Physics 137B Homework 8

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Question 1: Fermi's Golden Rule for Three Body Decays

(a) **Calculating the total rate:** Using the set up laid out above, we can integrate to get an expression for the total decay.

(b) **The lifetime of the neutron:** Assuming that $\mathcal{M} \sim 1$, calculate the lifetime of the neutron. Comment on the order of magnitude compared to the experimental value, which is $\tau \sim 887.7$s.

(c) **The lifetime of the muon:** Do the same with the muon, whose experimentaly measured lifetime is $\tau \sim 2.197 \times 10^{-6}$s.

Solution:

(a) The inital particle (neutron) is at rest, so by concservation of momentum we have

$$0 = \mathbf{P_f} + \mathbf{p_1} + \mathbf{p_2}$$

where P_f, p_1, p_2 denote the momenta of the proton, electron, and anti-neutrino after decay.

The protons is effectively static and we ignore its momentum, whereas we treat the proton and anti-neutrino relativistically. For simplicity, we asssume that $m_{\overline{\nu_e}} = 0$ since the mass of the anti-neutrino is much smaller than even that of the electron.

Now, to get the Decay Rate, we can integrate over the momenta of the electron and anti-neutrino:

$$W = \frac{2\pi}{\hbar} \left| \frac{G_F \mathcal{M}}{V} \right|^2 \int \frac{V d^3\mathbf{p_1}}{(2\pi\hbar)^3} \frac{V d^3\mathbf{p_2}}{(2\pi\hbar)^3} \delta\left(E_0 - E_1 - E_2\right)$$

Note that $d^3\mathbf{p_1} \to 4\pi p_1^2 \mathrm{d}p_1$ and similarly $d^3\mathbf{p_2} \to 4\pi p_2^2 \mathrm{d}p_2$. And since we're assuming $m_{\overline{\nu}} = 0$,

$$\begin{aligned}
& E_2^2 = (p_2 c)^2 + (\underbrace{m}_{=0} c^2)^4 \\
\implies & E_2^2 = c^2 p_2^2 \\
\implies & 2E_2 dE_2 = 2c^2 p_2 dp_2 \\
\implies & E_2 dE_2 = c^2 p_2 dp_2 \\
\implies & p_2 dp_2 = \frac{E_2}{c^2} dE_2 \\
\implies & p_2^2 dp_2 = \frac{E_2 p_2}{c^2} dE_2 = \frac{E_2 \cdot \left(\frac{E_2}{c}\right)}{c^2} dE_2 \\
\implies & p_2^2 \mathrm{d}p_2 = \frac{(E_2)^2}{c^3} \mathrm{d}E_2
\end{aligned}$$

Now,

$$\begin{aligned}
W &= \frac{2\pi}{\hbar}\left|\frac{G_F\mathcal{M}}{V}\right|^2 \iint \frac{V d^3\mathbf{p_1}}{(2\pi\hbar)^3}\frac{V d^3\mathbf{p_2}}{(2\pi\hbar)^3}\delta\left(E_0-E_1-E_2\right)\\
&= \frac{2\pi}{\hbar}\frac{|G_F\mathcal{M}|^2}{(2\pi\hbar)^6}\iint d^3\mathbf{p_1}d^3\mathbf{p_2}\delta\left(E_0-E_1-E_2\right)\\
&= \frac{2\pi}{\hbar}\frac{|G_F\mathcal{M}|^2}{(2\pi\hbar)^6}\iint 4\pi p_1^2\mathrm{d}p_1\cdot 4\pi p_2^2\mathrm{d}p_2\delta\left(E_0-E_1-E_2\right)\\
&= \frac{2\pi\cdot(4\pi)^2}{\hbar}\frac{|G_F\mathcal{M}|^2}{(2\pi\hbar)^6}\iint p_1^2\mathrm{d}p_1\cdot p_2^2\mathrm{d}p_2\ \ \delta\left(E_0-E_1-E_2\right)\\
&= \frac{|G_F\mathcal{M}|^2}{2\pi^3\hbar^7}\iint p_1^2\mathrm{d}p_1\cdot\frac{(E_2)^2}{c^3}\mathrm{d}E_2\ \ \delta\left(E_0-E_1-E_2\right)
\end{aligned}$$

The $\delta(E_0-E_1-E_2)$ factor selects only the contribution in which $E_0-E_1-E_2=0 \iff E_0 = E_1+E_2$ so we get

$$\begin{aligned}
W &= \frac{|G_F\mathcal{M}|^2}{2\pi^3\hbar^7}\int p_1^2\frac{(E_2)^2}{c^3}\mathrm{d}p_1\\
&= \frac{|G_F\mathcal{M}|^2}{2\pi^3\hbar^7c^3}\int p_1^2(E_0-E_1)^2\mathrm{d}p_1
\end{aligned}$$

