Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 4

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Question 1: Do the following exercises:
Exercise 7: Prove that the join of two nonempty path connected spaces is simply connected.

Exercise 8: Prove that the join of two nonempty spaces, of which one is path connected, is simply connected.

Solution: (Inspired by this answer by Kyle Miller)

Let's just do exercise 8, since it's a generalization of exercise 7. Suppose X, Y are non-empty spaces and X is path-connected and Y is any non-empty space.

Recall that the join of two spaces X, Y is the space of segments joining each point of X with each point of Y. The formal definition is

$$X * Y = (X \times Y \times I)/\sim$$

where we make the identifications

$$(x, y, 0) \sim (x, y', 0) \text{ for all } x \in X \text{ and } y, y' \in Y$$
$$(x, y, 1) \sim (x', y, 1) \text{ for all } x, x' \in X \text{ and } y \in Y$$

For example, the join of two lines would be

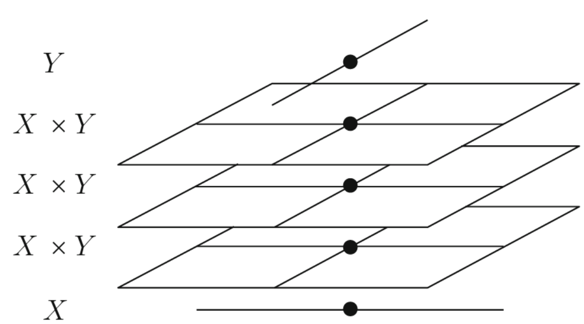


Fig. 8 The "horizontal sections" of a join

Source of figure: Fomenko and Fuchs, Homotopical Topology

Observation: We can assume Y is also path-connected. Why?

Y is not path-connected, so let's consider the collection of its path-connected components $\{Y_i\}_{i\in I}$ for some index set I. Then,

$$X * Y = \bigcup_{i\in I} X * Y_i$$

Let Z denote the open subset of $X * Y$ corresponding to $[0, 1/2) \subset I$. Now, let's define $A_i := Z \cup (X * Y_i)$. Since the Y_i's are disjoint, we have $A_i \cap A_j = Z$, $i \neq j$. So, if we are able to prove that each $X * Y_i$ is simply connected, $X * Y$ will be simply connected due to (inductive use of) Van-Kampen's theorem.

(Although it's true that each A_i is not necessarily open (which would be an issue for Van-Kampen's theorem), we don't have to worry about that here.

The role played by openness of each A_i in the proof of Van-Kampen's theorem is in showing surjectivity, which relies on $f^{-1}(A_i)$ for any continuous $f : [0, 1] \to X * Y$ being open in $[0, 1]$. This holds even though A_i is not open, as seen below:

Z is open in $X * Y$ so for any path $f : [0, 1] \to X * Y$ we have $f^{-1}(Z) \subseteq_{\text{open}} [0, 1]$. Now, take a point $s \in [0, 1]$ so that $f(s) = (x, y, t) \in X * Y$ has $t > 0$. Complete this.)

Thus, for the rest of the question, let's assume Y is also path-connected and show that $X * Y$ is simply connected.

Now, let's consider the two open sets U, V of $X * Y$ such that U corresponds to $[0, 2/3) \subset I$ and V corresponds to $(1/3, 1]$. Then, U deformation retracts onto X, V deformation retracts onto Y, and their intersection $U \cap V$ deformation retracts onto $X \times Y$. Let's take the base point of $X * Y$ to be $(x_0, y_0, 1/2) =: b$ where x_0, y_0 are the base-points of X and Y.

Then, Van-Kampen's theorem gives us

$$\pi_1(X * Y, b) = \pi_1(U, b) *_{\pi_1(U\cap V, b)} \pi_1(V, b)$$

Or, in other words, if we consider the inclusions and and induced homomorphisms as shown below,

$$\begin{tikzcd} & U \arrow[dr, "i_1"] & \\ U \cap V \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & X * Y \\ & V \arrow[ur, "i_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(U, b) \arrow[dr, "(i_1)_*"] & \\ \pi_1(U \cap V, b) \arrow[ur, "(j_1)_*"] \arrow[dr, "(j_2)_*"'] & & \pi_1(X * Y, b) \\ & \pi_1(V, b) \arrow[ur, "(i_2)_*"'] & \end{tikzcd}$$

then $\pi_1(X * Y, b)$ is the free product $\pi_1(U, b) * \pi_1(V, b)$ modulo the subgroup generated by all $i_{1*}(\gamma)(i_{2*}(\gamma))^{-1}$ for $\gamma \in \pi_1(U \cap V, b)$.

