Math H185 Homework 1

Keshav Balwant Deoskar

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Question 1

(a) Prove that if $z \in \mathbb{C}$ then $z\bar{z} = |z|^2$.

(b) Prove that if $z, w \in \mathbb{C}$ then $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.

(c) Use parts (a) and (b) to show that for complex numbers $w, z \in \mathbb{C}$,

$$|z \cdot w| = |z| \cdot |w|$$

<u>Proof:</u>

(a) Consider $z = a + bi \in \mathbb{C}$. Then, $\bar{z} = a - bi$, and

$$\begin{aligned} z \cdot \bar{z} &= (a + bi) \cdot (a - bi) \\ &= a \cdot a - a \cdot bi + bi \cdot a + (bi) \cdot (-bi) \\ &= a^2 - b^2 \cdot (-1) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

So,

$$\boxed{z \cdot \bar{z} = |z|^2}$$

(b) Now, suppose we have $z = a + bi$ and $w = c + di$. Then,

$$\begin{aligned} z \cdot w &= (a + bi) \cdot (c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

So,

$$\overline{z \cdot w} = (ac - bd) - (ad + bc)i \tag{1}$$

Now,

$$\begin{aligned} \overline{z} \cdot \overline{w} &= (a - bi) \cdot (c - di) \qquad (2) \\ &= (ac - bd) - (bc + ad)i \qquad (3) \end{aligned}$$

So, from equations (1) and (3), we see that

$$\boxed{\overline{z \cdot w} = \overline{z} \cdot \overline{w}}$$

(c) Now,

$$|z \cdot w|^2 = (z \cdot w) \cdot (\overline{z \cdot w}) = z \cdot w \cdot \overline{z} \cdot \overline{w}$$

(first equality due to part (a) and second due to part (b)).

Using commutativity and associativity of complex multiplication, we can write

$$\begin{aligned}(z \cdot w)(\overline{z \cdot w}) &= (z \cdot w)(\overline{w} \cdot \overline{z}) \\ &= z \cdot (w \cdot \overline{w}) \cdot \overline{z} \\ &= z \cdot \overline{z} \cdot (w \cdot \overline{w}) \\ &= |z|^2 \cdot |w|^2\end{aligned}$$

And since the norm of a complex number is guaranteed to be positive (or zero), we can take the square root on both sides without uncertainty and conclude that

$$\boxed{|z \cdot w| = |z| \cdot |w|}$$

Question 2

Draw a picture of the following subsets of $\mathbb{C}$.

(a) $\{z \in \mathbb{C} : |z - 6i| \leq 1\}$

(b) $\{z \in \mathbb{C} : \mathrm{Im}(z) = \mathrm{Re}(z)^2\}$

(c) $\{z \in \mathbb{C} : |z| = \mathrm{Re}(z) + 1\}$

Answer:

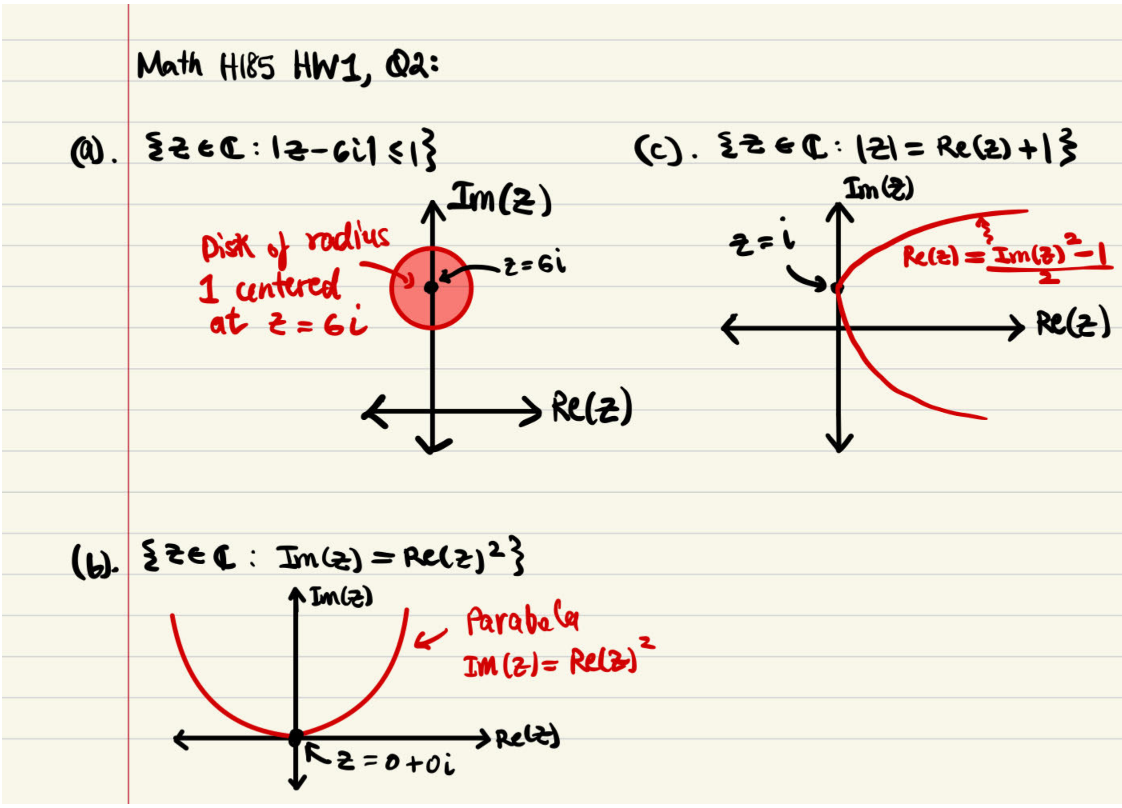
(a) This is the disk of radius 1 centered at the point $6i$.

