Physics 137B Lecture 8

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These are notes taken from lectures on Quantum Mechanics delivered by Professor Raúl A. Briceño for UC Berekley's Physics 137B class in the Sprng 2024 semester.

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Recap

- Last time we considered a system with two-fold degeneracy i.e. two orthonormal states ψ_a, ψ_b with the same energy E. The energies being the same causes our formula for the first-order correction obtained in non-degenerate PT to diverge.
- However, we found that if we consider linear combinations of the degenerate states $|\Psi\rangle_\pm = \alpha|\psi_a\rangle + \beta|\psi_b\rangle$ and solve the eigenvalue problem

$$\mathbf{W} \cdot \tilde{\mathbf{V}} = E^{(0)}\tilde{\mathbf{V}}$$

  where

$$\mathbf{W} = \begin{pmatrix} H'_{aa} & H'_{ab} \\ H'_{ba} & H'_{bb} \end{pmatrix}$$

  then we obtain α, β such that $|\Psi\rangle_\pm$ diagonalize the degenerate subspace and allow us to lift the degeneracy.

- We saw an example in the 2D Harmonic Oscillator, where we wrote the unperturbed hamiltonian in terms of raising and lowering operators as

$$\hat{H}_0 = \hbar\omega\left(\hat{a}_+\hat{a}_- + \hat{b}_+\hat{b}_- + 1\right)$$

  and then introduced the perturbation

$$\hat{H}' = \lambda m\omega^2\hat{x}\hat{y}$$

- We were able to solve this problem exactly by changing to *normal coordinates* and found the energy to be

$$E_{nn'} = \left(n + \frac{1}{2}\right)\hbar\omega\left(1+\lambda\right)^{1/2} + \left(n' + \frac{1}{2}\right)\hbar\omega\left(1-\lambda\right)^{1/2}$$

Today, we will solve for the first order correction using Perturbation Theory and check that it agrees with the exact solution.

1.1 Perturbation Theory approach to 2D Harmonic Oscillator

- The perturbation $\hat{H}' = \lambda m\omega^2\hat{x}\hat{y}$ can be written in terms of the raising and lowering operators as

$$\hat{H}' = \frac{\lambda\hbar\omega}{2}\left(\hat{a}_+ + \hat{a}_-\right)\left(\hat{b}_+ + \hat{b}_-\right)$$

- The unperturbed solutions can be written as

$$\begin{aligned} |n, n'\rangle^{(0)} &= |n^{(0)}\rangle_a \otimes |n'\rangle_b^{(0)} \\ &= \frac{(\hat{a}_+)^n}{\sqrt{n!}}\frac{(\hat{b}_+)^{n'}}{\sqrt{(n')!}}|00\rangle^{(0)} \end{aligned}$$

  where $|00\rangle^{(0)}$ is the unperturbed ground state.

- These are all the tools we need to approach the problem. Let's now calculate the first order energy correction to the first excited state.

Let's get to evaluating the $\mathbf{W}$ matrix.

$$\mathbf{W} = \begin{pmatrix} \langle 10|\hat{H}'|10\rangle & \langle 10|\hat{H}'|01\rangle \\ \langle 01|\hat{H}'|10\rangle & \langle 01|\hat{H}'|01\rangle \end{pmatrix}$$

The algebra here seems tedious but we can notice that the perturbation contains raising and lowering operators, which means the diagonal elements are all going to be zero.

Further,

$$\begin{aligned} \hat{H}'|10\rangle &= \frac{\hbar\omega}{2}\left(\hat{a}_+ + \hat{a}_-\right)\left(\hat{b}_+ + \hat{b}_-\right)|10\rangle \\ &= \frac{\hbar\omega}{2}\left(\hat{b}_+ + \hat{b}_-\right)\left(\sqrt{2}|20\rangle + |00\rangle\right) \\ &= \frac{\hbar\omega}{2}\left(\sqrt{2}|21\rangle + |01\rangle\right) \end{aligned}$$

So,

$$\begin{aligned} \langle 10|\hat{H}'|10\rangle &= 0 \\ \langle 01|\hat{H}'|10\rangle &= \frac{\hbar\omega}{2} \end{aligned}$$

Calculating the other column values, we find

$$\mathbf{W} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now that we've found $\mathbf{W}$, we can find the first-order energy corrections:

$$\begin{aligned} E_0^{(1)} &= \frac{1}{2}\left[H'_{aa} + H'_{bb} + \sqrt{(H'_{aa} - H'_{bb}) + 4|E'_{ab}|^2}\right] \\ &= \pm\frac{1}{2}\sqrt{4\cdot\left(\frac{\hbar\omega}{2}\right)^2} \\ &= \pm\frac{\hbar\omega}{2}\lambda \end{aligned}$$

So the energy corrections to the first excited state are

$$\boxed{E_{\pm}^{(1)} = \pm\frac{\hbar\omega}{2}\lambda}$$

Let's now compare this result to the exact solution, which is given by

$$E_{nn'} = \left(n + \frac{1}{2}\right)\hbar\omega\,(1+\lambda)^{1/2} + \left(n' + \frac{1}{2}\right)\hbar\omega\,(1-\lambda)^{1/2}$$

So, the exact $|01\rangle$ energy is

$$\begin{aligned} E_{0,1} &= \left(0+\frac{1}{2}\right)\hbar\omega \underbrace{(1+\lambda)^{1/2}}_{=\left(1+\frac{\lambda}{2}+\cdots\right)} + \left(1+\frac{1}{2}\right)\hbar\omega \underbrace{(1-\lambda)^{1/2}}_{=\left(1-\frac{\lambda}{2}+\cdots\right)} \\ &= \left(0+\frac{1}{2}+1+\frac{1}{2}\right)\hbar\omega + \lambda\left(\frac{1}{4}-\frac{1}{2}-\frac{1}{4}\right)\hbar\omega + \mathcal{O}(\lambda^2) \\ &\approx \underbrace{2\hbar\omega}_{=E_{01}^{(0)}} - \underbrace{\frac{\hbar\omega}{2}\lambda}_{E_{01,(-)}^{(1)}} \end{aligned}$$

and this matches up with the result we using the $\mathbf{W}$ matrix. Similiarly, we can check that the results match up for the $|10\rangle$ energy correction, or more generally the $|n,n'\rangle$ correction.

For the $|n,n'\rangle$ case, we have

$$\begin{aligned} E_{n,n'} &= \left(n+\frac{1}{2}\right)\hbar\omega \underbrace{(1+\lambda)^{1/2}}_{=\left(n'+\frac{\lambda}{2}+\cdots\right)} + \left(1+\frac{1}{2}\right)\hbar\omega \underbrace{(1-\lambda)^{1/2}}_{=\left(1-\frac{\lambda}{2}+\cdots\right)} \\ &= \left(n+\frac{1}{2}+n'+\frac{1}{2}\right)\hbar\omega + \lambda\left(\frac{n}{2}\frac{1}{4}-\frac{n'}{2}-\frac{1}{4}\right)\hbar\omega + \mathcal{O}(\lambda^2) \\ &\approx \underbrace{(n+n'+1)\hbar\omega}_{=E_{n,n'}^{(0)}} - \underbrace{(n-n')\frac{\hbar\omega}{2}\lambda}_{E_{n,n'}^{(1)}} \end{aligned}$$

Note: If the states $|a\rangle, |b\rangle$ are degenerate *but still* give us a $\mathbf{W}$ matrix whose off diagonals are trivial,

$$\mathbf{W} = \begin{pmatrix} H'_{aa} & 0 \\ 0 & H'_{bb} \end{pmatrix}$$

then the eigenvalue problem is solved by default! The $|a\rangle-$state energy correction and the $|b\rangle-$state energy correction are simply H'_{aa} and H'_{bb}

$$\begin{aligned} E_a^{(1)} &= H'_{aa} \\ E_b^{(1)} &= H'_{bb} \end{aligned}$$

In other words, the basis we started off with was already the "good" basis! We didn't need to solve the Eigenvalue problem to find a different degenerate-eigenspace-basis to diagonalize it.

1.2 Good States

Theorem: If $|a\rangle, |b\rangle$ are degenerate with respect to $\hat{H}$ and there exists a hermitian operator $\hat{A}$ such that

1. The states $|a\rangle, |b\rangle$ are eigenstates of $\hat{A}$ with distinct eigenvalues ($a \neq b$)

$$\hat{A}|a\rangle = a|a\rangle$$
$$\hat{A}|b\rangle = b|b\rangle$$

   and $\hat{A}$ commutes with the unperturbed hamiltonian as well as the perturbation,

2. $$[\hat{A}, \hat{H}'] = [\hat{A}, \hat{H}_0] = [\hat{A}, \hat{H}] = 0$$

then the eigenvectors of $\hat{A}$ form a "good basis" to use in the perturbation theory.

Proof: Let $\hat{H}(\lambda) = \hat{H}_0 + \lambda\hat{H}'$ and $\hat{A}$ commute.
[Finish later]