Math 214 Homework 1

Keshav Balwant Deoskar

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Q1. Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces with the standard topologies induced by their metrics. Show that $f : X_1 \to X_2$ is continuous as a map between topological spaces if and only if it is continuous in the $\epsilon - \delta$ sense as a map between metric spaces.

Proof:

- $\implies$ direction: Fix a point $f(x_0) \in X_2$ and for some $\epsilon > 0$ suppose we have another point $f(x) \in X_2$ so that $d_2((x), f(x_0)) < \epsilon$. We want to show there exists $\delta > 0$ such that $d_1(x, x_0) < \delta$.

  Since f is open in terms of topologies, the pre-image of the open ball $B := B_\epsilon(f(x_0)) \subseteq X_2$ is open in X_1. That is, $f^{-1}(B) \subseteq X_1$ is open. Further, since both $f(x), f(x_0) \in B$ we know that $x, x_0 \in f^{-1}(B)$. So, we can set $\delta = \sup\{d(a,b) : a, b \in f^{-1}(B)\}$. This gives us continuity in the $\epsilon - \delta$ sense.

- $\impliedby$ direction: We need to show, using $\epsilon - \delta$ continuity, that every open set in X_2 has open pre-image in X_1. In a metric space, open sets are generated by the basis formed by **open balls**, so it suffices to show the above holds for an open ball of arbitrary radius.

  Consider the open $\epsilon-$ball $B := B_\epsilon(f(x_0))$ centered around a point $f(x_0) \in X_2$ for some $\epsilon > 0$. We need to show that $f^{-1}(B)$ is open in X_1.

  Choose some other point $f(x) \in B$. Then, since $d_2(f(x), f(x_0)) < \epsilon$. By the hypothesis, we know $d_1(x, x_0) < \delta_x$ for some $\delta_x > 0$. So, letting $\delta = \frac{1}{2}\delta_x$, certainly we have

  $$\underbrace{B_\delta(x)}_{\text{open in } X_1} \subset f^{-1}(B)$$

  This argument works for any point $x \in f^{-1}(B)$, so every element of $f^{-1}(B)$ is an interior point. This shows the pre-image, $f^{-1}(B)$, of an open ball B is also open. As a result, any the pre-image of any open subset of X_2 is open in X_1.

Q2. Let $(X, \mathcal{O}_X)$ be a topological space and $Y \subset X$ a subset. Show that the subspace topology $\mathcal{O}_Y$ is the coarsest topology with respect to which the inclusion map $Y \hookrightarrow X$ is continuous (i.e. that if $\mathcal{O}'$ is another topology on Y such that $Y \hookrightarrow X$ is continuous, then $\mathcal{O}_Y \subset \mathcal{O}$).

Proof: Let's denote the coarsest topology on Y such that the inclusion $i : Y \to X$ is continuous as

$$\tau = \{i^{-1}(V) : V \in \mathcal{O}_X\}$$

Why is τ the coarsest topology on Y?

By definition, the inclusion is continuous if and only if $i^{-1}(V)$, for all $V \in \mathcal{O}_X$. So, if σ is a topology on Y, at the very least we must have

$$\{i^{-1}(V) : V \in \mathcal{O}_X\} \subset \sigma$$

Now, the subspace topology is defined as the collection

$$\mathcal{O}_Y = \{V \subseteq Y \mid V = U \cap Y, \;\; U \in \mathcal{O}_X\}$$

We want to show that $\tau = \mathcal{O}_Y$ i.e.

$$\{i^{-1}(V) : V \in \mathcal{O}_X\} = \{V \subseteq Y \mid V = U \cap Y, \;\; U \in \mathcal{O}_X\}$$

For any $V \subseteq X$, consider $v \in V$. Then, since i^{-1} is a function from X to Y, if $v \in Y$ then it gets mapped to itself. Otherwise, it gets mapped into the empty set. Thus, clearly, we have

$$i^{-1}(V) = V \cap Y$$

so the two topologies are equal.

Thus, the subspace topology is the coarsest topology such that the inclusion map in continuous.

