Math H185 Notes

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 January 17 - Introduction to Complex Numbers

1.1 Real Numbers

Before jumping into Complex Numbers, let's recall a property of Real Numbers - the set containing which is denoted $\mathbb{R}$.

Note: If $a \in \mathbb{R}$ then $a^2 \geq 0$. So, in this number system negative real numbers do not have square roots in $\mathbb{R}$.

This is a limitation of $\mathbb{R}$, which we can fix by enlargening our field. (Similar to how the set of rationals was enlargened to the set of reals in Real Analysis).

1.2 Imaginary Numbers

We can introduce a new kind of object called an "Imaginary number" such that imaginary numbers square to negatve (≤ 0) real numbers.

We write $i = \sqrt{-1}$.

Proposition: Any imaginary number can be expressed as bi, $b \in \mathbb{R}$.

Proof: Consider any $x \leq 0$. Then, $-x \geq 0$ which means there exists some $b \in \mathbb{R}^+$ such that $-x = b^2$. Then

$$(bi)^2 = b^2 i^2 = -b^2 = x$$

1.3 Complex Numbers

Complex Numbers

- A complex number is an expression $z = a + bi$ where $a, b \in \mathbb{R}$
- The set of complex numbers is denoted $\mathbb{C}$

Remark: $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

In a sense, this is saying that there are no more "deficiencies" - Unlike polynomials in the reals, *every* complex polynomials is guaranteed to have some complex roots. We will return to this statement later in the course when studying the Fundamental Theorem of Algebra.

Let $z = a + bi$ be a complex number. Then,

- The *real part* of z is $Re(z) = a \in \mathbb{R}$ and the *imaginary part* of z is $Im(z) = b \in \mathbb{R}$.
- The *complex conjugate* of z is $\bar{z} = a - bi$

1.4 Operations on Complex Numbers

"Addition is componentwise"

$$\begin{aligned} \text{Addtion: } z &= a + bi \\ +w &= c + di \\ z + w &= (a + c) + (b + d)i \end{aligned}$$

”Multiplication distributes”
For $z = (a + bi)$, $w = (c + di)$ we have

$$\begin{aligned} z \cdot w &= (a + bi) \cdot (c + di) \\ &= a \cdot (c + di) + bi \cdot (c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Addition and Multiplication satisfy the usual commutativity, associativity, and distributivity. However, Division is a bit more complicated.

Division: If $z \in \mathbb{C}$, $w \in \mathbb{C} \backslash \{0\}$, then $z/w \in \mathbb{C}$ is the unique complex number such that $w \cdot (z/w) = z$.

Examples: Write the following complex numbers as $a + bi$ where $a, b \in \mathbb{R}$

1. $(9 - 12i) + (12i - 16) = (9 - 16) + (-12i + 12i) = -7$
2. $(3 + 4i) \cdot (3 - 4i) = 9 - 12i + 12i - 16i^2 = 25$
3. $\frac{50+50i}{3-4i} = \frac{50+50i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{150+200i+150i+200i^2}{25} = \frac{-50+350i}{25} = -2 + 14i$