# Math H185 Homework 8

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**Question 1**

Where are the isolated singularities of the following functions? Classify them as "removable" "pole", or "essential" singularities.

(a) $1/(z^2+4z+3)$

(b) $\sin(z)/(z^3+z)$

(c) $\cos(1/\sin(z))$

(d) $e^{1/z}/\sin(z)$

**<u>Solution:</u>**

(a) We have

$$\frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$

So the function has singularities at $z=-3$ and $z=-1$. At both singularities, the function blows up to infinity, so both of them are poles.

(b) The function

$$f(z) = \frac{\sin(z)}{z^3+z} = \frac{\sin(z)}{z(z^2+1)}$$

has singularities at $z = 0, i, -i$.

However, if we analytically continue $f(z)$ by writing $\sin(z)$ in terms of its power series expansion, the $z$ in the denominator is cancelled. So, $z=0$ is a removable singularity. The other singularities are poles since the function blows up to infinity.

(c) Recall that

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

, so we have

$$\cos\left(\frac{1}{\sin(z)}\right) = \frac{e^{\frac{i}{\sin(z)}} + e^{-\frac{i}{\sin(z)}}}{2}$$

So, the function has singularities everywhere $\sin(z)=0$ i.e. for $z = 2n\pi, n \in \mathbb{Z}$. These singularities are all poles as the function blows up to infinity.

(d) For

$$f(z) = \frac{e^{1/z}}{\sin(z)}$$

, we have isolated singularities at $z = 2\pi n, n \in \mathbb{Z}$. These singularities are poles.

## Question 2

For the following functions, find the order of the pole at $z_0 = 0$, and then the residue.

(a) $f(z) = \frac{1-e^z}{z^3}$

(b) $f(z) = \frac{\sin(z^2)}{z^4}$

(c) $f(z) = \frac{1}{(2\cos(z)-2+z^2)^2}$

(d) $f(z) = \frac{z^2+1}{2z\cos(z)}$

**Solution:**

(a) For this function, the pole at the origin has order zero because of the $1/z^3$ factor. We can rewrite the function as

$$
\begin{aligned}
\frac{1-e^z}{z^3} &= \frac{1}{z^3}\left[1 - \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\right] \\
&= \frac{1}{z^3}\left[1 - \left(1 + z + \frac{z^2}{2!} + \cdots\right)\right] \\
&= \frac{1}{z^3}\left[\left(-\sum_{k=1}^{\infty} \frac{z^k}{k!}\right)\right] \\
&= \sum_{k=1}^{\infty} -\frac{z^{k-3}}{k!} \\
&= -\frac{z^{-2}}{1!} - \frac{z^{-1}}{2!} - \frac{z^0}{3!} - \frac{z^1}{4!} - \cdots \\
&= \frac{-1}{(z-0)^2} + \frac{(-1/2)}{(z-0)} - \sum_{j=0}^{\infty} \frac{z^j}{(j+3)!}
\end{aligned}
$$

and so we find the $\text{res}(f) = -1/2$.

(b) Again, writing $f(z)$ in terms of the power series for $\sin(z^2)$, we have

$$
f(z) = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^5}{5!} \pm \cdots
$$

so the pole at $z_0 = 0$ has order 2 and residue 0.

(c) The expansion for $\cos(z)$ is

$$
\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} \pm \cdots
$$

So,

$$
f(z) = \frac{1}{2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \pm \cdots\right) - 2 + z^2}
$$

There is no term of $z$ to the first power that appears in the denominator anywhere so we must have residue equal to zero. The order of the pole at $z_0 = 0$ is four because that's the least degree of $z$ in the denominator.

(d) We can find the residue of $f(z)$ at $z_0$ as

$$\lim_{z \to z_0} (z - z_0)\, f(z)$$

This gives us

$$\begin{aligned}
\lim_{z \to z_0} (z - z_0)\, f(z) &= \lim_{z \to z_0} (z - 0) \cdot \left( \frac{z^2 + 1}{2z \cos(z)} \right) \\
&= \lim_{z \to 0} \frac{z^2 + 1}{2 \cos(z)} \\
&= \lim_{z \to 0} \frac{z^2}{2 \cos(z)} + \frac{1}{2 \cos(z)} \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

So the residue of the pole at the origin is $1/2$. The pole has order 1 because when we expand $\cos(z)$ in $f(z)$ and simplify, there is a $1/z$ term.

