Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 13

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Question 1: Prove that the restriction to the zero section of the Thom class of an oriented bundle over a CW-complex coincides with the Euler class of the bundle. Derive from this that the degree-raising map from the cohomology of the base to itself, defined by composing the Thom isomorphism with the restriction to the zero section, coincides with the multiplication by the Euler class of the bundle.

Solution: (Referenced Chapter 12 of *Characteristic Classes* by Milnor and Stasheff for this question)

There are two things we want to show.

Claim #1: The restriction to the zero section of the Thom class of an oriented bundle over a CW-comples coincides with the Euler Class of the bundle.

Claim #2: The degree-raising map coincides with the multiplication by the Euler class of the bundle.

In class, we defined the Euler Class of an oriented (real) n-dimensional vector bundle ξ with base B as the obstruction to extending a section of the associated spherical fibration ξ_1^0 to ξ. More specifically,

Definition 0.1. *Let ξ be an n-dimensional (real) oriented vector bundle with CW Base B. The **Euler Class** $\mathfrak{o}_n(\xi) \in H^n(B;\mathbb{Z})$ is first obstruction to extending a section of the spherical fibration ξ_1^0 to ξ.*

(I'm using the notation $\mathfrak{o}_n(\xi)$ here because it is the top obstruction class)

Definition 0.2. *Given a vector bundle $\xi = (E, B, \mathbb{F}, p)$ we can create the associated (locally trivial) fibration $\xi_k = (E_k, B, R_k, p_k)$ where*

$$E_k = \{(x_1, \cdots, x_k) \in E \times \cdots \times E \mid p(x_1) = \cdots = p(x_k);\ x_1, \cdots, x_k \text{ are lin. ind.}\}$$

for $1 \le k \le n$ and R_k is the set of $k-$orthonormal frames in R^n.

*The **spherical fibrbation** ξ_1^0 is the fibration with base B and total space being the collection of orthonormal frames in the fibers of ξ.*

We defined the Thom class using Thom Spaces and Thom Isomorphisms. Specifically,

Definition 0.3. *Given a real $n-$dimensional oriented vector bundle ξ with base B, let $T(\xi) = D(\xi)/S(\xi)$ be the* ***Thom Space*** *where $D(\xi), S(\xi)$ are the Disk and Sphere bundles associated with the bundle ξ.*

Definition 0.4. *For $n-$dimensional real oriented vector bundle ξ with base B, an arbitrary $\alpha \in H^q(B; G)$, we have*

$$\boldsymbol{t}(\alpha) = \boldsymbol{t}(1) \smile \alpha$$

where the cohomology class $\boldsymbol{t}(1) \in H^n(T(\xi); G)$ is called the ***Thom class*** *of ξ, and t is the* ***Thom Isomorphism****,*

$$\mathbf{t} \,:\, H^q(B; G) \xrightarrow{\cong} \tilde{H}^{q+n}(T(\xi); G)$$

(Note: if $G = \mathbb{Z}_2$ then the orientability of ξ is not necessary)

Proof of Claim #1:

We want to show that $\mathfrak{o}_n(\xi)$ coincides with the pullback of $\mathbf{t}(1)$ under the zero section of ξ.

Question 2: Use the unoriented version of the previous exercise in order to show that $\mathbb{RP}^n$ cannot be embedded into $\mathbb{R}^{2n-1}$ when n is a power of 2.

Solution:

Method #2:
(Not quite what the question asks, but hopefully this is acceptable)

We'll use the following proposition:

Let E be a real bundle. If $w_{top}(E) \neq 0$ then E does not have a nowhere vanishing section.

Proof. Suppose we have a nowhere vanishing section of E denoted s. Then, s would span a line-subbundle of E which splits off i.e. we can write

$$E = \ell \oplus F$$

for $F = \ell^\perp$ being some subbundle of rank one less than E. Thus, we would have $w_i(E) = w_i(\ell \oplus F) = w_i(\ell)w_i(F) = 1 \cdot w_i(F)$ but $w_i(F)$ should vanish at the top rank. Thus, if there exists a nowhere vanishing section of E, $w_{top}(E) = 0$. □

It turns out that $T\mathbb{RP}^n \oplus \varepsilon \cong \tau^* \otimes \varepsilon^{n+1}$ where ϵ is the trivial bundle. Then, because $w(\varepsilon) = 1$, we have that $w(T\mathbb{RP}^n) = (1+h)^{n+1} \in H^2(\mathbb{RP}^n; \mathbb{Z}_2)$

Now, let $n = 2^k$ for some k and suppose there exists some embedding of $\mathbb{RP}^n$ into $\mathbb{R}^{2n-1}$. That would mean that $T\mathbb{RP}^n$ can be embedded into ε^{2n-2} and its complement, say Q, would have rank $n-2$. Then

$$w(T\mathbb{RP}^n)w(Q) = 1$$

Question 3: Diagonal quaternionic matrices define an inclusion of $(Sp_1)^n$ into Sp_n, which induces the homomorphism from $H^*(BSp_n)$ to $H^*((\mathbb{H}P^\infty)^n)$ between the cohomology of the classifying spaces. Yet, each $Sp_1 = SU_2$ contains the circle T^1 of diagonal SU_2-matrices. Compute the image of $H^*(BSp_n)$ in $H^*(BT^n) = \mathbb{Z}[x_1, \ldots, x_n]$ under map induced by the inclusion $T^n \subset (Sp_1)^n \subset Sp_n$, and find the images of the 1st obstruction classes to sections of the universal $\mathbb{H}V(n, n-m+1)$-bundles over $BSp_n = \mathbb{H}G(\infty, n)$.

Solution:

text

1 Appendix