Math H185 Homework 3

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Question 1

Let $r > 0$. For each $n \in \mathbb{Z}$, calculate
$$\int_{\partial B_r(0)} \overline{z}^n dz$$

Proof:

$$\begin{aligned}\int_{\partial B_r(0)} \overline{z}^n dz &= \int_{\partial B_r(0)} \overline{z}^n \frac{z^n}{z^n} dz \\ &= \int_{\partial B_r(0)} \frac{\left(|z|^2\right)^n}{z^n} dz\end{aligned}$$

But on $\partial B_r(0)$, we have $|z| = r$, so

$$int_{\partial B_r(0)} \overline{z}^n dz = r^{2n} \int_{\partial B_r(0)} \frac{1}{z^n} dz$$

And, from lecture, we know that

$$\begin{aligned}\int_{\partial B_r(0)} z^m dz &= \begin{cases} 0, m \neq -1 \\ 2\pi i, m = -1 \end{cases} \\ \implies \int_{\partial B_r(0)} \frac{1}{z^n} dz &= \begin{cases} 0, n \neq 1 \\ 2\pi i, n = 1 \end{cases}\end{aligned}$$

Thus, we have

$$\boxed{\int_{\partial B_r(0)} \overline{z}^n dz = \begin{cases} 0, & n \neq 1 \\ r^{2n} \cdot 2\pi i, n = 1 \end{cases}}$$

Question 2

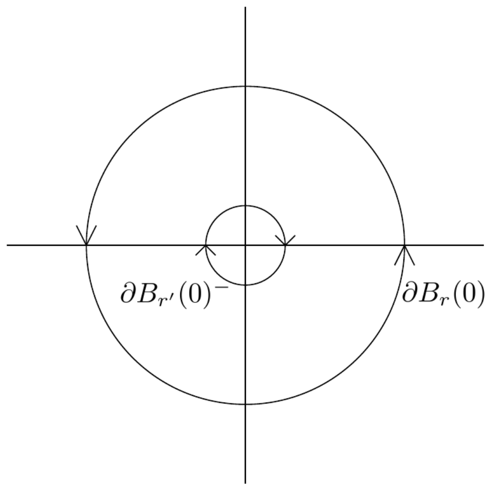
Show that if $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C} \setminus \{0\}$, then

$$\int_{\partial B_r(0)} f(z)dz$$

is independent of r.

Proof:

For any $r, r' > 0$ such that $r' < r$, let U be the annulus formed by the circles of radii r, r' around the origin. Then, the boundary ∂U is $\partial B_r(0) \cup \partial B_{r'}(0)^-$ where the $(-)$ superscript is meant to denote the reverse orientation.



We know from Cauchy's Theorem that if $f : \Omega \subseteq_{open} \to \mathbb{C}$ is holomorphic on Ω and γ is some path in Ω, then

$$\int_\gamma f(z)dz = 0$$

So, the integral we're interested in calculating is

$$\begin{aligned} 0 &= \int_{\partial U} f(z)dz \\ &= \int_{\partial B_r(0) \,\cup\, \partial B_{r'}(0)^-} f(z)dz \\ &= \int_{\partial B_r(0)} f(z)dz + \int_{\partial B_{r'}(0)^-} f(z)dz \\ &= \int_{\partial B_r(0)} f(z)dz - \int_{\partial B_{r'}(0)} f(z)dz \end{aligned}$$

$$\implies \boxed{\int_{\partial B_r(0)} f(z)dz = \int_{\partial B_{r'}(0)} f(z)dz}$$

This shows that the integral is independent of the radius of r of the ball around the origin, given that f is holomorphic on $\mathbb{C} \setminus \{0\}$.

Question 3

Let $f(z) = 3z^3 + z^2 + 4z + 1$. Calculate

$$\int_{\partial B_r(0)} f(z) z^n dz$$

for $r > 0$ and every $n \in \mathbb{Z}$.

Solution:
We already know that

$$\int_{\partial B_r(0)} z^m = \begin{cases} 0, m \neq -1 \\ 2\pi i, m = -1 \end{cases}$$

The $n \geq 0$ case is simple as the functions $f(z)$ and z^n are both holomorphic at all points $z_0 \in \mathbb{C}$, so their product is also holomorphic at all points in $\mathbb{C}$. Then, Cauchy's Theorem tells us

$$\int_{\partial B_r(0)} f(z) z^n dz = 0$$

For $n \leq -1$, it helps to break into cases. The integral we're trying to evaluate is

$$\begin{aligned} \int_{\partial B_r(0)} f(z) z^n dz &= \int_{\partial B_r(0)} 3z^3 \cdot z^n dz + \int_{\partial B_r(0)} z^2 \cdot z^n dz + \int_{\partial B_r(0)} 4z \cdot z^n dz + \int_{\partial B_r(0)} 1 \cdot z^n dz \\ &= 3 \int_{\partial B_r(0)} z^{n+3} dz + \int_{\partial B_r(0)} z^{n+2} dz + 4 \int_{\partial B_r(0)} z^{n+1} dz + \int_{\partial B_r(0)} z^n dz \end{aligned}$$

So, let's consider the following:

(a) $\underline{n = -1}$: In this case, the last integral survives while the others vanish, so

$$\int_{\partial B_r(0)} f(z) z^n dz = \int_{\partial B_r(0)} f(z) z^{-1} dz = 2\pi i$$

(b) $\underline{n = -2}$: In this case, the z^{n+1} integral survives whereas the other vanish, so

$$\int_{\partial B_r(0)} f(z) z^n dz = 4 \cdot \int_{\partial B_r(0)} f(z) z^{-1} dz = 4 \cdot (2\pi i)$$

(c) $\underline{n = -3}$: In this case, the z^{n+2} integral survives whereas the other vanish, so

$$\int_{\partial B_r(0)} f(z) z^n dz = \int_{\partial B_r(0)} f(z) z^{-1} dz = 2\pi i$$

(d) $\underline{n = -4}$: In this case, the z^{n+3} integral survives whereas the other vanish, so

$$\int_{\partial B_r(0)} f(z) z^n dz = 3 \cdot \int_{\partial B_r(0)} f(z) z^{-1} dz = 3 \cdot (2\pi i)$$

(e) $\underline{n \leq -5}$: In this case, all of the exponents are less than or equal to (-2). So, all of the integrals are separately equal to zero, making the total integral vanish.

In conclusion,

$$\boxed{\int_{\partial B_r(0)} f(z)z^n dz = \begin{cases} 0, n \geq 0 \text{ or } n \leq -5 \\ 2\pi i, n = -1 \\ 8\pi i, n = -2 \\ 2\pi i, n = -3 \\ 6\pi i, n = -4 \end{cases}}$$

Question 4

Calculate

$$I = \int_{\partial B_r(0)} \frac{e^{\sin(\cos(z))}}{z - \frac{\pi}{2}} dz$$

for $r = 1$, $r = 2$.

Solution:

(a) $\underline{r = 1}$: We notice that the integrand has a pole at $z = \pi/2 \approx 1.57 > 1$, so there exist no poles of the function in $B_1(0)$. The integrand is holomorphic in $B_1(0)$, so by Cauchy's Theorem, the integral evaluates to zero.

$$\boxed{I = 0}$$

(b) Cauchy's Formula tells us that if $f : \Omega \subseteq_{open} \mathbb{C} \to \mathbb{C}$ is holomorphic on Ω (or in fact, even just on $\Omega \setminus \{z_0\}$) then

$$f(w) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{w - z_0} dw$$

So,

$$\int_{\partial B_2(0)} \frac{e^{\sin(\cos(z))}}{\left(z - \frac{\pi}{2}\right)} dz = 2\pi i \cdot^{\sin(\cos(\pi/2))} = 2\pi i \cdot e^{sin(1)}$$

$$\boxed{I = 2\pi i e^{\sin(1)}}$$

Question 5

Suppose that $f(z)$ is analytic on a domain containing $\overline{B_r(z)}$. Using Cauchy's formulas, prove that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

This is called the *Mean Value Property*. More generally, prove also that

$$f^{(n)}(z) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z + re^{i\theta}) e^{-in\theta} d\theta$$

Proof:

Since we have a function $f(z)$ which is analytic on a domain containing $\overline{\mathbb{B}_r(z)}$, we can apply Cauchy's formula to find that

