Math 214 Homework 5

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Q4-5. Let $\mathbb{CP}^n$ denote the $n-$dimensional complex projective space.

(a) Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.

(b) Show that $\mathbb{CP}^n$ is diffeomorphic to $\mathbb{S}^n$.

Proof:

(a) To show that π is smooth, let's write it in terms of coordinates. Let $\tilde{U}_k = \{\left(z^1, \dots, z^{n+1}\right) \in \mathbb{C}^{n+1} \setminus \{0\}, z^k \neq 0\}$ and let $U_k = \pi(\tilde{U}_k)$. Let $z = (z^1, \dots, z^{n+1}) \in \tilde{U}_k$.

Then, $\{(U_k, \text{id})\}_{k=1,\dots,n}$ forms a collection of smooth chart which covers $\mathbb{CP}^n$ and $\{\left(\tilde{U}_k, \phi_k\right)\}_{k=1,\dots,n+1}$ form an open cover of $\mathbb{C}^{n+1} \setminus \{0\}$ where $\phi_i : U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ is defined as the map

$$[z^1 : \dots : z^{n+1}] \mapsto \left(\frac{z^1}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots \frac{z^{n+1}}{z^i}\right)$$

Then, the coordinate representation of π on each of these charts is given by

$$\begin{aligned}
(\phi_k \circ \pi \circ \text{id}) \big|_{\text{id}\left(U_k \cap \pi^{-1}(\tilde{U}_k)\right)} (z^1, \cdots, z^{n+1}) &= \phi_k \circ \pi(z^1, \cdots, z^{n+1}) \\
&= \phi_k \left([z^1 : \cdots : z^{n+1}]\right) \\
&= \left(\frac{z^1}{z^k}, \cdots, \frac{z^{k-1}}{z^k}, \frac{z^{k+1}}{z^k}, \cdots \frac{z^{n+1}}{z^k}\right)
\end{aligned}$$

which is smooth since $z^k \neq 0$ on the domain. From the above, π is smooth. Also, quotient maps are surjective by definition. Now, let's show π is a submersion. Let's denote $z^j = z^j + iy^j$. Then,

$$\frac{z^j}{z^k} = \frac{x^j x^k + y^j y^k}{(x^k)^2 + (y^k)^2} + i \frac{x^k y^j - x^j y^k}{(x^k)^2 + (y^k)^2}$$

In the coordinates of U_i, the differential $d\pi$ can be represented with the matrix

$$\begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & D \end{bmatrix}$$

where

$$D = \begin{pmatrix} \frac{x^k}{(x^k)^2+(y^k)^2} & \frac{-y^k}{(x^k)^2+(y^k)^2} \\ \frac{y^k}{(x^k)^2+(y^k)^2} & \frac{x^k}{(x^k)^2+(y^k)^2} \end{pmatrix}$$

Now, $det(D) = 1$ so $det(A) = 1$. This tells us that $d\pi$ has full rank and therefore π is a submersion. This proves part (a).

(b) To show $\mathbb{CP}^1 \cong_{diff} \mathbb{S}^2$, we define the following map $F : \mathbb{S}^2 \to \mathbb{CP}^1$

$$F(x,y,z) = \begin{cases} [1, \frac{x}{1-z} + i\frac{y}{1-z}], & \text{if } (x,y,z) \in \mathbb{S}^2 \setminus \{N\} \\ [\frac{x}{1+z} - i\frac{y}{1+z}, 1], & \text{if } (x,y,z) \in \mathbb{S}^2 \setminus \{S\} \end{cases}$$

Now, we note that $F\big|_{\mathbb{S}^2\setminus\{N\}} = \phi_2^{-1} \circ i \circ \sigma$ where i is the identification of $\mathbb{C}^1 \cong \mathbb{R}^2$, and σ is the stereographic projection from the north. Each of these is a diffeomorphism, thus so is $F\big|_{\mathbb{S}^2\setminus\{N\}}$ (on its image).

Similarly, $F\big|_{\mathbb{S}^2\setminus\{S\}} = \phi_1^{-1} \circ \tau \circ i \circ \tilde{\sigma}$ where $\tilde{\sigma}$ is the stereographic projection from the south and τ is complex conjugation. These are all diffeomorphisms, so $F\big|_{\mathbb{S}^2\setminus\{S\}}$ is a diffeomorphism onto its image.

Finally, we note that $U_1 \cup U_2$ cover $\mathbb{CP}^1$. So, $\mathbb{CP}^1 \cong_{diff} \mathbb{S}^1$.

Q4-6. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Proof: From LeeSM Proposition 4.28, We know that if $\pi : M \to N$ is a smooth submersion between smooth manifolds then π is an open map. Now, consider M to be a non-empty smooth compact manifold and let $N = \mathbb{R}^k$. $M \subseteq M$ is open when viewed as a subset of itself. However, $F(M)$ is a compact subset of $\mathbb{R}^k$ since F is a smooth map, and compact subsets of euclidean space are not open. Thus, we have a contradiction.

Q4-7. Suppose M and N are smooth manifolds, and $\pi : M \to N$ is an injective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29.

Proof:
From Theorem 4.28, we know that surjective smooth submersions are quotient maps. Then, from the uniqueness of the quotient topology, we know there is no other smooth manifold structure on N such that the conclusion of Theorem 4.29 holds.

