Math 214 Homework 5

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Q4-5. Let \mathbb{CP}^n denote the *n*-dimensional complex projective space.

- (a) Show that the quotient map $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.
- (b) Show that \mathbb{CP}^n is diffeomorphic to \mathbb{S}^n .

Proof:

Q4-6. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F: M \to \mathbb{R}^k$ for any k > 0.

Proof: From LeeSM Proposition 4.28, We know that if $\pi:M\to N$ is a smooth submersion between smooth manifolds then π is an open map. Now, consider M to be a non-empty smooth compact manifold and let $N=\mathbb{R}^k$. $M\subseteq M$ is open when viewed as a subset of itself. However, F(M) is a compact subset of \mathbb{R}^k since F is a smooth map, and compact subsets of euclidean space are not open. Thus, we have a contradiction.

Q4-7. Suppose M and N are smooth manifolds, and $\pi: M \to N$ is an injective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29.

Proof:

From Theorem 4.28, we know that surjective smooth submersions are quotient maps. Then, from the uniqueness of the quotient topology, we know there is no other smooth manifold structure on N such that the conclusion of Theorem 4.29 holds.

Q4-8. Let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x,y) = xy$. Show that π is surjective and smooth, and that for each smooth manifold P, a map $F: \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof:

For any $t \in \mathbb{R}$, we can simply choose x = t, y = 1. Then, $\pi(x, y) = \pi(t, 1) = t$, so the map is surjective. The map is also smooth since the partial derivatives with respect to $x^1, x^2 = x, y$ are smooth

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

However, π is not a smooth submersion since the differential of π

$$d\pi_{(0,0)} = \begin{pmatrix} x \\ y \end{pmatrix} \bigg|_{(0,0)} = \mathbf{0}$$

has rank zero at the origin, whereas it has rank 1 everywhere else on \mathbb{R}^2 . So, π is not a constant rank map.

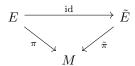
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Q4-9. Let M be a connected smooth manifold, and let $\pi: E \to M$ be a topological covering map. Complete the proof of proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map.

Proof:

Theorem 4.40: Suppose M is a connected smooth n-manifold and $\pi: E \to M$ is a topological covering map. Then E is a topological (n-1) manifold and there exsits a unique smooth structure on E such that π is a smooth covering map.

The book proves that E is a topological (n-1) manifold and that there exists a smooth structure on it such that π is a smooth covering map. Now, let's suppose \tilde{E} is the same set but with a different smooth structure on it, such that $\tilde{\pi}: \tilde{E} \to M$ is smooth. To show that the two smooth structures on E must be the same, let's prove that id : $E \to \tilde{E}$ is a diffeomorphism.



Every point in E is in the pre-image of some evenly covered $V \subseteq S$. Let U be the component of $\pi^{-1}(V)$ which contains p. Then, since V is evenly covered,

Q5-4. Show that the image of the curve $\beta:(-\pi,\pi)\to\mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of \mathbb{R}^2 .

Proof:

Q5-6. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all *unit* tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_xM, |v| = 1\}$$

This is called the *Unit Tangent Bundle of M*. Prove that UM is an embedded (2n-1)-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

Proof:

Q5-7. Let $F: \mathbb{R}^2 \to \mathbb{R}$ be defined as $F(x,y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of \mathbb{R}^2 ? For each level set, prove either that it is or that it is not an embedded submanifold.

Proof: