(Instructor: Chien-I Chiang)

Physics 105: Analytical Mechanics notes

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These are some notes taken from UC Berkeley's Physics 105 during the Summer '24 session, taught by Chien-I Chiang.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 First topic

text

2 July 3, 2024:

2.1 Finishing up discussion from last lecture

- Finish this from lecture recording

Continuing on with out discussion of when $H \neq E$, we can parametrize the position of a particle as $\vec{r} = \vec{r}(q_k, t)$

We have

$$\frac{\partial K}{\partial \dot{q_k}} = \frac{1}{2} m \left[2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_m} + \cdots \right]$$

Then,

$$\begin{cases} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} \right] \\ 2K = m \left[\left(\frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial q_m} \dot{q_k} \dot{q_m} \right) + 2 \frac{\partial \vec{r}}{\partial q_k} \cdot \frac{\partial \vec{r}}{\partial t} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \right] \end{cases}$$

(The expression for $2K$ is obtained by expandingo out

$$\boldsymbol{K = \frac{1}{2} m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}$$

in terms of indices – write this out explicitly later)

Which gives us the relation

$$\begin{aligned} \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} &= 2K - m \frac{\partial \vec{r}}{\partial t} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial q_k} \dot{q_k} + \frac{\partial \vec{r}}{\partial t} \right)}_{= \frac{d\vec{r}}{dt}} \\ &= 2K - \vec{p} \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

The question we were originally considering is **When is $\boldsymbol{H = E}$?**

Now,

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{q_k}} \dot{q} - L \\ &= \frac{\partial K}{\partial \dot{q_k}} \dot{q_k} = (K - V) \\ &= 2K - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} - K + V \\ &= K + V - \vec{p} \cdot \frac{\partial \vec{r}}{\partial t} \end{aligned}$$

So we see that $H = E = K + V$ only when

$$\frac{\partial \vec{r}}{\partial t} = 0$$

i.e. when $\vec{r} = \vec{r}(q_k, t)$ has no time dependence i.e. $\vec{r} = \vec{r}(q_k)$

Earlier, we considered the following setup:

and we showed that

$$H = E - m\omega^2\rho^2$$

So, let's check that

$$m\omega^2\rho^2 = \vec{p} \cdot \frac{\partial \vec{r}}{\partial t}$$

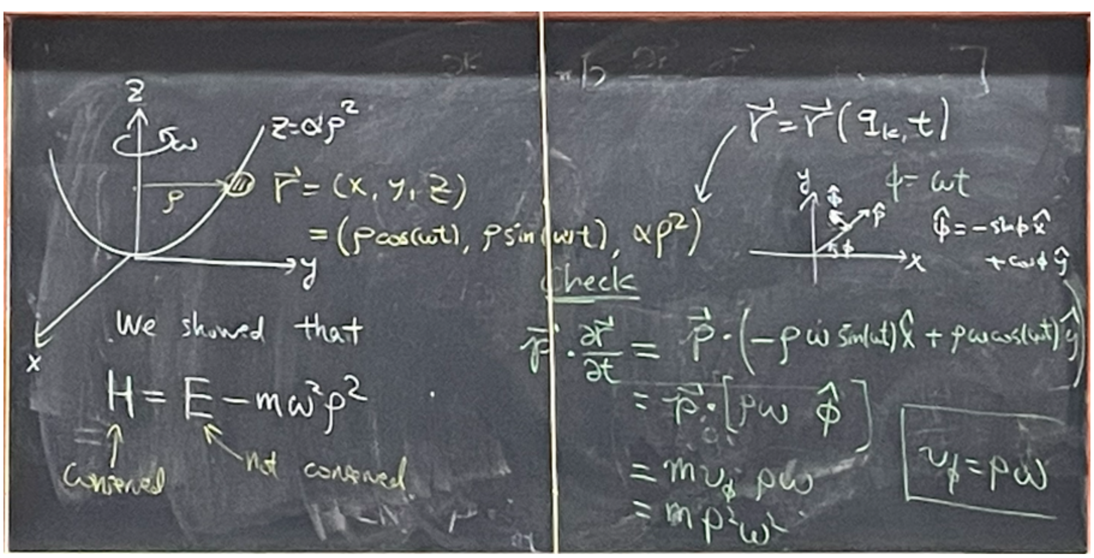
$$\begin{aligned}
\vec{p} \cdot \frac{\partial \vec{r}}{\partial t} &= \vec{p} \cdot (-\rho\omega \sin(\omega t)\hat{x} + \rho\omega \sin(\omega t)\hat{y}) \\
&= \vec{p} \cdot \left[\rho\omega\hat{\phi}\right] \\
&= m v_\phi \rho\omega \\
&= m\rho^2\omega^2
\end{aligned}$$

where $v_\phi = \rho\omega$

Since the hamiltonian itself has no time dependence, ***H* is conserved**. However, ***E* is not**. We can check that

$$dH = dE = d(m\omega^2\rho^2)$$

is indeed zero.



If we break the force on the bead into a normal force (denoted N) and a centripetal(?) force, then

$$
\begin{aligned}
dW &= \underbrace{\widehat{N\rho}}_{\text{torque about z-axis}} d\phi \\
&= \frac{dl_z}{dt} d\phi \\
&= d\left(\rho m \rho \omega\right)\omega \\
&= d\left(m\rho^2\omega^2\right)
\end{aligned}
$$

This is the energy that goes into the system.

By energy conservation, $dW = dE$.

$$\implies 0 = dE - dW = dE - d(m\rho^2\omega^2)$$

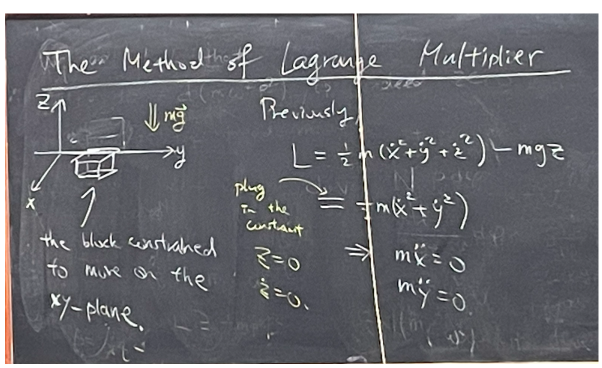
i.e. $E - m\rho^2\omega^2 = H$ is a conserved quantity.

So, the **Hamiltonian being conserved** and the **Hamiltonian being equal to Energy** are two different scenarios with two different conditions.

- The Lagrangian is time-independent i.e. $\frac{\partial L}{\partial t} = 0 \implies H$ is conserved.
- The position vector centered in an inertial frame $\vec{r} = \vec{r}(q_k, t)$ is time independent i.e $\frac{\partial \vec{r}}{\partial t} = 0 \implies H = E$

Now we move on to a powerful technique.

2.2 The Method of Lagrange Multipliers



We have a block constrained to move on the xy-plane, and we have gravity. Previously, we would say

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

i.e we would start with an unconstrained lagrangian, and then plug in the constraints $z = 0, \dot{z} = 0$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$\implies \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \end{cases}$$

Alternatively, we can implement the constraint $\ddot{z} = 0$ in the following way: We have the original lagrangian

$$L' = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz + \lambda z$$

where λ is the Lagrange multiplier and we can think of z as being the constraint function $f(z)$ and our constraint is $f(z) = 0$.

If we treat λ as an independent degree of freedom, we can write the Euler-Lagrange equation for λ as

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0 \implies z = 0 \text{ (constraint)}$$

On the other hand, if we look at the equation of motion for z, we get

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \implies \lambda - mg - m\ddot{z} = 0 \implies m\ddot{z} = \lambda - mg$$

and using the constraint $z = 0 \implies \ddot{z} = 0$ we get $-mg + \lambda = 0 \implies \lambda = mg$. Okay, but what physical meaning does λ have? It has to do with the **Normal force**. i.e. λ is encoding the **constraint** that the block can only move on the xy-plane due to the Normal force.

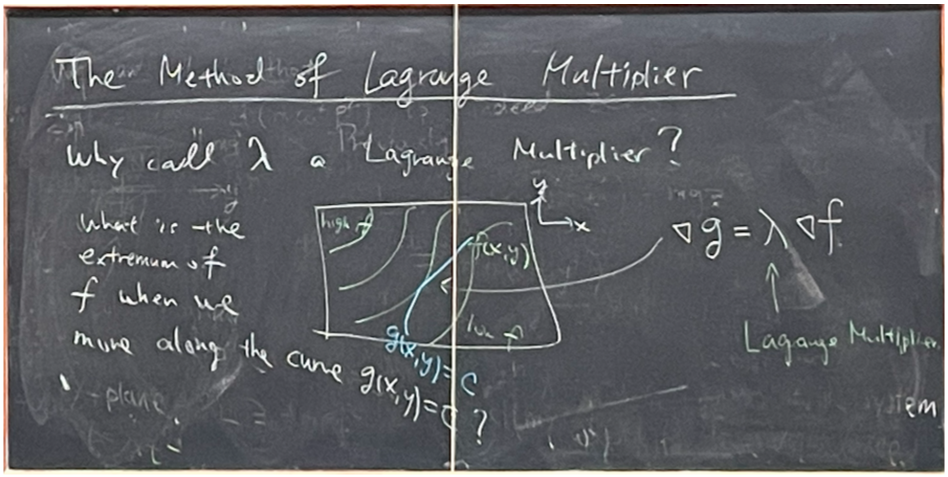
So, in general, for N constraints we have Lagrange Multipliers $\lambda_1, \cdots, \lambda_N$.

Why do we call λ a Lagrange Multiplier?

Recall from Calc 3 that if we have contours of a function $f(x, y)$ on the xy-plane and we are constrained to move along some other curve $g(x, y) = c$ on the plane, if we ask "What is the extremum of $f(x, y)$ as we move along the curve $g(x, y) = c$?" then visually we can tell that the extremum corresponds to the point where $g(x, y)$ intersects the contour of $f(x, y)$ only once. This is because at such a point, the gradients of the two functions are parallel:

$$\nabla g = \lambda \nabla f$$

This constant multiplier is the **Lagrange Multiplier**



So, in general, if we have a Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

we know that $\delta L = 0$ gives the Equations of Motion. But if we want to do this variation δL under some constraint $C(x, y, z) = 0$ then we need to consider

$$\delta L = \lambda \delta C \implies L' = L - \lambda C$$

Generally, if we have P constraints, $C_l(q_1, \cdots, t) = 0$, $l = 1, \cdots, P$ on the lagrangian L, we can write a new lagrangian

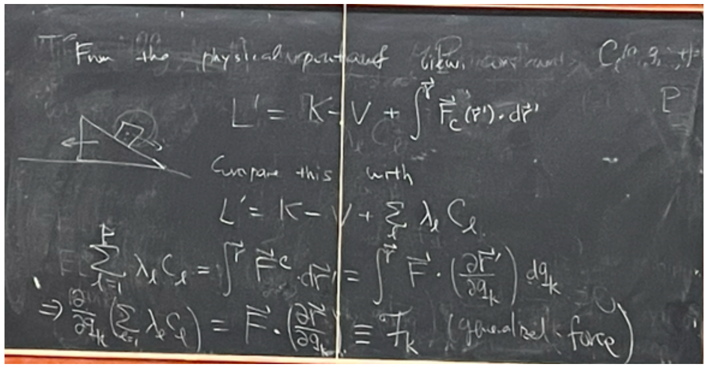
$$L' = L + \sum_{l=1}^{P} \lambda_l C_l$$

The Euler-Lagrange equation for λ_l leads to $C_l = 0$ and the Euler-Lagrange equation for the generalized coordinate q_k is

$$\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}\right) = 0$$

$$\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} + \underbrace{\sum_{l=1}^{P} \lambda_l \frac{C_l}{q_k}}_{\text{generalized force}}$$

On the physical point of view, consider the following system:



If we consider the system as a whole, the normal forces due to the block and the sledge are equal and opposite, so they cancel each other out - and so does the work that they do(?).

Howeverm if we consider the block only - we do have a normal force. The block is constrained the only move on the surface of the slope, so we can write

$$L' = K - V + \int^{\vec{r}} \vec{F}_C(\vec{r}) \cdot d\vec{r}'$$

(This is a bit handwavy - watch the lecture recording and think about this)

Then, if we conpare this with

$$L' = L - V + \sum_l \lambda_l C_l$$

we have

$$\sum_l \lambda_l C_l = \int^{\vec{r}} \vec{F}_C \cdot d\vec{r}' = \int^{\vec{r}} \vec{F} \cdot \left(\frac{\partial \vec{r}'}{\partial q_k} \cdot \mathrm{d}q_k\right)$$

$$\implies \frac{\partial}{\partial q_k}\left(\sum_l \lambda_l c_l\right) = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial q_k}\right) \equiv \mathcal{F}_k \text{ (generalized force)}$$

3 July 8, 2024: Lagrange Multipliers

3.1 More about Lagrange Multipliers

Last time, we saw that if we have constraints $C_l\left(\underbrace{q_1, \cdots, q_k}_{N}, t\right) = 0$ then we can write a constrained Lagrangian

$$L' = K - V + \sum_l \lambda_l C_l$$

These kinds of constraints, which are only constraints of the generalized coordinates are called **Holonomic constraints**. But these are not the most general constraints; we can have constaints which also depend on the derivatives $\dot{q_k}$. Those types of constraints are called **Non-holonomic constraints**.