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**Question 3**

Suppose that $f$ is holomorphic and has a pole of zero order $m$ at $z_0$. What is the order of the pole of the function $g(z) = f'(z)/f(z)$ at $z_0$, and what is the residue?

**Proof:**

Since $f$ is holomorphic on $\mathbb{C} \setminus \{z_0\}$, it is also analytic over $\mathbb{C} \setminus \{z_0\}$. As a result, we can express it as

$$f(z) = (z - z_0)^m h(z)$$

with some other analtyic function $h(z)$.

Then,

$$\begin{aligned}
g(z) &= \frac{f'(z)}{f(z)} \\
&= \frac{m\,(z - z_0)^{m-1}\, h(z) + (z - z_0)^m h'(z)}{(z - z_0)^m h(z)} \\
&= \frac{m}{(z - z_0)} + \frac{h'(z)}{h(z)}.
\end{aligned}$$

So, the order of the pole of $g(z)$ at $z_0$ is 1 and the residue is $m$.

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**Question 4**

Use the Residue Theorem to calculate

$$\int_{\partial B_3(0)} \frac{e^{iz}}{z^2(z-2)(z+5i)} dz$$

**Proof:**

The Residue theorem tells us that if $f$ is holomorphic in a neighborhood of $\overline{U}$ except for a finite set of isolated singularities then

$$\int_{\partial U} f(z)dz = 2\pi i \sum_j \mathrm{Res}_{z_j}(f)$$

Thus, to calculate the integral

$$\int_{\partial B_3(0)} \frac{e^{iz}}{z^2(z-2)(z+5i)}dz$$

we find the residues of at the poles $z = 0, 2, -5i$.

$\underline{z_0 = 0}$:

$$\frac{e^{iz}}{z^2(z-2)(z+5i)} = \frac{h(z)}{z^2}$$

where $h(z) = (e^{iz})/(z-2)(z+5i)$ is holomorphic at $z = 0$ so we can taylor expand it near $z = 0$ as

$$h(z) = A_0 + A_1(z-0) + A_2(z-0)^2 + \cdots$$

So,

$$f(z) = \frac{A_0}{(z-0)^2} + \frac{A_1}{(z-0)^1} + \cdots$$

where of course, $A_k = \frac{1}{k!} \cdot \left(\frac{d^k}{dz^k} h(z)\right)\Big|_{z=0}$.

So,

$$\begin{aligned}
A_1 &= \frac{1}{1!} \cdot \frac{d}{dx}\left[\frac{e^{iz}}{(z-2)(z+5i)}\right]\Bigg|_{z=0} \\
&= \frac{e^{iz}i(z-2)(z+5i) - (2x-2+5i)e^{iz}}{((z-2)(z+5i))^2}\Bigg|_{z=0} \\
&= \frac{1 \cdot i \cdot (-2)(5i) - (-2+5i) \cdot 1}{(-10i)^2} \\
&= -\frac{3}{25} + \frac{1}{20}i
\end{aligned}$$

$\underline{z_0 = 2}$: Similarly, for $z_0 = 2$ we have

$$\frac{e^{iz}}{z^2(z-2)(z+5i)} = \frac{g(z)}{(z-2)}$$

where $g(z) = (e^{iz})/(z^2)(z+5i)$ is holomorphic at $z = 2$ and we taylor expand it around $z_0 = 2$ as

$$g(z) = B_0 + B_1(z-2) + B_2(z-2)^2 + \cdots$$

So,

$$f(z) = \frac{B_0}{(z-2)} + B_1 + B_2 \cdot (z-2)^1 + \cdots$$

Then, the residue at $z = 2$ is

$$\begin{aligned} B_0 &= g(2) \\ &= \frac{e^{2i}}{4 \cdot (2+5i)} \\ &= \frac{e^{2i}}{164} \cdot (8 - 10i) \end{aligned}$$

$\underline{z_0 = -5i}$: By the exact same reasoning, since the pole at $-5i$ is a first order pole, the residue of $f(z)$ at $z = -5i$ is given by $h(-5i)$ where

$$h(z) = \frac{e^{iz}}{z^2(z-2)}$$

This comes out to

$$\begin{aligned} h(-5i) &= \frac{e^{-5}}{(-5i)^2\,(2-5i)} \\ &= \frac{e^{-5}}{-25 \cdot (2-5i)} \end{aligned}$$

Therefore, the integral is equal to

$$\int_{\partial B_3(0)} \frac{e^{iz}}{z^2(z-2)(z+5i)} dz = 2\pi i \left[ -\frac{3}{25} + \frac{1}{20} i + \frac{e^{2i}}{164} \cdot (8-10i) + \frac{e^{-5}}{-25 \cdot (2-5i)} \right]$$

[Come back to this question for simplification.]

