Math H185 Homework 7

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Question 1

Let $\{f_n\}$ be a sequence of holomorphic functions on an open subset $U \subseteq \mathbb{C}$ that converges uniformly to a function f on every compact subset of U. Show that the sequence $\{f_n'\}$ converges uniformly to f' on every compact subset of U. Then argue that the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

defines a holomorphic function in s for $\text{Re}(s) > 1$.

Proof:

We say a sequence of functions $\{f_n\}$ converges to function f on a subset $\Omega \subseteq \mathbb{C}$ if for every $\epsilon > 0$ there is some $N > 0$ so that whenever $z \in \Omega$ and $n > N$, we have

$$|f(z) - f_n(z)| < \epsilon$$

Let's first show that if $\{f_n\}$ is a sequence of holomorphic functions converging to f on every compact subset of U, then f is also holomorphic.

Let D be any disc whose closure is contained in U and let T be any triangle contained in D. Then, on D, $\{f_n\} \to f$. Since each f_n is holomorphic, Goursat's Theorem tells us:

$$\int_T f_n(z)dz = 0$$

for all n. Now, since $\{f_n\} \to f$ in the closure of D, f is continuous and we have

$$\int_T f_n(z)dz = \int_T f(z)dz$$

As a result,

$$\int_T f(z)dz = 0$$

Then, Morera's Theorem tells us that f is holomorphic on D. Since this holds for any D whose closure is contained in U, f is holomorphic on all of U.

Now, since the sequence $\{f_n\} \to f$ uniformly on any disc whose closure is contained in U, we can assume WLOG that the sequence converges uniformly on all of U. Now, given $\delta > 0$ let

$$\Omega_\delta = z \in U \ : \ \overline{D_\delta(z)} \subseteq U$$

be the set of points which are atleast a distance δ away from the boundary of U. To prove the theorem, it suffices to show that $\{f_n'\}$ converges uniformly to f' on each Ω_δ. We do so using the following inequality (for holomorphic F):

$$\sup_{z\in\Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{w\in U} |F(w)|$$

with $F = f_n - f$.

Proof of Inequality:

For every $z \in \Omega_\delta$, the closure of $D_\delta(z)$ is contained in U and Cauchy's Integral Formula tells us

$$F'(z) = \frac{1}{2\pi i} \int_{\partial D_\delta(z)} \frac{F(w)}{(w-z)^2} dw$$

Hence,

$$\begin{aligned} |F'(z)| &\leq \left|\frac{1}{2\pi i}\right| \int_{\partial D_\delta(z)} \frac{|F(w)|}{|w-z|^2} |dw| & (1) \\ &\leq \frac{1}{2\pi} \sup_{w\in U} |F(w)| \frac{1}{\delta^2} 2\pi\delta & (2) \\ &\leq \frac{1}{\delta} \sup_{w\in U} |F(w)| & (3) \end{aligned}$$

and this holds for any $z \in \Omega_\delta$ so of course,

$$\boxed{\sup_{z\in\Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{w\in U} |F(w)|}$$

Applying this with $F = f_n - f$, we have for any $z \in \Omega_\delta$ that

$$\begin{aligned} |F'(z)| &\leq \frac{1}{\delta} \sup_{w\in U} |F(w)| \\ \implies |f_n'(z) - f(z)| &\leq \frac{1}{\delta} \sup_{w\in U} |f_n(z) - f(z)| \end{aligned}$$

Since $\{f_n\} \to f$ uniformly on Ω_δ, for any $\epsilon > 0$ there exists $N > 0$ such that

$$|f_n(z) - f(z)| < \epsilon$$

for $z \in \Omega_\delta$ and $n > N$. Thus, for any ϵ, the same N guarantees that

$$|f_n'(z) - f'(z)| < \frac{\epsilon}{\delta} = \epsilon'$$

for $z \in \Omega_\delta$. Thus, the sequence $\{f_n'\} \to f'$ uniformly on each Ω_δ. Thus, the same holds on all of U.

Now, moving onto the Riemann Zeta function. We define the zeta function by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For $s = x + iy$, we have

$$\begin{aligned}
\left|n^{-s}\right| &= \left|n^{-(x+iy)}\right| \\
&= \left|n^{-x} \cdot n^{-iy}\right| \\
&= \left|n^{-x}\right| \cdot \left|e^{\ln(n^{-iy})}\right| \\
&= \left|n^{-x}\right| \cdot \underbrace{\left|e^{-iy\ln(n)}\right|}_{=1} \\
&= \left|n^{-\mathrm{Re}(s)}\right|
\end{aligned}$$

So,

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{\left|n^{\mathrm{Re}(s)}\right|}$$

