Spring '24

Math 185 Final Review

Keshav Balwant Deoskar

kdeoskar@berkeley.edu

Contents

1 January

2 February

3 March

4 April

4.1 April 1: Riemann Sphere

- We want to talk about functions that are holomorphic/meromorphic/have a pole "at" ∞. We do this by extending the complex plane and functions on it.

One point compactification

Real case

Consider the map $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{x}$$

This function is *almost* smooth. The only issue is that in some sense we have $f(0) = \infty \notin \mathbb{R}$. But what if we just add an "∞" point to $\mathbb{R}$?

Denote this space $\hat{\mathbb{R}}$ as the one-point compactification of $\mathbb{R}$ is $\mathbb{R} \cup \{\infty\} \cong \mathbb{S}^1$. Now, f can be continued to a map from $\mathbb{R}$ to $\hat{R}$ with the same formula.

We can further define

$$\frac{1}{\infty} = 0$$

to continue the function to a smooth map from $\hat{\mathbb{R}} \to \hat{\mathbb{R}}$!

$\hat{\mathbb{R}}$ is called the ***extended complex plane***.

Complex case

Similarly, we can define the ***extended complex plane*** $\hat{\mathbb{C}}$ by adding a point at infinity. $\hat{C} = \mathbb{C} \cup \{\infty\}$. Instead of $\mathbb{S}^1$, this space is topologically equivalent to $\mathbb{S}^2$ i.e. a sphere.

INSERT FIGURE.

The function $z \mapsto \frac{1}{z}$, $\mathbb{C} \to \mathbb{C}$ gets extended continuously to a function with the same formula defined on $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ under the convention

$$\begin{aligned} z &\mapsto \frac{1}{z} \\ 0 &\mapsto \infty \\ \infty &\mapsto 0 \end{aligned}$$

Under this map, a neighborhood of the 0 gets mapped into a neighborhood of ∞. More precisely, this map takes a chart $\mathbb{C} \subseteq U \ni 0$ and maps it to a chart $0 \in U \subseteq \mathbb{C}$ to a chart $\infty \in V \subseteq \hat{\mathbb{C}}$ and the two charts are equivalent in that we can go back and forth.

Def: We call a function $f : U \subseteq \mathbb{C} \to V \subseteq$ a ***biholomorphism*** if

- bijective
- holomorphic
- f^{-1} is also holomorphic

This is the natural notion of isomorphism in complex geometry. As such, a number of properties follow:

- Given a chain
$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}$$
with f holomorphic, we have g holomorphic if and only if $g \circ f$ is holomorphic.
- g has a removable singularity/pole/etc. at $z_0 \in V$ if and only if f has the same at $f^{-1}(z_0)$.
- $\mathrm{Res}_{z_0}(g) = \mathrm{Res}_{f^{-1}(z_0)}(f)$

We want to force $\mathrm{inv}(z), z \mapsto 1/z$ to be a biholomorphism. [write more later]

Meromorphic functions

Def: If $f(z)$ is holomorphic on $U \setminus \{z_0\}$, then $f(z)$ is said to be ***meromorphic*** if and only if it extends to holomorphic $\hat{f} : U \to \hat{\mathbb{C}}$.

The above is equivalent to saying f meromorphic if and only if

$$\lim_{z \to z_0} f(z) = \infty$$

A group homomorphism

$$\rho G \to \mathrm{Aut}(V)$$

is called a **representation** of G in V.

given a chain

$$U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X \xrightarrow{i} Y$$

4.2 April 3: