

Math H185 Lecture 2

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berkeley's Math H185 class in the Spring 2024 semester.

Contents

1	January 19 - The Complex Plane	2
1.1	What does \mathbb{C} look like geometrically?	2
1.2	Polar Coordinates	2

1 January 19 - The Complex Plane

1.1 What does \mathbb{C} look like geometrically?

[Insert Diagram of \mathbb{C} with Real and Imaginary Axes]

Def: The **Modulus** or **absolute value** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. It is the distance to 0.

There is a correspondence between the structure of \mathbb{C} and the vector structure of \mathbb{R}^2 .

$$\text{Addition in } \mathbb{C} \iff \text{Vector Addition in } \mathbb{R}^2$$

Exercise: Draw the subset of $z \in \mathbb{C}$ defined by

1. $\{|z| = 1\}$
2. $\{|z - 6| = 1\}$
3. $\{|z - 6i| \leq 1\}$
4. $\{\operatorname{Re}(z) \leq \operatorname{Im}(z)\}$
5. $\{|z| = \operatorname{Re}(z) + 1\}$

Answers: [Insert figures later.]

1. Circle with radius 1 centered at point $z = 0 + 0i$.
2. Circle with radius 1 centered at point $z = 6 + 0i$.
3. Disk with radius 1 centered at point $z = 0 + 6i$.
4. Everything above the line making 45-degree angle with the real axis.
5. (Horizontal) Parabola with vertex at $z = 0 + 1i$ since

$$\begin{aligned}\sqrt{x^2 + y^2} &= x + 1 \\ \implies x^2 + y^2 &= x^2 + 2x + 1 \\ \implies y^2 &= 2x + 1 \\ \implies x &= \frac{y^2 - 1}{2}\end{aligned}$$

So, we get a horizontal parabola.

1.2 Polar Coordinates

Rather than using the rectangular (cartesian) coordinates to describe \mathbb{C} , we can equivalently use **Polar coordinates** wherein we use the distance from the origin (modulus) and the angle made with the real axis (argument).

$$\begin{aligned}(x, y) &\rightarrow (r, \theta) \\ r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right)\end{aligned}$$

Notice however there is an ambiguity in the angle θ since $\theta + 2n\pi$ would describe the same point. Thus, we define the **Principal value branch** as the restriction $\theta \in [0, 2\pi]$.

Recall: The exponential Function is defined as below for $z \in \mathbb{C}$.

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

Now the usual definitions for the cosine and sine as in \mathbb{R} in terms of the unit circle doesn't quite work for complex numbers. However, we can still define them using the exponential!

$$\begin{aligned}\cos(z) &= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \dots \\ \sin(z) &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots\end{aligned}$$

Euler's Theorem: $e^{iz} = \cos(z) + i \sin(z)$, $z \in \mathbb{C}$.

Proof: Expanding the exponential, we have

$$e^{iz} = 1 + \frac{1}{1!}iz + \frac{1}{2!}(iz)^2 + \frac{1}{3!}(iz)^3 + \dots$$

and note that

$$\frac{1}{(2n)!}(iz)^{2n} = \frac{1}{(2n)!}z^{2n}(i^{2n}) = \frac{(-1)^n z^{2n}}{(2n)!}$$

Similarly,

$$\frac{1}{(2n-1)!}(iz)^{2n-1} = \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$$

Thus,

$$e^{iz} = \underbrace{\sum_{2n} \frac{(-1)^n z^{2n}}{(2n)!}}_{\cos(z)} + \underbrace{\left(\sum_{2n-1} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} \right) i}_{\sin(z)}$$

Additionally, we can convert back from Polar coordinates to Cartesian Coordinates as

$$\begin{aligned}(r, \theta) &\rightarrow (x, y) \\ x &= r \cos(\theta) \\ y &= r \sin(\theta)\end{aligned}$$

Multiplication: The multiplication of two complex numbers can be thought of in terms of polar coordinates as scaling by r and rotating by θ .

Topology of \mathbb{C}

The set \mathbb{C} is a **metric space**, and the metric on \mathbb{C} is $\text{dist}(z, w) = |z - w|$ (\iff Euclidean metric on \mathbb{R}^2).

Notation:

- Given a complex number $z_0 \in \mathbb{C}$ and $r > 0, r \in \mathbb{R}$, the set

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

is called the **open ball around z_0 of radius r** .

- Similarly, the set $\overline{B_r(z_0)}$ is the **closed ball**:

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

Open and Closed sets

- A subset $U \subseteq \mathbb{C}$ is **open** if for all $z \in U$, there exists $r > 0$ such that $B_r(z) \subset U$.
- A subset $V \subseteq \mathbb{C}$ is said to be **closed** if its complement V^c is open in \mathbb{C} .

Ex: The set $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ is closed. This set will be important when studying the complex logarithm.
[Insert figure.]

Note that while the closed ball is not open (since the points on the boundary i.e. points with $|z - z_0| = r$ don't satisfy the requirement), a set *can* be both open and closed. For example, the sets \mathbb{C} and \emptyset are both closed and open.