Math 214 Notes

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1.1 Defining Tangent Spaces via velocity vectors

For a point $p \in M$ on smooth manifold M, contained by chart (U, ϕ), lets define

$$\mathcal{J}_p(M) = \{\gamma : (-\epsilon, \epsilon) \to M : \gamma(0) = p, \epsilon > 0, smooth\}$$

and we set $\gamma_1 \sim \gamma_2$ if for any $f \in C^\infty(M)$,

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

Intuitively, this equvalence relation tellls us that all velocity vectors which are tangent to p are equivalent.

<u>Lemma:</u> γ is an equuivalence relation.

So, we can alternatively define the tangent space at p as

$$T_pM = \mathcal{J}_p(M) /_{\sim}$$

Relating this to T_pM defined using Derivations

We will basically show

$$\underbrace{T_pM}_{\text{curves}} \xrightarrow{\cong} \underbrace{T_pM}_{\text{derviations}}$$
$$[\gamma] \mapsto D[\gamma]$$

For example, one way to do this is to check on coordinate charts.

Note: Look up definition of tangent space using *germs*.

1.2 Relating the Tangent Spaces at different points

In $\mathbb{R}^n$

Consider $p, q \in \mathbb{R}^n, p \neq q$. There is a natural way to associate the the two *different* tangent space T_pM and T_qM simply by "translating" it, but on a general manifold, this isn't usually possible.

Note: Smooth structure isn't enough, we need more structure. For example, This is where *connections* and *parallel transport* come into play. Eg. For a manifold with Riemannian Metric, it is possible to do this "translation" of tangent spaces.

1.3 Tangent Bundle

Tangent Bundle

- Given a smooth manifold M^n, the ***Tangent Bundle*** is a smooth manifold of dimension $2n$.
- As a set, it is $TM \equiv \coprod_{p\in M} T_pM$ i.e. the disjoint union of the tangent spaces at all points.
- We have the associated projection $\pi : TM \to M$

$$v \in T_pM \mapsto p \in M$$

For example, $M = \mathbb{S}^1$, the at each point the tangent space is just a copy of $\mathbb{R}$. Therefore,

$$T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$$

[Insert figure – note that the image is just a depiction of the collection of objects, without any additional structure.]

1.3.1 Topology and Smooth Structure on TM

Given a smooth chart (U, ϕ) of M^n, we take $\left(\pi^{-1}(U), \tilde{\phi}\right)$ to be a chart on TM, where

$$\begin{aligned} \tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n \\ \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \left(\phi(p), v^1, \dots, v^n\right) \end{aligned}$$

That these charts generate a smooth structure on TM by the Smooth manifold construction lemma (*LeeSM Lemma 1.35*) since it satsfies all four conditions.

$\tilde{\phi}$ is a bijection from $\pi^{-1}(U)$ into an open subset of $\mathbb{R}^{2n}$

For $\pi^{-1}(U), \pi^{-1}(V)$

$$\begin{aligned} \tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) &= \tilde{\phi}(\pi^{-1}(U \cap V)) \\ &= \phi(U \cap V) \times \mathbb{R}^n \end{aligned}$$

and this is open in $\mathbb{R}^{2n}$.

$\tilde{\phi} \circ \tilde{\psi}^{-1} : \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) \to \tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V))$ is smooth as $\phi \circ \psi^{-1} = (\overline{y}^1, \dots, \overline{y}^n)$ so the mapping is

$$\left(\tilde{x}^1, \dots, \tilde{x}^n, v^1, \dots, v^n\right) \to \left(\phi \circ \psi^{-1}\right)(\vec{\tilde{x}}), \sum_{j=1}^{n} [completelater]$$

- Second countability: Countably many $\left(\tilde{\phi}, \pi^{-1}(U)\right)$ cover TM.
- Hausdorffness: For $pneqq$ in TM, either both lie in the same $\pi^{-1}(U)$ which is itself Hausdorff; or $p \in \pi^{-1}(U), q \in \pi^{-1}(V)$ which are disjonit and hence there exist open neighborhoods around the points p, q.