Then, the principle of stationary action gives us

$$0 = \delta S \implies \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l \underbrace{\lambda_l C_l}_{\vec{F}^C \frac{\partial \vec{r}}{\partial q_k}} \\ \\ C_l = 0 \end{cases}$$

Note that there are multiple ways to write the same constraint. And writing a constraint in a different manner changes the C_l, which further changes the λ_l. As such, the λ_l is not always a generalized force; it can also be a torque etc.

In total we have $N + P$ variables and $N + P$ equations, so we are able to solve the system if we know the initial conditions.

We got the above equation by varying the action, and in particular, by varying L with respect to q_k. But we can extend this a bit futher...

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_k} + \sum_l a_{lk}\lambda_l \\ a_{lk}\delta q_k + a_{lt}\delta t = 0 \end{cases}$$

(Here, the l index labels the **constraint** and the k labels the coordinate.)

In the case of Holonomic constraint,

$$a_{lk} = \frac{\partial C_l}{\partial q_l}$$
$$a_{lt} = \frac{\partial C_l}{\partial t}$$

For Holonomic constraint, we will have

$$\frac{\partial q_{lk}}{\partial t} = \frac{\partial q_{lt}}{\partial q_k}$$

3.2 Example: Tree log rolling down a ramp

Consider a tree log rolling down a (fixed) ramp without sliding.

[Include Figure]

To describe the motion of the log, generically, we need two degrees of freedom: X and θ.

But we also know the log is rolling **without sliding**. So if the tree moves a distance dx during rotation $d\phi$, then we know $Rd\phi = dx$ where R is the radius of the log. Or in other words,

$$Rd\phi - dx = 0$$

This constraint is of the general form we saw above: $\boxed{a_{lk}\delta q_k + a_{lt}\delta t = 0}$ with $a_{1,\theta} = R, a_{1,x} = -1$ and all the time components $a_{lt} = 0$.

Now, we can write the Lagrangian of this system:

$$L = \frac{1}{2}M(\dot{X}^2) + \frac{1}{2}I\dot{\theta}^2 + mgX\sin(\alpha)$$

Note that we're actually kind of mixing approaches here. Technically there should be *three* degrees of freedom because the log can move in (x, y) space and rotate, but we know that the log is constrained by the Normal force and we don't need both of x, y; just one will suffice.

Wait... so, why do we even bother using the Lagrange Multiplier stuff if we're gonna use the old method too?

The Lagrange multiplier method allows us to retain info about the contact forces so if we, say, want to find the magnitude of the tension in a string, we can still do so using the Lagrange Multiplier method. Whereas in the old method, contact forces are used to enfore constraints but we lose all information about them.

Anyway, after writing down the lagrangian, we can obtain the Equations of Motion (with the constraints):

$$\begin{cases} \frac{d}{dt}\left(m\dot{X}\right) = +mg\sin(\alpha) - \lambda_1 \\ \frac{d}{dt}\left(I\dot{\theta}\right) = \lambda_1 R \end{cases}$$

So, what exactly is $\boldsymbol{\lambda_1}$?

In the X equation of motion, we have $+mg\sin(\alpha)$ which is the component of gravity along the ramp. So, λ_1 has the same units as force. We can interpret λ_1 as the **frictional force**!

Then, in the θ equation of motion, we can interpret $\lambda_1 R$ as the **torque due to friction**!

Solving these further we have

$$\begin{cases} m\ddot{X} = mg\sin(\alpha) - \lambda_1 \text{ (1)} \\ I\ddot{\theta} = \lambda_1 R \text{ (2)} \\ R\dot{\theta} = \dot{X} \text{ (from the no-sliding condition)} \implies R\ddot{\theta} = \ddot{X} \text{ (3)} \end{cases}$$

Substituting (3) into (1) gives

$$\begin{cases} mR\ddot{\theta} = mg\sin(\alpha) - \lambda_1 \\ \frac{I}{R}\ddot{\theta} = \lambda_1 \end{cases}$$

$$\implies mR\left(\lambda_1 \frac{R}{I}\right) = mg\sin(\alpha) - \lambda_1$$

$$\implies \left(1 + \frac{mR^2}{I}\right)\lambda_1 = mg\sin(\alpha)$$

$$\implies \boxed{\lambda = \frac{mg\sin(\alpha)}{\left(1 + \frac{mR^2}{I}\right)}} \text{ This is the magnitude of friction!}$$

3.3 Example: A bead on a wire

We've seen this example before, but this time we want to calculate the normal force on the bead.

[Include Figure]

Using the Lagrange Multiplier method, we can write down the constrained Lagrangian as

$$L' = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right] - mgz - \lambda_1(\phi - \omega t) - \lambda_2\left(z - \alpha\rho^2\right)$$

So, the EL Equations look like

$$\begin{cases} m\ddot{\rho} = n\rho\dot{\phi}^2 - 2\lambda_2\alpha\rho \\ \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = \lambda_1 \\ m\ddot{z} = -mg + \lambda_2 \end{cases}$$

From the z EoM, we can tell that λ_1 is a force since it's being added with $-mg$. We can interpret it as the $\boldsymbol{z}$**−component** of the **Normal Force**.

Similarly, in the ϕ EoM we see that λ_1 is the derivative of the Angular Momentum, so λ_1 is the **torque**.

[Include figure]

Now, in the ρ equation, we know that $m\ddot{\rho}$ is also a force since ρ has units of length. So, $-2\lambda_2\alpha\rho$ must also be a force. Exactly which force is it? It's the **radial component** of the **Normal Force** (See the figure above.)

When it comes to actually solving for λ_1 and λ_2, we can solve for them after we solve for $\rho(t)$ using $z = \alpha\rho^2$ and other constraints.

[Add last bit from lecture recording - lots of figures]

4 July 9, 2024: Symmetries and Lagrangians

4.1 Note about the discussion from last time

[Write about clever method to find rolling constraint that Chien-I spoke about at the beginning of lecture]

4.2 Symmetries

Previously we discussed **Cyclic Coordinates**:

A coordinate q_k is cyclic if

$$\frac{\partial L}{\partial q_k} = 0$$

As a result, the EL equation gives us the result that

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \text{ is conserved}$$

This is a **symmetry** in the sense that when we change q_k, the Lagrangian does not change.

What exactly is a Symmetry? We define a symmetry of a system to be a **transformation** of the system such that the system behaves the same after transformation. For example, rotating a triangle by 120 degrees is a symmetry transformation of the triangle.

The study of symmetries falls under **Group Theory**, but in physics we're usually concerned specifically **continuous transformations**. Continuous symmetries often give rise to **conserved quantities**.

Example: θ independent lagrangian

We'll see this in more detail when we study Noether's Theorem.

4.3 Continuous Transformations

Usually, we have $L = L(q_k, \dot{q}_k, t)$. We can apply transformations on the q_k and t variables

$$\begin{aligned} q_k &\to q_k'(q_k, t) \\ t &\to t'(t) \end{aligned}$$

which in turn transform the lagrangian L

When we say a transformation is continuous, we mean that we can make a transformation parametrized by some small parameter ϵ such that when $\epsilon \to 0$, the transformation is just the identity transformation.

Since the mapping is continuous, we can expand the transformation as

$$\begin{aligned} q_k(t) \to q_k'(t') &= q_k(t) + \delta q_k \\ t \to t(t) &= t + \delta t \end{aligned}$$

Example: Continuous Rotation In the plane $\mathbb{R}^2$, we can rotate a vector $V = V_x\hat{x} + V_y\hat{y}$ using a standard rotation matrix:

$$\vec{V} \to \vec{V}' = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

where θ is a continuous parameter that represents the angle of rotation.

Examples: Non-continuous transformation

1. Dicrete rotation: Rotations where θ is only allowed to have specific values, for example $\theta = n\frac{\pi}{6}$
2. Parity: $(x, y, z) \to (-x, -y, -z)$

There are two ways to generate transformations in q_k.

1. With a fixed time, we can "mix" the coordinates:

$$q_k(t) \to q_k'(t) = q_k(t) + \underbrace{\Delta q_k(t)}_{\text{small transformation}}$$

For example, we can rotate a vector without messing with the time coordinate:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} = \begin{pmatrix} x(t) - \theta y(t) \\ y(t) + \theta x(t) \end{pmatrix}$$

We can represent this transformation concisely using the **levi-civita symbol, $\boldsymbol{\epsilon_{ij}}$**

$$\boxed{x_i' = x_i - \theta\epsilon_{ij}x_j}$$

2. We can generate a change in q_k by shifting the time: $t \to t + \delta t(t)$

$$q_k(t) \to q_k(t') = q_k(t + \delta t) = q_k(t) + \dot{q}_k\delta t$$

We can define the total (infinitessimal) transformation of q_k as

$$\begin{aligned} \delta q_k &\equiv q_k'(t') - q_k(t) = q_k'(t + \delta t) - q_k(t) \\ &\approx q_k'(t) + \dot{q}_k'\delta t = q_k(t) \\ \text{to first order} \to\ &\approx q_k(t) + \Delta q_k(t)\dot{q}_k(t)\delta - q_k(t) \end{aligned}$$

where we used

$$\dot{q}_k'(t) = \frac{d}{dt}(q_k + \Delta q_k) = \dot{q}_k(t) + \frac{d}{dt}(\Delta q_k)$$

Thus, to first order, we have

$$\boxed{\delta q_k(t) = \Delta q_k(t) + \dot{q}_k \delta t}$$

We say such a transformation by δq_k and/or δt is a symmetry if we have the same dynamics i.e. under the transformation, the **action, $\boldsymbol{\delta S = 0}$** does not change. ($\delta S$ is the change in S when we perform a particular transformation in terms of Δq_k and/or δt)

$$\begin{aligned}
0 &= \delta\left(\int dt L(q_k, \dot{q}_k, t)\right) \\
&= \int \delta(dt) L + \int dt \delta L \\
&= \int dt \frac{d(\delta t)}{dt} L + \int dt \left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\Delta q_k) + \frac{dL}{dt}\delta t\right)
\end{aligned}$$

where we should note that dL/dt is the **total** derivative

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

Continuing on and applying "Integration by Parts",

$$0 = \int dt \left[\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right)\Delta q_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k\right) + \frac{d}{dt}(L\delta t)\right]$$

If q_k satisfies the EoM,

$$0 = \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t\right)\right] = \left[\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t\right]_{t_i}^{t_f}$$

Therefore the quantity

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\delta t$$

is conserved! What we've shown is Noether's Theorem.

Noether's Theorem: If we have a continuous symmetry and the evolution of the system satisfies the EoM, then there is an associated conserved quantity given by

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k} \Delta q_k + L\delta t$$

which is called the **Noether Charge**.

In fact, we can extend this a little bit. The action can change, as long as it's of the form:

$$\delta S = \int dt \left(\frac{dK}{dt} \right)$$

because such a change just adds constant boundary terms $K(t_f) - K(t_i)$ which do not change the dynamics. So,

$$\frac{d}{dt}(Q - K) = 0$$

This $Q - K$ is a more general conserved charge.

4.4 Example: Spacial Translation

$$L = \frac{1}{2} m \dot{x}^i \dot{x}_i$$

This Lagrangian is invariant under the shift $\begin{cases} x_i \to x_i + \epsilon_i \text{ spatial translation} \\ t \to t \text{ no time translation} \end{cases}$.

So, $\delta x_i = \Delta x_i + \dot{x} \underbrace{\delta t}_{=0}$. As a result,

$$Q \equiv \frac{\partial L}{\partial \dot{x}_i} \Delta x_i = m\dot{x}_i \epsilon_i \implies m\dot{x}_i \text{ is conserved.}$$

4.5 Example: Time Translation

Consider the time translation

$$\begin{cases} x_i \to x_i \\ t \to t + \delta t \end{cases}$$

i.e. $\delta x_i = 0 = \Delta x_i + \dot{x}_i \delta t$ which implies

$$\Delta x_i = -\dot{x}_i \delta t$$

Consider the following Lagrangian under time translation

$$L = \frac{1}{2} m \dot{x}^i \dot{x}_i - V(x)$$

Since $L = L(x_i, \dot{x}_i, t)$, if the x_i's don't change then the change in L is just

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial x_i}\underbrace{\delta x_i}_{=0} + \frac{\partial L}{\partial \dot{x}_i}\underbrace{\delta dot x_i}_{=0} + \frac{\partial L}{\partial t}\delta t \\ &= \frac{\partial L}{\partial t}\delta t \end{aligned}$$

And, when $\frac{\partial L}{\partial t}\delta t = 0$, we have time translation symmetry, giving us the conserved current

$$\begin{aligned} Q &\equiv \frac{\partial L}{\partial \dot{x}_i}\delta x_i + L\delta t \\ &= -\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i \delta t + L\delta t \\ &= \left(\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L\right)(-\delta t) \end{aligned}$$

So when we have time translation symmetry, the **Hamiltonian**

$$H = \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L$$

is conserved.

