# Math 214 Notes

Keshav Balwant Deoskar

February 24, 2024

These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

## Contents

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## Recap

Some important results we've built in the last few lectures are:

- We built the incredibly handy *Rank Theorem*, which essentially tells us that the coordinate representations of constant rank maps between manifolds are canonical in a neighborhood of each point i.e. a map $F : M^m \to N^n$ of constant rank $r$ can be expressed, in some small neighborhood of any $x = (x^1, \dots, x^m)$, as

$$F(x^1, \dots, x^r, \dots, x^m) = (x^1, \dots, x^r)$$

- Last time, we applied this to $k-$slices. We argued that if we have $\mathbb{S}^k \hookrightarrow M^m$ and if locally $\mathbb{S}^k$ is a $k-$slice [finish later]
- Then, we spoke about Level Sets. Recall that if we have a smooth map $F : M^m \to N^n$, the inverse image of a point $q \in N$ i.e. $F^{-1}(q) \subseteq M$ is an embedded submanifold, given that it satisfies certain properties.

## 1.1 Constant Rank Level Set Theorem:

***Theorem:*** If $F : M^m \to N^n$ is smooth and of constant rank $r$, then for any $q \in N$, the level set $F^{-1}(q) \subset M$ is a proper submanifold of $M$ with dimension $(m - r)$.

*Proof:* We want to employ the rank theorem.

Let $S = F^{-1}(q)$. Applying the rank theorem, we obtain charts $(U, \psi)$ on $M$ and $(V, \psi)$ on $N$ such that $\psi \circ F \circ \phi^{-1}$ has the coordinate representation

$$(x^1, \dots, x^r, x^{r+1}, \dots, x^m) \mapsto (x^1, \dots, x^r, 0, \dots, 0)$$

So, if $\psi(q) = (c^1, \dots, c^r, 0, \dots 0)$, then the preimage in local coordinates on $M$ has the form $\{(c^1, \dots, c^r, x^{r+1}, x^m) : x^{r+1}, x^m \in \mathbb{R}\} \cap \phi(U)$ such that

But this is just a $k-$slice. So, around each point, we have a local k-slice. And we saw last time that a local $k$-slice is an embedded submanifold of $M$. Further, $F^{-1}(q)$ is closed in $M$ so it is a proper embedding.

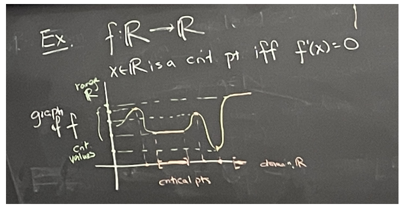
***Corollary:*** If $F : M \to N$ is a submersion, then $F^{-1}(q)$ is an $(m-n)$-dimensional submanifold.

## Regular Level sets

Given a smooth map $F : M^m \to N^n$

- a point $p \in M$ is ***regular*** if $dF_p$ is surjective at that point.
- Otherwse, the point $p$ is called a ***critical point***.
- A point $q \in N$ is a ***regular value*** if all points in $F^{-1}(q)$ are regular.
- Otherwise, $q$ is called a ***critical value***.

Example: For a smooth function $F : R \to R$, $x \in \mathbb{R}$ is a critcal point if and only if $f'(x) = 0$.




Just visually speaking, it looks like there are much fewer critical values than critical points. We'll say more about this in the next chapter when we deal with *Sard's Theorem*.

Remark (Proposition 4.1 in LeeSM):
The set of regular points in $M$ is open, and the restriction

$$F\big|_{\{\text{ Reg. pts. }\}}\{\text{Reg. pts.}\} \to N$$

is a submersion.

*Why is this set open?* See Example 1.28 in LeeSM.

*Why is it a submersion?* Because at a regular point $p$, $dF_p$ is surjective i.e. $\text{rank}(dF_p) = n$. This means the $m \times n$ matrix representation of $dF_p$ has an $n \times n$ submatrix with non-zero determinant. The determinant of this submatrix remains non-zero in a small neighborhood of $p$ due to continuity.

