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| **Physics 5C: Introductory Thermodynamics and Quantum Mechanics** | **Fall 2023** |

# Homework 8:

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**Q1.**

The range of visible wavelengths of light is roughly $\lambda_1 = 400nm$ to $\lambda_2 = 700nm$. The energy carried by a photon of light with wavelength $\lambda$ is given by

$$E = h\nu = \frac{hc}{\lambda}$$

Thus, the range of Photon Energies corresponding to visible wavelengths if $E_{min}$ up to $E_{max}$ where

$$\begin{aligned} E_{min} &= \frac{hc}{\lambda_2} \\ &= \frac{(6.62607015 \times 10^{-34} J \cdot Hz^{-1})(299792458 m \cdot s^{-1})}{700 \times 10^{-9} m} \\ &= 2.8377798 \times 10^{-19} J \\ &\approx 1.77 eV \end{aligned}$$

and the energy corresponding to $400nm$ is

$$\begin{aligned} E_{max} &= \frac{hc}{\lambda_1} \\ &= \frac{(6.62607015 \times 10^{-34} J \cdot Hz^{-1})(299792458 m \cdot s^{-1})}{400 \times 10^{-9} m} \\ &= 4.96611464 \times 10^{-19} J \\ &\approx 3.01 eV \end{aligned}$$

So, the range of Photon energies for visible light is roughly $[1.77, 3.01]$ eV.

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**Q2.**

The De-Broglie wavelength, $\lambda$, of a free electron of mass $m_e$ possessing momentum $p$ is given by

$$\lambda = \frac{h}{p}$$

Since the electron is free, the Momentum Operator commutes with the Hamiltonian, so the electron has a well defined energy (only kinetic, since there is no potential) which is related to momentum as

$$K = \frac{p^2}{2m_e}$$

or

$$p = \sqrt{2m_eK}$$

Thus, the De-Broglie Wavelength of a free electron with Kinetic Energy $K$ is

$$\lambda = \frac{h}{(2m_eK)^{1/2}}$$

The mass of an electron is $m_e = 9.10938356 \times 10^{-31}kg$, and Planck's constant is $6.62607015 \times 10^{-34}J \cdot s$, so

$$\frac{h}{(2m_e)^{1/2}} = \frac{6.62607015 \times 10^{-34}J \cdot s}{(2 \cdot 9.10938356 \times 10^{-31}kg)^{1/2}} = 4.90903962 \times 10^{-19}J \cdot s \cdot kg^{-1/2}$$

So, the De-Broglie Wavelength in SI units is

$$\begin{aligned}\lambda &= \frac{4.90903962 \times 10^{-19}}{\sqrt{K}} \cdot J^{1/2} \cdot s \cdot kg^{-1/2} \\ &= \frac{4.90903962 \times 10^{-19}}{\sqrt{K}} \cdot \frac{s \cdot kg^{-1/2}}{J^{-1/2}}\end{aligned}$$

The units work out to give back meters, $m$, since

$$\frac{s \cdot kg^{-1/2}}{J^{-1/2}} = \frac{s \cdot kg^{-1/2}}{(kg \cdot m^2 \cdot s^{-2})^{-1/2}} = m$$

Now, replacing one joule, $1J$, with the equivalent number of Electron Volts, which is $6.2415 \times 10^{18}$, should leave everything unchanged. So,

$$\lambda = \frac{4.90903962 \times 10^{-19}}{\sqrt{K}} \cdot \frac{s \cdot kg^{-1/2}}{(6.2415 \times 10^{18}eV)^{-1/2}}$$

and the wavelength obtained above is in meters, so for the answer to be in nanometers, we should multiply further by $10^9$, so that finally, the wavelength is given by

$$\begin{aligned}\lambda &= \frac{4.90903962 \times 10^{-19} \cdot 10^9}{(6.2415 \times 10^{18})^{-1/2}} \frac{1}{\sqrt{K}} \\ &= \frac{1.226425084}{\sqrt{K}} \\ &\approx \frac{1.23}{\sqrt{K}}\end{aligned}$$

Thus, if wavelength is to be found in nanometers and kinetic energy is expressed in electron volts, then we get the relation

$$\boxed{\lambda = \frac{1.23}{\sqrt{K}}}$$

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**Q3.**

Maximum energy transfer to the electron occurs when the change in frequency of the photon is maximum.

If the photon comes in with energy $E = h\nu_0 = \frac{hc}{\lambda_0}$, and gets scattered off it's original path by angle $\theta$, then the new frequency of the photon is given by

$$\nu = \frac{\nu_0}{1 + \left(\frac{h\nu_0}{mc^2}\right)(1 - \cos(\theta))}$$

So, the change in the energy of the photon is

$$\begin{aligned}
\Delta E_{photon} &= h(\nu - \nu_0) \\
&= h\nu_0 \left( \frac{1}{1 + \left(\frac{h\nu_0}{mc^2}\right)(1 - \cos(\theta))} - 1 \right) \\
&= E \left( \frac{1}{1 + \left(\frac{E}{mc^2}\right)(1 - \cos(\theta))} - 1 \right)
\end{aligned}$$

Now, the amount of energy transferred to the electron is equal in magnitude to $\Delta E_{photon}$, but is opposite in sign. So,

$$\boxed{T \equiv \Delta E_{electron} = E \left( 1 - \frac{1}{1 + \left(\frac{E}{mc^2}\right)(1 - \cos(\theta))} \right)}$$

This transfer-energy is maximized when the subtracted term is minimized. That is, when the denominator $1 + \frac{E}{mc^2}(1 - \cos(\theta))$ is at its greatest value.

This happens when $1 - \cos(\theta)$ is at its greatest, which is is $1 - (-1) = 2$.

