Math 214 Homework 4

Keshav Balwant Deoskar

February 15, 2024

Q3-1. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof:

<u>" $\implies$ " Direction:</u> Suppose the differential $dF_p : T_pM \to T_{F(p)}N$ is the zero map for every $p \in P$. We want to show that F is constant on the components of M.

Recall that if a function between euclidean spaces $f : \mathbb{R}^m \to \mathbb{R}^n$ has components $f = (f^1(x), \cdots, f^n(x))$ such that each component has partial derivatives equal to zero with respect to each of $x^1, \cdots, x^m$, then the function is constant.

For each point $p \in M$, let (U, ϕ) and (V, ψ) be smooth charts such that $p \in U, F(p) \in V, F(U) \subseteq V$. Let's consider the coordinate representation $\hat{F} = \psi \circ F \circ \phi^{-1}$. For any $p \in M$, using the chain rule, the differential of $\hat{F}$ is given by

$$\begin{aligned} d(\hat{F})_{\phi(p)} &= d\left(\psi \circ F \circ \phi^{-1}\right)_{\phi(p)} \\ &= d\psi_{F(p)} \circ dF_p \circ d\left(\phi^{-1}\right)_{\phi(p)} \\ &= 0 \end{aligned}$$

Since dF_p is the zero map for any point $p \in U$. So, the jacobian matrix for $\hat{F}$ is just the zero matrix, so the coordinate representation $\hat{F}$ is a constant map on U.

Since this holds for all points $p \in M$, F is locally constant so F is constant over the components of M.

<u>" $\impliedby$ " Direction:</u> Suppose F is constant on each component U of M. Since M is a manifold, it's locally path connected, implying that U is open in M. By hypothesis, we have $F\big|_U$ is constant. Now, for a point $p \in U$, let $v \in T_pM$ and $f \in C^\infty(N)$. Then,

$$d(F\big|_U)_p(v)(f) = v(f \circ F\big|_U) = 0$$

since $f \circ F\big|_U \in C^\infty$ is the constant map and the derivation of a constant map is zero.

Q3-3. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $T(M) \times T(N)$.

Proof:

By proposition 3.1, we know that the space $T_{(p,q)}(M, N)$ can be identified with the space $T_pM \oplus T_qN$ for every pair of points (p, q) where $p \in M, q \in N$.

To show that $T(M \times N)$ and $TM \times TN$, let's define the map

$$F : T(M \times N) \to TM \times TN$$

as

$$((p,q), u \oplus v) \mapsto ((p,u),(q,v))$$

Of course, the inverse of this map is $F^{-1} : TM \times TN \to T(M \times N)$ given by

$$((p,u),(q,v)) \mapsto ((p,q), u \oplus v)$$

Let's show that F is a diffeomorphism between the two spaces.

Let (U, ϕ) and (V, ψ) be smooth charts for M and N respectively. Let π_X denote the projection from $TX \to X$ for any smooth manifold X. We have a corresponding charts $\left(\pi^{-1}_{M \times N}(U \times V), \alpha\right)$ nad $\left(\pi^{-1}_M(U) \times \pi^{-1}_N(V), \beta\right)$ for $T(M \times N)$ and $TM \times TN$ where

$$\alpha : \left(\pi^{-1}_{M \times N}(U \times V)\right) \to \phi(U) \times \psi(V) \times \mathbb{R}^m \times \mathbb{R}^n, \quad \left((p,q), u^i \frac{\partial}{\partial x^i}\bigg|_p \oplus v^j \frac{\partial}{\partial y^j}\bigg|_q\right) \mapsto (\phi(p), \psi(q), u, v)$$

$$\beta : \left(\pi^{-1}_M(U) \times \pi^{-1}_N(V)\right) \to \phi(U) \times \mathbb{R}^m \times \psi(V) \times \mathbb{R}^n, \quad \left(\left(p, u^i \frac{\partial}{\partial x^i}\bigg|_p\right), \left(q, v^j \frac{\partial}{\partial y^j}\bigg|_p\right)\right) \mapsto (p, u, w, v)$$

where $u = \left(u^1, \ldots, u^m\right)$, $v = \left(v^1, \ldots, v^n\right)$. The coordinate representation of F is $\hat{F} = \beta \circ F \circ \alpha^{-1}(p,q,u,v) = (p,u,q,v)$ and the coordinate representation of $\hat{F}^{-1} = F^{-1}$ is $\alpha \circ F \circ \beta^{-1}(p,u,q,v) = (p,q,u,v)$.

