Physics 5C: Introductory Thermodynamics and Quantum Mechanics **Fall 2023**

Homework 5:

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Q1.

Show that in the high temperature, infrared limit (i. e. low frequency or long wavelength), Planck's Black-Body radiation formula reduces to Rayleigh-Jeans formula.

Sol:

Plack's black-body radiation formula states that the energy per unit volume per hertz radiated away by a black-body at frequency ν and absolute temperature T is given by

$$\boxed{u_\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{\exp(h\nu/k_bT) - 1}}$$

In the high temperature-infrared limit, we have $\frac{T}{\nu} >> h$, so $\frac{h\nu}{{}_BT}$ is a very small number.

Then, by taylor expanding to the first order, we have

$$\exp\left(\frac{h\nu}{k_BT}\right) \approx 1 + \frac{h\nu}{k_BT}$$

Therefore, u_ν is given by

$$\begin{aligned}
u_\nu &\approx \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{\left(1 + \frac{h\nu}{k_BT}\right) - 1} \\
&= \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{\frac{h\nu}{k_BT}} \\
&= \frac{8\pi h\nu^3}{c^3} \cdot \frac{k_BT}{h\nu} \\
&= \frac{8\pi h\nu^2}{c^3} \cdot k_BT
\end{aligned}$$

So, in the high-temperature, infrared limit, we have

$$\boxed{u_\nu = \frac{8\pi h\nu^2}{c^3} \cdot k_BT}$$

This expression is exactly the Rayleigh-Jeans formula!

Q2.

1. Consider an electron moving around a positively charged proton in a circular orbit. As it revolves, it emits energy in the form of synchrotron radiation.Write down the expression of energy loss per turn (any method will do, this will be a ball-park estimate!), in terms of the electron's energy, circulation radius, electron's rest mass and classical electron radius (2.82x10-15m):

   Sol:

   The Power radiated by an electron orbiting a proton can be found using the Larmor equation, and evaluates to

$$P = \frac{2cr_e}{3(m_0c^2)^3} \cdot \frac{E^4}{\rho^2}$$

   Now, the power is just the negative of the rate of change of energy. That is,

$$P = -\frac{dU}{dt}$$

   (Didn't complete Q2).

Q3.

What is the momentum of a proton with kinetic energy 1 GeV?

Sol: The Relativistic Kinetic Energy of a particle is given by

$$E_k = (\gamma - 1)m_0c^2$$

and the Relativistic momentum is given by

$$p = \gamma m_0 v$$

where m_0 is the rest-mass, v is the relative velocity of the particle's frame wrt the observer frame, and $\gamma = 1/\left(1 - \frac{v^2}{c^2}\right)$.

So, if we know E_k and m_0, we can find v as

$$\begin{aligned}
&\frac{E_k}{m_0c^2} + 1 = \gamma \\
\implies &\frac{E_k}{m_0c^2} + 1 = \frac{1}{1 - \frac{v^2}{c^2}} \\
\implies &1 - \frac{v^2}{c^2} = \frac{1}{\frac{E_k}{m_0c^2} + 1} \\
\implies &1 - \frac{1}{\frac{E_k}{m_0c^2} + 1} = \frac{v^2}{c^2} \\
\implies &v^2 = c^2\left[1 - \left(\frac{1}{\frac{E_k}{m_0c^2} + 1}\right)\right] \\
\implies &v = c\sqrt{1 - \left(\frac{1}{\frac{E_k}{m_0c^2} + 1}\right)} \\
\implies &v = c\sqrt{1 - \frac{1}{\gamma}}
\end{aligned}$$

Now, we know that the rest-mass of a proton is $m_0 = 1.6726231 \times 10^{-27} kg$ and the Kinetic Energy in this case is $E_k = 1GeV = 1.602176634 \times 10^{-10} J$.

So, we can now evaluate the Relativistic Momentum using $p = \gamma m_0 v$ and the expressions obtained for v and γ.

We then have

$$\gamma = \frac{E_k}{m_0c^2} + 1 = 2.065788175$$

$$v = c\sqrt{1 - \frac{1}{\gamma}} = 215334322.4 m/s$$

So,

$$\boxed{p = \gamma m_0 v = 7.44041459 \times 10^{-19} kg \cdot ms^{-1}}$$

Q4.

Consider Compton scattering of light of wavelength 1 micron off an electron at rest. What will be the wavelength of Compton-scattered light at right angles to the forward direction?

Sol:

We have light of wavelength 1 micron i.e. frequency $\nu_0 = \frac{1}{1 \times 10^{-6} m} = 10^6$ Hz which scatters off an electron at rest.

