Math H185 Homework 2

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Question 1

State the subsets of $\mathbb{C}$ on which the following functions are holomorphic, and calculate their derivatives on their domains of holomorphicity:

(a) $f(z) = e^{z^2+3z+4}$

(b) $f(z) = \frac{1}{e^z}$

(c) $f(z) = \frac{1}{z^2-3z+2}$

Solution:

A function $f(z)$ is holomorphic at z_0 if f is C^1 and the Cauchy-Riemann equations hold at z_0.

(a) The function $f(z) = e^{z^2+3z+4}$ is a composition of holomorphic functions as $f(z) = (g \circ h)(z)$ where $h(z) = z^2 + 3z + 4$ and $g(z) = e^z$.

Then, using the chain-rule,

$$\begin{aligned} f'(z) &= g'(h(z)) \cdot h'(z) \\ &= e^{z^2+3z+4} \cdot (2z+3) \end{aligned}$$

$$\implies \boxed{f'(z) = (2z+3)e^{z^2+3z+4}}$$

This function is holomorphic on all of $\mathbb{C}$.

(b) We can write $f(z) = \frac{1}{e^z}$ as being $f(z) = (g \circ h)(z)$ where $g(z) = \frac{1}{z}$ and $h(z) = e^z$.

Using the chain-rule, the derivative in the region of holomorphicity is:

$$\begin{aligned} f'(z) &= g'(h(z)) \cdot h'(z) \\ &= \left(\frac{-1}{(e^z)^2}\right) \cdot e^z \\ &= \frac{-1}{e^z} \end{aligned}$$

$$\implies \boxed{f'(z) = \frac{-1}{e^z}}$$

This function is also holomorphic on all of $\mathbb{C}$.

(c) Once again, we can express

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

as a composition of holomorphic functions $f(z) = (g \circ h)(z)$ where $g(z) = \frac{1}{z}$ and $h(z) = z^2 - 3z + 2$.

Its derivative in the region where it is holomorphic is

$$\begin{aligned} f'(z) &= g'(h(z)) \cdot h'(z) \\ &= \frac{1}{(z^2 - 3z + 2)^2} \cdot (2z + 3) \\ &= \frac{2z + 3}{z^2 - 3z + 2} \end{aligned}$$

The function is holomorphic as long as the denominator in $f'(z)$ is non-zero:

$$z^2 - 3z + 2 = (z - 1)(z - 2)$$

Thus, the function is holomorphic for $\mathbb{C} \setminus \{1, 2\}$

Question 3

Using Euler's Theorem, prove the angle addition formulas

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$$

and

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$

Proof:

Euler's theorem tells us that, for $t \in \mathbb{R}$,

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

Then, we have

$$\boxed{e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)} \quad (1)$$

But we can also write $e^{i(\theta_1+\theta_2)}$ as

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= e^{i\theta_1} \cdot e^{i\theta_2} \\ &= [\cos(\theta_1) + i\sin(\theta_1)] \cdot [\cos(\theta_2) + i\sin(\theta_2)] \\ &= \cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2) \\ &= [\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\cos(theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)] \end{aligned}$$

$$\boxed{e^{i(\theta_1+\theta_2)} = [\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)]} \quad (2)$$

So, we must have

$$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = [\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)] \quad (3)$$

And, taking the real and imaginary parts of equation (3), we get

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$
$$\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$$

which are exactly the angle addition formulas.

Question 3

Show that there is an everywhere holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \sin(z)/z$ for $z \neq 0$. What is $f(0)$? Write down $f'(z)$ as a power series.

Proof:

The function $\sin(z)/z$ is undefined, however we notice that $\sin(z)$ can be expressed as a power series:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

So, we can analytically continue $\sin(z)/z$ as

$$\frac{\sin(z)}{z} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$$

which has the value $f(0) = 1$ at the origin.

