Introduction to Smooth Manifolds - Some solutions

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Abstract

This is a collection of my personal solutions to some exercises from the book 'Introduction to Smooth Manifolds' (2nd edition) by John M. Lee.

These solutions are being written up and I work through the book, however I do not write up every exercise/problem and leave many to be updated later.

This has been written solely to deepen my own understanding of the material, but please feel free to contact me with corrections, concerns, or comments at kdeoskar@berkeley.edu.

Contents

1 Chapter 1: Smooth Manifolds

1.1 Topological Manifolds

Exercise 1.6: Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof:

To show Hausdorffness, consider two distinct points $x, y \in \mathbb{RP}^n, x \neq y$. Then,

To show second-countability, we use the following lemma:

For a topological space X with countable basis $\mathcal{B}$

(a) Any of its subsets $Y \subseteq X$ has a countable basis as well ($\mathcal{B}_Y = \{U : U = B \cap Y, B \in \mathcal{B}\}$)

(b) If $Z = {}^{X}/_{\sim}$ is a quotient space of X by an equivalence relation $\sim$, then

$$\{[x]_\sim : x \in B \in \mathcal{B}\}$$

will be a countable base for the topology induced on Z

<u>**Proof of Lemma:**</u> Do later.

Applying (a) tells us that $\mathbb{R}^{n+1} \setminus \{0\}$ as a subspace of $\mathbb{R}^n$ has a countable basis. Then, applying (b) with $Z = \mathbb{RP}^n = {}^{\mathbb{R}^{n+1} \setminus \{0\}}/_{\sim}$ where $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$, tells us that $\mathbb{RP}^n$ too has a countable basis.

Exercise 1.7: Show that $\mathbb{RP}^n$ is compact.

Proof:

Exercise 1.:

Proof:

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2 Chapter 4: Submersions, Immersions, and Embeddings

2.1 Maps of Constant Rank

2.1.1 *Local Diffeomorphisms*

Exercise 4.9: Show that the conclusions of Proposition 4.8 still hold if N is a manifold with boundary but not if M is.

Proof (Sketch):

- It follows from the result of Problem 4-2 that if $p \in M$ such that dF_p is non-singular, then $F(p) \in \text{Int}(N)$, which is an manifold without boundary, so naturally the results still apply.
- If, however, M has non-empty boundary, then the Inverse Function Theorem for manifolds breaks down (See result of Problem 4-1).

Exercise 4.10: Suppose M, N, P are manifolds with or without boundary and $F : M \to N$ is a local diffeomorphism. Prove the following:

(a) If $G : P \to M$ is countinuous, G is smooth iff $F \circ G$ is smooth.

(b) If F is surjective and $G : N \to P$ is any map, then G is smooth iff $G \circ F$ is smooth.

Proof:

(a) Suppose G is smooth, then at each point $p \in M$, $F \circ G$ is the composition of a two smooth maps (local diffeomorphisms restrict to smooth maps around each point). Conversely if $F \circ G$ is smooth, let $p \in P$ and let $U \subseteq M$ be an open set containing $G(p)$ such that $F\big|_U : U \to V \subseteq N$ is a local diffeomorphsm. Since G is continuous, $G^{-1}(U) \subseteq P$ is open. So, of course, $F \circ G\big|_{G^{-1}(U)}$ restricts to a smooth map on $G^{-1}(U)$. Then, $G\big|_{G^{-1}(U)} : G^{-1}(U) \to U$ can be written as

$$G\big|_{G^{-1}(U)} = (F \circ G)\big|_{G^{-1}(U)} \circ F^{-1}\big|_{F(U)}$$

so $G\big|_{G^{-1}(U)}$ is a composition of smooth maps, and is thus smooth itself. We have smooth restrictions for each point $p \in P$ which we can glue together using the Gluing Lemma for smooth functions, to find that G is continuous.

(b) Suppose G smooth, then $G \circ F$ is the composition of smooth maps and is thus smooth. Conversely, suppose $G \circ F$ is smooth. F is a surjective local diffeomorphism, so for any point $n \in N$ we can write

$$G = (G \circ F) \circ F^{-1}$$

So, G is the composition of two smooth maps and is itself a smooth map.

