(Instructor: Chien-I Chiang)

Physics 105: Analytical Mechanics notes

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These are some notes taken from UC Berkeley's Physics 105 during the Summer '24 session, taught by Chien-I Chiang.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 First topic

text

2 July 3, 2024:

2.1 Finishing up discussion from last lecture

- Finish this from lecture recording

Continuing on with out discussion of when $H \neq E$, we can parametrize the position of a particle as $\vec{r} = \vec{r}(q_k, t)$

We have

$$\frac{\partial K}{\partial \dot{q_k}} = \frac{1}{2} m \left[2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_m} + \cdots \right]$$

Then,

$$\begin{cases} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} \right] \\ 2K = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + 2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \right] \end{cases}$$

(The expression for $2K$ is obtained by expandingo out

$$K = \frac{1}{2} m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}$$

in terms of indices – write this out explicitly later)

Which gives us the relation

$$\begin{aligned} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} &= 2K - m \frac{\partial \vec{r}}{\partial t} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial q_k} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \right)}_{= \frac{d\vec{r}}{dt}} \\ &= 2K - \vec{p} \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

The question we were originally considering is **When is $H = E$?**

Now,

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{q}_k} \dot{q} - L \\ &= \frac{\partial K}{\partial \dot{q}_k} \dot{q}_k = (K - V) \\ &= 2K - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} - K + V \\ &= K + V - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

So we see that $H = E = K + V$ only when

$$\frac{\partial \vec{r}}{\partial t} = 0$$

i.e. when $\vec{r} = \vec{r}(q_k, t)$ has no time dependence i.e. $\vec{r} = \vec{r}(q_k)$

Earlier, we considered the following setup:

and we showed that

$$H = E - m\omega^2\rho^2$$

So, let's check that

$$m\omega^2\rho^2 = \vec{p}\cdot\frac{\partial \vec{r}}{\partial t}$$

$$\begin{aligned}\vec{p}\cdot\frac{\partial \vec{r}}{\partial t} &= \vec{p}\cdot(-\rho\omega\sin(\omega t)\hat{x} + \rho\omega\sin(\omega t)\hat{y})\\ &= \vec{p}\cdot\left[\rho\omega\hat{\phi}\right]\\ &= mv_\phi\rho\omega\\ &= m\rho^2\omega^2\end{aligned}$$

where $v_\phi = \rho\omega$

Since the hamiltonian itself has no time dependence, ***H* is conserved**. However, ***E* is not**. We can check that

$$dH = dE = d(m\omega^2\rho^2)$$

is indeed zero.

[Include figure]

If we break the force on the bead into a normal force (denoted N) and a centripetal(?) force, then

$$\begin{aligned} dW &= \underbrace{\overbrace{N\rho}}_{\text{torque about z-axis}} d\phi \\ &= \frac{dl_z}{dt} d\phi \\ &= d\left(\rho m \rho \omega\right)\omega \\ &= d\left(m\rho^2\omega^2\right) \end{aligned}$$

This is the energy that goes into the system.

By energy conservation, $dW = dE$.

$$\implies 0 = dE - dW = dE - d(m\rho^2\omega^2)$$

i.e. $E - m\rho^2\omega^2 = H$ is a conserved quantity.

So, the **Hamiltonian being conserved** and the **Hamiltonian being equal to Energy** are two different scenarios with two different conditions.

- The Lagrangian is time-independent i.e. $\frac{\partial L}{\partial t} = 0 \implies H$ is conserved.
- The position vector centered in an inertial frame $\vec{r} = \vec{r}(q_k, t)$ is time independent i.e $\frac{\partial \vec{r}}{\partial t} = 0 \implies H = E$

Now we move on to a powerful technique.

2.2 The Method of Lagrange Multipliers

We have a block constrained to move on the xy-plane, and we have gravity. Previously, we would say

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

i.e we would start with an unconstrained lagrangian, and then plug in the constraints $z = 0, \dot{z} = 0$

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ \implies &\begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \end{cases} \end{aligned}$$

Alternatively, we can implement the constraint $\ddot{z} = 0$ in the following way: We have the original lagrangian

$$L' = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz + \lambda z$$

where λ is the Lagrange multiplier and we can think of z as being the constraint function $f(z)$ and our constraint is $f(z) = 0$.

If we treat λ as an independent degree of freedom, we can write the Euler-Lagrange equation for λ as

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0 \implies z = 0 \text{ (constraint)}$$

On the other hand, if we look at the equation of motion for z, we get

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \implies \lambda - mg - m\ddot{z} = 0 \implies m\ddot{z} = \lambda - mg$$

and using the constraint $z = 0 \implies \ddot{z} = 0$ we get $-mg + \lambda = 0 \implies \lambda = mg$. Okay, but what physical meaning does λ have? It has to do with the **Normal force**. i.e. λ is encoding the **constraint** that the block can only move on the xy-plane due to the Normal force.

So, in general, for N constraints we have Lagrange Multipliers $\lambda_1, \cdots, \lambda_N$.

Why do we call $\boldsymbol{\lambda}$ a Lagrange Multiplier?

Recall from Calc 3 that if we have contours of a function $f(x, y)$ on the xy-plane and we are constrained to move along some other curve $g(x, y) = c$ on the plane, if we ask "What is the extremum of $f(x, y)$ as we move along the curve $g(x, y) = c$?" then visually we can tell that the extremum corresponds to the point where $g(x, y)$ intersects the contour of $f(x, y)$ only once. This is because at such a point, the gradients of the two functions are parallel:

$$\nabla g = \lambda \nabla f$$

This constant multiplier is the **Lagrange Multiplier**

So, in general, if we have a Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

we know that $\delta L = 0$ gives the Equations of Motion. But if we want to do this variation δL under some constraint $C(x, y, z) = 0$ then we need to consider

$$\delta L = \lambda \delta C \implies L' = L - \lambda C$$

Generally, if we have P constraints, $C_l(q_1, \cdots, t) = 0$, $l = 1, \cdots, P$ on the lagrangian L, we can write a new lagrangian

$$L' = L + \sum_{l=1}^{P} \lambda_l C_l$$

The Euler-Lagrange equation for λ_l leads to $C_l = 0$ and the Euler-Lagrange equation for the generalized coordinate q_k is

$$\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k} \right) = 0$$

$$\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} + \underbrace{\sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}}_{\text{generalized force}}$$

On the physical point of view, consider the following system:

[include picture of block and sledge which can both move]

If we consider the system as a whole, the normal forces due to the block and the sledge are equal and opposite, so they cancel each other out - and so does the work that they do(?).

