Q2. We have a particle in a 3D Potential given by $V(r) = \beta r^2$

$$= \beta(x^2+y^2+z^2)$$

So, TISE reads as

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]+\beta(x^2+y^2+z^2)\right\}\psi = E\psi$$

Let's look for separable solutions

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Assuming such a solution, TISE reads as

$$-\frac{\hbar^2}{2m}\left[YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}\right]$$

$$+ \left(\beta x^2 + \beta y^2 + \beta z^2\right)XYZ = E\cdot XYZ$$

Dividing through by XYZ, we have

$$\mathrm{I}\quad \left\{ -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}+\beta x^2 \right\}$$ only x dependence

$$\overset{\mathrm{II}}{+}\left\{ -\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2}+\beta y^2 \right\} = E$$ only y-dependence

$$\underset{\mathrm{III}}{+}\left\{ -\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2}+\beta z^2 \right\}$$ only z-dependence

In order for this to hold true for all (x,y,z), it must be the case that each of I, II, III is individually equal to some constant, say E_x, E_y, E_z resp.

Let us first consider I :

We have

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}+\beta x^2 = E_x$$

(see next pg.)

Multiplying through by X,

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + \beta x^2 X = E_x X$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \beta x^2\right] X(x) = E_x X(x). \quad \text{—— ①}$$

This looks just like a 1D QHO! The only difference is, for a 1D QHO, we would have $\beta = \frac{1}{2}m\omega^2$.

We can write ① in terms of $\hat{P}$ and $\hat{X}$ ($\hat{X}$ is the position operator while $X(x)$ is the wavefunction. For now, let's denote wave function $X(x)$ as $\psi(x)$, not to be confused with the overall wavefunction $\psi(x,y,z)$).

as $\left[\frac{1}{2m}\hat{P}^2 + \beta\hat{X}^2\right]\psi(x) = E\psi(x)$.

let $\beta = \frac{1}{2}m\omega^2$ with ω being the appropriately chosen frequency.

$\Rightarrow \omega = \sqrt{\frac{2\beta}{m}}$

Then, our system in the x-direction is modelled exactly by a 1D QHO.

This is a problem we have already solved! We can use results already obtained!

So, the allowed energies in the x-direction are given by $E_{n_x} = (n_x + \frac{1}{2})\hbar\omega$

and the energy eigenstates are

$$X_{n_x}(x) = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^n X_0(x)$$

raising operator, as defined usually.

($X_0(x)$ on next pg)

$$X_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

with $\omega = \sqrt{\frac{2\beta}{m}}$

Now, the exact same reasoning holds for the y and z directions

Therefore, the Energy of the system is given by

$$E = E_x + E_y + E_z$$

$$\Rightarrow \boxed{E_{n_x n_y n_z} = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega}$$

where $n_x, n_y, n_z = 0, 1, 2, \ldots$

(a). Thus, the Ground State Energy of the system, corresponding to $n_x = n_y = n_z = 0$ is

$$\boxed{E_{000} = \frac{3}{2}\hbar\omega}$$

(b). The Ground State Wavefunction of the particle is

$$\psi_{000}(x,y,z) = X_0(x)\,Y_0(y)\,Z_0(z)$$

$$\Rightarrow \boxed{\psi_{000}(x,y,z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)}}$$

with $\omega = \sqrt{\frac{2B}{m}}$

(c). To find the degeneracy of the n^{th} level, we want to find the number of pairs (n_x, n_y, n_z) which sum to n.

Using the stars-and-bars method,

$$d(n) = \binom{n+1}{2} = \frac{(n+1)!}{2!\,(n-1)!}$$

$$\rightarrow \boxed{d(n) = \frac{n(n+1)}{2}}$$

Empty Pg. Couldn't get rid of it.

