Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 4

kdeoskar@berkeley.edu

Question 1: Prove that all fibers of a Hurewicz Fibration with a path-connected base are homotopy equivalent to each other.

Solution:

Let's recall the definition. A triple (E, B, p) is a Hurewicz fibration if $p : E \to B$ is surjective, and the triple has the Covering Homotopy Property for any topological space X i.e. for mappings

1. $f : X \times [0,1] \to B, \quad F_0 : X \cong X \times \{0\} \to E$
2. with
$$f\big|_{X\times\{0\}} = p \circ F_0$$

there exists a homotopy

$$F : X \times [0,1] \to E$$

such that

1. $F\big|_{X\times\{0\}} = F_0$ and
2. $p \circ F = f$

Suppose the base space B is path-connected. Consider points $b, b' \in B$. Denote the inclusion of $p^{-1}(b)$ into E as g, and h be a continuous map from $p^{-1}(b) \times I$ to B.

$$\begin{tikzcd} p^{-1}(b) \arrow[r, "g"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ p^{-1}(b) \times I \arrow[r, "h"'] & B \end{tikzcd}$$

Since B is path-connected there must exist a path $\gamma(t) : I \to X$ between b and b'. Notice that $g \circ p$ sends every point in $p^{-1}(b)$ to $b \in B$ i.e. $h(x, 0) = b$ for any $x \in p^{-1}(b)$

Let's choose h to be

$$\begin{aligned} h : p^{-1}(b) \times I &\to B \\ (x, t) &\mapsto \gamma(t) \end{aligned}$$

where $\gamma(t)$ is a path connecting b and b'. Then, since (E, B, p) is a Hurewicz fibration, the CHP gives us $\tilde{H} : p^{-1}(b) \times I \to E$ which defines a homotopy between $p^{-1}(b)$ and $p^{-1}(b')$ (the continuous maps in either direction are obtained by varying the parameter t).

Question 2: Prove that if the fiber F of a Serre Fibration $f : E \to B$ is contractible in E, then $\pi_n(B)$ is the direct sum of $\pi_n(E)$ and $\pi_{n-1}(F)$.

Solution:

This follows directly from the following two results:

EXERCISE 14, SECTION 8.8: If A is contractible within X, then

$$\pi_n(X, A) \cong \pi_n(X) \oplus \pi_{n-1}(A)$$

LEMMA FROM SECTION 9.8: Let (E, B, p) be a Serre fibration, let $e_0 \in E$ be an arbitrary point, let $b_0 = p(e_0)$, and let $F = p^{-1}(b_0)$. Then the map

$$p_* : \pi_n(E, F, e_0) = \pi_n(B, b_0)$$

is an isomorphism for all n.

Proof of Exercise 14:

Suppose A is contractible in X. Let j denote the identity map $X \to X$ regarded as a map $(X, x_0) \to (X, A)$ and the induced homomorphism is $j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0)$ $i : A \to X$ is the inclusion map, ∂ denotes the connecting homomorphism.

Then, we have the homotopy sequence of the pair:

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0)$$
$$\cdots \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

which we know is exact.

Consider this part of the sequence:

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

Recall that we say A is contractible in X if the inclusion $i : A \to X$ is homotopic to a constant map $A \to X$. So, clearly, $i_* : \pi_n(A) \to \pi_n(X)$ is a zero homomorphism.

Now, j_* is injective and ∂ is surjective. Thus, we have $\pi_n(X, A, x_0) \cong \pi_n(X) \oplus \pi_{n-1}(A)$.

Question 3: Compute the third homotopy groups of the Unitary groups $U(n)$ for all n.

Solution:

Let's first find the 3rd homotopy groups for $U(1)$ and $U(2)$ using the fibration provided by the determinant map $\det : U(2) \to U(1)$. Under det, each $c \in U(1)$ has fiber $SU(2) \cong \mathbb{S}^2$ (intuitively, because we can rotate by any amount without impacting the determinant)

The fibration induces the homotopy sequence:

$$\pi_4(U(1)) \to \pi_3(\mathbb{S}^3) \to \pi_3(U(2)) \to \pi_3(U(1))$$

Note that since $U(1)$ is homeomorphic to $\mathbb{S}^1$, we have $\pi_3(U(1)) \cong \pi_3(\mathbb{S}^1) = 0$ and also $\pi_4(U(1)) \cong \pi_4(\mathbb{S}^1) = 0$ so our sequence is

$$0 \to \pi_3(\mathbb{S}^3) \to \pi_3(U(2)) \to 0$$

Thus, $\pi_3(U(2)) \cong \pi_3(\mathbb{S}^3) = \mathbb{Z}$.

Now for larger n, let's use the fact that

$$\mathbb{S}^{2n-1} = U(n)/U(n-1)$$

and so we have fiber bundle structure $U(n-1) \hookrightarrow U(n) \to \mathbb{S}^{2n-1}$ which induces the sequence

$$\pi_{k+1}(\mathbb{S}^{2n-1}) \to \pi_k(U(n)) \to \pi_k(U(n-1)) \to \pi_k(\mathbb{S}^{2n-1})$$

so, for $k < 2n$ we have

$$\pi_k(U(n)) = \pi_k(U(n-1))$$

Thus, $\pi_3(U(n)) = \mathbb{Z}$ for $n > 1$.