4.6 Example: Isotropic Harmonic Oscillator under rotation

Consider

$$L = \frac{1}{2}m\dot{x}^i\dot{x}_i - \frac{1}{2}kx^i x_i$$

under rotation

$$\begin{aligned} x_i &\to x_i - \theta\epsilon_{ij}x_j \\ t &\to t \end{aligned}$$

Then, the Lagrangian transforms into

$$\begin{aligned} L &\to \frac{1}{2}m\dot{x}^i\dot{x}_i - \frac{1}{2}kx^i x_i + m\dot{x}_i\left(-\theta\epsilon_{ij}\dot{x}_j\right) - kx_i\left(-\theta\epsilon_{ij}x_j\right) + \mathcal{O}(\epsilon^2) \\ &= L - \theta m\epsilon_{ij}\dot{x}_i\dot{x}_j + k\epsilon_{ij}x_i x_j \\ &= L \end{aligned}$$

where we used the fact that ϵ_{ij} is antisymmetric while $\dot{x}_i\dot{x}_j$, x_i, x_j are antisymmetric. Thus, the two terms other than L vanish.

In general if we have 2-d tensors $S_{ij} = S_{ji}$ (symmetric) and $A_{ij} = -A_{ji}$ (antisymmetric) then

$$\begin{aligned} S_{ij}A_{ij} &= -S_{ij}A_{ji} \\ &= -S_{ji}A_{ji} \\ &= -S_{ij}A_{ij} \text{ Since } S_{ij}A_{ij} \text{ is itself symmetric as a whole} \\ \implies S_{ij}A_{ij} &= 0 \end{aligned}$$

Here, the conserved current is

$$\begin{aligned} Q &\equiv \frac{\partial L}{\partial \dot{x}_i}\Delta x_i \\ &= m\dot{x}_i\left(-\theta\epsilon_{ij}x_j\right) \\ &= -\theta\epsilon_{ij}x_j m\dot{x}_i \\ &= -\theta\epsilon_{12}x_2 m\dot{x}_1 - \theta\epsilon_{21}x_1 m\dot{x}_2 \text{ (The terms with } i = j \text{ vanish because } e_{ii} = 0) \\ &= -\theta(ym\dot{x} - xm\dot{y}) \end{aligned}$$

which implies that **Angular momentum is conserved.** This marks the end of today's lecture.

. **(No lectures on July 10, 11 because of the midterm.)**

5 July 15, 2024: Obtaining Lagrangians from Symmetries

Today, we wrap up our discussion of Lagrangians. The goal today is to find the most general Lagrangian $L(x, \dot{x}, t)$ describing a 1D system which is **time-translation invariant** and **Galilean invariant**.

The framework we'll develop is what's done a lot in research (eg. in Effective Field Theory research), where we consider the symmetries we must enforce and then find the most general possible Lagrangian.

Also recall that a Lagrangian is called time-translation invariance if it has no eplicit time dependence.

Step 1: Consider the action

$$S = \int \mathrm{d}t L(x, \dot{x}, t)$$

and the Galilean transformation $x \to x + \Delta x = x - Vt$ (V is a constant) which causes the variation

$$\begin{aligned} \delta S &= \int \mathrm{d}t \left[\frac{\partial L}{\partial x}(-Vt) + \frac{\partial L}{\partial \dot{x}}(-V) \right] \\ &= \int \mathrm{d}t \left[(-V) \frac{d\tilde{K}}{dt} \right] \text{ (for some function } \tilde{K}) \end{aligned}$$

(as long as the change in action is of the form above, there is no change in dynamics.)

$$\implies \frac{\partial L}{\partial x} t + \frac{\partial L}{\partial \dot{x}} = \frac{d\tilde{K}}{dt} = \frac{\partial \tilde{K}}{\partial x} \dot{x} + \frac{\partial \tilde{K}}{\partial \dot{x}} \ddot{x} + \frac{\partial \tilde{K}}{\partial \ddot{x}} \dddot{x} + \cdots + \frac{\partial \tilde{K}}{\partial t}$$

But the LHS has derivatives up to $\dot{x}$ at most since we restrict our lagrangians. So, we must have $\tilde{K} = \tilde{K}(x, t)$ and

$$\frac{\partial L}{\partial x} t + \frac{\partial L}{\partial \dot{x}} = \frac{\partial \tilde{K}}{\partial x} \dot{x} + \frac{\partial \tilde{K}}{\partial t} \qquad (\star)$$

Step 2: Notice is that $\tilde{K} = \tilde{K}(x, t)$ means that the only place in equation $(\star)$ that we have $\dot{x}$-dependence is $\frac{\partial \tilde{K}}{\partial x} \dot{x} + \frac{\partial \tilde{K}}{\partial t}$ i.e. it is linear in $\dot{x}$.

$\implies$ at most, L can only be second order in $\dot{x}$, namely

$$L = f_2(x, t) \dot{x}^2 + f_1(x, t) \dot{x} + f_0(x, t)$$

Step 3: Finally, if we enfore time-translation symmetry then we have

$$\frac{\partial L}{\partial t} = 0$$

which tells us $f_i(x,t) = f_i(x)$ and so,

$$\implies L = f_2(x)\dot{x}^2 + f_1(x)\dot{x} + f_0(x) \qquad (\Delta)$$

Step 4: Let's look at equation ($\star$) again, which we obtained from enforcing Galilean invariance. With (Δ) we know

$$\frac{\partial L}{\partial x} \supseteq \frac{\partial f_2}{\partial x}\dot{x}^2$$

(sidenote: this is an abuse of notation that I love; it literally just means that $\frac{\partial L}{\partial x}$ includes a term $\frac{\partial f_2}{\partial x}\dot{x}^2$ it's so goofy fr)

But the RHS of equations ($\star$) has no $\dot{x}^2$ terms. Thus,

$$\frac{\partial f_2}{\partial x} = 0$$

i.e. f_2 is a constant.

Step 5: For the $f_1(x)\dot{x}$ term in the Lagrangian, we can say

$$S \supseteq \int_i^f f_1(x)\dot{x}dt = \int_i^f f_1(x)dx = F(x_f) - F(x_i) \text{ (constant)}$$

so this does not affect the dynamics and so we can safely throw it out of the Lagrangian and shove it to the curb.

Step 6: The only thing left is

$$L = f_2\dot{x}^2 + f_0(x)$$

Plugging this back into equation ($\star$), we get

$$\begin{aligned} \frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} &= \frac{\partial \tilde{K}}{\partial t} + \frac{\partial \tilde{K}}{\partial x}\dot{x} \\ \implies \frac{\partial f_0}{\partial x}t + 2\dot{x}f_2 &= \frac{\partial \tilde{K}}{\partial t} + \frac{\partial \tilde{K}}{\partial x}\dot{x} \qquad (\square) \end{aligned}$$

Matching the terms with and without $\dot{x}$ dependence, we get

$$\frac{\partial \tilde{K}}{\partial x} = 2f_2 = \text{ constant } \implies \tilde{K} = 2f_2x + g(t)$$

Step 7: Plugging the expression for $\tilde{K}$ into equation ($\square$) we get

$$\frac{\partial f_0}{\partial x}t + 2\dot{x}f_2 = \frac{\partial g}{\partial t} + 2f_2x = \frac{\partial g}{\partial t} + 2f_2\dot{x}$$

but $g(t)$ has no x-dependence $\implies$ $\frac{\partial f_0}{\partial x}$ cannot have x-dependence

$$\implies f_0 = c_1x + c_0$$

In summary, the most general Lagrangian for a 1D particle that is time-translation invariant and Galilean invariant is of the form

$$L = c_2\dot{x} + c_1 x + c_0$$

(where instead of writing f_2 we just wrote c_2). But also, constants don't affect the dynamics. Therefore, the most general Lagrangian we need is

$$L = c_2\dot{x}^2 + c_1 x$$

What kind of physics does this lagrangian describe?

That of a non-relativistic particle subject to a constant force field. For example, if the field is that of uniform gravity, then

$$L = \frac{1}{2}m\dot{x}^2 - mgx$$

(When people do this in Effective Field Theory, there are much more complicated symmetries to enfore such as $U(1), SU(2)$ etc. but the procedure is follows the same basic idea we've studied here.)

We were able to get a very specific form for our Lagrangian, but often symmetries are not strong enough to constrain our Lagrangian. In that case, we have to use other methods to determine the form of the Lagrangian for our system such as by experimental considerations.

Okay, now we go back to some more tradition stuff from classical mechanics.

5.1 Central Force Problem

A **Central Force** is one whose potential energy function has **Spherical symmetry** eg. Isotropic Harmonic Osciallators or a particle under uniform gravitational force.

Usually, we have a two-body problem with a Lagrangian of the form

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(r)$$

where $r = |\vec{r}_2 - \vec{r}_1|$. But, we can effectively change this into a one-body problem by using the **Center-of-Mass** coordinates.

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1, \quad \vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Using these coordinates and $M \equiv m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$ we can write the Lagrangian as

$$L = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{free motion of C.M.}} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r})}_{\text{non-trivial dynamics}}$$

We've essentially decoupled the two bodies. Now, if $U(\vec{r})$ has spherical symmetry i.e. $U(\vec{r}) = U(r)$ then it is natural to work in spherical coordinates.

So, the non-trivial part of the Lagrangian will be

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - U(r)$$

and ϕ is a cyclic coordinate of it. Thus,

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \mu r^2\sin^2(\theta) \equiv l_z$$

is conserved. Then,

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{l_z}{\mu^2 r^2\sin^2(\theta)}\right) - U(r)$$

The fact that the angular momentum l_z (and in fact, the *total* angular momentum l) is conserved means that the particle is moving on a fixed plane.

Why is total angular momentum conserved? A central force cannot cause torque, therefore angular momentum is conserved.

WLOG, we set the $z-axis$ to be perpendicular to the motion of the plane i.e. we set $\theta = \frac{\pi}{2} \implies \dot{\theta} = 0$ due to which $\sin(\theta) = 1$. So, our non-trivial Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{l_z}{2\mu r^2} - U(r)$$

Before we put the constraint on ϕ, the Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$

and the corresponding Hamiltonian is

$$\begin{aligned} H &= \frac{\partial L}{\partial\dot{q}_k}\dot{q}_k - L \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{l_z^2}{2\mu r^2} + U(r)}_{\text{effective potential}} \end{aligned}$$

Also, when we switch from Spherical Coordinates to Cartesian Coordinates we use

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

which have no time dependence, and so $H = E$. Aaaaaand... we're out of time so w'll continue tomorrow.

6 July 16, 2024: Central Forces and Orbital Shapes

6.1 Continuing our discussion of Central Forces

As we saw last time, when dealing with a 2-body central force, we can do a **change of coordinates** to make things simpler.

The dynamics of two bodies moving in a central force can be decoupled into the dynamics of a free body (moving with constant mass) and another mass subject to an effective potential.

[Include figure of 1-body and 2-body central forces]
[Write about the 2-body lagrangian being split into free and non-trivial parts.]

6.2 Aside: Ineratial Mass and Gravitational Mass

In Newton's Law of Gravitation,

$$|\vec{F}| = \frac{G\mu_1\mu_2}{r^2}$$

the quantities μ_1, μ_2 are called gravitational mass (though a better name would be gravitational charge since the force is of the same form as coulomb's law). They tell us how strong thee interaction is.

And, in Newton's First Law,

$$|\vec{F}| = m|\vec{a}|$$

the term m is the inertial mass. It tells us how hard it is to change the particles velocity.

Now, a priori, there is no reason for these quantities to be the same. However, if we conduct an experiment where a base-ball and a feather free-fall towards the earth, we know

$$\begin{aligned} a_{\text{baseball}} &= \frac{|\vec{F}|}{m_{\text{baseball}}} = \frac{G\mu_{\oplus}\mu}{R_{\oplus}^2 m_{\oplus}} \\ a_{\text{feather}} &= \frac{|\vec{F}|}{m_{\text{feather}}} = \frac{G\mu_{\oplus}\mu}{R_{\oplus}^2 m_{\oplus}} \end{aligned}$$

and we find via the experiment that $a_{\text{baseball}} = a_{\text{feather}} = g$
Then,

$$\left(\frac{G\mu_{\oplus}}{R_{\oplus}^2}\right)\left(\frac{\mu_{\text{baseball}}}{m_{\text{baseball}}}\right) = \left(\frac{G\mu_{\oplus}}{R_{\oplus}^2}\right)\left(\frac{\mu_{\text{feather}}}{m_{\text{feather}}}\right)$$

So the gravitational mass to the inertial mass ratio is the same across objects! Note that μ/m is analogous to e/m in electrostatics and, for example, the proton p^+ and electron e^- has *very different* e/m *ratio*. But miraculously, when it comes to gravity, μ/m is the same for all objects. Numerically, we can just set μ and m equal to each other.

