Physics 137B Homework 1

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Q1. Rotations about the z-axis:

(a) Let $\hat{R}_z(\delta)$ be the operator that rotates a function about the $z-$axis by an angle δ. It is defined by

$$\hat{R}_z(\delta)\psi(r,\theta,\phi) = \psi'(r,\theta,\phi) = \psi(r,\theta,\phi-\delta) \quad (1)$$

For an infinitessimal value of δ, show that

$$\hat{R}_z(\delta) \approx 1 - \frac{i\delta}{\hbar}\hat{L}_z$$

where $\hat{L}_z$ is the angular momentum operator about the z-axis.

(b) Using the taylor expansion for the right hand side of Equation (1), show that in general the operator $\hat{R}_z(\delta)$ is given by

$$\hat{R}_z(\delta) = \exp\left[-\frac{i\delta}{\hbar}\hat{L}_z\right]$$

(c) Consider a small value of δ, and evaluate the action of $\hat{R}_z(\delta)$ on the position 3D vector

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Draw a picture depicting this transformation and reconcile it with your result for $\hat{R}_z(\delta)\vec{r}$.

(d) We can readily write down the rotation operator for spin$-1/2$ particles, by replacing $\hat{L}_z$ with $\hat{S}_z = \frac{\hbar}{2}\sigma_z$ where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the Pauli matrix. With this, we find the rotation operator for a spin$-1/2$ particle to be

$$\hat{R}_{z,1/2}(\delta) = \exp\left[-\frac{i\delta}{2}\sigma_z\right]$$

Using the Taylor series of this operator and the fact $\sigma_z^2 = 1$, write $\hat{R}_{z,1/2}(\delta)$ explicitly as a two by two matrix. [Note: although you are using the Taylor series to arrive at your final expression, the final result should hold for any real value of δ]

(e) Let $\chi_+^(z)$ be an eigenvector of σ_z with eigenvalue $+1$. That is, $\sigma_z\chi_+^{(z)} = \chi_+^{(z)}$. Evaluate $\hat{R}_z(\delta)\chi_+^{(z)}$. Explain your result.

(f) Find the normalized eigenvectors ($\chi_\pm^{(z)}$) with eigenvalues $\pm$ of the σ_y Pauli matrix

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(g) Evaluate the action of the matrix you got above on the $\chi_+^{(y)}$, i.e. $\hat{R}_z(\delta)(\chi_+^{(y)})$. Explain your result.

Solutions:

(a) We define $\hat{R}_z(\delta)$ as

$$\hat{R}_z(\delta)\psi(r,\theta,\phi) = \psi(r,\theta,\phi-\delta)$$

Taylor expanding in terms of ϕ, we have

$$\psi(r,\theta,\phi-\delta) = \psi(r,\theta,\phi) - \delta\frac{\partial\psi}{\partial\phi} + \cdots$$

and the Angular Momentum operator about the $z-$axis can be expressed as

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$
$$\implies \frac{\partial}{\partial\phi} = \frac{i\hat{L}_z}{\hbar}$$

Thus,

$$\psi(r,\theta,\phi-\delta) = \psi(r,\theta,\phi) - \frac{i\delta\hat{L}_z}{\hbar}\psi(r,\theta,\phi) + \cdots$$

That is, to first order approximation, we have

$$\boxed{\hat{R}_z(\theta)\psi(r,\theta,\phi) = 1 - \frac{i\delta\hat{L}_z}{\hbar}\psi(r,\theta,\phi)}$$

(b) Using the Taylor expansion

$$\begin{aligned}
\psi(r,\theta,\phi-\delta) &= \psi - \delta\frac{\partial\psi}{\partial\phi} + \frac{1}{2!}\frac{\partial^2\psi}{\partial\phi^2}\delta^2 + \cdots \\
\implies \hat{R}_z(\delta)\psi(r,\theta,\phi) &= \left(\mathbf{1} - \delta\frac{\partial}{\partial\phi} + \frac{1}{2!}\frac{\partial^2}{\partial\phi^2}\delta^2 + \cdots\right)\psi(r,\theta,\phi) \\
\implies \hat{R}_z(\delta)\psi(r,\theta,\phi) &= \left(\mathbf{1} - \frac{i\delta}{\hbar}\hat{L}_z + \cdots\right)\psi(r,\theta,\phi) \\
\implies \hat{R}_z(\delta) &= \sum_{n=0}\frac{1}{n!}\left(-\frac{i\delta}{\hbar}\hat{L}_z\right)^n
\end{aligned}$$

Therefore,

$$\boxed{\hat{R}_z(\delta) = \exp\left[-\frac{i\delta}{\hbar}\hat{L}_z\right]}$$

(c) We consider an arbitrary 3D position vector

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix}$$

Now, acting on this vector with $\hat{R}_z(\delta)$ has the effect of carrying out the transformation $\phi \to \phi-\delta$, so

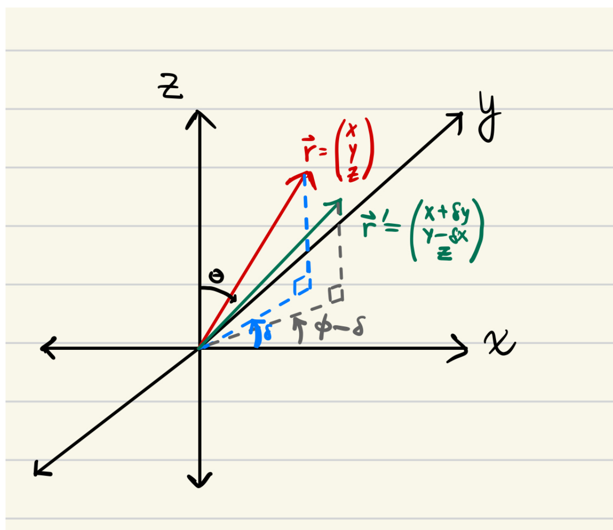
$$\begin{aligned}
\vec{r}' &= \begin{pmatrix} r\sin(\theta)\cos(\phi-\delta) \\ r\sin(\theta)\sin(\phi-\delta) \\ r\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} r\sin(\theta)\left[\cos(\phi)\cos(\delta) + \sin(\phi)\sin(\delta)\right] \\ r\sin(\theta)\left[\sin(\phi)\cos(\delta) - \cos(\phi)\sin(\delta)\right] \\ r\cos(\theta) \end{pmatrix}
\end{aligned}$$

Since δ is small, $\cos(\delta) \approx 1$ and $\sin(\delta) \approx \delta$. So,

$$
\begin{aligned}
\vec{r}' &= \begin{pmatrix} r\sin(\theta)[\cos(\phi)\cdot 1 + \sin(\phi)\cdot\delta] \\ r\sin(\theta)\,[\sin(\phi)\cdot 1 - \cos(\phi)\cdot\delta] \\ r\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} r\sin(\theta)\cos(\phi) + \delta\cdot r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) - \delta\cdot r\sin(\theta)\cos(\phi) \\ r\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} x+\delta y \\ y-\delta x \\ z \end{pmatrix}
\end{aligned}
$$

Thus, applying the $\hat{R}_z(\delta)$ operator with a small value of δ has the effect of the transformation:

$$
\begin{matrix} x & & x+\delta y \\ y & \rightarrow & y-\delta x \\ z & & z \end{matrix}
$$



(d) For a spin-1/2 particle, we have

$$
\hat{R}_{z,1/2}(\delta) = \exp\left[-\frac{i\delta}{2}\sigma_z\right]
$$

We now want to explicitly find the matrix representation of this operator, using the taylor series and the fact that $\sigma_z^2 = \mathbb{1}$.

Now,

$$
\begin{aligned}
\hat{R}_{z,1/2}(\delta) &= \exp\left[-\frac{i\delta}{2}\sigma_z\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\left(-\frac{i\delta}{2}\sigma_z\right)^n
\end{aligned}
$$

and

$$\sigma_z^2 = \mathbb{1} \implies \sigma_z^{2k} = \mathbb{1}$$
$$\sigma_z^2 = \mathbb{1} \implies \sigma_z^{2k+1} = \sigma_z$$

So,

$$\begin{aligned}
\hat{R}_{z,1/2}(\delta) &= \sum_{k\in\mathbb{N}} \left[\frac{1}{(2k)!}\left(-\frac{i\delta}{2}\right)^{2k} \mathbb{1}\right] + \sum_{k\in\mathbb{N}} \left[\frac{1}{(2k+1)!}\left(-\frac{i\delta}{2}\right)^{2k+1} \sigma_z\right] \\
&= \left[\sum_{k\in\mathbb{N}} \frac{1}{(2k)!}\left(-\frac{i\delta}{2}\right)^{2k}\right] \mathbb{1} + \left[\sum_{k\in\mathbb{N}} \frac{1}{(2k+1)!}\left(-\frac{i\delta}{2}\right)^{2k+1}\right] \sigma_z
\end{aligned}$$

