Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 6

kdeoskar@berkeley.edu

Question 1: Compute $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2)$ and the action of $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^2)$ on it.

Solution: (Inspired by this stackexchange post.)

To compute $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2)$ we'll use the following lemma:

Lemma 0.0.1. *For the universal cover $\widetilde{U} \to U$ of a CW Complex U, we have*

$$\pi_n(\widetilde{X}) \cong \pi_n(X)$$

for all $n \in \mathbb{N}$.

Proof.

□

Going back to the computation, we have

$$\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2) \cong \pi_2(\widetilde{\mathbb{S}^1 \vee \mathbb{S}^2})$$

where $\widetilde{\mathbb{S}^1 \vee \mathbb{S}^2}$ is the universal covering of $\mathbb{S}^1 \vee \mathbb{S}^2$, visualized as:

Include picture

Then, contracting each of the segments between the consecutive integers, we have

$$\widetilde{\mathbb{S}^1 \vee \mathbb{S}^2} \cong \bigvee_{k \in \mathbb{Z}} \mathbb{S}^2_k$$

where each $\mathbb{S}^2_k$ is a copy of $\mathbb{S}^2$, labelled by the integer k.

So, we have

$$\pi_2\left(\widetilde{\mathbb{S}^1 \vee \mathbb{S}^2}\right) \cong \pi_2\left(\bigvee_{k \in \mathbb{Z}} \mathbb{S}^2_k\right)$$

What is this space? **(Fill this in.)**

Thus,

$$\boxed{\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2) \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}}$$

<u>Action of π_1:</u> Generally, $\pi_1(X)$ acts on $\pi_n(X)$ $(n \geq 1)$ by "prepending" a loop i.e. move along a circle before an $n-$spheriod.

The inclusion of $\mathbb{S}^2 \hookrightarrow X\colon = \mathbb{S}^1 \vee \mathbb{S}^2$ gives us an element $\alpha \in \pi_2(X)$, which generates a cyclic subgroup of $\pi_2(X)$.

However, notice that if we consider a loop γ that goes around the $\mathbb{S}^1$ factor once in $\mathbb{S}^1 \vee \mathbb{S}^2$ then first moving along γ brings us back to the basepoint in $\mathbb{S}^1 \vee \mathbb{S}^2$ so following it up with some $\alpha \in \pi_2(X)$ is just another element of 2-spheroid i.e. $\alpha \cdot \gamma \in \pi_2(X)$. This $\gamma \cdot \alpha$ also generates a cyclic subgroup of $\pi_2(X)$. Continuing on with this patter we can see that $\gamma^n \circ \alpha \in \pi_2(X)$ for every $n \in \mathbb{N}$ and each of these generate (disjoint) cyclic subgroups.

Question 2: Compute the 2nd Homotopy Groups of Grassmannians $G(n,k)$ when $k, n-k > 1$

Solution:

Let $G(n,k)$ denote the set of $k-$dimensionl subspaces of $\mathbb{R}^n$. We know that

$$G(n,k) \cong O(n)/(O(n) \times O(n-k))$$

so we have a (serre) fibration

$$O(n-k) \hookrightarrow O(n) \to G(n,k)$$

which induces the exact sequence

$$\pi_2\left(O(n)\right) \to \pi_2(G(n,k)) \to \pi_1\left(O(n) \times O(n-k)\right) = \pi_1(O(n)) \times \pi_1(O(n-k)) \to \pi_1(O(n))$$

Now, to actually calculate $\pi_2(G(n,k))$ for the different n,k values we'll need to use the following results (common in the literature) which can be obtained using fibrations as well:

(a)

$$\pi_1(O(N)) = \begin{cases} \mathbb{Z}_2, & n = 1 \\ \mathbb{Z}, & n = 2 \\ \mathbb{Z}_2, & n \geq 3 \end{cases}$$

(b)

$$\pi_2(O(2)) = 0$$

(c)

$$\pi_2(O(3)) = 0$$

(d) When it comes to the homotopy groups of $O(N)$ for $N \in \mathbb{Z}$ we have, for $n \geq k+2$, by **Bott Periodicity**:

$$\pi_k(O(N)) \cong \pi_k(SO(N)) = \begin{cases} 0, & k = 2,4,5,6 \pmod 8 \\ \mathbb{Z}_2, & k = 0,1 \pmod 8 \\ \mathbb{Z}, & k = 7 \pmod 8 \end{cases}$$

