Math H185 Lecture 12

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 February 12 - Proof of Cauchy's Theorem (Sketch)

Recall that a primitive (i.e. an antiderivative) of $f : U \subseteq_{open} \mathbb{C} \to \mathbb{C}$ is a function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$.

Now, by the Fundamental Theorem of Calculus,

> If
>
> - f has a primitive on an open neighborhood of γ and
> - γ is closed
>
> Then
> $$\int_\gamma f(z)dz = 0$$

This is to be contrasted with the *Cauchy-Goursat Theorem* which states that

> If f is holomorphic on a neighborhood of U, then
> $$\int_{\partial U} f(z)dz = 0$$

Though the two statements above are very similar, they're not quite the same. The first one requires a primitive locally whereas the second requires it over an entire region.

We will sketch a proof of Cauchy's Theorem.

1.1 Sketch

- Step 1: Approximate the path γ by polygons.

  [Draw Image]

- Step 2: Subdivide into triangles.

  [Draw Image]

  Then the integral over the entire curve is the same as the sum of the integrals over triangles. We then want to show that
  $$\int_\Delta f = 0$$
  where Δ is a triangle. We can do so by taking the triangle Δ and carrying out the barycentric subdivision i.e. take the midpoints of all sides and draw lines between them. Then, we get
  $$\int_\Delta = \int_{\Delta_1} + \int_{\Delta_2} + \int_{\Delta_3} + \int_{\Delta_4}$$
  By the Triangle inequality $|A + B| \leq |A| + |B|$, we have

$$\implies \left|\int_{\Delta} f(z)dz\right| \leq 4\sup_i \left|\int_{\Delta_i} f(z)dz\right|$$

Then, we take the biggest Δ_i and do the same procedure again. Repeatedly subdividing and using the Triangle Inequality, we have

$$\implies \left|\int_{\text{original }\Delta} f(z)dz\right| \leq 4^n \sup_{\Delta^{(n)}} \left|\int_{\Delta^{(n)}} f(z)dz\right|$$

where $\Delta^{(n)}$ is the triangle obtained after subdividing $n-$times. So, $\Delta^{(1)} \supseteq \Delta^{(2)} \cdots \Delta^{(n)}$.

Now, there exists a limit point z_0 for this sequence of shrinking triangles. The limit point lies in the intersection of all the triangles

$$z_0 \in \bigcap_n \Delta^{(n)}$$

Key: Near a point z_0, the function f is well approximated by a linear function $f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)$ where $\lim_{z\to z_0} \frac{\epsilon(z)}{z-z_0} = 0$. We want to use this fact to control how big the integral over $\Delta^{(n)}$ can get.

$$\int_{\Delta^{(n)}} f = \underbrace{\int_{\Delta^{(n)}} f(z_0)}_{=0} + \underbrace{\int_{\Delta^{(n)}} f'(z_0)(z-z_0)}_{=0} + \int_{\Delta^{(n)}} \epsilon(z)$$

where the first two integrals are zero because constant and linear functions have primitives on $\mathbb{C}$ and thus their integrals over any closed curve is zero by the Fundamental Theorem of Calculus. So, what we really need to control is the *error term*.

We have

$$4^n \left|\int_{\Delta^{(n)}} \epsilon(z)dz\right| \leq 4^n \left|\int_{\Delta^{(n)}} |\epsilon(z)|\,|dz|\right| <<< 4^n \int_{\Delta^{(n)}} |z - z_0|\,|dz|$$

where $|dz| = |z'(t)|\,|dt|$ and $A_n <<< B_n$ denotes

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 0$$

Let's call the greatest possible distance between two points in a triangle as the *diameter* of the triangle, $\mathrm{diam}(\Delta^{(n)})$. So,

$$|z - z_0| \leq \mathrm{diam}(\Delta^{(n)}) = 2^{-n}\mathrm{diam}(\Delta^{(\text{original})})$$

[Get some more details from lecture recording]

Conclusion

$$\left|\int_{\Delta} f(z)dz\right| <<< C$$

where C is some fixed constant, and the LHS and RHS are constant sequences (but sequences none-the-less), meaning LHS/RHS $\to 0$ as $n \to \infty$ and so LHS $= 0$.

Summary

- Subdivide Δ into small triangles where f is well approximated by a linear function (because it's holomorphic).