Math H185 Homework 4

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Question 1

Let $w_1, w_2 \in \mathbb{C}$ and $r \in \mathbb{R}$ such that $|w_1| < r < |w_2|$. Find (with proof) the value of

$$\int_{\partial B_r(0)} \frac{1}{(z-w_1)(z-w_2)} dz$$

<u>Proof:</u>

The function $1/(z-w_1)(z-w_2)$ has a singularity at $z = w_1 \in B_r(0)$ but is holomorphic at all other points in the set. i.e. it is holomorphic on $B_r(0) \setminus \{w_1\}$.

Writing the integral as

$$\int_{\partial B_r(0)} \frac{f(z)}{(z-w_1)} dz, \quad f(z) = \frac{1}{z-w_2}$$

we realize we can apply Cauchy's Integral Formula, so

$$\int_{\partial B_r(0)} \frac{f(z)}{(z-w_1)} dz = 2\pi i f(w_1) \tag{1}$$

$$= \frac{2\pi i}{w_1 - w_2} \tag{2}$$

$$\implies \boxed{\int_{\partial B_r(0)} \frac{f(z)}{(z-w_1)} dz = \frac{2\pi i}{w_1 - w_2}}$$

Question 2

Use Cauchy's Integral Formula to calculate

$$\int_{\partial B_{10}(0)} \frac{\sin\left(\frac{\pi}{2} w\right)}{(w-1)(w-2)} dw$$

<u>Proof:</u>

This time, the function we are integrating has poles at $w = 1$ and $w = 2$ but is holomorphic everywhere else in $B_10(0)$. Now,

$$\int_{\partial B_{10}(0)} \frac{\sin\left(\frac{\pi}{2} w\right)}{(w-1)(w-2)} dw = \int_{\partial B_\delta(1)} \frac{\sin\left(\frac{\pi}{2} w\right)}{(w-1)(w-2)} dw + \int_{\partial B_\delta(2)} \frac{\sin\left(\frac{\pi}{2} w\right)}{(w-1)(w-2)} dw$$

where $\delta \in (0, 1)$ to ensure the two circles around our poles don't overlap.

In the "island" around $w = 1$, we can view the integrand as being

$$\frac{f(w)}{(w-1)}, \qquad f(w) = \frac{\sin\left(\frac{\pi}{2}w\right)}{w-2}$$

and similarly in the island around $w = 2$, we can view the integrand as

$$\frac{g(w)}{(w-2)}, \qquad g(w) = \frac{\sin\left(\frac{\pi}{2}w\right)}{w-1}$$

Then, applying Cauchy's Integral Formula on each island, we have

$$\begin{aligned}
\int_{\partial B_{10}(0)} \frac{\sin\left(\frac{\pi}{2}w\right)}{(w-1)(w-2)} dw &= 2\pi i \cdot f(1) + 2\pi i \cdot g(2) \\
&= 2\pi i \left[\frac{\sin\left(\frac{\pi}{2}\right)}{1-2} + \frac{\sin\left(\frac{\pi}{2}\cdot 2\right)}{2-1}\right] \\
&= 2\pi i \left[-1+0\right] \\
&= -2\pi i
\end{aligned}$$

Thus,

$$\boxed{\int_{\partial B_{10}(0)} \frac{\sin\left(\frac{\pi}{2}w\right)}{(w-1)(w-2)} dw = -2\pi i}$$

Question 3

Prove that if f_n is a sequence of functions on a finite interval $[a, b]$ converging uniformly, then

$$\int_a^b \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_a^b f_n(z)dz$$

__**Proof:**__

A sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to converge uniformly to a function f if for any $\epsilon > 0$ there exists a natural number N such that

$$|f(z) - f_n(z)| < \epsilon$$

Suppose we have such a collection of functions converging to the limit f. Then,

$$\begin{aligned}
\left|\left(\int_a^b f(z)dz\right) - \left(\int_a^b f(z)_n dz\right)\right| &= \left|\int_a^b \left(f(z) - f_n(z)\right) dz\right| \\
&\leq \int_a^b |f(z) - f_n(z)|\, dz \\
&\leq \int_a^b \epsilon\, dz \\
&\leq \epsilon \cdot (b-a)
\end{aligned}$$