This series expansion has infinite radius of convergence, and we proved in class that if $f(z)$ can be written as a power series of convergence radius R, then $f'(z)$ equals the $f(z)$ series differentiated term-by-term, and has the same radius of convergence.

Thus, $f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$ is everywhere holomorphic.

Question 4

Determine the radius of convergence of the power series

$$\sum_{z \geq 0} a_n z^n$$

in the following cases:

(a) $a_n = n$

(b) $a_n = \frac{1}{n^2}$

(c) $a_n = \frac{n^2-5}{4^n+3n}$

Proof:

We showed in class that for a power series of the form

$$\sum_{z\geq 0} a_n z^n$$

the radius of convergence is given by

$$\limsup_{n\to\infty} |a_n|^{1/n}$$

(a) For $a_n = n$,

$$\begin{aligned} R &= \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \\ &= \left(\limsup_{n\to\infty} |n|^{1/n}\right)^{-1} \\ &= 1 \end{aligned}$$

as $\lim_{n\to\infty} \sqrt[n]{n} = 1$ is a well known limit from real analysis.

(b) For $a_n = 1/n^2$,

$$\begin{aligned} R &= \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \\ &= \left(\limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{1/n}\right)^{-1} \\ &= \frac{1}{\left(\frac{1}{\limsup_{n\to\infty} \sqrt[n]{n}}\right)^2} \\ &= \frac{1}{1^2} \\ &= 1 \end{aligned}$$

(c) For $a_n = \frac{n^2-5}{4^n+3n}$, the radius of convergence is given by:

$$\begin{aligned} R &= \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \\ &= \left(\limsup_{n\to\infty} \left|\frac{n^2-5}{4^n+3n}\right|^{1/n}\right)^{-1} \end{aligned}$$

We use the following theorem and its corollary from Real Analysis:

Theorem 12.3 (Ross):
Let (s_n) be any sequence of non-zero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Note: If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists and equals L, then all four quantities in the expresson above are equal to L.

Corollary:
If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|$ exists and equals L, then $\lim_{n\to\infty} \left|s_n\right|^{1/n}$ exists and equals L.

Now,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\left(\frac{(n+1)^2-5}{4^{n+1}+3(n+1)}\right)\cdot\left(\frac{4^n+3n}{n^2-5}\right)\right| \\ &= \lim_{n\to\infty}\left|\frac{4^n}{4^{n+1}}\right| \text{ (Since the exponential dominates)} \\ &= \frac{1}{4}\end{aligned}$$

Since the limit exists, we have that

$$\limsup_{n\to\infty}|a_n|^{1/n} = \frac{1}{4}$$

so the radius of convergence comes out to be 4.

Question 5

Prove the following statements:

(a) The power series $\sum_n nz^n$ does not converge at any point of the unit circle.

(b) The power series $\sum_n \frac{z^n}{n^2}$ converges at every point of the unit circle.

(c)

Proof:

(a) If $|z| = 1$, we have $nz^n \to \infty$ so the series does not converge for any point on the unit circle.

(b) If $|z| = 1$, we have $\left|\frac{z}{n^2}\right| = \frac{|z|^n}{|n^2|} = \frac{1}{n^2}$, and we know from real analysis that $\sum_n \frac{1}{n^2}$ converges. So, the series $\sum_n \left|\frac{z^n}{n^2}\right|$ converges, which allows us to conclude that $\sum_n \frac{z^n}{n^2}$ converges absolutely.

Question 6

Calculate the derivative matrix

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}$$

for the function $f : \mathbb{C} \to \mathbb{C}, f(z) = |z|^2$, and using the Cauchy-Riemann equations, find all points $z_0 \in \mathbb{C}$ at which it is holomorphic.

Proof:

For $z = x + iy$ we can write the function $f(z) = |z|^2$ as

$$\begin{aligned}f(z) &= |z|^2 \\ &= z\cdot z^* \\ &= (x+iy)\cdot(x-iy) \\ &= x^2+y^2\end{aligned}$$

So, $\mathrm{Re}(f(z)) = u(x,y) = x^2 + y^2$ and $\mathrm{Im}(f(z)) = v(x,y) = 0$.