Q3. Let X and Y be two topologcal spaces and let $X' \subset X$, $Y' \subset Y$ be subspaces equipped with the subspace topology. Let $f : X \to Y$ be a continuous function such that $f(X') \subset Y$. Show that $f\big|_{X'} : X' \to Y'$ is continuous.

Proof:
Consider an open set $U \subseteq Y'$. Then, we can write $U = V \cap Y'$ for V open in Y.

$$\begin{aligned}
f\big|_{X'}{}^{-1}(U) &= f\big|_{X'}{}^{-1}(V \cap Y') \\
&= \left(f\big|_{X'}{}^{-1}(V)\right) \cap \left(f\big|_{X'}{}^{-1}(Y')\right) \quad \text{(By elementary set theory)} \\
&= \left(f^{-1}(V) \cap X'\right) \cap \left(f^{-1}(Y') \cap X'\right) \\
&= \tilde{V} \cap \tilde{Y'}
\end{aligned}$$

and this is open since it is a finite intersection of $\tilde{V} = f^{-1}(V) \cap X'$ and $\tilde{Y'} = f^{-1}(Y') \cap X'$ which are both open in X' with respect to the subspace topology.

Q4. Let X be a topological space with the property that for any $p, q \in X$ with $p \neq q$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(p) \neq f(q)$. Show that this implies X is Hausdorff.

Proof:
Since $f(p) \neq f(q)$, the distance $d = \frac{|p-q|}{3}$ is non-zero. Consider the open balls $B_1 := B_d(f(p))$ and $B_2 := B_d(f(q))$. Clearly, $B_1 \cap B_2 = \emptyset$, so their pre-images are also disjoint.
Now since f is continuous, their preimages $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are open in X. Thus, we have found disjoint open sets for each of the points x, y. The space X is Hausdorff.

Q5. Let X be a Hausdorff topological space and let $Y \subset X$ be a subset with the subspace topology. Show that Y is Hausdorff.

Proof: Consider points $p, q \in Y \subset X$. Since X is Hausdorff, there exist open disjoint sets O_p, O_q such that $p \in O_p, q \in O_q$, and $O_p \cap O_q = \emptyset$. Then the sets $U = O_p \cap Y$ and $V = O_q \cap Y$ are nonempty disjoint open sets in Y such that $p \in U, q \in V$, and $U \cap V = \emptyset$. Therefore, Y is Hausdorff.

Q6. Let X be a Hausdorff topological space and let $K \subset X$ be compact. Show that K is closed.

Proof: The subset K is closed if $K^c = X \setminus K$ is open. To show that K^c is open, let's consider a point $x \in K^c$ and show that it lies in the **interior** of K^c.

Since X is Hausdorff, for our chosen point x and any $y \in K$ there exist open sets $U_i \ni x, V_i \ni y$ such that $U_i \cap V_i = \emptyset$. The collection of open sets $\{V_i\}_i$ forms an open cover of K.

Since K is compact, there exists a finite subcover $\{V_i\}_{i \in I, |I| < \infty}$ of K. Let $\{U_i\}_{i \in I}$ be the collection of corresponding $x-$open sets. Now, let $U = \bigcap_{i \in I} U_i$. Then, certainly, $U \cap K = \emptyset$ and U is open since it's a finite intersection.

Therefore, $U \subset K^c$ is an open subset of x, showing any $x \in K^c$ is an interior point. Thus K^c is open, allowing us to conclude that K is closed.

Q7. Let $\phi : X \to Y$ be a continuous map between two topological spaces, and let $K \subset X$ be compact. Show that its image $\phi(K) \subset Y$ is compact as well.

Proof: Let $\{V_\alpha\}_{\alpha \in A}, V_\alpha \in Y$ be an open cover of $\phi(K)$. Then, the collection $\{U_\alpha\}_{\alpha \in A}$ where $U_\alpha = \phi^{-1}(V_\alpha)$ forms an open cover of K (each U_α is open due to the continuity of ϕ).

Now, since K is compact, there is a finite subcover $\{U_1, \ldots, U_n\}$ for some natural number n. Thus, the collection $\{V_1, \ldots, V_n\}$ where $V_k = \phi(U_k)$ forms an open cover of $\phi(K)$. The image of K is compact.

