Feynman Diagrams & QFT Notes

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These are notes taken from lectures on Feynman Diagrams and QFT delivered by Ivan Burbano. Any errors that may have crept in are solely my fault.

Contents

1 January 26 - First meeting! Setting goals.

1.1 The goal

This course wll be a very first-principles, barebones experience. Our goal for the next month will be to develop the tools to solve the integral

$$\int_{-\infty}^{\infty} dx\; e^{-S(x)} O(x)$$

where $S(x) = \frac{1}{2}ax^2 + \frac{1}{3!}gx^3 + \frac{1}{4!}\lambda x^2 + \cdots$, the constants a, g, λ etc. must be positive reals and are called *coupling constants* and $O(x)$ is a polynomial in x.

What is the physical motvation for this integral?

There are *two* sides to the physics related to this integral: Quantum (this is what we want!) and Statistical (this is what we do!).

[Insert picture]

1.2 The Quantum regime

In the Quantum regime, *fields* become *fuzzy!* We can't quite pin down what the configuration of the field is, rather we can tell what the *probability amplitude* of any given field configuration will be at a point in (space)time.

So, the fundamental question in QM is:

> If at time t_i we prepare a field ϕ_i, what is the probability amplitude that, at t_f, I measure ϕ_f?

We can say that

$$\phi_f = \langle \phi_f | U(t_f, t_i) | \phi_i \rangle$$

where $U(t_f, t_i)$ is called the *propagator* or the *time-evolution operator.*

According to feynman, the answer can be found by integrating the *action* over the space of all field configurations.

i.e.

$$= \int \mathcal{D}\Phi e^{iS(\Phi)}$$

where we are integrating over all field configurations $\Phi(x)$ such that $\Phi = \phi_i$ at t_i and $\Phi = \phi_f$ at t_f.

In more formal notation, the set over which we are integrating is

$$\{\phi \in C^\infty([t_i, t_f] \times \Sigma) \;\; : \;\; \phi\big|_{\Sigma_i} = \phi_i; \phi\big|_{\Sigma_f} = \phi_f\}$$

where Σ denoted the space we're working on.

1.3 The Statistical Side of Things

This is what we actually do!

Suppose we have some region, say a table, T populated by some *field* $\phi(x)$. For instance, it could be some spin distrbiution i.e. the field assigns each point on the table T with some spin.

Boltzmann showed that the probability (*not* probability amplitude) for the field to be in configuration ϕ is proportional to

$$e^{-S(\phi)}$$

where $S(\phi)$ is the energy of the field configuration.

In particular, the probablity is

$$\frac{e^{-S(\phi)}}{\mathcal{Z}}$$

where

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S(\phi)}$$

is the integral over the space (of all field configurations) of probabilities. It's called the **Partition Function**.

> Quick aside, what about dimensions?
>
> The partition function is dimensionless, whereas the feynman path integral we covered in the Quantum Regime *is* dimension-ful. This is our first hint that something is amiss with the path-integral. (Has to do with renomalization).

> Note: The ket $|x\rangle$ *does have units.* In particular, the completeness relation tells us
>
> $$\int dx\, |x\rangle\langle x| = 1$$
>
> So, $|x\rangle$ has units of $\frac{1}{\sqrt{\text{Length}}}$

1.4 An example from Stat. Mech: Kinetic Theory of gasses

Now, let's actually solve an integral! Let's compute

$$\int_{-\infty}^{\infty} d\phi e^{-S(\phi)} O(\phi)$$

where

$$S(\phi) = \frac{1}{2}\phi m^2 \phi - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$$

where $m^2 > 0$.

1. Partition Function : $O = 1$ For a free theory, we have $g = \lambda = \cdots = 0$

   So, our integral turns into

   $$\boxed{\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2}}$$

   This integral is intimately connected to the Kinetic Theory of Gasses.
   Sol:

$$
\begin{aligned}
\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2} &= \sqrt{\left(\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2}\right)^2} \\
&= \sqrt{\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2} \int_{-\infty}^{\infty} d\psi e^{-\frac{1}{2}m^2\phi^2}} \\
&= \sqrt{\int_{\mathbb{R}^2} d\phi d\psi \ e^{-\frac{1}{2}m^2(\phi^2+\psi^2)}}
\end{aligned}
$$

Now, we convert to polar coordinates with $r^2 = \psi^2 + \phi^2$, $u = \frac{1}{2}r^2m^2$, $du = drr$

$$
\begin{aligned}
&= \sqrt{\int_{\mathbb{R}^2} \underbrace{d\phi d\psi}_{=rdrd\theta} \ e^{-\frac{1}{2}m^2(\phi^2+\psi^2)}} \\
&= \sqrt{2\pi \int_{0}^{\infty} dr \ r \ e^{-\frac{1}{2}r^2m^2}} \\
&= \sqrt{\frac{2\pi}{m^2}} \int_{0}^{\infty} du \ e^{-u} \\
&= \sqrt{\frac{2\pi}{m^2}}
\end{aligned}
$$

Exercises!

1.
$$
\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2+J\phi}
$$

2.
$$
\int_{-\infty}^{\infty} d\phi_1 d\phi_2 e^{-\frac{1}{2}\vec{\phi^T}\cdot A\vec{\phi}}
$$

where A is any symmetric 2×2 matrix (can generalize to $n \times n$ matrices!).

2 February 2 -

2.1 Last time

- We computed the integral ($m^2 > 0$)

$$\mathcal{Z} = \int_{-\infty}^{\infty} d\phi e^{\phi m^2 \phi} = \sqrt{\frac{2\pi}{m^2}}$$

- We called this integral the ***partition function of our free theory*** (the "free" tells us that the action $S(\phi) = \frac{1}{2}\phi m^2 \phi$ is quadratic).

2.2 Today

- If we have a polynomial $\mathcal{O}(\phi)$, we want to calculate the expectation value

$$\langle \mathcal{O} \rangle := \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}\phi m^2 \phi} \mathcal{O}(\phi)$$

2.3 Schwinger-Dyson equation

Consider the following where $S(\phi)$ and $\mathcal{O}(\phi)$ are just some polunomails and $S(\phi) \to \infty$ as $\phi \to \pm\infty$

Theorem (Version 1)

$$\int_{-\infty}^{\infty} d\phi e^{-S(\phi)} \frac{d\mathcal{O}}{d\phi} = \int_{-\infty}^{\infty} d\phi e^{-S(\phi)} \frac{dS}{d\phi} \mathcal{O}$$

We can divide both integrals by $\mathcal{Z}$ to obtain the other form of the theorem:

Theorem (Version 2)

$$\left\langle \frac{d\mathcal{O}}{d\phi} \right\rangle = \left\langle \frac{dS}{d\phi} \mathcal{O} \right\rangle$$

Proof:

$$\int_{\infty}^{\infty} d\phi \frac{d}{d\phi} \left(e^{-S(\phi)} \mathcal{O}(\phi) \right) = \left[e^{-S(\phi)} \mathcal{O}(\phi) \right]_{-\infty}^{\infty} = 0$$

Now, why is this useful? Our whole goal today was to compute Expectation values but we haven't been calculating a whole lot of them.

Well, notice the following:

- $\langle 1 \rangle = 1$
- $\deg \frac{d\mathcal{O}}{d\phi}$ = $\deg \mathcal{O}$ - 1

- $\deg\left(\frac{dS}{d\phi}\cdot\mathcal{O}\right) = \deg S + \deg\mathcal{O} - 1$

s By proving the Schwniger-Dyson equation, we've gotten a relation between something of higher of degree and something of lower degree. So, we can go recursively until we reach a polynomial of degree 1, whose expectation value will be piss easy to calculate since $\langle 1\rangle = 1$.

Also, quick side note, $\frac{dS}{d\phi}$ give us the ***Lagrange Equations***, and so are called the ***Equations of motion!***

Note: We call anything that's a function of ϕ an operator. We'll relate this to the more familiar notion of an operator in Quantum Mechanics later.

2.4 Returning to our Free Theory

Now, returning to our free theory with action $S(\phi) = \frac{1}{2}\phi m^2\phi$ (Equations of motion $\frac{dS}{d\phi} = m^2\phi$). To caluclate the expectation value $\left\langle\phi^2\right\rangle$ we express it as

$$\begin{aligned}\left\langle\phi^2\right\rangle &= \frac{1}{m^2}\left\langle\underbrace{m^2\phi}_{dS/d\phi}\underbrace{\phi}_{\mathcal{O}}\right\rangle\\ &= \frac{1}{m^2}\left\langle\frac{dS}{d\phi}\mathcal{O}\right\rangle\end{aligned}$$

Then, applying the Schwinger-Dyson Equation, we get

$$\left\langle\phi^2\right\rangle = \underbrace{\frac{1}{m^2}}_{\langle\phi\phi\rangle}\left\langle\frac{d\phi}{d\phi}\right\rangle = \frac{1}{m^2}$$

This term $1/m^2$ is "contracted" from the $\langle\phi\phi\rangle$ term, and is called the ***propagator***.

2.5 ϕ^4 Free Theory

Let's now do something similar for ϕ^4.

$$\begin{aligned}\left\langle\phi^4\right\rangle &= \frac{1}{m^2}\left\langle m^2\phi\,\phi^3\right\rangle\\ &= \frac{1}{m^2}\left\langle\underbrace{\frac{d}{d\phi}(\phi^3)}_{\frac{d}{d\phi}(\phi\phi\phi)}\right\rangle \text{ (By Schwinger-Dyson)}\\ &= \frac{1}{m^2}\left\langle\frac{d\phi}{d\phi}\phi\phi\right\rangle + \frac{1}{m^2}\left\langle\phi\frac{d\phi}{d\phi}\phi\right\rangle + \frac{1}{m^2}\left\langle\phi\phi\frac{d\phi}{d\phi}\right\rangle\\ &= \langle\phi\phi\phi\rangle + \langle\phi\phi\phi\rangle + \langle\phi\phi\phi\rangle\\ &= \text{[Draw feynman diagrams]}\end{aligned}$$

[WATCH RECORDING AND ADD THE FEYNMAN DIAGRAM REPRESENTATIONS OF THESE TERMS – IMPORTANT

- Every ϕ is a dot with a line coming out of It

- When we contract two ϕ's we connect their lines

]

Exercises:

1. Show that
$$\langle \phi^4 \rangle = \frac{(n-1)!!}{m^n}$$
where the double exclaimation is the double-factorial.

2. Write the diagrams for $\langle \phi^6 \rangle$ in a few dfferent ways:

3. $$\begin{aligned} \langle \phi^6 \rangle &= \langle \phi\phi\phi\phi\phi\phi \rangle \\ &= \langle \phi^2 \phi^4 \rangle \\ &= \langle \phi^3 \phi^3 \rangle \\ &= \langle (\phi^6) \phi^0 \rangle \end{aligned}$$

4. Compute the partition function
$$\mathcal{Z} = \frac{1}{h} \int_0^L \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p \; e^{-\frac{-\beta p^2}{2m}}$$
explicitly (*Hint: Convert into the same form as we've solved in class using the substitution* $\phi = \frac{Lp}{h}$ *and then figure out what the action* $S(\phi)$ *should be.*)

5. Also Compute the free Energy F, where $e^{-\beta F} = \mathcal{Z}$.
   - Note that $F = E - TS, dF = -SdT - pdV$
   - $p = -\left(\frac{\partial F}{\partial V}\right)_\beta \implies$ Ideal Gas Law

3 February 9 - Wick's Theorem and Interacting Partition Functions

Recap

So far, we've

- Introduced the partition Function
$$\mathcal{Z} = \int_{\infty}^{\infty} \mathrm{d}\phi e^{-S(\phi)}$$
- Talked about the expectation values of operators
$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} \mathrm{d}\phi e^{-S(\phi)} \mathcal{O}(\phi)$$
also called correlation functions.
- Derived the ***Schwinger-Dyson Equation***
$$\left\langle \frac{dS}{d\phi} \mathcal{O} \right\rangle = \left\langle \frac{d\mathcal{O}}{d\phi} \right\rangle$$

Aside: The Schwinger-Dyson Equation tells us

$$\left\langle \frac{\delta S}{\delta \phi(x)} \mathcal{O} \right\rangle = \left\langle \frac{\delta cO}{\delta \phi(x)} \right\rangle$$

where $\frac{\delta S}{\delta \phi(x)}$ define the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

Suppose we have point x in space-time and an operator $\mathcal{O}$ which only depends on some other region which does *not* contain x (eg. $\mathcal{O}$ could just be the field at a point, in which case the value inside the other region wouldn't depend on x, or [insert other examples]).

[Insert image]

Then, the Schwinger-Dyson Equation tells us that

$$\begin{aligned} \left\langle \frac{\delta \mathcal{O}}{\delta \phi(x)} \right\rangle &= 0 \\ \implies \left\langle \frac{\delta S}{\delta \phi(x)} \mathcal{O} \right\rangle &= 0 \end{aligned}$$

i.e. the *classical* equations of motion are satisfied even in the quantum theory! (as long as there are no other operators acting on S which depend on $\phi(x)$ i.e. no other colliding operators).

Another Aside: The way that Symmetries work in QFT is a consequence of the Schwinger-Dyson equation. It extends Noether's Theorem from Classical Mechanics to the ***Ward-Takahashi Identities***.

It's also super useful in studying Non-perturbative theories! And induces the commutation relations! And is responsible for all joy and happiness in the world! The Schwinger-Dyson Equations are great.

[Write stuff related to Adarsh's Question and the ensuing discussion on the correlation function $\langle m\ddot{x}(t)x(0)\rangle$ for a free field theory with action given by $S = \int_{t_0}^{t_f} \mathrm{d}t \left(\frac{L}{2}m\dot{x}^2\right)$ – Couldn't hear well from recording]

- For a free theory, the action is very specific - it has to be quadratic in ϕ. For example, $S = \frac{1}{2}\phi m^2 \phi$ Then, the Equations of Motion are

  $$\frac{dS}{dt} = m^2\phi$$

  and the Schwinger-Dyson equation says

  $$\left\langle m^2\phi\ \mathcal{O}\right\rangle = \left\langle \frac{d\mathcal{O}}{dt} \right\rangle$$

  or equivalently that

  $$\langle \phi\ \mathcal{O}\rangle = \frac{1}{m^2}\left\langle \frac{d\mathcal{O}}{dt} \right\rangle$$

  So basically, if we have ϕ in an expectation value, we can kill it but the price we pay is that we have to replace it with a derivative (w.r.t. ϕ) and the propagator $\frac{1}{m^2}$.

- Last time, we calculated

  $$\langle \phi\phi\rangle = \frac{1}{m^2}\left\langle \frac{d}{d\phi}\phi \right\rangle = \frac{1}{m^2}$$

  This "contraction" is denoted as [include image]

  We can think of each of the ϕ's as being half-edges and the contraction operation as joining the two half-edges to get a single edge – which has a *cost* of $\frac{1}{m^2}$.

  We can also think of the above as $\left\langle \phi^2 \right\rangle$ which has slightly different notation. It is represented by two half-edges coming out of the *same* vertex. When we contract them, we get a *closed loop* rather than a single edge.