Since each of the component functions of $\hat{F}, \hat{F}^{-1}$ are smooth and bijective, the functions F and F^{-1} are themselves smooth. Thus F is a diffeomorphism between $T(M \times N)$ and $TM \times TN$.

Q3-4. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Proof:

Recall that we can cover $\mathbb{S}^1$ with *textbfangle charts*. Consider charts (U, θ) and (V, ψ) covering $\mathbb{S}^1$ where $U = \mathbb{S}^1 \setminus \{1\}$, $\theta : U \to (0, 2\pi)$ and $V = \mathbb{S}^1 \setminus \{-1\}$ (where we are viewing 1 an -1 as numbers in $\mathbb{C}$).

Then, let's define $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ as

$$F(z) = \begin{cases} F\left(z, v \frac{d}{d\theta}\Big|_z\right) = (z, v), \text{ if } z \in U \\ F\left(z, \hat{v} \frac{d}{d\phi}\Big|_z\right) = (z, v), \text{ if } z \in V \end{cases}$$

Note that for $z \in U \cap V$, we have

$$\frac{d}{d\theta}\bigg|_z = \frac{d}{d\phi}\bigg|_z$$

So, F is well defined. Now, we notice that $F\big|_{\pi^{-1}(U)}$ and $F\big|_{\pi^{-1}(U\mathbb{V})}$ are smooth bijections, so we have diffeomorphisms from $\pi^{-1}(U) \to U \times \mathbb{R}$ and $\pi^{-1}(V) \to V \times \mathbb{R}$ which agree on the overlap $\pi^{-1}(U) \cap \pi^{-1}(V)$.

Thus,

$$T\mathbb{S}^1 \cong_{\text{diff}} \mathbb{S}^1 \times \mathbb{R}$$

Q3-5. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square with side two centered at the origin: $K = \{(x, y) \ : \ \max\{|x|, |y|\} = 1\}$. Show that there is a homeomorphism $F : R^2 \to R^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property.

Proof:

For any point $p \in \mathbb{S}^1$, we can simply draw the line passing through the origin and p, and move p along the line until it hits the square. Points on the square can be traced back along the line. Points on the square lying in the same open neighborhood lie on lines which are close to each other, so their pre-images on the circle also lie in an open neighborhood. Thus, there exists a homeomorphism between $\mathbb{S}^1$ and K.

Suppose there is a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\left(\mathbb{S}^1\right)) = K$. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the path defined by

$$\gamma(t) = (\cos(t), \sin(t))$$

Then, for any $t \in \mathbb{R}$, we have $\gamma(t) \in \mathbb{S}^1$ and $\gamma'(t) \neq 0$. To show there exists no diffeomorphism, we'll use the corners of the square.

Let $t_c \in \mathbb{R}$ be such that $F \circ \gamma(t_c) = (1, 1)$, then for some $\epsilon > 0$ there exist intervals $I_- = (t_c - \epsilon, t_c)$ and $I_+ = (t_c, t_c + \epsilon)$ such that $F \circ \gamma(I_-) \subseteq \{1\} \times (-1, 1)$ (right edge of the square) and $F \circ \gamma(I_+) \subseteq (-1, 1) \times \{1\}$ (top edge of the square).

Then,

$$F \circ \gamma(t) = \Big\{ (1, y \circ F \circ \gamma(t)) \, , \text{ if } t \in I_- \; (x \circ F \circ \gamma(t), 1) \, , \text{ if } t \in I_+$$

and

$$(F \circ \gamma)'(t_c) = \frac{d\left(x \circ F \circ \gamma\right)}{dt}(t_c) \frac{\partial}{\partial x}\bigg|_{(1,1)} + \frac{d\left(y \circ F \circ \gamma\right)}{dt}(t_c) \frac{\partial}{\partial y}\bigg|_{(1,1)}$$

By continuity of $\frac{d(x \circ F \circ \gamma)}{dt}$ and $\frac{d(y \circ F \circ \gamma)}{dt}$, they must both vanish at t_c. Hence, $(F \circ \gamma)'(t_c) = 0$, but we know from LeeSM Proposition 3.24 that $(F \circ \gamma)'(t_c) = dF(\gamma'(t_c))$. Since F is a homeomorphism, dF_p is an isomorphism for any $p \in \mathbb{R}^2$. Since $\gamma'(t) \neq 0$, we have $dF\left(\gamma'(t_c)\right)$ so in particular $dF(\gamma'(t)) \neq 0$. Thus, F cannot be a diffeomorphism.

