Math 214 Notes

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1 February 6 - Tangent Vectors, Differentials of smooth maps

Recap

- Intuitively, we can visualize the tangent plane to a point on a sphere embedded in $\mathbb{R}^3$, however this doesn't generalize very well to hugher dimensions.
- For abstract manifolds, recall that we defined a map $v : C^\infty(M) \to \mathbb{R}$ to be a ***derivation at*** $p \in M$ if it satisfies two properties:
  - R-linearity:
$$v(af + bg) = av(f) + bv(g)$$
  - "Product Rule":
$$v(fg) = f(p)v(g) + v(f)g(p)$$
- The space of derivations at p is defined to be the Tangent Plane T_pM, and it has a natural vector space structure. Intuitively, each element of the tangent space is a direction in which we can take a directional derivative.
- Example: Take $M = \mathbb{R}^n$ and $\vec{p} \in \mathbb{R}^n$, $\vec{v}_0 = (v^1, \ldots, v^n) \in \mathbb{R}^n$

  Let
$$v(f) = \sum_{i=1}^{n} v^i \left(\frac{\partial}{\partial x^i} f \right)(p)$$

1.1 Properties of Derivations

Lemma: If $v \in T_pM$, then $v(c) = 0$ where $c \in \mathbb{R}$ is constant.

Proof: $v(c) = v(c\dot{1}) = c \cdot v(1)$, so it suffices to show that $v(1) = 0$. Now,

$$\begin{aligned} v(1) &= v(1 \cdot 1) \\ &= 1 \cdot v(1) + v(1) \cdot 1 \end{aligned}$$

$$\implies v(1) = 0$$

Lemma: If $f, g \in C^\infty(M)$ agree on a neighborhood $U \subseteq M$ containing p, and $v \in T_pM$, then $v(f) = v(g)$.

Proof: Find a precompact neighborhood $B \ni p$ such that $p \in \overline{B} \subseteq U$. There exists a bump function $\psi_n \in C^\infty$ such that

$$\begin{cases} \psi \equiv 1 \text{ on } \overline{B} \\ \operatorname{supp}(\psi) \subseteq U \end{cases}$$

Note that $\psi \cdot (f - g) \equiv 0$ on all of M, so

$$\begin{aligned} 0 &= v(\psi \cdot (f-g)) \\ &= \psi(p) \cdot v(f-g) + \underbrace{v(\psi) cdot (f-g)(p)}_{=0} \\ &= v(f-g) \\ &= v(f) - v(g) \end{aligned}$$

Now that we've proved some properties of derivations, we can consider other perspectives from which we can understand the tangent plane T_pM.

1.2 Tangent Space to $\mathbb{R}^n$

Lemma: At $\vec{a} = (a^1, dots, a^n) \in \mathbb{R}^n$, for each $\vec{=}(v^1, \ldots, v^n) \in \mathbb{R}^n$ we can define the map

$$D_{\vec{v}}\big|_{\vec{a}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}^n$$

as

$$f \mapsto \left(\partial_{\vec{v}} f\right)(\vec{a}) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}\bigg|_{\vec{a}}$$

which is a derivation at $\vec{a}$, so $D_{\vec{v}}\big|_{\vec{a}} \in T_{\vec{a}}\mathbb{R}^n$
and we wll prove that

$$D : \mathbb{R}^n \to T_{\vec{a}}\mathbb{R}^n$$

defined by

$$\vec{v} \to D_{\vec{v}}\big|_{\vec{a}}$$

is an isomorphism.

Proof:
Leaving linearity and injectivity as exercises, we show surjectivity. Take a derivation $v \in T_{\vec{a}}\mathbb{R}^n$, and let $u^i = v(x^i) \in \mathbb{R}, i = 1, \ldots, n$ (we're going to essentially do the reverse of D).

c We will show that

$$v = D_{\vec{u}}\big|_{\vec{a}} = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}\bigg|_{\vec{a}}$$

First note, for $f \in C^\infty(M)$, we have

$$\begin{aligned} f(\vec{x}) &= f(\vec{a}) + \int_0^1 \frac{d}{dt} f(\vec{a} + t(\vec{x} - \vec{a}))dt \\ &= f(\vec{a}) + \sum_{i=1}^{n} (x^i - a^i) \underbrace{\int_0^1 \frac{\partial f}{\partial x^i} \left(\vec{a} + t(\vec{x} - \vec{a})\right) dt}_{h^i(\vec{x})} \end{aligned}$$