***Theorem (Regular Level Set Theorem:)*** If $F : M^m \to N^n$ is a smooth map between manifolds and $q \in N$ is a regular value, then its inverse image $F^{-1}(q) \subseteq M$ is a smooth $(m-n)-$dimensional submanifold of $M$.

***Proof:*** Let $U = \{p \in M \ : \ rank(dF_p) = dimN\} \subseteq M$. i.e. the set of points where $F$ has full rank. This is an open subset of $M$ by Proposition 4.1, and if $q \in N$ is a regular value then $F^{-1}(q) \subseteq U$.

Then $F\big|_U$ is a submersion and by the Constant Rank Theorem for Level Sets, $F^{-1}(q) \hookrightarrow U$ is a smooth embedding, and $F^{-1}(q)$ is an $(m-n)$ dimensional submanifold of $U$. Finally, $U \hookrightarrow M$ is also a smooth embedding, so the composition $F\big|_U \hookrightarrow U \hookrightarrow M$ is also a smooth embedding. It follows that $F^{-1}(q)$ is an embedded submanifold of $M$.

Examples:

- Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

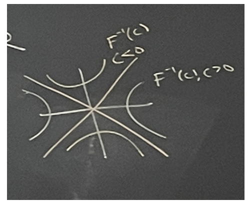
  $$(x, y) \mapsto x^2 - y^2$$

  Then,

  $$dF_{(x,y)} = \begin{bmatrix} 2x & -2y \end{bmatrix}$$

  which has rank 1 unless $x = y = 0$. i.e. the set of regular values if $\mathbb{R}^2 \setminus \{0\} \subseteq_{open} \mathbb{R}^2$.

  So, $F^{-1}(c)$ is an embedded $(2-1) = 1$ dimensional submanifold of $\mathbb{R}^2$.



- $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ can be thought of as a level set of

$$F : \mathbb{R}^{n+1} \to \mathbb{R}^1, \quad (x^1, \dots, x^{n+1}) \mapsto (x^1)^2 + \cdots + (x^{n+1})^2$$

So, $\mathbb{S}^n = F^{-1}(1)$. If we can check that $1 \in \mathbb{R}$ is a regular value of $F$ then our results above tell us that $\mathbb{S}^n = F^{-1}(1)$ is an embedded submanifold of $\mathbb{R}^2$.

To do so, we calculate the differential of $F$ and make sure it is of constant rank at any pre-image of 1.

The differential of $F$ is

$$df_{(x,y)} = \begin{bmatrix} 2x^1 & \cdots & 2x^{n+1} \end{bmatrix}$$

and this map has constant rank except at $(0, \dots, 0) \notin \mathbb{S}^n$. Thus, the Regular Level Set Theorem (RLST) tells us that

$$\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$$

is an $(n + 1 - 1) = n$ dimensional submanifold.

- (Converse not necessarily true) Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ which acts as

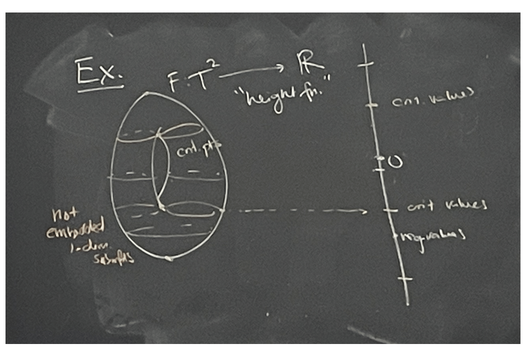
$$(x, y) \mapsto \left((x^1)^2 + (x^2)^2 - 1\right)^2$$

Then, the differential is

$$dF_{(x,y)} = \begin{bmatrix} 4x^1 \left((x^1)^2 + (x^2)^2 - 1\right) & 4x^2 \left((x^1)^2 + (x^2)^2 - 1\right) \end{bmatrix}$$

Then, all points of $\mathbb{S}^1$ are critical, but $\mathbb{S}^1$ is still an embedded submanifold of $\mathbb{R}^2$. So, the converse of RLST does not necessarily hold.

- Consider the "height function" on a torus $F : \mathbb{T}^2 \xrightarrow{\text{"height func."}} \mathbb{R}^2$




The pre-images of the critical values of this function are the figure-eight shaped sets shown in the figure.