The statement that μ/m is the same for all objects is called the **Equivalence Principle**. This has important implications in General Relativity.

Include discussion about curved space and hockey pucks on the earth.

6.3 Continuing on with the 2-body problem

Focusing on the dynamics of $\vec{r}$, the Lagrangian for that part is

$$\begin{aligned} L &= \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \\ &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - U(r) \end{aligned}$$

But notice that these dynamics are the same as those of the 1-body central force scenario as $\vec{L}$ is conserved and motion is on a plance. i.e. θ is fxed which we can WLOG set to $\pi/2$.

$$\begin{aligned} L &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \\ \implies H &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{l_z^2}{2\mu r^2} + U(r)}_{\text{effective potential for a 1D d.o.f. } r, U_{\text{eff}}} \end{aligned}$$

This is where we stopped last lecture. The position r is then subject to a force

$$\begin{aligned} F &= -\frac{\partial U_{\text{eff}}}{\partial r} \\ &= +\frac{l_z^2}{\mu r^2} - \frac{\partial U(r)}{\partial r} \end{aligned}$$

where

$$\frac{l_z^2}{\mu r^2} = \frac{\mu^2 r^2\dot{\phi}^2}{\mu r^2} = \frac{\mu v^2}{r} = \mu a_c$$

This looks like a "centrifugal force".

Let's think about centrifugal forces for a sec. Consider the following setup where Bob is at a distance r from the z-axis and is rotating about the axis a constant rate.

[Include figure]

In Bob's Frame, these three forces [complete this later]

[Include figure of spring extending from fixed point]

The angular momentum conservation keeps to r to be non-zero because when $r = 0$, we have $l_z = 0$. This is why we have a positive sign in front of the $(l_z)^2/\mu r^2$ term - the force is pushing outwards.

This U_{eff} also allows us to understand whether r is a fixed value or changes between some r_{min} and t_{max}. For example, if we consider a spring with $U(r) = \frac{1}{2}kr^2$ we get

$$U_{\text{eff}} = \frac{(l_z)^2}{2\mu r^2} + \frac{1}{2}kr^2$$

Plotting this, we have [Include figure]
[Write the discussion about the plot based on lecture recording and textbook.]

If we instead consider a gravitational system with $U(r) = -\frac{Gm_1m_2}{r^2}$ we have

$$U_{\text{eff}} = \frac{(l_z)^2}{2\mu r^2} - \frac{Gm_1m_2}{r^2}$$

which gives us the plot

[Include figure]

You've probably seen a very similar plot when studying Chemistry/Quantum Mechanics. The reason for that is because the effective potential for an atom with $l \neq 0$ has a very similar form.

[Include discussion of the plot based on lecture recording and textbook.]

If we want to find $r(t)$ we can use

$$\begin{aligned}
E &= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{(l_z)^2}{2\mu r^2} + U(r) \\
\implies \left(\frac{dr}{dt}\right)^2 &= \frac{2}{\mu}\left[E - \frac{(l_z)^2}{2\mu r^2} - U(r)\right] \\
\implies \int_{t_0}^{t} dt' &= \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{E - \frac{(l_z)^2}{2\mu r^2} - U(r)}}
\end{aligned}$$

and then after integrating, the inverse relation gives us $r(t)$.

6.4 Orbital Shape

We will just to pure 1-body problems here, but the procedure remains the same (just more tedious) in the 2-body case.

Now, what does it mean to find **Orbital Shape**? It refers to finding $r(\phi)$ i.e. we don't care about the time dependence here; just the dependence on ϕ.

Consider

$$E = \frac{1}{2}m\dot{r}^2 + \frac{(l_z)^2}{2mr^2} + U(r)$$

If we divide the entire equation by $l_z = mr^2\dot{\phi}$ we get

$$\frac{E}{l_z} = \frac{1}{2}m\frac{1}{m^2r^4}\frac{\dot{r}^2}{\dot{\phi}^2} + \frac{1}{2mr^2} + U(r)$$

Why is this useful? Well, notice that

$$\frac{\dot{r}}{\dot{\phi}} = \frac{dr/dt}{d\phi/dt} = \frac{dr}{d\phi}$$

So, we have

$$\begin{aligned}\frac{E}{l_z} &= \frac{1}{2}m\frac{1}{m^2r^2}\frac{\dot{r}^2}{\dot{\phi}^2} + \frac{1}{2mr^2} + U(r) \\ &= \frac{1}{2mr^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{2mr^2} + U(r)\end{aligned}$$

and now there is no time-dependence in the equation. Continuing on,

$$\begin{aligned}\left(\frac{dr}{d\phi}\right)^2 &= 2mr^4\left[\frac{E}{(l_z)^2} - \frac{1}{2mr^2} - \frac{U(r)}{(l_z)^2}\right] \\ \implies d\phi &= \pm\sqrt{\frac{1}{2m}}\frac{dr}{\sqrt{\frac{E}{l_z^2}r^4 - \frac{r^2}{2m} - \frac{U(r)}{l_z^2}r^4}} \\ \implies &\boxed{d\phi = \pm\frac{l_z}{\sqrt{2m}}\frac{1}{r^2}\frac{dr}{\sqrt{E - \frac{l_z^2}{2mr^2} - U(r)}}}\end{aligned}$$

Let's do a concrete calculation. For $U(r) = \frac{1}{2}kr^2$,

$$\int_{\phi_0}^{\phi} d\phi' = \pm\frac{l_z}{\sqrt{2m}}\int_{r_0}^{r}\frac{1}{r^2}\frac{dr}{\sqrt{E - \frac{l_z^2}{2mr^2} - \frac{1}{2}kr^2}}$$

Now, with the substitution $z = r^2, dz = 2rdr$

$$\begin{aligned}\int_{\phi_0}^{\phi} d\phi' &= \pm\frac{l_z}{\sqrt{2m}}\int_{z_0}^{z}\frac{1}{z}\frac{1}{2z}\frac{dz}{\sqrt{-\frac{l_z^2}{2mz} + E - \frac{1}{2}kz}} \\ &= \pm\frac{l_z}{\sqrt{2m}}\int_{z_0}^{z}\frac{1}{2z}\frac{dz}{\sqrt{-\frac{l_z^2}{2m} + Ez - \frac{1}{2}kz^2}}\end{aligned}$$

and using the Integral Formula

$$\int \frac{dz/z}{\sqrt{a+bz+cz^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bz+2a}{z\sqrt{b^2-4ac}}\right)$$

we get

$$\begin{aligned}
\phi - \phi_0 &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2-(l_z)^2/m}{r^2\sqrt{E^2-k(l_z)^2/m}}\right) \\
\implies \pm\sin\left[2(\phi-\phi_0)\right] &= \frac{Er^2-(l_z)^2/m}{r^2\sqrt{E^2-k(l_z)^2/m}} \\
\implies r^2 &= \frac{(l_z)^2/m}{\left\{E - \mp\sqrt{E^2-k(l_z)^2/m}\,\sin\left[2(\phi-\phi_0)\right]\right\}}
\end{aligned}$$

We're out of time now, but next time we'll see how this gives us an orbital shape which is an ellipse.

7 July 17, 2024: Orbital Shape continued

7.1 Orbital Shape for a spring with $\vec{F} = -k\vec{r}$

Yesterday, we found that

$$\begin{aligned}
\phi - \phi_0 &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2 - (l_z)^2/m}{r^2\sqrt{E^2 - k(l_z)^2/m}}\right) \\
\implies \sin(2[\phi - \phi_0]) &= \pm\frac{Er^2 - (l_z)^2/m}{r^2\sqrt{E^2 - k(l_z)^2/m}} \\
\implies & \boxed{r^2 = \frac{(l_z)^2/m}{E \mp \sqrt{E^2 - k(l_z)^2/m}\,\sin\left[2(\phi - \phi_0)\right]}} \qquad (\star)
\end{aligned}$$

Okay. Time for some high school maths. The equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we work in polar coordinates, the equation becomes

$$\begin{aligned}
\frac{r^2\cos^2(\phi)}{a^2} + \frac{r^2\sin^2(\phi)}{b^2} &= 1 \\
\implies r^2 &= \frac{a^2b^2}{a^2\sin^2(\phi) + b^2\cos^2(\phi)} \\
\implies r^2 &= \frac{a^2b^2}{a^2\left(\frac{1-\cos(2\phi)}{2}\right) + b^2\left(\frac{1+\cos(2\phi)}{2}\right)} \\
\implies r^2 &= \frac{2a^2b^2}{(a^2+b^2) + (b^2-a^2)\cos(2\phi)}
\end{aligned}$$

and using the identity $-\cos(\alpha) = \sin(\alpha - \frac{\pi}{2})$, we get

$$\boxed{r^2 = \frac{2a^2b^2}{(a^2+b^2) + (a^2-b^2)\sin(2\phi - \frac{\pi}{2})}} \qquad (\square)$$

Now, in equation ($\star$), if we set $\phi_0 = \frac{\pi}{4}$ and choose the + sign (doing this isn't cherry picking the result, but instead just choosing the coordinates we start off with), we get

$$r^2 = \frac{(l_z)^2/m}{E + \sqrt{E^2 - k(l_z)^2/m}\,\sin(2\pi - \frac{\pi}{2})}$$

So, comparing this with equation ($\square$), we see that the Orbital Shape for the Spring is an ellipse with

$$\begin{aligned}
a^2 &= \frac{1}{2k}\left[E + \sqrt{E^2 + k(l_z)^2/m}\right] \\
b^2 &= \frac{1}{2k}\left[E + \sqrt{E^2 - k(l_z)^2/m}\right]
\end{aligned}$$

The geometric properties of the Orbital Shape are determined by E and l.

7.2 Orbtial Shape corresponding to Gravitation

We found in a previous lecture that

$$\phi(r) = \pm\frac{l_z}{\sqrt{2m}}\int\frac{dr}{r^2\sqrt{E-\frac{(l_z)^2}{2mr^2}+\frac{GMm}{r}}}$$

$$\phi(r) = \pm\frac{l_z}{\sqrt{2m}}\int\frac{dr/r}{\sqrt{Er^2-\frac{(l_z)^2}{2m}+GMmr}}$$

This has the same form as the Spring case, where

$$\phi(r) = \pm\frac{l_z}{\sqrt{2m}}\int\frac{1}{2E}\frac{dr}{r^2\sqrt{Ez-\frac{(l_z)^2}{2m}+\frac{1}{2}kz^2}}$$

except we need to make the substitutions $z \to r,\ -\frac{1}{2}k \to E,\ E \leftrightarrow GMm$
So, using the solution found earlier with the appropriate substitutions, we get

$$\implies \phi-\phi_0 = \pm\sin^{-1}\left(\frac{GMMr-(l_z)^2/m}{r\sqrt{(GMm)^2+2E(l_z)^2/m}}\right)$$
$$\implies \sin(\phi-\phi_0) = \pm\frac{GMm^2r-l^2}{GMm^2re}$$

where

$$e = \sqrt{1+\frac{2E(l_z)^2}{G^2M^2m^2}}$$

(we'll see that the Orbital shape is once again an ellipse and this quantity e is the *eccentricity*.)

$$\implies r = \frac{(l_z)^2/(GMm^2)}{1\mp e\sin(\phi-\phi_0)}$$

Now, another way to define an Ellipse is in terms of its two foci.

[Include figure from lecture].

An ellipse has the relations $c = ae$ and a point on the ellipse has polar coordinates (r,ϕ) related by

$$r = \frac{a(1-e^2)}{1+e\cos(\phi)}$$

- When $e = 0$, $r = \frac{l_z^2}{GMm^2} \implies$ constant r i.e. circular orbit.
- When $0 < e < 1$, we have an elliplse, Namely, we can set $\phi_0 = \frac{\pi}{2}$ and choose the positive sign, giving us

$$r = \frac{l_z^2/GMm^2}{1+e\cos(\phi)}$$

which is of the form of $\frac{a(1-e^2)}{1+e\cos(\phi)}$ with

$$e = \sqrt{1 + \frac{2E(l_z)^2}{G^2M^2m^3}}$$

- We can also find a by

$$a = \frac{l_z^2/GMm^2}{(1-e^2)} = \frac{l_z^2}{-\frac{2E(l_z)^2}{G^2M^2m^2}}$$
$$\implies a = -\frac{GMm}{2E}$$

  The minus sign is fine because E is negative for a bound state, so a will be positive.

