Lecturer: Richard Borchers

Math 250A: Groups, Rings, and Fields

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These are some lecture notes taken from the Fall 2024 semester offering of Math 250A: Groups, Rings, and Fields taught by Richard Borcherds at UC Berkeley. The primary reference is [1]. This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 August 29, 2024:

1.1 Logistics:

First third of the course will be groups, next third will be rings, and we'll conclude with fields.

1.2 Groups

Definition 1.1. *(Group, Concrete): A Group os a* ***symmetry*** *of some math object.*

where by "symmetry" we mean a bijection $X \to X$ "preserving the structure of the object". This is all a bit vague, but we get the idea.

Definition 1.2. *The* ***Order*** *of a group is the number of elements.*

Definition 1.3. *(Group, Abstract): A set X along with a binary operation $\cdot : X \times X \to X$, Identity map $\mathbb{I}$, and inverse map $X \to X$ such that*

(a) A

(b) $aa^{-1} = a^{-1}a = 1$

(c) $(ab)c = a(bc)$

fill this in.

How do we show the equivalence of these two definitions? one direction (symmetry $\implies$ axiomatic definition) is obvious. **(Elaborate.)** But can we go the other way around?

1.3 Cayley Graphs

Problem: Given an abstract group G, find an object X where the symmetries are *isomorphic* to G.

The object X will turns out to be a **Cayley Graph** - which is a colored, directed graph whose points correspond to the elements of the abstract group G. So, the set of points S of the graph is really just the same as G.

What about the arrows? These will represent **left-actions of G on S**.

A **left-action** of a group G on a set S is a map $G \times S \to S$ such that

(a) $(g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$

(b) $1 \cdot s = s$

In the Cayley Graph, we consider the left-action

$$g(s) = g \underbrace{\cdot}_{\text{mult. in the grp.}} s$$

We notice that the Action is **Faithful** i.e. if $g(s) = s$ for all $s \in S$ then $g = 1$.

So, $\underline{G \subseteq \text{all symmetries of the set } S}$. What we've shown is a weak version of **Cayley's Theorem.**

To complete our goal we have to tackle the following:

Problem: Add extra structure to S in order to cut down the symmetry group to that of G.

The extra structure we'll be adding is a **right-action** of G on S defined by

$$s \cdot g = sg$$

We check that the left-action preserves structure. From the associative law of G, we have

$$(gs)h = g(sh)$$

i.e. left-actions **commutes** with right-actions.

Note that left-actions **do NOT** commute with left-actions as $g(hs) \neq h(gs)$.

So, finally, we define the **Cayley Graph** by the following rule:
For each $g \in G$, for a point $s \in S$, draw a lie from s to sg and demarcate it with the *color* g.

Eg. Cayley Graph of the Klein 4-group

Draw the cayley table and cayleg graph.

What we've shown above is that Abstract Groups are the same as Symmetries.

In showing this result we used the notion of group actions. A reasonable question to ask would be "What kind of actions does a group have on *itself*?"

1.4 Actions of G on itself

There are 8 ways in which a group G naturally acts on itself.

Include picture.

Note. *The adjoint action will come up again in a lecture or two.*

1.5 Goals

Okay, so we now have two definitions for groups. The ultimate goal would then be to:

(a) Classify all groups.

(b) Find all representations of a group.

It turns out this is far too difficult to be done! So, we restrict to all **simple groups**, and find that again the problem is too difficult!
But if we decide to classify all **abelian simple groups**, then we can get somewhere. There are other types of Groups that have been successfully classified, and we'll discuss some of them in this course.

1.6 What is a representation?

Write this soon.

1.7 An attempt at the impossible: Catalog of All Groups

Let's begin trying to classify groups according to their **orders**... though of course we'll eventually stop when it becomes too difficult to continue.

Our initial observations are kind of boring.

$$\begin{aligned} &\text{Order 1:} && \text{TRIVIAL} \\ &\text{Order 2:} && \mathbb{Z}/2\mathbb{Z} \\ &\text{Order 3:} && \mathbb{Z}/3\mathbb{Z} \end{aligned}$$

In fact, for any prime p, we can show that there exists precisely ***one*** group of order p up to isomorphism (consistent with our findings for $p = 2, 3$). The way to do this is via **Lagrange's Theorem**.

References

[1] Serge Lang. *Algebra, Revised Third Edition.* Springer Science+Business Media New York, 2002.