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Some exercises from Shankar

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Exercise 10.1.2:

We have basis vectors $|+\rangle, |-\rangle$ and operators

$$\sigma_1^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$\sigma_2^{(2)} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

1. We know that

$$\sigma^{(1)\otimes(2)} = \sigma^{(1)} \otimes \mathbb{1}^{(2)}$$

can be written in terms of how it acts on the basis vectors $|+\rangle\otimes|-\rangle$, $|+\rangle\otimes|+\rangle$, $|-\rangle\otimes|-\rangle$, and $|-\rangle\otimes|-\rangle$ in the sense that the element $\left(\sigma^{(1)} \otimes \mathbb{1}^{(2)}\right)_{11}$ is given by

$$\begin{aligned}
\left(\sigma^{(1)} \otimes \mathbb{1}^{(2)}\right)_{11} &= (\langle + | \otimes \langle + |) \left(\sigma_1^{(1)} \otimes \mathbb{1}^{(2)}\right) (|+\rangle \otimes |+\rangle) \\
&= (\langle + | \otimes \langle + |) \left(| \sigma_1^{(1)} | 1 \rangle \otimes | \mathbb{1} | + \rangle\right) \\
&= \left\langle + \middle| \sigma_1^{(1)} | + \right\rangle \left\langle + \middle| + \right\rangle \\
&= a
\end{aligned}$$

and so on.

Thus, carring out all such calculations, the matrix has 4×4 elements and is given by

$$\sigma_1^{(1)\otimes(2)} = \sigma^{(1)} \otimes \mathbb{1}^{(2)} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

2. Similar procedure for $\sigma_2^{(1)\otimes(2)}$. We find that

$$\sigma_2^{(1)\otimes(2)} = \mathbb{1}^{(1)} \otimes \sigma^{(2)} = \begin{bmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}$$

3. There is a better way to show parts (a) and (b). In this method, we prove a more general statement that if

$$A_1^{(1)} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$B_2^{(2)} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then, their direct product is given by

$$M = A_1^{(1)} \otimes B_2^{(2)} = \begin{bmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{bmatrix}$$

Proof: The matrix element $M_{(i_1 \cdot i_2),(j_1 \cdot j_2)}$ is related to entries in A and B as

$$\begin{aligned} M_{(i_1 \cdot i_2),(j_1 \cdot j_2)} &= (\langle i_1 \mid \otimes \langle i_2 \mid) \left(A_1^{(1)} \otimes B_2^{(2)} \right) (\mid j_1 \rangle \otimes \mid j_2 \rangle) \\ &= (\langle i_1 \mid \otimes \langle i_2 \mid) \left(A_1^{(1)} \mid j_1 \rangle \otimes B_2^{(2)} \mid j_2 \rangle \right) \\ &= \left\langle i_1 \middle| A_1^{(1)} \middle| j_1 \right\rangle \left\langle i_2 \middle| B_2^{(2)} \middle| j_2 \right\rangle \\ &= (A_1^{(1)})_{i_1,j_1} \cdot (B_2^{(2)})_{i_2,j_2} \end{aligned}$$

and this is equivalent to the matrix form written above.

Thus, we can directly apply this result in finding the tensor product between $\sigma_1^{(1)}$ and $\sigma_2^{(2)}$:

$$\begin{aligned} (\sigma_1 \sigma_2)^{(1)\otimes(2)} &= \sigma_1^{(1)} \otimes \sigma_2^{(2)} \\ &= \begin{bmatrix} a\begin{pmatrix} e & f \\ g & h \end{pmatrix} & b\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ c\begin{pmatrix} e & f \\ g & h \end{pmatrix} & d\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{bmatrix} \\ &= \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix} \end{aligned}$$

DO SECOND METHOD OF PART 3 LATER

Problem