Howeverm if we consider the block only - we do have a normal force. The block is constrained the only move on the surface of the slope, so we can write

$$L' = K - V + \int^{\vec{r}} \vec{F}_C(\vec{r}) \cdot d\vec{r}'$$

(This is a bit handwavy - watch the lecture recording and think about this)
Then, if we conpare this with

$$L' = L - V + \sum_l \lambda_l C_l$$

we have

$$\sum_l \lambda_l C_l = \int^{\vec{r}} \vec{F}_C \cdot d\vec{r}' = \int^{\vec{r}} \vec{F} \cdot \left(\frac{\partial \vec{r}'}{\partial q_k} \cdot \mathrm{d}q_k \right)$$

$$\implies \frac{\partial}{\partial q_k}\left(\sum_l \lambda_l c_l\right) = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial q_k}\right) \equiv \mathcal{F}_k \text{ (generalized force)}$$

3 July 8, 2024:

3.1 More about Lagrange Multipliers

Last time, we saw that if we have constraints $C_l\left(\underbrace{q_1, \cdots, q_k}_{N}, t\right) = 0$ then we can write a constrained Lagrangian

$$L' = K - V + \sum_l \lambda_l C_l$$

These kinds of constraints, which are only constraints of the generalized coordinates are called **Holonomic constraints**. But these are not the most general constraints; we can have constaints which also depend on the derivatives $\dot{q_k}$. Those types of constraints are called **Non-holonomic constraints**.

Then, the principle of stationary action gives us

$$0 = \delta S \implies \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l \underbrace{\lambda_l C_l}_{\vec{F}^C \frac{\partial \vec{r}}{\partial q_k}} \\ \\ C_l = 0 \end{cases}$$

Note that there are multiple ways to write the same constraint. And writing a constraint in a different manner changes the C_l, which further changes the λ_l. As such, the λ_l is not always a generalized force; it can also be a torque etc.

In total we have $N + P$ variables and $N + P$ equations, so we are able to solve the system if we know the initial conditions.

We got the above equation by varying the action, and in particular, by varying L with respect to q_k. But we can extend this a bit futher...

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l a_{lk}\lambda_l \\ a_{lk}\delta q_k + a_{lt}\delta t = 0 \end{cases}$$

(Here, the l index labels the **constraint** and the k labels the coordinate.)

In the case of Holonomic constraint,

$$a_{lk} = \frac{\partial C_l}{\partial q_l}$$
$$a_{lt} = \frac{\partial C_l}{\partial t}$$

For Holonomic constraint, we will have

$$\frac{\partial q_{lk}}{\partial t} = \frac{\partial q_{lt}}{\partial q_k}$$

3.2 Example: Tree log rolling down a ramp

Consider a tree log rolling down a (fixed) ramp without sliding.

[Include Figure]

To describe the motion of the log, generically, we need two degrees of freedom: X and θ.

But we also know the log is rolling **without sliding**. So if the tree moves a distance dx during rotation $d\phi$, then we know $Rd\phi = dx$ where R is the radius of the log. Or in other words,

$$Rd\phi - dx = 0$$

This constraint is of the general form we saw above: $\boxed{a_{lk}\delta q_k + a_{lt}\delta t = 0}$ with $a_{1,\theta} = R, a_{1,x} = -1$ and all the time components $a_{lt} = 0$.

Now, we can write the Lagrangian of this system:

$$L = \frac{1}{2}M(\dot{X}^2) + \frac{1}{2}I\dot{\theta}^2 + mgX\sin(\alpha)$$

Note that we're actually kind of mixing approaches here. Technically there should be *three* degrees of freedom because the log can move in (x, y) space and rotate, but we know that the log is constrained by the Normal force and we don't need both of x, y; just one will suffice.

Wait... so, why do we even bother using the Lagrange Multiplier stuff if we're gonna use the old method too?

The Lagrange multiplier method allows us to retain info about the contact forces so if we, say, want to find the magnitude of the tension in a string, we can still do so using the Lagrange Multiplier method. Whereas in the old method, contact forces are used to enfore constraints but we lose all information about them.

Anyway, after writing down the lagrangian, we can obtain the Equations of Motion (with the constraints):

$$\begin{cases} \frac{d}{dt}\left(m\dot{X}\right) = +mg\sin(\alpha) - \lambda_1 \\ \frac{d}{dt}\left(I\dot{\theta}\right) = \lambda_1 R \end{cases}$$

So, what exactly is $\boldsymbol{\lambda_1}$?

In the X equation of motion, we have $+mg\sin(\alpha)$ which is the component of gravity along the ramp. So, λ_1 has the same units as force. We can interpret λ_1 as the **frictional force**!

Then, in the θ equation of motion, we can interpret $\lambda_1 R$ as the **torque due to friction**!

Solving these further we have

$$\begin{cases} m\ddot{X} = mg\sin(\alpha) - \lambda_1 \ (1) \\ I\ddot{\theta} = \lambda_1 R \ (2) \\ R\dot{\theta} = \dot{X} \text{ (from the no-sliding condition)} \implies R\ddot{\theta} = \ddot{X} \ (3) \end{cases}$$

Substituting (3) into (1) gives

$$\begin{cases} mR\ddot{\theta} = mg\sin(\alpha) - \lambda_1 \\ \frac{I}{R}\ddot{\theta} = \lambda_1 \end{cases}$$
$$\implies mR\left(\lambda_1 \frac{R}{I}\right) = mg\sin(\alpha) - \lambda_1$$
$$\implies \left(1 + \frac{mR^2}{I}\right)\lambda_1 = mg\sin(\alpha)$$
$$\implies \boxed{\lambda = \frac{mg\sin(\alpha)}{\left(1 + \frac{mR^2}{I}\right)}} \text{ This is the magnitude of friction!}$$

3.3 Example: A bead on a wire

We've seen this example before, but this time we want to calculate the normal force on the bead.