Let's denote $\mathrm{Re}(s) = \sigma$. Now,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{|n^\sigma|} \leq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\sigma} dx \\
\implies &\sum_{n=1}^{\infty} \frac{1}{|n^\sigma|} \leq \int_1^{\infty} \frac{1}{x^\sigma} dx
\end{aligned}$$

For $\sigma > 1$, the integral converges:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^\sigma} dx &= \left[\frac{1}{1-\sigma} x^{1-\sigma}\right]_{x=1}^{x=\infty} \\
&= \frac{1}{1-\sigma} \left[\frac{1}{x^{\sigma-1}}\right]_{x=1}^{x=\infty} \\
&= \frac{1}{1-\sigma} [0 - 1] \\
&= \frac{1}{\sigma - 1}
\end{aligned}$$

Therefore, for $\sigma > 1$, the sum converges absolutely. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

defines a holomorphic function on the half place $\mathrm{Re}(s) > 1$.

Question

Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} + s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx$$

where $\{x\}$ denotes the fractional part of x. Prove that the right hand side defines a holomorphic function in s for $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$.

Proof:

Let's look at the integral on the right hand side:

$$\begin{aligned}\int_1^\infty \frac{\{x\}}{x^{s+1}}dx &= \sum_{n=1}^\infty \int_n^{n+1} \frac{x-n}{x^{s+1}}dx \\ &= \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{x^s} - \frac{n}{x^{s+1}}dx\end{aligned}$$

If $|s| > 1$, this can be written as

$$\begin{aligned}\int_1^\infty \frac{\{x\}}{x^{s+1}}dx &= \int_1^\infty \frac{1}{x^s}dx - \sum_{n=1}^\infty \int_n^{n+1} \frac{n}{x^{s+1}}dx \\ &= \frac{1}{s-1} - \sum_{n=1}^\infty n \cdot \left[\frac{x^{-s}}{-s}\right]_{x=n}^{x=n+1} \\ &= \frac{1}{s-1} + \sum_{n=1}^\infty \frac{n}{s}\left[(n+1)^{-s} - n^{-s}\right] \\ &= \frac{1}{s-1} - \frac{1}{s}\sum_{n=1}^\infty n\left[n^{-s} - (n+1)^{-s}\right]\end{aligned}$$

Now, we can re-express the sum by combining terms cleverly:

$$\begin{aligned}\sum_{n=1}^\infty n\left[n^{-s} - (n+1)^{-s}\right] &= 1 \cdot \left(1^{-s} - 2^{-s}\right) + 2\left(2^{-s} - 3^{-s}\right) + 3 \cdot \left(3^{-s} + 4^{-s}\right) + \cdots \\ &= 1^{-s} + (2-1) \cdot 2^{-s} + (4-3) \cdot 3^{-s} \cdots \\ &= 1^{-s} + 2^{-s} + 3^{-s} + \cdots \\ &= \sum_{n=1}^\infty \frac{1}{n^s} \\ &= \zeta(s)\end{aligned}$$

This rearranging of terms is only guaranteed to be valid when the sum converges absolutely, and that happens for $\operatorname{Re}(s) > 1$.

Thus, for $\operatorname{Re}(s) > 1$, we have

$$\begin{aligned}\int_1^\infty \frac{\{x\}}{x^{s+1}}dx &= \frac{1}{s-1} - \frac{1}{s}\zeta(s) \\ \implies &\boxed{\zeta(s) = s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx + \frac{s}{s-1}}\end{aligned} \tag{1}$$

Earlier, we proved that if we have a sequence of holomorphic functions $\{f_n\}$ which converge uniformly to a function f on an open subset $U \subseteq_{\text{open}} \mathbb{C}$, then f is holomorphic.

Consider the sequence of functions $\{f_n\}_{n=1}^\infty$ defined on $U = \{s \in \mathbb{C} \;:\; \operatorname{Re}(s) > 0\} \setminus \{1\}$ where

$$f_n(z) = \frac{s}{s-1} + s\int_1^n \frac{\{x\}}{x^{s+1}}dx$$

Each of these functions is holomorphic on U because $s/(s-1)$ is a rational function whose denominator does not vanish in U and the integral evaluates to

