

# Physics 198: Differential Geometry and Lie Groups

# Notes

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These are some short (**and really informal!**) notes that will be used to complement the lectures for the UC Berkeley DeCal 'Physics 198: Differential Geometry and Lie Groups for Physics Students'.

The primary reference for the class is "*Differential Geometry and Lie Groups for Physicists*" by Marián Fecko, but there are number of recommended texts as well.

I'm writing this document to flesh out my own understanding of the topics we'll be covering - borrowing explanations and intuitions from all of these books.

Of course, not all of the content in this document will be covered in class. Sections that are likely not going to be covered are indicated with a \* in the index. Further, this is a class for *physicists*, so to any mathematicians lurking around - please forgive my sloppiness in some parts of the text. This is very much a work in progress.

Many thanks to Ori J. Ganor for being our faculty sponsor, to Michelle Jing Dong and Finn Fraser Grathwol for allowing me to join them as a facilitator, as well as to Philip LaPorte who guided me through my first experience with Differential Topology - and to whom I still owe a Math DRP Term Paper (I hope this will suffice).

This template is based heavily off of the one produced by [Kevin Zhou](#).

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# 1 Topological Spaces

## 1.1 Motivation: Continuous functions in $\mathbb{R}^n$

## **2 Manifolds and Differential structure**

### **2.1**

### 3 Tangent, Cotangent, and Tensor Spaces (+Bundles)

When considering the motion of a particle along some curve  $\gamma(t)$  in  $\mathbb{R}^3$ , we have a very natural idea of what the tangent to the curve at each position (its velocity vector) looks like, and even what its acceleration vector might be. We might even have an intuitive understanding of these notions for a particle moving along the surface of a sphere  $\mathbb{S}^2$ . But what about a particle moving in  $\mathbb{RP}^n$ ? How do we generalize these notions to all smooth manifolds?

To generalize velocity vectors, we introduce tangent space, which is the "linear approximation" of the manifold at a given point. We'll see how to generalize acceleration later when we discuss connections and covariant derivatives.

(The approach I'll present here roughly follows Chapter 3 of Lee's Introduction to Smooth Manifolds [2].)

#### 3.1 Tangent Vectors

To get started, let's define the set of **Geometric Tangent Vectors** at a point  $a \in \mathbb{R}^n$  to be the set  $\{a\} \times \mathbb{R}^n$ , denoted as  $\mathbb{R}_a^n$  i.e. we literally just attach a copy of  $\mathbb{R}^n$  to that point (**Of course, this means  $\mathbb{R}_a^n$  is isomorphic to  $\mathbb{R}^n$  as a vector space**). The reason we do this is so that the spaces of geometric tangent vectors at distinct points  $a$  and  $b$  are disjoint.

This might seem like a goofy thing to do, but suppose  $a$  is a point on the surface of  $\mathbb{S}^{n-1}$  embedded within  $\mathbb{R}^n$ . Then, we can think of the set of vectors *tangent to  $\mathbb{S}^{n-1}$*  which we might denote as  $T_a\mathbb{S}^{n-1}$  as some subset of  $\mathbb{R}_a^n$  (for  $\mathbb{S}^{n-1}$  it would be the 2-dimensional plane tangent to the surface at point  $a$ ).

There are two issues with this:

- This depends on the manifold we're interested in being embedded in some  $\mathbb{R}^n$ .
- We haven't figured out a way to explicitly find  $T_a\mathbb{S}^{n-1}$ .

Notice however, that given some  $v \in \mathbb{R}_a^n$ , we have an operator  $D_v|_a : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  which takes a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and returns the **directional derivative** of  $f$  in the direction  $v$  **at the point  $a$**  as

$$\left. \frac{d}{dt} \right|_{t=0} f(a + tv)$$

In general, we can have all sorts of operators that act on functions  $f \in C^\infty$ , but differential operators in  $\mathbb{R}^n$  satisfy a very specific rule:

$$\text{Leibniz Rule: } d(fg) = (df)g + f(dg)$$

Let's call any operator  $w$  that satisfies the Leibniz Rule a **Derivation**, and let's denote the set of all derivations **at a point  $a$**  as  $T_a\mathbb{R}^n$  i.e.

$$T_a\mathbb{R}^n = \{\text{Linear maps, } w : \text{for any } f, g \in C^\infty(\mathbb{R}^n), w(f \cdot g) = w(f) \cdot g + f \cdot w(g)\}$$

Here's where things get interesting.

**Theorem 3.1.** *We have vector space isomorphism between  $R_a^n$  and  $T_a\mathbb{R}^n$  given by the map  $v_a \in R_a^n \mapsto D_v|_a \in T_a\mathbb{R}^n$  where*

$$D_v|_a : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$$

*is the differential operator defined (as earlier) as*

$$D_v|_a(f) = \left. \frac{d}{dt} \right|_{t=0} f(a + tv)$$

*for  $f \in C^\infty(\mathbb{R}^n)$ .*

This might look complicated, but what essentially what we're saying is that the collection of geometrical tangent vectors available is identical to the collection of directional derivatives we can take on any function  $f \in C^\infty(\mathbb{R}^n)$ .

**Proof.** (Complete this soon; Follow [2, proposition 3.2]) □

So, we've proven that thinking geometrically about tangent vectors is equivalent to thinking about derivations of functions in  $C^\infty(\mathbb{R}^n)$ . We use this intuition to **define** the set of tangent vectors at a point  $a \in M$  (where  $M$  is some smooth manifold) to be  $T_aM$  i.e.

$$T_aM = \{\text{Derivations, } w \mid \text{for any } f, g \in C^\infty(M), w(fg) = w(f) \cdot g + f \cdot w(g)\}$$

Since the two vector spaces are isomorphic, we immediately get the following

**Corollary:** Let  $\{x_1, \dots, x_n\}$  denote the standard coordinate basis of  $\mathbb{R}^n$ . Then for any point  $a \in \mathbb{R}^n$ , the collection

$$\{D_{x_1}|_a, \dots, D_{x_n}|_a\} = \left\{ \left. \frac{\partial}{\partial x^1} \right|_a, \dots, \left. \frac{\partial}{\partial x^n} \right|_a \right\}$$

forms a basis for  $T_a\mathbb{R}^n$ .

**Isomorphisms between Geometric Tangent Vectors, Derivations, and Velocity Vectors on a Curve**

(Decide whether to include this here or later; and whether or not to talk about differentials just yet)

### 3.2 Cotangent Space

Now that we've introduced the Tangent space  $T_x M$ , we can also talk about the **Cotangent space**,  $T_x^* M$  which is defined to be the **vector space dual** to  $T_x M$  i.e.