- When $e = 1$, physically we have

$$r = \frac{l_z^2/GMm^2}{1+\cos(\phi)}$$

  This matches the geometric parametrization

$$r = \frac{2a}{1+\cos(\phi)}$$

  which describes a parabola.

- Also, note that since

$$e \equiv \sqrt{1 + \frac{2El_z^2}{G^2M^2m^3}}$$

  we have $e = 0 \Leftrightarrow E = 0$. This matches up with our discussion yesterday. [Flesh this out more]

- When $e > 1$, physically we still have $r = \frac{l_z^2/GMm^2}{1+e\cos(\phi)}$. But this will match with the geometric parametrization of a hyperbola.

7.3 Coulomb Scattering

(discussed in Ch. 8 of H&S) In the case where we have two charges of the same sign,

$$U = \frac{kq_1q_2}{r} > 0$$

$$\implies U_{\text{eff}} = \frac{l_z^2}{2mr^2} + \frac{kq_1q_2}{r}$$

If we plot U_{eff} we see there is no potential well and thus no bound states. [Include figure]

If we imagine charge q_2 of mass m moving towards charge q_1 (which we fix in place) with initial velocity $\vec{v}_0$ and impact parameter b, we note that the impact parameter gives us the angular momentum of q_2:

$$l = bmv_0$$

Goal: We want to find the relation between b and how much the particle q_2 is deflected.

[include picture of two charges with same mass same charge moving towards fixed target, but with different impact parameters]

Intuitively it makes sense that the smaller the impact parameter, the greater the deflection. We will make this more precise.

Compare

$$H = \frac{1}{2}m\dot{r}^2 + \frac{l_z^2}{2mr^2} + \frac{kq_1q_2}{r}$$

for the Coulomb case with

$$H = \frac{1}{2}m\dot{r}^2 + \frac{l_z^2}{2mr^2} - \frac{GMm}{r}$$

for the gravitational case. All we need to do is substitute

$$-GMm \to +kq_1q_2$$

and we'll get the result.

In the gravitational case we had

$$\pm\sin(\phi - \phi_0) = \frac{GMm^2r - l_z^2}{GMm^2r\sqrt{1 + \frac{2El_z^2}{G^2M^2m^2}}}$$

and doing the substitution gives us

$$\pm\sin(\phi - \phi_0) = \frac{-kq_1q_2mr - l^2}{-kq_1q_2mr\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}}$$

(here l is the angular momentum with reference to the fixed point charge)

$$\implies r = \frac{l^2}{kq_1q_2m\left[\pm\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}\sin(\phi - \phi_0) - 1\right]}$$

Comparing this with the geometric parametrization of a parabola, if we choose $phi_0 = \frac{\pi}{2}$ and take the + sign (so that the bottom overall has a negative sign) we get

$$r = \frac{l^2}{-kq_1q_2m\left[\sqrt{1 + \frac{2El^2}{k^2q_1^2q_2^2m}}\cos(\phi) + 1\right]}$$

Recall that in the gravity case we defined

$$e \equiv \sqrt{1 + \frac{2El^2}{G^2M^2m^3}}$$

In the coulomb case this turns into

$$e \equiv \sqrt{1 + \frac{2El}{k^2q_1^2q_2^2m}}$$

and

$$1 - e^2 = -\frac{2El^2}{k^2q_1^2q_2^2m}$$

With these, for the coulomb case, we get

$$\boxed{r = \frac{(1-e^2)\left(\frac{kq_1q_2}{2E}\right)}{1 + e\cos(\phi)}}$$

(There might be a sign error here - go through derivation)
This describes a hyperbola, and $r \to \infty$ at ϕ_1, ϕ_2 such that

$$\cos(\phi_{1,2}) = -\frac{1}{e}$$

We can use ϕ_1, ϕ_2 to find the deflection angle and write it in terms of b (which is our goal).

8 July 18, 2024: Scattering Theory

Yesterday... we found that

$$r = \frac{-(e^2-1)a}{1+e\cos\phi}, \quad a \equiv \left(\frac{kq_1q_2}{2E}\right)$$

wher e is the eccentricity

$$e = \sqrt{1+\frac{2El^2}{k^2q_1^2q_2^2m}} > 1 \text{ (for } E>0 \text{ which corresponds to the repulsive case)}$$

What is this parametrization (r, ϕ) telling us?

[Include figure]

- For $\phi = 0$,

  $$r = -\frac{(e^1-1)a}{1+2} = -(e-1)a < 0$$

  So, we start off at a point to the left of the origin on the x-axis.

- The numerator of r is a constant so the time evolution is completely determined by the denominator of r. Now, the denominator reaches zero when $\cos(\phi) = -\frac{1}{e}$ and so that point the r goes to $-\infty$.

- Once we cross this point, $\cos(\phi)$ becomes even more negative than $-1/e$ so r just to being positive.

- (Write about each of the arcs)

We made an incorrect comment earlier where we claimed that the $\pm$ in

$$r = \frac{(e^2-1)a}{1+\cos\phi}$$

correspond to the two arcs of the hyperbola respectively, but this is not the case.

In fact, we can choose either one (plus or minus) and it will describe the whole parabola, but with a reversed orientation (if we think about parametrization). **(verify that this is true.)**

Does this make sense? There's an issue: Physically, this trajectory doesn't make sense. The (red+blue part doesn't make sense as we have a repulsive force instead of attractive). Physically, only the left branch (green+yellow) of the Hyperbola

Back to finding the deflection

Let's return to our main goal: to relate b with the deflection angle Θ.

[Include figure]

Finally,

$$\begin{aligned}\Theta &= \pi - (\phi_2 - \phi_1) \\ &= \pi - \left(2\pi - \cos^{-1}\left(-\frac{1}{e}\right) - \cos^{-1}\left(-\frac{1}{e}\right)\right) \\ &= 2\cos^{-1}\left(-\frac{1}{e}\right) - \pi\end{aligned}$$

Let's try to find a nicer expression for Θ.

$$\begin{aligned}\cos(\Theta) &= \cos\left[2\cos^{-1}\left(-\frac{1}{e}\right) - \pi\right] \\ &= \cos\left[2\cos^{-1}\left(-\frac{1}{e}\right)\right]\cos(\pi) - \sin\left[\cos^{-1}\left(-\frac{1}{e}\right)\right]\underbrace{\sin(\pi)}_{=0} \\ &= -\cos\left[2\phi_1\right] \\ &= -\left(\cos^2\phi_1 - \sin^2\phi_1\right) \\ &= -\left(2\cos^2\phi - 1\right) \\ &= 1 - \frac{2}{e^2} \\ &= \frac{\frac{2El^2}{k^2q_1^2q_2^2 - 1}}{\frac{2El^2}{k^2q_1^2q_2^2 + 1}}\end{aligned}$$

[Include figure]

and we can express it in terms of the impact factor using

$$\begin{aligned}l &= b(mv_0) \\ &= bP \\ &= b\sqrt{2mE}\end{aligned}$$

where we used $E = \frac{P^2}{2m}$ to get the last equality.

Doing so, we get

$$\cos\Theta = \frac{4b^2E^2 - k^2q_1^2q_2^2}{4b^2E^2 + k^2q_1^2q_2^2}$$

For later convenience, let's make a further simplification by noting that

$$\begin{aligned}\cot^2(\Theta/2) &= \frac{\cos^2(\Theta/2)}{\sin^2(\Theta/2)} \\ &= \frac{1+\cos\theta}{1-\cos\theta}\end{aligned}$$

and we know

$$1 + \cos\Theta = 1 + \frac{4b^2E^2 - k^2q_1^2q_2^2}{4b^2E^2 + k^2q_1^2q_2^2} = \frac{8b^2E^2}{4b^2E^2 + k^2q_1^2q_2^2}$$

$$1 - \cos\Theta = 1 - \frac{4b^2E^2 - k^2q_1^2q_2^2}{4b^2E^2 + k^2q_1^2q_2^2} = \frac{2k^2q_1^2q_2^2}{4b^2E^2 + k^2q_1^2q_2^2}$$

giving us a very neat expression

$$\cot^2(\Theta/2) = \frac{4b^2E^2}{k^2q_1^2q_2^2}$$
$$\Longrightarrow \boxed{\cot(\Theta/2) = \frac{2bE}{kq_1q_2}}$$

In some ways, we've achieved our goal here. Namely, given (b, E) we can calculate the deflection angle Θ. But we can do more.

For instance, think about Ruthorford's experiment in which gold atoms were bombarded by α particles. In such an experiment, it's practically impossible to measure b for each alpha particle. So, we must find another way. We do this using **Scattering Amplitudes** and **Cross Sections**.

8.1 Cross Sections

In practice, we can't know the exact value for b, and probably only have a range for its possible values.

[include figure - from recording; I forgot to take a picture in lecture]

The particles passing through the yellow cross section will pass through the corresponding blue solid angle which is given by $d\Omega = 2\pi \sin\Theta d\Theta$ for scattering angle Θ.

We define the **(differential) scattering cross-section** $\sigma(\Theta)$ such that

$$\sigma(\Theta)\, d\Omega = \frac{\text{\# of particles passing through the blue solid angle per time}}{\text{Incident Intensity, } I}$$

where

$$I \equiv \frac{\text{\# of incident particles}}{\text{area} \cdot \text{time}}$$

Often in experiments, the thing we control is I. The scattering cross section $\sigma(\Theta)$ carries the information of *probability*. Namely,

$\sigma(\Theta)d\Omega \propto$ The probability for an inclident particle to get scattered with deflection angle Θ

In experiments like that of the Large Hadron Collider, these scattering cross sections are the quantities that we measure experimentally. Though, when we're dealing with quantum effects, the theory agrees with classical scattering only to first order and we need some corrections.

$$\begin{aligned}
\sigma(\Theta)d\Omega &= \frac{\text{\# of particles passing through yellow}}{I} \\
&= \frac{I \cdot 2\pi n \cdot db}{I} \\
\implies \sigma(\Theta) &= \frac{2\pi b db}{d\Omega} \\
&= \frac{2\pi b db}{2\pi \sin\Theta d\Theta} \\
&= \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|
\end{aligned}$$

(We put absolute values around the derivative because $\frac{db}{d\Theta} < 0$ as increasing b decreases Θ, but in our formula we're really dealing with positive valued geometric quantities.)

(Note that the dimension of $\sigma(\Theta)$ is $[length]^2$.) (Explain why.)

So, to find $\sigma(\Theta)$ we need to solve for $\left|\frac{db}{d\Theta}\right|$

$$\begin{aligned}
\cot(\Theta/2) &= \frac{2bE}{kq_1q_2} \\
\implies \frac{d}{d\Theta}\left[\cot\left(\frac{\Theta}{2}\right)\right] &= \frac{2E}{kq_1q_2}\frac{db}{d\Theta} \\
\implies -\frac{1}{2}\csc^2\left(\frac{\Theta}{2}\right) &= \frac{2E}{kq_1q_2}\frac{db}{d\Theta} \\
\implies \left|\frac{db}{d\Theta}\right| &= \frac{kq_1q_2}{4E}\csc^2\left(\frac{\Theta}{2}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma(\Theta) &= \frac{b}{\sin(\Theta)}\left|\frac{db}{d\Theta}\right| \\
&= \frac{b}{2\sin\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Theta}{2}\right)} \cdot \frac{kq_1q_2}{4E}\csc^2\left(\frac{\Theta}{2}\right)
\end{aligned}$$

and to eliminate b from the expression, recall that

$$\begin{aligned}
\cot(\Theta/2) &= \frac{2bE}{kq_1q_2} \\
\implies b &= \frac{kq_1q_2\cot\left(\frac{\Theta}{2}\right)}{2E}
\end{aligned}$$

Plugging this into the equation above gives us

$$\sigma(\Theta) = \frac{\frac{kq_1q_2\cot\left(\frac{\Theta}{2}\right)}{2E}}{2\sin\left(\frac{\Theta}{2}\right)\cos\left(\frac{\Theta}{2}\right)} \cdot \frac{kq_1q_2}{4E}\csc^2\left(\frac{\Theta}{2}\right)$$

$$\implies \boxed{\sigma(\Theta) = \left(\frac{kq_1q_2}{4E}\right)^2 \csc^4\left(\frac{\Theta}{2}\right)}$$

This expression

$$\sigma(\Theta) = \left(\frac{kq_1q_2}{4E}\right)^2 \csc^4\left(\frac{\Theta}{2}\right)$$

is called the **Rutherford Scattering Cross-Section**

Note that there is a type in the book when it comes to the charges q_1, q_2 in the expression (find the type and mention it explicitly.)

Looking at the behavior of $\csc^4\left(\frac{\Theta}{2}\right)$ vs $\frac{\Theta}{2}$, we see that *most* particles just travel straight.