Q8. Let $\phi : X \to Y$ be a continuous and bijective map between a compact topological space X and Hausdorff topological space Y. Show that its inverse $\phi^{-1} : Y \to X$ is continuous; hence ϕ is a homeomorphism.

Proof:
Before we begin let's prove two quick lemmas:

> <u>Lemma:</u> A map $\phi : X \to Y$ is continuous if and only if for every closed $V \subseteq Y$ the pre-image $f^{-1}(V) \subseteq X$ is closed.
>
> <u>Proof:</u> Suppose $f : X \to Y$ is continuous and consider closed $V \subseteq Y$. Then $Y \setminus V$ is open, so $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is open. So, $f^{-1}(V)$ is closed.
>
> Next, suppose for every closed subset of Y, the pre-image ni X is closed. Then for open $A \subseteq Y$, $Y \setminus A$ is closed so $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is closed. Thus, $f^{-1}(A)$ is open and f is continuous.

> <u>Lemma:</u> If X is compact and $K \subseteq X$ is closed, then K is compact.
>
> Since X is compact, it has a finite subcover $X \subseteq \{U_\alpha\}_\alpha$. But $K \subseteq X$, so $\{U_\alpha\}_\alpha$ is also a finite-subcover of K. Thus, K is compact.

Let's get to it. We want to show that ϕ^{-1} is continuous. So, we need to show that for any open/closed set $U \subseteq X$, the pre-image $(\phi^{-1})^{-1}(U) = \phi(V) \subseteq Y$ is open/closed.

Consider a closed set $U \subseteq X$. Since X is compact and U is closed, U itself is compact as well. Now, X is a compact set and $\phi : X \to Y$ is continuous, so $\phi(U) = V \subseteq Y$ is compact as well (by Q7). Finally, Y is Hausdorff and $V \subseteq Y$ is compact, so Y is closed (by Q6). Thus ϕ^{-1} is continuous!

Q9. Let M be a topological $0-$dimensional manifold. Show that M is a countable subset with the discrete topology.

Proof: Suppose M is a topological manifold of dimension 0. Let $p \in M$ and let U be a neighborhood of p. Now, U is homeomorphic to an open subset of $R^0 = \{0\}$. The only subset of R^0 is R^0 itself, so U contains only one point. As a result, every singleton subset of M is open, which then makes *any* subset of M an open set. So, M has the discrete topology.

Now, every singleton set U is open in M, and every open set is a union of basis sets. Thus, each of the singleton sets must also be in the basis $\mathcal{B}$ (If not, then it would have to be a union of basis sets which somehome combine to give a set with only one point). Finally, M is second-countable by definition, so $\mathcal{B}$ is countable i.e the set of all singleton sets is countable. So, it must be the case that M contains a countable number of points.

Q10. Let M be a topological, $n-$dimensional manifold and $M' \subseteq M$ is an open subset. Show that M', equipped with the subspace topology, is also a topological $n-$dimensional manifold. So, any open subset of $\mathbb{R}^n$ is a topological $n-$manifold.

Proof: In order for M' to be a topological $n-$ dimensional manifold, it must be

- Hausdorff
- Second countable
- Locally Euclidean

Hausdorff: Consider any two points $p, q \in M' \subseteq M$. Since they are points in M, which is Hausdorff, there exist sets U, V open in M such that $p \in U, q \in V$ and $U \cap V = \emptyset$.

Then, the sets $\tilde{U} = U \cap M', \tilde{V} = V \cap M'$ which are open in M' with respect to the subspace topology satisfy $p \in \tilde{U}, q \in \tilde{V}$, and $\tilde{U} \cap \tilde{V} = \emptyset$. Thus, M' is Hausdorff.

Second Countable: Since M is a manifold, it is second countable by definition. Let's denote its countable basis as $\mathcal{B}$. Then, consider the collection

$$\mathcal{B}' = \{B \cap M' : B \in \mathcal{B}\}$$

This collection $\mathcal{B}'$ is also countable, and its sets form a basis for the subspace topology since any generic open subset $\tilde{W} \subset M'$ can be expressed as

$$\begin{aligned}
\tilde{W} &= \underbrace{W}_{\text{open in } M} \cap M' \\
&= \left(\bigcup_{i \in I} \underbrace{B_i}_{\in \mathcal{B}} \right) \cap M' \\
&= \bigcup_{i \in I} \left(\underbrace{B_i \cap M'}_{\in \mathcal{B}'} \right)
\end{aligned}$$

Thus, M' is also second-countable.