Q3-7. Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of $C_p^\infty(M)$ of derivtaions of $C_p^\infty(M)$. Define a map $\Phi : \mathcal{D}_pM \to T_pM$ by $\left(\Phi_v\right) f = v\left([f]_p\right)$. Show that Φ is an isomorphism.

Proof:

First, let's verify that Φ_v is a derivation at point p. For functions $f, g \in C^\infty(M)$, we have

$$\begin{aligned} \Phi_v\left(fg\right) &= v[fg]_p \\ &= f(p)v[g]_p + v[f]_p g(p) \\ &= f(p)\left(\Phi_v\right)_g + \left(\Phi_v\right)_f \end{aligned}$$

So, Φ_v is a derivation at p.

To check that it's an isomorphism, we note that Φ is linear so we just need to show injectivity and surjectivity.

For injectivity, let $Phi_v = 0$. Let $[f]_p$ be the germ of some pair (f, U), where U is open an contains p. By the extension lemma for smooth functions, there is a smooth function supported in U such that $\psi \equiv 1$ on U. Then $\tilde{f} = \psi f$ is a smooth function such that $[\tilde{f}]_p = [f]p$. Then $v[f]_p = v[\tilde{f}]_p = (\Phi_v)(\tilde{f}) = 0$. Since $[f]_p$ was arbitrary, we conclude that $v = 0$, so the map Φ injective.

For surjectivity, let $w \in T_pW$ be an arbitrary derivation. Then, define $v \in \mathcal{D}_pM$ by $v[f]_p = wf$. This is well defined because of Proposition 3.8 (On a neighborhood of p, $f = g \implies wf = wg$). Then, $(\Phi_v) f = v[f]_p = wf$ for any $f \in C^\infty M$, so $\Phi v = w$. This proves surjectivity.

Since Φ is a linear bijection, it is an isomorphism between vector spaces.

Q3-8.

Proof:
To show that the map $\Psi : \mathcal{V}_P M \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective, consider curves γ_1, γ_2 starting at p such that $[\gamma_1] = [\gamma_2]$.

' Let (U, ϕ) be a chart containing p. Then, in the coordinates of this chart, we have

$$\gamma_1'(0) = \left(x^i \circ \gamma_1\right)'(0) \frac{\partial}{\partial x^i}\bigg|_p = \left(x^i \circ \gamma_2\right)'(0) \frac{\partial}{\partial x^i}\bigg|_p = \gamma_2'(0)$$

because $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any smoot function $f \in C^\infty(M)$ in a neighborhood of p.

For injectivity, suppose $\gamma_1'(0) = \gamma_2'(0)$ and consider any smooth real valued function f on a neighborhood of p. Then,

$$(f \circ \gamma_1)'(0) = d\left(f \circ \gamma_1\right)_0 \left(\frac{\partial}{\partial t}\bigg|_0\right) = df_{\gamma(0)} \circ d\gamma_0 \left(\frac{\partial}{\partial t}\bigg|_0\right) = df_{\gamma(0)} \circ \gamma_1'(0) = df_{\gamma(0)} \circ \gamma_1'(0) = (f_2 \circ \gamma)'(0)$$

for some other path γ. Thus, $[\gamma_1] = [\gamma_2]$ and the map is injective.

For surjectivity, let v be an arbitrary derivation at p and let (U, ϕ) be a chart containing p such that $\phi(p) = 0$. Then, we can write

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p$$

in the coordinates of (U, ϕ). For some $\epsilon > 0$, we can define a curve

$$\tilde{\gamma} : [0, \epsilon), \quad t \mapsto (tv^1, \cdots tv^n)$$

and then let $\gamma = \phi^{-1} \circ \tilde{\gamma}$. Then, γ is a smooth curve starting at p, since it is the composition of two smooth functions and

$$\gamma'(0) = \frac{\partial \gamma^i}{\partial t}(0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = \frac{\partial(\phi^{-1} \circ \gamma)}{\partial t}(0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = \frac{\partial(tv^i)}{\partial t}(0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = v^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = v$$
