Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 February 22 - Onto Chapter 6! Sard's Theorem.

Recap

- Last time, we saw how

1.1 Measure zero sets in manifolds

Measure

We say a subset $A \subseteq \mathbb{R}^n$ has ***(n-dimensional) measure zero*** if for any $\delta > 0$ there are $X_1, X_2, \dots \in \mathbb{R}^n$ and $r_1, r_2, \dots > 0$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} B_{r_i}(X_i)$$

and

$$\sum_{i=1}^{\infty} r_i^n < \delta$$

Properties (in $\mathbb{R}^n$):

1. If $\subseteq \mathbb{R}^n$ is compact and has measure zero, $B \subseteq A \implies B$ has measure zero.
2. If $A_1, A_2, \dots$ have measure zero, then $\bigcup_{i=1}^{\infty} A_i$ has measure zero.
3. If $A \cap \left(\{c\} \times \mathbb{R}^{n-1}\right) \subseteq \{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dim measure zero for all $c \in \mathbb{R}$, then A has n-dim measure zero. (Version of Fubini's Theorem)
4. If $f : A \subseteq \mathbb{R}^{n-1} \to R$ is continuous, then its graph $\Gamma(f) = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n$ has measure zero as well (follows from property 3).
5. Every non-trivial affine (linear subspace + constant translation) subspace of $\mathbb{R}^n$ has measure zero.
6. If $A \subseteq \mathbb{R}^n$ has measure zero, then $A^c = \mathbb{R}^n \setminus A$ is dense in $\mathbb{R}^n$.
7. If A is measure zero, then for all $p \in A$ there exsts a neighborhood $p \in U_p \subseteq_{open} \mathbb{R}^n$ such that $A \cup U_p$ has measure zero.
8. If $A \subseteq \mathbb{R}^n$ is measure zero and $F : A \to \mathbb{R}^n$ is ***Lipschitz***, then $F(A)$ is also measure zero.

   Lipschitz: There exists $K > 0$ such that, for all x, y, we have $|F(x) - F(y)| \leq K|x - y|$.

9. If $S^{k<n} \subseteq \mathbb{R}^n$ is a submanifold, then it has n-dim measure zero since it is covered by $k - slice$ charts, each of measure zero.

An example of a function which can map a set of measure 0 onto a set of non-zero measure is the ***Cantor Function***. Write more later.

1.2 Measure on smooth manifolds

- Given a smooth manifolds M^n, a subset $A \subseteq M$ is said to have measure zero if for any smooth chart (U, ϕ) of M, the set $\phi(U \cap A) \subseteq \mathbb{R}^n$ has measure zero.
- The above is equivalent to saying there exist smooth charts $(U_i, \phi_i)_{i \in I}$ that cover M i.e. $M = \bigcup_{i \in I} U_i$ such that $\phi_i(U_i \cap A)$ has measure zero for all $i \in I$.

Exercise: Check that the $A \subseteq \mathbb{R}^n$ has measure zero in the usual sense if and only if A has measure zero when viewing $\mathbb{R}^n$ as a manifold.

Some of the properties from $\mathbb{R}^n$ are carried over to the setting wth manifolds. Namely,

1. $B \subseteq A$, A has measure 0 miplieis B has measure zero.
2. Each A_i has measure zero implies $\bigcup_{i=1}^{\infty} A_i$ has measure zero.

(6), (7), (9) also hold for smooth manifolds.

1.3 Motivation for Sard's Theorem

Consider $F : R^2 \to \mathbb{R}$ defined as $(x, y) \mapsto x^2 - y^2$ and its level sets.

- We notice that 0 is the only critical value, while $\mathbb{R} \setminus \{0\}$ is the set of regular values for this function.
- Also, if we consider the function from last class, we notice that there are visually very few critcal values as compared to regular values.

Sards theorem makes this intuition more rigorous.

