Math 214 Notes

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These are notes taken from lectures on Differential Topology delivered by Eric C. Chen for UC Berekley's Math 214 class in the Sprng 2024 semester. Any errors that may have crept in are solely my fault.

Contents

1 January 25 - Smooth structure, Local coordinates

Recall that a smooth manifold is a pair $(M^n \mathcal{A})$ where M^n is a topological manifold of dimension n, and $\mathcal{A}$ s a maxmial smooth atlas on M.

Remarks:

- If $(U, \phi) \in \mathcal{A}$ then for $U' \subset U$ we have $(U, \phi\big|_{U'}) \in \mathcal{A}$
- If $(U, \phi) \in \mathcal{A}$ and $\mathcal{X} : \phi(U) \to \mathcal{X}(\phi(U)) \subset \mathbb{R}^n$ is a diffeomorphism then $(U, \mathcal{X} \circ \phi) \in \mathcal{A}$
- If $\phi : U \to \mathbb{R}^n$ is injective and $U \subset_{open} M$, then for

Let's see some examples:

- $\mathbb{R}^n$ with $\mathcal{A} =$ max smooth atlas containing $\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$

  Theorem: If $(M^n, \mathcal{A})$ is a smooth manifold then for $M' \subset_{open} M$ and $\mathcal{A}' = \{(U, \phi) | U \subset M'\}$, the pair $(M', \mathcal{A}')$ is also a smooth manifold. **So, open subsets of $\mathbb{R}^\backslash$ have smooth manifold structure.**

- $(\mathbb{S}^n, \mathcal{A})$ where

$$\begin{aligned} \mathcal{A} &= \text{max smooth atlas containing } \{(U_i^{\pm}, \phi_i)\} \\ &= \text{max smooth atlas containnig stereographic projection from N, S pole} \end{aligned}$$

- where V $n-$dimensional vector space over $\mathbb{R}$ and

$$\mathcal{A}' = \{(V, \phi) | \phi : V \to \mathbb{R}^n \text{ is an isomorphism}\}$$

  can be enlarged to a maximal smooth atlas $\mathcal{A}$. $(V, \mathcal{A})$ s a smooth manifold. (Missing some detail, fill from picture)

- $M = \mathbb{R}$, $\mathcal{A} =$ max smooth atlas containing $\{(\mathbb{R}, id_R)\}$ and $\mathcal{A}' =$ max smooth atlas containing $\{(R, \phi)\}, \phi : \mathbb{R} \to \mathbb{R}$ defined as $x \mapsto x^3$.

  These are distinct smooth atlases $\mathcal{A} \neq \mathcal{A}'$ since the charts $(\mathbb{R}, id_{\mathbb{R}}), (\mathbb{R}, \phi)$ are not compatible.

$$(\phi \circ id_{\mathbb{R}})(x) = x^3 \text{ s smooth, but}$$

$$(id_{\mathbb{R}} \circ \phi^{-1})(x) = \sqrt[3]{x} \text{ not diff at } x = 0 \text{ so this map is } \underline{\text{not smooth}}$$

  Read about more examples lie $GL_n(\mathbb{R})$, cartesian product, etc.

Note: We'll see more examples of multiple maximal smooth atlases on a manifold later.

Additional Discussion

So far we've defined two major classes of objects: topological manifolds and smooth manifolds.

Q: Does every topological manifold M^n admit a smooth structure?

A:

- If the dimension is $n \leq 3$ or lower, then yes & they are unque up to diffeomorphism (Moise1952).
- For $n \geq 4$, not necessarily & even if they do, they may be nonunique (M^{10}, Kervaire 1960; M^4, Donaldson, Friedman, & Kirby, 80s)

Q: Are there exotic (not diffeo to standard smooth structure) smooth structures on $\mathbb{S}^n$.

A:

- For $n \leq 3$, no (Relevant to Poincaré Conjecture).
- For $n = 4$, unknown (Smooth Poincaré Conjecture).
- For $n \geq 5$, depends on n (See ManifoldAtlas - Exotic spheres; first one ($n = 7$) constructed by Milnor).