Thus,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

and

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

Thus, the derivative matrix is

$$\begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}$$

Now, we know that if a function $f(z)$ is C^1 and the Cauchy-Riemann equations hold at z_0, then $f(z)$ is holomorphic at z_0. We're already found each of the first derivatives, and observed that they're continuous real functions. So, $f(z) = |z|^2$ is certainly C^1.

Now at a point $z_0 = x_0 + iy_0 \in \mathbb{C}$, in order for the CR equations to hold, we need

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \implies 2x_0 = 0$$
$$\text{and}$$
$$\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0) \implies 2y_0 = 0$$

So, the only point where $f(z) = |z|^2$ satisfies the Cauchy-Riemann equations is $z = \mathbf{0}$. However, $\{\mathbf{0}\} \subseteq \mathbb{C}$ is not an open subset. So, there are no open subsets $\Omega \subseteq_{open} \mathbb{C}$ where $f(z) = |z|^2$ is C^1 and satisfies the Cauchy-Riemann equations.

Therefore, the function $f(z) = |z|^2$ is ***holomorphic nowhere***.

Question 7

For a piece-wise smooth parametrized curve $\gamma : [a,b] \to \mathbb{C}$, let $\gamma^- : [a,b] \to \mathbb{C}$ be the curve with the reverse orientation, defined by $\gamma^-(t) = \gamma(a+b-t)$. Show that

(a) $\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$

(b) Let $z_0 \in \mathbb{C}$ and r be a real positive number with $r < |z_0|$. For every $n \in \mathbb{Z}$, find (with proof) the value of

$$\int_{\partial B_r(z_0)} z^n dz$$

<u>**Proof:**</u>

(a) The integral of $f(z)$ along the curve γ is defined to be

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

So, the integral over γ^- is

$$\begin{aligned}\int_{\gamma^-} f(z)dz &= \int_a^b f(\gamma^-(t))(\gamma^-)'(t)dt \\ &= \int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)dt\end{aligned}$$

Introduce the substitution $u = a+b-t$. Then, $\gamma'(u)du = (-\gamma'(t))(-dt) = \gamma'(t)dt$, and

$$\begin{cases} t = a \implies u = b \\ t = b \implies u = a \end{cases}$$

Then,

$$\begin{aligned}\int_{\gamma^-} f(z)dz &= \int_b^a f(\gamma(u))\gamma'(u)(du) \\ &= -\int_a^b f(\gamma(u))\gamma'(u)(du) \\ &= -\int_a^b f(\gamma(t))\gamma'(t)(dt) \quad \text{(Since } u \text{ and } t \text{ are just dummy variables)} \\ &= -\int_\gamma f(\gamma(t))\gamma'(t)dt\end{aligned}$$

Thus, reversing the orientation of the path introduces a negative sign!

$$\boxed{\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz}$$

(b) We can parametrize the circle of radius $r < |z_0|$ around the point z_0 i.e. $\partial B_r(z_0)$ as

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad \gamma(t) = z_0 + re^{it}$$

Now, we want to integrate $f(z) = z^n$ over this curve.

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^{2\pi} f(\gamma(t))\ \gamma'(t)dt \\ &= \int_0^{2\pi} (z_0 + re^{it})^n \cdot (ire^{it})dt\end{aligned}$$

The easiest way to do so is to use the ***Fundamental Theorem of Contour Integrals***:

> Theorem: If a continuous function f has primitive F on open set Ω and $\gamma : [a, b] \to \mathbb{C}$ is a piece-wise smooth continuous curve in Ω with $\gamma(a) = w_1, \gamma(b) = w_2$ then
>
> $$\int_\gamma f(z)dz = F(w_2) - F(w_1)$$

