(Instructor: Chien-I Chiang)

Physics 105: Analytical Mechanics notes

Keshav Balwant Deoskar

kdeoskar@berkeley.edu

These are some very terse notes taken from UC Berkeley's Physics 105 during the Summer '24 session, taught by Chien-I Chiang.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 First topic

text

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2.1 Finishing up discussion from last lecture

- Finish this from lecture recording

Continuing on with out discussion of when $H \neq E$, we can parametrize the position of a particle as $\vec{r} = \vec{r}(q_k, t)$

We have

$$\frac{\partial K}{\partial \dot{q_k}} = \frac{1}{2} m \left[2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_m} + \cdots \right]$$

Then,

$$\begin{cases} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} \right] \\ 2K = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + 2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \right] \end{cases}$$

(The expression for $2K$ is obtained by expandingo out

$$\boldsymbol{K = \frac{1}{2} m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$$

in terms of indices – write this out explicitly later)

Which gives us the relation

$$\begin{aligned} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} &= 2K - m \frac{\partial \vec{r}}{\partial t} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial q_k} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \right)}_{= \frac{d\vec{r}}{dt}} \\ &= 2K - \vec{p} \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

The question we were originally considering is **When is $\boldsymbol{H = E}$?**

Now,

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{q_k}} \dot{q} - L \\ &= \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = (K - V) \\ &= 2K - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} - K + V \\ &= K + V - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

So we see that $H = E = K + V$ only when

$$\frac{\partial \vec{r}}{\partial t} = 0$$

i.e. when $\vec{r} = \vec{r}(q_k, t)$ has no time dependence i.e. $\vec{r} = \vec{r}(q_k)$

Earlier, we considered the following setup:

and we showed that

$$H = E - m\omega^2\rho^2$$

So, let's check that

$$m\omega^2\rho^2 = \vec{p} \cdot \frac{\partial \vec{r}}{\partial t}$$

$$\begin{aligned}
\vec{p} \cdot \frac{\partial \vec{r}}{\partial t} &= \vec{p} \cdot (-\rho\omega \sin(\omega t)\hat{x} + \rho\omega \sin(\omega t)\hat{y}) \\
&= \vec{p} \cdot \left[\rho\omega\hat{\phi}\right] \\
&= m v_\phi \rho\omega \\
&= m\rho^2\omega^2
\end{aligned}$$

where $v_\phi = \rho\omega$

Since the hamiltonian itself has no time dependence, ***H* is conserved**. However, ***E* is not**. We can check that

$$dH = dE = d(m\omega^2\rho^2)$$

is indeed zero.

[Include figure]

If we break the force on the bead into a normal force (denoted N) and a centripetal(?) force, then

$$\begin{aligned} dW &= \overbrace{N\rho}_{\text{torque about z-axis}} d\phi \\ &= \frac{dl_z}{dt} d\phi \\ &= d\left(\rho m \rho \omega\right)\omega \\ &= d\left(m\rho^2\omega^2\right) \end{aligned}$$

This is the energy that goes into the system.

By energy conservation, $dW = dE$.

$$\implies 0 = dE - dW = dE - d(m\rho^2\omega^2)$$

i.e. $E - m\rho^2\omega^2 = H$ is a conserved quantity.

So, the **Hamiltonian being conserved** and the **Hamiltonian being equal to Energy** are two different scenarios with two different conditions.

- The Lagrangian is time-independent i.e. $\frac{\partial L}{\partial t} = 0 \implies H$ is conserved.
- The position vector centered in an inertial frame $\vec{r} = \vec{r}(q_k, t)$ is time independent i.e $\frac{\partial \vec{r}}{\partial t} = 0 \implies H = E$

Now we move on to a powerful technique.

2.2 The Method of Lagrange Multipliers

We have a block constrained to move on the xy-plane, and we have gravity. Previously, we would say

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

i.e we would start with an unconstrained lagrangian, and then plug in the constraints $z = 0, \dot{z} = 0$

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ \implies &\begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \end{cases} \end{aligned}$$

Alternatively, we can implement the constraint $\ddot{z} = 0$ in the following way: We have the original lagrangian

$$L' = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz + \lambda z$$

where λ is the Lagrange multiplier and we can think of z as being the constraint function $f(z)$ and our constraint is $f(z) = 0$.

