Math 214 Homework 6

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Q5-22. Prove Theorem 5.48 (existence of defining functions for regular domains).

> ***Theorem 5.48:*** If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.

Proof: We know $M = \text{Int}(D) + \partial D + (M \setminus D)$. By the boundary slice condition, at every point of $\partial D/\text{Int}D/(M \setminus D)$ (resp.) we can find a chart (U, ϕ) such that the n^{th} coordinate of ϕ is zero/negative/positive (resp.).

We can take our charts to be such that $\phi(c_1, \ldots, c_{n-1}, x_n)$ is an increasing function of x_n for each point $(c_1, \cdots, c_{n-1})$. Morevover, we can assume that $\frac{\partial}{\partial x_n}\phi > 0$ at every point, since if it is not then we can replace ϕ with $\phi + x_n$.

Now, cover Int$M$ with charts whose n-th coordinate is negative and $(M \setminus D)$ with charts whose n-th coordinate is positive. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ be the collection of all charts mentioned above, and let $\{\psi\alpha\}_\alpha$ be a partition of unity subordinate to the cover by charts.

Define the function $f : M \to \mathbb{R}$ by

$$f(x) = \sum_\alpha (x_n \circ \phi_\alpha) \cdot \psi_\alpha$$

Then 0 is a regular value of f by the derivative condition and $f^{-1}([-\infty, 0]) = D$. Thus, f os a defining function for D.

For the case where D is compact, we take the same charts as above however we also assume that they are all precompact, that the cover is countable, and that there are only finitely many charts intersecting D. (Paracompact?)

Let $\{(U_{-t}, \phi_{-t}), \ldots, (U_0, \phi_0)\}$ be the charts intersecting D, and let $\{(U_j, \phi_j)\}_{j \in \mathbb{N}}$ be the charts in $(M \setminus D)$. Let $\{\psi_{-t}, \ldots, \psi_0\} \cup \{\psi_j\}_{j \in \mathbb{N}}$ be a partition of unity subordinate to the cover. Define

$$g(x) = \sum_{j=-t}^{0} (x_n \circ \psi_j) \cdot \psi_j + \sum_{j=1}^{\infty} j\psi_j$$

Then, g is a defining function for D, and $g^{-1}([-\infty, 0])$ is compact for any $c \in \mathbb{R}$ by precompactness of the charts U_j and compactness of D.

Q6-5. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Show that M has a tubular neighborhood U with the following property: for each $y \in U$, $r(y)$ is the unique point in M closest to y, where $r : U \to M$ is the retraction defined in Proposition 6.25.

Proof: We do so in two major steps:

1. Show that if $y \in \mathbb{R}^n$ has a closest point $x \in M$, then $(y - x) \perp T_x M$.
2. Then, for each $x \in M$, show that it is possible to choose $\delta > 0$ such that every $y \in E(V_\delta(x))$ has a closest point in M, and that point is equal to $r(y)$.

Fix a point $y \in \mathbb{R}^n$. Let's begin with step 1. If we define the function $f : \mathbb{R}^n \to \mathbb{R}$ to be distance from y i.e.

$$f(x) = |y - x| = \sqrt{\sum_{i=1}^{n} (y^i - x^i)}$$

the directional derivative of f in the direction v is zero for any $v \in T_x M$.

The gradient of $f(x)$ of course points along the direction $(y - x)$ that's the direction of fastest change. Therefore, for $v \in T_x M$ we have

$$\begin{aligned} &\nabla f \cdot v = 0 \\ \implies &c(y - x) \cdot v = 0 \text{ where } c \text{ is some constant} \\ \implies &(y - x) \cdot v = 0 \end{aligned}$$

Thus, if there is a closest point $x \in M$ to $y \in \mathbb{R}^n$, the vector $(y - x) \perp T_x M$.

For step 2, let U be the tubular neighborhood of M defined via $\rho : M \to \mathbb{R}$ in the proof of Theorem 6.24. Then, define $\tilde{\rho} = \frac{1}{2}\rho$ and let it define another tubular neighborhood $\tilde{U}$.

Suppose there is a point $x \in M$ and $y \in \tilde{U}$ such that $|x - y| < |x - r(y)|$ and x is the closest point in M to y. Then,

$$|x - y| \le |x - y| + |y - r(y)| < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$$

giving that $y \in V_\delta(x)$. But $r(y)$ is the unique point of $M \cap V_\delta(x)$ such that $(x - p)$ is orthogonal to $T_x M$. Thus, $r(y) = x$ is the unique closest point in M to y.

Q6-6. Suppose $M \subseteq \mathbb{R}^n$ is a compact embedded submanifold. For any $\epsilon > 0$, let M_ϵ be the set of points in $\mathbb{R}^n$ whose distance from M is less than ϵ. SHow that for sufficiently small ϵ, ∂M_ϵ is a compact embedded hypersurface in $\mathbb{R}^n$, and $\overline{M}_\epsilon$ is a compact regular domain in $\mathbb{R}^n$ whose interior contains M.

Proof:
We know by the Tubular Neighborhood Theorem that M is guaranteed to have a tubular neighborhood U. Suppose $\delta : M \to \mathbb{R}$ is the positive continuous function on M defining U by $U = E(V)$ where $V \subseteq NM$ and $E : NM \to \mathbb{R}^n$ are defined by

$$V = \{(x, v) \in NM \ : \ |v| < \delta(x)\} \tag{1}$$
$$E(x, v) = x + v \tag{2}$$

Since M is compact, δ achieves a maximum value $\epsilon > 0$. Then, we can just take $\delta(x)$ to be the constant function ϵ. Doing so gives us $V_\epsilon = \{(x, v) \in NM \ : \ |v| < \epsilon\}$. Thus, $U = E(V) = \{x + v : x \in M, |v| < \epsilon\} = M_\epsilon$

It follows that ∂M_ϵ is the image of the subset

$$\partial V_\epsilon = \{(x, v) \in NM : |v| = \epsilon\}$$

under the map E.

Now, ∂V_ϵ is clearly an embedded submanifold in V_ϵ and E is a diffeomorphism from V_ϵ onto U. Thus, it follows that ∂M_ϵ is an embedded submanifold in U (and hence in $\mathbb{R}^n$). ∂M_ϵ is a closed subset of $\mathbb{R}^n$ since it's a boundary, and it is bounded since M_ϵ is bounded. Therefore, ∂M_ϵ is a compact hypersurface in $\mathbb{R}^n$.

For the last part, note that the closure $\overline{M}_\epsilon$ is a proper submanifold since it is of codimension and is defined as a subset of $\mathbb{R}^n$. Its interior is M_ϵ, which contains M. Therefore, $\overline{M}_\epsilon$ is a compact regular domain in $\mathbb{R}^n$ whose interior contains M.

Q6-9. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $F(x,y) = (e^y \cos(x), e^y \sin(x), e^{-y})$. For which positive numbers r is F transverse to the sphere $S_r(0) \subseteq \mathbb{R}^3$? For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?

Proof:

A smooth map $F : N \to M$ between smooth manifolds is said to be transverse to an embedded submanifold $S \subseteq M$ if at every point $x \in F^{-1}(S)$ we have

$$dF_x(T_xN) + T_{F(x)}S = T_{F(x)}M$$

The differential of F at point $(x,y) \in \mathbb{R}^2$ is

$$dF_{(x,y)} = \begin{pmatrix} -e^y \sin(x) & e^y \cos(x) \\ e^y \cos(x) & e^y \sin(x) \\ 0 & -e^{-y} \end{pmatrix}$$

At a point $x \in S_r(0)$, the preimage $F^{-1}(x)$ is a point (x,y) satisfying

$$\begin{aligned} &\left(e^y \cos(x)\right)^2 + \left(e^y \sin(x)\right)^2 + \left(e^{-y}\right)^2 = r^2 \\ \implies &e^{2y} + e^{-2y} = r^2 \end{aligned}$$

So, the map will *not* intersect transversely with $S_r(0)$ only at a point (x,y) such that

$$\sqrt{(e^{2y} + e^{-2y})} = r$$

and the "vectors"

$$\partial_x F = (-e^y \sin(x), e^y \cos(x), 0) \text{ and } \partial_y F = \left(e^y \cos(x), e^y \sin(x), -e^{-y}\right)$$

are tangent to the sphere at the point $F(x,y)$ because then $dF_x(T_xN)$ and $T_{F(x)}S$ "coincide" and do not span the entire $T_{F(x)}M$ tangent space.

The condition that $\partial_x F$ and $\partial_y F$ be tangent to the sphere at $F(x,y)$ is expressed as the following orthogonality conditions:

$$\begin{aligned} \partial_x F \cdot F(x,y) = 0 \\ \partial_y F \cdot F(x,y) = 0 \end{aligned}$$

where $F(x,y)$ is viewed as the vector from the origin to the point $F(x,y)$.

Substituting in the coordinates, the first condition is

$$\begin{aligned} &(-e^y \sin(x), e^y \cos(x), 0) \cdot \left(e^y \cos(x), e^y \sin(x), e^{-y}\right) = 0 \\ \implies &- e^{2y} \sin(x)\cos(x) + e^{2y} \cos(x)\sin(x) + 0 = 0 \end{aligned}$$

This is always true.

and the second is equivalent to

$$\begin{aligned}
&\left(e^y\cos(x), e^y\sin(x), -e^{-y}\right)\cdot\left(e^y\cos(x), e^y\sin(x), e^{-y}\right)=0\\
\implies& e^{2y}\cos^2(x)+e^{2y}\sin^2(x)-e^{-2y}=0\\
\implies& e^{2y}-e^{-2y}=0\\
\implies& e^{2y}=e^{-2y}\\
\implies& e^{4y}=1\\
\implies& y=0
\end{aligned}$$

So, we have $\sqrt{e^{2\cdot(0)}+e^{-2\cdot(0)}}=r \implies r=\sqrt{2}$. Therefore, F is transverse to $S_r(0)$ for all positive r other than $r=\sqrt{2}$.

For which r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$? Theorem 6.30 tells us that as long as $F: N \to M$ is transverse to S, the pre-image $F^{-1}(S)$ is an embedded submanifold of N, so $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$ for all positive $r \neq \sqrt{2}$.

Q6-10. Suppose $F: N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subseteq M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}\left(T_{F(p)}X\right)$. Conclude that if two embedded submanifolds $X, X' \subseteq M$ intersect transversely, then $T_p(X\cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$.

Proof: We have smooth map $F: N \to M$ transverse to embedded submanifold $X \subseteq M$ and $W = F^{-1}(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
N \arrow[r, "F"] & M \\
W \arrow[u, hook, "i_W"] \arrow[r, "F|_W"'] & X \arrow[u, hook, "i_X"']
\end{tikzcd}$$

Then, by the functoriality of the differential, we also have the following commutative diagram for a given point $p \in W$:

$$\begin{tikzcd}
T_pN \arrow[r, "dF_p"] & T_{F(p)}M \\
T_pW \arrow[u, "d(i_W)_p"] \arrow[r, "d(F|_W)_p"'] & T_{F(p)}X \arrow[u, "d(i_X)_{F(p)}"']
\end{tikzcd}$$

Let dim$M = m$, dim$X = k$. Now, X is an embedded submanifold of M, so each point $x \in X$ has a neighborhood U such that $X \cap U$ is the regular level set of a defining function $\phi: U \to \mathbb{R}^{m-k}$. So, for each point $p \in F^{-1}(X\cap U)$, we have

$$\ker d\Phi_{F(p)} = T_{F(p)}X$$

Since F is transverse to X, for any point $p \in F^{-1}(X)$ we have

$$dF_p(T_pN) + T_{F(p)}X = T_{F(p)}M$$

Similarly, since W is an embedded submanifold of N we know $\Phi \circ F: F^{-1}(U) \to R^{m-k}$ is the local defining map for W. For each p
$inF^{-1}(X\cap U)$, we have

$$T_pW = \ker\left(d\Phi_{F(p)} \circ dF_p\right)$$

Then, consider a tangent vector $v \in T_pW$. We have $v \in T_p(W)$ if and only if $dF_p \in \ker d\Phi_{F(p)} = T_p(X)$ which means $v \in (dF_p)^{-1}\left(T_{F(p)}X\right)$. Hence,

$$T_pW = (dF_p)^{-1}\left(T_{F(p)}X\right)$$

Q6-16. Suppose M and N are smooth manifolds. A class $\mathcal{F}$ of smooth maps from N to M is said to be ***stable*** if it has the following property: whenever $\{F_s : s \in S\}$ is a smooth family of maps from N to M, and $F_{s_0} \in \mathcal{F}$ for some $s_0 \in S$, then there is a neighborhood U of s_0 in S such that $F_s \in \mathcal{F}$ for all $s \in U$. Prove that if N is compact, then the following classes of smooth maps from N to M are stable:

(a) immersions

(b) submersions

(c) embeddings

(d) diffeomorphisms

(e) local diffeomorphisms

(f) maps that are transverse to a given properly embedded submanifold $X \subseteq M$.

Proof: Cases (a), (b), (e) all depend on *every* member of the smooth family $\mathcal{F}$ having full rank, given that one of the funcitons in the family has full rank. The family of functions is parametrized by $F : N \times S \to M$. Let's choose charts for N, S, and M. Now, if we we find the coordinate representation of $F_s \in \mathcal{F}$, we find that the jacobian of F_s varies smoothly with s.

Thus, if F_{s_0} has full rank at point p, so does F_s at x for all $(x, s) \in O \times U$, where O and U are open neighborhoods of p and s_0 respectively. Since N is compact it can be covered by finitely many open sets O_i, where F_s has full rank at x for all $(x, s) \in O_i \times U_i$. Then, F_s has full rank on all of N for $s \in \bigcap_i U_i$. This proves cases (a), (b), (e).

(c). Now we want to show that embeddings are stable. We already know that immersions are stable, and injective immersions on compact domains are embeddings, so all we need to show is that injectivity is stable.

Suppose there is no such neighborhood U of s_0 such that F_s is injective for all $s \in U$. Then, there is a sequence (t_i) converging to s_0 in S such that F_{t_i} is not injective. Thus, there are pairs $(x_i, y_i) \in N \times N \setminus \Delta_N$ such that $F_{t_i}(x_i) = F_{t_i}(y_i)$. By compactness of N, we can assume that the sequences (x_i) and (y_i) converge.

Since F_{s_0} is injective, the sequences must converge to a common point x. Consider the map $G : N \times S \to M \times S$ defined as $(p, s) \mapsto (F_s(p), s)$. Its differential is of the form

$$dG_{(x,s_0)} = \begin{pmatrix} d\left(F_{s_0}\right)_x & * \\ 0 & \mathrm{id} \end{pmatrix}$$

so $dG_{(x,s_0)}$ is injective, since $d\left(F_{s_0}\right)_x$ is injective. But this implies that G is injective in a neighbourhood of (x, s_0), contradicting the work above. Thus it must be the case that such a neighborhood U exists and the class of embeddings is stable.

(d). Let F_{s_0} be a diffeomorphism. Then, by part (c), F_s is an embedding for all s in some open neighborhood of s_0. Then $F_s(N)$ is a compact (and hence closed) submanifold of M. Since F_{s_0} is a diffeomorphism, N and M have equal dimensions. Since F_s is an embedding, $F_s(N)$ is of codimension

0 in M, and is therefore open. The only nonempty open and closed subset of M is M itself, so the maps F_s are bijective embeddings, and therefore they are diffeomorphisms.

(f). Now, let X be the properly embedded submanifold to which F_{s_0} is transverse. Let $p \in F_{s_0}^{-1}(X)$ and let U be a chart centered at $F_{s_0}(p)$ that is a slice chart for X. More specifically, give U coordinates $(x_1, \ldots, x_m)$ and let $X \cap U$ correspond to the subset of points of the form $(x_1, \ldots, x_k, 0 \ldots, 0)$. The transversality assumption guarantees that $d\left(F_{s_0}\right)_p\left(T_p N\right)$ projects onto the $n-k$ last coordinate of $T_{F_{s_0}} M$. This is an open condition. Since $d\left(F_{s_0}\right)_p$ depends smoothly on s_0 and p, we conclude that there are open neighbourhoods U and O of s_0 and p, respectively, such that this holds. This implies that $F_s\big|_O$ is transverse to X. By compactness of N, we can cover it with finitely many such neighbourhoods O and thus conclude that F_s is transverse to X for all s in some open neighborhood s_0.
