Math H185 Homework 5

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Question 1

Are the following subsets of $\mathbb{C}$ simply connected? Answer "yes" or "no".

(a) $\mathbb{R}$

(b) $\mathbb{C} \setminus \mathbb{R}$

(c) $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$

(d) $\mathbb{C} \setminus B_r(z_0)$ where $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_{\geq 0}$

(e) $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$

Proof:

(a) Yes

(b) No

(c) Yes

(d) No

(e) Yes

Question 2

Let $U = C \setminus i\mathbb{R}_{\leq 0}$. Let $\log_U$ be the logarithm function on U. Express the following complex numbers in the form $x + iy$ where $x, \in \mathbb{R}$.

(a) $\log_U(1 + \sqrt{3}i)$

(b) $\log_U(-e)$

(c) $\log_U(1 - i)$

(d) $\log_U(1 - \sqrt{3}i)$

Proof:

The logarithm with the branch cut along the negative imaginary axis is defined as

$$\log(z) = \log(r) + i\theta$$

where $z = re^{i\theta}$ with $\theta \in (-\pi/2, +3\pi/2)$.

(a) Let's begin by expressing $z = 1 + \sqrt{3}i$ in polar form:

$$\begin{aligned}
|1+\sqrt{3}| &= \sqrt{1^2 + \left(\sqrt{3}\right)^2} \\
&= \sqrt{1+3} = \sqrt{4} = 2 \\
\implies r &= 2
\end{aligned}$$

and

$$\begin{aligned}
\theta &= \arctan\left(\frac{\sqrt{3}}{1}\right) \\
\implies \theta &= \frac{\pi}{3}
\end{aligned}$$

Thus,

$$\boxed{\log_U\left(1+\sqrt{3}i\right) = \log(2) + i\frac{\pi}{3}}$$

(b) Now, $z = -e$ can be expressed in polar form as $z = e \cdot e^{i\pi}$, so

$$\log_U(-e) = \log(e) + i\pi$$

$$\boxed{\log_U(-e) = 1 + i\pi}$$

(c) In polar form, $1 - i = \sqrt{2}e^{i\cdot(-\pi/4)}$ so

$$\boxed{\log_U(1-i) = \log\left(\sqrt{2}\right) - i\frac{\pi}{4}}$$

(d) In polar form, $1 - \sqrt{3}i = 2e^{i\cdot -pi/3}$, so

$$\boxed{\log_U\left(1-\sqrt{3}i\right) = \log(2) - i\frac{\pi}{3}}$$

Question 3

Give an example of a simply connected open subset $U \subset \mathbb{C}$ and $z_1, z_2 \in U$ such that $z_1 z_2 \in U$ but

$$\log_U(z_1 z_2) \neq \log_U(z_1) + \log_U(z_2)$$

<u>Proof:</u>
Consider the open subset $U \equiv \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and consider $z_1 = 2e^{i\pi/2}$, $z_2 = 2e^{i3\pi/4}$. Then,

$$z_1 z_2 = 4e^{i(\pi/2 + 3\pi/4)} = 4e^{i(5\pi/4)} = 4e^{-i3\pi/4}$$

where we must re-express the argument θ because the logarithm on $U\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is defined as

$$\log_U(z) = \log(r) + i\theta$$

where $z = r + i\theta$ and $|\theta| < \pi$

and

$$\begin{aligned}\log_U(z_1 z_2) &= \log_U(4e^{i\cdot(-3\pi/4)}) \\ &= \log(2) - i\frac{3\pi}{4}\end{aligned}$$

whereas

$$\begin{aligned}\log_U(z_1) + \log_U(z_1) &= \log_U\left(2e^{0\pi/2}\right) + \log_U\left(2e^{i3\pi/4}\right) \\ &= \left(\log(2) + i\frac{\pi}{2}\right) + \left(\log(2) + i\frac{3\pi}{4}\right) \\ &= 2\cdot\log(2) + i\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) \\ &= \log(4) + i\frac{5\pi}{4}\end{aligned}$$

So we notice that when $z_1 z_2$ crosses the branch cut, as in our case, we get

$$\log_U(z_1 z_2) \neq \log_U(z_1) + \log_U(z_2)$$

Question 4

Let $f = \frac{1}{z^2 - z}$, viewed as a function on $U \equiv \mathbb{C} \setminus \{0, 1, -1\}$. For the following curves, evaluate the quantity

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz$$

(a) Draw image later

(b) Draw image later

(c) Draw image later

Proof:

(a) We want to evaluate

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{-\gamma_0} f(z)dz$$

Integrating over γ_1 and then integrating over $-\gamma_0$ amounts to integrating over a closed loop. So,

$$\begin{aligned}\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz &= \int_{\gamma} f(z)dz \\ &= \int_{\gamma} \frac{1}{z(z-1)}dz \\ &= \int_{\gamma} \frac{g(z)}{z-0}dz\end{aligned}$$

where $g(z) = 1/(z-1)$ and γ is a closed loop centered at the origin. Then, applying the Cauchy Integral formula,

$$\begin{aligned}\int_{\gamma} \frac{g(z)}{z-0}dz &= 2\pi i \cdot g(0) \\ &= 2\pi i \cdot (-1) \\ &= -2\pi i\end{aligned}$$

So,

$$\boxed{\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz = -2\pi i}$$

(b) Carrying out the same procedure, we have

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz = \int_{\gamma} \frac{g(z)}{z-0} + \frac{h(z)}{z-1} dz$$

where $g(z) = \frac{1}{z-1}$, $h(z) = \frac{1}{z-0}$, and γ is the closed curve enclosing 0 and 1 formed by traversing γ_1 and then $-\gamma_0$.