Q4-8. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x,y) = xy$. Show that π is surjective and smooth, and that for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof:
For any $t \in \mathbb{R}$, we can simply choose $x = t, y = 1$. Then, $\pi(x,y) = \pi(t,1) = t$, so the map is surjective. The map is also smooth since the partial derivatives with respect to $x^1, x^2 = x, y$ are smooth

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

However, π is not a smooth submersion since the differential of π

$$d\pi_{(0,0)} = \begin{pmatrix} x \\ y \end{pmatrix}\Bigg|_{(0,0)} = \mathbf{0}$$

has rank zero at the origin, whereas it has rank 1 everywhere else on $\mathbb{R}^2$. So, π is not a constant rank map.

Q4-9. Let M be a connected smooth manifold, and let $\pi : E \to M$ be a topological covering map. Complete the proof of proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map.

Proof:

> ***Theorem 4.40:*** Suppose M is a connected smooth $n-$manifold and $\pi : E \to M$ is a *topological* covering map. Then E is a topological $(n-1)$ manifold and there exsits a unique smooth structure on E such that π is a smooth covering map.

The book proves that E is a topological $(n-1)$ manifold and that there exists a smooth structure on it such that π is a smooth covering map. Now, let's suppose $\tilde{E}$ is the same set but with a different smooth strucuture on it, such that $\tilde{\pi} : \tilde{E} \to M$ is smooth. To show that the two smooth structures on E must be the same, let's prove that $\mathrm{id} : E \to \tilde{E}$ is a diffeomorphism.

$$\begin{tikzcd} E \arrow[rr, "\mathrm{id}"] \arrow[dr, "\pi"'] & & \tilde{E} \arrow[dl, "\tilde{\pi}"] \\ & M & \end{tikzcd}$$

Every point $x \in E$ lies in the pre-image (under π) of some evenly covered subset $V \subseteq M$ which is the domain of a chart $\phi : V \to \mathbb{R}^m$.

Then, let U be an open neighborhood of x on which π restricts to a homeomorphism from U to V

$$\pi\big|_U : U \to V$$

Then, it follows that $\phi \circ \left(\pi\big|_U\right) : U \to \mathbb{R}^m$ is a smooth map with respect to both atlases $\mathcal{T}_1, \mathcal{T}_2$ on $E, \tilde{E}$. Doing this for all points $x \in E$, we have a cover of E in both atlases. Thus, the two atlases are the same.

Q5-4. Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$.

Proof:
If we denote the image of the curve as S and let U be a small open neighborhood in $\mathbb{R}^2$ centered around the origin, then $S \cap U$ is open in S with the subspace topology. However, for small enough S, the set $(S \cap U) \setminus \{\mathbf{0}\}$ (open set in S with just the origin deleted, so still open in S) has four connected components whereas any open ball in $\mathbb{R}^n$ after deleting a point has either two connected components (in the $n = 1$ case) or one connected component ($n \neq 1$). Thus, it is impossible for this open subset of S to be homeomorphic to any open set in $\mathbb{R}^n$. So, the image of the curve cannot be an embedded submanifold of $\mathbb{R}^2$.

Q5-6. Suppose $M \subseteq \mathbb{R}^n$ is an embedded $m-$dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all *unit* tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n \ : \ x \in M, v \in T_xM, \ |v| = 1\}$$

This is called the ***Unit Tangent Bundle of*** M***.*** Prove that UM is an embedded $(2n-1)-$dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

Proof:
Consider the map $\Phi : TM \to \mathbb{R}$ defined, for $x \in M$ and $v \in T_x M$, as

$$(x, v) \mapsto |v|^2 = (v^1)^2 + \cdots + (v^n)^2$$

Then, $UM = \Phi^{-1}(1)$ and Φ is a smooth map of constant rank $(= 1)$. The differential of Φ is never singular because $v \neq 0$ so $\dim \ker \Phi = 0$ and so the Rank Nullity Theorem tells us $\dim \mathbb{R} = \dim \mathrm{Im}\Phi = 1$.

Then UM forms a regular level set of Φ, so by Corollary 5.14 in LeeSM, it is an embedded submanifold whose codimension is equal to 1. Thus, its dimension is $\dim T\mathbb{R}^n - 1 = 2n - 1$.

Q5-7. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Proof:
The differential of $F(x, y)$ is given by

$$dF_{(x,y)} = \begin{bmatrix} 3x^2 + y & x + 3y^2 \end{bmatrix}$$

This differential is non-singular at all points in $\mathbb{R}^2$ other than $(x, y) = (0, 0)$ and $(x, y) = \left(-\frac{1}{3}, -\frac{1}{3}\right)$.

- $F(0, 0) = 0$
- $F\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}$

So, for any $c \in \mathbb{R} \setminus \{0, \frac{1}{27}\}$, the level set $F^{-1}(c)$ is an embedded submanifold by the Regular Level Set Theorem.

Now, when it comes to $F^{-1}(0)$, the level set is the singleton $\{(0, 0)\} \subseteq \mathbb{R}^2$. This is a $0-$dimensional submanifold because the inclusion of a point into $\mathbb{R}^2$ is smooth. The same argument holds for the level set $F^{-1}(1/27) = \{(-1/3, -1/3)\}$