For the sake of convenience, let's denote the two sums as

$$A \equiv \sum_{k\in\mathbb{N}} \frac{1}{(2k)!}\left(-\frac{i\delta}{2}\right)^{2k}$$
$$B \equiv \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)!}\left(-\frac{i\delta}{2}\right)^{2k+1}$$

Recall that the matrix representations of $\mathbb{1}$ and σ_z are

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then,

$$\begin{aligned}
\hat{R}_{z,1/2}(\delta) &= A\mathbb{1} + B\sigma_z \\
&= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \\
&= \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}
\end{aligned}$$

But notice that

$$\begin{aligned}
A + B &= \sum_{k\in\mathbb{N}}^{\infty} \frac{1}{k!}\left(-\frac{i\delta}{2}\right)^{k} \\
&= e^{-\frac{i\delta}{2}}
\end{aligned}$$

and

$$\begin{aligned}
A - B &= \sum_{k\in\mathbb{N}}^{\infty} \frac{1}{k!} \cdot (-1)^k \left(-\frac{i\delta}{2}\right)^{k} \\
&= \sum_{k\in\mathbb{N}}^{\infty} \frac{1}{k!} \cdot \left(-1 \times -\frac{i\delta}{2}\right)^{k} \\
&= \sum_{k\in\mathbb{N}}^{\infty} \frac{1}{k!} \left(\frac{i\delta}{2}\right)^{k} \\
&= e^{+\frac{i\delta}{2}}
\end{aligned}$$

Therefore, the final expression we get for the rotation operator is

$$\boxed{\hat{R}_z(\delta) = \begin{pmatrix} e^{-\frac{i\delta}{2}} & 0 \\ 0 & e^{+\frac{i\delta}{2}} \end{pmatrix}}$$

(e) We let $\chi_+^{(z)}$ be the eigenvector of σ_z having eigenvalue $+1$ i.e. $\sigma_z \chi_+^{(z)} = \chi_+^{(z)}$.

$$\begin{aligned} \sigma_z \chi_+^{(z)} &= \chi_+^{(z)} \\ \Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} \\ \Longrightarrow \begin{cases} a = a \\ b = -b \end{cases} \end{aligned}$$

So, $b = 0$ and a can be any real number (before normalization), but as per convention let's choose $a = 1$. So,

$$\boxed{\chi_+^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Applying the rotation operator then gives us

$$\begin{aligned} \hat{R}_z(\delta)\chi_+^{(z)} &= \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{+i\delta/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\delta/2} \\ 0 \end{pmatrix} \\ &= e^{-i\delta/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Thus,

$$\boxed{\hat{R}_{z,1/2}(\delta)\chi_+^{(z)} = e^{-i\delta/2}\chi_+^{(z)}}$$

So, applying the rotation matrix does not change the physical system, which makes sense.

$$\begin{aligned} \sigma_z \left(\hat{R}_{z,1/2}(\delta)\chi_+^{(z)}\right) &= \sigma_z \left(e^{-i\delta/2}\chi_+^{(z)}\right) \\ &= e^{-i\delta/2}\left(\sigma_z \chi_+^{(z)}\right) \\ &= e^{-i\delta/2}\chi_+^{(z)} \\ &= \hat{R}_{z,1/2}(\delta)\chi_+^{(z)} \end{aligned}$$

So, the rotated spinor is *also* an eigenspinor of σ_z with eigenvalue $+1$. Intuitively this makes sense, because $\chi_+^{(z)}$ was already completely "aligned" with the $z-$axis, so rotation about the $z-$axis should make no observable difference.

(f) We want to find the normalized eigenvectors ($\chi_\pm^{(y)}$) with eigenvalue ± 1 of the σ_y Pauli Matrix,

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Let $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector of σ_y with eigenvalue ± 1. Then,

$$
\begin{aligned}
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \pm\begin{pmatrix} a \\ b \end{pmatrix} \\
\implies \begin{pmatrix} -ib \\ ia \end{pmatrix} &= \pm\begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}
$$

<u>For the $+1$ case:</u>

$$
\begin{aligned}
-ib &= a \\
ia &= b
\end{aligned}
$$

$$
\begin{aligned}
&\implies -i(ia) = a \\
&\implies \boxed{a = a}
\end{aligned}
$$

and

$$
\begin{aligned}
b &= -\frac{1}{i}a \\
\implies &\boxed{b = ia}
\end{aligned}
$$

i.e we can arbitrarily choose a, get b from $b = ia$, and then normalize the spinor as $\chi^\dagger\chi = 1$.

