Math H185 Lecture 8

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 February 2 - Cauchy's Formula, Infinite Differentiability

Recall

- Last time, we saw Cauchy's formula, which states that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open neighborhood $\supseteq \overline{B_r(z_0)}$
$$f(w) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{w-z}dw$$
This integral can be thought of as kind of a "weighted average".
- What we'll see soon is that this integral gives us the infinitely differentiability of a holomorphic function.
- Something to keep in mind is *when we're allowed to interchange an integral and derivative* i.e. when the following equation is valid
$$\frac{\partial}{\partial z}\int_\gamma g(z,w)dw = \int_\gamma \frac{\partial g(z,w)}{\partial z}dw$$
- We can interchahnge them when $g(z,w)$ and all of its derivatives are both continuous (jointly on both z, w).

1.1 Infinite differentiability of a holomorphic function

Consider a function f which is holomorphic on an open neighborhood $\supseteq \overline{B_r(z_0)}$. Then, by Cauchy's formula, for $z \in B_r(z_0)$ we have
$$f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{w-z}dw$$

Then, differentiating both sides of the equation
$$\begin{aligned} f'(z) &= \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{d}{dz}\frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z^2)}dw \end{aligned}$$

But notice that all functions in the above expression are still contniuous! So, we can differentiate again
$$f''(z) = \frac{2}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z)^3}dz$$
...and again!
$$f''(z) = \frac{6}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z)^4}dz$$

..and we can keep going! In general, the n^{th} derivative is given by

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}dz}$$

So, Given f is holomorphic i.e. it is once complex-differentiable, Cauchy's formula tells us that its derivative also holomorphic, and then that functions derivate is holomorphic, and... so on.

Corollary: If f is holomorphic at z_0, then f is infinitely $\mathbb{C}-$ differentiable in at z_0.

It is impossible for a once-differentiable complex function to not be infinitely differentiable! This is in stark contrast to real valued functions where we can have once differentiable functions.

1.2 Where Now? Contour Integration

Next, we use Cauchy's Theorem + Cauchy's formula to do ***contour integration***. [Write snazzy section intro]

Example: Consider the Integral

$$\int_{\partial B_4(0)} \frac{z}{z^2+1} dz$$

The integrand has singularities at $z = \pm i$ as $z^2 + 1 = (z - i)(z + i)$, so the integrand doesn't immediately vanish. How do we actually calculate the integral then? We have three main methods.

Method A: By Cauchy's Theorem
[Insert image and explain why the following is true]

$$\int_{\partial B_4(0)} f(z)dz = \int_{\partial B_{1/2}(i)} f(z)dz + \int_{\partial B_{1/2}(-i)} f(z)dz$$

Now, the function $f(z)$ can be viewed from two different perspectives:

$$f(z) = \frac{z}{(z-i)(z+i)} = \frac{g(z)}{(z-i)} = \frac{h(z)}{(z+i)}$$

where $g(z) =$ and $h(z) =$

Viewing $f(z)$ as $g(z)/(z - i)$ on the island where $(z - i)$ has a pole, and as $h(z)/(z + i)$ on the island where $(z + i)$ has a pole, we can then apply Cauchy's formula so that

$$\begin{aligned}\int_{\partial B_4(0)} f(z)dz &= \int_{\partial B_{1/2}(i)} f(z)dz + \int_{\partial B_{1/2}(-i)} f(z)dz \\ &= 2\pi i \left(\frac{1}{2i}\right) + 2\pi i \left(\frac{-i}{-2i}\right) \\ &= 2\pi i\end{aligned}$$

Method B: Partial Fraction Decomposition
Another way to do the integral is to recognize

$$\frac{z}{z^2+1} = \frac{1}{2i}\left(\frac{z}{z-i} - \frac{z}{z+1}\right)$$

So,

$$\begin{aligned}\int_{\partial B_4(0)} \frac{z}{z^2+1} dz &= \frac{1}{2i}\left(\int \frac{z}{z-i} dz - \int \frac{z}{z+i} dz\right) \\ &= 2\pi i\end{aligned}$$

Method C: U-substitution

Carry out the substitution $u = z^2 + 1$. Then, $du = 2zdz$. This gives us

$$\int_{\partial B_4(0)} \frac{z}{z^2+1} dz =^? \int_{B_{16}(i)} \frac{1}{2}\frac{du}{u} = \frac{1}{2}(2\pi i)$$

Why are we off by a factor of 2?

> We changed the contour of integration as needed, but *not* the "rate" at which we traverse the contour. [Give more careful explanation with graph]

So, really, the expression after $u-$ substitution should be

$$\begin{aligned}\int_{\partial B_4(0)} \frac{z}{z^2+1} dz &= 2 \times \int_{B_{16}(i)} \frac{1}{2}\frac{du}{u} = 2 \times \frac{1}{2}(2\pi i)\\ &= 2\pi i\end{aligned}$$

and this is consistent with our previous methods.