Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 3

kdeoskar@berkeley.edu

Question 1: Prove that the fundamental group of a loop space ΩX of any base point space (X, x_0) is abelian. The same for any topological group.

Solution:

Let's consider the case for a topological group G first.

Recall that a hausdorff topological space G is called a **topological group** if it is equipped with **continuous maps**

$$* : G \times G \hookrightarrow G, \quad (x, y) \mapsto xy$$
$$i : G \to G, \quad x \mapsto x^{-1}$$

First, observe that for a topological group G, the base-point doesn't mapper for the fundamental group.

Consider any two points $a, b \in G$. Then, we have a multiplication map $m_{a^{-1}b} : G \to G$, $x \mapsto x(a^{-1}b)$ obtained from $* : G \times G \to G$ by fixing the argument from the second G factor to be $a^{-1}b$. This map $m_{a^{-1}b}$ sends $a \mapsto b$ and is a homeomorphism on G, so it can be viewed as a homeomorphism between the base-pointed spaces (G, a) and (G, b).

Since it's a homeomorphism, it induces an isomorphism

$$\pi_1(G, a) \overset{(m_{a^{-1}b})_*}{\cong} \pi_1(G, b)$$

So, for convenience, we can study $\pi_1(G, e)$ where e is the identity element.

Let's show that for a topological group G, the fundamental group $\pi_1(G, e)$ is abelian. Consider two paths $c : [0, 1] \to G, \ s \mapsto c(s) \in G$ and $d : [0, 1] \to G, \ t \mapsto d(t) \in G$ starting and ending at $e \in G$. The goal is to show that cd is homotopic to dc (where cd refers to the concatenation where we do c, then d).

Draw out the 2D plane spanned by perpendicular s, t axes and consider the square $[0,1]\times[0,1]$ with $(s,t)=(0,0)$ being the bottom-left corner of the square and $(s,t)=(0,1)$ being the top-left corner. Then we can interpret a point $(s,t)\in[0,1]\times[0,1]$ as $c(s)*d(t)$ where $*$ denotes group multiplication.

Then, the path cd corresponds to traversing horizontally along the bottom-edge of the square, following by traversing vertically up the right-edge. The path dc corresponds to traversing vertically the left-edge and then horizontally the top-edge of the square.

Then, to find a homotopy between the two paths is the same as deforming one of these curves into the other while making sure we start at the bottom-left and end at the top-right, which we can certainly do thanks to the presence of the group multiplication map.

Now, recall that an **H-space** is a topological space X with an element $e\in X$ and continuous map $\mu : X\times X\to X$ such that $\mu(e,e)=e$ and the maps $x\mapsto\mu(e,x)$ and $x\mapsto\mu(x,e)$ are both homotopic to the identity map Id_X.

The exact same argument as the one provided for Topological Groups works for H-spaces except with the group multiplication replaced with the map μ.

Now, note that for a base-pointed space (X,x_0), the loop space ΩX is an H-space. Thus, the fundamental group of ΩX is abelian.

Proof that ΩX is an H-space:
The loop space ΩX of base-pointed space (X,x_0) consists of loops starting and ending at x_0 i.e.

$$\Omega X=\{\gamma \mid \gamma : [0,1]\to X \text{ is continuous and } \gamma(0)=\gamma(1)=x_0\}$$

This space has a "multiplication" map $\mu:\Omega X\times\Omega X\to\Omega X$ defined by concatenation of loops i.e. for two loops $\alpha,\beta\in\Omega X$ we have

$$\mu(\alpha,\beta)=\begin{cases}\alpha(2t), & 0\le t\le\frac{1}{2}\\ \beta(2t-1), & \frac{1}{2}<t\le 1\end{cases}$$

Certainly, we have $\mu(e,e)=e$. The maps $\alpha\mapsto\mu(\alpha,e)$ and $\alpha\mapsto\mu(e,\alpha)$ are homotopic to the identity

(Complete this soon using the pasting lemma and stuff; see https://math.stackexchange.com/questions/1755653/every-loop-space-omega-y-w-0-has-the-structure-of-an-h-group for inspiration.)

Question 2: Every non-orientable connected manifold has the *oriented* double-cover, whose fiber over a given point consists of the two orientations of (of the tangent space) at that point. Find out which of the double covers $G_+(n,k)$ over $G(n,k)$ (k other than 0, n) are orienting.

Solution: (Inspired by these Math.StackExchange posts: Fundamental groups of Grassmann and Stiefel Manifolds, Oriented Grassmannian is a 2-sheeted covering space of Grassmannian, The Oriented Grassmannian is simply connected for $n > 2$)

We know that there is a 2-to-1 projection from $G_+(n,k)$ to $G(n,k)$ for each n, k. So, the problem of figuring out which double covers $G_+(n,k)$ over $G(n,k)$ are orienting boils down to figuring out which of the $G(n,k)$ are non-oriented.

Question 3:

(a) Show that if X is locally path-connected, then the projection from $E(X, x_0)$ to X is *open* (i.e. the image of every open set is open)

(b) If, in addition, X is semi-locally simply connected, then $E(X, x_0)$ is locally path-connected.

Solution:

(a) Consider a locally path-connected base-pointed space (X, x_0). Recall that "locally path-connected" means that for every point $x \in X$ and every open neighborhood $U \ni x$ there exists an open neighborhood $V \ni x$ such that

  (a) $\overline{V} \subset U$ and

  (b) Any two points in V can be connected via a path in U.

  The space (X, x_0) has path-space $E(X, x_0)$ and projection

$$\begin{aligned} p : E(X, x_0) &\to X \\ \gamma &\mapsto \gamma(1) \end{aligned}$$

  Recall that the topology on $E(X, x_0)$ has the basis consisting of sets of the form $U(K, O)$ where

$$U(K, O) = \{\gamma \mid \gamma : K \subseteq_{cpt} I \to O \subseteq_{open} X \text{ is continuous}\}$$

  Consider any such basis open-set $U(K, O)$