Knot 176: Introduction to Low-Dimensional Topology **Fall 2023**

Homework 1: Due Date: September 08

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Metric Spaces

Q1.1.1: Prove that for any metric space (Z, d_Z), a function between the metric spaces $f : (\mathbb{R}^n, d_E) \to (Z, d_Z)$ is continuous if and only if $f : (\mathbb{R}^n, d_T) \to (Z, d_Z)$ is continuous.

Proof: Recall that a function between two metric spaces $f : (X, d_X) \to (Y, d_Y)$ is said to be continuous if for any open set $V \subseteq Y$, the pre-image $f^{-1}(V)$ is open in X.

<u>Forward Direction:</u> Suppose we have a continuous function $f : (\mathbb{R}^n, d_E) \to (Z, d_Z)$. That means, for any open set $V \subset Z$ (wrt d_Z), the pre-image $f^{-1}(V) \subset \mathbb{R}^n$ is open with respect to the Euclidean Metric.

That is, for any point $p \in f^{-1}(V)$, there exists an open ball $B(p, r_p)$ centered around p with radius r_p such that $B(p, r_p) \subset f^{-1}(V)$ where

$$B(p, r_p) = \left\{ y \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (p_i - y_i)^2} < r_p \right\}$$

We have a set of points y for which

$$\begin{aligned} \sqrt{\sum_{i=1}^{n} (p_i - y_i)^2} &< r_p \\ \implies \sum_{i=1}^{n} (p_i - y_i)^2 &< r_p^2 \end{aligned}$$

But notice that $(p_i - y_i)^2 = |p_i - y_i|^2$. So,

$$\sum_{i=1}^{n} |p_i - y_i|^2 < r_p^2$$

But again, notice that $|p_i - y_i| \leq |p_i - y_i|^2$. So,

$$\boxed{\sum_{i=1}^{n} |p_i - y_i| < r_p^2}$$

So, around any point $p \in f^{-1}(V)$, if there exists an open ball $B_E(p, r_p)$ then there also exists an open ball $B_T(p, r_p^2)$ – which means that if a set $f^{-1}(V)$ is open with respect to the Euclidean Metric, it is also open with respect to the Taxicab Metric.

Thus, if $f : (\mathbb{R}^n, d_E) \to (Z, d_Z)$ is continuous, then $f : (\mathbb{R}^n, d_T) \to (Z, d_Z)$ is also continuous.

Reverse Direction: Now suppose we have a continuous function $f : (\mathbb{R}^n, d_T) \to (Z, d_Z)$. That means, for any open set $V \subset Z$ (wrt d_Z), the pre-image $f^{-1}(V) \subset \mathbb{R}^n$ is open with respect to the Taxicab Metric.

That is, for any point $q \in f^{-1}(V)$, there exists an open ball $B(q, r_q)$ centered around q with radius r_q such that $B(q, r_q) \subset f^{-1}(V)$ where

$$B(q, r_q) = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} |q_i - y_i| < r_q \right\}$$

So we have a set of points y for which

$$\begin{aligned}
&\sum_{i=1}^{n} |q_i - y_i| < r_q \\
\implies &\left(\sum_{i=1}^{n} |q_i - y_i| \right)^2 < r_q^2 \\
\implies &(|q_1 - y_1| + \cdots + |q_n - y_n|)^2 < r_q^2 \\
\implies &\sum_{i=1}^{n} |q_i - y_i|^2 + \sum_{i,j<n} |q_i - y_i| \cdot |q_j - y_j| < r_q^2 \\
\implies &\sum_{i=1}^{n} |q_i - y_i|^2 < r_q^2
\end{aligned}$$

But once again, $|q_i - y_i|^2 = (q_i - y_i)^2$. So,

$$\boxed{\sum_{i=1}^{n} (q_i - y_i)^2 < r_q^2}$$

Therefore, for any point $q \in f^{-1}(V)$, if there is an open ball with respect to the Taxicab metric $B_T(q, r_q)$ there is also an open ball with respect to the Euclidean metric $B_E(q, r_q^2)$. That is, if a set is open in the Taxicab Metric it is also open in the Euclidean Metric.

As a result, if a function is continuous with respect to the Taxicab metric, it is also continuous with respect to the Euclidean Metric.