Q3. The wavefunction ψ_{nlm} of a Hydrogen atom is given by

$$\psi_{nlm} = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

where $Y_l^m(\theta,\phi)$ is a Spherical Harmonic and $R_{nl}(r)$ is the Radial wavefunction, given by

$$R_{nl}(r) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\right]$$

So, to find $\psi_{433}(r,\theta,\phi)$, we want to find the functions

$$R_{43}(r) \quad \text{and} \quad Y_3^3(\theta,\phi).$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\phi} P_l^m(\cos\theta)$$

$$\Rightarrow Y_3^3(\theta,\phi) = \sqrt{\frac{7}{4\pi}\cdot\frac{1}{6!}}\, e^{i3\phi} P_3^3(\cos(\theta))$$

$$\Rightarrow \boxed{Y_3^3(\theta,\phi) = \sqrt{\frac{35}{64\pi}}\; e^{i3\phi} \sin^3(\theta)}$$

and

$$L_q^p(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_{p+q}(x).$$

where

$$L_q(x) = \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q (e^{-x}x^q)$$

Thus,

$$L_0^7(x) = (-1)^7 \left(\frac{d}{dx}\right)^7 L_7(x)$$

and

$$L_7(x) = \frac{e^x}{7!}\left(\frac{d}{dx}\right)^7 (e^{-x} x^7).$$

Carrying the appropriate calculations out, we find that

$$\boxed{L_0^7(x) = 1}$$

Thus, $\boxed{L_0^7\left(\frac{r}{2a}\right) = 1}$

So, the radial wavefunction, $R_{43}(r)$ is

$$R_{43}(r) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}}\, e^{-r/na}\left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\right]$$

$$= \sqrt{\left(\frac{2}{4a}\right)^3 \frac{(4-3-1)!}{2(4)(4+3)!}}\, e^{-r/4a}\left(\frac{2r}{4a}\right)^l L_{4-3-1}^{2(3)+1}\left(\frac{2r}{4a}\right)$$

$$\Rightarrow R_{43}(r) = \sqrt{\frac{1}{(2a)^3}\frac{0!}{8\times 7!}}\, e^{-r/4a}\left(\frac{1}{2a}\right)^{l} \underbrace{L_0^7\left(\frac{r}{2a}\right)}_{1}$$

$$\Rightarrow R_{43}(r) = \frac{1024\sqrt{35}}{105(16a)^{3/2}}\left[\frac{r^3}{(16a)^3}\right]\exp\left(\frac{-4r}{16a}\right)$$

Thus, the wavefunction is

$$\psi_{433}(r,\theta,\phi) = R_{43}(r)\, Y_3^{-3}(\theta,\phi)$$

$$= \frac{1024\sqrt{35}}{105(16a)^{3/2}}\left[\frac{r^3}{(16a)^3}\right] e^{-\frac{4r}{16a}}$$

$$\times \sqrt{\frac{35}{64\pi}}\, e^{i3\phi}\sin^3(\theta)$$

$$\Rightarrow \psi_{433}(r,\theta,\phi) = \frac{1024\times 35}{105\,(8\sqrt{\pi})}\cdot\frac{1}{(16a)^{3/2}}\left[\frac{r^3}{(16a)^3}\right]$$

$$\times e^{-\frac{4r}{16a}}\, e^{+i3\phi} \times \sin^3(\theta)$$

$$\Rightarrow \boxed{\psi_{433}(r,\theta,\phi) = \frac{1}{6144\sqrt{\pi}}\cdot\frac{r^3}{a^{9/2}}\cdot e^{-\frac{r}{4a}}\, e^{i3\phi}\sin^3(\theta)}$$