[Include Figure]

Using the Lagrange Multiplier method, we can write down the constrained Lagrangian as

$$L' = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right] - mgz - \lambda_1(\phi - \omega t) - \lambda_2\left(z - \alpha\rho^2\right)$$

So, the EL Equations look like

$$\begin{cases} m\ddot{\rho} = n\rho\dot{\phi}^2 - 2\lambda_2\alpha\rho \\ \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = \lambda_1 \\ m\ddot{z} = -mg + \lambda_2 \end{cases}$$

From the z EoM, we can tell that λ_1 is a force since it's being added with $-mg$. We can interpret it as the $\boldsymbol{z}$**−component** of the **Normal Force**.

Similarly, in the ϕ EoM we see that λ_1 is the derivative of the Angular Momentum, so λ_1 is the **torque**.

[Include figure]

Now, in the ρ equation, we know that $m\ddot{\rho}$ is also a force since ρ has units of length. So, $-2\lambda_2\alpha\rho$ must also be a force. Exactly which force is it? It's the **radial component** of the **Normal Force** (See the figure above.)

When it comes to actually solving for λ_1 and λ_2, we can solve for them after we solve for $\rho(t)$ using $z = \alpha\rho^2$ and other constraints.

[Add last bit from lecture recording - lots of figures]

4 July 9, 2024: Symmetries and Lagrangians

4.1 Note about the discussion from last time

[Write about clever method to find rolling constraint that Chien-I spoke about at the beginning of lecture]

4.2 Symmetries

Previously we discussed **Cyclic Coordinates**:

A coordinate q_k is cyclic if

$$\frac{\partial L}{\partial q_k} = 0$$

As a result, the EL equation gives us the result that

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \text{ is conserved}$$

This is a **symmetry** in the sense that when we change q_k, the Lagrangian does not change.

What exactly is a Symmetry? We define a symmetry of a system to be a **transformation** of the system such that the system behaves the same after transformation. For example, rotating a triangle by 120 degrees is a symmetry transformation of the triangle.

The study of symmetries falls under **Group Theory**, but in physics we're usually concerned specifically **continuous transformations**. Continuous symmetries often give rise to **conserved quantities**.

Example: θ independent lagrangian

We'll see this in more detail when we study Noether's Theorem.

4.3 Continuous Transformations

Usually, we have $L = L(q_k, \dot{q}_k, t)$. We can apply transformations on the q_k and t variables

$$q_k \to q'_k(q_k, t)$$
$$t \to t'(t)$$

which in turn transform the lagrangian L

When we say a transformation is continuous, we mean that we can make a transformation parametrized by some small parameter ϵ such that when $\epsilon \to 0$, the transformation is just the identity transformation.

Since the mapping is continuous, we can expand the transformation as

$$
\begin{aligned}
q_k(t) \to q'_k(t') &= q_k(t) + \delta q_k \\
t \to t(t) &= t + \delta t
\end{aligned}
$$

Example: Continuous Rotation In the plane $\mathbb{R}^2$, we can rotate a vector $V = V_x\hat{x} + V_y\hat{y}$ using a standard rotation matrix:

$$
\vec{V} \to \vec{V}' = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}
$$

where θ is a continuous parameter that represents the angle of rotation.

Examples: Non-continuous transformation

1. Dicrete rotation: Rotations where θ is only allowed to have specific values, for example $\theta = n\frac{\pi}{6}$
2. Parity: $(x, y, z) \to (-x, -y, -z)$

There are two ways to generate transformations in q_k.

1. With a fixed time, we can "mix" the coordinates:

$$
q_k(t) \to q'_k(t) = q_k(t) + \underbrace{\Delta q_k(t)}_{\text{small transformation}}
$$

For example, we can rotate a vector without messing with the time coordinate:

$$
\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} = \begin{pmatrix} x(t) - \theta y(t) \\ y(t) + \theta x(t) \end{pmatrix}
$$

We can represent this transformation concisely using the **levi-civita symbol, $\boldsymbol{\epsilon_{ij}}$**

$$
\boxed{x'_i = x_i - \theta \epsilon_{ij} x_j}
$$

2. We can generate a change in q_k by shifting the time: $t \to t + \delta t(t)$

$$
q_k(t) \to q_k(t') = q_k(t + \delta t) = q_k(t) + \dot{q}_k \delta t
$$

We can define the total (infinitessimal) transformation of q_k as

$$
\begin{aligned}
\delta q_k &\equiv q'_k(t') - q_k(t) = q'_k(t + \delta t) - q_k(t) \\
&\approx q'_k(t) + \dot{q}'_k \delta t = q_k(t) \\
\text{to first order} \to\ &\approx q_k(t) + \Delta q_k(t) \dot{q}_k(t) \delta - q_k(t)
\end{aligned}
$$

where we used

$$\dot{q}'_k(t) = \frac{d}{dt}(q_k + \Delta q_k) = \dot{q}_k(t) + \frac{d}{dt}(\Delta q_k)$$

Thus, to first order, we have

$$\boxed{\delta q_k(t) = \Delta q_k(t) + \dot{q}_k \delta t}$$

We say such a transformation by δq_k and/or δt is a symmetry if we have the same dynamics i.e. under the transformation, the **action, $\boldsymbol{\delta S = 0}$** does not change. ($\delta S$ is the change in S when we perform a particular transformation in terms of Δq_k and/or δt)

$$\begin{aligned}
0 &= \delta\left(\int dt L(q_k, \dot{q}_k, t)\right) \\
&= \int \delta(dt) L + \int dt \delta L \\
&= \int dt \frac{d(\delta t)}{dt} L + \int dt \left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\Delta q_k) + \frac{dL}{dt}\delta t\right)
\end{aligned}$$

where we should note that dL/dt is the **total** derivative

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

Continuing on and applying "Integration by Parts",

$$0 = \int dt \left[\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right)\Delta q_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k\right) + \frac{d}{dt}(L\delta t)\right]$$

If q_k satisfies the EoM,

$$0 = \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t\right)\right] = \left[\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t\right]_{t_i}^{t_f}$$

Therefore the quantity

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t$$

is conserved! What we've shown is Noether's Theorem.

Noether's Theorem: If we have a continuous symmetry and the evolution of the system satisfies the EoM, then there is an associated conserved quantity given by

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t$$

which is called the **Noether Charge**.

