Math 214 Homework 5

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Q4-5. Let $\mathbb{CP}^n$ denote the $n-$dimensional complex projective space.

(a) Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.

(b) Show that $\mathbb{CP}^n$ is diffeomorphic to $\mathbb{S}^n$.

Proof:

Q4-6. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Proof: From LeeSM Proposition 4.28, We know that if $\pi : M \to N$ is a smooth submersion between smooth manifolds then π is an open map. Now, consider M to be a non-empty smooth compact manifold and let $N = \mathbb{R}^k$. $M \subseteq M$ is open when viewed as a subset of itself. However, $F(M)$ is a compact subset of $\mathbb{R}^k$ since F is a smooth map, and compact subsets of euclidean space are not open. Thus, we have a contradiction.

Q4-7. Suppose M and N are smooth manifolds, and $\pi : M \to N$ is an injective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29.

Proof:
From Theorem 4.28, we know that surjective smooth submersions are quotient maps. Then, from the uniqueness of the quotient topology, we know there is no other smooth manifold structure on N such that the conclusion of Theorem 4.29 holds.

Q4-8. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and that for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof:
For any $t \in \mathbb{R}$, we can simply choose $x = t, y = 1$. Then, $\pi(x, y) = \pi(t, 1) = t$, so the map is surjective. The map is also smooth since the partial derivatives with respect to $x^1, x^2 = x, y$ are smooth

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

However, π is not a smooth submersion since the differential of π

$$d\pi_{(0,0)} = \begin{pmatrix} x \\ y \end{pmatrix}\Bigg|_{(0,0)} = \mathbf{0}$$

has rank zero at the origin, whereas it has rank 1 everywhere else on $\mathbb{R}^2$. So, π is not a constant rank map.

Q4-9. Let M be a connected smooth manifold, and let $\pi : E \to M$ be a topological covering map. Complete the proof of proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map.

Proof:

> ***Theorem 4.40:*** Suppose M is a connected smooth $n-$manifold and $\pi : E \to M$ is a *topological* covering map. Then E is a topological $(n-1)$ manifold and there exsits a unique smooth structure on E such that π is a smooth covering map.

The book proves that E is a topological $(n-1)$ manifold and that there exists a smooth structure on it such that π is a smooth covering map. Now, let's suppose $\tilde{E}$ is the same set but with a different smooth strucuture on it, such that $\tilde{\pi} : \tilde{E} \to M$ is smooth. To show that the two smooth structures on E must be the same, let's prove that $\text{id} : E \to \tilde{E}$ is a diffeomorphism.

$$\begin{tikzcd} E \arrow[rr, "\text{id}"] \arrow[dr, "\pi"'] & & \tilde{E} \arrow[dl, "\tilde{\pi}"] \\ & M & \end{tikzcd}$$

Every point in E is in the pre-image of some evenly covered $V \subseteq S$. Let U be the component of $\pi^{-1}(V)$ which contains p. Then, since V is evenly covered,

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Q5-4. Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$.

Proof:

Q5-6. Suppose $M \subseteq \mathbb{R}^n$ is an embedded $m-$dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all *unit* tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n \ : \ x \in M, v \in T_xM, \ |v| = 1\}$$

This is called the ***Unit Tangent Bundle of*** M***.*** Prove that UM is an embedded $(2n-1)-$dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

Proof:

Q5-7. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Proof:
