# Research Notes

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**Abstract**

These are some random notes written while reading up on exponentially-suppressed effects in finte-volume matrix elements. These are mainly just for my own understanding, but if you've stumbbled upon these notes I hope they prove useful in some capacity.

# Contents

# 1 Finite-volume functions

The current objective is to understand Appendix C of [1] and reproduce results on page 18 of the paper. The paper studies $2 \to 2$ and $2 + \mathcal{J} \to 2$ transition amplitudes. We are interested in the **analytic-continuations of the finite-volume functions $F(P, L)$, $G(P, L)$, and $G^{\mu=0}(P, L)$ below threshold**.

**Analytic Continuation Below threshold - What does this mean?**

***THIS IS A TEMPORARY EXPLANATION. RE-WRITE THIS.

- We are studying interactions between two particles, so the energy possessed by their combined system *must be greater than* their rest-mass energy due to $E^2 = m^2c^4 + p^2c^2$. This minimum energy corresponding to the sum of their rest-masses is the **(kinematic) threshold** being referred to.

- The finite-volume functions are complex-functions with the energy-momentum of the pair being the input-parameter, and we are interested in the anayltic continuation of these functions below the energy/mass threshold described above.

Note that here $L$ is the finite-box length and $P = (E, \mathbf{P})$ is the total energy-momentum of the two particle system. We can boost to the Center-of-Momentum Frame (CMF), where we define $P^\star = (E^\star, \mathbf{0})$. We have the relation

$$(E^\star)^2 \equiv s \equiv P_\mu P^\mu = E^2 - \mathbf{P}^2$$

where we have generic $P$ and the Mandelstam variable $s$.

**Mandelstam Variables**

- Numerical quantities used to encode energy, momentum, and scattering angle in $2 \to 2$ interactions.
- If the Minkowski metric is chosen to be $diag(+, -, -, -)$ then the three Mandelstam variables are

$$\begin{aligned} s &= (p_1 + p_2)^2 c^2 = (p_3 + p_4)^2 c^2 \\ t &= (p_1 - p_3)^2 c^2 = (p_4 - p_2)^2 c^2 \\ u &= (p_1 - p_4)^2 c^2 = (p_3 - p_2)^2 c^2 \end{aligned}$$

  where $p_1, p_2$ and $p_1, p_2$ are the four-momenta of the incoming and outgoing particles.
- $s$ and $t$ are the squares of CMF Energy and Momentum respectively. (Check this) source.

Initial and final 3-momentum states in the Center-of-Momentum Frame are denoted as $\mathbf{k_i}^\star$ and $\mathbf{k_f}^\star$. With this notation we can introduce the partial-wave expansion of the Elastic Scattering Matrix $\mathcal{M}$:

$$\mathcal{M}(s, \hat{\mathbf{k}}_i^\star, \hat{\mathbf{k}}_f^\star) = 4\pi \sum_{l,m_l} Y_{lm_l}(\hat{\mathbf{k}}_f^\star)\mathcal{M}(s)Y_{lm_l}^*(\hat{\mathbf{k}}_i^\star)$$

Which we can express in terms of the $K-$matrix as

$$\mathcal{M}(s) = \mathcal{K}(s)\frac{1}{1 - i\rho(s)\mathcal{K}(s)}$$

$\rho(s)$ is the two-body phase space

$$\rho(s) = \frac{q^\star}{8\pi E^\star} = \frac{1}{16\pi}\sqrt{1 - \frac{4m^2}{s}}$$

where $q^\star$ is the relative momentum of the two particles in the CMF, $q^\star = \sqrt{s/4 - m^2}$ – the square root introduces a branch cut in the complex $s$ plane.

### 1.1 Analytic Continuation of $c_{JM}^{(n)}$

The functions $F(P,L), G(P,L), G^{\mu=0}(P,L)$ can be expressed in terms of the function in terms of a class of functions

$$c_{JM}^{(n)}(P,L) = \left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_k\right] \frac{\omega_k^\star}{\omega_k} \frac{\sqrt{4\pi}k^{\star\,J} Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{(q^{\star 2} - k^{\star 2} + i\epsilon)^n}$$

The relations we require are given by

$$\begin{aligned}
F(P,L) &= \frac{1}{2E^\star} c_{00}^{(}1)(P,L) \\
G(P,L) &= \frac{1}{4E^\star} c_{00}^{(}2)(P,L) \\
G^{\mu=0}(P,L) &= -\frac{E}{4E^{\star 3}} c_{00}^{(}1)(P,L)
\end{aligned}$$

