# Math H185 Homework 9

Keshav Balwant Deoskar

May 13, 2024

**Question 1**

For the following functions, classify the singularity at $\infty$ and calculate the residue at $\infty$.

(a) $f(z) = z\sin(1/z)$

(b) $f(z) = e^z$

(c) $f(z) = 3z^4 + z^3 + 4z^2 + z + 5$

**Solution:**

(a) We have $f(z) = z\sin(1/z)$ so

$$\begin{aligned} F(z) &= f(1/z) \\ &= \frac{\sin(z)}{z} \\ &= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right] \end{aligned}$$

$F(z)$ has a removable singularity at 0 since it can be extended to the function

$$F(z) = 1 - \frac{z^2}{3!} + \frac{z^2}{5!} + \cdots$$

Therefore, $f(z)$ has a removable singularity at $\infty$ and the residue is zero.

(b) We have $f(z) = e^z$ so

$$F(z) = f(1/z) = e^{1/z}$$

$F(z)$ has an essential singularity at $z = 0$ so $f(z)$ has an essential singularity at $\infty$. To find the residue let's expand $F(z)$ out:

$$\begin{aligned} F(z) &= e^{1/z} \\ &= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \end{aligned}$$

This has a residue of 1 at zero, so $\text{Res}_{z_0=\infty} f(z) = 1$.

(c) We have $f(z) = 3z^4 + z^3 + 4z^2 + z + 5$ so

$$\begin{aligned} F(z) &= f(1/z) \\ &= \frac{3}{z^4} + \frac{1}{z^3} + 4\frac{1}{z^2} + \frac{1}{z} + 5 \end{aligned}$$

which has a pole of order one and residue equal to one at 0. Therefore, $f(z)$ has a pole of order 1 at $\infty$ and $\text{Res}_{z_0=\infty} f(z) = 1$.

**Question 2**

Let $f(z) = \frac{z(z-2)^3}{\sin(z^2)(z-4)^5}$. Use the argument principle to calculate

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{f'(z)}{f(z)} dz$$

for $r = 1, 3, 5$

__**Proof:**__
The Argument Principle tells us that

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{f'(z)}{f(z)} dz = \#\text{ of zeroes of } f \text{ in } B_r(0) - \#\text{ of poles of } f \text{ in } B_r(0)$$

We have

$$f(z) = \frac{z(z-2)^3}{\sin(z^2)(z-4)^5}$$

which can be written as

$$\begin{aligned}\frac{z(z-2)^3}{\sin(z^2)(z-4)^5} &= \frac{z}{\sin(z^2)} \cdot \frac{(z-2)^3}{(z-4)^5} \\ &= \frac{z}{z^2 - \frac{(z^2)^3}{3!} + \frac{(z^2)^5}{5!} + \cdots} \cdot \frac{(z-2)^3}{(z-4)^5} \\ &= \frac{1}{z - \frac{z^5}{3!} + \frac{z^9}{5!} + \cdots} \cdot \frac{(z-2)^2}{(z-4)^5}\end{aligned}$$

The zeroes of this function occur at $z = 0$ (multiplicity 1) $z = 2$ (with multiplicity 3). The poles of this function occur at $z = 0$ (with multiplicity 2) and at $z = 4$ (with multiplicity 5).

(a) __$r = 1$:__ The only pole or zero lying in this ball are the ones at the origin, so in this region,

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{f'(z)}{f(z)} dz = 1 - 2 = -1$$

(b) __$r = 3$:__ In this region we also have the zero at $z = 2$. So,

$$\frac{1}{2\pi i}\int_{\partial B_3(0)} \frac{f'(z)}{f(z)} dz = 4 - 6 = -2$$

(c) __$r = 5$:__ In this region we have all of the zeroes and poles, so

$$\frac{1}{2\pi i}\int_{\partial B_5(0)} \frac{f'(z)}{f(z)} dz = 4 - 21 = -17$$

Question 3

Do the following problem from Stein-Shakarchi. Recall that an entire function is a function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic at all $z \in C$:

Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

**Proof:** Consider an entire and injective function $f$. Then, $f$ is non-constant. Let's define $g(z) = f(1/z)$.

If the singularity of $g(z)$ at 0 were an essential singularity, then the Casorati-Weierstass theorem would imply that the image $g\left(B_1(0) \setminus \{0\}\right)$ is dense in $\mathbb{C}$. However, $g\left(B_{1/2}(2)\right)$ is an open set by the open mapping theorem and these two maps intersect, which shows that $g(z)$ (and hence $f(z)$) is not injective.