Q1.1.2: Consider the following definition of continuity: a map between metric spaces $f : X \to Y$ is continuous if for every $x \in X$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Prove that a map is continuous in this sense if and only if for every open set $V \subset Y$ we have $f^{-1}(V)$ is open in X.

Proof:

"Forward" direction: Suppose we have a function which is continuous in the $\epsilon - \delta$ sense. Now, suppose for contradiction that we have an open set $V \subset Y$ such that $f^{-1}(V)$ is not open in X.

That means, there is some point $f(x_0) \in V$ whose pre-image $x_0 \in f^{-1}(V)$ does not have an open ball around it which is contained in $f^{-1}(V)$. That is, there is no δ such that $B(x_0, \delta) \subset f^{-1}(V)$, so there is no δ such that $f(B(x_0, \delta)) \subset V$, which means that there exist $\epsilon > 0$ for which there are no δ such that $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$ – but this contradicts our assumption! So, the $\epsilon - \delta$ definition of continuity implies the Open Pre-image definition of continuity.

"Reverse" direction: Suppose the function $f : X \to Y$ is continuous in the sense that for every open set $V \in Y$, the pre-image $f^{-1}(V)$ is open in X.

Now, let's fix a point $x \in X$ and some $\epsilon > 0$. Then, $B_Y(f(x_0), \epsilon)$ – the epsilon ball around $f(x_0)$ – is an open subset of Y.

Since f is continuous, if $B_Y(f(x_0), \epsilon)$ is open in Y then $f^{-1}(B_Y(f(x_0), \epsilon))$ is open in X.

Now, we also know that $x_0 \in f^{-1}(B_Y(f(x_0), \epsilon))$ and since $f^{-1}(B_Y(f(x_0), \epsilon))$ is open – so by the definition of an open set in a metric space, there exists an open ball with some radius $\delta > 0$ containing x such that $B_X(x, \delta) \subset f^{-1}(B_Y(f(x_0), \epsilon))$

That is,

$$\boxed{f(B_X(x, \delta)) \subset B_Y(f(x_0), \epsilon)}$$

So, the open-set definition of continuity implies the $\epsilon - \delta$ definition of continuity for maps between metric spaces.

Topologies

Q1.2.1: Let $\{\tau_i\}_{i \in I}$ be a collection of topologies on X. Show that $\cap_{i \in I} \tau_i$ is a topology on X.

Proof: In order to show that $\mathcal{O} = \cap_{i \in I} \tau_i$ is a topology on X, we need to show that

(a) Arbitrary unions of sets in $\mathcal{O}$ are also in $\mathcal{O}$.

(b) Finite intersections of sets in $\mathcal{O}$ are in $\mathcal{O}$.

(c) X and $\emptyset$ are in $\mathcal{O}$.

Let's show that these conditions are satisfied:

(a) The collection $\mathcal{O}$ consists of all the sets that are common to all of $\{\tau_i\}_{i \in I}$. Consider an arbitrary union of open sets $A_j \in \mathcal{O}$.

Now, each of these sets A_j is open in every one of τ_i, the union $\cup_j A_j$ is also open in each one of τ_i (since each τ_i is itself a topology).

Therefore, the union $\cup_j A_j$ is an open set which is common to all of τ_i. Therefore,

$$\bigcup_j A_j \in \mathcal{O}$$

(b) Consider a finite intersection of open sets in the intersection $V = A_1 \cap A_2 \cap \cdots \cap A_n$, where $A_j \in \mathcal{O}$ i.e. each of A_j is common to every τ_i. Then, since each τ_i is a topology and A_j are open sets in each τ_i, their finite intersection V is also an open set in each τ_i. Now, since $V \in \tau_i$ for all $i \in I$, we have

$$V \in \bigcap_{i \in I} \tau_i = \mathcal{O}$$

(c) The empty set is, by definition, an element of every set so

$$\emptyset \in \mathcal{O}$$

And since every τ_i is a topology, we have $X \in \tau_i$ for all $i \in I$. Thus,

$$X \in \bigcap_{i \in I} \tau_i = \mathcal{O}$$

Hence, the intersection of a family of topologies on X is itself a topology on X.