We found that the $(+1)$-eigenspinor has the form $\chi_+^{(y)} = \begin{pmatrix} a \\ ia \end{pmatrix}$.

$$
\begin{aligned}
(\chi_+^{(y)})^\dagger\chi_+^{(y)} &= 1 \\
\implies \begin{pmatrix} a & -ia \end{pmatrix}\begin{pmatrix} a \\ ia \end{pmatrix} &= 1 \\
\implies a^2 + a^2 &= 1 \\
\implies a^2 &= \frac{1}{2} \\
\implies &\boxed{a = \frac{1}{\sqrt{2}}}
\end{aligned}
$$

So, the $+1$ eigenspinor is

$$
\boxed{\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}
$$

<u>For the -1 case:</u>

$$
\begin{aligned}
-ib &= -a \\
ia &= -b
\end{aligned}
$$

$$
\begin{aligned}
&\implies i(-ia) = a \\
&\implies \boxed{a = a}
\end{aligned}
$$

and

$$
\boxed{b = -ia}
$$

So, following the same argument as before, $(\chi_-^{(y)})^\dagger \chi_-^{(y)} = 1$ gives

$$\begin{aligned}
\begin{pmatrix} a & ia \end{pmatrix} \begin{pmatrix} a \\ -ia \end{pmatrix} &= 1 \\
\implies a^2 + a^2 &= 1 \\
\implies \boxed{a = \frac{1}{\sqrt{2}}}
\end{aligned}$$

So, the (-1)-eigenspinor is

$$\boxed{\chi_-^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

> To summarise, the normalized eigenspinors of the σ_z Pauli matrix are $\chi_+^{(y)}$ and $\chi_-^{(y)}$ given by
>
> $$\chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$
> $$\chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(g) Now, let's evaluate the rotation matrix on $\chi_+^{(y)}$.

$$\begin{aligned}
\hat{R}_z(\delta)\chi_+^{(y)} &= \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{+i\delta/2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\delta/2} \cdot 1 + 0 \cdot i \\ 0 \cdot 1 + e^{+i\delta/2} \cdot i \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\delta/2} \\ ie^{+i\delta/2} \end{pmatrix}
\end{aligned}$$

Is this rotated spinor still an eigenspinor of σ_y?

$$\begin{aligned}
\sigma_y \left(\hat{R}_z(\delta)\chi_+^{(y)} \right) &= \sigma_y \left(\frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\delta/2} \\ ie^{+i\delta/2} \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{2}} \sigma_y \begin{pmatrix} e^{-i\delta/2} \\ ie^{+i\delta/2} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} \\ ie^{+i\delta/2} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \cdot e^{-i\delta/2} - i \cdot ie^{+i\delta/2} \\ ie^{-i\delta/2} + 0 \cdot e^{+i\delta/2} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{+i\delta/2} \\ ie^{-i\delta/2} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} e^{+i\delta} \begin{pmatrix} e^{-i\delta/2} \\ ie^{-3i\delta/2} \end{pmatrix}
\end{aligned}$$

So, $\left(\hat{R}_z(\delta)\chi{+}^{(y)} \right)$ is not an eigenspinor of σ_y. Intuivitely, this makes sense since rotating the eigenspinor $\chi_+^{(y)}$ (which was orginally "completely oriented in the y-direction" in some sense) about the $z-$axis would seem to introduce some $x-$axis alignment.

Q2. Unitary Operators: We have now seen two examples of continuous unitary transformations that are of the form $\hat{U}(\delta) = \exp(-\hat{M}\delta)$ where $\hat{M}$ is hermitian. Prove that any operator of this form is unitary as long as $\hat{M}$ is hermitian.

<u>Proof:</u> Let $\hat{M}$ be a Hermitian operator i.e. $\hat{M} = \hat{M}^\dagger$, and consider the operator

$$\hat{U}(\delta) = \exp(-i\hat{M}\delta)$$

We want to show that $\hat{U}$ is unitary i.e.

$$\hat{U}(\delta)^\dagger \hat{U}(\delta) = \mathbb{1}$$

What is $\hat{U}^\dagger$?

$$\begin{aligned}
\hat{U}^\dagger &= \exp(-i\hat{M}\delta) \\
&= \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\hat{M}\delta\right)^n\right]^\dagger \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\left(-i\hat{M}\delta\right)^n\right]^\dagger \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n \left[(-i\hat{M})^n\right]^\dagger \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n (\hat{M}^n)^\dagger \left[(-i)^n\right]^\dagger \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n \hat{M}^n \left[(-i)^n\right]^\dagger
\end{aligned}$$

Let's break into cases and see what $[(-i)^n]^\dagger = [(-1)^n(i)^n]^\dagger$ evaluates to in each case.