1.4 Sard's Theorem

Theorem: If $F : M \to N$

Proof:
If the $m = n$ case, we may assume $M = N = \mathbb{R}^n$.
[Complete from image]

Now the general case. It suffices to show that for all $p \in M$ there exists a neighborhood $p \in U \subseteq_{open} M$ such that the set of cirtical values of $F\big|_U$ has measure zero. So, we may assume that $F : U \subseteq \mathbb{R}^m \to N$. By further restricting, we may also assume $N = \mathbb{R}^n$. We proceed by induction.

Base case: $m = 0$.

All differentials $dF_p : \{\vec{0}\} \to T_{F(p)}\mathbb{R}^n$. Then, if $n = 0$, there are ***no critical points/values*** whereas if $n > 0$ then ***all*** points are critical, $F(M) \subseteq N$. (In fact, if $n = 0$ then dF_p is surjective so we have no critical points/values)

Now, we induct on m for $m, n \geq 1$. Let $C = \{\text{crit. pts of } F \subseteq U \subseteq \mathbb{R}^m\}$ and

$$C_k = \{p \in U \ : \ \frac{\partial^l F}{\partial\left(x^1\right)^l \cdots \partial\left(x^i\right)^l}\bigg|_{p=0} = 0 \text{ for all } l \leq k, i_1, \cdots, i_l \in \{1, \ldots, m\}\}$$

i.e. C_k is the set of points where all partial derviatives upto orer k vanish.

Note that $C \supset C_1 \supset C_2 \supset \cdots$ and each of them are closed in U. Now, write

$$F(C) = \underbrace{F(C - C_1)}_{(i)} \cup \underbrace{F(C_1 - C_2) \cdots \cup F(C_k - C_{k+1})}_{(ii)} \cup \underbrace{F(C_k)}_{(iii)}$$

We will show that each of (i), (ii), (iii) has measure zero.

Part (i) : $F(C - C_1)$ has measure zero

Pick $p \in C - C_1$, then replace U by $U - C_1$ and C by $C - C_1$ since C_1 is closed. Let's work on these sets.

[insert image]

WLOG, we have

$$\frac{\partial F^1}{\partial x^1} \neq 0$$

Set $y^1 = F^1$. We can choose smooth functions $y^2, \cdots, y^m : U \to \mathbb{R}$ such that

$$\left(\frac{\partial y^i}{\partial x^j}\right)_{i,j=1,\ldots,m}$$

is invertible (for example, we could choose the sums of coordinate functions).

[Write matrix form from picture]

Then, $\Phi = \left(y^1, \ldots, y^m\right)$ is a local diffeomorphism at p so there exsits a neighborhood $p \in U^1 \subseteq U$ such that $\Phi\left(U^1\right)$ is open and $\Phi\big|_{U'} : U' \to \Phi(U')$ is invertible with smooth inverse.

Let $\tilde{F} = F \circ \left(\Phi\big|_{U'}\right)^{-1} : \Phi(U') \to \mathbb{R}^n$. Note that if $q \in U'$ is a critcal point of F if and only if $\Phi(q)$ is a critical point of $\tilde{F}$. We need to show

$$\{\text{Critical points of } F\big|_{U'}\} = \{F(U' \cap C)\} =$$

has measure zero.

Now,

$$\begin{aligned} F(x^1, \ldots, x^m) &= \left(F^1(x^1, \ldots, x^m), \ldots, \right) \\ &= \tilde{F} \circ \Phi(x^1, \ldots, x^m) \\ &= \left(y^1(x^1, \ldots, x^m), y^2, \ldots, y^m\right) \end{aligned}$$

i.e. $\tilde{F}(x^1, \ldots, x^m) = \left(x^1, \tilde{F}^2, \ldots, \tilde{F}^m\right)$ and

$$d\tilde{F} = \begin{bmatrix} 1 & 0 \cdots 0 \\ * & \\ \vdots & \left(\frac{\partial \tilde{F}^j}{\partial x^i}\right)_{i,j=2,\ldots,m} \\ * & \end{bmatrix}$$

This matrix has dimension $m \times n$ and it is surjective if and only if the smaller matrix

$$\left(\frac{\partial \tilde{F}^j}{\partial x^i}\right)_{i,j=2,\ldots,m}$$

is surjective. Define $\tilde{C}_S = C \cap$