Locally Euclidean: Since M is locally euclidean, each point $p \in M$ is associated with a chart (U, ϕ) where $U \ni p$ is an open subset of M and $\phi : U \to V \subseteq_{\text{open}} \mathbb{R}^n$ is a homeomorphism.

For a point $m \in M'$ there is a chart $(U, \phi : U \to V \subseteq_{open} \in \mathbb{R}^n)$ such that $\phi(m) = 0$.

(Every point is guaranteed to have an associated chart, and if the map sends $m \mapsto p \neq 0 \in \mathbb{R}^n$, we can get the desired map by just appending a "$-p$" to the existing map.)

Now, consider the set $U \cap M' \subseteq_{open} M'$. This is open in M so $\phi(U \cap M')$ is open in $\mathbb{R}^n$. Choose a real number $r > 0$ such that the $r-$ball around zero is $B_r(0) \subset \phi(U \cap M')$. Since ϕ is a homeomorphism, $\tilde{U} = \phi^{-1}(B_r(0))$ is open in M'.

Then, $(\tilde{U}, \phi\big|_{\tilde{U}})$ forms a chart of M' around the point m. Therefore $M' \subseteq M$ is also a topological $n-$ dimensional manifold.

Q11. Generalize the example from class that S^1 is a topological, $1-$dimensional manifold to

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

To do this, consider the projection maps

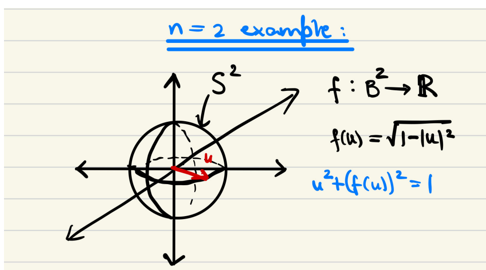
$$\phi_i^{\pm} : U_i^{\pm} := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \pm x_i \geq 0\} \to B_1 := \{x_1^2 + \cdots + x_n^2 \leq 1\} \subseteq \mathbb{R}^n$$

Proof: To show that S^n is a topological $n-$dimensional manifold, we note that Hausdorffness and Second-countability are directly inherited from $\mathbb{R}^n$ as s^n is a subspace.
To show it is locally euclidean, consider the propjection maps

$$\phi_i^{\pm} : U_i^{\pm} := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \pm x_i \geq 0\} \to B_1 := \{x_1^2 + \cdots + x_n^2 \leq 1\} \subseteq \mathbb{R}^n$$

Consider the continuous function $f : \mathbb{B}^n \to \mathbb{R}$ given by

$$f(u) = \sqrt{1 - |u|^2}$$



Then, the set $U_i^+ \cap \mathbb{S}^n$ is the graph of

$$x^i = f(x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

where the hat above x^i indicates that it is omitted, since

$$\begin{aligned} x^i &= 1 - \left[(x^1)^2 + \cdots + (x^{i-1})^2 + (x^{i+1})^2 + \cdots + (x^{n+1})^2\right] \\ \implies (x^1)^2 &+ \cdots + (x^{i-1})^2 + (x^i)^2 + (x^{i+1})^2 + \cdots + (x^{n+1})^2 = 1 \end{aligned}$$

Similarly, the set $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f(x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

Thus, each of the sets $U_i^{\pm} \cap \mathbb{S}^n$ is locally euclidean on dimension n. The coordinates on $\mathbb{S}^n$ are given by

$$\phi_i^{\pm}(x^1, \dots, x^{n+1}) = \pm f(x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

Since each point on $\mathbb{S}^n$ must necessarily lie in one of the sets $U_i^{\pm} \cap \mathbb{S}^n$, each point lies in the domain of at least one chart. Hence, $\mathbb{S}^n$ is a topological $n-$dimensional manifold.