Now, since $U, V, U \cap V$ respectively deformation retract onto $X, Y, X \times Y$ we have

$$\begin{aligned} &\pi_1(U, b) \cong \pi_1(X, x_0) \\ &\pi_1(V, b) \cong \pi_1(Y, y_0) \\ &\pi_1(U \cap V, b) \cong \pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \end{aligned}$$

Thus, when we evaluate the amalgamated project, the fact that $\pi_1(U \cap V, b) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ forces the amalgamated product to be trivial. i.e. $\pi_1(X * Y) \cong \{1\}$. Thus, the join $X * Y$ is simply connected.

Question 2: Let X be the space obtained by attaching one end of a 2-dimensional cylinder to the other end by the double cover map of the circle. (This space is sometimes called the "mapping torus" of the doble-cover map of $\mathbb{S}^1$ to $\mathbb{S}^1$.) Compute all homotopy groups of X. *Hint:* First solve the problem for the regular Z-covering of X obtained by unwinding the mapping cylinder along the generator.

Solution: (Inspired by Johnathan Evans's UCL Topology and Groups course.)

Let's do this by thinking about Mapping tori. Given a space X and a continuous map $\phi : X \to X$, we can define the **Mapping Torus**

$$MT(\phi) = (X \times I)/\sim$$

where $(\phi(x), 0) \sim (x, 1)$.

Let's prove a general-ish theorem.

Theorem 0.1. *Let X be a CW Complex, and $\phi : X \to X$ be a cellular map. Then, the mapping torus $MT(\phi)$ has CW Structure where each $k-$cell e in X*

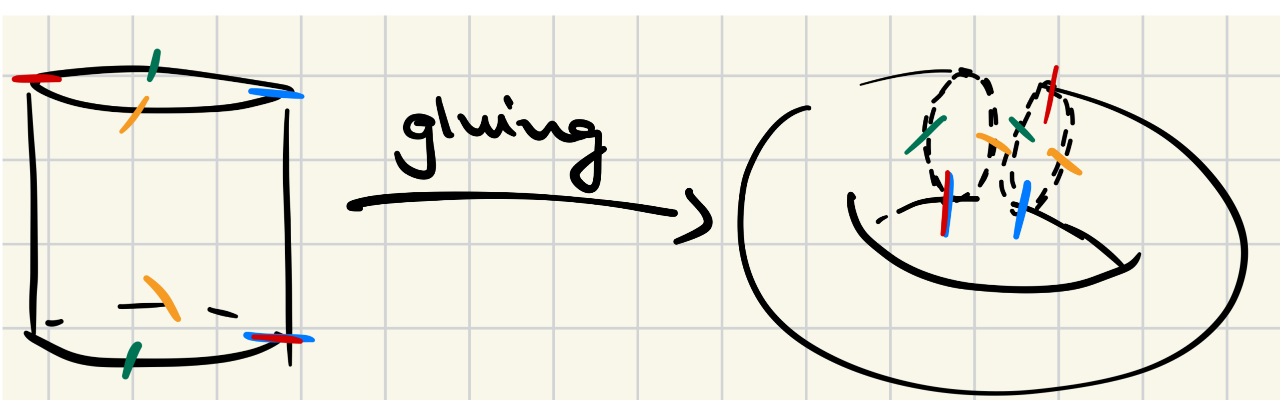
- *gives a $k-$cell $e \times \{0\} \in X \times \{0\}$*
- *gives a $(k+1)-$cell, $e \times [0, 1]$, in $(X \times I)/\sim$*