The frequency of the Compton-scattered light in the direction making angle θ with the forward direction is given by

$$\boxed{\nu = \frac{\nu_0}{1 + \frac{h\nu_0}{mc^2}(1 - \cos(\theta))}}$$

where m is the mass of the electron (this expression agrees with experiment.)

So, the frequency of the Compton-scattered light at right angles ($\theta = m\frac{\pi}{2},\ m \in \mathbb{Z}$) to the forward direction will be

$$\begin{aligned}
\nu &= \frac{\nu_0}{1 + \frac{h\nu_0}{mc^2}(1 - 0)} \\
&= \frac{10^6 Hz}{1 + \frac{(6.62607015\times 10^{-34} J\cdot Hz^{-1})(10^6 Hz)}{(9.1093837\times 10^{-31} kg)(299792458^2 m^2\cdot s^{-2})}} \\
&= 999997.5737 Hz \\
&\approx 10^5 Hz
\end{aligned}$$

Q5.

Consider the wave-function: $\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$ where A, λ, ω are positive and real.

1. Normalize Ψ.

   Sol: We use the normalization condition to find the appropriate value of A.

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{\infty}^{\infty} \Psi(x,t)^* \Psi(x,t) dx = 1 \\
\implies &\int_{-\infty}^{\infty} \left(Ae^{-\lambda|x|}e^{+i\omega t}\right) \cdot \left(Ae^{-\lambda|x|}e^{-i\omega t}\right) dx = 1 \\
\implies &\int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx = 1 \\
\implies &A^2 \left[\int_{-\infty}^{0} e^{2\lambda x} dx + \int_{0}^{\infty} e^{-2\lambda x}\right] dx = 1 \\
\implies &A^2 \left[\frac{e}{2\lambda} + \frac{e}{2\lambda}\right] = 1 \\
\implies &A^2 = \frac{\lambda}{e}
\end{aligned}$$

   So, we have

$$\boxed{A = \sqrt{\frac{\lambda}{e}}}$$

2. Determine the expectation values: $\langle x \rangle$, $\langle x^2 \rangle$

   Sol: The expectation values are given by

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi(x,t)^* x \Psi(x,t) dx$$

and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi(x,t)^* x^2 \Psi(x,t) dx$$

So, we have

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \left(Ae^{-\lambda|x|}e^{+i\omega t}\right) \ x \ \left(Ae^{-\lambda|x|}e^{-i\omega t}\right) dx \\
&= A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|} dx
\end{aligned}$$

Now, x is an odd function while $e^{-2\lambda|x|}$ is an even function. So their product is an odd functon which means the integral over the range $[-\infty, \infty]$ is zero.

Therefore,

$$\boxed{\langle x \rangle = A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|} dx = 0}$$

Now, the expectation value of x^2 is

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \left(Ae^{-\lambda|x|}e^{+i\omega t}\right) \ x^2 \ \left(Ae^{-\lambda|x|}e^{-i\omega t}\right) dx \\
&= A^2 \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx \\
&= A^2 \left[\int_{-\infty}^{0} x^2 e^{2\lambda x} dx + \int_{0}^{\infty} x^2 e^{-2\lambda x} dx\right]
\end{aligned}$$

Now, we could use integration by parts for the two integrals but alternatively we can use *Integration under the Integral sign.*

Notice that

$$\begin{aligned}
\int_{-\infty}^{0} x^2 e^{2\lambda x} &= \int_{-\infty}^{0} \frac{1}{4} \frac{\partial^2}{\partial \lambda^2} \left(e^{2\lambda x}\right) dx \\
&= \frac{\partial^2}{\partial \lambda^2} \left(\int_{-\infty}^{0} \frac{e^{2\lambda x}}{4} dx\right) \\
&= \frac{\partial^2}{\partial \lambda^2} \left[\frac{e^{2\lambda x}}{8\lambda}\right]_{-\infty}^{0} \\
&= \frac{\partial^2}{\partial \lambda^2} \left(\frac{e}{8\lambda}\right) \\
&= \frac{e}{4\lambda^3}
\end{aligned}$$

Similarly, we can evaluate the other integral to find

$$\begin{aligned}
\int_0^\infty x^2 e^{-2\lambda x} dx &= \int_0^\infty \frac{1}{4}\frac{\partial^2}{\partial\lambda^2}\left(e^{-2\lambda x}\right) dx \\
&= \frac{\partial^2}{\partial\lambda^2}\left(\int_0^\infty \frac{e^{-2\lambda x}}{4} dx\right) \\
&= \frac{\partial^2}{\partial\lambda^2}\left[\frac{-e^{-2\lambda x}}{8\lambda}\right]_0^\infty \\
&= \frac{\partial^2}{\partial\lambda^2}\left(\frac{e}{8\lambda}\right) \\
&= \frac{e}{4\lambda^3}
\end{aligned}$$