Rank Theorem:

Proof: See https://math.colorado.edu/ macz9339/math6230/outline.pdf.

2.2 Embeddings

Exercise 4.16: Show that every composition of smooth embeddings is a smooth embedding.

Proof:

Consider manifolds without boundaries M, N, P and smooth embeddings $F : M \to N, G : N \to P$. Then, F and G are individually homeomorphisms onto their images, thus their composition $G \circ F$ is as well.

Also, since both F, G are smooth immersions, they are both smooth maps whose differentials are injective. So, their composition $G \circ F$ is smooth too, and its differential (given by the chain rule as) at a point $p \in M$

$$d\left(G \circ F\right)_p = dG_{F(p)} \circ dF_p$$

is the composition of two injective linear maps, thus it is also injective.

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3 Chapter 6: Sard's Theorem

Exercise 6.1:

Proof: [Write formal proof later]

Exercise 6.7: Let M be a smooth manifold with or without boundary. Show that a countable union of sets of measure zero in M has measure zero.

Proof: A subset $A \subseteq M$ has measure zero in M if for every smooth chart (U, ϕ) for M, the subset $\phi(U \cap A)$ has n-dimensional measure zero.

If we have a countable collection of measure zero sets $\{U_\alpha\}$ in M, each of them has the above property and for a given chart (U, ϕ), we have $\phi\left(U \cap \left(\bigcup_\alpha U_\alpha\right)\right) = \bigcup_\alpha U \cap U_\alpha$.

This set has $n-$dimensional measure zero since it is a countable union of sets with $n-$dimensional measure zero. [Prove this later maybe]

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4 Chapter 7: Lie Groups

5 Chapter 8: Vector Fields

Exercise 8.9:

Proof:

(a) Since $f, g \in C^\infty(M)$, fX, gY are defined by

$$(fX)_p = f(p)X_p, \qquad (gY)_p = g(p)Y_p$$

So $fX + gY$ is defined at each point $p \in M$ as

$$(fX + gY)_p = f(p)X_p + g(p)Y_p$$

and this is again a smooth vector field since $f(p), g(p) \in \mathbb{R}$. [Come back to this].

(b) We've already shown that $\mathfrak{X}(M)$ is closed under addition with the ring $C^\infty(M)$. We also showed Multiplication because if $X \in \mathfrak{X}(M)$ then for any $f \in C^\infty(M)$ then (fX) is defined at each point $p \in M$ as

$$f(p)X_p$$

This is a smooth function because if we choose any chart $\big(U, (x^1, x^2, \cdots, x^n)\big)$ for M and chart $\big(\pi^{-1}(U), \big(x^1, \cdots, x^n, v^1, \cdots, v^n\big)\big)$ for T_pU then the function $f(p)X(p)$ can be written in coordinates as

$$\big(f(p)X_p^1, \cdots, f(p)X_p^n\big)$$

where each X_p^i is a smooth function and so is $f(p)$. The product of two smooth functions is again smooth so every component function of $f(p)X_p$ is smooth.

Problem 8-5: Prove Proposition 8.11 (Completion of Local Frames).

Proof:

(a) Suppose $(X_1, \cdots, X_k)$ is a linearly independent $k-$tuple of smooth vector fields defined over an open subset $U \subseteq M$. At each $p \in U$, $\{X_1\big|_p, \cdots, X_k\big|_p\}$ is a set of k linearly independent vectors in T_pM. Choose $v_{k+1}, \cdots, v_n$ so that $\{X_1\big|_p, \cdots, X_k\big|_p, v_{k+1}, \cdots, v_n\}$ form a basis for T_pM.

Then, in a neighborhood V of p, we can extend $v_{k+1}, \cdots, v_n$ to constant vector fields $\{X_{k+1}, \cdots, X_n\}$ (which are then smooth). So, in the neighborhood V, $\{X_1\big|_p, \cdots, X_k\big|_p, X_{k+1}\big|_p, \cdots, X_n\big|_p\}$ forms a basis for T_pM. [Complete this]