If we treat λ as an independent degree of freedom, we can write the Euler-Lagrange equation for λ as

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0 \implies z = 0 \text{ (constraint)}$$

On the other hand, if we look at the equation of motion for z, we get

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \implies \lambda - mg - m\ddot{z} = 0 \implies m\ddot{z} = \lambda - mg$$

and using the constraint $z = 0 \implies \ddot{z} = 0$ we get $-mg + \lambda = 0 \implies \lambda = mg$. Okay, but what physical meaning does λ have? It has to do with the **Normal force**. i.e. λ is encoding the **constraint** that the block can only move on the xy-plane due to the Normal force.

So, in general, for N constraints we have Lagrange Multipliers $\lambda_1, \cdots, \lambda_N$.

Why do we call $\boldsymbol{\lambda}$ a Lagrange Multiplier?

Recall from Calc 3 that if we have contours of a function $f(x, y)$ on the xy-plane and we are constrained to move along some other curve $g(x, y) = c$ on the plane, if we ask "What is the extremum of $f(x, y)$ as we move along the curve $g(x, y) = c$?" then visually we can tell that the extremum corresponds to the point where $g(x, y)$ intersects the contour of $f(x, y)$ only once. This is because at such a point, the gradients of the two functions are parallel:

$$\nabla g = \lambda \nabla f$$

This constant multiplier is the **Lagrange Multiplier**

So, in general, if we have a Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

we know that $\delta L = 0$ gives the Equations of Motion. But if we want to do this variation δL under some constraint $C(x, y, z) = 0$ then we need to consider

$$\delta L = \lambda \delta C \implies L' = L - \lambda C$$

Generally, if we have P constraints, $C_l(q_1, \cdots, t) = 0$, $l = 1, \cdots, P$ on the lagrangian L, we can write a new lagrangian

$$L' = L + \sum_{l=1}^{P} \lambda_l C_l$$

The Euler-Lagrange equation for λ_l leads to $C_l = 0$ and the Euler-Lagrange equation for the generalized coordinate q_k is

$$\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}\right) = 0$$

$$\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} + \underbrace{\sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}}_{\text{generalized force}}$$

On the physical point of view, consider the following system:

[include picture of block and sledge which can both move]

If we consider the system as a whole, the normal forces due to the block and the sledge are equal and opposite, so they cancel each other out - and so does the work that they do(?).

Howeverm if we consider the block only - we do have a normal force. The block is constrained the only move on the surface of the slope, so we can write

$$L' = K - V + \int^{\vec{r}} \vec{F}_C(\vec{r}) \cdot d\vec{r}'$$

(This is a bit handwavy - watch the lecture recording and think about this)

Then, if we conpare this with

$$L' = L - V + \sum_l \lambda_l C_l$$

we have

$$\sum_l \lambda_l C_l = \int^{\vec{r}} \vec{F}_C \cdot d\vec{r}' = \int^{\vec{r}} \vec{F} \cdot \left(\frac{\partial \vec{r}'}{\partial q_k} \cdot \mathrm{d}q_k\right)$$

$$\implies \frac{\partial}{\partial q_k}\left(\sum_l \lambda_l c_l\right) = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial q_k}\right) \equiv \mathcal{F}_k \text{ (generalized force)}$$

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3.1 More about Lagrange Multipliers

Last time, we saw that if we have constraints $C_l\left(\underbrace{q_1, \cdots, q_k}_{N}, t\right) = 0$ then we can write a constrained Lagrangian

$$L' = K - V + \sum_l \lambda_l C_l$$

These kinds of constraints, which are only constraints of the generalized coordinates are called **Holonomic constraints**. But these are not the most general constraints; we can have constaints which also depend on the derivatives $\dot{q_k}$. Those types of constraints are called **Non-holonomic constraints**.

Then, the principle of stationary action gives us

$$0 = \delta S \implies \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l \underbrace{\lambda_l C_l}_{\vec{F}^C \frac{\partial \vec{r}}{\partial q_k}} \\ \\ C_l = 0 \end{cases}$$

Note that there are multiple ways to write the same constraint. And writing a constraint in a different manner changes the C_l, which further changes the λ_l. As such, the λ_l is not always a generalized force; it can also be a torque etc.

In total we have $N + P$ variables and $N + P$ equations, so we are able to solve the system if we know the initial conditions.

We got the above equation by varying the action, and in particular, by varying L with respect to q_k. But we can extend this a bit futher...

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l a_{lk}\lambda_l \\ a_{lk}\delta q_k + a_{lt}\delta t = 0 \end{cases}$$

In the case of Holonomic constraint,

$$a_{lk} = \frac{\partial C_l}{\partial q_l}$$
$$a_{lt} = \frac{\partial C_l}{\partial t}$$

For Holonomic constraint, we will have

$$\frac{\partial q_{lk}}{\partial t} = \frac{\partial q_{lt}}{\partial q_k}$$

3.2 Example: Tree log rolling down a ramp

Consider a tree log rolling down a (fixed) ramp without sliding.