9 July 22, 2024: Virial Theorem

This is a theorem about average kinetic energy $\langle K \rangle$ and average potential energy $\langle U \rangle$ of a system. Consider the quantity,

$$G \equiv \sum_i \vec{P}_i \cdot \vec{r}_i$$

with respect to some reference point $\mathcal{O}$. Then,

$$\begin{aligned}\frac{dG}{dt} &= \sum_i \left(\vec{F}_i \cdot \vec{r}_i + \vec{P}_i \cdot \vec{v}_i\right) \\ &= \sum_i \left(\vec{F}_i \cdot \vec{r}_i + 2K_i\right)\end{aligned}$$

If we take the time average over a period of time τ,

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = \langle \sum_i \left(\vec{F}_i \cdot \vec{r} + 2K_i\right)\rangle$$

- If the motion of the system is periodic and we choose τ such that it's equal to the period, then

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right] = 0$$

- If the system is not periodic, then the LHS is equal to $\frac{1}{\tau}\left[G(\tau) - G(0)\right]$. Since both $G(\tau)$ and $G(0)$ should be finite, the LHS goes to zero if we take the limit $\tau \to \infty$.

In either case, we have

$$\langle \sum_i \vec{F}_i \cdot \vec{r}_i \rangle + 2\langle \sum_i K_i \rangle = 0$$

This is called the **Virial Theorem**.

In the case of a single particle subject to $\vec{F} = -\frac{GMm}{r^2}\hat{r}$, moving in the radial direction, we have

$$\begin{aligned}&\langle -\frac{GMm}{r^2}\hat{r} \cdot r\hat{r}\rangle + 2\langle K \rangle = 0 \\ \implies &\langle -\frac{GMm}{r^2}\rangle + 2\langle K \rangle = 0 \\ \implies &\langle U \rangle + 2\langle K \rangle = 0\end{aligned}$$

We can check that for circular orbit,

[include picture]

$$\begin{aligned} \frac{GMm}{R^2} &= ma_c = m\frac{v^2}{R} \\ \implies \frac{GMm}{R} &= mv^2 \\ \implies K &= \frac{1}{2}mv^2 = -\frac{1}{2}U \end{aligned}$$

which lines up with the result from the Virial Theorem.

Let's consider N particles interacting gravitationally. For $N = 3$,

$$\begin{aligned} \sum_i \langle \vec{F}_i \cdot \vec{r}_i \rangle &= \langle \vec{F}_{2\to1} \cdot \vec{r}_1 + \vec{F}_{3\to1} \cdot \vec{r}_1 + \vec{F}_{1\to2} \cdot \vec{r}_2 + \vec{F}_{3\to2} \cdot \vec{r}_2 + \vec{F}_{1\to3} \cdot \vec{r}_3 + \vec{F}_{2\to3} \cdot \vec{r}_3 \rangle \\ &= \langle \vec{F}_{1\to2} \cdot (\vec{r}_2 - \vec{r}_1) + \vec{F}_{1\to3} \cdot (\vec{r}_3 - \vec{r}_1) + \vec{F}_{2\to3} \cdot (\vec{r}_3 - \vec{r}_2) \rangle \text{ (By Newton's third law)} \\ &= \text{ two equations missing here; fill later} \\ &= \langle U_{12} + U_{13} + U_{23} \rangle \\ &= \langle U_{\text{tot}} \rangle \end{aligned}$$

Though we've done this for $N = 3$ in particular, the procedure is exactly the same for any value of N.

Thus, the Virial Theorem implies that

$$\begin{aligned} \langle U_{\text{tot}} \rangle + 2\langle K \rangle &= 0 \\ \implies \langle K \rangle &= -\frac{1}{2}\langle U_{\text{tot}} \rangle \end{aligned}$$

This is useful is astronomy where we can experimentally approximate the mass (how? fill in) and velocity (using Doppler Shift) of a galaxy, giving us $\langle K \rangle$, and then calculate $\langle U_{\text{tot}} \rangle$.

In some cases, there is a mismatch. These mismatches led to the conception of ideas like **Dark Energy** and **Dark Mass**.

Note this the Virial Theorem only holds in the non-relativistic case since the derivation depends on relations like

$$\begin{aligned} \vec{p} &= m\vec{v} \\ K &= \frac{1}{2}m\,|\vec{v}|^2 \end{aligned}$$

With this, we've completed chapter 7. We only cover Coulomb Scattering from Chapter 8 (which we've done already), so now we'll move onto Chapter 9.

9.1 Fictitious Force and Linear Accelerating Frames

Suppose Bob is standing in a bus with acceleration $\vec{a}$, and there's a pendant hanging from the ceiling. If we draw the Force Diagram in *Bob's reference frame*, there will be three forces on the pendant:

[Include force diagram]

However, if we look at it from the perspective of Alice (who is standing at rest on the ground outside the Bus), the pendant appears to be moving and have only two forces acting on it.

[Include picture]

[Include equations]
We end up with the result,

$$\vec{F}_{fic} = m\vec{a}$$

In some sense, the fictition force is just a result of relative motion. For example, to a driver sitting in a car with acceleration $\vec{a}$, objects on the ground outside of his window have a relative acceleration $-\vec{a}$, so if the second law holds, they should experience (in the drivers reference frame),

$$\vec{F}_{fix} = m\vec{a}$$

[Include figure]

Though we call it a *fictitious* force, if you sit in a car you certainly do feel the force. This is a manifestation of the Equivalence pricniple.

[Include figure]

9.2 Fictitious Forces in the Lagrangian Formalism

Now, how do we arrive at the Equations of Motion with fictitious forces in the Lagrangian Formulation?

Consider an inertial frame S and a frame S' moving with constant acceleration $\vec{a}$ such that the two origins coincide at $t = t' = 0$ and the S'-frame starts from rest.

[Include picture]

Then at time t, the distance between their origins measured in the S-frame will be $\frac{1}{2}at^2$. So, the point described by position x in S-frame has coordinate

$$x' - \frac{1}{2}at^2 \implies x(x') = x' + \frac{1}{2}at^2$$

in the S'-frame.

Now, S-frame is an inertial frame so for a particle subject to potential $V(x)$, we have

$$\begin{aligned} L &= \frac{1}{2}m\dot{x}^2 - V(x) \\ &= \frac{1}{2}m\left(\dot{x}' + at\right)^2 - V(x(x', t)) \end{aligned}$$

Sidenote: There is explicit time dependence so the Hamiltonian is not conserved.

So, the Equations of Motion are found as

$$\begin{aligned} &\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} = 0\right) \\ \implies &-\frac{\partial x}{\partial x'}\frac{\partial V}{\partial x} - \frac{d}{dt}\left(m\left(x' + at\right)\right) = 0 \\ \implies &-\frac{\partial V}{\partial x} - m\ddot{x} - ma = 0 \\ \implies &m\ddot{x} = \underbrace{-\frac{\partial V}{\partial x}}_{\text{real force}} \underbrace{-ma}_{\text{fic. force}} \end{aligned}$$

9.3 Rotating Forces

For simplicity, let's first consider the following system in outer space and subject to no gravity:

[Include picture]

Let's say there's an observer "Bob" in the spaceship and another observer "Alice" outside the space station. Suppose Bob throws a ball straight up with respect to himself. For Bob, the ball appears to go straight up, but for Alice, it has a horizontal component as well due to the rotation ω.

For Alice, the ball has an initial velocity

$$\vec{v}_0 = \omega R\hat{x} + \omega R\hat{y}$$

For Alice, the ball travels with a trajectory of constant velocity.

[Include figure]

The time it takes to reach the ground again is

$$\Delta t = \frac{\sqrt{2}R}{\sqrt{2}\omega R} \text{ (dist./velocity) } = \frac{1}{\omega}$$

How much does Bob rotate in this time?

$$\begin{aligned} \Delta\theta &= \omega \cdot \delta t \\ &= \omega \cdot \frac{1}{\omega} \\ &= 1 \text{ (radians)} \end{aligned}$$

So, what Bob will see is that he tosses the ball up but it lands to his right at a distance

$$d - R \cdot \frac{\pi}{4} - R \cdot 1 = R\left(\frac{\pi}{4} - 1\right)$$

We don't have gravitational forces, so why the turn? This is due to Coriolis force.

9.4 Rotations

[Include picture]

$$\begin{aligned} \vec{V}' = \begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \\ \text{(when } \theta \text{ is small)} &\approx \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \end{aligned}$$

10 July 23, 2024:

Last time we ended off by discussing what an infinitessimal rotation about the z-axis look like:

[Include picture]

$$\vec{V}' \approx \begin{pmatrix} 1 & -\theta_z & 0 \\ \theta_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left[1 - \theta_z \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Similarly, if we rotate about the y-axis in the "positive direction" (given by the right hand rule)

[Include picture]

$$\vec{V}' \approx \begin{pmatrix} 1 & 0 & +\theta_y \\ 0 & 1 & 0 \\ -\theta_y & 0 & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left[1 - \theta_y \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

and an infinitessimal rotation about the x-axis looks like

[Include picture]

$$\vec{V}' \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_x \\ 0 & \theta_x & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left[1 - \theta_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix} \right] \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

We can compactly write these relations as

$$V_i' = \delta_{ij} - \theta_k \epsilon_{kij} V_j$$

where ϵ_{ijk} is the Levi-Civita symbol,

$$\epsilon_{ijk} = \begin{cases} +1, & \text{for cyclic permutations of (123)} \\ +1, & \text{for cyclic permutations of (132)} \\ 0, & \text{otherwise} \end{cases}$$

Note that the Levi-Civita symbol is totally antisymmetric in its indices i.e.

$$\epsilon_{ijk} = -\epsilon jik = -\epsilon_{kij} = -\epsilon_{ikj}$$

Also note that

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

10.1 Rotating Frame

Consider an intertial frame S', and a rotating frame S. Given a vector $\vec{r}$ in the S' frame, we want to find its expression in the S frame.

[Include picture]

Since $\vec{r}$ is a vector - a physical quantity - it is always $\vec{r}$ regardless of which frame (coordinate basis) we choose. So, really, the question we want to ask is "What do the ***components*** of the $\vec{r}$ vector in the S' frame look like?".

$$\vec{r} = r'^i \hat{e}'_i = r^i \hat{e}_i$$

We can think that the S' frame is obtained by actively rotating the inertial frame. When we think of the **basis vectors rotating**, we have

[include picture]

For a rotation by angle θ,

$$\hat{e}_x = \cos\theta \hat{e'}_x - \sin\theta \hat{e'}_y$$
$$\hat{e}_y = \sin\theta \hat{e'}_x + \cos\theta \hat{e'}_y$$

More generally, for a small rotation,

$$\hat{e}_i = (\delta_{ij} + \theta_k \epsilon_{kij})\, \hat{e'}_j$$

That is, we can think of the rotation of a frame over time being defined by the relations

$$\hat{e}_i(t) = R_{ij}(t) \hat{e'}_j$$
$$\hat{e}_i(t+dt) = (\delta_{ij} + d\theta_k \epsilon_{kij})\, \hat{e'}_j(t)$$

So,

$$\begin{aligned}\dot{\hat{e}}_i &= \frac{\hat{e}_i(t+dt) - \hat{e}_i(t)}{dt} \\ &= \frac{\hat{e}_i(t) + d\theta_k \epsilon_{kij} \hat{e}_j(t)}{dt} \\ &= \dot{\theta}_k \epsilon_{kij} \hat{e}_j(t)\end{aligned}$$

Now, also,

$$\begin{aligned}\dot{\vec{r}} &= \dot{r'}^i \dot{\hat{e}}'_i \\ &= \dot{(r')}^i \hat{e}_i + (r')^i \underbrace{\dot{\hat{e}}'_i}_{=0} \\ &= \dot{r}^i \text{ Rename } j \text{ as } i \text{ and } i \text{ as } j \\ &\text{a couple of equations} \\ &= \left(\dot{r}_i + \epsilon_{ikj}\dot{\theta}_k r_j\right)\hat{e}_i\end{aligned}$$

Fill this in later.

In the textbook, they used a more visual approach and wrote that for any vector $\vec{A}$,

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{iner.}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{rot.}} + \vec{\omega} \times \vec{A}$$

This form can be a little bit confusing because the vector components in the S and S' frames are really being written in different bases, so we CANNOT say

$$\dot{r'}_i = \dot{r}_i + \epsilon_{ijk}\dot{\theta}_k r_j$$

The expression obtained by the textbook includes not just the components but also the basis vectors themselves.

Fill in the rest of this crap later.

11 July 24, 2024: More involved examples

Last time we discussed the intuition behind the Euler Term and Centrifugal force. Let's talk about the Coriolis force now, and then move on to some examples.

11.1 Coriolis Force

We found, last time, that for a scenario as pictured below, the coriolis force is given by

$$F_{\text{coriolis}} = -2m\left(\vec{\omega} \times \vec{v}\right)$$

Include picture

Suppose the observer throws a ball towards the center such that it has a straight trajectory seen by Alice.