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**Question 5**

(a) Find the residue of $f(z) = 1/\sin(z)$ at $z_0 = 0$ and use this to calculate

$$\int_{B_1(0)} \frac{1}{\sin(z)} dz$$

(b) Calculate

$$\int_{B_4(0)} \frac{1}{\sin(z)} dz$$

**Proof:**

(a) To find the residue of $1/sin(z)$, we use the fact that the residue of $f$ at a simple pole $a \in \mathbb{C}$ is equal to

$$\lim_{z \to a} (z-a) f(z)$$

So, we find the residue at $z_0 = 0$ to be

$$\begin{aligned} \lim_{z \to 0} (z - 0) \frac{1}{\sin(z)} &= \lim_{z \to 0} \frac{z}{\sin(z)} \\ &= \frac{\lim_{z \to 0} 1}{\lim_{z \to 0} \frac{\sin(z)}{z}} \end{aligned}$$

the limit in the denominator is known to equal 1, so

$$\lim_{z\to 0}\frac{z}{\sin(z)}=1$$

Then, using the Residue Theorem,

$$\boxed{\int_{B_1(0)}\frac{1}{\sin(z)}dz=2\pi i}$$

(b) We can be more general and find the residue at each pole of $\frac{1}{\sin(z)}$ i.e. at every $z=n\pi$, $n\in\mathbb{Z}$ as

$$\begin{aligned}\lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z)}&=\lim_{z\to n\pi}\frac{1}{\cos(z)}\\&=(-1)^n\end{aligned}$$

The poles of $f(z)$ that lie within $B_4(0)$ are $z_0=-3\pi,-2\pi,-1\pi,0,\pi,2\pi,3\pi$ Then, using the Residue Theorem,

$$\begin{aligned}&\int_{B_4(0)}\frac{1}{\sin(z)}dz=2\pi i\,(-1+1-1+1-1+1-1)\\\implies&\boxed{\int_{B_4(0)}\frac{1}{\sin(z)}dz=2\pi i}\end{aligned}$$

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**Question 6**

Calculate

$$\int_{-\infty}^{\infty}\frac{\sin^3(x)}{x^3}dx$$

**Proof:**

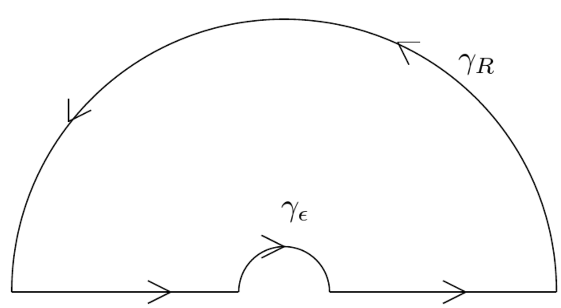
Motivated by the identity $\sin^3(x)=\frac{1}{4}\left(3\sin(x)-\sin(3x)\right)$ (which also holds for $x\in\mathbb{C}$), let's define the complex function

$$f(z)=\frac{1}{4}\left(3e^{iz}-e^{3iz}\right)$$

Expanding this out,

$$\begin{aligned}f(z)&=\frac{1}{4}\left[3\cdot(\cos(z)+i\sin(z))-(\cos(3z)+i\sin(3z))\right]\\\implies\operatorname{Im}(f(z))&=\frac{1}{4}\left[3\sin(z)-\sin(3z)\right]\\\implies\operatorname{Im}(f(z))&=\sin^3(z)\end{aligned}$$

Let's integrate this over the following contour $\Gamma$ which excludes the origin:



Since $f(z)$ has no singularities in the region bounded by the countour $\Gamma$, Cauchy's theorem gives us

$$\begin{aligned}
&\int_{\Gamma} \frac{f(z)}{z^3} dz = 0 \\
\implies &\int_{-R}^{-\epsilon} \frac{f(z)}{z^3} dz + \int_{\gamma_\epsilon} \frac{f(z)}{z^3} dz + \int_{+\epsilon}^{+R} \frac{f(z)}{z^3} dz + \int_{\gamma_R} \frac{f(z)}{z^3} dz = 0 \\
\implies &\lim_{\epsilon \to 0, R \to \infty} \left( \int_{-R}^{-\epsilon} \frac{f(z)}{z^3} dz + \int_{\gamma_\epsilon} \frac{f(z)}{z^3} dz + \int_{+\epsilon}^{+R} \frac{f(z)}{z^3} dz + \int_{\gamma_R} \frac{f(z)}{z^3} dz \right) = 0
\end{aligned}$$

