Physics 137B Lecture 3

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These are notes taken from lectures on Quantum Mechanics delivered by Professor Raúl A. Briceño for UC Berekley's Physics 137B class in the Sprng 2024 semester.

Contents

1 January 22 - More about symmetries

1.1 Last time

- Symmetries are transformations which leave a system invariant.
- Can be continuous or discrete.
- Active: $|\psi\rangle \to |\psi'\rangle = \hat{U}|\psi\rangle$ Passive: $\hat{\Theta} \to \hat{\theta}' = \hat{U}^\dagger \hat{\Theta} \hat{U}$
- Unitary operators preserve norm:

$$\begin{aligned} &\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle \\ \implies &\hat{U}^\dagger \hat{U} = 1 = \hat{U}\hat{U}^\dagger \end{aligned}$$

- If $\hat{U}$ transformations are a symmetry, $\langle\hat{U}\rangle$ is a conserved quantity since

$$\frac{d}{dt}\langle\hat{U}\rangle = \frac{i}{\hbar}\langle \underbrace{\left[\hat{U}, \hat{H}\right]}_{=0,\text{symmetry}} \rangle + \langle \cancelto{0}{\frac{\partial \hat{U}}{\partial t}} \rangle \text{ Ehrenfest's Theorem}$$

$$\implies \boxed{\frac{d}{dt}\langle U\rangle = 0}$$

2 Parity $\hat{\Pi}$

This operator "flips" the coordinate system i.e.

$$\hat{\Pi} \underbrace{|\vec{r}\rangle}_{|x\rangle\otimes|y\rangle\otimes|z\rangle} = |-\vec{r}\rangle = |-x\rangle \otimes |-y\rangle \otimes |-z\rangle$$

Note that

$$\hat{\Pi}^2|\vec{r}\rangle = \hat{\Pi}|-\vec{r}\rangle = |\vec{r}\rangle$$

Thus, $\hat{\Pi}^2 = \mathbf{1} \implies \boxed{\hat{\Pi}^{-1} = \hat{\Pi}}$ This tells us that the **eigenvalues of** $|\hat{pi}$ **are** $\lambda_\pm = \pm 1$

Proof: Suppose

$$\hat{\Pi}|n\rangle = \lambda_n|n\rangle$$

Then

$$\begin{aligned} &\hat{\Pi}^2|n\rangle = \lambda_n\hat{\Pi}|n\rangle - \lambda_n^2|n\rangle \\ \implies &\lambda_n^2 = 1 \\ \implies &\lambda_n = \pm 1 \end{aligned}$$

Parity states can only have positive or negatve parity.

Claim: $\hat{\Pi}$ is Hermitian. i.e. $\hat{\Pi}^\dagger = \hat{\Pi}$.

Proof: Consider some arbitrary position states $|f\rangle, |g\rangle$ and

$$\begin{aligned}
\langle f|\hat{\Pi}|g\rangle &= \int d\vec{r}\langle f|\hat{\Pi}|\vec{r}\rangle\langle\vec{r}|g\rangle \\
&= \int d\vec{r}\langle f|| - \vec{r}\rangle g(\vec{r}) \\
&= \int_{\infty}^{\infty} d\vec{r} f^*(-\vec{r'})g(\vec{r'}) \\
&= -\int_{\infty}^{-\infty} d\vec{r'} f^*(\vec{r'})g(-\vec{r'}) \text{ (Change of variables } \vec{r'} = -\vec{r}) \\
&= \int_{-\infty}^{\infty} d\vec{r'} f^*(\vec{r'})g(-\vec{r'}) \text{ (Change of variables } \vec{r'} = -\vec{r}) \\
&= \int d\vec{r'}\langle f|\vec{r'}\rangle\langle -\vec{r'}|g\rangle \\
&= \int d\vec{r'}\langle f|\vec{r'}\rangle\langle\vec{r'}|\hat{\Pi}^\dagger|g\rangle \\
&= \langle f|\hat{\Pi}^\dagger|g\rangle
\end{aligned}$$

So, we have

$$\boxed{\hat{\Pi} = \hat{\Pi}^\dagger}$$

Symmetries make our calculations very simple.

Example: Let $\hat{\Theta}$ be odd under parity i.e.

$$\hat{\Pi}\hat{\Theta} = -\hat{\Theta}$$

Then, $\langle n|\hat{\Theta}|n\rangle = 0$.

