Physics 137B Homework 5

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Question 1: WKB Approximations in an infinite square well

(a) Find the WKB spectrum for $f(x) = \kappa x$ where κ is a positive constant.

(b) FInd the WKB spectrum for

$$f(x) = \begin{cases} 0, 0 < x < a/2 \\ V_0, a/2 < x < a \end{cases}$$

where V_0 is a positive constant.

Solution:

We found in class that the approximate quantization condition for potentials of the form

$$V(x) = \begin{cases} f(x), 0 < x < a \\ \infty, \text{otherwise} \end{cases}$$

can be written as

$$\int_0^a p(x) dx = n\pi\hbar$$

where $p(x) = \sqrt{2m(E - V(x))}$ and $E > V(x)$.

(a) For $f(x) = \kappa x$, $\kappa \in \mathbb{R}^+$ the quantization condition can be written as

$$\begin{aligned} &\int_0^a \sqrt{2m(E - \kappa x)} dx = n\pi\hbar \\ \implies &\sqrt{2m} \int_0^a (E - \kappa x)^{1/2} dx = n\pi\hbar \\ \implies &\sqrt{2m} \left[-\frac{2}{3\kappa} (E - \kappa x)^{3/2} \right] \Big|_{x=0}^{x=a} = pi\hbar \\ \implies &\sqrt{2m} \cdot \left(-\frac{2}{3\kappa} \right) \cdot \left[(E - \kappa a)^{3/2} - E^{3/2} \right] = n\pi\hbar \end{aligned}$$

This quantization condition gives us the eigen-energy spectrum.

(b) For

$$f(x) = \begin{cases} 0, 0 < x < a/2 \\ V_0, a/2 < x < a \end{cases}$$

where V_0 is a positive constant, we have the condition

$$\begin{aligned}
&\int_0^a p(x)dx = n\pi\hbar \\
\implies &\int_0^{a/2} p(x)dx + \int_{a/2}^a p(x)dx = n\pi\hbar \\
\implies &\int_0^{a/2} \sqrt{2m(E-0)}dx + \int_{a/2}^a \sqrt{2m(E-V_0)}dx = n\pi\hbar \\
\implies &\sqrt{2mE}\cdot\frac{a}{2} + \sqrt{2m(E-V_0)}\cdot\frac{a}{2} = n\pi\hbar \\
\implies &\frac{\sqrt{2m}a}{2}\left[\sqrt{E_n}+\sqrt{E_n-V_0}\right] = n\pi\hbar
\end{aligned}$$

This is the quantization condition which gives us the eigen-energy spectrum in this case.

Question 2: WKB with a finite barrier

Consider the finite-barrier potential define by

$$V(x) = \begin{cases} V_0, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

Consider the scenario where $V_0 > E$.

(a) Calculate the transmission probability $T = |F/A|^2$ exactly.

(b) Calculate the transmission probability using WKB and the approximations made in class, namely $T \sim e^{-2\gamma}$, where $\gamma = \frac{1}{\hbar}\int_x^a p(x')dx'$

(c) Compare these two results, and comment on which limit the scaling o these two solutions are expected to agree. We are just looking for exponential behavior.

Solution:

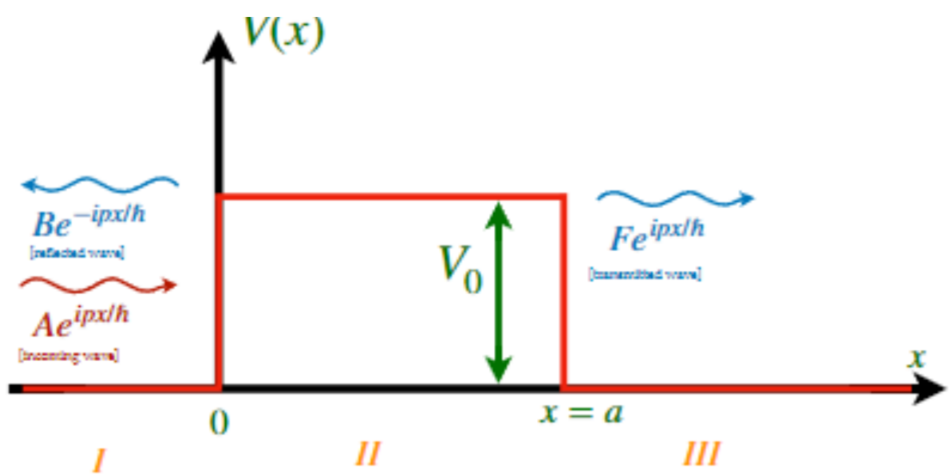

FIG. 1: Shown is a graphical depiction of the finite barrier potential.