So, the inequality

$$\left|\left(\int_a^b f(z)dz\right) - \left(\int_a^b f(z)_n dz\right)\right| \leq \epsilon'$$

is satisfied for any $n \geq N'$ where N' is the natural number which gives us $|f(z) - f_n(z)| < \frac{\epsilon}{(b-a)}$, and N' is guaranteed to exist by uniform convergence.

By definition, this means

$$\int_a^b \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty}\int_a^b f_n(z)dz$$

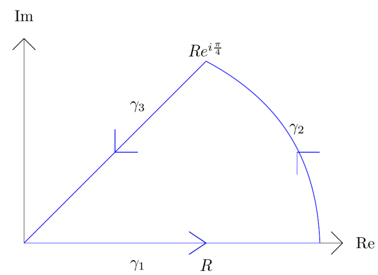
Question 4

Prove that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}$$

Proof:

Consider the integral of the function $f(z) = e^{-z^2}$ over the following contour γ. We will take the limit $R \to \infty$ to evaluate the integral.



Then,

$$\int_\gamma e^{-z^2}dz = \int_{\gamma_1} e^{-z^2}dz + \int_{\gamma_2} e^{-z^2}dz + \int_{\gamma_3} e^{-z^2}dz$$

But also, the function e^{-z^2} is holomorphic on all of $\mathbb{C}$ and γ is a closed curve. Then, by the Cauchy-Goursat theorem,

$$\int_\gamma e^{-z^2}dz = 0$$

So, in particular, we have

$$\lim_{R\to\infty}\int_\gamma e^{-z^2}dz = \lim_{R\to\infty}\left(\int_{\gamma_1} e^{-z^2}dz + \int_{\gamma_2} e^{-z^2}dz + \int_{\gamma_3} e^{-z^2}dz\right) = 0$$

Along the circular part, γ_2:

We have that

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \leq \int_{\gamma_2}\left|e^{-z^2}\right|dz \tag{3}$$

where $x, y \geq 0$.

In this region, we have

$$e^{-z^2} = e^{-(x+iy)^2} = e^{-(x^2-2i(xy)+y^2)} \tag{4}$$
$$\implies e^{-z^2} = e^{-(x^2+y^2)}\cdot e^{-i(2xy)} \tag{5}$$
$$\implies \left|e^{-z^2}\right| = \left|e^{-(x^2+y^2)}\right|\cdot\underbrace{\left|e^{-i(2xy)}\right|}_{=1} \tag{6}$$

As $R \to \infty$, we have $\sqrt{x^2+y^2} \to \infty$ so certainly $(x^2+y^2) \to \infty$ and thus $e^{-(x^2+y^2)} \to 0$.

So,

$$\left| \int_{\gamma_2} e^{-z^2} dz \right| \xrightarrow{R\to\infty} 0$$

This allows us to conclude that in the limit as R goes to infinity, the contribution due to γ_2 goes to zero.

For γ_3, the parametrization starting at $Re^{i\frac{\pi}{4}}$ and going to the origin is slightly annoying to integrate, so we work with the reverse orientation and just introduce a negative sign.

We parametize the line going from the origin to the point $Re^{i\frac{\pi}{4}}$ as $\gamma_3^{(-)}(t) = e^{i\frac{\pi}{4}}t$ for $t \in [0, R]$. Thus,