Thus, the greatest amount of energy that can be transferred to the electron in Compton Scattering is

$$\begin{aligned}
T &= E \left( 1 - \frac{1}{1 + \left(\frac{2E}{mc^2}\right)} \right) \\
&= E \left( \frac{1 + \left(\frac{2E}{mc^2}\right)}{1 + \left(\frac{2E}{mc^2}\right)} - \frac{1}{1 + \left(\frac{2E}{mc^2}\right)} \right) \\
&= E \left( \frac{\frac{2E}{mc^2}}{1 + \left(\frac{2E}{mc^2}\right)} \right) \\
&= E \left( \frac{1}{\frac{mc^2}{2E}\left[1 + \left(\frac{2E}{mc^2}\right)\right]} \right) \\
&= E \left( \frac{1}{\frac{mc^2}{2E} + 1} \right)
\end{aligned}$$

Thus, the max energy that can be transferred is

$$\boxed{T_{max} = E\left(\frac{1}{1+\frac{mc^2}{2E}}\right)}$$

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**Q4.**

In the Free Electron Gas (Fermi Gas) model of a solid, if we have a rectangular solid of dimensions $(l_x, l_y, l_z)$, the Free Electron Gas potential

$$V(x) = \begin{cases} 0, & \text{for } 0 < x < l_x, 0 < y < l_y, 0 < z < l_z \\ \infty, & \text{otherwise} \end{cases}$$

The wavefunction is zero in the region with infinite potential. In the region with zero potential, the Schroedinger Equation,

$$-\frac{\hbar}{2m}\nabla^2\psi = E\psi$$

which we can solve by separation of variables exactly as we did in Question 1 of this HW.

Following the exam same steps as the solution for question 1, but with the length in each dimension now being $l_x$, $l_y$, or $l_z$ respectively, we find the wavefunction to be

$$\psi_{n_x n_y n_z} = \sqrt{\frac{8}{l_x l_y l_z}} \sin\left(\frac{n_x\pi}{l_x}x\right) \sin\left(\frac{n_y\pi}{l_y}y\right) \sin\left(\frac{n_z\pi}{l_z}z\right)$$

and the allowed energies are

$$E_{n_x n_y n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2}\right) = \frac{\hbar^2 k^2}{2m}$$

where $k$ is the magnitude of the wave vector $\vec{k} = (k_x, k_y, k_z)$.

Then, in the $k$-space, if we imagine the 3-d space with axes along $k_x$, $k_y$, $k_z$ with lines drawn at each integer value of $n_x$, $n_y$, $n_z$, each intersection point represents a distinct stationary state.

Each one of these blocks (and so each one of the states) occupies a volume

$$\frac{\pi^2}{l_x l_y l_z} = \frac{\pi^3}{V}$$

of $k$-space, where $V$ is the volume of the actual physical solid.

Suppose our sample contains $N$ atoms, and each one contributes $d$ free atoms. Now, due to the Pauli Exclusion Principle, each block in $k$-space can accomodate two electrons.

Now, because $N$ is a huge number (on the order of $10^{23}$ due to the value of Avogadro's constant), the rectangular volume in $k$-space can be approximated by a sphere of some radius $k_F$ (called the Fermi Radius). The radius is determined by the requirement that each pair of electrons requires a volume of $\frac{\pi^3}{V}$.

$$\frac{1}{8}\left(\frac{4}{3}\pi^2 k_F\right) = \frac{Nd}{2}\left(\frac{\pi^3}{V}\right)$$

Thus,

$$k_F = \left(3\frac{Nd}{V}\pi^2\right)^{1/3}$$

The corresponding energy, $E_F$, is called the Fermi Energy

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\frac{Nd}{V}\pi^2\right)^{2/3}$$

So, we have

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\frac{Nd}{V}\pi^2\right)^{2/3}}$$

We can calculate the **total energy** of the degenerate Fermi Gas by considering spherical shells of thickness $dk$ in the $k$-space.

A shell of thickness $dk$ has volume

$$dV = \frac{1}{8}\left(4\pi k^2\right) dk$$

So, the number of electron states in the shell is

$$\frac{2[(1/2)\pi k^2 dk]}{(\pi^3/V)} = \frac{V}{\pi^2}k^2 dk$$

Each one of these electron states carries energy $\frac{\hbar^2 k^2}{2m}$, so the energy of the shell of radius $k$ in $k$-space is

$$dE = \frac{\hbar^2 k^2}{2m}\cdot\frac{V}{\pi^2}k^2 dk$$

Therefore, the Total Energy is given by

$$\begin{aligned} E_{tot} &= \frac{\hbar^2 k^2 V}{2m\pi^2}\int_0^{k_F} k^4 dk \\ &= \frac{\hbar^2 V}{2m\pi^2}\left[\frac{k^5}{5}\right]_0^{k_F} \\ &= \frac{\hbar^2 V}{2m\pi^2}\cdot\frac{k_F^5}{5} \\ &= \frac{\hbar^2 V}{10m\pi^2}\cdot\left(3\frac{Nd}{V}\pi^2\right)^{5/3} \end{aligned}$$

So, the total energy *per electron* is

$$\begin{aligned}\frac{E_{tot}}{Nd} &= \frac{\hbar^2 V}{10m\pi^2} \cdot \left(3\frac{Nd}{V}\pi^2\right)^{5/3} \cdot \frac{1}{Nd} \\ &= \frac{3}{5} \cdot \frac{\hbar^2}{2m}\left(3\frac{Nd}{V}\right)^{2/3} \\ &= \frac{3}{5}E_F\end{aligned}$$

Thus, the energy per electron of a 3D Degenerate Fermi Gas is related to the Fermi Energy as $(3/5)E_F$.