So, we have

$$\begin{aligned}
\langle x^2\rangle &= A^2\left[\int_{-\infty}^0 x^2 e^{2\lambda x} dx + \int_0^\infty x^2 e^{-2\lambda x} dx\right] \\
&= A^2\left[\frac{e}{4\lambda^3} + \frac{e}{4\lambda^3}\right] \\
&= \left(\frac{\lambda}{e}\right)\cdot\left(\frac{e}{2\lambda^3}\right) \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

3. Find the standard deviation of x.

Sol:

The standard deviation of x is given by

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{1}{2\lambda^2} - 0^2} \\
&= \frac{1}{\lambda\sqrt{2}}
\end{aligned}$$

$$\boxed{\sigma_x = \frac{1}{\lambda\sqrt{2}}}$$

4. Plot $|\Psi|^2$ and mark points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$.

Sol:

The amplitude squared of the wavefunction is

$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)^*\Psi(x,t) \\
&= \left(Ae^{-\lambda|x|}e^{+i\omega t}\right)\left(Ae^{-\lambda|x|}e^{-i\omega t}\right) \\
&= A^2 e^{-2\lambda|x|} \\
\implies |\Psi(x)|^2 &= \frac{\lambda}{e}\cdot e^{-2\lambda|x|} \quad \text{(No time dependence)}
\end{aligned}$$

The graph of the wavefunction amplitude looks like:

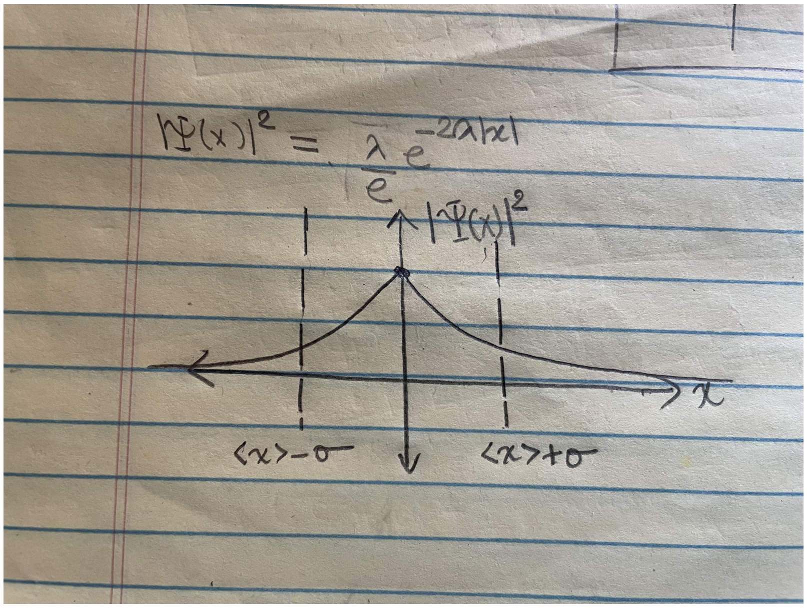

Figure 5.1: Wavefunction Amplitude Squared

5. What is the probability of finding the particle outside this range?

We know that $\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1$, and probability of finding the particle within a range $[a, b]$ is

$$\mathbb{P}(a \leq x \leq b) = \int_a^b |\Psi(x)|^2 dx$$

The probability of finding the particle with the range $[\langle x\rangle - \sigma, \langle x\rangle + \sigma] = [-\sigma, \sigma]$ is

$$\begin{aligned}
\int_{-\sigma}^{\sigma} |\Psi(x)|^2 dx &= \frac{\lambda}{e} \int_{-\sigma}^{\sigma} e^{-2\lambda|x|} dx \\
&= \frac{\lambda}{e} \cdot \left[\int_{-\sigma}^{0} e^{2\lambda x} dx + \int_{0}^{\sigma} e^{-2\lambda x} dx \right] \\
&= \frac{\lambda}{e} \left(\left[\frac{e^{2\lambda x}}{2\lambda} \right]_{-\sigma}^{0} + \left[\frac{-e^{-2\lambda x}}{2\lambda} \right]_{0}^{\sigma} \right) \\
&= \frac{\lambda}{e} \left(\frac{1}{2\lambda} - \frac{e^{-2\lambda\sigma}}{2\lambda} + \frac{-e^{-2\lambda x}}{2\lambda} + \frac{1}{2\lambda} \right) \\
&= \frac{1}{e}
\end{aligned}$$

Therefore the probability of finding the particle *outside* of this range is

$$1 - \frac{1}{e}$$