Proof: If γ is smooth, the proof is a simple application of the Chain rule and Fundamental

Theorem of Calculus (over the real numbers):

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_{w_1}^{w_2} f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b F'(\gamma(t))\gamma'(t)dt \\
&= \int_a^b \left(\frac{d}{dt}F(\gamma(t))\right) dt \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= F(w_2) - F(w_1)
\end{aligned}$$

If γ is only piece-wise smooth, with points $a = a_0 < a_1 < \cdots < a_n = b$ such that it's smooth on each $[a_k, a_{k+1}]$ for $0 \leq k \leq n-1$, then we can argue exactly as before each interval and then obtain the entire integral from a telescoping series:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \\
&= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(\gamma(t)) \cdot \gamma'(t)dt \\
&= \sum_{k=0}^{n-1} F(\gamma(a_{k+1})) - F(\gamma(a_k)) \\
&= F(\gamma(a_n)) - F(\gamma(a_0)) \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= F(w_2) - F(w_1)
\end{aligned}$$

Corollary: If γ is a path such that $\gamma(b) = \gamma(a)$, then

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

Now returning to the integral

$$\int_\gamma z^n dz$$

If $n \neq -1$, then z^n has a primitive (namely $z^{n+1}/(n+1)$) and the path we're integrating is such that the stard and end points are the same. Thus, by the Corollary to the Fundamental Theorem of Contour Integrals, we have

$$\int_\gamma z^n dz = 0, \quad n \neq -1$$

If $n = -1$, using the curve parametrization $\gamma(t) = z_0 + re^{it}$ where $t \in [0, 2\pi]$ we have

$$\begin{aligned}\int_\gamma \frac{1}{z}dz &= \int_0^{2\pi} \frac{1}{\gamma(t)} \cdot \gamma'(t)dt \\ &= \int_0^{2\pi} \frac{1}{z_0 + re^{it}} \cdot (ire^{it})dt \\ &= \frac{ir}{z_0} \int_0^{2\pi} \frac{e^{it}}{1 + \frac{r}{z_0}e^{it}} dt\end{aligned}$$

Note that because our integral doesn't contain the origin, we have $r < |z_0| \implies |\frac{r}{z_0}e^{it}| < 1$. This allows us to expand the $1/(1 + \frac{r}{z_0}e^{it})$ in the integrand as a power series

$$\begin{aligned}\frac{ir}{z_0} \int_0^{2\pi} \frac{e^{it}}{1 + \frac{r}{z_0}e^{it}} dt &= \frac{ir}{z_0} \int_0^{2\pi} e^{it} \sum_{k=0}^{\infty} \left(\frac{-r}{z_0}e^{it}\right)^k dt \\ &= -i \sum_{k=0}^{\infty} \left(\frac{-r}{z_0}\right)^{k+1} \int_0^{2\pi} e^{i(k+1)t} dt \\ &= -i \sum_{k=0}^{\infty} \left(\frac{-r}{z_0}\right)^{k+1} \cdot \frac{e^{i(k+1)t}}{i(k+1)}\Big|_0^{2\pi} \\ &= 0 \text{ Since } e^{i0} = e^{i2\pi}\end{aligned}$$

where we were able to switch the integral and sum in the second equality because the sum converges absolutely as the sum of abolute values

$$\begin{aligned}\sum_{k=0}^{\infty} \Big| - \frac{r}{z_0} eit\Big|^k &= \sum_{k=0}^{\infty} 1 \cdot \underbrace{\Big|\frac{r}{z_0}\Big|^k}_{<1} \\ &= \frac{1}{1 - |\frac{r}{z_0}|} \quad \text{(Geometric series of real numbers)}\end{aligned}$$

In conclusion, we found that

$$\int_{\partial B_r(z_0)} z^n dz = 0$$

for $r < |z_0|$ and all $n \in \mathbb{Z}$.