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{w-z} dw$$

Now, the $r-$ball can be parametrized as $\gamma(\theta) = z + re^{i\theta}$ for $\theta \in [0, 2\pi]$. So,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\theta})}{(z+re^{i\theta})-z} \times \left(ie^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\theta})}{+re^{i\theta}} \times \left(ie^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(z+re^{i\theta}) d\theta \end{aligned}$$

This gives us the Mean Value Property. More generally, we know from Cauchy's Integral formula that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{(w-z)^{n+1}} dw$$

So, once again, parametrizing the ball using $\gamma(\theta) = z + re^{i\theta}$ where $\theta \in [0, 2\pi]$ we have

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\partial B_r(z)} \frac{f(w)}{(w-z)^{n+1}} dw \\ &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\theta})}{(z+re^{i\theta}-z)^{n+1}} \cdot \left(ire^{i\theta}\right) d\theta \\ &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\theta})}{(re^{i\theta})^{n+1}} \cdot \left(ire^{i\theta}\right) d\theta \\ &= \frac{n!}{2\pi} \int_0^{2\pi} f(z+re^{i\theta}) \frac{\left(re^{i\theta}\right)}{(re^{i\theta})^{n+1}} d\theta \\ &= \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z+re^{i\theta}) \cdot e^{-in\theta} d\theta \end{aligned}$$

as desired.

Question 6

Calculate

$$\int_{\partial B_5(0)} \frac{\overline{z}}{z-1} dz$$

Warning: Recall that $\overline{z}$ is not holomorphic, so Cauchy's formula does not directly apply to it. Nevertheless, a clever trick will save the day.

Solution:

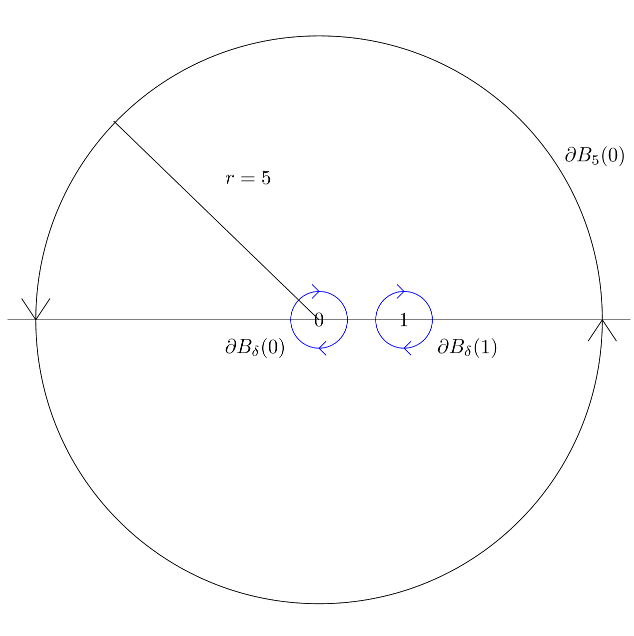
Note that, although $\overline{z}$ is not holomorphic anywhere – making the entire integrand not holomorphic anywhere – we can multiple the numerator and denominator with z to get:

$$\int_{\partial B_5(0)} \frac{\overline{z}z}{z(z-1)}dz = \int_{B_5(0)} \frac{|z|^2}{z(z-1)}dz$$

and on $\partial B_5(0)$, we have $|z| = 5$. So the integral is

$$\int_{\partial B_5(0)} \frac{\overline{z}}{z-1}dz = 25\int_{B_5(0)} \frac{1}{z(z-1)}dz$$

The integrand has poles at $z = 0$ an $z = 1$. Rather than integrating over $\partial B_5(0)$, we can intgrate over "islands" surrounding the poles and apply Cauchy's Formula as appropriate:



$$\begin{aligned}
25\int_{\partial\partial B_5(0)} \frac{1}{z(z-1)}dz &= 25\left[\int_{\partial B_\delta(0)} \frac{1/(z-1)}{(z-0)}dz + \int_{\partial B_\delta(1)} \frac{1/(z)}{(z-1)}dz\right] \\
&= 25\left[2\pi i \cdot \left(\frac{1}{0-1}\right) + 2\pi i \cdot \left(\frac{1}{1}\right)\right] \\
&= 50\pi i\,(-1+1) \\
&= 0
\end{aligned}$$

Question 7

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is smooth (i.e. infinitely differentiable at $x = 0$), but that f is not analytic in any neighborhood of 0.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined as

$$f(z) = \begin{cases} {}^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Show that f is not even continuous at $z = 0$.

