Math 214 Homework 7

Keshav Balwant Deoskar

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Q7-2. Let G be a Lie Group.

(a) Let $m : G \times G \to G$ denote the multiplication map. Using proposition 3.14 to identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$, show that the differential $dm_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$dm_{(e,e)}(X, Y) = X + Y$$

(b) Let $i : G \to G$ denote the invversion map. Show that $di_e : T_eG \to T_eG$ is given by $di_e(X) = -X$.

Proof:

(a)

Q7-4. Let $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ denote the determinant function. Use Corollary 3.25 to compute the differential of det, as follows.

(a) For any $A \in M(n, \mathbb{R})$, show that

$$\left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) = \mathrm{tr}A$$

, where $\mathrm{tr}\left(A^i_j\right) = \sum_i A^i_i$ is the trace of A.

(b) For $X \in GL(n, \mathbb{R})$ and $B \in T_X GL(n, \mathbb{R}) \cong M(n, \mathbb{R})$, show that

$$d\left(\det\right)_X(B) = (\det X)\,\mathrm{tr}\left(X^{-1}B\right)$$

.

Proof:

Q7-6. Suppose G is a Lie Group and U is any neighborhood of the identity. Show that there exists a neighobrhood V of the identity such that $V \subseteq U$ and $gh^{-1} \in U$ whenever $g, h \in V$.

Proof:

Q7-11. Repeat Problem 7-9 for $GL(n+1, \mathbb{C})$ and $\mathbb{CP}^n$.

Proof:

Q7-22.

(a) Show that quaternionic multiplication is associative but not commutative.

(b) Show that $(pq)^* = q^*p^*$ for all $p, \in \mathbb{H}$

(c) Show that $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$ is an iner product on $\mathbb{H}$, whose associated norm satisfies $|pq| = |p||q|$.

(d) Show that every nonzero quaternion has a two-sided multiplicative inverse given by $p^{-1} = |p|^{-2}p^*$.

(e) Show that the se t$\mathbb{H}^*$ of nonzero quaternions is a Lie group under quaternionic multiplication.

Proof:

Q7-23. Let $\mathbb{H}^*$ be the Lie Group of nonzero quaternions and let $\mathcal{S} \subseteq \mathbb{H}^*$ be the set of unit quaternions. Show that $\mathcal{S}$ is a properly embedded Lie subgroup of $\mathbb{H}^*$, isomorphic to $SU(2)$.

Proof:
