# Math H185 Lecture 3

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# 1 January 24 - Power Series

**Power Series**

A **Power Series** is a formal expression

$$\sum_{n\geq 0} a_n z^n, z_n \in \mathbb{C}$$

for which operations are defined as:

- Addition:

$$\sum_{n\geq 0} a_n z^n$$

- multiplication

- "Formal" here means we temporarily ignore whether it makes sense to plug in complex numbers into such formulae.

## 1.1 Convergence

Defining these formal expressions is cool, but when does a power seres actually define a function? It does so when the seres **converges**.

Example: Geometric series:
Let $a \in \mathbb{C}$, then for the geometric series we have $a_n = a^n$. So,

$$\sum_{n\geq 0} a^n z^n = 1 + az + a^2z^2 + \cdots$$

converges if $S_n = \sum_{n\geq 0}^{N-1} a^n z^n$ has a limit.

By the same argument as in the reals, we can get a closed form expression for $S_N$:

$$S_N = \frac{1-(az)^n}{1-(az)}$$

To deal with convergence, we break into cases and take the limit.

- $|az| < 1$ **case:**

$$|az| < 1 \implies |az|^N \to^{N\to\infty} 0$$

So,

$$\lim_{N\to\infty} S_N = \frac{1}{1-az}$$

- $|az| > 1$ **case:**

$$|az| > 1 \implies |az|^N \to^{N\to\infty} \infty$$

so

$$\lim_{N\to\infty} S_N \text{ diverges}$$

- $|az| = 1$ **case:**

  Diverges if $|az| \neq 1$ and if $|az| = 1$. In this case, both diverge, but in general there may be more complicated behavior.

**Conclusion:** The geometric series converges (absolutely) for when $|z| < |a|$.

Recall that
$$\sum_{n \geq 0} z_n, z_n \in \mathbb{C}$$
coverges absolutely if
$$\sum_{n \geq 0} |z_n|$$
converges.

So, we notice that the series converges for any $z$ such that $|z| < 1/|a|$. This region is just an open disk of radius $|a|$. In general, power series has radii of convergence.

**Radius of Convergence**

**Def:** A complex Power Series
$$\sum_{n \geq 0} a_n z^n$$
has Radius of Convergence
$$r = \left( \lim_{n \to \infty} \sup |a_n|^{1/n} \right)^{-1} \in \mathbb{R}$$

**Example:** For $a_n = a^n$, we have
$$r = \left( \lim_{n \to \infty} \sup |a_n|^{1/n} \right)^{-1} = \frac{1}{|a|}$$
which matches with the result obtained earlier.

**Theorem:**

1. If $|z| < r$, then $f(z)$ converges absolutely.
2. If $|z| > r$, then it diverges.
3. At $|z| = r$, more care is needed.

**Proof Sketch:**

1. Consider $z$ such that $|z| < (1-\epsilon)r$ for some $\epsilon > 0$.

$$\begin{aligned} \implies |a_n z^n| &< |a_n|(1-\epsilon)^n r^n \\ &\leq |a_n|(1-\epsilon)^n \left(\frac{1}{|a_n|^{1/n}}\right)^n \quad \text{Assume } a_n \neq 0 \\ &\leq (1-\epsilon)^n \quad (\text{If } a_n = 0\text{, this inequality is true trivially}) \end{aligned}$$

$$\implies \text{Convergence by Geometric series}$$

Term by term, the series is smaller than the geometric series (which converges), thus it also converges(Dominated Convergence Theorem).

2. If $|z| > r$, then $|z| > r/(1-\epsilon)$ for some $\epsilon > 0$ while

$$|a_n|^{1/n} > \left(\lim_{k\to\infty} \sup |a_k|^{1/k}\right)(1-\epsilon)$$

for infinitely many $n$.

$$\begin{aligned} &\implies |a_n z^n| > \frac{1}{r^n}(1-\epsilon)^n \cdot \frac{r^n}{(1-\epsilon)^n} > 1 \quad \text{for inf. many } n \\ &\implies \text{the sum diverges} \end{aligned}$$

**Example:** Consider a polynomial

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

$$\begin{aligned} &\implies \lim_{n\to\infty} \sup |a_n|^{1/n} = 0 \\ &\implies r = \infty \end{aligned}$$

**Example:** Consider the exponential function

$$\exp(z) = e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

$$\implies r = \lim_{n\to\infty} (n!)^{1/n} =_{\text{claim}} \infty$$

Proof of claim: We wll show that for any $b$, $\frac{1}{2}b < (n!)^{1/n}$ for all $n >> 0$.

Note that we can shift the center of the series from 0 to some point $z_0$.
So, instead of

$$\sum_{n\geq 0} a_n z^n$$

We have

$$\sum_{n\geq 0} a_n (z - z_0)^n$$

The radius of convergence is given by the same expression.

### 1.2 Differentiation of Power Series

Knowing convergence properties of a series is also useful for the purposes of *differentiation*.

**Theorem:** Let $r$ be the radius of convergence of $f(z) = \sum_{n\geq 0} a_n(z - z_0)^n$. Then, $f$ is (complex) differentiable on $B_r(z_0)$ and

$$f'(z_0) = \sum_{n\geq 0} n a_n (z - z_0)^{n-1}$$

**Proof:** The proof follows the same argument as that in real analysis. Write the proof later. One proof can be found here.

We will break the proof into two parts:

(a) First, we show that if $\sum_{n\geq 0} a_n(z - z_0)^n$ has radius of convergence $r$, then so does $\sum_{n\geq 0} n a_n(z - z_0)^{n-1}$.

(b) Second, we show that if $f(z) = \sum_{n\geq 0} a_n(z - z_0)^n$ then the derivative at $z_0$ is indeed

$$f(z_0) = \sum_{n\geq 0} n a_n (z - z_0)^{n-1}$$

**Proof:** (Check if this is correct in Office Hours)

(a) Earlier, we proved that for any complex power series there exists a number $r$ called the radius of convergence such that the series converges for any $z \in \mathbb{C}$ such that $|z| < r$. We also showed that $r$ is given by

$$r = \limsup_{n\to\infty} |a_n|^{1/n}$$

So, what we want to do here is show that the radii of convergence for the two series $R, R'$ are equal.

Now,

$$\begin{aligned} R' &= \limsup_{n\to\infty} |n a_n|^{1/n} \\ &= \limsup_{n\to\infty} |n|^{1/n} \cdot |a_n|^{1/n} \\ &= \limsup_{n\to\infty} |n^{1/n}| \cdot |a_n|^{1/n} \\ &= \limsup_{n\to\infty} |1| \cdot |a_n|^{1/n} \\ &= R \end{aligned}$$

So the two series have equal radii of convergence ($= R$).

(b) Next, we want to show that if

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

and

$$g(z) = \sum_{n \geq 0} n a_n (z - z_0)^{n-1}$$

Then, in fact, $f'(z) = g(z)$.