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Physics c191: Introduction to Quantum Computing

Homework 1

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Question 1:

We want to show that a single cubit pure state can be described by only two real parameters.

An arbitrary single cubit pure state is an element of $\mathcal{H} = \mathbb{C}^2$. We can write it as a linear combination of the basis states $|0\rangle, |1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Writing the complex coefficients in polar form, $\alpha = r_1 e^{i\phi_1},\ \beta = r_2 e^{i\phi_2}$ we can write

$$\begin{aligned} |\psi\rangle &= r_1 e^{i\phi}|0\rangle + r_2 e^{i\phi}|1\rangle \\ &= e^{i\phi_1}\left[r_1|0\rangle + r_2 e^{i(\phi_2-\theta_1)}|1\rangle\right] \end{aligned}$$

Physically, any two states related by a global phase are equivalent, because only the amplitude squared matter when it comes to measurements and $|e^{i\phi}|^2 = 1$ for any angle ϕ. Thus, we can ignore the global phase ϕ_1 in our expression for $|\psi\rangle$.

We also have the normalization condition

$$\begin{aligned} &|r_1|^2 + |r_2 e^{i(\phi_2-\phi_1)}|^2 \\ \implies &r_1^2 + r_2^2 = 1 \end{aligned}$$

Thus, an arbitrary single cubit pure state (up to global phase) is completely described by the parameters r_1, r_2 such that $r_1^2 + r_2^2 = 1$ i.e. $(r_1, r_2) \in \mathbb{S}^1$ $\phi{:}= (\phi_2 - \phi_1) \in [0, 2\pi]$.

But since (r_1, r_2) must be a point on the unit circle, we can instead just use the angle corresponding to the point (r_1, r_2) given by $\phi = \arctan(r_2/r_1) \in [-\pi/2, \pi/2]$.

So, our single cubit pure state is completely described by two angles, $\phi \in [0, 2\pi]$ and $\theta \in [-\pi/2, \pi/2]$ i.e. states correspond to points on the unit sphere $\mathbb{S}^2$.

Equivalently, working with $\theta \in [0, \pi]$ (same interval as earlier), we can imagine using the usual spherical coordinates to describe points on $\mathbb{S}^2$. Taking $|0\rangle$ and $|1\rangle$ to be the North and South poles respectively i.e.

$$|0\rangle = \begin{pmatrix} x = 0 \\ y = 0 \\ z = 1 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} x = 0 \\ y = 0 \\ z = -1 \end{pmatrix}$$

we can write an arbitrary state $|\psi\rangle$ as a linear point on the sphere.
(Finish this.)
Thus a general single cubit pure state can be written as

$$|\psi\rangle = e^{i\gamma}\left[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right]$$

Question 1

1. We want to find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices I, X, Y, Z and show that $X^2 = Y^2 = Z^2 = I$