Math H185 Lecture 9

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 February 9 -

Exercise

Show that
$$\int_0^\infty \frac{1-\cos(x)}{x^2}dx = \frac{\pi}{2}$$

Recall Euler's theorem

$$\begin{aligned}
e^{iz} &= \cos(z) + i\sin(z) \\
\implies e^{-iz} &= \cos(-z) - i\sin(z) \\
\implies \cos(z) &= \frac{e^{iz}+e^{-iz}}{2}
\end{aligned}$$

and the series expansion of cosine is

$$\cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$

Now,

$$\begin{aligned}
\int 0^\infty \frac{1-\cos(x)}{x^2}dx &= \int_0^\infty \frac{1-\frac{(e^{iz}+e^{-iz})}{2}}{x^2}dx \\
&= \frac{1}{2}\int_0^\infty \frac{1-e^{iz}}{x^2}dz + \frac{1}{2}\int_0^\infty \frac{1-e^{-iz}}{x^2}dz
\end{aligned}$$

By u-sub in the second integral we find that this is equal to

$$\begin{aligned}
&= \frac{1}{2}\int_0^\infty \frac{1-e^{iz}}{x^2}dz + \frac{1}{2}\int_{-\infty}^0 \frac{1-e^{iz}}{x^2}dz \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1-e^{iz}}{x^2}dz
\end{aligned}$$

If we consider the following contour C in the complex plane, [insert image], and take the integral of the function $f(z) = \frac{1-e^{iz}}{z^2}$ then by Cauchy's theorem we have

$$\int_C f(z)dz = 0$$

However, we can split the integral over the entire contour into integrals over different parts of the contour, and notice that the integral on line superimposed with the real line is exactly the integral we need.

<u>Claim:</u> The integral over the semi-cicle of radius R in the complex plane goes to zero as $R \to \infty$.

$$\int_{\gamma_2} \frac{1-e^{iz}}{2}dz \xrightarrow{R\to\infty} 0$$

<u>Proof:</u>

$$
\begin{aligned}
\left|\int_{\gamma_2} \frac{1-e^{iz}}{z^2} dz\right| &\leq \int_{\gamma_2} \left|\frac{1-e^{iz}}{z^2}\right| d\left|z\right| \\
&= \int_{\gamma_2} \frac{\left|1-e^{iz}\right|}{\underbrace{\left|z^2\right|}_{R^2}} d\left|z\right| \\
&\leq \frac{2}{R^2} \times \pi R \\
&\xrightarrow{R\to\infty} 0
\end{aligned}
$$

where for the last inequality we are that

$$
\begin{aligned}
\left|1-e^{iz}\right| \leq \left|1\right| + \left|e^{iz}\right| \\
\leq 1+1=2
\end{aligned}
$$

since

$$
\begin{aligned}
z = x+iy \implies \left|e^{iz}\right| &= \left|e^{x}e^{iy}\right| \\
&\leq 1 \text{ if } y \geq 0
\end{aligned}
$$

This is why we chose the circle in the upper half plane rather than the lower half.

For the integral over γ_4, we can use the taylor expansion. Since the contour lies extremely close to zero, the later terms in the series expansion are bounded near $z = 0$.

Writing the expansion,

$$
\begin{aligned}
f(z) &= \frac{1-e^{iz}}{z^2} \\
&= \frac{1-(1+iz+\frac{1}{2!}(iz)^2+\cdots)}{z^2} \\
&= \frac{-i}{z} + \underbrace{Otherterms}_{boundednearz=0}
\end{aligned}
$$

So, we can get the integral over γ_4 as

[Complete the rest of the proof over weekend when recording comes out]

1.1 Jordan's Lemma:

Jordan's Lemma states that if we have a function of the form $f(z) = e^{aiz}g(z)$, $a \in \mathbb{R}_{\geq 0}$ and we integrate over a semi-circular contour of radius R in the upper plane, then

$$
\int_{C_R} f(z)dz \leq \left(\sup_{C_R} \left|g(z)\right|\right) \cdot \frac{\pi}{a}
$$

Why is this lemma useful? If we use our naive approach, as we did in the earlier integral, then

$$
\left|e^{iaz}\right| \left|g(z)\right| \leq 1 \cdot \left|g(z)\right|
$$

But this is a *super lossy estimation* because the exponential dies off really quickly and is only close to 1 in magnitude in a small region. Jordan's Lemma give us a much tighter bound.

[Complete the proof from recording]