(b). To find the expected value of r in this state, we find

$$\langle r \rangle = \langle \psi_{433} | r | \psi_{433} \rangle$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \psi_{433}^{*}(r,\theta,\phi)\, r\, \psi_{433}(r,\theta,\phi) \cdot r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$\Rightarrow \langle r \rangle = \frac{1}{(6144)^2 \pi a^9} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \left\{ \begin{array}{l} \left(r^3 e^{-\frac{r}{4a}} e^{-i3\phi} \sin^3\theta\right) r \\ \times \left(r^3 e^{-\frac{r}{4a}} e^{i3\phi} \sin^3\theta\right) \end{array} \right\} \times r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$\Rightarrow \langle r \rangle = \frac{1}{6144^2 \pi a^9} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} r^9 e^{-\frac{r}{2a}} \sin^7(\theta)\, dr\, d\phi\, d\theta$$

$$= \frac{1}{6144^2 \pi a^9} \times (2\pi) \times \int_0^{\infty} r^9 e^{-\frac{r}{2a}} dr \times \int_0^{\pi} \sin^7(\theta)\, d\theta$$

Let $I_1 = \int_0^{\pi} \sin^7(\theta)\, d\theta.$

Then $I_1 = \int_0^{\pi} (\sin^6\theta) \sin\theta \, d\theta$

$$= \int_0^{\pi} (1-\cos^2\theta)^3 \sin\theta \, d\theta$$

Let $u = \cos\theta \Rightarrow du = -\sin\theta \, d\theta$

When $\theta = 0$, $u = 1$
$\theta = \pi$, $u = -1$

$$\Rightarrow I_1 = \int_{-1}^{1} (1-u^2)^3 \, du$$

$$(1-u^2)^3 = (1-u^2)(1-u^2)(1-u^2)$$
$$= (1-u^2) + (-u^2+u^4) + (-u^2+u^4) + (u^4-u^6)$$

$$\Rightarrow I_1 = \int_{-1}^{1} (1 - 3u^2 + 2u^4 - u^6) \, du$$

This is easy to calculate, and we get

$$\boxed{I_1 = \frac{32}{35}}$$

Next, let $I_2 = \int_0^\infty r^9 e^{-r/2a} \, dr$

Let $\frac{r}{2a} = t \Rightarrow r = t(2a)$

$\Rightarrow dr = 2a \, dt$

When $r = 0$, $t = 0$.
$r = \infty$, $t = \infty$.

Thus, $I_2 = (2a)^{10} \int_0^\infty t^9 e^{-t} \, dt$

At this point, I gave up and used an Integral calculator:

$$\boxed{I_2 = (2a)^{10} \times 362880}$$

So, $\langle r \rangle = \frac{2}{6144^2 a^9} \times I_1 \times I_2$

$= \frac{2}{6144^2 a^9} \times \frac{32}{35} \times (2a)^{10} \times 362880$

$$\Rightarrow \boxed{\langle r \rangle = 18a}$$, where a is the Bohr Radius

(c). Now, $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

So, $\hat{L}_x^2 + \hat{L}_y^2 = \hat{L}^2 - \hat{L}_z^2$

and since $[\hat{L}, \hat{L}_z] = 0$, we can measure definite values for $\hat{L}, \hat{L}_z$ simultaneously.

- $\hat{L} |n, l, m\rangle = \hbar l(l+1) |n, l, m\rangle$
- $\hat{L}_z |n, l, m\rangle = \hbar m |n, l, m\rangle$

So, $\hat{L} \psi_{433} = \hbar 3(3+1) \psi_{433}$

$\hat{L}_z \psi_{433} = \hbar 3 \psi_{433}$

So, if we were to measure $(\hat{L}_x + \hat{L}_y)$, we would get

$$(\hat{L}_x + \hat{L}_y)\psi_{433}(r,\theta,\phi) = (\hat{L} - \hat{L}_z)\psi_{433}(r,\theta,\phi)$$
$$= 3(4)\hbar\,\psi_{433}(r,\theta,\phi) - 3\hbar\psi_{433}(r,\theta,\phi)$$

$$\Rightarrow \boxed{(\hat{L}_x + \hat{L}_y)\psi_{433}(r,\theta,\phi) = 3\hbar\psi_{433}(r,\theta,\phi)}$$

Thus, if we were to measure the quantity $(L_x + L_y)$ of the state $\psi_{433}(r,\theta,\phi)$, we would obtain the value $3\hbar$.