Math 214 Notes

Keshav Balwant Deoskar

March 12, 2024

These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 March 12 -

Recap

- Last time, we defined Lie Groups (ossessing both group and smooth manifold structure) and Lie Group Homomorphisms.
- This time we'll define Lie Subgroups.

1.1 Lie Subgroups

Given a Lie group G, $H \subseteq G$ is a ***Lie subgroup*** if it is the image of an injective Lie Group Homomorphism.

Remark: If $H \subseteq G$ is an embedded(immersed? check later) submanifold and subgroup, then H is a Lie subgroup.

Examples:

- $GL(n, \mathbb{R})$ has lie subgroups $SK(n, \mathbb{R})$ and $O(n)$.
- Taking $G = T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ consider the map $F : \mathbb{R} \to T^2$ defined by

$$t \mapsto \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right)$$

  - If $\alpha \in \mathbb{Q}$, then F is periodic and thus not injective, so $\ker F$ is nontrivial.

    If we find the period k, we can define $\tilde{F} : \mathbb{S}^1 \to T^2$ by

$$e^{2\pi i t} \mapsto \left(e^{2\pi i t \cdot k}, e^{2\pi i t k \alpha}\right)$$

    With this map, we see that $\mathbb{S}^1 \subseteq \mathcal{T}^2$ is a Lie Subgroup.

  - If $\alpha \in \mathbb{Q}$, then $F : \mathbb{R} \to \mathcal{T}^2$ s injective, so $F(\mathbb{R}) \subseteq T^2$ is a Lie Subgroup. This is an example of a lie subgroup which is not embedded, but it immersed.

Lemma: If $H \subseteq G$ is an open subgroup, then H is the union of connected components of G.

Proof:

$$G = \bigcup_{g \in G} gH = \bigcup_{g \in G} \underbrace{L_g(H)}_{\text{open}}$$

Theorem: If G is connected, then any neighborhood of $e \in G$ generates G.

Proof: Let $H \subseteq G$ be the subgroup generated by a neighborhood U of the identity i.e. $e \in U$. Thus, $U \subseteq H$.

Now, notice that for any $g \in H$, $L_g(U) \subseteq H$ and $L_g(U)$ is open so H is open.

Then, by the previous lemma, H is the union of connected subgroups of G. This implies $H = G$. [Add more detail later]

There are many examples of Lie Groups which are *not* connected eg. $GL(n, \mathbb{R})$ (Look at the determinant map).

T

he ***Identity component*** $G_0 \subseteq G$ is the connected component of G containing $e \in G$.

Theorem: G_0 is a Lie group.

Proof:

$$e \in$$

[Finish later]

Theorem: If $H \subseteq G$ is a Lie subgroup which is also an embedded submanifold, then H is closed in G i.e. it is a proper submanifold.

(Read in textbook; also in Chapter 20 we'll show that $H \subseteq G$ closed $\implies$ H is a Lie Subgroup submanifold)

1.2 Group actions

If G is a group and M is a set, then ***Left Group Action*** (denoted as [fill later]) is the map

$$\begin{aligned} \mathcal{O} : & G \times M \to M \\ & (g,p) \mapsto g \cdot p \end{aligned}$$

such that

$$e \cdot p = p$$

and

$$(g_1 \star g_2) \cdot p = g_1 \cdot (g_2 \cdot p)$$

Similarly, a ***Right group action*** (denoted as [fill later]) is te map

$$\begin{aligned} \mathcal{O} : & G \times M \to M \\ & (p,g) \mapsto p \cdot g \end{aligned}$$

such that

$$p \cdot e = p$$

and

$$p \cdot (g_1 \star g_2) = (p \cdot g_1) \cdot g_2$$

If G is a Lie Group and M is a smooth manifold, then these actions are smooth if $\mathcal{O}$ is a smooth map.

Remark: [Fill from image]

Examples: [fill from image]

-

Let's discuss some more notions related to group actions.

- Suppose Lie Group G acts on M. The ***Orbit*** of $p \in M$ is

$$G_p = \{gp \ : \ g \in G\}$$

- Note that two orbits are eitherr equal or disjoint.
- We denote the set of orbits as G/M.
- the ***Isotropy group*** is
- The action is ***transitive*** if $G \cdot p = M$
- The action is ***free*** if $G_p = \{e\}$ for all $p \in M$.

Examples:

- Fill later from image.

1.3 Equivariant Maps

Suppose Lie Group G acts on manifolds M, N and we have a map $F : M \to N$. We say that F is ***G-equivariant*** if

$$F\left(g \cdot p\right) = g \cdot F(p) \text{ (For left actions)}$$
$$F\left(p \cdot g\right) = F(p) \cdot g \text{ (For right actions)}$$

Examples:

- Let V be a vector space and let $GL(V)$ act on it from the left. Define the left action of $GL(V)$ on the tensor product space $V \otimes V$ as

$$g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)$$

  Then, $F : V \to V \otimes V$ defined by $v \mapsto v \otimes v$ is $G-$equivariant.

Begin addition properties, equiv. rank theorem, and orbits.

1.4 Representations

A representation of a Lie group G is a Lie Group homomorphism $\rho : G \to GL(V)$ where V is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

The representations of F correspond to smooth actions of [G acting on V] such that $v \mapsto g \cdot v$, which are linear for all $g \in G$.

Examples:

- Fill later.

Theorem: Every compact Lie Group has a faihtful (injective) representation

$$\rho : G \to GL(V)$$

for some V.

As a result, every compact Lie Group is isomorphic to the Lie subgroup of some $GL(V)$.

Lie Groups are especially useful in physics. Let's take a look at some low dimensional examples which are useful in physics.

1.5 The groups $SO(3)$, $SU(2)$