$$\begin{aligned}
\sum_{k=1}^{n}\int_{k}^{k+1}\frac{x-k}{x^{s+1}}dx &= \sum_{k=1}^{n}\left[\frac{x^{1-s}}{1-s}+\frac{k}{s}x^{-s}\right]_{k}^{k+1} \\
&= \sum_{k=1}^{n}\left[\left(\frac{(k+1)^{1-s}}{1-s}+\frac{k}{s}(k+1)^{-s}\right)-\left(\frac{k^{1-s}}{1-s}+\frac{k}{s}k^{-s}\right)\right] \\
&= \sum_{k=1}^{n}\left[\frac{1}{1-s}\left((k+1)^{1-s}-k^{1-s}\right)+\frac{k}{s}\left((k+1)^{-s}-k^{-s}\right)\right] \\
&= \frac{1}{1-s}\sum_{k=1}^{n}\left[\underbrace{(k+1)^{1-s}-k^{1-s}}_{\text{telescoping}}\right]+\frac{k}{s}\sum_{k=1}^{n}\left[(k+1)^{-s}-k^{-s}\right] \\
&= \frac{(n+1)^{1-s}-1^{1-s}}{1-s}-\frac{1}{s}\left[1\cdot\left(2^{-s}-1^{-s}\right)+2\cdot\left(3^{-s}-2^{-s}\right)+\cdots+n\left((n+1)^{-s}-n^{-s}\right)\right] \\
&= \frac{(n+1)^{1-s}-1}{1-s}-\frac{1}{s}\left[-1^{-s}-2^{-s}-3^{-s}-\cdots-n^{-s}+n(n+1)^{-s}\right] \\
&= \frac{(n+1)^{1-s}-1}{1-s}-\frac{1}{s}\sum_{k=1}^{n}\frac{1}{n^{s}}+\frac{1}{s}n(n+1)^{-s}
\end{aligned}$$

Each of these terms in holomorphic on U, so the entire integral term $s\int_1^n \frac{\{x\}}{x^{s+1}}dx$ is holomorphic on U.

We also have uniform convergence of $\{f_n\}$ to $f(z)$ where

$$f(z) = s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx + \frac{s}{s-1}$$

on U. Therefore, the right hand side of equation (1) is holomorphic on $U = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$

Question 3

For $x \in \mathbb{R}$, let $Q_0(x) \equiv \{x\} - 1/2$. Prove by induction that there exist for all $k \geq 0$, bounded functions $Q_k(x)$ satisying all of the following conditions:

(a) $\int_0^1 Q_k(x)dx = 0$

(b) $\frac{dQ_{k+1}(x)}{dx} = Q_k(x)$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$

(c) $Q_k(x+1) = Q_k(x)$ for all $x \in \mathbb{R}$.

Proof:

Base Case:

For $k = 0$, we have $Q_0 = \{x\} - 1/2$. This function is bounded as $\sup |Q_0(x)| = 1/2$. Define $Q_1(x)$ as:

$$Q_1(x) \equiv \begin{cases} F_1(x) + C_1, & x \in [0,1) \\ Q_1(\{x\}), & x \notin [0,1) \end{cases}$$

where $F_1(x)$ is the anti-derivative of $Q_0(x)$ and C_1 is a constant chosen such that property (a) is satisfied i.e. $C_1 = -\int_0^1 F_1(x)dx$.

Let's verify that it satisfies the three properties:

(a) The integral over the unit interval is

$$\begin{aligned}\int_0^1 Q_0(x)dx &= \int_0^1 \{x\} - \frac{1}{2}dx \\ &= \int_0^1 x - \frac{1}{2}dx \\ &= \left[\frac{x^2}{2} - \frac{x}{2}\right]_0^{\to 1} \\ &= \left(\frac{1}{2} - \frac{1}{2}\right) - (0 - 0) \\ &= 0\end{aligned}$$

(b) For $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $x \in [0,1)$ we have $Q_1(x) = F_1(x) + C_1$ so

$$\begin{aligned}\frac{dQ_1(x)}{dx} &= \frac{dF_1(x)}{dx} + 0 \\ &= Q_0(x)\end{aligned}$$

and for $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $x \notin [0,1)$ we have $Q_1(x) = Q_1(\{x\})$, and so

$$\begin{aligned}\frac{dQ_1(x)}{dx} &= \frac{dF_1(\{x\})}{dx} + 0 \\ &= Q_0(\{x\}) \\ &= \{x\} - \frac{1}{2} \\ &= Q_0(x)\end{aligned}$$

(c) For $x \in \mathbb{R}$, we have

$$Q_1(x+1) = Q_1(\{x+1\}) = Q_1(\{x\}) = Q_1(x)$$

Inductive Hypothesis and Step

Okay, now suppose functions $Q_i(x)$ where $0 \leq i \leq n$ are defined such that properties (a), (b), and (c) are satisfied. Let's define $Q_{n+1}(x)$ as

$$Q_{n+1}(x) = \begin{cases} F_{n+1} + C_{n+1}, & x \in [0,1) \\ Q_{n+1}(\{x\}), & x \notin [0,1) \end{cases}$$

where F_{n+1} is the antiderivative of Q_n and C_{n+1} is a constant chosen so that property (a) is satisfied i.e. $C_{n+1} = -\int_0^1 F_{n+1}(x)dx$.

