# Math H185 Lecture (not sure)

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

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# 1 April 8 - The Agument Principle

To start, let's recall a lemma from the homework.

> ***Lemma:*** Let $f$ be meromorphic at $z_0$ with zero of order $z \in \mathbb{Z}$. Then, $f'/f$ has a simple pole at $z_0$ if $n \neq 0$ or a removable pole at $z_0$ if $n = 0$ with residue $n$.

***Proof: (outline)*** By the structure theorem,

$$f(z)(z - z_0)^n h(z)$$

where $h(z)$ is holomorphic and non-vanishing at $z_0$. Then,

$$\begin{aligned} \frac{f'}{f} &= \frac{n(z - z_0)^{n-1} h(z) + h'(z)(z - z_0)^n}{(z - z_0)^n h(z)} \\ &= \frac{n}{z - z_0} + \underbrace{\frac{h'(z)}{h(z)}}_{\text{holomorphic}} \end{aligned}$$

The Argument Principle arises from combining this observation with the Residue theorem.

> ***Theorem: (The Arguement Principle)*** Let $f$ be meromorphic on a neighborhood $\overline{U}$. Then,
>
> $$\frac{1}{2\pi i} \int_{\partial U} \frac{f'(z)}{f(z)} dz = \begin{array}{l} \#\text{ of zeroes of } f \text{ in } U \\ -\#\text{ of poles of } f \text{ in } U \end{array}$$
>
> where the zeroes and poles are counted *with* multiplicity.

***Proof:*** Using the Residue Theorem,

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f'(z)}{f(z)} dz = \sum_{z_j} \text{Res}_{z_j} \left( \frac{f'(z)}{f(z)} \right)$$

where the $z_j$'s are zeros/poles of $f$. By the lemma above, $\text{Res}_{z_j} \left( \frac{f'(z)}{f(z)} \right)$ is just the order of each zero (or "minus" the order if it is a pole). So, we have arrived at the desired result.

## 1.1 Some Consequences the Argument Principle

> ***Rouche's Theorem:*** Suppose $f, g$ are holomorphic on a neighborhood of $\overline{U} \subseteq \mathbb{C}$. Assume that $|f(z)| > |g(z)|$ for all $z \in \partial U$. Then, the number of zeroes of $f$ in $U$ equals the number of zeroes of $f + g$ in $U$.

***Proof:***
Let $f_\lambda = f + \lambda g$ for $\lambda \in [0, 1]$.

- $f_\lambda$ is holomorphic around $\overline{U}$.
- $f_0 = f_1$, $f_1 = f + g$.

By the argument prnciple,

$$\#\text{ of zeroes of } f_\lambda = \frac{1}{2\pi i} \int_{\partial U} \frac{f'_\lambda(z)}{f_\lambda(z)} dz$$

The RHS is continuous in $\lambda$ (why?) and is never zero on the boundary because $|f| > |g|$ on $\partial U$. The RHS is a continuous function, whereas the LHS is just an integer. So, if the RHS to be a continuous function $[0,1] \to \mathbb{Z}$ then it must be a constant function.

***Cor:*** A polynomial of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad n > 0$$

has $n$ roots (with multiplicity).

***Proof:*** Let $f(z) = z^n$ and $g(z) = a_{n-1}z^{n-1} + \cdots + a_0$. Then, $P(z) = f(z) + g(z)$. Let's try to apply Rouche's Theorem. In order to do so let's choose $R > 0$ such that for all $r \geq R$,

$$r^n > |a_{n-1}|r^{n-1} + \cdots + a_0$$
$$\implies |f| > |g| \text{ on } \partial B_r(0) \forall r \geq R$$

Then, Rouche's Theorem tells us that $n = \#$ of zeros(f) in $B_r(0) = \#$ of zeros(P) in $B_r(0)$.

Another consequence of Rouche's Theorem is the Open Mapping Theorem.

**Open maps**

***Definition:*** A map $f : U \to V$ between topological spaces $U, V$ is said to be ***open*** if $f$(Open set in $U$) is open in $V$.

Ex: $V = \mathbb{C}$ and $f =$ constant is an open map.
Ex: [Insert image from lecture.] Idea is that any map that sends an open interval to a single point is *not* open.

***Theorem:*** Let $f : U \to \mathbb{C}$ be nonconstant and holomorphic. If $U$ is connected, then $f$ is open.

***Proof:*** Let $z_0 \in U$. It suffices to show that $\forall$ sufficiently small $\epsilon > 0$, $f(B_\epsilon(z_0))$ is open. Moreover it suffices to show that $f(B_\epsilon(z_0))$ contains $B_r(w_0)$ for some $r > 0$ where $w_0 = f(z_0)$.

[Insert image from lecture]

i.e. we want: $\forall w \in B_r(w_0), \ \exists z \in B_\epsilon(z_0)$ such that $f(z) = w$, i.e. $f(z) - w = 0$.

"Aha! A zero counting problem" - Tony.

Now, since $f(z_0) = w_0 \neq w$ we can pick $\epsilon$ small enough so that $f(z) \neq w_0$ on $\partial B_\epsilon(z_0)$.

$$\implies |f(z) - w_0| > \delta > 0 \ \ \forall z \in \partial B_\epsilon(z_0)$$
$$\implies |f(z) - w| > \frac{\delta}{2} > 0 \ \ \forall w \in \partial B_{\delta/2}(w_0)$$

Take $r = \delta/2$ Then,

$$\implies |f_w(z_0)| > 0 \ \ \forall z \in \partial B_\epsilon(z_0)$$
$$\implies \frac{f_w'(z)}{f_w(z)} \text{ is continuous in } z \in \partial B_\epsilon(z_0) \text{ i.e. } B_r(w_0)$$
$$\# \text{ zeroes of } f_w = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f_w'(z)}{f_w(z)} dz \text{ is continuous in } w$$

Then, by the same reasoning as Rouche's Theorem, both sides are constant $> 0$.