Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 11

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Question 1: Prove the **Morse Lemma:** *A smooth function in a neighborhood of a non-degenerate critical point with zero critical value is locally diffeomorphic to the quadratic form defined by its second differential.*

Solution:

The approach:

1. Let f_T be a linear interpolation between the function f and the quadratic form f_0.

2. Apply the homotopy method:

   - Look for a family of local diffeomorphisms g_t such that $f_t(g_t(x)) = f_0(x)$. Differentiate this relation in t in order to obtain an infinitesimal version of the equation, which would require finding a time-dependent family of vector fields v_t from which g_t can be recovered using the uniqueness and existence theorem for solutions of ordinary differential equations.
   - To solve that infinitesimal equation for v_t, the following Hadamard's lemma can be useful: In $\mathbb{R}^n$ with coordinates $y_1, ..., y_n$, a smooth function vanishing at the origin can be written as $y_1 G_1(y) + ... + y_n G_n(y)$ where G_i are some smooth functions. If you use it, prove it too (of find its proof somewhere).

Question 2: In $\mathbb{CP}^2 \times \mathbb{CP}^2$, consider the hypersurface defined F given by $x_1y_1+x_2y_2+x_3y_3 = 0$ where (x_1, x_2, x_3) and (y_1, y_2, y_3) are homogeneous coordinates on the left and right projective planes respectively. Identify F with the manifold of complete flags in $\mathbb{C}^3$, find the Kernel of the homomorphism

$$\mathbb{Z}[u, v]/(u^3, v^3)$$

where u, v are the generators in the cohomology algebras of the left and right projective planes Poincaré-dual to projective lines therein, and show the isomorphism is surjective.

<u>**Solution:**</u>

text

Question 3: Use Intersection Theory to prove the classical **Borsuk-Ulam Theorem:** *$n-$odd continuous functions on $\mathbb{S}^n$ have a common zero.*

Solution:

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