Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 February 15 - Embedded Submanifolds, Slices

Recap

- Last time we proved the Constant Rank Theorem and discussed some of its implications.
- We also looked at Embedded Submanifolds. Today, we'll be talking more about these objects.
- To recap, recall that for a smooth manifolds M, a subset $S \subseteq M$ is an ***embedded submanifold*** of M if endowing the subspace topology on it makes it a smooth manifold with smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth map.

1.1 More useful definitions

Now, we have not only a notion of embedded manifolds, but also *properly* embedded manifolds.

A submanifold $S \subseteq M$ is ***properly embedded*** if $S \hookrightarrow M$ is *proper*. i.e. the inverse image of compact sets in M are compact in S.

Example: The 2-sphere $\mathbb{S}^2$ with stereographic projection is *not* a properly embedded manifold in $\mathbb{R}^2$ since the pre-image of $\mathbb{S}^2 \setminus \{N\}$ is the entire $\mathbb{R}^2$ plane (which is not compact).

Proposition: A submanifold $S \subseteq M$ is properly embedded f and only if $S \subseteq_{closed} M$.

Proof:

" $\Longleftarrow$ " Direction: If $K \subseteq M$ is compact, then $K \cap S$ is compact (closed subset of compact set).

" $\Longrightarrow$ " Direction: Let's check that S contains all limit points (to show S is closed). Taek a sequence $\{X_i\} \in S$ such that $X_i \to X_\infty \in M$. This implies that the set $\{X_1, X_2, \cdots, X_\infty\} \subseteq M$ is compact, so $\{X_1, X_2, \cdots, X_\infty\} \cap S$ is compact by properness. But this implies that $S_\infty \in S$ as otherwise we could construct an open cover of [complete this proof later].

1.2 Slice Charts

Consider a smooth manifolds M^n with smooth chart (U, ϕ) where $\phi = (x^1, \cdots, x^n)$. A $k-$***slice*** of the chart (U, ϕ), $k \leq n$ is the set

$$S = \{p \in U :\}$$

[Complete def later]

A chart (U, ϕ) is a $k-$slice chart for $S \subseteq U$ if $S = S \cap U$ is a $k-$slice.

A subset $S \subseteq M$ is said to satisfy the ***local*** $k-$***slice condition*** if every point $p \in S$ is in the domain of a (local) $k-$slice chart (U, ϕ).

Example: For a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$, the ***graph***

$$S = \Gamma(f) = \{(\vec{x}, f(\vec{x})) \in \mathbb{R}^m \times \mathbb{R}^n\}$$

is a global $m-$slice chart [Complete this later]

1.3 $k-$slice theorem

Theorem: Given a smooth manifold M^n and subset $S \subseteq M$, S is a $k-$dimensional submanifold of M^n if and only if S satsifies the local $k-$slice condition.

Proof:

" $\implies$ " Direction: We have a smooth structure on S such that the inclusion mao $F : S \hookrightarrow M$ is an embedding. For each point $p \in S$, we need to find a $k-$slice chart.

[Insert image]

The constant rank theorem tells us that there exist smooth charts (U, ϕ) and (V, ψ) such that the coordinate representation of F i.e. $\hat{F} = \psi \circ F \circ \phi^{-1}$, $\hat{F}(x^1, \cdots, x^k) = (x^1, \cdots, x^k, \underbrace{0, \cdots, 0}_{n-k})$.

Let $\hat{U} = \phi(U), \hat{V} = \psi(V)$ and let's observe that $\hat{F}(\hat{U}) = \hat{U} \times \{\vec{0}\}$. Let's further define $\hat{V}' = \hat{V} \cap \left(\hat{U} \times \mathbb{R}^{(n-k)}\right)$ and define $V' = \psi^{-1}(\hat{V}')$.

By the subspace topology, there exists some $U' \subseteq_{open} M$ such that $U \subseteq_{open} S$ is $U = U' \cap S$.

Set $V'' = V' \cap U'$ and set $\psi'' = \psi\big|_{V''}$. We now claim that (V'', ψ'') is a lcoal $k-$slice chart for S at point p. To show this, we need to check

$$V'' \cap S = (\psi'')^{-1} \left(\mathbb{R}^k \times \{\vec{0}\}\right)$$

Let's check the inclusions in both directions:

- if $p \in V'' \cap S$ then since

$$V'' \cap S =$$

  we have $p \in U$

$$\implies \psi''(p) = \psi(p) \in \mathbb{R}^k \times \{\vec{0}\}$$

- Conversely, if $p \in (\psi''^{-1})(\mathbb{R}^k \times \{\vec{0}\})$ then

$$\begin{aligned} &\psi(p) \in \mathbb{R}^k \times \{\vec{0}\}, p \in V'' = V' \cap U' \\ \implies &\psi(p) \in \psi(V') = \hat{V} \cap \left(\hat{U} \times \mathbb{R}^{n-k}\right) \subseteq \hat{U} \times \mathbb{R}^{n-k} \\ &\text{and } \psi(p) \in \mathbb{R}^k \times \{\vec{0}\} \\ \implies &\psi(p) \in \hat{U} \times \{\vec{0}\} = F\left(\hat{\ } U\right) \\ \implies &p \in \phi^{-1}(\hat{U}) \in S \end{aligned}$$

This concludes the $\implies$ direction of the theorem.

<u>" $\impliedby$ " Direction:</u> Now, let;s suppose S satisfies the local $k-$slice condition. We want to define a smooth manifold structure on S and check that the embedding is smooth.

$S \hookrightarrow M$ is a bijection so take subspace topology on S, then it is second countable and hausdorff. In particular, the inclsion is bijective so it is an injective map. [rewrite all of this later, gave up on LiveTexing this proof]

<u>Coroallary:</u> The smooth structure on S si uniquely determined by requiring either

- $S \hookrightarrow M$ to be an embedding
- for any $k-$slice chart $(U, \phi = (x^1, \cdots, x^n))$ of S

$$\left(U \cap S, (x^1, \cdots, x^n)\right)$$

is a smooth chart of S.

1.4 Level sets

An important class of submanifolds is the class of submanifolds which form ***level sets*** of functions from one smooth manifold to another.

Examples:

- The level sets $f^{-1}(r^2)$ of $f(x, y) = x^2 + y^2$ are circles of radius r, where we follow the convention $f^{-1}(0) = \{\vec{0}\}$ and $a > 0 \implies f^{-1}(-a) = \emptyset$.
- $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2 - y^2$. The level sets are hyperbolas embedded in $\mathbb{R}^2$.

<u>Remark:</u> Given any closed $K \subseteq M$, there exists $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$ (application of partitions of unity.)

1.5 Constant Rank Level Set Theorem

Theorem: If $F : M^m \to N^n$ is a smooth map of constant rank r, then for any $q \in N$ the level set $F^{-1}(q) \subseteq M$ is a proper $(m - r)$ dimensional submanifold (codim $F^{-1}(q) = r$).

Example:

- $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $(x, y) \mapsto x^2 - y^2$ has Differential
- $$df\big|_{(x,y)} = \begin{bmatrix} 2x & -2y \end{bmatrix}$$

  which has constant rank 1 unless $x = y = 0$. Then, the map

  $$F = f\big|_{\mathbb{R}^2 \setminus \{0\}}$$

  has constant rank which implies $F^{-1}(q)$ is a proper $1-$dimensional submanifold.

We'll see a proof of this theorem in the next lecture.