Note that

$$h^i(\vec{a}) = \frac{\partial f}{\partial x^i}(\vec{a})$$

Apply the derivation v to $f(\vec{x})$:

$$\begin{aligned} v(f) &= v(f(\vec{a})) + \sum_{i=1}^{n} v(x^i - a^i) h^i(\vec{a}) + \sum_{i=1}^{n} v(a^i - a^i) v(h^i) \\ &= \sum_{i=1}^{n} u^i \cdot \frac{\partial f}{\partial x^i}(\vec{a}) \end{aligned}$$

1.3 The Differential of a Smooth Map

Give smooth manifolds M^m, N^n and a smooth map $F : M \to N$, the ***differential of F*** at $p \in M$ is defined as

$$\begin{aligned} dF_p : T_p M &\to T_{F(p)} N \\ v &\mapsto dF_p(v) \end{aligned}$$

where $(dF_p(v))(f) = v(f \circ F)$, for $f \in C^\infty(N)$.

[Insert image]

One way to inperpret this is as follows: If we think of a derivative (derivation at point $p \in M$) to be a tangent/velocity curve then dF_p tells us how this velocity curve changes under the map $F : M \to N$.

Example: Take $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and

$$F(\vec{x}) = \left(F^1(x^1, \ldots, x^m), \ldots, F^n(x^1, \ldots, x^m)\right)$$

Then, for $f \in C^\infty(N)$,

$$\begin{aligned} \left(dF_p \left(\left. \frac{\partial}{\partial x_i} \right|_p \right) \right)(p) &= \left(\left. \frac{\partial}{\partial x_i} \right|_p \right)(f \circ F) \\ &= [finishafterlec] \end{aligned}$$

Read: More properties of differential (Maybe prove later).

- Composition of diffeomorphisms, differential of identity is isomorphism, etc.

Now that we've studied the differental of a map between euclidean spaces, let's work with more general manifolds.

1.4 Tangent Space of Smooth Manfiolds (with or without boundary)

Proposition: Let M^n be a smooth manifold, $p \in U \subseteq_{open} M$ and consider the inclusion map $\iota : U \to M$. Then,

$$\begin{aligned} d\iota_p : T_pU &\to T_pU \\ v &\mapsto d\iota_p(v) \end{aligned}$$

is an isomorphism [Recall that by definition $(d\iota_p(v))(f) = v\left(f \circ \iota\right)$].

This is essentially an application of the property we proved earlier that the derviation only depends on a small neighborhood at each point.

Proof:

Injectivity: Since it's a linear map, it is injective if

$$\begin{aligned} d\iota_o(v) = 0 &\iff d\iota_p(v)(f) = 0 \text{ for all} f \in C^\infty(M) \\ &\iff v\left(f \circ \iota\right) = 0 \\ &\iff v\left(f\big|_U\right) = 0 \end{aligned}$$

We want to show $f(\tilde{f}) = 0$ for all $\tilde{f} \in C^\infty(U)$. Choose $f \in C^\infty(M)$ such that $\tilde{f} = f$ near p. Then,

$$\begin{aligned} 0 &= v\left(f\big|_U\right) \\ &= v(\tilde{f}) \text{ (by locality of derivations)} \end{aligned}$$

This shows injectivity.

Surjectivity: Given a derivation $\tilde{v} \in T_pM$, we want to find $v \in T_pU$ such that $\tilde{v}(f) = v\left(\tilde{f}\Big|_U\right)$ for all $\tilde{f} \in C^\infty$. Given a function $f \in C^\infty(U)$, define $v \in T_pU$ by

- Choose some (doesn't matter which) extension $\tilde{f} \in C^\infty(M)$ such that $f = \tilde{f}$ on a neighborhood of p.
- Set $v(f) \equiv \tilde{v}(\tilde{f})$ and check that this is well defined i.e. independet of $\tilde{f}$ choice (results from locality of derivations)

Corollary: $\dim(T_pM^m) = m$

Proof: Given $p \in M^m$, choose a smooth chart (U, ϕ) with $p \in U$. Then, from the above result, we know

$$T_pM \xleftarrow[d\iota_p\text{,by prop}]{\cong} T_pU \xrightarrow[d\phi_p\text{,isom.}]{\cong} T_{\phi(p)}\phi(U) \xrightarrow[\text{by incl. map}]{\cong} T_{\phi(p)}\mathbb{R}^m = \mathbb{R}^m$$

1.5 Coordinates

[Write from image and include graphic]

Note that

$$\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^m}\right|_p$$

is a basis for T_pM.

<u>Read:</u> Differential of a smooth map in coordinates.