Let's verify that each of the properties hold.

(a) Holds by contruction of $Q_{n+1}(x)$

(b) Again, due to the construction, For $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $x \in [0,1)$ we have $Q_1(x) = F_1(x) + C_1$ so

$$\begin{aligned}\frac{dQ_{n+1}(x)}{dx} &= \frac{dF_{n+1}(x)}{dx} + 0 \\ &= Q_n(x)\end{aligned}$$

and for $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $x \notin [0, 1)$ we have $Q_1(x) = Q_1(\{x\})$, and so

$$\begin{aligned}\frac{dQ_1 = n+1(x)}{dx} &= \frac{dF_{n+1}(\{x\})}{dx} + 0 \\ &= Q_n(\{x\}) \\ &= Q_n(x)\end{aligned}$$

(c) For $x \in \mathbb{R}$, we have

$$Q_{n+1}(x+1) = Q_{n+1}(\{x+1\}) = Q_n(\{x\}) = Q_n(x)$$

Question 4

With $Q_k(x)$ as in the previous problem, prove the formula

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty \left(\frac{d^k}{dx^k}Q_k(x)\right) x^{-s-1}dx$$

for $\mathrm{Re}(s) >> 1$, and deduce that there is an analytic continuation of $\zeta(s)$ to $\{s \in \mathbb{C} : \mathrm{Re}(s) > -k\} \setminus 1$

Proof:

We found in Question 2 that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$

We can rewrite this in terms of $Q_0(x) = \{x\} - 1/2$ and then simplify to get

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} - s\int_1^\infty \frac{Q_0(x) + 1/2}{x^{s+1}}dx \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{1/2}{x^{s+1}}dx - s\int_1^\infty \frac{Q_0}{x^{s+1}}dx \\ &= \frac{s}{s-1} - \frac{s}{2}\int_1^\infty x^{-s-1}dx - s\int_1^\infty Q_0(x)\cdot x^{-s-1}dx \\ &= \frac{s}{s-1} - \frac{s}{2}\left[\frac{x^{-s}}{-s}\right]_1^\infty - s\int_1^\infty Q_0(x)\cdot x^{-s-1}dx \\ &= \frac{s}{s-1} - \frac{s}{2}\left[0 - \frac{-1}{s}\right] - s\int_1^\infty Q_0(x)\cdot x^{-s-1}dx \\ &= \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty Q_0(x)\cdot x^{-s-1}dx\end{aligned}$$

Also, recall that in Question 3, that we found for $x \in \mathbb{R} \setminus \mathbb{Z}$

$$\frac{dQ_{k+1}(x)}{dx} = Q_k(x)$$

This relation only fails at the integers, which are a set of measure zero, so they don't contribute to the integral.

Therefore, we can write

$$\int_1^\infty Q_0(x) \cdot x^{-s-1} dx = \int_1^\infty \left(\frac{d^k Q_k(x)}{dx^k} \right) \cdot x^{-s-1} dx$$

Therefore,

$$\boxed{\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty \left(\frac{d^k Q_k(x)}{dx^k} \right) \cdot x^{-s-1} dx}$$

Now, the terms other than the integral are all holomorphic on $\mathbb{C} \setminus \{1\}$, so let's think about where the integral converges absolutely and is holomorphic.

For now, instead of Q_k, let's think about Q_0. Using Integration by Parts, we have

$$\begin{aligned}\int_1^\infty \frac{Q_0(x)}{x^{s+1}} dx &= \left[\frac{Q_1(x)}{x^{s+1}} \right]_1^\infty - \int_1^\infty \frac{Q_1(x)}{x^{s+2}} dx \\ &= Q_1(1) - \int_1^\infty \frac{Q_1(x)}{x^{s+2}} dx\end{aligned}$$

The first term is just a constant and is holomorphic everywhere. The integral on the other hand is absolutely convergent for $\text{Re}(s) + 2 > 1 \implies \text{Re}(s) > -1$.

If further apply Integration by Parts on the integral with Q_1 in it, we will get another constant term and an integral which converges absolutely for $\text{Re}(s) > -2$.

We can do this k times until we get the integral with $\frac{Q_k(x)}{x^{s+k}}$, so the zeta function $\zeta(s)$ under this continuation is equal to a bunch of constants + an integral which is absolutely convergent on $\{s \in \mathbb{C} \ : \ \text{Re}(s) > -k\} \setminus 1$. Therefore, we find that this continuation of $\zeta(s)$ is holomorphic on $\{s \in \mathbb{C} : \text{Re}(s) > -k\} \setminus 1$.