Thus, the singularity at $z = 0$ must be a pole, implying that $f(z)$ is a polynomial. Suppose $f(z)$ is a polynomial of degree $m$. Then $f$ has $m$ zeroes, accounting for multiplicity. But if $f$ were to have any number of distinct roots greater than 1, then it would not be injective since the roots would both be mapped to zero. So, $f$ must have the form $f(z) = c(z - z_0)^m$ for $c, z_0 \in \mathbb{C}$. But for $m \geq 2$ this is *also* not necessarily injective as $f(z_0 + 1) = f(z_0 + e^{2\pi i/m})$. Thus, we must have $m = 1$ meaning $f(z)$ is a linear polynomial i.e. it is of the form

$$f(z) = az + b$$

for $a, b \in \mathbb{C}, a \neq 0$.

---

Question 4

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

Prove that if $f$ is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$,, and for some integer $k \geq 0$ and some constants $A, B > 0$, then $f$ is a polynomial of degree $\leq k$.

**Proof:**
We have an entire function $f(z)$ such that

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

Since $f$ is entire, it is holomorphic (thus analytic) on all of $\mathbb{C}$ and can be written in terms of a power series expansion about the origin:

$$f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot z^n$$

Now, since $f(z)$ is entire, so are each of its derivatives. Let $g(z) = f^{(k+1)}(z)$. Cauchy's inequality on the disc of radius $R$, $D_R$, tells us

$$g^{(l)}(0) \leq \frac{(l+k+1)!}{R^{l+k+1}} \sup_{\partial D_R} |f(z)|, \quad l = 0, 1, 2, \cdots$$

Then, by the assumption,

$$g^{(l)}(0) \leq \frac{(l+k+1)!}{R^{l+k+1}} \sup_{\partial D_R} AR^k + B$$

Now, in the limit as $R \to \infty$, $\frac{(l+k+1)!}{R^{l+k+1}} \sup_{\partial D_R} AR^k + B \to 0$ so $g^{(l)}(0) = f^{(l+k+1)}(0)$ for $l = 0, 1, 2, \cdots$ must be equal to zero. i.e. all of the coefficients the power series expansion beyond the $z^k$ term are zero. Therefore, $f(z)$ is a polynomial of degree $\leq k$.

---

**Question 5**

Let $w_1, \cdots, w_n$ be the points on the unit circle in the complex plane. Prove that there exists a point $z$ on the unit circle such that the product of the distances from $z$ to the points $w_j$, $1 \leq j \leq n$, is at least 1. Conclude that there exists a point $w$ on the unit circle such that the product of the distances from $w$ to the points $w_j$, $1 \leq j \leq n$, is exactly equal to 1.

**Proof:**
Define a function $g(z) : B_1(0) \to \mathbb{C}$ as

$$g(z) = \prod_{j=1}^{n} (z - w_j)$$

This function is holomorphic on the unit disc. Then, by the Maximum Modulus Principle, $g(z)$ can only attain its maximum (modulus) on the unit circle. But note that

$$g(0) = \prod_{j=1}^{n} w_j \implies |g(0)| = \prod_{j=1}^{n} \underbrace{|w_j|}_{=1} = 1$$

So, the maximum modulus of $g(z)$ must at least be greater than 1. So, there exists a point $z$ on the unit circle such that $g(z)$ is atleast 1.

Now, if we define $f(z)$ to be

$$f(z) = \prod_{j=1}^{n} |z - w_j|$$

Then, $f(z)$ can be zero (if $z = w_j$) and it is at least 1 from the argument above. Therefore, by the Intermediate Value Theorem, there must exist some $w$ on the unit circle such that $f(w)$ is exactly 1.

---

## Question 6

Suppose $f$ and $g$ are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that $f$ has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z)$$

Show that if $\epsilon$ is sufficiently small, then

(a) We know that $f$ has a unique zero at $z = 0$. Now, $f_\epsilon(z)$ has a unique zero in $|z| \leq 1$, and

(b) if $z_\epsilon$ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

**Proof:**

(a) The functions $f, g$ are holomorphic on a region $\Omega$ containing the unit disc. We can assume WLOG that $\Omega$ is an open set (otherwise we could simply replace it with a suitable open set.)