(b) This is a parabola.

(c) Consider $z = a + bi \in \mathbb{C}$. We know that $|z| = \sqrt{a^2 + b^2}$, so this region is

$$\begin{aligned}& a^2 + b^2 = (a+1)^2 \\ \implies & a^2 + b^2 = a^2 + 2a + 1 \\ \implies & b^2 = 2a + 1 \\ \implies & a = \frac{b^2 - 1}{2}\end{aligned}$$

That is, the set is a horizontal parabola opening up to the right, with vertex at $z = i$.



Question 3

Express the following complex numbers in the form $a + bi$ where $a, b \in \mathbb{R}$.

(a) $(12 + 15i) + (-5 - 8i)$

(b) $(4 + 7i)(5 - 2i)$

(c) $\frac{3}{i}$

(d) $\frac{169}{5+12i}$

Answer:

(a)

$$\begin{aligned}(12 + 15i) + (-5 - 8i) &= (12 - 5) + (15 - 8)i \\ &= 7 + 7i\end{aligned}$$

(b)

$$\begin{aligned}(4 + 7i)(5 - 2i) &= 4(5) + 4(-2i) + (7i)(5) + (7i)(-2i) \\ &= 20 - 8i + 35i + 14 \\ &= 34 + 17i\end{aligned}$$

(c)

$$\begin{aligned}\frac{3}{i} &= \frac{3}{i}\cdot\frac{i}{i}\\ &= \frac{3i}{-1}\\ &= 0-3i\end{aligned}$$

(d)

$$\begin{aligned}\frac{169}{5+12i} &= \frac{169}{5+12i}\cdot\frac{5-12i}{5-12i}\\ &= \frac{169\cdot(5-12i)}{25+144}\\ &= \frac{169}{169}\cdot(5-12i)\\ &= 5-12i\end{aligned}$$

Question 4

Express the following complex numbers in the form $a+bi$ where $a,b\in\mathbb{R}$.

(a) $4e^{\pi i/4}$

(b) $2e^{\pi i/2}+4e^{4\pi i/3}$

(c) $(6e^{\pi i/6})\cdot(6e^{\pi i/4})$

Answer:

(a) For $4e^{\pi i/4}$, we have:

$$\begin{aligned}4e^{\pi i/4} &= 4\cdot\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\\ &= 4\cdot\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\boxed{\implies 4e^{\pi i/4} = 2\sqrt{2}+2\sqrt{2}\cdot i}$$