In fact, we can extend this a little bit. The action can change, as long as it's of the form:

$$\delta S = \int dt \left(\frac{dK}{dt} \right)$$

because such a change just adds constant boundary terms $K(t_f) - K(t_i)$ which do not change the dynamics. So,

$$\frac{d}{dt}(Q - K) = 0$$

This $Q - K$ is a more general conserved charge.

4.4 Example: Spacial Translation

$$L = \frac{1}{2} m \dot{x}^i \dot{x}_i$$

This Lagrangian is invariant under the shift $\begin{cases} x_i \to x_i + \epsilon_i \text{ spatial translation} \\ t \to t \text{ no time translation} \end{cases}$.

So, $\delta x_i = \Delta x_i + \dot{x} \underbrace{\delta t}_{=0}$. As a result,

$$Q \equiv \frac{\partial L}{\partial \dot{x}_i} \Delta x_i = m \dot{x}_i \epsilon_i \implies m \dot{x}_i \text{ is conserved.}$$

4.5 Example: Time Translation

Consider the time translation

$$\begin{cases} x_i \to x_i \\ t \to t + \delta t \end{cases}$$

i.e. $\delta x_i = 0 = \Delta x_i + \dot{x}_i \delta t$ which implies

$$\Delta x_i = -\dot{x}_i \delta t$$

Consider the following Lagrangian under time translation

$$L = \frac{1}{2} m \dot{x}^i \dot{x}_i - V(x)$$

Since $L = L(x_i, \dot{x}_i, t)$, if the x_i's don't change then the change in L is just

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial x_i} \underbrace{\delta x_i}_{=0} + \frac{\partial L}{\partial \dot{x}_i} \underbrace{\delta dot x_i}_{=0} + \frac{\partial L}{\partial t} \delta t \\ &= \frac{\partial L}{\partial t} \delta t \end{aligned}$$

And, when $\frac{\partial L}{\partial t} \delta t = 0$, we have time translation symmetry, giving us the conserved current

$$\begin{aligned}
Q &\equiv \frac{\partial L}{\partial \dot{x}_i}\delta x_i + L\delta t \\
&= -\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i \delta t + L\delta t \\
&= \left(\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L\right)(-\delta t)
\end{aligned}$$

So when we have time translation symmetry, the **Hamiltonian**

$$H = \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L$$

is conserved.

4.6 Example: Isotropic Harmonic Oscillator under rotation

Consider

$$L = \frac{1}{2}m\dot{x}^i\dot{x}_i - \frac{1}{2}kx^i x_i$$

under rotation

$$\begin{aligned}
x_i &\to x_i - \theta\epsilon_{ij}x_j \\
t &\to t
\end{aligned}$$

Then, the Lagrangian transforms into

$$\begin{aligned}
L &\to \frac{1}{2}m\dot{x}^i\dot{x}_i - \frac{1}{2}kx^i x_i + m\dot{x}_i\left(-\theta\epsilon_{ij}\dot{x}_j\right) - kx_i\left(-\theta\epsilon_{ij}x_j\right) + \mathcal{O}(\epsilon^2) \\
&= L - \theta m\epsilon_{ij}\dot{x}_i\dot{x}_j + k\epsilon_{ij}x_i x_j \\
&= L
\end{aligned}$$

where we used the fact that ϵ_{ij} is antisymmetric while $\dot{x}_i\dot{x}_j$, x_i, x_j are antisymmetric. Thus, the two terms other than L vanish.

In general if we have 2-d tensors $S_{ij} = S_{ji}$ (symmetric) and $A_{ij} = -A_{ji}$ (antisymmetric) then

$$\begin{aligned}
S_{ij}A_{ij} &= -S_{ij}A_{ji} \\
&= -S_{ji}A_{ji} \\
&= -S_{ij}A_{ij} \text{ Since } S_{ij}A_{ij} \text{ is itself symmetric as a whole} \\
\implies S_{ij}A_{ij} &= 0
\end{aligned}$$

Here, the conserved current is

$$\begin{aligned}
Q &\equiv \frac{\partial L}{\partial \dot{x}_i}\Delta x_i \\
&= m\dot{x}_i\left(-\theta\epsilon_{ij}x_j\right) \\
&= -\theta\epsilon_{ij}x_j m\dot{x}_i \\
&= -\theta\epsilon_{12}x_2 m\dot{x}_1 - \theta\epsilon_{21}x_1 m\dot{x}_2 \text{ (The terms with } i = j \text{ vanish because } e_{ii} = 0) \\
&= -\theta(ym\dot{x} - xm\dot{y})
\end{aligned}$$

which implies that **Angular momentum is conserved.** This marks the end of today's lecture.

. **(No lectures on July 10, 11 because of the midterm.)**

5 July 15: Obtaining Lagrangians from Symmetries

Today, we wrap up our discussion of Lagrangians. The goal today is to find the most general Lagrangian $L(x, \dot{x}, t)$ describing a 1D system which is **time-translation invariant** and **Galilean invariant**.

The framework we'll develop is what's done a lot in research (eg. in Effective Field Theory research), where we consider the symmetries we must enforce and then find the most general possible Lagrangian.

Also recall that a Lagrangian is called time-translation invariance if it has no eplicit time dependence.

Step 1: Consider the action

$$S = \int \mathrm{d}t L(x, \dot{x}, t)$$

and the Galilean transformation $x \to x + \Delta x = x - Vt$ (V is a constant) which causes the variation

$$\begin{aligned}\delta S &= \int \mathrm{d}t \left[\frac{\partial L}{\partial x}(-Vt) + \frac{\partial L}{\partial \dot{x}}(-V)\right] \\ &= \int \mathrm{d}t \left[(-V)\frac{d\tilde{K}}{dt}\right] \text{ (for some function } \tilde{K}\text{)}\end{aligned}$$

(as long as the change in action is of the form above, there is no change in dynamics.)

$$\implies \frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} = \frac{d\tilde{K}}{dt} = \frac{\partial \tilde{K}}{\partial x}\dot{x} + \frac{\partial \tilde{K}}{\partial \dot{x}}\ddot{x} + \frac{\partial \tilde{K}}{\partial \ddot{x}}\dddot{x} + \cdots + \frac{\partial \tilde{K}}{\partial t}$$

But the LHS has derivatives up to $\dot{x}$ at most since we restrict our lagrangians. So, we must have $\tilde{K} = \tilde{K}(x, t)$ and

$$\frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} = \frac{\partial \tilde{K}}{\partial x}\dot{x} + \frac{\partial \tilde{K}}{\partial t} \qquad (\star)$$

Step 2: Notice is that $\tilde{K} = \tilde{K}(x, t)$ means that the only place in equation ($\star$) that we have $\dot{x}$-dependence is $\frac{\partial \tilde{K}}{\partial x}\dot{x} + \frac{\partial \tilde{K}}{\partial t}$ i.e. it is linear in $\dot{x}$.