We want to think about the bound states of this system ($V_0 > E$). We have the following wave-functions in the different regions:

$$\text{Region I: } \psi(x) = Ae^{ikx} + Be^{-ikx}$$
$$\text{Region III: } \psi(x) = Fe^{ikx}$$

where $k = \sqrt{2mE}/\hbar$

but we don't know what the function is in Region II. Let's find out what it is, and then use boundary conditions to find the relation between A and F to calculate the transmission coefficient.

(a) In region II, we have potential V_0, so the Schrödinger Equation reads as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

Or equivalently,

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi = \frac{2m}{\hbar^2}(V_0 - E)\psi$$

If we write $l \equiv \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$, then the equation has the form

$$\frac{d^2\psi}{dx^2} = l^2\psi$$

and thus has general solution of the form

$$\boxed{\psi(x) = Ce^{lx} + De^{-lx}}$$

So, we have

$$\psi(x) = \begin{cases} \text{Region I: } Ae^{ikx} + Be^{-ikx} \\ \text{Region II: } Ce^{lx} + De^{-lx} \\ \text{Region III: } Fe^{ikx} \end{cases}$$

along with the boundary conditions

$$\begin{aligned}
&\text{Continuity of } \psi \text{ at 0: } \psi_I(0) = \psi_I(0) \\
&\text{Continuity of } d\psi/dx \text{ at 0: } \frac{d\psi}{dx}(0^-) = \frac{d\psi}{dx}(0^+) \\
&\text{Continuity of } \psi \text{ at } a\text{: } \psi_{II}(a) = \psi_{III}(a) \\
&\text{Continuity of } d\psi/dx \text{ at } a\text{: } \frac{d\psi}{dx}(a^-) = \frac{d\psi}{dx}(a^+)
\end{aligned}$$

The first condition tells us

$$\begin{aligned}
&Ae^{ik(0)} + Be^{-ik(0)} = Ce^{l(0)} + De^{-l(0)} \\
\implies &A + B = C + D
\end{aligned}$$

the second tells us

$$\begin{aligned}
&ik\left[Ae^{ik(0)} - Be^{-ika(0)}\right] = l\left[Ce^{l(0)} - De^{-l(0)}\right] \\
\implies &ik(A - B) = l(C - D)
\end{aligned}$$

the third tells us

$$Ce^{l(a)} + De^{-l(a)} = Fe^{ika}$$

the fourth tells us

$$l\left[Ce^{l(a)} - De^{-l(a)}\right] = ikFe^{ika}$$

Now, via a whole bunch of algebra, we find that

$$\frac{F}{A} = \frac{e^{-ika}}{\cosh(la) + i(\gamma/2)\sinh(la)}$$

where $\gamma \equiv l/k - k/l$.

Now, the transmission coefficient is given by

$$\begin{aligned} T &= \left(\frac{F}{A}\right)^* \left(\frac{F}{A}\right) \\ &= \frac{e^{+ika}}{\cosh(la) - i(\gamma/2)\sinh(la)} \cdot \frac{e^{-ika}}{\cosh(la) + i((\gamma/2)\sinh(la))} \\ &= \frac{1}{\cosh^2(la) + (\gamma^2/4)\sinh^2(la)} \end{aligned}$$

$$\implies \boxed{T = \frac{1}{\cosh^2(la) + (\gamma^2/4)\sinh^2(la)}}$$

where

$$\begin{aligned} \gamma &= \frac{l}{k} - \frac{k}{l} \\ &= \frac{\sqrt{V_0 - E}}{\sqrt{E}} - \frac{\sqrt{E}}{\sqrt{V_0 - E}} \end{aligned}$$

So,

$$\begin{aligned} \frac{\gamma^2}{4} &= \frac{1}{4}\left(\frac{V_0 - E}{E} + \frac{E}{V_0 - E} - 2\right) \\ &= \frac{1}{4}\left(\frac{1 - E/V_0}{E/V_0} + \frac{E/V_0}{1 - E/V_0} - 2\right) \end{aligned}$$