Proof:

(a) Away from $x = 0$ the function is clearly smooth, so we just need to show smoothness at $x = 0$. First off, $f(x)$ is continuous at $x = 0$ since the left limit is clearly equal to zero and the right limit is

$$\lim_{x \to 0+} e^{-\frac{1}{x^2}} = 0$$

whch agrees with the left limit.

In fact, by standard applications of L'Hopital's Rule and Induction, we find that

Let's show using induction that the kth derivative is given by

$$f^{(k)}(x) = p_{2k}(x) \frac{e^{-1/x^2}}{(x^2)^{2k}}$$

for $x > 0$ and $x \leq 0$, where $p_{2k}(x)$ is a polynomial of degree at most $2k$.

The $x \leq 0$ part clearly follows since the function itself is identically zero on that region.

Now, for $x > 0$, the first derivative of $f(x)$ is

$$f'(x) = \frac{-1}{2x^3} \cdot e^{-1/x^2} = \left(-\frac{1}{2}x\right) \frac{e^{-1/x^2}}{x^4} = \left(-\frac{1}{2}x\right) \frac{e^{-1/x^2}}{(x^2)^{2 \cdot 1}}$$

which is exactly the form we need. This establishes the base case.

Now suppose the claim holds for the k-th derivative. Then,

$$\begin{aligned} f^{(k+1)}(x) &= p'_{2k}(x) \cdot \frac{e^{1/x^2}}{(x^2)^{2k}} + p_{2k}(x) \cdot \frac{\frac{-1}{2x^3} e^{-1/x^2}}{x^2} + p_{2k}(x) \cdot e^{-1/x^2} \cdot \left(\frac{-4k}{x^{4k+1}}\right) \\ &= \left[x^2 \cdot p'_{2k}(x) - \frac{x}{2} \cdot p_{2k}(x) - 4k \cdot p_{2k}(x)\right] \frac{e^{-1/x^2}}{x^{4k+2}} \\ &= \underbrace{\left[x^2 \cdot p'_{2k}(x) - \frac{x}{2} \cdot p_{2k}(x) - 4k \cdot p_{2k}(x)\right]}_{deg=2k+1} \frac{e^{-1/x^2}}{(x^2)2k+1} \end{aligned}$$

This proves the claim. Further, we can prove by induction that $f^{(k)}(0) = 0$. For the base case, this is true because of the definition of the function. Now, assume it holds for the k-th case. To show that $f^{(k+1)}(0)$ exists at the origin, we need to show that $f^{(k)}$ has one sided limits which agree at 0.

The left-limit of $f^{(k)}$ is just zero, from the definition of the function. The right hand limit is

$$\begin{aligned}
\lim_{x \to 0} \frac{p_{2k}(x) \frac{e^{-1/x^2}}{(x^2)^{2k}} - 0}{x - 0} &= \lim_{x \to 0} p_{2k}(x) \frac{e^{-1/x^2}}{x^{2(k+1)}} \\
&= p(0) \lim_{x \to 0} \frac{e^{-1/x^2}}{x^{2k+2}} \\
&= 0
\end{aligned}$$

where the last equality follows because we can prove

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^{2k+2}} = 0$$

for any k using Induction and by repeatedly applying L'Hopital's rule.

Therefore, $f(x)$ is smooth.

But we very obviously have a problem! The taylor expansion for $f(x)$ is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

and we've alredy shown that $f^{(k)}(0) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

whereas the values of $f(x)$ for any $x > 0$ are strictly non-zero. Thus, the function is not analytic in any neighborhood of 0.

(b) Now consider the complex function

$$f(z) = \begin{cases} e^{-1/z^2}, z \neq 0 \\ 0, z = 0 \end{cases}$$

This function is not even continuous, because if we take the limit $z = x + iy \to 0$ along the imaginary axis i.e. $x = 0$, we find

$$\begin{aligned}
\lim_{z \to 0} f(z) &= \lim_{x=0, y \to 0} e^{-1/z^2} \\
&= \lim_{x=0, y \to 0} e^{-1/(iy)^2} \\
&= \lim_{y \to 0} e^{1/y^2}
\end{aligned}$$

But this limit diverges!