Applying Cauchy's Integral Formula,

$$\begin{aligned}\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz &= 2\pi i g(0) + 2\pi i h(1) \\ &= 2\pi i \left[-1+1\right] \\ &= 0\end{aligned}$$

(c) This time, the contour $\gamma_1 + (-\gamma_0)$ is a closed contour enclosing -1, So

$$\begin{aligned}\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz &= \int_{\gamma} \frac{1}{z^2 - z} dz \\ &= \int_{\gamma} \frac{1}{z^2 - z} \cdot \frac{z^2+z}{z^2+z} dz \\ &= \int_{\gamma} \frac{z^2+z}{z^4 - z^2} dz \\ &= \int_{\gamma} \frac{z^2+z}{z^2\left(z^2-1\right)} dz \\ &= \int_{\gamma} \frac{j(z)}{z^2-1} dz\end{aligned}$$

where $j(z) = \frac{z^2+z}{z} = 1 + \frac{1}{z}$.

Then, applying Cauchy's Integral Formula,

$$\begin{aligned}\int_{\gamma_1} f(z)dz - \int_{\gamma_0} f(z)dz &= 2\pi i j(-1) \\ &= 2\pi i \cdot (1-1) \\ &= 0\end{aligned}$$

Question 5

Assume the result that if γ_0, γ_1 are homotopic curves in a subset $U \subseteq \mathbb{C}$ and f is a holomorphic function on U, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Using this, show that if f is any holomorphic function on a simply connected open subset $U \subseteq \mathbb{C}$, then f has a primitive on U.

Proof:

Choose any $z_0 \in U$, and define the function for any $z \in U$ as $F(z) = \int_\gamma f(w)dw$ where γ is a path between z and z_0. This map is well defined because on a simply connected open subset, any two curves γ_0, γ_1 between z_0, z will be homotopic and so by our assumption we have

$$\int_{\gamma_0} f(w)dw = \int_{\gamma_1} f(w)dw$$

We claim that $F(z)$ is the primitive of $f(z)$ on U. Let's now prove this.

For small enough $h \in \mathbb{C}$, we will have $\overline{B_h(z_0)} \in U$ i.e. $z, z+h \in \mathbb{C}$. Then,

$$\begin{aligned}
F(z+h) - F(z) &= \int_{z_0}^{z} f(w)dw - \int_{z_0}^{z+h} f(w)dw \\
&= \int_{z}^{z+h} f(w)dw \\
\implies \frac{F(z+h)-F(z)}{h} &= \frac{1}{h}\int_{z}^{z+h} f(w)dw \\
\implies \lim_{h\to\infty} \frac{F(z+h)-F(z)}{h} &= \lim_{h\to 0}\frac{1}{h}\int_{z}^{z+h} f(w)dw
\end{aligned}$$

We can calculate the limit by integrating along the straight line connecting z and $z+h$ which we can parametrize as

$$\gamma(t) = z + th$$

for $t \in [0,1]$

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\int_{z}^{z+h} f(w)dw &= \lim_{h\to 0}\frac{1}{h}\int_{0}^{1} f(z+th)\cdot(h)\,dt \\
&= \lim_{h\to 0}\int_0^1 f(z+th)dt \\
&= f(z)
\end{aligned}$$

Therefore,

$$\boxed{\lim_{h\to\infty} \frac{F(z+h)-F(z)}{h} = f(z)}$$

So, $F(z)$ is indeed the primitive of $f(z)$!

Question 6

Let $U \subseteq \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ be a function such that $\int_T f(z)dz = 0$ for every (parametrized) triangle $T \subseteq U$. Prove that f is holomorphic on all of U.

Proof:

Consder a point $a \in U$. Since U is an open set, there is some $r > 0$ such that $B_r(a) \subseteq U$. The restriction of f to this open ball, $f\big|_{B_r(a)}$, is continuous and satisfies the property that

$$\int_\gamma f(z)dz = 0$$

for all triangular contours contained in $B_r(a)$.

Let's define $F : B_r(a) \to \mathbb{C}$ as

$$F(z) = \int_{[a,z]} f(z)dz$$

where $[a, z]$ is the line segment from a to z in $\mathbb{C}$. This function is well defined because $B_r(a)$ is simply-connected.

Now,

$$\begin{aligned}
F'(z) &= \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} \\
&= \lim_{h\to 0} \frac{\int_{[a,z+h]} f(w)\,\mathrm{d}w - \int_{[a,z]} f(w)\,\mathrm{d}w}{h} \\
&= \lim_{h\to 0} \frac{\overbrace{\int_{[a,z+h]} f(w)\,\mathrm{d}w + \int_{[z+h,z]} f(w)\,\mathrm{d}w + \int_{[z,a]} f(w)\,\mathrm{d}w}^{=\int_\gamma f(w)\,\mathrm{d}w = 0} + \int_{[z,z+h]} f(w)\,\mathrm{d}w}{h} \\
&= \lim_{h\to 0} \frac{\int_{[z,z+h]} f(w)\,\mathrm{d}w}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^1 f(z+th)\cdot h\,\mathrm{d}t \\
&= \lim_{h\to 0} \int_0^1 f(z+th)\,\mathrm{d}t \\
&= \int_0^1 \lim_{h\to 0} f(z+th)\,\mathrm{d}t \\
&= \int_0^1 f(z)\,\mathrm{d}t \qquad (f \text{ continuous}) \\
&= f(z)
\end{aligned}$$

So, $F(z)$ is holomorphic on $B_r(a)$ with derivative $f(z)$. But we know that holomorphic functions are infinitely differentiable – meaning $f(z)$ is also holomorphic on $B_r(a)$.

Since $a \in U$ was chosen arbtrarily, the above argument holds for all points in U so we have arrived at the desired result.