(b) In class, we found that under the WKB approximation, the transmission coefficient is roughly given by

$$T \sim e^{-2\gamma}, \quad \gamma = \frac{1}{\hbar}\int_0^a |p(x')|dx'$$

where $p(x) = \sqrt{2m(E - V_0)}$

We find γ to be

$$\begin{aligned} \gamma &= \frac{1}{\hbar}\int_0^a \left|\sqrt{2m(E - V_0)}\right| dx \\ &= \frac{1}{\hbar}\int_0^a \sqrt{2m(V_0 - E)}dx \\ &= \frac{\sqrt{2m(V_0 - E)}a}{\hbar} \\ &= l \cdot a \end{aligned}$$

where $l \equiv \sqrt{2m(V - 0 - E)}/\hbar$

Thus, the transmission coefficient is given by

$$T \sim e^{-2al} = e^{-2\frac{\sqrt{2m(V_0 - E)}a}{\hbar}}$$

(c) Let's now compare our two results. We obtained the WKB approximation by making the assumption that the barrier was very high and very broad. This would mean that

$$\sqrt{2m(V_0 - E)} >> \sqrt{2mE}$$
$$\implies l >> k$$
$$\implies \gamma^2 = \left(\frac{l}{k} - \underbrace{\frac{k}{l}}_{\approx 0}\right)^2 \approx \left(\frac{l}{k}\right)^2$$

Then, the transmission coefficient is approximately

$$T \approx \frac{1}{\cosh(la)^2 + \frac{(l/k)^2}{24}\sinh(la)}$$

We find that this matches pretty well with e^{-la} when $l >> k$.

Question 3: Half Harmonic Oscillator

Consider the half-harmonic oscillator potential in 1D,

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, x > 0 \\ \infty, \text{ otherwise} \end{cases}$$

(a) Find the exact eigenvalues to this problem.

(b) Find the quantization condition that the half-harmonic oscillator satisfie according to the WKB approximation and compare the results to what we found in class.

Solution:

(a) Since the potential is infinite for $x < 0$, the wavefunction must disappear in that region, which means only the normal QHO wavefunctions which vanish at the origin i.e. which are odd functions can survive.

The n-th QHO eigenfunction has the same parity (odd or even) as the parity of n itself. So, the odd n eigenfunctions survive while the even n eigenfunctions are killed off.

Thus, the eigenvalues of the half-harmonic oscillator have the form

$$E_n = \left(2n - 1 + \frac{1}{2}\right)\hbar\omega = \left(2n - \frac{1}{2}\right)\hbar\omega$$

for $n = 1, 2, 3, \ldots$

(b) In the case that there are two turning points located at $x = 0$ and $x = x_2$, the wavefunction satisfies

$$\psi(x) = \begin{cases} \frac{2D}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_x^{x_2} p(x')dx' + \frac{\pi}{4}\right], x < x_2 \\ \frac{D}{\sqrt{p(x)}} \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|dx'\right], x > x_2 \end{cases}$$

assuming that $E > V(x)$ for $x < x_2$ and $E < V(x)$ for $x > x_2$.

For any system whose potential has a vertical wall, we have

$$\frac{1}{\hbar}\int_0^x p(x)dx + \frac{\pi}{4} = n\pi$$

In the half-harmonic oscillator potential, we have

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2, x > 0 \\ \infty, \text{ otherwise} \end{cases}$$

so

$$p(x) = \sqrt{2m(E - V(x))} = \sqrt{2m(E - (1/2)m\omega^2x^2)} = m\omega\sqrt{x_2^2 - x^2}$$

The first turning point is $x_1 = 0$ and the second turning point x_2 is $x_2 = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$. So,

$$\begin{aligned} \int_0^{x_2} p(x)dx &= m\omega\int_0^{x_2}\sqrt{x_2^2 - x^2}dx \\ &= \frac{\pi}{4}m\omega x_2^2 \\ &= \frac{\pi E}{2\omega} \end{aligned}$$

Then, the quantization condition

$$\frac{1}{\hbar}\int_0^x p(x)dx + \frac{\pi}{4} = n\pi$$

forces the eigen-energies to have the form

$$E_n = \left(2n - \frac{1}{2}\right)\hbar\omega$$

for $n = 1, 2, 3\ldots$

This matches with the exact eigenenergies found in part (a).