$\implies$ at most, L can only be second order in $\dot{x}$, namely

$$L = f_2(x, t)\dot{x}^2 + f_1(x, t)\dot{x} + f_0(x, t)$$

Step 3: Finally, if we enfore time-translation symmetry then we have

$$\frac{\partial L}{\partial t} = 0$$

which tells us $f_i(x,t) = f_i(x)$ and so,

$$\implies L = f_2(x)\dot{x}^2 + f_1(x)\dot{x} + f_0(x) \qquad (\Delta)$$

Step 4: Let's look at equation ($\star$) again, which we obtained from enforcing Galilean invariance. With (Δ) we know

$$\frac{\partial L}{\partial x} \supseteq \frac{\partial f_2}{\partial x}\dot{x}^2$$

(sidenote: this is an abuse of notation that I love; it literally just means that $\frac{\partial L}{\partial x}$ includes a term $\frac{\partial f_2}{\partial x}\dot{x}^2$ it's so goofy fr)

But the RHS of equations ($\star$) has no $\dot{x}^2$ terms. Thus,

$$\frac{\partial f_2}{\partial x} = 0$$

i.e. f_2 is a constant.

Step 5: For the $f_1(x)\dot{x}$ term in the Lagrangian, we can say

$$S \supseteq \int_i^f f_1(x)\dot{x}dt = \int_i^f f_1(x)dx = F(x_f) - F(x_i) \text{ (constant)}$$

so this does not affect the dynamics and so we can safely throw it out of the Lagrangian and shove it to the curb.

Step 6: The only thing left is

$$L = f_2\dot{x}^2 + f_0(x)$$

Plugging this back into equation ($\star$), we get

$$\begin{aligned} \frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} &= \frac{\partial \tilde{K}}{\partial t} + \frac{\partial \tilde{K}}{\partial x}\dot{x} \\ \implies \frac{\partial f_0}{\partial x}t + 2\dot{x}f_2 &= \frac{\partial \tilde{K}}{\partial t} + \frac{\partial \tilde{K}}{\partial x}\dot{x} \qquad (\square) \end{aligned}$$

Matching the terms with and without $\dot{x}$ dependence, we get

$$\frac{\partial \tilde{K}}{\partial x} = 2f_2 = \text{ constant } \implies \tilde{K} = 2f_2x + g(t)$$

Step 7: Plugging the expression for $\tilde{K}$ into equation ($\square$) we get

$$\frac{\partial f_0}{\partial x}t + 2\dot{x}f_2 = \frac{\partial g}{\partial t} + 2f_2x = \frac{\partial g}{\partial t} + 2f_2\dot{x}$$

but $g(t)$ has no x-dependence $\implies \frac{\partial f_0}{\partial x}$ cannot have x-dependence

$$\implies f_0 = c_1x + c_0$$

In summary, the most general Lagrangian for a 1D particle that is time-translation invariant and Galilean invariant is of the form

$$L = c_2\dot{x} + c_1 x + c_0$$

(where instead of writing f_2 we just wrote c_2). But also, constants don't affect the dynamics. Therefore, the most general Lagrangian we need is

$$L = c_2\dot{x}^2 + c_1 x$$

What kind of physics does this lagrangian describe?

That of a non-relativistic particle subject to a constant force field. For example, if the field is that of uniform gravity, then

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

(When people do this in Effective Field Theory, there are much more complicated symmetries to enfore such as $U(1), SU(2)$ etc. but the procedure is follows the same basic idea we've studied here.)

We were able to get a very specific form for our Lagrangian, but often symmetries are not strong enough to constrain our Lagrangian. In that case, we have to use other methods to determine the form of the Lagrangian for our system such as by experimental considerations.

Okay, now we go back to some more tradition stuff from classical mechanics.

5.1 Central Force Problem

A **Central Force** is one whose potential energy function has **Spherical symmetry** eg. Isotropic Harmonic Osciallators or a particle under uniform gravitational force.

Usually, we have a two-body problem with a Lagrangian of the form

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(r)$$

where $r = |\vec{r}_2 - \vec{r}_1|$. But, we can effectively change this into a one-body problem by using the **Center-of-Mass** coordinates.

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1, \quad \vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Using these coordinates and $M \equiv m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$ we can write the Lagrangian as

$$L = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{free motion of C.M.}} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r})}_{\text{non-trivial dynamics}}$$

We've essentially decoupled the two bodies. Now, if $U(\vec{r})$ has spherical symmetry i.e. $U(\vec{r}) = U(r)$ then it is natural to work in spherical coordinates.

So, the non-trivial part of the Lagrangian will be

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - U(r)$$

and ϕ is a cyclic coordinate of it. Thus,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2(\theta) \equiv l_z$$

is conserved. Then,

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{l_z}{\mu^2 r^2 \sin^2(\theta)}\right) - U(r)$$

The fact that the angular momentum l_z (and in fact, the *total* angular momentum l) is conserved means that the particle is moving on a fixed plane.

Why is total angular momentum conserved? A central force cannot cause torque, therefore angular momentum is conserved.

WLOG, we set the $z-axis$ to be perpendicular to the motion of the plane i.e. we set $\theta = \frac{\pi}{2} \implies \dot{\theta} = 0$ due to which $\sin(\theta) = 1$. So, our non-trivial Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{l_z}{2\mu r^2} - U(r)$$

Before we put the constraint on ϕ, the Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$

and the corresponding Hamiltonian is

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{l_z^2}{2\mu r^2} + U(r)}_{\text{effective potential}} \end{aligned}$$

Also, when we switch from Spherical Coordinates to Cartesian Coordinates we use

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

which have no time dependence, and so $H = E$. Aaaaaand... we're out of time so w'll continue tomorrow.