(b) For $2e^{\pi i/2}+4e^{4\pi i/3}$, we have:

$$\begin{aligned}2e^{\pi i/2}+4e^{4\pi i/3} &= 2\cdot\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)+4\cdot\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)\\ &= 2\cdot\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)+4\cdot\left(\cos\left(\pi+\frac{\pi}{3}\right)+i\sin\left(\pi+\frac{\pi}{3}\right)\right)\\ &= 2\cdot(0+i\cdot 1)+4\cdot\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\\ &= (2-2)-2\sqrt{3}i\end{aligned}$$

$$\boxed{\implies 2e^{\pi i/2}+4e^{4\pi i/3} = 0-2\sqrt{3}i}$$

(c) For $(6e^{\pi i/6}) \cdot (6e^{\pi i/4})$, we have:

$$\begin{aligned}(6e^{\pi i/6}) \cdot (6e^{\pi i/4}) &= 36e^{\pi i \cdot \left(\frac{1}{6}+\frac{1}{4}\right)} \\ &= 36e^{\pi i \cdot \left(\frac{4}{24}+\frac{6}{24}\right)} \\ &= 36e^{\pi i \cdot \frac{10}{24}} \\ &= 36 \cdot \left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]\end{aligned}$$

$$\boxed{\implies (6e^{\pi i/6}) \cdot (6e^{\pi i/4}) = 9.317485624 + 34.77332975i}$$

Question 5

Express the following complex numbers in the form $re^{i\theta}$ where $r \in \mathbb{R}_{\geq 0}$ and $\theta \in [0, 2\pi]$

(a) $1 + \sqrt{3}i$

(b) $\sqrt{3} - i$

(c) $1 + i$

(d) $\frac{1}{1-i}$

Answer:

(a) For $1 + \sqrt{3}$, we have:

$$\begin{aligned}r &= \sqrt{1^2 + (\sqrt{3})^2} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

and

$$\theta = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

Thus,

$$\boxed{1 + \sqrt{3} = 2e^{\frac{1}{3}\pi i}}$$

(b) For $\sqrt{3} - i$, we have:

$$\begin{aligned}r &= \sqrt{(\sqrt{3})^2 + (-i)^2} \\ &= \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2 \cdot (i)^2} \\ &= \sqrt{3 - 1} \\ &= \sqrt{2}\end{aligned}$$

and

$$\theta = \arctan\left(\frac{-1}{3}\right) \approx -0.32175055$$

So,

$$\boxed{\sqrt{3} - i = \sqrt{2}e^{-0.32175055i}}$$

(c) For $1 + i$, we have:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and

$$\theta = \arctan(1/1) = \frac{\pi}{4}$$

$$\boxed{1 + i = e^{\frac{\pi i}{4}}}$$

(d) For $\frac{1}{1-i} = \frac{1+i}{2}$, we have:

$$\begin{aligned} r &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{2 \cdot \frac{1}{4}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} \theta &= \arctan((1/2)/(-1/2)) \\ &= \arctan(-1) \\ &= 2\pi - \frac{\pi}{4} \\ &= \frac{7\pi}{4} \end{aligned}$$

$$\boxed{1 + i = \frac{1}{\sqrt{2}}e^{\frac{7\pi i}{4}}}$$

Question 6

Let $w = re^{i\theta}$ with $r > 0$, and $n > 0$ be a positive integer. Write down in terms of r, θ, and n the polar form of all complex numbers z such that $z^n = w$.

Answer:

In terms of the cartesian form $a + bi$, two complex numbers are the same if they have equal Real and Imaginary parts. In terms of polar coordinates, this means that they must have

- the same modulus, since $|z|^2 = \sqrt{a^2 + b^2}$ and the two equal complex numbers must both have $\text{Re}(z) = a, \text{Re}(z) = b$.
- their arguments, Θ_1, Θ_2 must be related as

$$\Theta_1 - \Theta_2 = 2\pi \cdot k$$

for some $k \in \mathbb{Z}$ because rotating around the origin by any integer multiple of 2π brings us back to the same point.

Let's denote z in polar form as $z = Le^{i\tau}$. Then, in order to have $z^n = w$ we need

$$\begin{aligned} (Le^{i\tau})^n &= re^{i\theta} \\ \implies L^n e^{i\cdot n\tau} &= re^{i\theta} \\ \implies \frac{L^n}{r} e^{i\cdot(n\tau-\theta)} &= 1 \end{aligned}$$

Now, $|z \cdot w| = |z| \cdot |w|$, so

$$\begin{aligned} |1| &= |\frac{L^n}{r} e^{i\cdot(n\tau-\theta)}| \\ \implies 1 &= |\frac{L^n}{r}| \cdot |e^{i(n\tau-\theta)}| \\ \implies 1 &= \frac{|L^n|}{|r|} \cdot |e^{i(n\tau-\theta)}| \end{aligned}$$

But $|e^{iz}| = 1$ for any $z \in \mathbb{C}$. Thus, $\frac{|L^n|}{|r|} = \frac{|L|^n}{|r|} 1$ and since $L, r \in \mathbb{R}$ this simply means

$$\boxed{L = r^{1/n}}$$

As for the argument, we know that $\frac{L}{r} e^{i\cdot(n\tau-\theta)} = 1 + 0i$. That is,

$$\underbrace{(n\tau - \theta)}_{\Theta_1} - \underbrace{0}_{\Theta_2} = 2\pi k$$

for $k \in \mathbb{Z}$.

