Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 January 23 - More examples, Transition Maps, Smooth Atlases

So far, we've studied topological manifolds whch are topologcal spaces with some additional properties, namely, they are

- Hausdorff
- Locally Euclidean
- Second Countable

Some important properties of topologcial manifolds: They are

- locally compact
- admit compact exhaustions
- paracompact

1.1 Charts

Let M^n be an $n-$dimensional manifoold. At each point $p \in M$, there exists a neighborgood U and homeomorphism $\phi : U \to \tilde{U} \subseteq_{open} \mathbb{R}^n$.

[Insert Figure]

Then, the pair (U, ϕ) is called a **chart**. Also, the map $\phi(q)$ can be thought of as $\phi(q) = (\psi^1(q, \ldots, \phi^n(q)))$ where the ϕ^i are called **coordinate functions**.

Example: The unit circle $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an $n-$dimensional manifold which can be covered by charts $(U_i^{\pm}, \phi_i^{\pm})$ where

$$U_i^{\pm} =$$
$$\phi_i^{\pm} =$$

Example: Projective Space is defined as

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} / \sim)$$

with the equivalence relation $\vec{x} \sim \vec{y}$ f $\vec{x} = \lambda \vec{y}$ for $\lambda \in \mathbb{R}_{\neq 0}$.

Projectvie space is the same as the equivalence class of lines $\{\text{Lines } L \subset \mathbb{R}^{n+1}, \vec{0} \in L\}$ with the quotient Topology endowed by the quotient map

$$\pi : (\mathbb{R}^{n+1} - \{\vec{0}\}) \to \mathbb{RP}^n$$

where we say $A \subset \mathbb{RP}^n$ is open ff $\pi^{-1}(A)$ is open.

To show that Projective Space is a manifold by coordinate charts, wrirte

$$[(x_1, \ldots, x_{n+1})] = [x_1 : \cdots : x_{n+1}]$$

Note that

$$[x_1 : \cdots : x_{n+1}] = [\lambda x_1 : \cdots : \lambda x_{n+1}]$$

for any $\lambda \neq 0$.

Then, define $U_i = \{[x_1 : \cdots : x_{n+1}] : x_i =\neq 0\} \subset_{open} \mathbb{RP}^n$ and the map $\phi_i : U_i \to \mathbb{R}^n$ as

$$[x_1 : \cdots : x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$$

[write the rest after class from photo taken]

1.2 Smooth Manifolds

1.2.1 Transition Maps

Suppose M^n is a topological manifold.

Transition Map

Def: If (U, ϕ), (V, ψ) are charts of M, then

$$\psi \circ \phi^{-1}\big|_{\psi(U\cap V)} : \phi(U \cap V) \to \psi(U \cap V)$$

is a transition map or a change of coordinates map.

[Insert Image later]

Theorem: Transition maps are homeomorphisms.

Proof: $\psi \circ \phi^{-1}\big|_{\psi(U\cap V)}$ and $\left(\psi \circ \phi^{-1}\big|_{\phi(U\cap V)}\right)^{-1} = \phi \circ \psi^{-1}\big|_{\psi(U\cap V)}$ are both continuous since they are the compositions (and then restrictions) of continuous functions.

For example, consider $M = \mathbb{R}^n$.

- We obtain one chart (U, ϕ) from Polar Coordinates:

$$U = \mathbb{R}^2 \setminus \{\mathbb{R}_{\geq 0} \times \{0\}\}$$
$$\phi : U \to \mathbb{R}_+ \times (0, 2\pi) \text{ defined by}$$
$$\phi(\vec{z}) = (|\vec{z}|, \arg(\vec{z}))$$

- And another chart (V, ψ) from Euclidean coordinates:

$$V = \mathbb{R}^2$$
$$\psi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by}$$
$$\psi(\vec{z}) = (z_1, z_2)$$

- We can then understand the transition map between them: [Write from picture taken]

Another example is $M = \mathbb{S}^2 \subset \mathbb{R}^3$ and its open charts $(U_i^{\pm}, \phi_i^{\pm})$.

