Math 250A Homework 1

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Question I.1

Show that every group of order ≤ 5 is abelian.

Proof: Consider a group with order $|G| \leq 5$ and elements $x, y \in G$ such that $xy \neq yx$ (making G non-abelian). Due to closure under inversion we know $(xy)^{-1}, (yx)^{-1} \in G$ and the inverses are not equal to each other since inverses are unique. Then $x, y, xy, yx, (xy)^{-1}, (yx)^{-1} \in G$ so $|G| \geq 6$. This contradicts our assumption. Thus every group of order ≤ 5 must be abelian. ■

Question I.2

Show that there are two non-isomorphic groups of order 4, namely the cyclic one, and the product of two cyclic groups of order 2.

Proof: Consider the cyclic group

$$C := \mathbb{Z}/4\mathbb{Z} \cong \{0, 1, 2, 3\}$$

with modular arithemetic addition as the composition law and

$$D := (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \cong \{(0,0), (1,0), (0,1), (1,1)\}$$

with componentwise addition as the composition law. Both of these groups have order 4. However, the element $3 \in C$ has order 4 because $3 + 3 + 3 + 3 = 12 \equiv 0 \pmod 4$. However, all elements of D have order ≤ 2. Thus, the two groups cannot be isomorphic. ■

Question I.9

(a) Let G be a group and H be a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H also of finite index.

(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Show that $H_1 \cap H_2$ is also of finite index.

Proof: . ■

Question I.10

Let G be a group and H be a subgroup of finite index. Show that there are only finitely many right cosets of H and that the number of right cosets equals the number of left cosets.

<u>**Proof:**</u> . ■