For a point  $x \in M$  in the smooth manifold  $M$ , the Cotangent space is defined as

$$T_x^* M \equiv (T_x M)^* = \{\omega \mid \omega : T_x M \rightarrow \mathbb{R} \text{ is a linear map}\}$$

The elements of  $T_x^* M$  are called **cotangent vectors** or **covectors**.

You may be wondering why we bother defining this space, and that's a valid question! **(Try to come up with a concise explanation here instead of just relegating to the example at the end.)** We'll discuss some physical examples of vectors and covectors towards the end of the chapter once we've discussed bundles.

### 3.3 Tangent Bundles and Vector Fields

So far, we've specified definitions for tangent and cotangent spaces **at a particular point** of a smooth manifold  $M$ . But what if we're interested in, say, a *vector field* on  $M$  (for instance, the electric field)? Then, intuitively, it feels like we'd need a space which

1. Specifies one vector for each point  $p \in M$ .
2. Has the same level of structure as  $M$ .

Enter, **The Tangent Bundle**. We define a new space denoted by  $TM$  as the disjoint union of the tangent bundles at each of the points in  $M$ .

For smooth manifold  $M$ , we define the Tangent Bundle  $TM$  as

$$TM \equiv \coprod_{p \in M} T_p M$$

i.e. the elements of  $TM$  are pairs of tangent vectors and the point they're associated with,  $(v, p)$ . Additionally, by default, we equip it with a **projection map** down to  $M$  defined by

$$\begin{aligned} \pi : TM &\rightarrow M \\ (v, p) &\rightarrow p \end{aligned}$$

Some features of  $TM$  to note are:

**Theorem 3.2.** *For a smooth manifold  $M$  of dimension  $n$ , there is a natural topology we can endow  $TM$  with in order to make it a smooth manifold of dimension  $2n$ . With respect to this topology,  $\pi : TM \rightarrow M$  is a smooth map.*

**Proof.** **(Complete this soon.)**

□



Okay. So, we have a collection of the tangent spaces at each point and a map that, given a tangent vector, tells us which point we're dealing with. How do we go the other way? i.e. How do we take a point and assign to it a vector?

(If it hasn't been done already, write sections about submersions immersions and embeddings of topological + smooth manifolds)

Recall that given a continuous map between topological spaces  $\pi : M \rightarrow N$ , any **continuous right inverse** to  $\pi$  i.e. a map  $\sigma$  such that

$$\pi \circ \sigma = \text{Id}_N$$

is called a **section** of  $\pi$ .

Given a smooth manifold  $M$ , a **Vector Field**  $X$  on  $M$  is a **smooth section** of the tangent-space projection map  $\pi : TM \rightarrow M$ . i.e. it is a smooth map  $X : M \rightarrow TM$  such that

$$\pi \circ X = \text{Id}_M$$

### 3.4 Cotangent Bundle and Covector Fields

### 3.5 Pullbacks and Pushforwards

### 3.6 Tensor Spaces, Tensor Fields, and Tensor Bundles

### 3.7 Example: Vectors and Covectors Electrodynamics

### 3.8 Example: Vectors and Covectors in Quantum Mechanics

## 4 Lie Groups and Lie Algebras

In physics, we can learn a great deal from studying the symmetries of *continuous and smooth* systems. For instance, knowing that a system is invariant under smooth rotation tells us that its angular momentum is conserved (by Noether's Theorem). Group Theory is the study of symmetries, whereas (continuity + smoothness) falls in the domain of smooth manifolds. **Lie Groups** lie in the intersection.

They are ubiquitous in Physics. Regardless of which branch of physics you study, you will likely benefit from studying Lie Group **Actions** and **Representations**.

The exposition here is mostly a combination of the approaches and content covered in [2] and [3].

### 4.1 Lie Groups

**Definition 4.1.** A **Lie Group** is a smooth manifold  $G$  equipped with smooth maps

$$\begin{aligned} \text{Multiplication, } m : G \times G &\rightarrow G \\ (g, h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} \text{Inversion, } i : G &\rightarrow G \\ a &\mapsto b \equiv a^{-1} \end{aligned}$$

which endow it with group structure.

**Example:** The group  $O(n)$  is the *three dimensional orthogonal group*. One way to think about this group (or to **represent** it \*wink wink\*) is as the set of  $n \times n$  matrices,  $A$ , which satisfy the equation

$$AA^T = \text{Id}_n$$

Here, the "multiplication" and "inversion" maps are just matrix multiplication and inversion.

But this is just one way to represent the group. In truth, we define it by the way it **acts** (\*wink wink\*) on  $\mathbb{R}^n$ . Namely, it's the group of transformations such that any two vectors  $v_1, v_2$  that *started off orthogonal* to each other will *remain orthogonal* after transformation.

**(Write about the other prominent matrix Lie Groups + highlighting the Lorentz and Poincare Groups)**

### The Left Translation map

**Definition 4.2.** Every element  $g \in G$  of the Lie Group induces a map

$$\begin{aligned} L_g : G &\rightarrow G \\ h &\mapsto gh \end{aligned}$$

called **left translation by  $g$** .

**Theorem 4.1.** For each  $g \in G$ , the left translation  $L_g$  is a **diffeomorphism**.

**Proof.**

- The map  $L_g$  is bijective on  $G$ :  
*Injectivity:* If  $g = g' \in G$  then  $L_g(h) = gh = g'h = L_{g'}(h)$  for any  $h \in G$ .  
*Surjectivity:* For any  $h \in G$ ,  $L_g(g^{-1}h) = g(g^{-1}h) = (gg^{-1})h = h$

- The map  $L_g$  is a composition of smooth maps (inclusion and multiplication)

$$G \xrightarrow{\iota} G \times G \xrightarrow{m} G$$

and so it's a smooth map. The inverse map is  $(L_g)^{-1} = L_{g^{-1}}$  and so the same reasoning holds.

□

Similarly, we have the **Right Translation map**,  $R_g : G \rightarrow G$  defined as  $R_g(h) = hg$ , which has pretty much the same properties as left-translations. However, conventionally, we use left-translation more often.

Now, since  $L_g$  is a diffeomorphism (and thus a smooth map), we also have the push-forward  $(L_g)_* : TG \rightarrow TG$ . Just to be explicit, let's spell things out clearly.

- $L_g : G \rightarrow G$  is a diffeomorphism.
- So, the differential of  $L_g$  at some point  $h \in G$  is the map between tangent spaces at the points  $h$  and  $L_g(h)$

$$d(L_g)_h : T_h G \rightarrow T_{L_g(h)} G = T_{(gh)} G$$

- So,  $d(L_g)$ , also denoted as  $(L_g)_*$  is a map from  $TG \rightarrow TG$ .