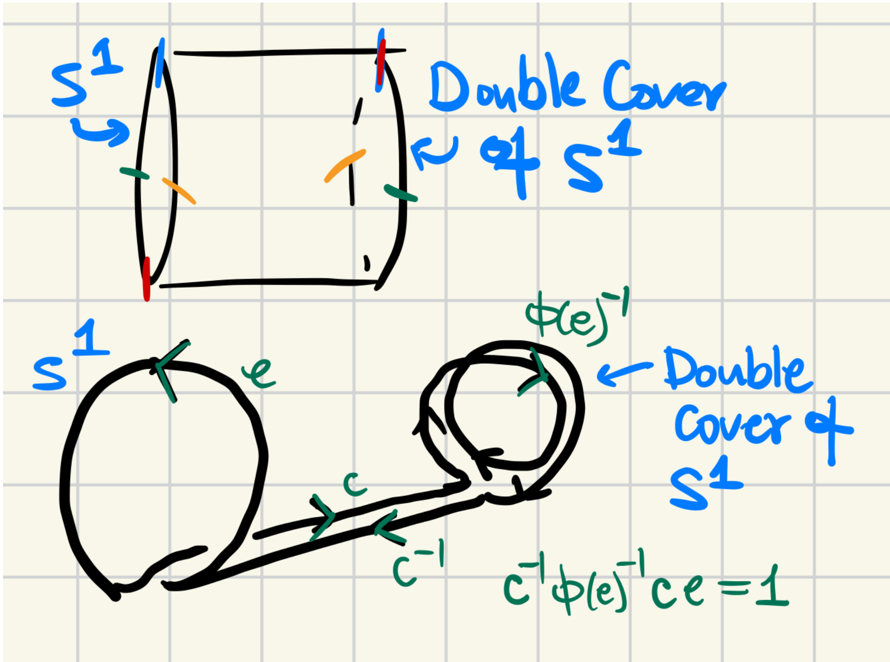
Theorem 0.2. *Suppose X has only one $0-$cell, e and $\phi : X \to X$ is a cellular map. Then,*

$$\pi_1(MT(\phi), (x, 0)) = \left\langle \begin{matrix} \text{generators of } \pi_1(X, x), \\ \text{and a new generator } c \end{matrix} \;\middle|\; \begin{matrix} \text{relations in } \pi_1(X, x), \\ \text{and a new relation for each 1-cell in } X \end{matrix} \right\rangle$$

where c is a new generator coming from the 1-cell $\{x\} \times [0, 1]$. For each $1-$cell e in X, we get a new relation $c^{-1}\phi(e)^{-1}ce = 1$.

The space that X we're considering in this question is a mapping torus of $\mathbb{S}^1$ with $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ being the double-cover map $\phi(z) = z^2$ for $z \in \mathbb{S}^1$ viewing $\mathbb{S}^1$ as $U(1) \subseteq \mathbb{C}$.





The relation c^{-1}

$\mathbb{S}^1$ is indeed a CW Complex with only one $0-$cell, and ϕ is indeed a cellular map. Denote the basepoint of $\mathbb{S}^1$ as x_0 (this is the 0-cell). The fundamental group of $\mathbb{S}^1$ is $\mathbb{Z} = \langle a \rangle$. So, the fundamental group is given by (since the double cover map sends $a \mapsto a^2$),

$$\pi_1(X, (x_0, 0)) = \left\langle a, c \,\middle|\, c^{-1}a^{-2}ca = 1 \right\rangle$$
$$\implies \pi_1(X, (x_0, 0)) = \left\langle a, c \,\middle|\, a^{-1}ba = ab \right\rangle$$

Since X has no $3-$cells, $\pi_n(X)$ is trivial for $n \geq 3$.

(Collaborated with Finn Fraser Grathwol) For the $n = 2$ case, we can use the long exact sequence of the fibration $\mathbb{S}^2 \to X \to \mathbb{S}^1$, which gives

$$\cdots \to \pi_2(\mathbb{S}^1) \to \pi_1(X) \to \pi_2(\mathbb{S}^1) \to \cdots$$

and $\pi_2(\mathbb{S}^1)$ is trivial, which tells us $\pi_2(X)$ is trivial.

Question 3: If one removes the arrow φ_3 from the diagram in the five-lemma, leaving all other assumptions intact, will it be true that $A_3 \cong B_3$? i.e. if we have the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "f_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5
\end{tikzcd}$$

where the rows are exact, φ_5 is a monomorphism, φ_2, φ_4 are epimorphisms, will it be true that $A_3 \cong B_3$?

Solution:

No. It's not necessarily true that $A_3 \cong B_3$ if we remove φ_3. For a counterexample, consider the following:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

It's well known that $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ because $\mathbb{Z}_4$ has an element of order 4 while $\mathbb{Z}_2 \times \mathbb{Z}_2$ has elements of order only up to 2.