Math H185 Lecture 28

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 April 10 - Geometric Interpretation of the Argument Principle

- Recall that for a complex number $z = re^{i\theta}$, the *argument* of z is "$\arg(z) = \theta$".
- Though it is not defined well when $r = 0$ and in general can be changed by a muliple of 2π, in a small neighborhood we can make a choice and ensure the argument is locally well-defined.
- Also recall "$\log(z) = \log(r) + i\theta$". The logarithm isn't well-defined because the argument isn't well defined.
- However, the *difference* between the arguments of two complex numbers is indeed well-defined.

Now, let's spell out the connection with the Argument Principle.

$$\frac{1}{2\pi i}\int_{\partial U} \underbrace{\frac{f'}{f}}_{\frac{d}{dz}\log(f(z))} dz = \#\text{ zeros in U} - \#\text{ poles in U}$$

So, the integral on the LHS is measuring the range of angles as f goes around the boundary of U because the change in logs is approximately the change in argument or angle.

Listening in class. Write notes from recording. [Fill stuff in]

Lemma: Let $F : U \to V$ $(U, V \subseteq \mathbb{C})$ be holomorphic and $z_0 \in U$ such that $f'(z_0) \neq 0$. Then f is a biholomorphism to its image (locally) near z_0.

<u>Ex:</u> $f(z) = z^2$ [Write more later]

Proof:
1^{st} step: <u>f injective</u> on some $U' \ni z_0$. Let $w_0 = f(z_0)$. The zeros of $f(z) - w_0$ are isolated (why?) so on some $\overline{B_r(z_0)}$,

$$|f(z) - w_0| \geq \epsilon > 0$$

and for $w \in B_{\delta/2}(w_0)$,

$$|w - w_0| \leq \frac{\epsilon}{2}$$