$$\begin{aligned}
\int_{\gamma_3} e^{-z^2} dz &= -\int_{\gamma_3^{(-)}} e^{-z^2} dz && (7)\\
&= -\int_0^R e^{-\left(e^{i\frac{\pi}{4}}t\right)^2} \cdot e^{i\frac{\pi}{4}}\ dt && (8)\\
&= -\int_0^R e^{-\overbrace{e^{i\pi/2}}^{=i} t^2} \cdot e^{i\pi/4} dt && (9)\\
&= -e^{i\pi/4} \int_0^R e^{-it^2} dt && (10)\\
&= -e^{i\pi/4} \int_0^R \left[\cos(-t^2) + i\sin(-t^2)\right] dt && (11)\\
&= -e^{i\pi/4} \int_0^R \left[\cos(t^2) - i\sin(t^2)\right] dt && (12)\\
&= -\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \int_0^R \left[\cos(t^2) - i\sin(t^2)\right] dt && (13)\\
&= -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \int_0^R \left[\cos(t^2) - i\sin(t^2)\right] dt && (14)\\
& && (15)
\end{aligned}$$

So, in the limit as $R \to \infty$ we have

$$\lim_{R\to\infty} \int_{\gamma_3} e^{-z^2} = -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \int_0^R \left[\cos(t^2) - i\sin(t^2)\right] dt$$

For γ_1, the curve is superimposed with the real axis so $z = x + 0y$ and the integral is

$$\int_{\gamma_1} e^{-z^2} dz = \int_0^R e^{-x^2} dx$$

In the limit, this is

$$\begin{aligned}
\lim_{R\to\infty} \int_{\gamma_1} e^{-z^2} dz &= \int_0^\infty e^{-x^2} dx \\
&= \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} dx \text{ (Since this is an even function)} \\
&= \frac{1}{2}\sqrt{\pi}
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
&\lim_{R\to\infty}\left(\int_{\gamma_1} e^{-z^2}dz+\int_{\gamma_2} e^{-z^2}dz+\int_{\gamma_3} e^{-z^2}dz\right)=0\\
\implies &\lim_{R\to\infty}\int_{\gamma_1} e^{-z^2}dz+\lim_{R\to\infty}\int_{\gamma_2} e^{-z^2}dz+\lim_{R\to\infty}\int_{\gamma_3} e^{-z^2}dz=0\\
\implies &\frac{1}{2}\sqrt{\pi}+0-\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\int_0^R\left[\cos(t^2)-i\sin(t^2)\right]dt=0\\
\implies &\int_0^\infty\left[\cos(t^2)-i\sin(t^2)\right]dt=\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}\\
\implies &\int_0^\infty\left[\cos(t^2)-i\sin(t^2)\right]dt=\sqrt{\pi}\cdot\frac{1}{(\sqrt{2}+i\sqrt{2})}\\
\implies &\int_0^\infty\left[\cos(t^2)-i\sin(t^2)\right]dt=\frac{\sqrt{\pi}}{(\sqrt{2}+i\sqrt{2})}\cdot\frac{(\sqrt{2}-i\sqrt{2})}{(\sqrt{2}-i\sqrt{2})}\\
\implies &\int_0^\infty\left[\cos(t^2)-i\sin(t^2)\right]dt=\frac{\sqrt{2\pi}}{4}-i\frac{\sqrt{2\pi}}{4}
\end{aligned}$$

Thus, taking the Real and Imaginary parts of this last equation, we find that

$$\boxed{\int_0^\infty\cos\left(t^2\right)dt=\int_0^\infty\sin\left(t^2\right)dt=\frac{\sqrt{2\pi}}{4}}$$

Question 5

Prove that

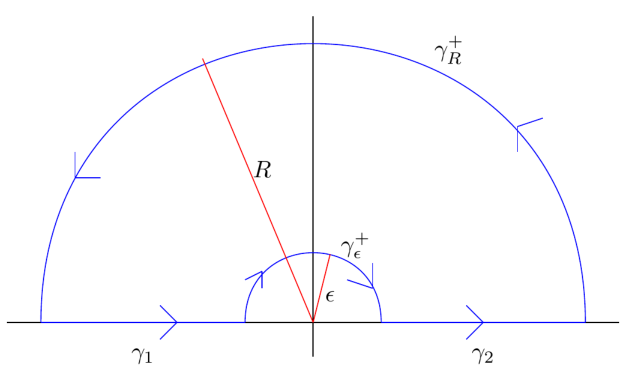
$$\int_{-\infty}^\infty\frac{\sin(x)}{x}dx=\frac{\pi}{2}$$