<u>For $n = 4k + 0, k \in \mathbb{Z}$:</u>

$$\begin{aligned}
(-1)^n \cdot i^n &= 1 \cdot 1 \\
\implies [(-i)^n]^\dagger &= 1^\dagger = 1
\end{aligned}$$

<u>For $n = 4k + 1, k \in \mathbb{Z}$:</u>

$$\begin{aligned}
(-1)^n \cdot i^n &= -1 \cdot i \\
\implies [(-i)^n]^\dagger &= (-i)^\dagger = i
\end{aligned}$$

<u>For $n = 4k + 2, k \in \mathbb{Z}$:</u>

$$\begin{aligned}
(-1)^n \cdot i^n &= 1 \cdot (-1) \\
\implies [(-i)^n]^\dagger &= (-1)^\dagger = -1
\end{aligned}$$

<u>For $n = 4k + 3, k \in \mathbb{Z}$:</u>

$$\begin{aligned}
(-1)^n \cdot i^n &= (-1) \cdot (-i) \\
\implies [(-i)^n]^\dagger &= i^\dagger = -i
\end{aligned}$$

We notice that these coincide exactly with the powers of i, so

$$\begin{aligned}
\hat{U}^\dagger &= \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n \hat{M}^n i^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(i\hat{M}\delta\right)^n \\
&= \exp\left(+i\hat{M}\delta\right)
\end{aligned}$$

So, if $\hat{U} = \exp(-i\hat{M}\delta)$ then $\hat{U}^\dagger = \exp(+i\hat{M}\delta)$.

To show that $\hat{U}^\dagger\hat{U} = \mathbb{1}$, we use the following well-known identity:

For linear operators $\hat{A}$ and $\hat{B}$,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \mathbb{1} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots$$

<u>Proof Sketch:</u> Through some brute force, we can show

$$\begin{aligned}
e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \left(\mathbb{1} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^3 + \cdots\right) \hat{B} \left(\mathbb{1} - \hat{A} + \frac{1}{2!}\hat{A}^2 - \frac{1}{3!}\hat{A}^3 + \cdots\right) \\
&= \mathbb{1} + \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) + \frac{1}{2!}\hat{A}^2\hat{B} - \hat{A}\hat{B}\hat{A} + \frac{1}{2!}\hat{B}\hat{A}^2 + \cdots \\
&= \mathbb{1} + \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] - \frac{1}{2!}[\hat{A}, \hat{B}]\hat{A} + \cdots \\
\implies e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \mathbb{1} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots
\end{aligned}$$

So, applying this formula with $\hat{A} = i\hat{M}\delta$ and $\hat{B} = \mathbb{1}$ we get

$$e^{i\hat{M}\delta}\mathbb{1}e^{-i\hat{M}\delta} = \mathbb{1} + [e^{i\hat{M}\delta}, \mathbb{1}] + \frac{1}{2!}[e^{i\hat{M}\delta}, [e^{i\hat{M}\delta}, \mathbb{1}]] + \cdots$$

Now, the identity operator $\mathbb{1}$ commutes with every linear operator, so the only non-vanishing term on the RHS is the first term i.e. $\mathbb{1}$ itself.

Therefore,

$$\begin{aligned}
e^{i\hat{M}\delta}e^{-i\hat{M}\delta} &= \mathbb{1} \\
\implies \boxed{\hat{U}^\dagger\hat{U} = \mathbb{1}}
\end{aligned}$$

We can also show $\hat{U}\hat{U}^\dagger$ in exactly the same way. So, U is unitary.

Q3. Conservation Laws: For each of the Hamiltonians below, determine which of the following are conserved quantities:

$$\{p_x, p_y, p_z, p^2, S_x, S_y, S_z, L_x, L_y, L_z, L^2\}$$

where p_j, S_j, L_j are the *expectation* values of the momentum, spin, and orbital angular momentum along the j^{th} direction.

(a) The Hamiltonian of a Free Electron: $\hat{H} = \frac{\hat{p}^2}{2m}$, where $\hat{p}^2 = \hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2$.