Okay, let's calculate $\langle\phi\phi\phi\phi\rangle$ now.

$$\begin{aligned}
\langle\phi\phi\phi\phi\rangle &= \frac{1}{m^2}\left\langle \frac{d}{d\phi}(\phi\phi\phi) \right\rangle \\
&= \frac{1}{m^2}\left\langle \cdot\ \ \frac{d\phi}{d\phi}\phi\phi \right\rangle + \frac{1}{m^2}\left\langle \cdot\ \ \phi\frac{d\phi}{d\phi}\phi \right\rangle + \frac{1}{m^2}\left\langle \cdot\ \ \phi\phi\frac{d\phi}{d\phi} \right\rangle
\end{aligned}$$

where the $\cdot$ just represents the ϕ which we killed off and replaced by the $\frac{1}{m^2}$ and derivative. Again, we can write this in Contraction Notation and Diagrammatic Notation as :

Insert figures.

Now, we can apply the Schwinger-Dyson equation *again!* i.e. contract again. In terms of contraction notation and diagrammation notation, this would be :

Insert figures.
and each of these gives us $\frac{1}{m^4}$, so in total we have

$$\langle\phi\phi\phi\phi\rangle = \frac{3}{m^4}$$

If we thought of this as $\left\langle \phi^4 \right\rangle$ instead then we would end up with the diagrammatic representation :

Insert figure.

What we just did is apply ***Wick's Lemma***, which can be expressed in many ways – such as :

- $$\boxed{\langle \phi \mathcal{O} \rangle = \frac{1}{m^2} \left\langle \frac{d\mathcal{O}}{d\phi} \right\rangle}$$

- $$\langle \phi\phi\phi \cdots \phi \rangle = \sum_{\text{all contractions}} (\cdots)$$

- $$\langle \mathcal{O} \rangle = e^{\frac{1}{2}\frac{1}{m^2}\frac{d^2}{d\phi^2}} \mathcal{O}\Big|_{\phi=0}$$

All of these are equivalent and contain the same information as the Schwinger-Dyson Equation for a free field theory.

What's the difference between Wick's Lemma and the Scwinger-Dyson Equation then? Ask Ivan.

How do we interpret the third form of Wick's Lemma? Let's do an example. Consider the operator $\mathcal{O} = \phi^2$. Then,

$$e^{\frac{1}{2}\frac{1}{m^2}\frac{d^2}{d\phi^2}} \phi^2\Big|_{\phi=0} = \left[1 + \frac{1}{2}\frac{1}{m^2}\frac{d^2}{d\phi^2} + \frac{1}{2!}\left(\frac{1}{2}\frac{1}{m^2}\frac{d^2}{d\phi^2}\right)^2 + \cdots\right] \phi^2\Big|_{\phi=0}$$

- But only the first two terms survive since all terms after that contain more than two derivatives and kill off the ϕ^2.
- The first term also gives zero because it gives us $1(\phi^2) = \phi^2$ but we are evaluating this at $\phi = 0$. So, we end up with
$$\langle \mathcal{O} \rangle = \frac{1}{m^2}$$
- We don't want to differentiate too many times, but if we don't differentiate enough, the terms are killed off. This perfectly represents the contractions: Insert image

  The factor of $\frac{1}{2}$ cancels out the symmetry in contractions (we can contract in two different ways: the first onto the second *or* the second onto the first).

Exercise: Verify $\langle \phi\phi\phi\phi \rangle = \frac{3}{m^4}$ using the third version of Wick's Lemma.

3.1 Aside: Is there any topological significance to the shapes of the terms in our Feynman Diagrams?

No; not in the theory we are dealing with. If we're dealing with matrix models, however, we need to put in a little more theory.

If our fields ϕ are matrices, then we need to *thicken up* our feynman diagrams. For example, the $\langle\phi^4\rangle$ diagrams would be

[Insert image]

Note that the first two diagrams can be drawn on a sphere while the third cannot – it must be drawn on a torus. The diagrams now have *genus*, and they are organized by genus. The terms in the expansion will have some *genus coefficient*, and the higher the genus coefficient of a diagram, the higher its genus will be. This is how diagrams in string theory are organized.

An example such a theory is *Yang-Mills Theory* which is the theory underlying Quantum Chromodynamics. QCD actually only has 3 colors but something people do to get a good approximation while making calculations easier is to pretend it has N colors and take the $N \to \infty$ limit. In this limit, the higher genus terms are killed off and only *planar diagram* terms remain.

Write about the O(3) model stuff Ivan was mentioning.

3.2 Interacting Theories

Let's now move on from *free theories* to *interacting theories.* Suppose we have an interacting $\phi 4$ theory whose action is

$$S = \frac{1}{2}\phi m^2\phi - \frac{\lambda}{4!}\phi^4$$

where $S_0 = \frac{1}{2}\phi m^2\phi$ is the *free action.*

Let's try to calculate the quotient

$$\frac{\mathcal{Z}}{\mathcal{Z}_0}$$

where

$$\mathcal{Z} = \int_{-\infty}^{\infty} \mathrm{d}\phi e^{-S(\phi)}$$

is the partition function of our interacting theory and

$$\mathcal{Z}_0 = \int_{-\infty}^{\infty} \mathrm{d}\phi e^{-S_0(\phi)} = \sqrt{\frac{2\pi}{m^2}}$$

is the free partition function.

Then, this ratio is given by

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{1}{\mathcal{Z}_0}\int_{-\infty}^{\infty} \mathrm{d}\phi e^{-\frac{1}{2}\phi m^2\phi} e^{\frac{\lambda}{4!}\phi^4} = \left\langle e^{\frac{\lambda}{4!}\phi^4}\right\rangle_0$$

i.e. it is equal to the expectation value of $e^{\frac{\lambda}{4!}\phi^4}$ in the free theory.

This is why we study the ratio! We wanted to study $\mathcal{Z}$, but we already know $\mathcal{Z}_0$ and the ratio looks just like an expectation value in the free theory, so we are able to express $\mathcal{Z}$ immediately as

$$\mathcal{Z} = \mathcal{Z}_0 \times \left\langle e^{\frac{\lambda}{4!}\phi^4}\right\rangle_0$$

We can then expand the exponential in the expectation value and use linearity to write

$$\mathcal{Z} = \mathcal{Z}_0 \times \left[\frac{\langle 1\rangle_0}{1} + \frac{\lambda}{4!}\left\langle\phi^4\right\rangle_0 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2\left\langle\phi^4\phi^4\right\rangle_0 + \cdots\right]$$

This is Perturbation Theory! Let's write this in diagrams now.

Insert Image

The three diagrams here are really the same, so we can write the total contribution from the $\frac{\lambda}{4!}\left\langle\phi^4\right\rangle_0$ term as $\lambda \times \frac{1}{8} \times (\infty)$ where ∞ is not infty, but the diagram. The claim is that we could have *guessed* the number 8. How so? (Brace yourself - this is gonna be cool as HELL)

8 is the ***number of symmetries of the diagram!***

Doing this for the other terms too, we have...

[Insert Diagrams]

Wait wait wait... how are we defining symmetries of diagrams? Consider the diagram below.

Insert figure.

It has a vertix in the center and then four *half-edges* – rememeber, these four half-edges are the four fields we have contracted together.

A *symmetry* of the diagram is a mapping of the half-edges into themselves such that

1. Whenever two half-edges belong to the same vertex *before mapping*, they must still belong to the same vertex after.
2. Whenever two half-edges form an edge before mapping, they must still form an edge after mapping.

A cool consequence of this is that a mathematician trying to compute the number of symmetries of some graph might just be able to get the job done by hiring their physicist friend to compute the correlation function of some appropriate field theory! Because, at the end of the day,

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{\substack{\text{All diagrams } \Gamma \\ \text{with 4 valent vertices}}} \frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|}$$

where $w(\Gamma)$ is a weightage, given by the Feynman Rules, equal to the propagator.

Exercise: Compute

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \left\langle e^{\frac{g}{3!}\phi^3}\right\rangle_0 = \cdots + \mathcal{O}(\phi^6)$$

i.e. compute the perturbations up to the 6th order in ϕ, using the third version of Wick's Lemma