Math H185 Lecture 3

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

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1.1 Sequences and Series

Recall that a sequence of complex numbers $\{z_n \in \mathbb{C}\}$ is said to *converge* to $z \in \mathbb{C}$ if for all $\epsilon > 0$ there exists a natural number $N \geq 1$ such that

$$|z_n - z| < \epsilon$$

for all $n \geq N$.

Equivalently,

$$\lim_{n \to N} |z_n - z| = 0$$

In HW1, we show that if $z_n = x_n + iy_n$ and $z = x + iy$ where $x, y, x_n, y_n \in \mathbb{R}$ then

$$\lim_{n\to\infty} |z_n - z| = 0 \iff \begin{cases} \lim_{n\to\infty} |x_n - x| = 0 \\ \lim_{n\to\infty} |y_n - y| = 0 \end{cases}$$

1.2 Complex Dfferentiability

Let $f : U \subseteq_{\text{open}} \mathbb{C} \to \mathbb{C}$.

Holomorphic Functions

f is Holomorphic at $z_0 \in U$ if

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If so, call the limit at $f'(z_0)$.

Note: Keep in mind that h is a complex number.

- This means that for any sequence $h_n \to 0$ <u>or</u> $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$|h| < \delta \implies |\frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0)| < \epsilon$$

- This is the most important definition of the course.

<u>Remark:</u> Although $\mathbb{C}$ is the same as $\mathbb{R}^2$ as a metric space, Holomorphicity is much stronger than differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ because the limits along every path to a point are required to be equal.

<u>Example:</u> Consider the function $f(z) = \overline{z}$.

We observe that

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{\overline{z+h} - \overline{z}}{h} \\ &= \frac{\overline{h}}{h} \end{aligned}$$

Now, if we take the limit as $h \to 0$ along the real line, we get $\frac{\overline{h}}{h} = \frac{h}{h} = 1$, however if we take the limit along the imaginary line we get $\frac{\overline{h}}{h} = \frac{-h}{h} = -1$

On the other hand if we consider the counterpart of this function in $\mathbb{R}^2$ as $f(x, y) = (x, -y)$, this function is *smooth everywhere.* In contrast to this, the complex function $f(z) = \overline{z}$ is *not holomorphic at any* $z_0 \in \mathbb{C}$.

We will see that holomorphic functions have strong rigidity properties not shared by real differentiable functions. For instance,

- If f, g are holomorphic on a connected open set $U \subseteq \mathbb{C}$ and $f = g$ on a line segment in U, then in fact they agree at *all* points in U: $f(z) = g(z) \forall z \in U$. This is the **Principle of Analytic Continuation.**
- Another example of surprising rigidity is that if f is holomorphic on U i.e. it is one differentiable on U, then in fact it is *infinitely* differentiable on U.

Examples:

1. $f(z) = z^n$

   Calculate:

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{(z+h)^n - z^n}{h} \\ &=_{\text{binom thm.}} \left[\frac{1}{h} \left(z^n + nz^{n-1}h + \cdots + nzh^{n-1} + h^n \right) - z^n \right] \\ &= nz^{n-1} + h(\cdots) \\ &\implies \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = nz^{n-1} \end{aligned}$$

1.3 Stability Properties

While holomorphicity is different from real Dfferentiability, there are a number of properties which are justified by the same $\epsilon - \delta$ proofs as those from $\mathbb{R}$ analysis.

- If $f, g : U \to \mathbb{C}$ are holomorphic at $z_0 \in U$ then
  - $(f + g)$ is holomorphic at z_0 and
  $$(f + g)'(z_0) = f'(z_0) + g'(z_0)$$
  - fg is holomorphic at z_0 and
  $$(fg)'(z_0) = f'g(z_0)g(z_0) + f(z_0)g'(z_0)$$
  - Chain Rule: $(f \circ g)$ is holomorphic at z_0 and
  $$(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$$
  - Division: (f/g) is holomorphic at z_0 if $g(z_0) \neq 0$ and
  $$\left(\frac{f}{g} \right) = \frac{f'(z_0)g(z_0) + f(z_0)g(z_0)}{g(z_0)^2}$$

Polynomials: Finite sum of monomials.

$$f(z) = a_n z^n + \cdots + a_0$$

By linearity (Stability property 1), *all* Polynomials are holomorphic on C.

Rational Functions: Ratios of Polynomials.

$$h(z) = \frac{f(z)}{g(z)}$$

By Stability property 3, all rational functions are holomorphic on $\{z \in \mathbb{C} : g(z) \neq 0\} \subseteq_{\text{open}} \mathbb{C}$.

Warm-down examples: Where are the following functions holomorphic, and what are their derivatives in those regions?

1. $f(z) = \frac{1}{z}$
2. $f(z) = z^2 + 3z + \frac{1}{2}$
3. $f(z) = \text{Re}(z)$
4. $f(z) = i \cdot \text{Im}(z)$
5. $f(z) = \text{Re}(z) + i \cdot \text{Im}(z)$

Answers:

1. Holomorphic on $\mathbb{C} \setminus \{0\}$, and derivative in the region is
$$\frac{-1}{z^2}$$
2. Holomorphic on $\mathbb{C}$, and derivative in the region is
$$2z + 3$$
3. Not holomorphic *anywhere*, snice limit vertically is always zero but limit horizontally will be non-zero.
4. Not holomorphic *anywhere*, snice limit horizontally is always zero but limit vertically will be non-zero.
5. Holomorphic on $\mathbb{C}$, and derivative in the region is 1 ($f(z) = z$, so $f'(z) = 1$ at all points).