Include picture

In Bob's frame, the ball has an initial velocity with a leftward component (so that the net velocity seen by *Alice* is straight upwards).

$$\vec{v}_o' = \vec{v}_{\text{left}} + \vec{v}_y$$

such that

$$\vec{v}_{\text{left}} + \omega R\hat{x} = 0$$

What does the trajectory of the ball look like in Bob's reference frame?

Include picture

Both Bob and Alice will observe that the ball reaches the center. In Alice's frame the trajectory was a straight line, but in Bob's frame, the balls initial velocity was *towards the left*.

So, Bob sees the ball go left but then *curve* so that the ball is dragged back to the center. The explanation for this in Bob's frame is that there is a **force** dragging the ball back. This is the Coriolis force.

11.2 Fictitious Force on Earth

Consider the earth, rotating about its axis with angular velocity Ω, and a coordinate system on it as described by the picture below.

In the Rotating Frame,

$$\vec{\Omega} = (0, \Omega\cos\lambda, \Omega\sin\lambda)$$

If we have an object moving on Earths surface with velocity

$$\vec{v} = (v_x, v_y, v_z) \approx (v_x, v_y, 0)$$

then, the Coriolis force experienced by the object is

$$\begin{aligned}\vec{F}_{\text{coriolis}} &= -2m\vec{\Omega} \times \vec{v} \\ &= -2m \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ v_x & v_y & 0 \end{bmatrix} \\ &= -2m\Omega \left(-v_y \sin\lambda \hat{x} + v_x \sin\lambda \hat{y} - v_x \cos\lambda \hat{z}\right)\end{aligned}$$

We can ignore the z-component since it's much smaller than $\vec{F}_g = -mg\vec{z}$ and so won't impact the dynamics of the object in the z-direction much. So,

$$F_{\text{coriolis}} = -2m\Omega \sin\lambda \left(-v_y \hat{x} + v_y \hat{y}\right)$$

- Note that $\sin\lambda = 0$ when $\lambda = 0$ i.e. $\vec{F}_{\text{coriolis}} = 0$ at the **equator**. In contrast, $\left|\vec{F}_{\text{coriolis}}\right|$ is the greatest at the **poles**.
- Apparently, $\vec{F}_{\text{coriolis}} \cdot \vec{v}_{\text{surface}} = 0$ because

$$(-v_y \hat{x} + v_x \hat{y}) \cdot (v_x \hat{x} + v_y \hat{y}) = 0$$

- Also, let's think: What should be the **direction** of the Coriolis force?

  Include picture

  The way we chose λ, it takes positive and negative values in the northern and southern hemispheres respectively. Thus, the $\sin\lambda$ prefactor has positive/negative value in the northern/-souther hemisphere.

  Thus in the **Northern** Hemisphere Coriolis Force acts to the **right** whereas in the **South** it acts to the **left**!

  We can observe this in the **direction of rotation of hurricanes**.

  Include picture

  Beware however. If you've ever seen the demonstration of water swirling in different directions in the northern and southern hemispheres, that has **nothing to do with Coriolis Forces**! At those small scales, the direction of swirling just depends on the initial conditions. We don't see the effects on the Coriolis force on such small scales.

Okay. Let's do some more quantitative work now.

11.3 Foucalt's Pendulum

Consider a pendulum at some point of the earth which is swinging as usual. Due to the rotation of the Earth, the *plane of oscillation* is also rotating. This effect can be observed anywhere other than the poles or the equator.

Our goal is to find the rotational period/frequency of the oscicllating plane. Further, we'd like to find the motion $(x(t), y(t), 0)$ of the pendulum on the surface of the earth.

Include picture

In the Inertial Reference Frame (Universe), the Lagrangian of the system is

$$L = \frac{1}{2}m\underbrace{\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}_{\dot{\vec{r}}\cdot\dot{\vec{r}}}|_{\text{in}} - U(\vec{r})|_{\text{in}}$$

and we discussed yesterday that we can write $\dot{\vec{r}}$ in the rotating frame as

$$\begin{aligned}\dot{\vec{r}} &= \dot{r}_i^{(in)}\hat{e}_i^{(in)} \\ &= \underbrace{\dot{r}_i^{(rot.)}\hat{e}_i^{(rot.)}}_{\vec{v}^{(rot.)}} + \underbrace{\epsilon_{ijl}\dot{\theta}_j r_k \hat{e}_i^{(rot.)}}_{\vec{\omega}^{(rot.)}\times\vec{r}^{(rot.)}}\end{aligned}$$

So,

$$\dot{\vec{r}}\cdot\dot{\vec{r}} = \left(\dot{r}_i^{(rot.)}\hat{e}_i^{(rot.)} + \epsilon_{ijk}\dot{\theta}_j r_k \hat{e}_i^{(rot.)}\right)\cdot\left(\dot{r}_m^{(rot.)}\hat{e}_m^{(rot.)} + \epsilon_{mjk}\dot{\theta}_j r_k \hat{e}_m^{(rot.)}\right)$$

Now, both $\hat{e}_i^{(in.)}$ and $\hat{e}_i^{(rot.)}$ form orthonormal bases. So, $\hat{e}_i^{(rot.)}\cdot\hat{e}_j^{(rot.)} = \delta_{ij}$

So, in the rotating frame, the Lagrangian is basically (dropping the superscripts $(rot.)$ because all quantities are now in the rotating frame)

$$L = \frac{1}{2}m\left(\vec{v} + \vec{\Omega}\times\vec{r}\right)\cdot\left(\vec{v} + \vec{\Omega}\times\vec{r}\right) - U(\vec{r}).$$

where $\vec{\Omega}$ is the angular velocity of the earth.

Since the rotation of the earth is slow, the Coriolis force is weak and we only need to keep things to first order in Ω. Then,

$$L \approx \frac{1}{2}m\vec{v}\cdot\vec{v} + m\vec{v}\cdot\left(\vec{\Omega}\times\vec{r}\right) - U(\vec{r})$$

and as we discussed earlier $\vec{\Omega} = (0, \Omega, \cos\lambda, \Omega\sin\lambda)$ so

$$\begin{aligned}\vec{v}\cdot\left(\vec{\Omega}\times\vec{r}\right) &= \begin{vmatrix} v_x & v_y & v_z \\ 0 & \Omega\cos\lambda & \Omega\sin\lambda \\ x & y & z \end{vmatrix} \\ &= \Omega\cos\lambda\,(zv_x - xv_z) - \Omega\sin\lambda\,(yv_x - xv_y)\end{aligned}$$

making the Lagrangian

$$L \approx \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\dot{x}\Omega\left(z\cos\lambda - y\sin\lambda\right) + 2\dot{y}x\Omega\sin\lambda - 2\dot{z}x\Omega\cos\lambda\right] - mgz$$

So far we've written the Lagrangian in our generalized coordinates of the rotating frame, but we still need to enfore the **constraint** provided by the fixed length of the pendulum.

Include picture

Our constraint is that

$$x^2 + y^2 + z^2 = R^2$$
$$\implies z = \underbrace{-}_{\text{sign conv.}} R\sqrt{1 - \frac{x^2}{R^2} - \frac{y^2}{R^2}}$$

(The signs are a lil sketch; check with Chien-I)

We make an approximation (effectively a small angle approximation):

$$\frac{x^2}{R^2} \sim \frac{y^2}{R^2} << 1$$

Then,

$$z \approx -R\left(1 - \frac{x^2}{2R^2} - \frac{y^2}{2R^2}\right) \text{ (using Binomial Theorem)}$$
$$\implies \dot{z} \approx \frac{x\dot{x}}{R^2} + \frac{y\dot{y}}{R^2}$$

Putting these into the Lagrangian and only keeping terms to first order in x^2/R^2 and y^2/R^2 we have

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}mg\left(+\frac{x^2}{R^2} + \frac{y^2}{R^2}\right) - m\Omega\sin\lambda\left(\dot{x}y - \dot{y}x\right)$$

(dropping constant terms and terms that are 2nd order in small parameters)

Then, the E-L Equations look like:

$$\underline{x} \quad m\ddot{x} - m\Omega\sin\lambda\dot{y} + mg\frac{X}{R} - m\Omega\sin\lambda\dot{y} = 0$$
$$\underline{y} \quad m\ddot{y} + m\Omega\sin\lambda\dot{x} + mg\frac{y}{R} + m\Omega\sin\lambda\dot{x} = 0$$

We have a set of coupled second order differential equations. This is annoying! To guess the answer (ansatz) let's think physically. At the north pole i.e. $\lambda = \frac{\pi}{2}$ we expect that we should have a simple solution for (x', y') measured in the inertial frame and (x', y') is related to (x, y) in the rotating frame by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

We'll continue this tomorrow, but the approach will be to solve for the system's evolution in terms of (x', y') and then use a rotation to get (x, y).

12 July 25, 2024: Continuing with Foucault's Pendulum

(Write a recap if there's time for it)

$$L \approx \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}mgR\left[\frac{x^2}{R^2}+\frac{y^2}{R^2}\right] - m\omega\sin\lambda(\dot{x}y-\dot{y}x)$$
$$\Longrightarrow \begin{cases} \ddot{x} - 2\omega\sin\lambda\dot{y} + \omega_0^2 x = 0 & (1) \\ \ddot{y} + 2\omega\sin\lambda\dot{x} + \omega_0^2 y = 0 & (2) \end{cases}$$

where $\omega_0^2 = \frac{g}{R}$

These look like the equations of motion of a 2D Isotropic Harmonic Oscillator, but with an interaction term containing $(\dot{x}y - \dot{y}x)$. This makes the EL equations *coupled* second order differential equations.

To guess the ansatz, we expect that the solution when $\lambda = \frac{\pi}{2}$ should have the simple form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where x' and y' should satisfy the Equations of Motion

$$\begin{aligned} \ddot{x}' + \omega_0^2 x' &= 0 \\ \ddot{y}' + \omega_0^2 y' &= 0 \end{aligned}$$

The reason we expect this to be the case is that our inertial reference frame (x', y', z') in outer space **(Watch the recording again. Didn't understand why we should expect this)**
So, we can very easily find $\begin{pmatrix} x' \\ y' \end{pmatrix}$ which satisfy these simply EoM, and then obtain the $\begin{pmatrix} x \\ y \end{pmatrix}$ in the rotating frame using a rotation matrix transformation.

Okay. Let's check to see if our guess is true.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \Omega \begin{pmatrix} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \dot{x}' \\ \dot{y}' \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} (\ddot{x}' + \ddot{y}') + 2\Omega \begin{pmatrix} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \mathcal{O}(\Omega^2)$$

(We only worry about terms that are linear in Ω)

Plugging this into Equation (1) gives us

$$0 = \underbrace{\left(\cos(\Omega t)\ddot{x}' + \sin(\Omega t)\ddot{y}'\right) + 2\Omega\left(-\sin(\Omega t)\dot{x}' + \cos(\Omega t)\dot{y}'\right)}_{\ddot{x}} - 2\Omega \underbrace{\sin\lambda}_{=1 \text{ as } \lambda=\frac{\pi}{2}} \underbrace{\left(-\sin(\Omega t)\dot{x}' + \cos(\Omega t)\dot{y}'\right)}_{\dot{y} \text{ terms, but only those linear in } \Omega} +$$

$$\omega_0^2 \underbrace{\left(\cos(\Omega t)x' + \sin(\Omega t)y'\right)}_{x}$$

$$= \cos(\Omega t)\ddot{x}' + \sin(\Omega t)\ddot{y}' + \omega_0^2\cos(\Omega t)x' + \omega_0^2\sin(\Omega t)y' \qquad (3)$$

Similarly, if we plug it into Equation (2) we get

$$0 = -\sin(\Omega t)\ddot{x}' + \cos(\Omega t)\ddot{y}' - \omega_0^2\sin(\Omega t)x' + \omega_0^2\cos(\Omega t)y' \qquad (4)$$

These equations give us a way to decouple x and y. For eg. If we want to eliminate $\ddot{x}$ we can do $[(3)\times\sin(\Omega t) + (4)\times\cos(\Omega t)]$ because then the coefficients multiplying $\ddot{x}$ are $\pm\cos(\Omega t)\sin(\Omega t)$ and thus they cancel.

Eliminating $\ddot{x}'$ and x', we find that indeed

$$\ddot{y}' + \omega_0^2 y' = 0$$

Similarly, we can eliminate $\ddot{y}'$ and y'. Doing so, we would find

$$\ddot{x}' + \omega_0^2 x' = 0$$

(Write this stuff out explicitly if there's time for it.)
For $\lambda = \frac{\pi}{2}$ we indeed have the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where $\begin{pmatrix} x' & y' \end{pmatrix}$ is the solution for

$$\begin{cases} \ddot{x}' + \omega_0^2 x' = 0 \\ \ddot{y}' + \omega_0^2 y' = 0 \end{cases}$$

So we've solved the system for $\lambda = \frac{\pi}{2}$.
For other values of λ, formally, we just have a "new" $\tilde{\Omega}$ such that x, y satisfy equations of motion

$$\ddot{x} - 2\tilde{\Omega}\dot{y} + \omega_0^2 x = 0$$
$$\ddot{y} + 2\tilde{\Omega}\dot{x} + \omega_0^2 y = 0$$

where $\tilde{\Omega} = \Omega\sin\lambda$. This has the same form as the equation of motion when $\lambda = \frac{\pi}{2}$. We just need to swap $\Omega \to \tilde{\Omega}$.