We have $f_\epsilon(z) - f(z) = \epsilon g(z)$. For sufficiently small $\epsilon$, we have

$$|f_\epsilon(z) - f(z)| = |\epsilon g(z)| < f(z)$$

on $\partial\mathbb{D}$. $\epsilon g(z)$ is still holomorphic on $\Omega$, so Rouche's theorem applies. It tells us that $f(z)$ and $f(z) + \epsilon g(z)$ have the same number of zeros in the unit circle. Therefore, $f_\epsilon(z)$ has a unique zero in the unit disc.

(b) We have $f_\epsilon(z) = f(z) + \epsilon g(z)$ as in part (a). Now, choose some $\epsilon_0 < \epsilon$ and the associated function $f_{\epsilon_0} = f(z) + \epsilon_0 g(z)$ whose root is denoted $z_{\epsilon_0}$.

Since $f$ has a unique zero at $z = 0$, we can choose $r > 0$ such that $f$ such that $\overline{B}_r(z_{\epsilon_0}) \subseteq \mathbb{D}$ and $f$ is non-vanishing on $\partial\overline{B}_r(z_{\epsilon_0})$. Then, we can appropriately choose $\delta > 0$ so that

$$\min_{z \in \partial\overline{B}_r(z_{\epsilon_0})} |f(z)| > \delta \max_{z \in \partial\overline{B}_r(z_{\epsilon_0})} |g(z)|$$

Then, applying Rouche's Theorem again, we see that $f_\epsilon$ has only 1 one zero inside $B_r(z_{\epsilon_0})$. But we saw earlier that $f_\epsilon$ has a unique zero. Thus, it must be the case that $z_\epsilon \in B_r(z_{\epsilon_0})$. i.e. $|z_\epsilon - z_{\epsilon_0}| < r$. The above is exactly the statement that $\epsilon \mapsto z_\epsilon$ is continuous at $\epsilon_0$.

---

## Question 7

Let $f$ be non-constant and holomorphic in an open set containing the closed unit disc.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of $f$ contains the unit disc.

(b) If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of $f$ contains the unit disc.

**Proof:**

(a) The goal is to show that $f$ takes on every value in $\mathbb{D}$ i.e. $f(z) - w_0 = 0$ has a root for every $w_0 \in \mathbb{D}$.

Since $f$ is holomorphic on an open set containing the unit circle with $|f(z)| = 1$ for $|z| = 1$ and $|w_0| < 1$, so is $f - w_0$. Thus, Rouche's Theorem applies, meaning $f$ and $f - w_0$ have the same number of zeros inside the unit circle for any $w_0 \in \mathbb{D}$.

So, if we show that $f - 0 = f$ has a root on $\mathbb{D}$, the rest will follow. Suppose for contradiction that $f$ has no roots on $\mathbb{D}$. Then $1/f : \mathbb{D} \to \mathbb{C}$ is holomorphic. So, by the Maximum Modulus principle, $|1/f| \leq 1$ or equivalently $|f(z)| \geq 1$ on $\mathbb{D}$.

However, we know that for any $z \in \partial\mathbb{D}$, $|f(z)| = 1$. If $U$ is a small neighborhood of $z \in \partial\mathbb{D}$ contained in the domain of $f$ (the open set $U$), then $f(U)$ is an open neighborhood of $f(z)$ whic contradicts $1 \leq |f(z)|$ for $z \in \mathbb{D}$ as there must be some values of $z$ for which $f(z) > 1$ in order for $f(U)$ to be open.

This is a contradiction. Therefore, there must exist a root for $f(z)$ that lies in $\mathbb{D}$. The desired result follows from our discussion above.

(b) The exact same reasoning works for part (b).

---

**Question 8**

Suppose $f$ is a non-vanishing continuous function on $\overline{B}(0)$ that is holomorphic in $B_1(0)$. Prove that if $|f(z)| = 1$ whenever $|z| = 1$ then $f$ is constant.

**Proof:**
We have $f$ holomorphic on $B_1(0)$ and $|f(z)| = 1$ on $\partial B_1(0)$. By the maximum modulus principle, $f(z)$ cannot attain a maximum modulus on $\operatorname{int}(B_1(0)) = B_1(0)$, so it must be the case that $|f(z)| \leq 1$.

However, since $f(z)$ is non-vanishing on $\overline{B}_1(0) = \mathbb{D}$, the function $\frac{1}{f(z)}$ satisfies all of the above conditions too, meaning that $\left|\frac{1}{f(z)}\right| \leq 1$ or equivalently that $|f(z)| \geq 1$.

Therefore, it must be the case that $|f(z)| = 1$ on all of $\mathbb{D}$ i.e. it is constant.

---