It takes in a pair

$$(v, p), \quad v \in T_p G, \quad p \in G$$

as the input and sends the pair to

$$(d(L_g)_p(v), L_g(p)), \quad d(L_g)_p(v) \in T_{L_g(p)} G, \quad L_g(p) = gp \in G$$

This gives us a means to "transport" ourselves back from any element  $g \in G$  to the identity element  $e \in G$  via  $(L_{g^{-1}})_* = (L_g^{-1})_*$ .

Intuitively, this gives us the impression that a lot of information about the group as a whole is contained in the Identity element. Turns out this is more true than one might expect, as we can actually reconstruct the entire Lie Group given a neighborhood around the identity! (**Write about Proposition 7.14 from [2]**) We'll explore this idea more after defining the Lie Algebra of a Lie Group.

## 4.2 Lie Algebras

In general, a  $\mathbb{K}$ -**Algebra**  $(V, +, \cdot, \times)$  in math refers to a vector space  $(V, +, \cdot)$  over the field  $\mathbb{K}$  (where  $\cdot$  denotes scalar multiplication) with a multiplication operation  $\times : V \times V \rightarrow V$ .

Usually if the  $\mathbb{K}$  isn't mentioned it's assumed to be  $\mathbb{R}$ .

**Example:** The collection of  $n \times n$  matrices  $\text{Mat}_n(\mathbb{K})$  is a  $\mathbb{K}$ -algebra where the  $\times$  operation is the usual matrix multiplication operation.

**Example:** The set  $\text{Hom}_{\mathbb{K}}(V, W)$  of  $\mathbb{K}$ -linear maps between the  $\mathbb{K}$ -vector spaces  $V$  and  $W$  form a  $\mathbb{K}$ -algebra where the  $\times$  operation is *composition* of maps.

(Add more examples.)

A general **Lie Algebra** is an Algebra whose  $\times$  operation satisfies specific properties. In this case, the  $\times$  operation is referred to as the **Lie Bracket** and denoted as  $[\cdot, \cdot]$ . In order to be a lie bracket, the multiplication operation must be:

- *Antisymmetric:* Switching inputs introduces a negative sign i.e.  $[v, w] = -[w, v]$
- *Bilinear:* Linear in both arguments i.e.

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[X, aZ + bW] = a[X, Z] + b[X, W]$$

- *Satisfy the Jacobi Identity* i.e.

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The Jacobi Identity is a sort-of substitute for associativity. **(Provide more motivation for these properties)**

Let's develop some more results related to the Bracket.

### Some properties of the Bracket

**(Include results about coordinate expression and such)**

**Definition 4.3.** Given smooth manifolds  $M, N$ , a smooth map  $F : M \rightarrow N$  between them, and vector fields  $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$  we say  $X$  and  $Y$  are  **$F$ -related** if for every  $p \in M$ ,

$$dF_p(X_p) = Y_{F(p)}$$

**(Include proposition 8.16 from [2])**

**Theorem 4.2. (Naturality of the Bracket):** Consider smooth manifolds  $M, N$  with or without boundary, and a smooth map  $F : M \rightarrow N$ . If  $X_1, X_2 \in \mathfrak{X}(M)$  are  $F$ -related, respectively, to  $Y_1, Y_2 \in \mathfrak{X}(N)$  then their Lie Bracket  $[X_1, X_2]$  is  $F$ -related to  $[Y_1, Y_2]$

**Proof.** By Proposition 8.16 from [2] We have

$$X_1 X_2 (f \circ F) = X_1 (Y_2 f \circ F) = (Y_1 Y_2 f) \circ F$$

and similarly,

$$X_2 X_1 (f \circ F) = (Y_2 Y_1 f) \circ F$$

Then for  $f \in C^\infty(N)$ , which gives us  $(f \circ F) \in C^\infty(M)$ , we have

$$\begin{aligned} [X_1, X_2](f \circ F) &= X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F) \\ &= (Y_1 Y_2 f) \circ F - (Y_2 Y_1 f) \circ F \\ &= ((Y_1 Y_2 f) - (Y_2 Y_1 f)) \circ F \\ &= (Y_1 Y_2 - Y_2 Y_1)(f) \circ F \\ &= (Y_1 Y_2 - Y_2 Y_1) f \circ F \\ &= ([Y_1, Y_2] f) \circ F \end{aligned}$$

□

**(Explain explicitly how this means they're F-related.)**

Note, the word "Naturality" here has a specific categorical meaning. Check the category theory appendix to learn more. **(give me a chop on the neck if it's not there yet by the time you're reading this.)**

**Corollary. (Pushforwards of the Bracket):** Suppose  $F : M \rightarrow N$  is a diffeomorphism and  $X_1, X_2 \in \mathfrak{X}(M)$ . Then,  $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$

### Lie Algebra of a Lie Group

Earlier, we defined the pushforward  $(L_g)_*(X)$  for a vector field  $X$  defined on a lie group  $G$ . It turns out that the set of **Left Invariant Vector Fields** on  $G$  forms a Lie Algebra. i.e.

**Theorem 4.3.** Given a Lie Group  $G$ , the set of vector fields invariant under left-translation by all  $g \in G$ ,

$$\mathcal{L}(G) = \mathfrak{g} \equiv \{X \in \Gamma(TG) \mid (L_g)_*(X) = X \text{ for all } g \in G\}$$

with the Lie Bracket of two vector fields defined as

$$[X, Y]f = XYf - YXf$$

for  $f \in C^\infty(G)$  is a Lie Algebra.

This object  $\mathcal{L}(G)$  is called the **Lie Algebra of the Lie Group  $G$** . The proof that this set is indeed a Lie Algebra is sketched below, but see [2] for more detail.

**Proof:** We already know that  $\Gamma(TG)$  is a vector space. The reason the subspace  $\mathcal{L}(G) \subset \Gamma(TG)$  is also a vector space is... left as an exercise for the reader (sorry).

(Hint: First show that given vector fields  $X, Y$  that  $[X, Y]$  is also a vector field. Then show that  $X + Y$  is invariant under left-translation, given  $X, Y \in \mathcal{L}(G)$  i.e it is closed under vector addition.)

To show that  $\mathcal{L}(G)$  is a Lie Algebra, we need to demonstrate that it is closed under the Bracket, and that the bracket satisfies the Jacobi Identity.

The second task is just a straightfoward calculation, so it's left as an exercise (sorry again). (Maybe just include it instead of leaving it to the reader)

For the first task, note that for  $X, Y \in \mathcal{L}(G)$  we have

$$\begin{aligned} (L_g)_*[X, Y] &= [(L_g)_*X, (L_g)_*Y] \text{ (Since } L_g \text{ is a diffeomorphism)} \\ &= [X, Y] \text{ (Since } X, Y \text{ are Left-Invariant)} \end{aligned}$$

So  $\mathcal{L}(G)$  is closed under the bracket. This proves the claim.