Now, in our question, we have the following cases (we're considering $n, n-k > 1$)

1. n, k such that $n, (n-k) \geq 3$ $n-k > 2$ and:

$$\pi_2(O(n)) \to \pi_2(G(n,k)) \to \pi_1(O(n)) \times \pi_1(O(n-k)) \to \pi_1(O(n))$$

Question 3: Let X be a $K(G,n)$ and Y a cellular $K(H,n)$. Show that the map of Y to X inducing a given group homomorphism $\phi : H \to G$ exists, and is unique up to homotopy.

Solution:

We know there exists group homomorphism $\phi : H \to G$ and recall that an Eilenberg-Maclane Space $X = K(F,n)$ is one for which

$$\pi_k(X) = \begin{cases} F, & k = n \\ 0, & k \neq n \end{cases}$$

Now, we have $X = K(G,n)$, $Y = K(H,n)$, and Y is known to be a CW complex. We can assume that Y is a CW Complex obtained from $\mathrm{sk_n}Y := \bigwedge_\alpha \mathbb{S}^n_\alpha$ where each α corresponds to a generator of H and attaching all cells (attach $(n+1)-$cells according to the relations in H and **(Fill in some more.)**)

We can define $f_n : \mathrm{sk}_n Y \to X$ by mapping each S^n_α to a corresponding spheroid $f_n(\alpha) \in \pi_n(X, x_0)$.

Also, the attaching maps $\partial D^{n+1} \to \mathrm{sk}_n Y$ for $(n+1)-$dimensional cells represent the identity element in H (so its image under f_n is trivial in G +), which we can use to extend f_n to $f_{n+1} : \mathrm{sk}_{n+1} Y \to X$.

We can do this inductively to extend f_{n+1} to $f_{n+k} : \mathrm{sk}_{n+k} X \to Y$ with $k > 1$.

We want to show that any two maps $f, g : X \to Y$ which induce a given homomorphism $\phi : H \to G$ are the same up to homotopy. To do so, let's use cell induction.

Suppose y_0 and x_0 are the basepoints on Y and X respectively. For the base case, let's assume there exists some path in X between $f(y_0)$ and $g(y_0)$. This path then gives us the homotopy between $f\big|_{\mathrm{sk}_0 Y} : \mathrm{sk}_0 Y \to X$ and $g\big|_{\mathrm{sk}_0 Y} : \mathrm{sk}_0 Y \to X$.

For the induction step, suppose we already have a homotopy $h_{k-1} \times [0,1] : \mathrm{sk}_{k-1} Y \to X$ between $f\big|_{\mathrm{sk}_{k-1} Y}$ and $g\big|_{\mathrm{sk}_{k-1} Y}$ and a $k-$cell D^k of Y with characteristic map $\Phi : D^k \to \mathrm{sk}_k Y$. We want to extend this to a map $D^k \times [0,1] \to X$ such that the map agrees with $f \circ \Phi$ on $D^k \times \{0\}$, with $g \circ \Phi$ on $D^k \times \{1\}$, and with $h \circ \Phi\big|_{partial D^k}$ on $\partial D^k \times [0,1]$.

Note that $(D^k \times \{0\}) \cup (D^k \times \{1\}) \cup (\partial D^k \times [0,1]) \approx \partial\left(D^k \times [0,1]\right)$, so all of the conditions listed above together encode a $k-$spheroid $\partial\left(D^k \times [0,1]\right) \to X$, and the extension to $D^k \times [0,1]$ we desired can be define if the spheroid can be contracted in X. Since we're working with Eilenberg-Maclane spaces, we have $\pi_k(X) = 0$ for $k \neq n$, so we can make the extension. For the $k = n$ case we can argue that the claim holds because $f\big|_{\mathrm{sk}_n Y}$ and $g\big|_{\mathrm{sk}_n Y}$ represent the same element in $G = \pi_n(X)$ and are thus homotopic.