In the relativistic limit ($E_1\approx p_1c$), we have

$$\begin{aligned}
\int_0^{p_1^{max}} p_1^2(E_0-E_1)^2\mathrm{d}p_1 &= \int_0^{p_1^{max}} p_1^2(E_0-p_1c)^2\mathrm{d}p_1\\
&= \int_0^{p_1^{max}} p_1^2(E_0^2+p_1^2c^2-2E_0p_1c)^2\mathrm{d}p_1\\
&= \int 0^{p_1^{max}} E_0^2p_1^2+p_1^4c^2-2E_0p_1^3c\mathrm{d}p_1\\
&= \left[\frac{E_0^2\left(p_1^{max}\right)^3}{3}+\frac{\left(p_1^{max}\right)^5c^2}{5}-2E_0\frac{\left(p_1^{max}\right)^4c}{4}\right]
\end{aligned}$$

And recall that $p_1^{max}=\frac{E_0}{c}$. Thus,

$$\int_0^{p_1^{max}} p_1^2(E_0-E_1)^2\mathrm{d}p_1 = \frac{E_0^5}{30c^3}$$

so,

$$\boxed{W = \frac{|G_F\mathcal{M}|^2}{60\pi^3\hbar^7c^6}\cdot E_0^5}$$

or, in terms of the neutron and proton masses,

$$\begin{aligned}
W &= \frac{|G_F\mathcal{M}|^2}{60\pi^3\hbar^7c^6}\cdot\left[c^2\cdot(m_n-n_p)\right]^5\\
\implies &\boxed{W = \frac{|G_F\mathcal{M}|^2}{60\pi^3\hbar^7c^4}\cdot(m_n-n_p)^5}
\end{aligned}$$

(b) Assuming $\mathcal{M} \sim 1$, the lifetime of a neutron, τ, is given by

$$\tau = \frac{1}{W} = \frac{60\pi^3\hbar^7 c^6}{|G_F \mathcal{M}|^2} \cdot \frac{1}{(E_0)^5} = \frac{60\pi^3\hbar^7 c^4}{|G_F \mathcal{M}|^2} \cdot \frac{1}{(m_n - m_p)^5} \sim 2500s = 2.5 \times 10^3 s$$

The experimentally measured value is $\tau \sim 887.7s$, so our estimate is nearly on the same order of magnitude. It's still a very crude approximation because we're off by a factor of about 3.

(c) One way in which the muon decays is

$$\mu \to e + \overline{\nu}_e + \nu_\mu$$

This decay follows mechanics similar to the beta decay we studied earlier, except in this case the light particles are $\overline{\nu}_e, \nu_\mu$ while the heavy particles are μ, e. Thus in this case $E_0 = c^2 (m_\mu - m_e)$. Then, using the formula found earlier, we calculate the lifetime of the muon to be

$$\begin{aligned} \tau &= \frac{60\pi^3\hbar^7 c^4}{|G_F \mathcal{M}|^2} \cdot \frac{1}{(m_\mu - m_e)^5} \\ &\sim 6.5 \times 10^{-7} s \end{aligned}$$

This is one order of magnitude off from the experimental value of $\tau \approx 2.197 \times 10^{-6} s$.

Question 2: Sudden Approximation

A particle of mass m is in the ground state of a harmonic oscillator with the standard Hamiltonian $\hat{H}_{HO} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$. At time $t = 0$, the value of k changes suddenly to $k' = 4k$. Find the probability that the oscillator remains in its ground state.

Solution:

$$\begin{aligned} &\underline{\text{Before time } t = 0:}\ \hat{H}|\psi_n\rangle = \hbar\omega \left(n + \frac{1}{2}\right) |\psi_n\rangle \text{ with } \hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} \\ &\underline{\text{After time } t = 0:}\ \hat{H}'|\phi_{n'}\rangle = \hbar\omega' \left(n' + \frac{1}{2}\right) |\phi_{n'}\rangle \text{ with } \hat{H} = \frac{\hat{p}^2}{2m} + \frac{4k\hat{x}^2}{2} \end{aligned}$$

where ϕ_n is the n^{th} state of the new harmonic oscillator with spring constant $4k$.

The particle starts off in the ground state, and the probability that we find it *still* in the ground state after the sudden change is

$$\begin{aligned} P_{(n'=0)} &= |\langle \phi_n(t) | \psi(t) \rangle|^2 \\ &= \left| d_{(n'=0)} \right|^2 \end{aligned}$$

In the position basis,

$$\begin{aligned} \psi_0(x) &= \langle x | \psi_0 \rangle \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \\ &= \left(\frac{m}{\pi\hbar}\right)^{1/4} \left(\frac{k}{m}\right)^{1/8} e^{-x^2\sqrt{km}/2\hbar} \end{aligned}$$