So, we have

$$\boxed{\tau = \frac{(2\pi k + \theta)}{n}}$$

However, notice that we have some redundancy again. Both k and n are integers (and $n > 0$), so the values k and $k' = k + m \cdot n$ where m is any integer give the same value of τ. So, really, the set of values for k which give us **distinct** τ values is the Integers modulo n.

Thus, in terms of r, θ, n the complex number z must be of the form

$$\boxed{z = r^{1/n} e^{i\cdot(2\pi k+\theta)/n}}$$

where $k \in \mathbb{Z}/n\mathbb{Z}$, in order for the equation $z^n = w$ to hold.

Question 7

Let $\{z_n = x_n + iy_n\}_n$ be a sequence of complex numbers. Prove that $\{z_n\}_n$ converges to $z \equiv x + iy$ if and only if $\lim_{x\to\infty} x_n = x$ and $\lim_{y\to\infty} y_n = y$. Here, $x_n, y_n, x, y \in \mathbb{R}$.

Answer:

- $\underline{\implies \text{ Direction:}}$ Suppose the sequence $\{z_n = x_n + iy_n\}$ converges to $z = x + iy$. Then, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|z_n - z| < \epsilon$$

$$\implies |(x_n + iy_n) - (x + iy)| < \epsilon$$
$$\implies |(x_n - x) + i(y_n - y)| < \epsilon$$

Now, we know for all $z \in \mathbb{C}$ that $|\text{Re}(z)| < |z|$ and $|\text{Im}(z)| < |z|$.

For $z = x + iy$,

$$x^2 < |z|^2 = x^2 + y^2 \quad \text{since } x^2, y^2 > 0$$
$$\implies x^2 < |z|$$

and similarly for $y < |z|$

Therefore,

$$|x_n - x| < |(x_n - x) + i(y_n - y)| < \epsilon$$
$$|y_n - y| < |(x_n - x) + i(y_n - y)| < \epsilon$$

Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ also converge.

- $\underline{\impliedby \text{ Direction:}}$ Suppose the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. Then for any $\epsilon > 0$ there exist integers $N, M \geq 0$ such that for all $n \geq N$, $|x_n - x| < \epsilon$ and for all $m \geq M$, $|y_n - y| < \epsilon$.

Find the values of N, M corresponding to $\frac{\epsilon}{2} > 0$ and set $L = \max\{N, M\}$. Then, for all $l \geq L$ we have $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon$.

Now, note that $|i(y_n - y)| = |i| \cdot |y_n - y| = |y_n - y|$. Using the Triangle Inequality, we have

$$|(x_n - x) + i(y_n - y)| \leq |x_n - x| + \underbrace{|i(y_n - y)|}_{=|y_n - y|} < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$\implies |(x_n + iy_n) - (x + iy)| < \epsilon$$

Thus,

$$\boxed{|z_n - z| < \epsilon}$$

So, the sequence $\{z_n\}$ converges to z.

Question 8

Let $f(z) = |z|$ be viewed as a function $f : \mathbb{C} \to \mathbb{C}$. Show that f is not holomorphic at any $z \in \mathbb{C}$.

Answer:

For complex number $z = a + bi$, consider the function $f(z) = |z| = \sqrt{a^2 + b^2}$. The difference quotient is

$$\frac{f(z+h) - f(z)}{h} = \frac{|z+h| - |z|}{h}$$

Notice that the numerator is always a real number, whereas h in the denominator may have an imaginary part. bh1

Thus, for any $z \in \mathbb{C}$, if we approach the point z horizontally i.e. $Im(h) = 0$ throughout the approach, then the difference quotient is a real number. However if we approach z vertcally i.e. $Re(z) = 0$ throughout the approach, then the difference quotient is an imaginary number. Therefore, the function is not holomorphic anywhere.
