Math H185 Lecture 2

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 January 19 - The Complex Plane

1.1 What does $\mathbb{C}$ look like geomertrically?

[Insert Diagram of $\mathbb{C}$ with Real and Imaginary Axes]

Def: The **Modulus** or **absolute value** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. It is the distance to 0.

There is a correspondence between the structure of $\mathbb{C}$ and the vector structure of $\mathbb{R}^2$.

$$\text{Addition in } \mathbb{C} \iff \text{Vector Addition in } \mathbb{R}^2$$

Exercise: Draw the subset of $z \in \mathbb{C}$ defined by

1. $\{|z| = 1\}$
2. $\{|z - 6| = 1\}$
3. $\{|z - 6i| \leq 1\}$
4. $\{\text{Re}(z) \leq \text{Im}(z)\}$
5. $\{|z| = \text{Re}(z) + 1\}$

Answers: [Insert figures later.]

1. Circle with radius 1 centered at point $z = 0 + 0i$.
2. Circle with radius 1 centered at point $z = 6 + 0i$.
3. Disk with radius 1 centered at point $z = 0 + 6i$.
4. Everything above the line making 45-degree angle with the real axis.
5. (Horizontal) Parabola with vertex at $z = 0 + 1i$ since

$$\begin{aligned} \sqrt{x^2 + y^2} &= x + 1 \\ \implies x^2 + y^2 &= x^2 + 2x + 1 \\ \implies y^2 &= 2x + 1 \\ \implies x &= \frac{y^2 - 1}{2} \end{aligned}$$

So, we get a horizontal parabola.

1.2 Polar Coordinates

Rather than using the rectangular (cartesian) coordinates to describe $\mathbb{C}$, we can equvalently use **Polar coordinates** wherein we use the distance from the origin (modulus) and the angle made wtih the real axis (argument).

$$\begin{aligned} &(x, y) \to (r, \theta) \\ &r = \sqrt{x^2 + y^2} \\ &\theta = \arctan\left(\frac{y}{x}\right) \end{aligned}$$

Notice however there is an amibiguity in the angle θ since $\theta + 2n\pi$ would describe the same point. Thus, we define the **Principal value branch** as the restriction $\theta \in [0, 2\pi]$.

Recall: The taylor expansion of the the exponential for any real number $x \in \mathbb{R}$ is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

For $z \in \mathbb{C}$, the exponential function is defined as below.

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

Now the usual definitions for the cosine and sine as in $\mathbb{R}$ in terms of the unit circle doesn't quite work for complex numbers. However, we can still define them using the exponential!

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4}z^4 + \cdots$$
$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$$

Euler's Theorem: $e^{iz} = \cos(z) + i\sin(z)$, $z \in \mathbb{C}$.

Proof: Expanding the exponential, we have

$$e^{iz} = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

and note that

$$\frac{1}{(2n)!}(iz)^{2n} = \frac{1}{(2n!)}z^{2n}(i^{2n}) = \frac{(-1)^n z^{2n}}{(2n)!}$$

Similarly,

$$\frac{1}{(2n-1)!}(iz)^{2n-1} = \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$$

Thus,

$$e^{iz} = \underbrace{\sum_{2n} \frac{(-1)^n z^{2n}}{(2n)!}}_{\cos(z)} + \underbrace{\left(\sum_{2n-1} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} \right) i}_{\sin(z)}$$

Additionally, we can convert back from Polar coordinates to Cartesian Coordinates as

$$(r, \theta) \to (x, y)$$
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Multiplication: The multiplication of two complex numbers can be thought of in terms of polar coordniates as scaling by r and rotating by θ.

For $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ we have

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= (r_1 r_2)\, e^{i\theta_1 + \theta_2} \end{aligned}$$

So, the complex number obtained after multiplication has length equal to the product of the lengths of z_1, z_2 and has argument equal to the sum of the arguments of z_1, z_2.

1.3 Topology of $\mathbb{C}$

The set $\mathbb{C}$ is a **metric space**, and the metric on $\mathbb{C}$ is $\text{dist}(z, w) = |z - w|$ ($\iff$ Eucldiean metric on $\mathbb{R}^2$, so as metric spaces they are equivalent $\mathbb{C} \cong \mathbb{R}^2$).

Notation: On any metric space, we have a notion of open sets and open balls.

- Given a complex number $z_0 \in \mathbb{C}$ and $r > 0, r \in \mathbb{R}$, the set

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

  is called the **open ball around z_0 of radius r**.

- Similarly, the set $\overline{B_r(z_0)}$ is the **closed ball**:

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

Open and Closed sets

- A subset $U \subseteq \mathbb{C}$ is **open** if for all $z \in U$, there exists $r > 0$ such that $B_r(z) \subset U$.
- A subset $V \subseteq \mathbb{C}$ is said to be **closed** if its complement V^c is open in $\mathbb{C}$.

Ex: The set $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ is closed. This set will be important when studying the complex logarithm. [Insert figure.]

Note that while the closed ball is not open (since the points on the boundary i.e. points with $|z - z_0| = r$ don't satisfy the requirement), a set *can* be both open and closed. For example, the sets $\mathbb{C}$ and $\emptyset$ are both closed and open.