Above, what we did was find the conditions where a level set of a function is a submanifold. Now, given a manifold, how do we *define* a function such that the submanifold forms one of its level sets?

### 1.2 Defining Functions

Given a submanifold $S^k \subseteq M^m$ of $M$,

- a smooth map $F : M^m \to N^n$ such that $S = F^{-1}(q)$ for some regular value $q \in N$ is a ***defining function for*** $S$
- if $p \in S$ and $p \in U \subseteq_{open} M$, then a smooth map $F : U \to N$ is a ***local defining function*** for $S$ at $p$ if $S \cap U = F^{-1}(q)$ for some regular value $q \in N$.

*Theorem:* Given a subset $S \subseteq M^m$, $S$ is a $k-$dimensional submanifold if and only if there exsit local defining functions $F : U \to \mathbb{R}^{m-k}$ at every point $\in S$.

(Proof follows from k-slicing and constant rank level set theorem)

### 1.3 Tangent Space to a Submanifold

[Write from image]

In a local k-slice chart for $M$, $\left(U, \left(x^1, \ldots, x^m\right)\right)$ we have

$$S \cap U = \{x^{+1} = c^{k+1}, \ldots, x^m = c^m\}$$

[Write from image]

If $F : U \subseteq M \to N$ is a local defining function for $S$, i.e. $U \cap S = F^{-1}(q)$ for a regular value $q$, then the extrnisic tangent space to $S$ is

$$T_p^{\text{extrnisic}} S = \ker dF_p$$

### Example: The group of orthogonal matrices

The orthogonal group is

$$O(n) = \{A \in \mathbb{R}^{n\times n} \ : \ A^T A = \mathrm{Id}_{n\times n}\} \subseteq \mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$$

Defining the map $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times N}_{\text{symm.}} \left(\cong \mathbb{R}^{n(n+1)/2}\right)$ defined as

$$A \mapsto A^T A$$

*Note:* $\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$

Then, the orthogonal group is just the pre-image of the identity i.e. $O(n) = R^{-1}\left(\mathrm{Id}_{n\times n}\right)$. We want to check that $\mathrm{Id}_{n\times n}$ is a regular value of $F$.

Let $A \in O(n)$, and compute the differential which maps tangent vectors as

$$dF_A : T_A \mathbb{R}^{n\times n} \to T_{F(A)} \mathbb{R}^{n\times n}_{\text{Symm.}}$$
$$B \mapsto \left.\frac{d}{dt}\right|_{t=0} F(A + tB)$$

Now,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} F(A+tB) &= \left.\frac{d}{dt}\right|_{t=0} (A+tB)^T(A+tb) \\
&= \left.\frac{d}{dt}\right|_{t=0} A^TA + t\left(B^TA + A^TB\right) + t^2B^TB \\
&= B^TA + A^TB
\end{aligned}$$

We need to check that $dF_A$ is surjective, which is equivalent to checking that the kernal satisfies the Rank Nullity Theorem.

$$\begin{aligned}
\dim\ker dF_A &= \dim \mathbb{R}^{n\times n} - \dim \mathbb{R}^{n\times n}_{\text{symm.}} \\
&= n^2 - \frac{n(n+1)}{2} \\
&= \frac{n(n-1)}{2}
\end{aligned}$$

Notice that this is exactly the dimension of the space of $n \times n$ anti-symmetric matrices which maes sense as

$$\begin{aligned}
\ker(dF_A) &= \{B \in \mathbb{R}^{n\times n} \ : \ B^TA = -A^TB\} \\
&= \{B \in \mathbb{R}^{n\times n} \ : \ \left(A^TB\right)^T = -A^TB\} \\
&= \{B \in \mathbb{R}^{n\times n} \ : \ B = AC, \text{ for some antsymmetric } C\}
\end{aligned}$$

[Fill in from images]

Also, a very often used tangent space is

$$T_{\mathrm{Id}_{n\times n}}$$

$O(n)$ is a lie group and $\mathbb{R}^{n\times n}_{\text{antisymm.}}$ is its Lie Algebra.