Okay, so that's cool, but why do we care about left-invariant vector fields? How exactly do they help us study Lie Groups? It turns out that  $\mathcal{L}(G)$  is isomorphic as a vector space to  $T_e G$  i.e. the tangent space at the identity.

This is a big deal!  $T_e G$  is a "linear approximation" of the manifold at the identity, and we proved earlier that we can reconstruct the entire Lie Group (assuming it's connected) given a neighborhood of the identity.

This begs the question, is there a method to reconstruct the Lie Group given the Lie Algebra? **Yes!** Or at least, sort of. We'll study this method (using something called the exponential map) after learning about Integral Curves and Flows of Vector Fields.

### **4.3 \*Classification of Lie Groups**

### **4.4 \*Dynkin Diagrams**

### **4.5 Reconstruction of Lie Groups from Lie Algebras**

**(Cover this section after introducing Integral Curves and Flows)**

## **5 Flows and the Lie Derivative**

### **5.1 Flow of a field**

### **5.2 Lie Derivative**



## **6 Exterior Derivative and Exterior Algebra**

## 7 Representations of Lie Groups and Lie Algebras

### 7.1 What is a Representation?

- To understand what a group really *is*, it can be very enlightening to study what the group *does* i.e. to study **group actions**. Further, linear algebra is easier than abstract algebra, so if we can study the action of a group in terms of linear algebraic objects, we'll get a lot more mileage.
- How exactly do we do this? We can use some sort of map to assign a *linear operator* over that vector space to each group element to describe (or represent) the action of each group element on the vector space elements. The map that we use is a **representation** of the group.

Recall that the space of all linear operators  $\rho : V \rightarrow V$  is denoted  $\text{End}(V)$ . The subset of these operators which are invertible (isomorphisms on  $V$ ) is denoted  $\text{Aut}(V)$ .

Notably,  $\text{Aut}(V)$  has a group structure! **(Check this!)** On the other hand,  $\text{End}(V)$  becomes an (Associative, and later Lie) Algebra if we define the commutator for  $A, B \in \text{End}(V)$  as

$$[A, B] = AB - BA$$

Now the formal definition.

**Definition 7.1.** *Group Representation* Given a group  $G$  and vector space  $V$ , a group homomorphism

$$\rho : G \rightarrow \text{Aut}(V)$$

is called a **representation** of  $G$  in  $V$ .

**Example.** Complete this later

We can use the same idea to define the representation of an algebra, but this time with  $\text{End}(V)$ .

**Definition 7.2.** *Lie Algebra Representation* Given a Lie algebra  $\mathcal{G}$  and vector space  $V$ , an algebra homomorphism

$$\rho' : \mathcal{G} \rightarrow \text{End}(V)$$

is called a representation of the Lie algebra  $\mathcal{G}$  over  $V$ .

**Note.** The representations  $\rho$  and  $\rho'$  of a lie group and its lie algebra are related! so  $\rho'$  is called the **derived representation**.

#### Fecko, Exercise 12.1.4

Consider a Lie algebra  $\mathcal{G}$  whose basis elements  $\{E_i\}$  satisfy the commutation relations

$$[E_i, E_j] = c_{ij}^k E_k$$

and a representation  $f : \mathcal{G} \rightarrow \text{End}(V)$ . Then, define  $\mathcal{E}_i \equiv f(E_i)$ . Since  $f$  is a homomorphisms

between algebra, it is linear and respects the commutator i.e. for  $A, B \in \mathcal{G}$

$$f([A, B]) = [f(A), f(B)]$$

Thus,

$$\begin{aligned} [\mathcal{E}_i, \mathcal{E}_j] &= [f(E_i), f(E_j)] \\ &= f([E_i, E_j]) \\ &= f(c_{ij}^k E_k) \\ &= c_{ij}^k f(E_k) \\ &= c_{ij}^k \mathcal{E}_k \end{aligned}$$

(The basis elements of the representation satisfy the same commutation relation as those of the Lie Algebra!)

#### Fecko, Exercise 12.1.5

Do this one later

- The assignment from Lie Group to Lie Algebra  $G \mapsto \mathcal{G}$  is nice and unique, but the other way around can get messy.
- Similarly, given a Lie group representation  $\rho$  there is a unique Lie algebra representation  $\rho'$ , but not necessarily the other way around.

#### Fecko, Exercise 12.1.6

- (i) Consider the Lie Group  $H = \text{Aut}(V) \equiv \text{GL}(V)$ . Recall that the Lie Algebra of  $H$  is

Write about  $\rho$ -invariant inner products.

## 7.2 Reducible and Irreducible Representations

## 8 Covariant Derivatives, Connections, and Parallel Transport

So far we've : Studied Topological Spaces and Manifolds  $\rightarrow$  endowed them with Smooth Structures to turn them into Smooth Manifolds  $\rightarrow$  studied natural structures on them such as Tangent, Cotangent, Vector Spaces and Bundles.

### 8.1 Motivation for the Covariant Derivative and Connection

Suppose a particle is moving along some path  $\gamma : (0, 1) \rightarrow M$  on a manifold  $M$ . Its **velocity** at a point  $\gamma(t)$  would be some vector in the tangent space at that point  $V_{\gamma(t)} \in T_{\gamma(t)}M$ . What if we wanted to find the particles **acceleration**?

Well, for that, we'd want to study how  $V_{\gamma(t)}$  changes over the path  $\gamma(t)$  i.e. we'd need to find the difference quotient

$$\lim_{dt \rightarrow 0} \frac{V(\gamma(t+dt)) - V(\gamma(t))}{dt}$$

The issue is that  $V(\gamma(t+dt)) \in T_{\gamma(t+dt)}M$  whereas  $V(\gamma(t)) \in T_{\gamma(t)}M$  i.e. they lie in different tangent spaces which, a priori, have nothing to do with each other, and so taking the difference

$$V(\gamma(t+dt)) - V(\gamma(t))$$

really makes no sense for general manifolds.

This wasn't an issue in  $\mathbb{R}^n$  because  $T_p\mathbb{R}^n \cong \mathbb{R}^n$  for any  $p \in \mathbb{R}^n$ , which allowed us to compare tangent vectors at different points. Equivalently this is why, in  $\mathbb{R}^n$ , we can pick up a vector  $v$  with its tail fixed at point  $p$  and move it to any other point  $q$  keeping it facing the same direction, without changing the vector. This procedure of moving a tangent vector (or more generally any element of a tensor space) from one point to another while keeping it "parallel" to itself is called **Parallel Transport**.