and

$$
\begin{aligned}
\phi_0(x) &= \langle x|\phi_0\rangle \\
&= \left(\frac{m\omega'}{\pi\hbar}\right)^{1/4} e^{-m\omega' x^2/2\hbar} \\
&= \left(\frac{m}{\pi\hbar}\right)^{1/4} \left(\frac{4k}{m}\right)^{1/8} e^{-x^2\sqrt{4km}/2\hbar}
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_0|\psi_0\rangle &= \int_{-\infty}^{\infty} dx \langle\phi_0|x\rangle\langle x|\psi_0\rangle \\
&= \int_{-\infty}^{\infty} dx \phi_0^*(x)\psi_0(x) \\
&= \left(\frac{m}{\pi\hbar}\right)^{1/4} \left(\frac{m}{\pi\hbar}\right)^{1/4} \left(\frac{k}{m}\right)^{1/8} \left(\frac{4k}{m}\right)^{1/8} \int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2\sqrt{km}}{2\hbar}\right) \exp\left(-\frac{x^2\sqrt{4km}}{2\hbar}\right) \\
&= \left(\frac{m}{\pi\hbar}\right)^{1/2} \cdot \underbrace{\left(\frac{k}{m}\right)^{1/4}}_{\sqrt{\omega}} \cdot (4)^{1/8} \int_{-\infty}^{\infty} dx\, e^{-\frac{3x^2\sqrt{km}}{2\hbar}} \\
&= 2^{1/4} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt[root]{\frac{2}{2}\frac{\pi\hbar}{m\omega}} \\
&= 2^{1/4} \left(\frac{2}{3}\right)^{1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{-1/2} \\
&= \frac{2^{3/4}}{3^{1/2}}
\end{aligned}
$$

So,

$$
\begin{aligned}
d_{(n'=0)} &= |\langle\phi_0|\psi_0\rangle|^2 \\
&= \frac{2^{3/2}}{3} \\
&= 2\frac{\sqrt{2}}{3}
\end{aligned}
$$

Question 3: More Sudden Approximation

If the atom is initially in the groun state, what is the probability that the $^3He^+$ ion remains in the ground state after the transition?

Solution:

The effect of the nuclear decay is to change the nuclear charge at $t = 0$ without affecting the orbital electrons. We're interested in the probability that the ion remains in the ground state after the transition.

$$
\begin{aligned}
&\underline{\text{Before time } t = 0 :}\hat{H}|\psi_{nlm}\rangle = -\frac{E_1}{n^2}|\psi_{nlm}\rangle \\
&\underline{\text{After time } t = 0 :}\hat{H}'|\phi_{nlm}\rangle = -\frac{2^2 E_1}{n^2}|\phi_{nlm}\rangle
\end{aligned}
$$

where $\hat{H}$ is the hydrogen atom hamiltonian wherein the nuclear charge is $Z = 1$ and $\hat{H}'$ is the same hamiltonian modified with $Z = 2$ in this case.

The ground state solution for the a Hydrogenic Hamiltonian is

$$\psi_{100}(r) = \left(\frac{2}{\sqrt{4\pi a_0^3}}\right) e^{-r/a_0}$$

with the Bohr radius being

$$a_0 = \frac{\hbar}{mZe^2}$$

So, the Bohr radii for Tritium is

$$a_0^t = \frac{\hbar}{me^2}$$

and since the Helium ion has $Z = 2$, its bohr radius is

$$a_0^+ = \frac{\hbar}{2me^2} = \frac{a_0^t}{2}$$

the probability that the ion remains in the ground state after the sudden change is given by $|\langle\phi_{100}|\psi_{100}\rangle|^2$.

$$\begin{aligned}
\langle\phi_{100}|\psi_{100}\rangle &= \iiint \mathrm{d}^3\vec{r}\phi^*_{100}(r)\psi_{100}(r) \\
&= 4\pi \int_0^\infty r^2 \mathrm{d}r\ \phi^*_{100}(r)\psi_{100}(r) \\
&= 4\pi \int_0^\infty r^2 \mathrm{d}r\ \left(\frac{2}{\sqrt{4\pi(a_0^t/2)^3}} e^{-2r/a_0^t}\right)^* \left(\frac{2}{\sqrt{4\pi(a_0^t)^3}} e^{-r/a_0^t}\right) \\
&= 4\pi \int_0^\infty r^2 \mathrm{d}r \left(\frac{4}{4\pi} \cdot \frac{2^{3/2}}{(a_0^t)^3}\right) e^{-3r/a_0^t} \\
&= 4\pi \frac{\sqrt{8}}{(a_0^t)^3\pi} \int_0^\infty r^2 \mathrm{d}r\ e^{-3r/a_0^t}
\end{aligned}$$

Let $y = r/a_0^t$. Then,

$$\begin{aligned}
\langle\phi_{100}|\psi_{100}\rangle &= \frac{4\sqrt{8}}{(a_0^t)^3} \int_0^\infty (a_0^t y)^2 (a_0^t \mathrm{d}y) e^{-3y} \\
&= 4\sqrt{8} \int_0^\infty \mathrm{d}y\ y^2 e^{-3y} \\
&= 4\sqrt{8} \cdot \left(\frac{2}{27}\right) \text{ Using Integral calculator} \\
&= \frac{8\sqrt{8}}{27} \\
&= \frac{16\sqrt{2}}{27} \\
&= 0.838
\end{aligned}$$

Thus the probability that we find the ion still in the ground state is $|\langle\phi_{100}|\psi_{100}\rangle|^2 = (0.838)^2 \approx 0.703$