[Include Figure]

To describe the motion of the log, generically, we need two degrees of freedom: X and θ.

But we also know the log is rolling **without sliding**. So if the tree moves a distance dx during rotation $d\phi$, then we know $Rd\phi = dx$ where R is the radius of the log. Or in other words,

$$Rd\phi - dx = 0$$

This constraint is of the general form we saw above: $\boxed{a_{lk}\delta q_k + a_{lt}\delta t = 0}$ with $a_{1,\theta} = R, a_{1,x} = -1$ and all the time components $a_{lt} = 0$.

Now, we can write the Lagrangian of this system:

$$L = \frac{1}{2}M(\dot{X}^2) + \frac{1}{2}I\dot{\theta}^2 + mgX\sin(\alpha)$$

Note that we're actually kind of mixing approaches here. Technically there should be *three* degrees of freedom because the log can move in (x, y) space and rotate, but we know that the log is constrained by the Normal force and we don't need both of x, y; just one will suffice.

Wait... so, why do we even bother using the Lagrange Multiplier stuff if we're gonna use the old method too?

The Lagrange multiplier method allows us to retain info about the contact forces so if we, say, want to find the magnitude of the tension in a string, we can still do so using the Lagrange Multiplier method. Whereas in the old method, contact forces are used to enfore constraints but we lose all information about them.

Anyway, after writing down the lagrangian, we can obtain the Equations of Motion (with the constraints):

$$\begin{cases} \frac{d}{dt}\left(m\dot{X}\right) = +mg\sin(\alpha) - \lambda_1 \\ \frac{d}{dt}\left(I\dot{\theta}\right) = \lambda_1 R \end{cases}$$

So, what exactly is $\boldsymbol{\lambda_1}$?

In the X equation of motion, we have $+mg\sin(\alpha)$ which is the component of gravity along the ramp. So, λ_1 has the same units as force. We can interpret λ_1 as the **frictional force**!

Then, in the θ equation of motion, we can interpret $\lambda_1 R$ as the **torque due to friction**!

Solving these further we have

$$\begin{cases} m\ddot{X} = mg \sim (\alpha) - \lambda_1 \ (1) \\ I\ddot{\theta} = \lambda_1 R \ (2) \\ R\dot{\theta} = \dot{X} \text{ (from the no-sliding condition)} \implies R\ddot{\theta} = \ddot{X} \ (3) \end{cases}$$

Substituting (3) into (1) gives

$$\begin{cases} mR\ddot{\theta} = mg\sin(\alpha) - \lambda_1 \\ \frac{I}{R}\ddot{\theta} = \lambda_1 \end{cases}$$
$$\implies mR\left(\lambda_1 \frac{R}{I}\right) = mg\sin(\alpha) - \lambda_1$$
$$\implies \left(1 + \frac{mR^2}{I}\right)\lambda_1 = mg\sin(\alpha)$$
$$\implies \boxed{\lambda = \frac{mg\sin(\alpha)}{\left(1 + \frac{mR^2}{I}\right)}} \text{ This is the magnitude of friction!}$$

3.3 Example: A bead on a wire

We've seen this example before, but this time we want to calculate the normal force on the bead.

[Include Figure]

Using the Lagrange Multiplier method, we can write down the constrained Lagrangian as

$$L' = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right] - mgz - \lambda_1(\phi - \omega t) - \lambda_2\left(z - \alpha\rho^2\right)$$

So, the EL Equations look like

$$\begin{cases} m\ddot{\rho} = n\rho\dot{\phi}^2 - 2\lambda_2\alpha\rho \\ \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = \lambda_1 \\ m\ddot{z} = -mg + \lambda_2 \end{cases}$$

From the z EoM, we can tell that λ_1 is a force since it's being added with $-mg$. We can interpret it as the ***z*−component** of the **Normal Force**.

Similarly, in the ϕ EoM we see that λ_1 is the derivative of the Angular Momentum, so λ_1 is the **torque**.

[Include figure]

Now, in the ρ equation, we know that $m\ddot{\rho}$ is also a force since ρ has units of length. So, $-2\lambda_2\alpha\rho$ must also be a force. Exactly which force is it? It's the **radial component** of the **Normal Force** (See the figure above.)

When it comes to actually solving for λ_1 and λ_2, we can solve for them after we solve for $\rho(t)$ using $z = \alpha\rho^2$ and other constraints.

[Add last bit from lecture recording - lots of figures]