6 July 16:

6.1 Continuing our discussion of Central Forces

As we saw last time, when dealing with a 2-body central force, we can do a **change of coordinates** to make things simpler.

The dynamics of two bodies moving in a central force can be decoupled into the dynamics of a free body (moving with constant mass) and another mass subject to an effective potential.

[Include figure of 1-body and 2-body central forces]
[Write about the 2-body lagrangian being split into free and non-trivial parts.]

6.2 Aside: Ineratial Mass and Gravitational Mass

In Newton's Law of Gravitation,

$$|\vec{F}| = \frac{G\mu_1\mu_2}{r^2}$$

the quantities μ_1, μ_2 are called gravitational mass (though a better name would be gravitational charge since the force is of the same form as coulomb's law). They tell us how strong thee interaction is.

And, in Newton's First Law,

$$|\vec{F}| = m|\vec{a}|$$

the term m is the inertial mass. It tells us how hard it is to change the particles velocity.

Now, a priori, there is no reason for these quantities to be the same. However, if we conduct an experiment where a base-ball and a feather free-fall towards the earth, we know

$$\begin{aligned} a_{\text{baseball}} &= \frac{|\vec{F}|}{m_{\text{baseball}}} = \frac{G\mu_\oplus \mu}{R_\oplus^2 m_\oplus} \\ a_{\text{feather}} &= \frac{|\vec{F}|}{m_{\text{feather}}} = \frac{G\mu_\oplus \mu}{R_\oplus^2 m_\oplus} \end{aligned}$$

and we find via the experiment that $a_{\text{baseball}} = a_{\text{feather}} = g$
Then,

$$\left(\frac{G\mu_\oplus}{R_\oplus^2}\right)\left(\frac{\mu_{\text{baseball}}}{m_{\text{baseball}}}\right) = \left(\frac{G\mu_\oplus}{R_\oplus^2}\right)\left(\frac{\mu_{\text{feather}}}{m_{\text{feather}}}\right)$$

So the gravitational mass to the inertial mass ratio is the same across objects! Note that μ/m is analogous to e/m in electrostatics and, for example, the proton p^+ and electron e^- has *very different e/m ratio.* But miraculously, when it comes to gravity, μ/m is the same for all objects. Numerically, we can just set μ and m equal to each other.

The statement that μ/m is the same for all objects is called the **Equivalence Principle**. This has important implications in General Relativity.

Include discussion about curved space and hockey pucks on the earth.

6.3 Continuing on with the 2-body problem

Focusing on the dynamics of $\vec{r}$, the Lagrangian for that part is

$$\begin{aligned} L &= \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \\ &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - U(r) \end{aligned}$$

But notice that these dynamics are the same as those of the 1-body central force scenario as $\vec{L}$ is conserved and motion is on a plance. i.e. θ is fxed which we can WLOG set to $\pi/2$.

$$\begin{aligned} L &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \\ \implies H &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{l_z^2}{2\mu r^2} + U(r)}_{\text{effective potential for a 1D d.o.f. } r, U_{\text{eff}}} \end{aligned}$$

This is where we stopped last lecture. The position r is then subject to a force

$$\begin{aligned} F &= -\frac{\partial U_{\text{eff}}}{\partial r} \\ &= +\frac{l_z^2}{\mu r^2} - \frac{\partial U(r)}{\partial r} \end{aligned}$$

where

$$\frac{l_z^2}{\mu r^2} = \frac{\mu^2 r^2\dot{\phi}^2}{\mu r^2} = \frac{\mu v^2}{r} = \mu a_c$$

This looks like a "centrifugal force".

Let's think about centrifugal forces for a sec. Consider the following setup where Bob is at a distance r from the z-axis and is rotating about the axis a constant rate.

[Include figure]

In Bob's Frame, these three forces [complete this later]

[Include figure of spring extending from fixed point]

The angular momentum conservation keeps to r to be non-zero because when $r = 0$, we have $l_z = 0$. This is why we have a positive sign in front of the $(l_z)^2/\mu r^2$ term - the force is pushing outwards.

This U_{eff} also allows us to understand whether r is a fixed value or changes between some r_{min} and t_{max}. For example, if we consider a spring with $U(r) = \frac{1}{2}kr^2$ we get

$$U_{\text{eff}} = \frac{(l_z)^2}{2\mu r^2} + \frac{1}{2}kr^2$$

Plotting this, we have [Include figure]
[Write the discussion about the plot based on lecture recording and textbook.]

If we instead consider a gravitational system with $U(r) = -\frac{Gm_1m_2}{r^2}$ we have

$$U_{\text{eff}} = \frac{(l_z)^2}{2\mu r^2} - \frac{Gm_1m_2}{r^2}$$

which gives us the plot

[Include figure]

You've probably seen a very similar plot when studying Chemistry/Quantum Mechanics. The reason for that is because the effective potential for an atom with $l \neq 0$ has a very similar form.

[Include discussion of the plot based on lecture recording and textbook.]

If we want to find $r(t)$ we can use

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{(l_z)^2}{2\mu r^2} + U(r) \\ \implies \left(\frac{dr}{dt}\right)^2 &= \frac{2}{\mu}\left[E - \frac{(l_z)^2}{2\mu r^2} - U(r)\right] \\ \implies \int_{t_0}^{t} dt' &= \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r} \frac{dr}{\sqrt{E - \frac{(l_z)^2}{2\mu r^2} - U(r)}} \end{aligned}$$

and then after integrating, the inverse relation gives us $r(t)$.

6.4 Orbital Shape

We will just to pure 1-body problems here, but the procedure remains the same (just more tedious) in the 2-body case.

Now, what does it mean to find **Orbital Shape**? It refers to finding $r(\phi)$ i.e. we don't care about the time dependence here; just the dependence on ϕ.

