Math H185 Lecture 14

Keshav Balwant Deoskar

February 16, 2024

These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 February 16 - The Logarithm

Today we'll be discussing the logarithm. Before getting to it though, lte's revisit the exponential function, which we defined pretty early in the course.

1.1 Exponential Function

The exponential function

$$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \cdots + \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

The series expansion has infinite radius of convergence, making the exponential a holomorphic function, and further allowing us to obtain the derivative by just differentiating the series term by term.

$$\begin{aligned} \implies \exp'(z) &= \sum_{n=1}^{\infty} \frac{1}{n!} n z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \exp(z) \end{aligned}$$

Note: There exists a power series f with radius of convergence ∞ such that

$$f'(z) = f(z), \quad f(0) = 1$$

This property actually haracterizes the exponential.

1.2 Range of the Exponential Function

[Complete by watching recording]

So, it turns out that every complex number other than the origin is a target of the exponential map.

In showing this, we used the property $e^{a+b} = e^a e^b$. Strictly speaking, we must prove this holds for complex numbers. Let's do so quickly.

Lemma: For $z_1, z_2 \in \mathbb{C}$ we have

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

Proof: If we fix z_1 and view the exponential as a function of z_2, we have

$$\frac{\partial}{\partial z_2} \left(e^{z_1+z_2} \right) = e^{z_1+z_2}$$

and similarly if we fix z_2 and view it as a function of z_1, we have

$$\frac{\partial}{\partial z_1} \left(e^{z_1+z_2} \right) = e^{z_1+z_2}$$

[Finish this proof later]

Corollary: $e^z \neq 0$ since for any e^z there exists an inverse e^{-z} such that $e^z e^{-z} = 1$.

1.3 The Logarithm Function

We now want to define the logarithm, $\log(z)$, such that

- $e^{\log(z)} = z$
- Logarithm is continuous.

But we have a problem! If we start at $\theta = 0$, rotate around, and eventualy hit $\theta = 2\pi$ which is the same point as $\theta = 0$, we will have a discontinuity in the neighborhood of $\theta = 0$ since the function will be small if we move coounterclockwise but large if we move clockwise.

So, let's tackle the issue using the following theorem:

Theorem: Let $U \subseteq_{open} \mathbb{C}$ be simply connected such that $0 \neq U, 1 \neq U$. Then, there exists a unique function

$$\log_U : U \to \mathbb{C}$$

such that

- it is holomorphic
- $\log_U(1) = 0$
- $e^{\log_U(z)} = z$ for all $z \in U$.

Proof: Define $\log_U(z)$ to be the primitive of $\frac{1}{z}$ i.e.

$$\log_U(z) = \int_1^z \frac{1}{w} dw$$

This is well define since U is simply connected and certaintly satisfies the condition $\log_U(1) = 0$.

To check the last condition, let's verify that

$$\frac{d}{dz} \log_U(z) = \frac{1}{z}$$

[complete after lecture]

To define the logarithm, we can choose a curve containing the origin and consider its *complement* since the complement is guaranteed to be simply connected. The curve we choose is called a ***branch cut***.

Examples of different branch cuts

- ***Principal Branch cut:*** $U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Insert picture

  This is kind of like the "canonical" branch cut since, even for real numbers, we usually don't bother defining the logarithm for negative numbers.

Midterm Monday March 4

- 10 true/false questions
- 3-4 calculations
- No proof on the first midterm.
- A sample midterm will be released.