Math H185 Lecture 34

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1 Normal Famlies

Normal Families

A family of functions $\mathcal{F}$ on $U \subseteq \mathbb{C}$ is said to be ***normal*** if for all sequences $f_1, f_2, f_3, \dots \in \mathcal{F}$, there exists a convergent subsequence.

This definition expresses that a family of functions is compact in a sense.

Theorem (Arzela-Ascoli): If $\mathcal{F}$ is Uniformly bounded and Equicontinuous on all compact subsets, then it is normal.

- Here, **uniformly bounded** on a (compact) subset $K \subseteq U$ means there exists B such that $|f(z)| < B$ for all $f \in \mathcal{F}$ and $z \in K$. i.e. the same bound B applies to *all* functions in the family.

  Ex: Each function of the form $f_n(z) = n$ is bounded, but the family $\{f_n\}$ is not uniformly bounded.

- A function being **Equicontinuous** on K means that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|z_1 - z_2| < \delta$ then $|f(z_1) - f(z_2)| < \epsilon$ for all $z_1, z_2 \in K$.

  Ex: Suppose $\mathcal{F}$ is a family of functions on $[0, 1]$. If $\{f'(z)\}_{f \in \mathcal{F}}$ is uniformly bounded, then $\mathcal{F}$ is equicontinuous.

  Ex: $f_n(x) = x^n$ on $[0, 1]$ is *not* equicontinuous. We can see this by lettnig $x_1 = 1, x_2 = 1 - \delta$. Then,

  $$|f_n(x_1) - f_n(x_2)|$$

  [Complete this example later]

To prove the Arzela-Ascoli Theorem, the key idea we'll use is *diagonalization* (to arrange countably many conditions).

Principle of Diagonalization: Given countably many conditions on a sequence $\mathrm{cond}_1, \mathrm{cond}_2, \mathrm{cond}_3, \cdots$ which are inherited on subsequences and sequence $f_1, f_2, f_3, \cdots$ such that for all j any subsetequence has a further subsequence which condition cond_j. Then, there exists a subsequence $f_1^{(\infty)}, f_2^{(\infty)}, \cdots$ satisfying all cond_j.

Proof-ish: Suppose we have

$$\begin{array}{lllll} f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & \cdots & \text{satisfying } \mathrm{cond}_1 \\ f_1^{(2)} & f_2^{(2)} & f_3^{(2)} & \cdots & \text{satisfying } \mathrm{cond}_2 \\ f_1^{(3)} & f_2^{(3)} & f_3^{(3)} & \cdots & \text{satisfying } \mathrm{cond}_3 \\ \vdots & \vdots & \vdots & \ddots & \end{array}$$

Then, taking the diagonal gives us a subsequence of $\mathcal{F}$ satisfying all conditions cond_j.

[Write the rest from recording]