When defining the Lie Derivative along some other field  $W$ , we solved this problem using a **trick**. We found the flow  $\theta$  generated by  $w$  connected the points and then took the pullback of  $V(\gamma(t+dt))$  under  $\theta$ .

However, that makes the Lie Derivative dependent on the behavior of the other vector field  $W$ . Whereas, we want a derivative of  $V$  that only depends on the path taken.

**Connections** provide us with an alternative method to compare objects in tensor spaces based at different points.

**What properties should this new derivative have?** (This motivation has been borrowed from [1] and [4]).

In Cartesian Coordinates, the partial derivative operator is a map  $\partial_\mu$  from  $(k, l)$  to

$(k, l + 1)$  tensor fields on  $\mathbb{R}^n$  [1] which also satisfies

1. Linearity in all arguments
2. Leibniz Rule on Tensor Products.
3. Commutativity with Contraction
4. Acts as the normal partial derivative operator on scalar functions:  $\nabla_\mu \phi = \partial_\mu \phi$  for  $f \in C^\infty(M)$

If we want to generalize  $\partial_\mu$  as a map between collections of tensor spaces on a general (smooth) manifold  $M$ , our notion of a derivative should at least satisfy these properties. Let's call such a generalization a **Covariant Derivative** and denote it as  $\nabla$ .

It turns out that any covariant derivative can be written in the form of a partial derivative  $\partial_\mu$  + a correction matrix,  $(\Gamma_\mu)_\sigma^\rho$  (**We'll see why in the next section**). So, for example, the action of  $\nabla_\mu$  on a vector  $V^\nu$  would be given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + (\Gamma_\mu)_\sigma^\lambda V^\sigma$$

The entries of the  $n \times n$  matrix  $\Gamma_\mu$  are called the **connection coefficients**. We often drop the parentheses and just denote them as  $\Gamma_{\mu\sigma}^\rho$ .

**Note:** The object that we get on covariant differentiation is a tensor,  $\nabla_\mu V$ . The notation  $\nabla_\mu V^\nu$  is *really* referring to the  $\nu^{\text{th}}$  component of this tensor  $(\nabla_\mu V)^\nu$ , but we often drop the parentheses.

## 8.2 Looking at the connection more closely

So far I've primarily used the notation  $\Gamma_{\mu\lambda}^\nu$  to refer to the connection coefficients, but another common notation is to simply use  $\nabla$  without any subscript. This is because we can view the connection as a map  $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ .

Suppose we are working in a basis  $\{e_\mu\}$  of  $\mathfrak{X}(M)$ . Then, the connection coefficients are *defined* as

$$\nabla_{e_\mu} e_\lambda = \Gamma_{\mu\lambda}^\nu e_\nu$$

and the notation  $\nabla_\mu$  is really shorthand for  $\nabla_{e_\mu}$ .

Given vector fields  $X = X^\mu e_\mu$  and  $Y = Y^\nu e_\nu$  then the connection maps  $(X, Y)$  to  $\nabla_X Y$  and the operator  $\nabla_X$  is the covariant derivative in the direction  $X$  with respect to the connection.

Using the properties of the connection that we've declared it to possess, we have

$$\begin{aligned}
 \nabla_X Y &= \nabla_X (Y^\mu e_\mu) \\
 &= X(Y^\mu) e_\mu + Y^\mu \nabla_X (e_\mu) \\
 &= X^\nu e_\nu (Y^\mu) e_\mu + Y^\mu X^\nu \nabla_\nu (e_\mu) \\
 &= X^\nu e_\nu (Y^\mu) e_\mu + X^\nu Y^\mu \Gamma_{\nu\mu}^\lambda e_\lambda
 \end{aligned}$$

But note that

$$Y^\mu \Gamma_{\nu\mu}^\lambda e_\lambda = Y^\lambda \Gamma_{\nu\lambda}^\mu e_\mu$$

since we're just swapping dummy variables which take on the same values. So,

$$\begin{aligned}
 \nabla_X Y &= X^\nu e_\nu (Y^\mu) e_\mu + X^\nu Y^\lambda \Gamma_{\nu\lambda}^\mu e_\mu \\
 &= X^\nu \left( e_\nu (Y^\mu) + Y^\lambda \Gamma_{\nu\lambda}^\mu \right) e_\mu
 \end{aligned}$$

Since, in the last step, we were able to pull out  $X^\nu$  and  $e_\mu$ , it makes sense to define the  $\mu^{\text{th}}$  component of  $\nabla_\nu Y$  as

$$(\nabla_\nu Y)^\mu = e_\nu(Y^\mu) + \Gamma_{\nu\lambda}^\mu Y^\lambda$$

This might look a bit different from our claim earlier that a connection is a partial derivative + a correction, but  $\{\partial_\mu\}$  is a particularly nice basis for  $\mathfrak{X}(M)$  and if we choose to use it, then,

$$\boxed{(\nabla_\nu Y)^\mu = \partial_\nu Y^\mu + \Gamma_{\nu\lambda}^\mu Y^\lambda}$$

### Covariant Derivatives of Vector Fields

We already saw above that for a vector field  $V \in \mathfrak{X}(M)$ , the covariant derivative can be written as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + (\Gamma_\mu)^\nu_\lambda V^\lambda$$

We wanted  $\nabla_\mu$  to be a map from the space of  $(k, l)$  tensors to the space of  $(k, l + 1)$  tensors. So, the  $(1, 0)$  tensor  $V$  should get mapped to a  $(1, 1)$  tensor by  $\nabla_\mu$ . If it is indeed a  $(1, 1)$  tensor, it should transform like

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \nabla_\mu V^\nu$$

In the new coordinates, the LHS transforms as

$$\begin{aligned}
 \nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma_{\mu'\lambda'}^{\nu'} V^{\lambda'} \\
 &= \left( \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_{\mu'} \right) \left( \frac{\partial x^{\nu'}}{\partial x^{\nu}} V^\nu \right) + \Gamma_{\mu'\lambda'}^{\nu'} \left( \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \right) \\
 &= \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_\mu V^\nu + \Gamma_{\mu'\lambda'}^{\nu'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \quad (\text{product rule})
 \end{aligned}$$

The RHS can be expanded out as

$$\begin{aligned}\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \nabla_\mu V^\nu &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left( \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial}{\partial x^\mu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma_{\mu\lambda}^\nu V^\lambda\end{aligned}$$

Setting the two expressions equal to each other (and cancelling out terms in common), we have

$$\frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma_{\mu'\lambda'}^{\nu'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma_{\mu\lambda}^\nu V^\lambda$$

So,

$$\boxed{\Gamma_{\mu'\lambda'}^{\nu'} = \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\mu\lambda}^\nu - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}}$$

Thus in order for  $\nabla_\mu$  to be a valid covariant derivative, the connection coefficients need to transform as in the box above. Note that they transform **non-tensorially**.