Mathematically, this just means that with

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \cos(\tilde{\Omega} t) & \sin(\tilde{\Omega} t) \\ -\sin(\tilde{\Omega} t) & \cos(\tilde{\Omega} t) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

then the coordinate $\begin{pmatrix} x' \\ y' \end{pmatrix}$ will have the same simple EoM

$$\begin{cases} \ddot{x}' + \omega_0^2 x' = 0 \\ \ddot{y}' + \omega_0^2 y' = 0 \end{cases}$$

Caveat: This is a mathematical trick. The $\begin{pmatrix} x' \\ y \end{pmatrix}$ here ***is no longer the initial position in the initial reference frame above the north pole.***

in this case, the plane of oscillation rotates with an angular speed $\tilde{\Omega}$.

Since we don't have much time and we're already taking about rotation, we will continue on with Rigid Body Dynamics and cover Hamiltonian Mechanics towards the end.

12.1 Rigid Body Dynamics

[include picture]

Question: How do we describe rigid body motion?

First off, it's good to have an inertial reference frame and a tagged point (on the body). We can look at the position $\vec{R} = (x, y, z)$ (so we have three d.o.f's) of the tagged point as it moves in time.

Usually we choose the Center-of-Mass as the tagged point, or if the object is fixed we use the the well... fixed point.

Include picture

Question: How do we describe rigid body rotation?

We can choose another (primed) coordinate system with its origin at the Tagged point which is fixed with respect to the point. As the body rotates, so does the primed coordinate system.

Suppose we have a particle whose position vector in the primed coordinates is

$$\vec{r'} = r'_i \hat{e'}_i$$

This $\vec{r'}$ is time-independent (since the body is rigid). The position $\vec{r}$ of the same point in the body (in the original interial frame) is given by

$$\begin{aligned} \vec{r} &= \vec{R} - \vec{r'} \\ \implies r_i \hat{e}_i &= R_i \hat{e}_i + r'_i \hat{e'}_i \text{ (Note, we do not have } r_i = R_i + r'_i \text{)} \end{aligned}$$

We know that $\hat{e'}_i$ is related to $\hat{e}_i$ via a rotational matrix.

$$\hat{e}_i = \mathcal{R}_{ji}\hat{e}_j$$

(Ask Chien-I about this.)
Question: What properties do we have for this rotational matrix $\mathcal{R}_{ij}$ and how many d.o.f's do we have?

These rotational matrices are elements of $SO(3)$ i.e. the Special Orthogonal Group of dimension 3. The matrices must satisfy the following conditions:

1. *Orthogonal:* $\delta_{ij}R_{im}R_{jn} = \delta_{mn}$

2. *Special:* $\det(R_{ij}) = 1$

Why?

- No matter how we rotate the frame, we do NOT change the lengths and angles $\implies$ the dot product $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$ should be invariant under rotations. If we write the vectors in component form, $\vec{A} = A_i\hat{e}_i$, $\vec{B} = B_j\hat{e}_j$ then

$$\vec{A}\cdot\vec{B} = A_iB_j\underbrace{\hat{e}_i\cdot\hat{e}_j}_{\delta_{ij}} = A_iB_i$$

Thus, if under rotation, $A_i \to A'_i = \mathcal{R}_{ij}A_k$, $B_j \to B'_j = \mathcal{R}_{jm}B_m$, then requiring that the dot product remains invariant means

$$\begin{aligned}
&A_iB_i = A_iB_i \\
\implies &\delta_{ij}A'_iB'_j = \delta_{ij}A_iB_j \\
\implies &\delta_{ij}\mathcal{R}_{ik}A_k\mathcal{R}_{jm}A_m = \delta_{ij}A_iB_j \\
\implies &\delta_{ij}\mathcal{R}_{ik}\mathcal{R}_{jm}A_kB_m = \delta_{km}A_kB_m
\end{aligned}$$

(We can do this since we're dealing with components, not the actual matrices.)

And requiring that this relation holds true for any $\vec{A}, \vec{B}$ means that $\mathcal{R}_{ik}, \mathcal{R}_{jm}$ must satisfy the equation

$$\boxed{\delta_{ij}\mathcal{R}_{ik}\mathcal{R}_{jm} = \delta_{km}}$$

Why do we call this relation orthogonal? We don't change the lengths or (relative) angles, so vectors that start off being orthogonal remain orthogonal. **(Chien-I included more explicit calculations. Include if there's time.)**

- Secondly, why do we require $\det(\mathcal{R}_{ij}) = 1$?

The orthogonal condition implies that our rotation matrices must have determinant of either $+1$ or -1. However, if we want to maintain the **chirality** of our coordinate system i.e. if we want a right-handed system to remain right handed after rotation, we must restrict to those matrices whose determinant is $+1$.

13 July 29, 2024: Laptop died, writing on paper. Type it up later

14 July 30, 2024: Inertia Tensor and Examples

14.1 Moment of Inertia Matrix

Yesterday we ended off by speaking about Spin Angular Momentum. We said that if we have a rigid body viewed in a *lab frame* and *body frame*,

Include picture

Then the Spin Angular Momentum, defined as

$$\vec{L}_{\text{spin}} = \int dm \vec{\mathscr{r}} \times (\vec{\omega} \times \vec{\mathscr{r}})$$

can be expressed (in index notation) as

$$\begin{aligned}
L_{\text{spin},\, i} &= \int dm \epsilon_{ijk} \mathscr{r}_j (\omega \times \mathscr{r})_k \\
&= \int dm \epsilon_{ijk} \mathscr{r}_j (\epsilon_{kmn} \omega_m \mathscr{r}_n) \\
&= \int dm \epsilon_{ijk} \epsilon_{mnk} \mathscr{r}_j \mathscr{r}_n \omega_m \\
&= \int dm \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) \mathscr{r}_j \mathscr{r}_n \omega_m \\
&= \int dm \left(\mathscr{r}^2 \delta_{im} - \mathscr{r}_i \mathscr{r}_m\right) \omega_m
\end{aligned}$$

where in the last step we're using the notation $\mathscr{r}^2$ to denote $\sum_j \mathscr{r}_j \mathscr{r}_j$ which we got because of the δ_{jn} factor.

We have

$$\begin{aligned}
L_{\text{spin},\, i} &= I_{im}\omega_m \text{ where} \\
I_{im} &\equiv \int dm \left(\mathscr{r}^2 \delta_{im} - \mathscr{r}_i \mathscr{r}_m\right)
\end{aligned}$$

This I_{im} matrix is called the **Moment of Inertia matrix**. This is a generalization of $\vec{L} = I\vec{\omega}$ (which is only true for symmetric objects rotating about the symmetry axis).

Generally, $\vec{L_{\text{spin}}} = \hat{I}\vec{\omega}$ and $\vec{L}$ is generally NOT parallel to $\vec{\omega}$.

Now, if we look at the definition of the inertia matrix, $\delta_{im} = \delta_{mi}$ and $\mathscr{r}_i \mathscr{r}_m = \mathscr{r}_m \mathscr{r}_i$, so the

$$I_{ab} = \int dm \left(\mathscr{r}^2 \delta_{ab} - \mathscr{r}_a \mathscr{r}_b\right)$$

matrix is **Real and Symmetric**. Thus, its eigenvectors form an **orthonormal basis**. We can always choose the direction of these eigenbasis vectors $\hat{e}'_i$ such that $\{\hat{e}'_i\}$ forms a **right-handed coordinate system**.

$\implies$ We can **always** go from a body frame to another frame by rotation such that, in the new frame, the I_{ab} matrix is diagonal. We call this frame the **Principle Axis (Body) Frame**.

In the principal axis frame,

$$L_i = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Note that $\vec{L}$ is only parallel to $\vec{\omega}$ if $I_{xx} = I_{yy} = I_{zz}$.

14.2 Example:

If we have a rigid ring of radius R in $\mathbb{R}^3$,

Include picture

a position on the ring is of the form

$$\vec{\mathscr{r}} = (x, y, z) = (R\cos\phi, R\sin\phi, 0)$$

Complete this

15 July 31, 2024: Continuing on from yesterday

15.1 Torque-free Dynamics of a Symmetric Top

Yesterday, we wrote down the Lagrangian, and found that the non-trivial part is

$$L = \frac{1}{2}\omega_i I_{ij} \omega_j$$

Note that this quantity is a **scalar**, so its value should be invariant if we rotate our frame. Thus, we can write it in the principle axis body frame in terms of

$$\hat{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

where $I_{xx} = I_{yy} = I$ because of the symmetry of the object.

We also know that, in terms of Euler Angles, the angular velocity in the body frame is

$$\vec{\omega}' = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{pmatrix}$$

[Include picture reminding us what the Euler Angles θ, ϕ, ψ are]

So, expanding the Lagrangian out in the Principle Axis body frame,

$$\begin{aligned} L &= \frac{1}{2}\omega'_i I'_{ij} \omega'_j \\ &= \frac{1}{2} I {\omega'_x}^2 + \frac{1}{2} I {\omega'}_y^2 + \frac{1}{2} I_3 {\omega'}_z^2 \\ &= \frac{1}{2} I \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \end{aligned}$$

Note that we have two cyclic coordinates: ϕ and ψ

$$\begin{aligned} \implies p_\psi &\equiv \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) = \text{ const} \\ \implies p_\varphi &\equiv \frac{\partial L}{\partial \dot{\varphi}} = I\dot{\varphi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\cos\theta \\ &= I\dot{\varphi}\sin^2\theta + p_\psi\cos\theta = \text{ const.} \end{aligned}$$

Algebraically, we can solve for (p_ψ, p_φ) in terms of $(\dot{\psi}, \dot{\varphi})$ and so they're basically interchangeable.

Since $\frac{\partial L}{\partial t} = 0$, $H = K = \text{const}$ as well i.e.

$$H = \frac{1}{2} I \dot{\phi}^2 \sin^2\theta + \frac{1}{2} I \dot{\theta}^2 + \frac{p_\psi^2}{2I_3} = \text{ const}$$

15.2 Some Physical Considerations

- There is no torque wrt the C.M $\implies$ the Spin Angular Momentum $\vec{L}$ is constant in the inertial frame.
- On the other hand, in the Pricniple Axis body frame,

$$\begin{pmatrix} L'_x \\ L'_y \\ L'_z \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} I\omega'_x \\ I\omega'_y \\ I_3\omega'_z \end{pmatrix}$$

  Note that

$$\begin{pmatrix} L'_x \\ L'_y \end{pmatrix} = I \begin{pmatrix} \omega'_x \\ \omega'_y \end{pmatrix}$$

  so, **for just the $\boldsymbol{x', y'}$ components**, we have $\vec{L}$ parallet to $\vec{\omega'}$

- In the body frame, (include picture).
  The x', y' components of $\vec{\omega}$ are parallel to the x', y' components fo $\vec{L}$. This is **always true** $\implies$ as $\vec{L}$ evolves $\vec{\omega'}$ evolves in a way such that the $z'-$axis, $\vec{L}$, $\vec{\omega'}$ lie in the same plane.

On the other hand, we have

$$\begin{aligned} \text{const.} &= p_\psi = I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) \\ &= I_3\omega'_z \\ &= L'_z \end{aligned}$$

(The $z-$component of the angular momentum in the body frame)

Fill in from lecture and pictures

16 Hamiltonian Formulation

Recall that when $\frac{\partial L}{\partial t} = 0$, the quantity

$$H \equiv \frac{\partial L}{\partial \dot{q}_k} dotq_k - L$$

is a conserved quantity. We can also think of this as a **Legendre Transformation** where we go from $L(q_k, \dot{q}_k, t)$ (a function of generalized coordinates and velocities) to $H(q_k, p_k, t)$ where $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$ (so H is a function of generalized coordinates and *momenta*).

What do we mean by Legendre Transformation?
It's basically when we go from one set of independent coordinates that describe our system to another. Consider the function $H = p_k\dot{q}_k - L$

Complete this

So, naturally, we should think about H as $H(q_k, p_k, t)$ i.e. a functions whose independent variables are coordinates and momenta.

Such transforms show up a ton in Thermodynamics.

But if we do think about H this waty, we also have

$$dH = \frac{\partial H}{\partial q_k} d(q_k) + \frac{\partial H}{\partial p_k} d(p_k) + \frac{\partial H}{\partial t} dt \qquad (2)$$

So, (1) and (2) give us

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k}$$

If q_k satisfies the E-L equations, then

$$\frac{\partial H}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \dot{p_k}$$

So,

$$\dot{q}_k = \frac{\partial H}{\partial p_k},$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

These are known as **Hamilton's Equation**.