Over $\gamma_R$, we have $\left|\frac{1}{z^3}\right| = \frac{1}{R^3}$. Now,

$$\left| \int_{\gamma_R} \frac{f(z)}{z^3} \right| \leq \int_{\gamma_R} \left| \frac{f(z)}{z^3} \right| = \int_{\gamma_R} \frac{|f(z)|}{R^3} dz$$

In the $R \to \infty$, we have $\int_{\gamma_R} \frac{|f(z)|}{R^3} dz \to 0$. Thus, $\left| \int_{\gamma_R} \frac{f(z)}{z^3} \right| \to 0$ which means

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{f(z)}{z^3} = 0$$

Now, the function

$$\frac{f(z)}{z^3} = \frac{1}{z^3} \cdot \frac{1}{4} \left( 3e^{iz} - e^{3iz} \right)$$

can be expanded using the taylor series expansion for the exponential.

$$\begin{aligned}
\frac{f(z)}{z^3} &= \frac{1}{4z^3} \left[ (3-1) + z(3i - 3i) + z^2 \left( 3\frac{i^2}{2} - \frac{(3i)^2}{2} \right) + \cdots \right] \\
&= \frac{1}{4z^3} \cdot \left[ 3z^2 + \cdots \right] \\
&= \frac{(3/4)}{z} + \cdots
\end{aligned}$$

Therefore, the function $f(z)/z^3$ has a first order pole at the origin and its residue there is $3/4$. Now, in the $\epsilon \to 0$ limit. we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \frac{f(z)}{z^3} dz &= \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{f(z)}{z^3} dz \\
&= \frac{1}{2} \cdot \left( -2\pi i \mathrm{Res}_0 \left( \frac{f(z)}{z^3} \right) \right) \\
&= -\pi i \frac{3}{4} \\
&= -\frac{3}{4} \pi i
\end{aligned}$$

where $C_\epsilon$ is the circle (reverse orientation) around the origin with radius $\epsilon$, rather than just the half circle we've considered. This allows us the apply the Residue theorem.

So,

$$\lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{\epsilon}\frac{f(z)}{z^3}dz+\int_{+\epsilon}^{+R}\frac{f(z)}{z^3}dz+\int_{\gamma_\epsilon}\frac{f(z)}{z^3}dz\right)=0$$

$$\implies \int_{-\infty}^{\infty}\frac{f(x)}{x^3}dx-\frac{3}{2}\pi i=0 \text{ (Where the first integral is just over the real line)}$$

$$\implies \int_{-\infty}^{\infty}\underbrace{\frac{(3\cos(x)-\cos(3x))}{4}}_{\mathrm{Re}(f(z))}\cdot\frac{1}{x^3}dz+i\int_{-\infty}^{\infty}\frac{\sin^3(x)}{x^3}dz=\frac{3}{4}\pi i$$

The first integral vanishes because it's an odd function benig integrated over an interval symmetric about the origin.

Therefore,

$$\boxed{\int_{-\infty}^{\infty}\frac{\sin^3(x)}{x^3}dx=\frac{3}{4}\pi}$$

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**Question 7**

Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0)\setminus\{z_0\}$. Suppose also that

$$|f(z)|\le A\,|z-z_0|^{-1+\epsilon}$$

for some $\epsilon>0$, and all $z$ near $z_0$. Show that the singularity of $f$ at $z_0$ is removable.

**Proof:**
We have $f(z)$ such that

$$|f(z)|\le A\,|z-z_0|^{-1+\epsilon}$$

near $z_0$.

Now if $z_0$ is a pole of $f$, then we can write $f(z)=\frac{h(z)}{(z-z_0)^k}$ for $z$ near $z_0$, $l\ge 1$, and $h(z)\neq 0$ near $z_0$. So,

$$\left|\frac{h(z)}{(z-z_0)^k}\right|\le A\,|z-z_0|^{-1+\epsilon}$$

or equivalently,

$$|h(z)|\le A\,|z-z_0|^{k-1+\epsilon}$$

In order for this to be true for arbitrary $\epsilon$, we would require $h(z_0)=0$, which is a contradiction. Thus, $z_0$ must be a removable singularity of $f$.

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