If it were just the first term on the RHS then the connection coefficients would also transform like a  $(1,1)$  tensor, but they do not because of the presence of the second term.

The second term makes  $\Gamma_\mu$  non-tensorial in such a way that it cancels the non-tensoriality of  $\partial_\mu$ , making  $\nabla_\mu V^\nu$  a  $(1,1)$  tensor as desired.

### Covariant Derivatives of Covector Field

We've found an expression for the covariant derivative of a 1-vector field  $V \in \mathfrak{X}(M)$ . How about a 1-form,  $\omega \in \mathfrak{X}^*(M)$ ? Well, we argued earlier that **any** covariant derivative should have the form  $\nabla_\mu = \partial_\mu + \Gamma_\mu$ , so for  $\omega$  as well we can write that the components of

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}_{\mu\nu}^\lambda \omega_\lambda$$

where  $\tilde{\Gamma}_\mu$  is some other set of connection coefficients.

Also, notice the slight change in the placement of the indices on  $\tilde{\Gamma}$  because we're dealing with the components of a covector rather than a vector this time. **(Explain this more explicitly)**

**Can we find a relationship between the components of  $\Gamma$  (for vector fields) and  $\tilde{\Gamma}$  (for covector fields)?** Yes we can!

Given the one-form  $\omega$  and vector field  $V$ ,  $\omega_\lambda V^\lambda$  is a scalar. So, by the Leibniz property, we have

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= \nabla_\mu(\omega_\lambda) V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) \\ &= \left[ \partial_\mu \omega_\lambda + \tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma \right] V^\lambda + \omega_\lambda \left[ \partial_\mu V^\lambda + \Gamma_{\mu\rho}^\lambda V^\rho \right] \\ &= (\partial_\mu \omega_\lambda) V^\lambda + \tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma V^\lambda + \omega_\lambda (\partial_\mu V^\lambda) + \Gamma_{\mu\rho}^\lambda \omega_\lambda V^\rho\end{aligned}$$

But since  $\omega_\lambda V^\lambda$  is just a scalar function, the covariant derivative should basically just act as a partial derivative:

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= \partial_\mu (\omega_\lambda V^\lambda) \\ &= (\partial_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\partial_\mu V^\lambda) \text{ (Leibniz rule in } \mathbb{R}^n)\end{aligned}$$

So equating the two expressions above and cancelling common terms, only the  $\Gamma$  and  $\tilde{\Gamma}$  terms survive. In fact,

$$\tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma V^\lambda + \Gamma_{\mu\rho}^\lambda \omega_\lambda V^\rho = 0$$

But  $\lambda, \sigma, \rho$  are all dummy variables (we're summing over all possible values for these variables) - the only one that is fixed is  $\mu$ . So, we can rename all the dummy variables to get

$$\begin{aligned}\tilde{\Gamma}_{\mu\lambda}^\sigma \omega_\sigma V^\lambda + \Gamma_{\mu\lambda}^\sigma \omega_\sigma V^\lambda &= 0 \\ \Rightarrow \boxed{\tilde{\Gamma}_{\mu\lambda}^\sigma = -\Gamma_{\mu\lambda}^\sigma}\end{aligned}$$

Thus, once we know have a connection for vector fields, we automatically have the connection for one-forms.

### Covariant derivatives of General Tensors

The result from above is splendid! Arbitrary  $(r, s)$ -type tensors are formed using  $r$  one-forms and  $s$  vector fields. Therefore, we have the connection for any tensor space i.e. we can take the covariant derivative of any tensor!

To recap:

- (a) Given the connection coefficients  $\Gamma_{\mu\lambda}^\sigma$  we can find the covariant derivative of  $V \in \mathfrak{X}(M)$  with respect to the connection by

$$(\nabla_\mu V)^\lambda = \partial_\mu V^\lambda + \Gamma_{\mu\sigma}^\lambda V^\sigma$$



(b) The covariant derivative of any *one-form* with respect to the connection is

$$(\nabla_\mu \omega)_\lambda = \partial_\mu \omega_\lambda - \Gamma_{\mu\lambda}^\sigma \omega_\sigma$$

(c) For a general  $(r, s)$  tensor  $T \in \underbrace{\mathfrak{X}(M) \otimes \cdots \otimes \mathfrak{X}(M)}_r \otimes \underbrace{\mathfrak{X}^*(M) \otimes \cdots \otimes \mathfrak{X}^*(M)}_s$  whose components are denoted as  $T_{\nu_1 \cdots \nu_s}^{\mu_1 \cdots \mu_r}$ , the covariant derivative can be found as

$$\begin{aligned} (\nabla_\sigma T)_{\nu_1 \cdots \nu_s}^{\mu_1 \cdots \mu_r} = & \partial_\sigma T_{\nu_1 \cdots \nu_s}^{\mu_1 \cdots \mu_r} \\ & + \Gamma_{\sigma\lambda}^{\mu_1} T_{\nu_1 \cdots \nu_s}^{\lambda \mu_2 \cdots \mu_r} + \cdots + \Gamma_{\sigma\lambda}^{\mu_r} T_{\nu_1 \cdots \nu_s}^{\mu_1 \mu_2 \cdots \mu_{r-1} \lambda} \\ & - \Gamma_{\sigma\nu_1}^\lambda T_{\lambda \cdots \nu_s}^{\mu_1 \mu_2 \cdots \mu_r} - \cdots - \Gamma_{\sigma\nu_s}^\lambda T_{\nu_1 \cdots \nu_{s-1} \lambda}^{\mu_1 \mu_2 \cdots \mu_r} \end{aligned}$$

i.e. for each  $\mu_i$  index for  $\mathfrak{X}(M)$ , we add a  $+\Gamma$  term, and for each  $\nu_j$  index for  $\mathfrak{X}(M)$  we add a  $-\Gamma$  term.

So, to specify a covariant derivative is equivalent to specifying the **Connection** i.e.  $n^3$  coefficients  $\Gamma_{\mu\lambda}^\nu$  for a manifold of dimension  $n$ . Different Connections will give rise to different Covariant Derivatives.

### 8.3 Torsion

We noticed earlier that the connection coefficients don't transform tensorially due to the second term in their transformation equation.

$$\Gamma_{\mu'\lambda'}^{\nu'} = \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\mu\lambda}^\nu - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

But since mixed partial derivatives commute (Clairaut's theorem) for the kinds of functions we're dealing with, we have

$$\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} = \frac{\partial^2 x^{\nu'}}{\partial x^\lambda \partial x^\mu}$$

Notice that the transformation of  $\Gamma_{\lambda\mu}^\nu$  (instead of  $\Gamma_{\mu\lambda}^\nu$ ) is given by

$$\begin{aligned} \Gamma_{\lambda'\mu'}^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\lambda\mu}^\nu - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\lambda \partial x^\mu} \\ &= \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma_{\lambda\mu}^\nu - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \end{aligned}$$

i.e. the non-tensorial term is identical in both of them.

Thus, the object

$$T_{\mu\lambda}^\nu \equiv \Gamma_{\mu\lambda}^\nu - \Gamma_{\lambda\mu}^\nu$$

transforms tensorially as

$$\begin{aligned} \left( \Gamma_{\mu'\lambda'}^{\nu'} - \Gamma_{\lambda'\mu'}^{\nu'} \right) &= \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left( \Gamma_{\mu\lambda}^\nu - \Gamma_{\lambda\mu}^\nu \right) \\ \implies T_{\mu'\lambda'}^{\nu'} &= \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left( T_{\mu\lambda}^\nu \right) \end{aligned}$$

This object  $T_{\mu\lambda}^\nu$  is called the **Torsion Tensor** associated with the connection  $\Gamma_{\mu\lambda}^\nu$ .

If a connection is symmetric in its lower indices i.e.  $\Gamma_{\mu\lambda}^\nu = \Gamma_{\lambda\mu}^\nu$  then the torsion tensor vanishes and the connection is said to be **torsion-free**.

(Add more intuition about the Torsion Tensor; explain the name)

#### 8.4 Levi-Civita Connection

- We noted earlier that different connections give rise to different covariant derivatives.
- This raises the question, "Is there a specific connection associated to each manifold that can be considered as a 'standard' or 'canonical' one?"
- It turns out that for **manifolds endowed with a metric**, there is indeed a unique *torsion-free* and *metric-compatible* connection called the **Levi-Civita Connection**.

A Connection is said to be **compatible with a metric  $g$**  if the covariant derivative of the metric with respect to the connection is zero everywhere i.e.

$$\nabla_\rho g_{\mu\nu} = 0$$

or, equivalently,

$$\nabla_X g = 0$$

for any vector field  $X$ .

**Theorem 8.1. (Fundamental Theorem of Riemannian Geometry):** *There is exactly one torsion-free, metric compatible connection for a given metric  $g_{\mu\nu}$  on a Riemannian manifold  $M$ .*

**Proof.** If we are able to get a unique expression for the connection coefficients in terms of the metric, that will show both existence and uniqueness.

Recall that  $g_{\mu\nu}$  is a  $(0, 2)$  tensor (because it acts on two vectors  $x^\mu, y^\nu$  as  $g_{\mu\nu}x^\mu y^\nu$ ), so we have two  $-\Gamma$  terms in the covariant derivative (one for each lower index).

Let's find the covariant derivatives for three cyclic indices:

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma_{\sigma\mu}^\lambda g_{\lambda\nu} - \Gamma_{\sigma\nu}^\lambda g_{\mu\lambda} = 0 \quad (1)$$

$$\nabla_\mu g_{\nu\sigma} = \partial_\mu g_{\nu\sigma} - \Gamma_{\mu\nu}^\lambda g_{\lambda\sigma} - \Gamma_{\mu\sigma}^\lambda g_{\nu\lambda} = 0 \quad (2)$$

$$\nabla_\nu g_{\sigma\mu} = \partial_\nu g_{\sigma\mu} - \Gamma_{\nu\sigma}^\lambda g_{\lambda\mu} - \Gamma_{\nu\mu}^\lambda g_{\sigma\lambda} = 0 \quad (3)$$

where all of the covariant derivatives are zero because of metric-compatibility. Now, subtracting equation (2) and (3) from equation (1) gives us

$$\begin{aligned} 0 &= \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu} - \Gamma_{\sigma\mu}^\lambda g_{\lambda\nu} - \Gamma_{\sigma\nu}^\lambda g_{\mu\lambda} + \Gamma_{\mu\sigma}^\lambda g_{\nu\lambda} + \Gamma_{\mu\nu}^\lambda g_{\lambda\sigma} + \Gamma_{\nu\sigma}^\lambda g_{\lambda\mu} + \Gamma_{\nu\mu}^\lambda g_{\sigma\lambda} \\ &= \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu} + 2\Gamma_{\mu\nu}^\lambda g_{\lambda\sigma} \end{aligned}$$

where in the first equation, because the connection is assumed to be torsion-free ( $\Gamma_{\sigma\mu}^\lambda = \Gamma_{\mu\sigma}^\lambda$  and so on) and the metric is assumed to be symmetric i.e.  $g_{\mu\nu} = g_{\nu\mu}$ , the red/blue terms connection terms cancel while the black terms add.

So we have

$$\Gamma_{\mu\nu}^\lambda g_{\lambda\sigma} = -\frac{1}{2} (\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\mu})$$

and then multiplying by  $g^{\rho\sigma}$  turns the LHS into

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda g_{\lambda\sigma} g^{\rho\sigma} &= \Gamma_{\mu\nu}^\rho \underbrace{g_{\rho\sigma} g^{\rho\sigma}}_{=1} \text{ (Since } \lambda \text{ is just a dummy variable)} \\ &= \Gamma_{\mu\nu}^\rho \end{aligned}$$

Therefore, we have a manifestly unique expression for the Connection Coefficients in terms of the metric, given by

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

This defines the Levi-Civita Connection Coefficients. □

In General Relativity, we pretty much always work with the Levi-Civita connection, and the **Christoffel Symbols** that you've come across if you've studied GR before are precisely the **Connection Coefficients of the Levi-Civita connection**.

(Maybe talk about the Contorsion tensor)

## 8.5 Curvature

## 8.6 Parallel Transport and Geodesics

## **9 Appendix: Some important results from Multivariable Calculus**

### **9.1 Inverse Function Theorem**

### **9.2 Implicit Function Theorem**

## 10 Appendix: Linear Algebra

## 11 Appendix: Group Theory

This branch of math is essentially the study of **symmetries** i.e. transformations that leave a system unchanged or **invariant**.

### Formal definition of a Group

A group is a pair  $(G, \star)$ , where  $G$  is a set and  $\star : G \rightarrow G$  is a *bilinear operation*, satisfying three properties. Namely,

(a) Associativity:  $a \star (b \star c) = (a \star b) \star c$

(b) Identity: There exists an element  $e \in G$  such that, for all  $g \in G$ ,

$$g \star e = e \star g = g$$

(c) Inverses: For each  $a \in G$ , there must exist an *inverse* element denoted  $a^{-1}$  such that

$$a \star a^{-1} = a^{-1} \star a$$

For example,

- Any vector space  $V$  with vector addition being the bilinear product  $\star$  is a group.
- The set of permutations of three objects, called the **Symmetric group of order 3**, denoted  $S_3$  is a group.

Note that in a vector space,  $v \star w = w \star v$ . This is a very special property called **commutativity** i.e. the order in which we operate group elements does not matter. Such a group is said to be **commutative** or **abelian**. In contrast to this, the symmetric group of order 3,  $S_3$  is **non-abelian** (in fact, it's the smallest such group!)

**Note:** The number of elements in a group  $G$  is called its *order* and is denoted by  $|G|$ .

### 11.1 Subgroup

#### Subgroup

- For a group  $G$ , a subset  $H \subset G$  is called a subgroup if

(a)  $e \in H$

(b)  $a, b \in H \implies ab \in H$  (this is called *closure*)

(c)  $a \in H \implies a^{-1} \in H$

- If  $H$  is a subgroup of  $G$ , we write  $H \leq G$ .

## 11.2 Coset

### Coset

- For a group  $G$  and subgroup  $H \subset G$ , we can consider an element  $g \in G, g \notin H$  and define the *left coset of  $H$*  to be

$$gH \equiv \{gh : h \in H\}$$

- There is a one to one correspondence between a subgroup  $H$  and any coset of  $H$ .

## 11.3 Lagrange's Theorem

**Theorem 11.1.** *If  $G$  is a finite group and  $H \subset G$  is a subgroup, then  $H$  divides  $G$ .*

**Proof Sketch:** Let's define the equivalence relation  $g_1 \sim g_2$  iff  $g_1^{-1}g_2 \in H$ . Suppose

$$g_2H = g_1H$$

Then

$$g_1^{-1}g_2H = H$$

Any two distinct cosets are disjoint, thus all cosets are of equal order since they are disjoint equivalence classes. So,

$$|G| = |g_1H| + |g_2H| + c \dots \quad (4)$$

$$= \underbrace{k}_{\# \text{ of cosets}} \cdot |H| \quad (5)$$

Thus,  $|H|$  divides  $|G|$ .

## 11.4 Normal subgroups

Some subgroups have the very special property that their *left and right cosets are equal*. This is unusual since group element multiplication is not generally guaranteed to be commutative.

### Normal Subgroup

A subset of a group  $G$  is said to be a **normal subgroup** if

- It is a subgroup
- $gN = Ng$

A subgroup is denoted as  $N \trianglelefteq G$ .

Normal subgroups allow us to produce a very important class of groups called *Quotient Groups*.

## 11.5 Quotient Groups

### Quotient Group

Given a group  $G$  and normal subgroup  $N \trianglelefteq G$ , we can define the **quotient group** as

$$G/N \equiv \{[a] = aN : a \in G\}$$

**Exercise:** Verify that the bilinear product on the quotient group

$$[a] \star [b] = [ab]$$

is **well defined** i.e. does not depend on the specific elements  $a, b$  chosen to represent the equivalence classes  $[a], [b]$ .

## 11.6 Group Homomorphisms

So far we've spoken about a group and its subgroups, all stuff that was relatively self-contained. However, in math, we often want to study *maps between objects*. The "natural" map between groups is called a *homomorphism*.

Now, the numbers 1 to 10 in english are "One", "Two", ..., "Ten" whereas in say Spanish they are "Unos", "Dos", ..., "Diez". Furthermore, to convey the idea of adding numbers in English, we can say "One **plus** Two" whereas the same idea in Spanish would be "Uno más Dos".

Clearly the numbers 1 to 10 are the same regardless of the names we use to describe them in different languages, and similarly for the process of adding them (which is a *really* just a bilinear product). So, there is a sort of mapping between the two.



Drawing from this (imperfect) analogy, a homomorphism between two groups is a like a dictionary, which maps the words between the two languages and allows us to translate.

### Group Homomorphisms and Isomorphisms

- For groups  $(G, \cdot)$  and  $(H, \star)$ , a map  $\phi : G \rightarrow H$  is a homeomorphism if

$$\phi(a \cdot b) = \phi(a) \star \phi(b)$$

- Further, if the map is *bijective* then it is called an **Isomorphism**. We then say  $G$  and  $H$  are isomorphic, denoted as  $G \cong H$ .

- For example, a group  $G$  is isomorphic to *itself*. The map  $G \rightarrow G$  is then called an **automorphism**

- An **inner automorphism** is of the form

$$\phi_h(g) = h^{-1}gh$$

i.e. it conjugates each element  $g \in G$  with respect to some particular  $h \in G$ .

Group Homomorphisms are *structure preserving*.

### 11.7 Kernel, Image, and the First Isomorphism Theorem:

Similar to vector spaces, we define the *Kernel* and *image* of a map  $\phi : G \rightarrow H$  to be

$$\text{Ker}(\phi) \equiv \{g \in G : \phi(g) = e\}$$

$$\text{Im}(\phi) \equiv \{h \in H : h = \phi(g), g \in G\}$$

Then, the first isomorphism theorem is as follows:

**Theorem 11.2. First Isomorphism Theorem:** For groups  $G, H$  and map  $\phi : G \rightarrow H$

- (a)  $\text{Ker}(\phi)$  is a normal subgroup of  $G$  i.e.  $\text{Ker}(\phi) \trianglelefteq G$ .
- (b)  $\text{Im}(\phi)$  is a subgroup of  $H$  i.e.  $\text{Im}(\phi) \leq H$ .
- (c) The quotient of  $G$  by  $\text{ker}(\phi)$  is isomorphic to  $\text{Im}(\phi)$  i.e.

$$G/\text{Ker}(\phi) \cong \text{Im}(\phi)$$

**Proof. (Write this up)**

□

## 11.8 Group Presentations (TO DO)

## 12 Extra: Measure Theory

## 13 Extra: Category Theory

## 14 Extra: Homotopy and Homology

## 15 Extra: Cohomology and Spectral Sequences

## References

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- [2] John M. Lee. *Introduction to Smooth Manifolds, 2nd edition*. Springer, 2013.
- [3] Frederic P. Schuller Simon Rea. Lectures on the geometric anatomy of theoretical physics. <https://mathswithphysics.blogspot.com/2016/07/lectures-on-geometric-anatomy-of.html>, 2014.
- [4] Robert M. Wald. *Lecture Notes on General Relativity*. The University of Chicago Press, 1984.