Physics 198: Differential Geometry and Lie Groups

Notes

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These are some short notes that will be used to complement the lectures for the UC Berkeley DeCal 'Physics 198: Differential Geometry and Lie Groups for Physics Students'. The primary reference for the class is "*Differential Geometry and Lie Groups for Physicists*" by Marián Fecko, and so this document will cover topics in roughly the same order - though with some additional and alternative explanations.

These notes assume the reader is familiar with results from Linear Algebra and Multivariable Calculus, at the level of Math 54 and Math 53 at UC Berkeley.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 Lie Groups

In physics, we can learn a great deal from studying the symmetries of *continuous* systems.

2 Representations of Lie Groups and Lie Algebras

2.1 What is a Representation?

- To understand what a group really *is*, it can be very enlightening to study what the group *does* i.e. to study ***group actions***. Further, linear algebra is easier than abstract algebra, so if we can study the action of a group in terms of linear algebraic objects, we'll get a lot more mileage.
- How exactly do we do this? We can use some sort of map to assign a *linear operator* over that vector space to each group element to describe (or represent) the action of each group element on the vector space elements. The map that we use is a **representation** of the group.

Recall that the space of all linear operators $\rho : V \to V$ is denoted $\text{End}(V)$. The subset of these operators which are invertible (isomorphisms on V) is denoted $\text{Aut}(V)$.

Notably, $\text{Aut}(V)$ has a group structure! **(Check this!)** On the other hand, $\text{End}(V)$ becomes an (Associative, and later Lie) Algebra if we define the commutator for $A, B \in \text{End}(V)$ as

$$[A, B] = AB - BA$$

Now the formal definition.

Definition. *Group Representation Given a group G and vector space V, a group homomorphism*

$$\rho : G \to \text{Aut}(V)$$

is called a ***representation*** *of G in V.*

Example. *Complete this later*

We can use the same idea to define the representation of an algebra, but this time with $\text{End}(V)$.

Definition. *Lie Algebra Representation Given a Lie algebra $\mathcal{G}$ and vector space V, an algebra homomorphism*

$$\rho' : \mathcal{G} \to \text{End}(V)$$

is called a representation of the Lie algebra $\mathcal{G}$ over V.

Note. *The representations ρ and ρ' of a lie group and its lie algebra are related! so ρ' is called the* ***derived representation****.*

Fecko, Exercise 12.1.4

Consider a Lie algebra $\mathcal{G}$ whose basis elements $\{E_i\}$ satisfy the commutation relations

$$[E_i, E_j] = c^k_{ij} E_k$$

and a representation $f : \mathcal{G} \to \text{End}(V)$. Then, define $\mathcal{E}_i \equiv f(E_i)$. Since f is a homomorphisms between algebra, it is linear and respects the commutator i.e. for $A, B \in \mathcal{G}$

$$f([A, B]) = [f(A), f(B)]$$

Thus,

$$\begin{aligned}
[\mathcal{E}_i, \mathcal{E}_j] &= [f(E_i), f(E_j)] \\
&= f([E_i, E_j]) \\
&= f\left(c_{ij}^k E_k\right) \\
&= c_{ij}^k f(E_k) \\
&= c_{ij}^k \mathcal{E}_k
\end{aligned}$$

(The basis elements of the representation satisfy the same commutation relation as those of the Lie Algebra!)

Fecko, Exercise 12.1.5

Do this one later

- The assignment from Lie Group to Lie Algebra $G \mapsto \mathcal{G}$ is nice and unique, but the other way around can get messy.
- Similarly, given a Lie group representation ρ there is a unique Lie algebra represenation ρ', but not necessarily the other way around.

Fecko, Exercise 12.1.6

(i) Consider the Lie Group $H = \mathrm{Aut}(V) \equiv \mathrm{GL}(V)$. Recall that the Lie Algebra of H is

Write about ρ-invariant inner products.

2.2 Reducible and Irreducible Representations

References for the chapter

3 Connections and Parallel Transport on a Manifold

References for the chapter