We will end up using the **Poisson Summation Formula** in the following form:

$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq 0}\int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})$$

**Proof:**

- We want to analytically continue each of these functions below the kinematic threshold i.e. $P < (2m)^2$.
- At a sub-threshold momentum $P_\kappa$, we have

  $$m^2 - P_\kappa^2/4 = \kappa^2$$

  (why?)
- We apply the Poisson Summation Formula

  $$\boxed{\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq 0}\int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

  to $c_{JM}^{(n)}$. (source: https://arxiv.org/pdf/hep-lat/0507006.pdf)
- Since we are dealing with sub-threshold momenta, we don't need to worry about the singularity, so the $i\epsilon$ vanishes (check this logic...)

Then

$$\left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right] g(\vec{k}) = \sum_{\vec{m}\neq 0} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{m}\cdot\vec{k}} g(\vec{k})$$

$$\implies \left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right] \frac{\omega_k^\star}{\omega_k} \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{({q^\star}^2 - {k^\star}^2)^n} = \sum_{\vec{m}\neq 0} \int \frac{d^3k}{(2\pi)^3} \cdot \frac{\omega_k^\star}{\omega_k} \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{({q^\star}^2 - {k^\star}^2)^n} \cdot e^{iL\vec{m}\cdot\vec{k}}$$

$$\implies c_{JM}^{(n)}(P_\kappa, L) = \sum_{\vec{m}\neq 0} \int \frac{d^3k}{(2\pi)^3} \cdot \frac{\omega_k^\star}{\omega_k} \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{({q^\star}^2 - {k^\star}^2)^n} \cdot e^{iL\vec{m}\cdot\vec{k}}$$

Now, we use the facts that

- $q^\star = \sqrt{s/4 - m^2}$
- $m^2 - P_\kappa^2/4 = \kappa^2$
- The Integration measure is Lorentz Invariant i.e.

$$\frac{d^3\mathbf{k}^\star}{\omega_{\mathbf{k}}^\star} = \frac{d^3\mathbf{k}}{\omega_{\mathbf{k}}} \implies d^3\mathbf{k}^\star = d^3\mathbf{k} \cdot \frac{\omega_{\mathbf{k}}^\star}{\omega_{\mathbf{k}}}$$

$$\implies c_{JM}^{(n)}(P_\kappa, L) = \sum_{\vec{m}\neq 0} \int \frac{d^3\mathbf{k}^\star}{(2\pi)^3} \cdot \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{(s/4 - m^2 - {k^\star}^2)^n} \cdot e^{iL\vec{m}\cdot\vec{k}}$$

We're integrating with respect to $d^3\mathbf{k}^\star/(2\pi)^3$ i.e. working in the Center-of-Momentum Frame, where $\mathbf{P}_\kappa^\star = (E_\kappa^\star, \mathbf{0})$ and so ${E^\star}^2 = P_\kappa^2 = s$. Then we have

$$s/4 - m^2 - {k^\star}^2 = \frac{P_\kappa^2}{4} - m^2 - {k^\star}^2 = -\kappa^2 - {k^\star}^2$$

Thus,

$$\implies c_{JM}^{(n)}(P_\kappa, L) = \sum_{\vec{m}\neq 0} \int \frac{d^3\mathbf{k}^\star}{(2\pi)^3} \cdot \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{(-\kappa^2 - {k^\star}^2)^n} \cdot e^{iL\vec{m}\cdot\vec{k}}$$

$$\implies \boxed{c_{JM}^{(n)}(P_\kappa, L) = (-1)^n \sum_{\vec{m}\neq 0} \int \frac{d^3\mathbf{k}^\star}{(2\pi)^3} \cdot \frac{\sqrt{4\pi} {k^\star}^J Y_{JM}\left(\hat{\mathbf{k}}^\star\right)}{(\kappa^2 + {k^\star}^2)^n} \cdot e^{iL\vec{m}\cdot\vec{k}}}$$

## 1.2 Volume Effect on Energy and Phase Shift $\delta s$

## References

[1] Andrew W. Jackura Raúl Briceño Maxwell T. Hansen. "Consistency checks for two-body nitevolume matrix elements: I. Conserved currents and bound states". In: (2019). URL: https://arxiv.org/pdf/1909.10357.pdf.