Consider

$$E = \frac{1}{2}m\dot{r}^2 + \frac{(l_z)^2}{2mr^2} + U(r)$$

If we divide the entire equation by $l_z = mr^2\dot{\phi}$ we get

$$\frac{E}{l_z} = \frac{1}{2}m\frac{1}{m^2r^4}\frac{\dot{r}^2}{\dot{\phi}^2} + \frac{1}{2mr^2} + U(r)$$

Why is this useful? Well, notice that

$$\frac{\dot{r}}{\dot{\phi}} = \frac{dr/dt}{d\phi/dt} = \frac{dr}{d\phi}$$

So, we have

$$\begin{aligned}\frac{E}{l_z} &= \frac{1}{2}m\frac{1}{m^2r^2}\frac{\dot{r}^2}{\dot{\phi}^2} + \frac{1}{2mr^2} + U(r)\\ &= \frac{1}{2mr^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{2mr^2} + U(r)\end{aligned}$$

and now there is no time-dependence in the equation. Continuing on,

$$\begin{aligned}\left(\frac{dr}{d\phi}\right)^2 &= 2mr^4\left[\frac{E}{(l_z)^2} - \frac{1}{2mr^2} - \frac{U(r)}{(l_z)^2}\right]\\ \implies d\phi &= \pm\sqrt{\frac{1}{2m}}\frac{dr}{\sqrt{\frac{E}{l_z^2}r^4 - \frac{r^2}{2m} - \frac{U(r)}{l_z^2}r^4}}\\ \implies &\boxed{d\phi = \pm\frac{l_z}{\sqrt{2m}}\frac{1}{r^2}\frac{dr}{\sqrt{E - \frac{l_z^2}{2mr^2} - U(r)}}}\end{aligned}$$

Let's do a concrete calculation. For $U(r) = \frac{1}{2}kr^2$,

$$\int_{\phi_0}^{\phi} d\phi' = \pm\frac{l_z}{\sqrt{2m}}\int_{r_0}^{r}\frac{1}{r^2}\frac{dr}{\sqrt{E - \frac{l_z^2}{2mr^2} - \frac{1}{2}kr^2}}$$

Now, with the substitution $z = r^2, dz = 2rdr$

$$\begin{aligned}\int_{\phi_0}^{\phi} d\phi' &= \pm\frac{l_z}{\sqrt{2m}}\int_{z_0}^{z}\frac{1}{z}\frac{1}{2z}\frac{dz}{\sqrt{-\frac{l_z^2}{2mz} + E - \frac{1}{2}kz}}\\ &= \pm\frac{l_z}{\sqrt{2m}}\int_{z_0}^{z}\frac{1}{2z}\frac{dz}{\sqrt{-\frac{l_z^2}{2m} + Ez - \frac{1}{2}kz^2}}\end{aligned}$$

and using the Integral Formula

$$\int \frac{dz/z}{\sqrt{a+bz+cz^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bz+2a}{z\sqrt{b^2-4ac}}\right)$$

we get

$$\begin{aligned}
\phi - \phi_0 &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2-(l_z)^2/m}{r^2\sqrt{E^2-k(l_z)^2/m}}\right) \\
\implies \pm\sin\left[2(\phi-\phi_0)\right] &= \frac{Er^2-(l_z)^2/m}{r^2\sqrt{E^2-k(l_z)^2/m}} \\
\implies r^2 &= \frac{(l_z)^2/m}{\left\{E - \mp\sqrt{E^2-k(l_z)^2/m}\,\sin\left[2(\phi-\phi_0)\right]\right\}}
\end{aligned}$$

We're out of time now, but next time we'll see how this gives us an orbital shape which is an ellipse.

7 July 17: Orbital Shape continued

7.1 Orbital Shape for a spring with $\vec{F} = -k\vec{r}$

Yesterday, we found that

$$\begin{aligned}
\phi - \phi_0 &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2 - (l_z)^2/m}{r^2\sqrt{E^2 - k(l_z)^2/m}}\right) \\
\implies \sin(2[\phi - \phi_0]) &= \pm\frac{Er^2 - (l_z)^2/m}{r^2\sqrt{E^2 - k(l_z)^2/m}} \\
\implies & \boxed{r^2 = \frac{(l_z)^2/m}{E \mp \sqrt{E^2 - k(l_z)^2/m}\sin\left[2(\phi - \phi_0)\right]}} \qquad (\star)
\end{aligned}$$

Okay. Time for some high school maths. The equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we work in polar coordinates, the equation becomes

$$\begin{aligned}
\frac{r^2\cos^2(\phi)}{a^2} + \frac{r^2\sin^2(\phi)}{b^2} &= 1 \\
\implies r^2 &= \frac{a^2b^2}{a^2\sin^2(\phi) + b^2\cos^2(\phi)} \\
\implies r^2 &= \frac{a^2b^2}{a^2\left(\frac{1-\cos(2\phi)}{2}\right) + b^2\left(\frac{1+\cos(2\phi)}{2}\right)} \\
\implies r^2 &= \frac{2a^2b^2}{(a^2+b^2) + (b^2-a^2)\cos(2\phi)}
\end{aligned}$$

and using the identity $-\cos(\alpha) = \sin(\alpha - \frac{\pi}{2})$, we get

$$\boxed{r^2 = \frac{2a^2b^2}{(a^2+b^2) + (a^2-b^2)\sin(2\phi - \frac{\pi}{2})}} \qquad (\square)$$

Now, in equation $(\star)$, if we set $\phi_0 = \frac{\pi}{4}$ and choose the + sign (doing this isn't cherry picking the result, but instead just choosing the coordinates we start off with), we get

$$r^2 = \frac{(l_z)^2/m}{E + \sqrt{E^2 - k(l_z)^2/m}\sin(2\pi - \frac{\pi}{2})}$$

So, comparing this with equation $(\square)$, we see that the Orbital Shape for the Spring is an ellipse with

$$\begin{aligned}
a^2 &= \frac{1}{2k}\left[E + \sqrt{E^2 + k(l_z)^2/m}\right] \\
b^2 &= \frac{1}{2k}\left[E + \sqrt{E^2 - k(l_z)^2/m}\right]
\end{aligned}$$

The geometric properties of the Orbital Shape are determined by E and l.