1.1 Smooth Manifold Lemma

Smooth Manifold Lemma: Let M be a set, $\{(\underbrace{U_\alpha}_{\text{subset of } M}, \phi_\alpha)\}_{\alpha \in I}$ where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is injective such that

- $\phi(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ is open for all $\alpha, \beta \in I$ *(Gives topology and locally euc.)*
- $\phi_\alpha \circ \phi_\beta^{-1}\Big|_{\phi_\beta(U_\alpha \cap U_\beta)}$ is smooth for all $\alpha, \beta \in I$ *(Gives smooth transition maps)*
- M is covered by countably many U_α *(Gives second countability)*
- For all $p, q \in M$ where $p \neq q$ there are
  1. $\alpha \in I$ such that $p, q \in U_\alpha$ OR
  2. α, β such that $p \in U_\alpha, q \in U_\beta$ such that $U_\alpha \cap U_\beta = \emptyset$

  (Gives Hausdorffness)

then M has a unique topological and smooth structure such that (U_α, ϕ_α) are smooth charts.

"Proof:" Define a topology on M by

$A \subset M$ is open if and only if

$$\phi_\alpha(A \cap U_\alpha) \subset \mathbb{R}^n$$

is open for all $\alpha \in I$.

(Add more detail later).

Example: Grassmann Manifolds

$$M = \mathrm{Gr}_k(\mathbb{R}^n) = \{V \subset \mathbb{R}^n \text{ linear subspaces } : \dim(V) = k\}$$

To prove that the Grassmann Manifold is indeed a smooth manifoold, let

$I = \{(P, Q) : P, Q \subset \mathbb{R}^n$ linear subspaces such that $\mathbb{R}^n = P \oplus Q,\ \dim(P) = k,\ \dim(Q) = n - k\}$

For $\alpha = (P, Q) \in I$ define

$$U_\alpha = \{V \in \mathrm{Gr}_k(\mathbb{R}^n) : V \cap Q = \{0\}\}$$

[Insert figure]

Then, for any $V \in U_\alpha$, there exists a unique linear map $A_{P,Q,V} : P \to Q$ st

$$V = \{\}$$

[Complete this later]

Note: As to the four parts of the Smooth Manifold Lemma,

- (1) can be checked directly
- (2) we can check the transition maps are smooth from the above
- (3) we can cover M by finitely many charts
- (4) given P, P' we can find Q such that $P \cap Q = P' \cap Q = \{0\}$

1.2 Smooth Manifolds with Boundary

So far, we have been describing spaces like the open disk. But intuitively, the *closed* disk should also be a manifold. So, we define a new kind of manifold:

Topological Manifold with Boundary

A topological manifold M^n **with boundary** is a topological space such that it is

1. Hausdorff
2. Second-Countable
3. For any $p \in M$ there exsits an open neighborhood $U \subseteq M$ and homeomorphism

$$\phi : U \to \phi(U) \subset \mathbb{H}^n = \{(x_1, \dots, x_n) : x_n \geq 0\}$$

Def:

- A point $p \in M$ is a **boundary point** if there exists a chart (U_2, ϕ_2) such that $\phi_2(p) \in \partial\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n\}$.

- A point $q \in M$ is an **interior point** if there exists a chart (U_1, ϕ_1) such that $\phi_1(U)$ is open in $\mathbb{R}^n$.

  Remark: We will verify later that a point cannot be both an interior and boundary point.

  Remark: Topological Manifolds are also topological manifolds with boundary ($\partial M = \emptyset$)

Now, usually when we discuss smoothness we work with *open* sets. So, what about *smooth manifolds with boundary*? How do we need to modify our notion of smoothness to account for the boundary?

1.3 Smoothness and Transition Maps

Def: For an arbitrary subset $A \subset \mathbb{R}^n$, we say $f : A \to \mathbb{R}^m$ is *smooth* if we can extend to a smooth $\overline{f} : V \to \mathbb{R}^m$ where $A \subset V \subset_{open} \mathbb{R}^n$ and $\overline{f}\big|_A = f$.

Seeley's Theorem: ϕ is smooth if all partial derivatives exist in the interor and can be extended countinuously to the boundary.