<u>**Proof:**</u>

Consider the function

$$f(z)=\frac{e^{iz}}{z}$$

being integrated over the indented semicircle:



The function is holomorphic everywhere other than the origin, so over this contour, we have

$$\begin{aligned}
&\int_{\gamma} f(z)dz = 0 \\
\implies &\int_{\gamma_1} f(z)dz + \int_{\gamma_\epsilon^+} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_R^+} f(z)dz = 0 \\
\implies &\lim_{R\to\infty, \epsilon\to 0} \left(\int_{\gamma_1} f(z)dz + \int_{\gamma_\epsilon^+} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_R^+} f(z)dz \right) = 0
\end{aligned}$$

where $\gamma_\epsilon^+, \gamma_R^+$ denote the semicircles of radii ϵ and R being traversed clockwise and counter-clockwise respectively.

Let's first consider the integral γ_R^+. Notice that

$$\left| \frac{e^{iz}}{z} \right| = \frac{\overbrace{|e^{iz}|}^{=1}}{|z|} = \frac{1}{|z|} \xrightarrow{R\to\infty} 0$$

and

$$\left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right| \leq \int_{\gamma_R^+} \left| \frac{e^{iz}}{z} \right| dz \xrightarrow{R\to\infty} 0$$

$$\lim_{R\to\infty} \int_{\gamma_R^+} \frac{e^{iz}}{z} dz = 0$$

Next, let's consider the integral over γ_ϵ^+. We have

$$\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz = -\int_{\gamma_\epsilon^-} \frac{e^{iz}}{z} dz$$

where γ_ϵ^- is the same semicircle, just traversed in the counter-clockwise direction. We can parametrize it as $\gamma_\epsilon^-(t) = \epsilon e^{it}$ where $t \in [0, \pi]$. Then,

$$\begin{aligned}
\int_{\gamma_\epsilon^-} \frac{e^{iz}}{z} dz &= \int_0^\pi \frac{e^{i\left(\epsilon e^{it}\right)}}{\epsilon e^{it}} \cdot \left(i\epsilon e^{\epsilon it}\right) dt \\
&= \int_0^\pi ie^{\left(i\epsilon e^{it}\right)} dt
\end{aligned}$$

So,

$$\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz = -\int_0^\pi ie^{\left(i\epsilon e^{it}\right)} dt$$

In the limit $\epsilon \to 0$, this becomes

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz &= -\int_0^\pi ie^{(0)} dt \\
&= -\int_0^\pi i \cdot 1 dt \\
&= -\pi i
\end{aligned}$$

Next, notice that in the limit $(R \to \infty, \epsilon \to 0)$, we have

$$\int_{\gamma_1} \frac{e^{iz}}{z} dz + \int_{\gamma_1} \frac{e^{iz}}{z} dz \to \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \int_{-\infty}^{\infty} \frac{\cos(x) + i\sin(x)}{x} dx$$

since the countours γ_1, γ_2 coincide with the real axis, making the imaginary part of z equal to zero.

Bringing everything together, we have

$$\lim_{R\to\infty,\epsilon\to 0} \left(\int_{\gamma_1} f(z)dz + \int_{\gamma_\epsilon^+} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_R^+} f(z)dz \right) = 0$$

$$\Longrightarrow \lim_{R\to\infty,\epsilon\to 0} \left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \right) + \lim_{R\to\infty,\epsilon\to 0} \left(\int_{\gamma_\epsilon^+} f(z)dz + \int_{\gamma_R^+} f(z)dz \right) = 0$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{\cos(x) + i\sin(x)}{x} dx + (-\pi i + 0) = 0$$

$$\Longrightarrow \underbrace{\int_{-\infty}^{\infty} \frac{\cos(x)}{x} dx}_{=0,\text{odd function}} + i \int_{-\infty}^{\infty} \underbrace{\frac{\sin(x)}{x}}_{\text{even function}} dx = i\pi$$

$$\Longrightarrow 2i \int_0^{\infty} \frac{\sin(x)}{x} dx = i\pi$$

$$\Longrightarrow \boxed{\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}}$$