(b) Hamiltonian of an electron in the presence of a constant background electric field pointing along the $z-$ axis: $\hat{H}_E = \frac{\hat{p}}{2m} + eE_0\hat{z}$, where E is the value of the electric field intensity.

(c) Hamiltonian of an electron in the presence of a weak constant magnetic field (B_z) along the $z-$axis: $\hat{H}_E = \frac{\hat{p}}{2m} + B_z\left(e\frac{\hat{L}_z}{2m} - \gamma\hat{S}_z\right)$, where γ is the gyromagnetic ratio of the electron, i.e. it is constant.

We know that for an operator $\hat{Q}$, the expectation value $\langle\hat{Q}\rangle$ is a conserved quantity if $\hat{Q}$ *commutes with the Hamiltonian* i.e. $[\hat{H}, \hat{Q}] = 0$.

The Angular Momentum operator in the i^{th} direction is

$$\hat{L}_i = \epsilon_{ijk}\hat{X}_j\hat{P}_k$$

where we are using the Einstein Summation Convention, rather than explicitly writing the summations out.

Now, starting from the canonical commutation relations

$$[\hat{X}_i, \hat{X}_j] = 0 = [\hat{P}_i, \hat{P}_j]; \quad [\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij};$$

we can obtain some useful relations:

1. Between $\hat{L}_i$ and $\hat{X}_l$:

$$\begin{aligned}
[\hat{L}_i, \hat{X}_l] &= [\epsilon_{ijk}\hat{X}_l\hat{P}_k, X_l] \\
&= \epsilon_{ijk}[X_jP_k, X_l] \\
&= \epsilon_{ijk}\left([X_j, X_l]P_k + X_j[P_k, X_l]\right) \\
&= \epsilon_{ijk}\left(0\cdot P_k + X_j(-i\hbar\delta_{lk})\right) \\
&= -i\hbar\epsilon_{ijk}X_j\delta_{lk} \\
&= -i\hbar\epsilon_{ijl}X_j \\
\implies [\hat{L}_i, \hat{X}_l] &= i\hbar\epsilon_{ilj}X_j
\end{aligned}$$

$$\boxed{[\hat{L}_i, \hat{X}_l] = i\hbar\epsilon_{ilj}X_j}$$

2. Between $\hat{L}_i$ and $\hat{P}_l$:

$$\begin{aligned}
[\hat{L}_i, \hat{P}_l] &= [\epsilon_{ijk}X_jP_k, P_l] \\
&= \epsilon_{ijk}[X_jP_k, P_l] \\
&= \epsilon_{ijk}\left([X_j, P_l]P_k + X_j[P_k, P_l]\right) \\
&= \epsilon_{ijk}\left(i\hbar\delta_{jl}P_k + X_j\cdot 0\right) \\
&= i\hbar\epsilon_{ilk}P_k
\end{aligned}$$

$$\boxed{[\hat{L}_i, \hat{P}_l] = i\hbar\epsilon_{ilk}P_k}$$

3. Between $\hat{L}_i$ and $\hat{L}_j$:

$$\begin{aligned}
[L_i, L_j] &= [\epsilon_{iab}\hat{X}_a\hat{P}_b, \epsilon_{jmn}\hat{X}_m\hat{P}_n] \\
&= \epsilon_{iab}\epsilon_{jmn}[\hat{X}_a\hat{P}_b, \hat{X}_m\hat{P}_n] \\
&= \epsilon_{iab}\epsilon_{jmn}\left([\hat{X}_a\hat{P}_b, \hat{X}_m]\hat{P}_n + \hat{X}_m[\hat{X}_a\hat{P}_b, \hat{P}_n]\right) \\
&= \epsilon_{iab}\epsilon_{jmn}\left([\hat{X}_a\hat{P}_b, \hat{X}_m]\hat{P}_n + \hat{X}_m[\hat{X}_a\hat{P}_b, \hat{P}_n]\right) \\
&= \epsilon_{iab}\epsilon_{jmn}\left(\underbrace{[\hat{X}_a, \hat{X}_m]}_{0}\hat{P}_b\hat{P}_n + \hat{X}_a[\hat{P}_b, \hat{X}_m]\hat{P}_n + \hat{X}_m[\hat{X}_a, \hat{P}_n]\hat{P}_b + \hat{X}_m\hat{X}_a\underbrace{[\hat{P}_b, \hat{P}_n]}_{0}\right) \\
&= \epsilon_{iab}\epsilon_{jmn}\left(\hat{X}_a[\hat{P}_b, \hat{X}_m]\hat{P}_n + \hat{X}_m[\hat{X}_a, \hat{P}_n]\hat{P}_b\right) \\
&= \epsilon_{iab}\epsilon_{jmn}\left(\hat{X}_a(-i\hbar\delta_{bm})\hat{P}_n + \hat{X}_m(i\hbar\delta_{an})\hat{P}_b\right) \\
&= -i\hbar\epsilon_{iab}\epsilon_{jnb}\hat{X}_a\hat{P}_n + i\hbar\epsilon_{iab}\epsilon_{jma}\hat{X}_m\hat{P}_b \\
&= i\hbar\epsilon_{iab}\epsilon_{jnb}\hat{X}_a\hat{P}_n - i\hbar\epsilon_{iba}\epsilon_{jma}\hat{X}_m\hat{P}_b
\end{aligned}$$

