows from the Hausdorffness of the range.

Theorem: If $F: M \to N$ is an imersion, $p \in M$ then there exists a neighborhood U such that $p \in U \subseteq_{open} M$ and $F\big|_U$ is an embedding.

Proof: (Prove this later; follows from Rank Theorem which we proved earlier).

Read:

Submersions, Smooth Covering Maps.

1.3 On to Chapter 5! Submanifolds.

1.3.1 Embedded Smooth Manifolds

- Given a smooth manifold M, we say a subset $S \subseteq M$ is an ***embedded smooth submanifold*** if it is a topological manifold when endowed with the subspace topology and has smooth structure such that the inclusion map $S \hookrightarrow M$ is a smooth embedding.
- Equivalently, we can characterize S as being an embedded smooth submanifold if there exists an embedding $F: N \to M$ such thta $F(N) = S$.

Examples:

- $S \subseteq_{open} M$ such that dim$S =$ dim$M$ (codim$S = 0$).
- $S \subseteq_{open} M$ such that dim$S = 0$ (codim$S =$dim$M$).
- $F: \mathbb{R}_+ \to \mathbb{R}^2$ defined as
$$t \mapsto (t, \sin(1/t))$$
is a 1-D submanifold of $\mathbb{R}^2$.