Question 6

Prove that

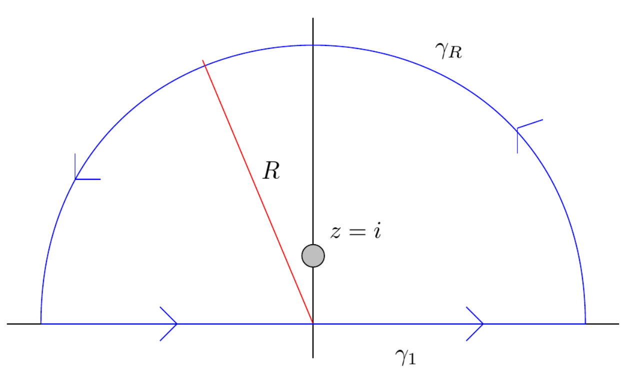
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} dx = \frac{\pi}{e}$$

Proof:

Consider the function

$$f(z) = \frac{e^{iz}}{z^2+1}$$

being integrated over the following contour, denoted γ:



Of course,

$$\int_\gamma f(z)dz = \int_{\gamma_r} f(z)dz + \int_{\gamma_R} f(z)dz$$

The function

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}$$

has poles at $z = i, z = -i$ but if holomorphic everywhere else. Only the pole at $z = i$ lies within the area enclosed by our contour. As a result, Cauchy's Integral formula tells us that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \frac{\left(e^{iz}/(z+i)\right)}{(z-i)}dz \\ &= 2\pi i \cdot \left(\frac{e^{iz}}{z+i}\bigg|_{z=i} \right) \\ &= 2\pi i \cdot \left(\frac{e^{-1}}{2i} \right) \\ &= \frac{\pi}{e}\end{aligned}$$

In particular, this value is fixed even when we take the limit $R \to \infty$

$$\begin{aligned}&\lim_{R\to\infty} \int_\gamma f(z)dz = \frac{\pi}{e} \\ \implies &\lim_{R\to\infty} \left(\int_{\gamma_1} f(z)dz + \int_{\gamma_R} f(z)dz \right) = \frac{\pi}{e} \\ \implies &\lim_{R\to\infty} \int_{\gamma_1} f(z)dz + \lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \frac{\pi}{e}\end{aligned}$$

Let's consider the integral of $f(z)$ over the semi-circular arc γ_R. First off,

$$\left| \frac{e^{iz}}{z^2+1} \right| = \frac{|e^{iz}|}{|z|} = \frac{1}{|z^2+1|} \xrightarrow{R\to\infty} 0$$

and

$$\left| \int_{\gamma_R} \frac{e^{iz}}{z} dz \right| \leq \int_{\gamma_R} \left| \frac{e^{iz}}{z} \right| dz \xrightarrow{R\to\infty} 0$$

Therefore,

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z} dz = 0$$

Also note that

$$\begin{aligned}\lim_{R\to\infty} \int_{\gamma_1} \frac{e^{iz}}{z} dz &= \int_{-\infty}^{\infty} \frac{\cos(x) + i\sin(x)}{x^2+1} dx \\ &= \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} dx + i \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1} dx \\ &= \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} dx + 0\end{aligned}$$

where

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1} dx = 0$$

because it is the integral of an odd function over a symmetric interval.

Putting everything together, we have

$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_1} f(z)dz + \lim_{R\to\infty} \int_{\gamma_R} f(z)dz &= \frac{\pi}{e} \\ \implies \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} dx + 0 + 0 &= \frac{\pi}{e} \end{aligned}$$

So, we arrive at the desired result:

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1} dx = \frac{\pi}{e}}$$