7.2 Orbtial Shape corresponding to Gravitation

We found in a previous lecture that

$$\phi(r) = \pm \frac{l_z}{\sqrt{2m}} \int \frac{dr}{r^2\sqrt{E - \frac{(l_z)^2}{2mr^2} + \frac{GMm}{r}}}$$

$$\phi(r) = \pm \frac{l_z}{\sqrt{2m}} \int \frac{dr/r}{\sqrt{Er^2 - \frac{(l_z)^2}{2m} + GMmr}}$$

This has the same form as the Spring case, where

$$\phi(r) = \pm \frac{l_z}{\sqrt{2m}} \int \frac{1}{2E} \frac{dr}{r^2\sqrt{Ez - \frac{(l_z)^2}{2m} + \frac{1}{2}kz^2}}$$

except we need to make the substitutions $z \to r,\ -\frac{1}{2}k \to E,\ E \leftrightarrow GMm$
So, using the solution found earlier with the appropriate substitutions, we get

$$\implies \phi - \phi_0 = \pm \sin^{-1}\left(\frac{GMMr - (l_z)^2/m}{r\sqrt{(GMm)^2 + 2E(l_z)^2/m}}\right)$$
$$\implies \sin(\phi - \phi_0) = \pm \frac{GMm^2r - l^2}{GMm^2re}$$

where

$$e = \sqrt{1 + \frac{2E(l_z)^2}{G^2M^2m^2}}$$

(we'll see that the Orbital shape is once again an ellipse and this quantity e is the *eccentricity*.)

$$\implies r = \frac{(l_z)^2/(GMm^2)}{1 \mp e\sin(\phi - \phi_0)}$$

Now, another way to define an Ellipse is in terms of its two foci.

[Include figure from lecture].

An ellipse has the relations $c = ae$ and a point on the ellipse has polar coordinates (r, ϕ) related by

$$r = \frac{a(1 - e^2)}{1 + e\cos(\phi)}$$

- When $e = 0$, $r = \frac{l_z^2}{GMm^2} \implies$ constant r i.e. circular orbit.
- When $0 < e < 1$, we have an elliplse, Namely, we can set $\phi_0 = \frac{\pi}{2}$ and choose the positive sign, giving us

$$r = \frac{l_z^2/GMm^2}{1 + e\cos(\phi)}$$

which is of the form of $\frac{a(1-e^2)}{1+e\cos(\phi)}$ with

$$e = \sqrt{1 + \frac{2E(l_z)^2}{G^2M^2m^3}}$$

- We can also find a by

$$a = \frac{l_z^2/GMm^2}{(1-e^2)} = \frac{l_z^2}{-\frac{2E(l_z)^2}{G^2M^2m^2}}$$
$$\implies a = -\frac{GMm}{2E}$$

The minus sign is fine because E is negative for a bound state, so a will be positive.

- When $e = 1$, physically we have

$$r = \frac{l_z^2/GMm^2}{1+\cos(\phi)}$$

This matches the geometric parametrization

$$r = \frac{2a}{1+\cos(\phi)}$$

which describes a parabola.

- Also, note that since

$$e \equiv \sqrt{1 + \frac{2El_z^2}{G^2M^2m^3}}$$

we have $e = 0 \Leftrightarrow E = 0$. This matches up with our discussion yesterday. [Flesh this out more]

- When $e > 1$, physically we still have $r = \frac{l_z^2/GMm^2}{1+e\cos(\phi)}$. But this will match with the geometric parametrization of a hyperbola.

7.3 Coulomb Scattering

(discussed in Ch. 8 of H&S) In the case where we have two charges of the same sign,

$$U = \frac{kq_1q_2}{r} > 0$$

$$\implies U_{\text{eff}} = \frac{l_z^2}{2mr^2} + \frac{kq_1q_2}{r}$$

If we plot U_{eff} we see there is no potential well and thus no bound states. [Include figure]

If we imagine charge q_2 of mass m moving towards charge q_1 (which we fix in place) with initial velocity $\vec{v}_0$ and impact parameter b, we note that the impact parameter gives us the angular momentum of q_2:

$$l = bmv_0$$

Goal: We want to find the relation between b and how much the particle q_2 is deflected.

[include picture of two charges with same mass same charge moving towards fixed target, but with different impact parameters]

Intuitively it makes sense that the smaller the impact parameter, the greater the deflection. We will make this more precise.

Compare

$$H = \frac{1}{2}m\dot{r}^2 + \frac{l_z^2}{2mr^2} + \frac{kq_1q_2}{r}$$

for the Coulomb case with

$$H = \frac{1}{2}m\dot{r}^2 + \frac{l_z^2}{2mr^2} - \frac{GMm}{r}$$

for the gravitational case. All we need to do is substitute

$$-GMm \to +kq_1q_2$$

and we'll get the result.

In the gravitational case we had

$$\pm\sin(\phi - \phi_0) = \frac{GMm^2r - l_z^2}{GMm^2r\sqrt{1 + \frac{2El_z^2}{G^2M^2m^2}}}$$

and doing the substitution gives us

$$\pm\sin(\phi - \phi_0) = \frac{-kq_1q_2mr - l^2}{-kq_1q_2mr\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}}$$

(here l is the angular momentum with reference to the fixed point charge)

$$\implies r = \frac{l^2}{kq_1q_2m\left[\pm\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}\sin(\phi - \phi_0) - 1\right]}$$

Comparing this with the geometric parametrization of a parabola, if we choose $phi_0 = \frac{\pi}{2}$ and take the + sign (so that the bottom overall has a negative sign) we get

$$r = \frac{l^2}{-kq_1q_2m\left[\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}\cos(\phi) + 1\right]}$$

Recall that in the gravity case we defined

$$e \equiv \sqrt{1 + \frac{2El^2}{G^2M^2m^3}}$$

In the coulomb case this turns into

$$e \equiv \sqrt{1 + \frac{2El}{k^2q_1^2q_2^2m}}$$

and

$$1 - e^2 = -\frac{2El^2}{k^2q_1^2q_2^2m}$$

With these, for the coulomb case, we get

$$\boxed{r = \frac{(1-e^2)\left(\frac{kq_1q_2}{2E}\right)}{1 + e\cos(\phi)}}$$

(There might be a sign error here - go through derivation)
This describes a hyperbola, and $r \to \infty$ at ϕ_1, ϕ_2 such that

$$\cos(\phi_{1,2}) = -\frac{1}{e}$$

We can use ϕ_1, ϕ_2 to find the deflection angle and write it in terms of b (which is our goal).