- Consider the charts (U_1^+, ϕ_1^+) and (U_3^+, ϕ_3^+). [Draw diagram from pciture taken]
- The transition map between these charts is

$$\begin{aligned}\phi_3^+ \circ (\phi_1^+)^{-1}(x_2, x_3) &= \phi_3^+(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3) \\ &= (\sqrt{1 - x_2^2 - x_3^2}, x_2)\end{aligned}$$

1.2.2 Smoothness and Atlases

Smooth compatibility

- **Def:** Given a topological manifold M, two of its charts (U, ϕ), (V, ψ) are **smoothly compatible** if both transition maps are **smooth**

$$\psi \circ \phi^{-1}\big|_{\phi(U \cap V)}, \phi \circ \psi^{-1}\big|_{\psi(U \cap V)}$$

- When we say both transitions are smooth, we mean infinitely differentiable.
- Remark: These transition maps are in fact *diffeomorphisms*.

Atlases

- **Def:** Given a topological manifold M, an atlas $\mathcal{A}$ of M is a collection of charts such that

$$M = \bigcup_{(U,\phi) \in \mathcal{A}} U$$

- $\mathcal{A}$ is **smooth** if all charts of $\mathcal{A}$ are smoothly compatible.
- $\mathcal{A}$ is a **maximal smooth atlas** if there is no smooth atlas $\mathcal{A}'$ such that $\mathcal{A} \subset \mathcal{A}$.

Theorem: Every smooth atlas $\mathcal{A}$ of M is contained in a unique maximal smooth atlas.

Proof: Let $\overline{\mathcal{A}} = \{(U,\phi) \text{ charts on } M : (U,\phi) \text{ smoothly compatible with all } (V,\psi) \in \mathcal{A}\} \supset A$.

1. Then, $\overline{\mathcal{A}}$ is a smooth atlas on M.

   We want to check $(U_1,\phi_1),(U_1,\phi_1) \in \overline{\mathcal{A}}$ are smoothly compatible i.e. the smoothness of the transition maps $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$.

   Now, we may not know whether these charts are compatible. What we *do* know is that for some point $p \in U_1 \cap U_2$ there is a chart $(V,\psi) \in \mathcal{A}$ such that $p \in V$.

   Now, by definition of $\overline{\mathcal{A}}$ we know that $\phi_2 \circ \psi^{-1}$ and $\psi \circ \phi_1^{-1}$ are smooth (since those charts are smoothly compatible). Thus,

   $$\phi_2 \circ \phi_1^{-1} = (\phi_2 \circ \psi^{-1}) \circ (\psi \circ \phi_1^{-1})$$

   is smooth as a composition of smooth maps on appropriate domains.

   [Draw Diagram]

2. Next, we want to show that $\overline{\mathcal{A}}$ is maximal.

   Claim: Suppose $\mathcal{A}' \supset \mathcal{A}$ where $\mathcal{A}'$ s a smooth atlas. Then, $\mathcal{A}' \subset \overline{\mathcal{A}}$.

   Note that fi $(U',\phi') \in \mathcal{A}'$ then this chart is compatble wth every chart in $\mathcal{A}'$. Since $\mathcal{A} \subset \mathcal{A}'$ we have that (U',ϕ') is compatible with all charts in $\mathcal{A}$. So, $(U',\phi') \in \overline{\mathcal{A}}$.

$\implies$ Maximality and Uniqueness.

Remark: If smooth atlases $\mathcal{A}_1$, $\mathcal{A}_2$ are such that any $(U_1,\phi_1) \in \mathcal{A}_1$ is compatible with any $(U_2,\phi_2) \in \mathcal{A}_2$, then $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$.

Proof: $\mathcal{A}_{12} := \mathcal{A}_1 \cup \mathcal{A}_2$. Then, $\overline{\mathcal{A}}_{12}$ is a maximal smooth atlas containing $\mathcal{A}_{12}$ and thus also containing both $\mathcal{A}_1$ and $\mathcal{A}_2$.

1.2.3 Smooth manifolds

Smooth Structure and Smooth Manifolds

- **Def:** A maximal smooth atlas $\mathcal{A}$ on a topological manifold M s a smooth structure on M.
- **Def:** A smooth manifold is a pair $(\underbrace{M^n}_{\text{top. mfd.}}, \underbrace{\mathcal{A}}_{\text{smooth structure on } M})$

Remark: The above are C^∞ manifolds but we can make similar definitions for $C^k, C^{k,\alpha}, \underbrace{C^w}_{\text{analytic}}$ or complex manifolds.