Using the well known identity for the product of two levi-civita symbols, we can express $\epsilon_{iab}\epsilon_{jbn}$ as

$$\epsilon_{iab}\epsilon_{jbn} = \delta_{ij}\delta_{an} - \delta_{in}\delta_{aj}$$

and $\epsilon_{iab}\epsilon_{jma}$ as

$$\epsilon_{iab}\epsilon_{jma} = \delta_{ij}\delta_{bm} - \delta_{im}\delta_{bj}$$

So,

$$\begin{aligned}
[\hat{L}_i, \hat{L}_j] &= i\hbar(\delta_{ij}\delta_{an} - \delta_{in}\delta_{aj})\hat{X}_a\hat{P}_n - i\hbar(\delta_{ij}\delta_{bm} - \delta_{im}\delta_{bj})\hat{X}_m\hat{P}_b \\
&= \cancel{i\hbar\delta_{ij}\hat{X}_a\hat{P}_a} - i\hat{X}_j\hat{P}_i - \cancel{i\hbar\hat{X}_b\hat{P}_b} + i\hbar\hat{X}_i\hat{P}_j \\
&= i\hbar\left(\hat{X}_i\hat{P}_i - \hat{X}_j\hat{P}_i\right) \\
&= i\hbar\epsilon_{ijk}\hat{L}_k
\end{aligned}$$

Thus,

$$\boxed{[L_i, L_k] = i\hbar\epsilon_{ijk}\hat{L}_k}$$

Also, the Spin Operators follow exactly the same algebra as the Orbital Angular Momentum Operators i.e.

$$[\hat{S}_i, \hat{S}_k] = i\hbar\epsilon_{ijk}\hat{S}_k$$

However, for an electron, spin is an internal degree of freedom and is completely independent of the particle's position or momentum.

We can describe the Hilbert space of the Electron $\mathbb{V}_e$, as being a tensor product of an infinite-dimensional hilbert space $\mathbb{V}_o$ describing its orbital degrees of freedom and a two-dimensional space $\mathbb{V}_s$ describing its spin degrees of freedom:

$$\mathbb{V}_e = \mathbb{V}_o \otimes \mathbb{V}_s$$

Thus, the Spin operators $\hat{S}_i$ (which act on $\mathbb{V}_s$) act on a different hilbert space than the position and momentum operators (which act on $\mathbb{V}_o$), and thus commute with them in all directions:

$$\boxed{[\hat{S}_i, \hat{X}_j] = 0}$$

$$\boxed{[\hat{S}_i, \hat{P}_j] = 0}$$

for all i, j. As a result, **the expected values associated with the spin operators are always conserved**.

Now that we have all of our basic commuatition relations in hand, lets tackle each Hamiltonian:

(a) **Free Electron**:

$$\hat{H} = \frac{\hat{P}^2}{2m}$$

- Since the Hamiltonian only has the momentum squared operator in it, certainly it commutes with $\hat{P}^2$

$$[\hat{H}, \hat{P}^2] = [\frac{\hat{P}^2}{2m}, \hat{P}^2] = \frac{1}{2m}[\hat{P}^2, \hat{P}^2] = 0$$

