Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 January 16 - Topology review, topological manifolds

This class is on Differential Topology, so the objects we wll study are *smooth manifolds*. Manifolds are topological spaces which are locally euclidean, and endowing a smooth structure onto a manfilod allows us to conduct some sort of *calculus on the manifold*. For instance, using Smoothness we will define actions such as differentiation, integration and objects such as vector fields and their flows.

Some examples of manifolds are

- Submanfolds
- Lie groups (these show up a lot in physics)
- Quotient manifolds

In addition to the global structure of manifolds, we will also study some local structures such as Remannian metrics and curvature, differential forms, vector/tensor bundles, and de Rham cohomology.

1.1 Topological Spaces

Recall,

Topological Space

A topological space is a pair (X, O_X) where $O_X \subseteq \mathcal{P}(X)$ satisfying

- $\emptyset, X \in O_X$
- Finite intersections are also in O_X i.e. for each $U_\alpha \in O_X$, the intersection $\bigcap_{\alpha \in A, |A| < \infty} U_\alpha \in O_X$
- Arbitrary unions are also in O_X i.e. for each $U_\alpha \in O_X$, the union $\bigcup_{\alpha \in A} U_\alpha \in O_X$

We say that $A \subseteq X$ is open if $A \in O_X$ $B \subseteq X$ is closed if $X \setminus B \in O_X$, $U \subseteq X$ is a neighborhood of $p \in X$ iff $p \in U, U \in O_X$.

Ex: Any metrci space (X, d) has a topology induced by its metric defined as (X, O_X) and for $A \subseteq X$, $A \in O_X$ iff for all $p \in A$ there exists $r > 0$ such that $p \in B_r(p) \subseteq A$.

Basis of a Topology

Given (X, O_X), we say $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for the topology if

$$A \in O_X \;\; iff \;\; \forall p \in A, \exists B \in \mathcal{B} \; st \; p \in B \subseteq A$$

Ex: The set of open ball with rational radii is a countable basis for the usual topology on $\mathbb{R}^n$.

1.2 Continuous maps between Topological Spaces

Continuous Map

Given topological spaces (X, O_X) and (Y, O_Y), the map $\phi : X \to Y$ is continuous if for every open $B \subseteq Y$ the pre-image $\phi^{-1}(B) \subseteq X$ is open.

Homeomorphism

$\phi : X \to Y$ s a Homeomorphism if ϕ and ϕ^{-1} are both continuous.

Ex: The map from $[0, 2\pi)$ to the circule is continuous but its inverse is not, so the two spaces are not Homeomorphic.

1.3 Subspace Topology

Subspace Topology

Gven a topological space (X, O_X), a subset $Y \subseteq X$ can be endowed with the subspace topology defined as

$$O_Y = \{A \cap Y : A \in O_X\}$$

1.4 Compactness

Compactness

- An **open cover** of X is a collection of open sets U_α such that $X \subseteq \bigcap_{\alpha \in A} U_\alpha$.
- A subset $K \subseteq X$ of topological space X is compact if every open cover $\{U_\alpha\}_\alpha$ has a **finite** subcover.

Add some more intuition regarding this later.

Hausdorff

A topological space X is Hausdorff if for any ponits $p, q \in X$ there exist open sets U and V such that $p \in U, q \in V$ and $U \cap V \emptyset$.

Insert figure later.

1.5 Topological Manifolds

The following statement seems innoccuous enough, but it requires the heavy machinery of de Rham Cohomology to prove:

Lemma: (Topological Invariance of Dimension)
If $\phi : U \to^{\text{homeo}} Y$ where $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ then $n = m$.

Def: A Topological Space X is locally eucldiean of dimension n at $p \in X$ if there exists an open set $U \subseteq X$ such that $p \in U \subseteq X$ is Homeomorphic to some open $\tilde{U} \subseteq \mathbb{R}^n$.
(Insert figure later, for instance of homeo. between sphere and open set in $\mathbb{R}^2$)

Exercise: Can requrie $\tilde{U} = B_1(0) \subseteq \mathbb{R}^n$

Lemma: The Dmiension n in the defintiion above is uniquely determined by $p \in X$.
(Insert figure later and write up proof) Basically, compose homeomorphisms ϕ_1^{-1}, ϕ_2 then use Invariance of Dimension to show $m = n$.

Topological Manifold

A Topological Space M is an n-dimensional topological manifold if it is

- Hausdorff.
- Second-countable (has a countable basis for its topology).
- Locall Euclidean of dimension n at all points.

For ex. $\mathbb{R}^n$ is an n-dimensional manifold.

Some non-examples of manifolds:

- Not Hausdorff: $X = \mathbb{R} \times \{0,1\} \sim$ i.e. two copies of R with $(X,0) \sim (X,1)$ if $x < 0$ and the topology induced by
$$\pi : \mathbb{R} \times \{0,1\} \to X$$
and $A \subseteq X$ iff $\pi^{-1}(A) \subseteq \mathbb{R} \times \{0,1\}$ is open. (This is a standard example – include a figure later.)
- Not Locally Euclidean: Same as before, but $(X,0) \sim (X,1)$ if $x \leq 0$. We have a problem at $[(0,0)] = [(0,1)]$.
- Not Second Countable: S uncountable and having discrete topology, then the space $S \times \mathbb{R}$ is not a manifold because it doesn't have a countable basis.
- Not Second Countable: "Long Line"
<u>Claim:</u> There exists an uncountable, well ordered set S such that the maximal element $\Omega \in S$ satsfies for all $\alpha \in S, \alpha \neq \Omega$, $\{x \in S : x < \alpha\}$ is countable.

  Consider the set $X = {(S \times [0,1])} \Big/ {\{\alpha_0\} \times \{0\}}$ where α_0 is minimal in S.

  We order lexicographically i.e.
$$(\alpha, s) < (\tilde{\alpha}, \tilde{s}) \text{ if } \alpha < \tilde{\alpha} \text{ or } \alpha = \tilde{\alpha}, s < \tilde{s}$$

  And we endow the long line with the *order topology* to be generated by the following basis

  This space is both Hausdorff and Locally Eucldean, but *not second countable.* complete this one later.

<u>Some examples of Topological Manfifolds:</u>

- The unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$
  - We can cover the circle by maps $\phi^+ : U_i^+ \to (-1,1)$ defined by $(x_1, x_2) \mapsto x_2$ with inverse $(\phi^+)^{-1} : (-1,1) \to U_i^+$ given by $(x_1, x_2) \mapsto \sqrt{arg}$ [Finish writing this later]

2 January 18 - Topological Properties of Manifolds

Connectivity

Def: A topologcial space X is connected if $\emptyset, X$ are the only two subsets of X which are both open and closed.

Path-connectedness: If for any two points $p, q \in X$ there exsits a path i.e. a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = p, \gamma(1) = q$ then X is path-connected.