Is there a nice basis for this topology?

1.4 Examples

- $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ (i.e. if a tangent bundle splits as a direct product as $TM^n \cong M \times \mathbb{R}^n$ then it's called a "trival bundle")
- $T\mathbb{S}^n$, $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ but $T\mathbb{S}^n \neq \mathbb{S}^2 \times \mathbb{R}^2$ (by Hairy-ball theorem, also for n even in general), $T\mathbb{S}^3 = \mathbb{S}^3 \times \mathbb{R}^3$, $T\mathbb{S}^7 = \mathbb{S}^7 \times \mathbb{R}^7$, otherwsie $T\mathbb{S}^n = \mathbb{S}^n \times \mathbb{R}^n$, (Adams 1962).

1.5 Chapter 4 Begins! Submersions, Immersions, and Embeddings

Before we begin, we revew some useful theorems.

1.5.1 Inverse Function Theorem.

In the $\mathbb{R}^n$ case, the Inverse Function Theorem tells us that if the derivative at a point x is non-singular, then we can invert the function in the locality of x.

Theore: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, then if $n \times n$ matrix representing the linear map

$$df_{x_0} : T_{x_0}\mathbb{R}^n \to T_{f(x_0)}\mathbb{R}^n$$

is invertible, then

$$f\big|_U : U \to f(U)$$

is a diffeomorphsm (smooth, with smooth inverse).

1.5.2 Inverse Function Theorem for smooth manifolds (without boundary)

Theorem: For smooth manifolds M^n, N^n and a smooth map $F : M \to N$, at a given point $p \in M$ if the map

$$dF_p : T_pM \to T_{F(p)N}$$

is invertble, then there exsits a neighborhood $U \subseteq_{open} M$ such that $p \in U$ and $F\big|_U : U \to F(U)$ is a dffeomorphism.

For the proof, we work on smooth charts and apply the IFT between euclidean spaces.

Remark:

- In fact, F is a ***local diffeomorphism*** at p if and only if dF_p is invertible.
- If F is a local diffeomorphsm at all $p \in M$ and F is invertible, then F is a global diffeomorphism.

Example:

For an example of something which is a local, but not global, diffeomorphism we can think of the Covering map $F : \mathbb{R} \to \mathbb{S}^1 \subseteq \mathbb{C}, t \mapsto e^{2\pi it}$. Locally, the differential is an isomorphism so the map is a local diffeomorphism. However, it isn't injectve! So it can't be a global diffeomorphism.

Read: Properties of dffeomorphisms.

1.6 Maps of Constant Rank

A smooth map $F : M^m \to N^n$ is an

- ***Immersion***: if dF_p is injective for all $p \in M$.
- ***Submersion***: if dF_p is surjective for all $p \in M$.
- ***Full rank***: if the rank $dF_p = \min\{m, n\}$ $p \in M$.
- ***Constant rank***: if the rank of dF_p is contant at all $p \in M$.

We will see later that immersions and submersions act, locally, like intjective and surjective maps.

Theorem: If dF_p has full rank, then there exists a neighborhood $p \in U \subseteq_{open} M$ such that $F\big|_U$ has full rank.

(dF_p has an invertble $\min\{m, n\} \times \min\{m, n\}$ submatrix with non-zero determinant. This is an open condition.)

Examples:

- A map $M_1 \to M_1 \times M_2$ defined by fixing some $x_1 \in M_2$ and then mapping

$$x_1 \mapsto (x_1, x_2)$$

  is an immersion.
- A map $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is smooth and has $\gamma(t) \neq 0$ for all $t \in \mathbb{R}$ is an immersion.
- The map $M_1 \times M_2 \to M_1$, $(x_1, x_2) \to x_1$ is a submersion.
- The projection $\pi : TM^n \to M^n$ is a submersion.