- Now, since $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2$ and $[\hat{P}_i, \hat{P}_j] = 0$ for $i, j = x, y, z$ we immediately find that

$$[\hat{H}, \hat{P}^2] = 0 \implies [\hat{H}, \hat{P}_i^2] = 0$$

- The Hamiltonian is rotationally invariant, so $\hat{H}$ commutes with each $\hat{L}_i$ and with $\hat{L}^2$. We can show this explicitly as

$$\begin{aligned}
\left[\frac{\hat{P}^2}{2m}, \hat{L}_m\right] &= -\frac{1}{2m}\left[\hat{L}_m, \hat{P}_i\hat{P}_i\right] \quad \text{(Using Einstein Summation Convention)} \\
&= -\frac{1}{2m}\left([\hat{L}_m, \hat{P}_i]\hat{P}_i + \hat{P}_i[\hat{L}_m, \hat{P}_i]\right) \\
&= -\frac{1}{2m}\left[(\epsilon_{mik} i\hbar \hat{P}_k)\hat{P}_i + \hat{P}_i(\epsilon_{mik} i\hbar \hat{P}_k)\right] \\
&= -\frac{i\hbar}{2m}\epsilon_{mik}[\hat{P}_k, \hat{P}_i] \\
&= 0 \quad \text{(Momentum components always commute)}
\end{aligned}$$

$$\boxed{[\hat{H}, \hat{L}_m] = 0}$$

So, the Hamiltonian commutes with each $\hat{L}_m$ and thus also commutes with $\hat{L}^2$ as

$$\begin{aligned}
[\hat{H}, \hat{L}^2] &= [\hat{H}, \hat{L}_m\hat{L}_m] \\
&= [\hat{H}, \hat{L}_m]\hat{L}_m + \hat{L}_m[\hat{H}, \hat{L}_m] \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Summary: The Free Electron Hamiltonian commutes with all the operators associated with the quantities we're interested in, so **all of their expectation values are conserved.**

(b) **Electron in Constant Electric Field**:

$$\hat{H}_E = \frac{\hat{p}^2}{2m} + eE_0\hat{z}$$

which can be re-written in slightly different notation as

$$\hat{H}_E = \frac{\hat{P}^2}{2m} + eE_0\hat{X}_3$$

We know of course that the free part of the hamiltonian commutes with all the operators, so we are interested in the behavior of $eE_0\hat{X}_3$.

- We know that $[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}$ so

$$[eE_0\hat{X}_3, \hat{P}_j] = eE_0 i\hbar\delta_{i3}$$

  This means that the Hamiltonian commutes with the momentum operators *other than* $\hat{P}_3 = \hat{p}_z$.
- Since it doesn't commute with the momenta in every direction, it also doesn't commute with $\hat{P}^2$.
- We know that $[\hat{X}_l, \hat{L}_i] = -i\hbar\epsilon_{ilj}X_j$.
  So,

$$[eE_0\hat{X}_3, \hat{L}_i] = -i\hbar\epsilon_{i3j}X_j$$

  When $i = 3$, the levi-civita tensor becomes zero for all terms, so we have $[eE_0\hat{X}_3, \hat{L}_3] = 0$ i.e. the Hamiltonian commutes with the $\hat{L}_z$ operator. However, it *doesn't commute with the other directions.*
- Since the Hamiltonian doesn't commute with the Angular Momentum with respect to all axes, it doesn't commute with the total angular momentum $\hat{L}^2$.

  Summary: The p_x, p_y, l_z and spin expected values are conserved. But the p_z, p^2, L_x, L_y, L^2 values are not.

(c) **Electron in a weak Constant Magnetic Field**:

$$\hat{H}_E = \frac{\hat{P}^2}{2m} + B_z\left(e\frac{\hat{L}_z}{2m} - \gamma\hat{S}_z\right)$$

Again, we know the free part $\hat{P}^2/2m$ commutes with all of the operators, so we only discuss the other components of the Hamiltonian

- Using the commutation relations between $\hat{L}_i$ and $\hat{P}_j$ found earlier, we know that $\hat{L}_z$ only commutes with $\hat{P}_z$. The $\hat{S}_z$ present in the hamiltonian commutes with all momentum operators since they act on different hilbert spaces.
- Now, the spin matrices $\hat{S}_x$ and $\hat{S}_y$ commute with $\hat{P}^2$ and $\hat{L}_z$, but do not commute with $\hat{S}_z$ due to the commutation relation

$$[\hat{S}_i, \hat{S}_k] = i\hbar\epsilon_{ijk}\hat{S}_k$$

  The operator $\hat{S}_z$ does commute with itself so it commutes with the hamiltonian as a whole.
- The angular momentum operators all commute with the spin operators since they act on different hilbert spaces. However, $\hat{L}_x$ and $\hat{L}_y$ do not commute with $\hat{L}_z$, so only $\hat{L}_z$ commutes with the hamiltonian as a whole.

Summary: The conserved expected values are: p_z, s_z, l_z. All other expected values are not conserved.