Proof:

$$\begin{aligned}
\langle n|\hat{\Theta}|n\rangle &= \langle n|\hat{\Pi}^\dagger\hat{\Theta}\hat{\Pi}|n\rangle \\
&= \left(\langle n|\hat{\Pi}\right)\left(\hat{\Pi}\hat{\Theta}\hat{\Pi}\ \right)\left(\hat{\Pi}|n\rangle\right) \\
&= -\lambda_n^2\langle n|\hat{\Theta}|n\rangle \\
&= \langle n|\hat{\Theta}|n\rangle \\
&= 0
\end{aligned}$$

Vectors like Position and Momentum have odd parity, while pseudoscalars like the dot product and pseudovectors like angular momentum have positive parity.

- $\hat{\Pi}\vec{r}\hat{\Pi} = -\vec{r}$
- $\hat{\Pi}\vec{p}\hat{\Pi} = -\vec{p}$
- $\hat{\Pi}\vec{r} \cdot \vec{r}\hat{\Pi} = |\vec{r}|^2$

- $$\begin{aligned}\hat{\Pi}\vec{L}\hat{\Pi} &= \hat{\Pi}\vec{r}\times\vec{p}\hat{\Pi}\\ &= (-1)^2\vec{r}\times\vec{p}\\ &= \vec{L}\end{aligned}$$

- $\hat{\Pi}\vec{S}\hat{\Pi} = \vec{S}$

2.1 Continuous Transformations

One Example of a continuous transformations is **translation in 1D** whose operator is denoted as $\hat{\mathcal{T}}$.

[Insert Figure]

The Translation operator is defined by the action

$$\hat{T}(a)\psi(x) = \psi'(x) = \psi(x-a)$$

Taylor expanding the last expression, we have

$$\begin{aligned}\hat{T}(a)\psi(x) &= \psi(x-a)\\ &= \psi(x) - a\frac{d}{dx}\psi(x) + \cdots\end{aligned}$$

We can relate this to the momentum operator since

$$\hat{P} = -i\hbar\frac{d}{dx} \iff \frac{d}{dx} = \frac{i}{\hbar}\hat{P}$$

Which gives us

$$\begin{aligned}\hat{T}(a)\psi(x) &\approx \left(\mathbf{1} - \frac{ia\hat{P}}{\hbar} + \cdots\right)\psi(x)\\ \implies \hat{T}(a) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-ia\hat{P}}{\hbar}\right)^2\\ &= \exp\left(\frac{-ia\hat{P}}{\hbar}\right)\end{aligned}$$

One thing we can deduce from this is that the **Translation operator is Unitary**.

$$\begin{aligned}(\hat{T}(a))^\dagger(\hat{T}(a)) &= \exp\left(\frac{+ia\hat{P}}{\hbar}\right)\exp\left(\frac{-ia\hat{P}}{\hbar}\right)\\ &= \exp\left(\frac{ia\hat{P - P}}{\hbar}\right) \quad \text{(This is valid because } \hat{P} \text{ commutes with itself)}\\ &= \mathbf{1}\end{aligned}$$

2.2 Momentum conservation

If $\hat{T}(a)$ is a symmetry of the system we have

$$[\underbrace{\hat{T}(a)}_{e^{\frac{-ia\hat{P}}{\hbar}}}, \hat{H}] = 0$$

$$\begin{aligned} &\implies \left[\exp\left(-\frac{ia\hat{P}}{\hbar}\right), \hat{H}\right] = 0 \\ &\implies \left[\hat{P}, \hat{H}\right] = 0 \end{aligned}$$

But then, Ehrenfest's Theorem tells us that $\langle\hat{P}\rangle = 0$ as

$$\frac{d}{dt}\langle\hat{P}\rangle = \frac{i}{\hbar}\langle\left[\hat{P}, \hat{H}\right]\rangle 0$$

So, the value of momentum measured is conserved.
Momentum conservation is the result of Translation Symmetry.

2.3 Time Translation ($\hat{U}(\Delta)$)

$$\begin{aligned} \hat{U}(\Delta)\psi(t) &= \psi(t-\Delta) \\ &= \left(1 - \Delta\frac{d}{dt} + \cdots\right)\psi(t) \\ &= \left(1 + \frac{i\Delta\cdot\hat{H}}{\hbar}\cdots\right)\psi(t) \\ &= \exp\left(\frac{i\Delta\cdot\hat{H}}{\hbar}\right)\psi(t) \end{aligned}$$

where in the third equality, we used that

$$\hat{H} = i\hbar\frac{d}{dt} \implies \frac{d}{dt} = -i\frac{\hat{H}}{\hbar}$$

If Time Translation is a symmetry, then this is equivalent to saying that the energy of the system is conserved. **So, Energy Conservation is a result of Time Translation.**