Math H185 Lecture 3

Keshav Balwant Deoskar

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Contents

1 January 24 - Power Series

Power Series

A **Power Series** is a formal expression

$$\sum_{n\geq 0} a_n z^n, z_n \in \mathbb{C}$$

for which operations are defined as:

- Addition:

$$\sum_{n\geq 0} a_n z^n$$

- multiplication

- "Formal" here means we temporarily ignore whether or not it makes sense to plug in complex numbers into such formulae.

1.1 Convergence

Defining these formal expressions is cool, but when does a power seres actually define a function? It does so when the series **converges**.

Example: Geometric series:
Let $a \in \mathbb{C}$, then for the geometric series we have $a_n = a^n$. So,

$$\sum_{n\geq 0} a^n z^n = 1 + az + a^2 z^2 + \cdots$$

converges if $S_n = \sum_{n\geq 0}^{N-1} a^n z^n$ has a limit.

By the same argument as in the reals, we can get a closed form expression for S_N:

$$S_N = \frac{1-(az)^n}{1-(az)}$$

To deal with convergence, we break into cases and take the limit.

- $|az| < 1$ **case:**

$$|az| < 1 \implies |az|^N \xrightarrow{N\to\infty} 0$$

So,

$$\lim_{N\to\infty} S_N = \frac{1}{1-az}$$

- $|az| > 1$ **case:**

$$|az| > 1 \implies |az|^N \xrightarrow{N\to\infty} \infty$$

so

$$\lim_{N\to\infty} S_N \text{ diverges}$$

- $|az| = 1$ **case:**

  Now, if $|az| = 1$ but $az \neq 1$, then $(az)^N = \underbrace{(az) \times \cdots \times (az)}_{N \text{ times}}$ just means we rotate around on the unit circle without converging to any point in particular. So, the sum diverges.

  If instead we have $az = 1$, then the denominator $1 - az$ vanishes and the sum diverges again.

  Note: This time it happened to be the case that both case falling under $|az| = 1$ diverged, but in general we can have more complicated behavior.

Conclusion: The geometric series converges (absolutely) for when $|z| < |a|$.

Recall that

$$\sum_{n \geq 0} z_n, z_n \in \mathbb{C}$$

coverges absolutely if

$$\sum_{n \geq 0} |z_n|$$

converges.

So, we notice that the series converges for any z such that $|z| < 1/|a|$. This region is just an open disk of radius $|a|$. In general, power series have radii of convergence.

Radius of Convergence

Def: A complex Power Series

$$\sum_{n \geq 0} a_n z^n$$

has Radius of Convergence

$$r = \left(\lim_{n \to \infty} \sup |a_n|^{1/n} \right)^{-1} \in \mathbb{R}$$

Example: For $a_n = a^n$, we have

$$r = \left(\lim_{n \to \infty} \sup |a_n|^{1/n} \right)^{-1} = \frac{1}{|a|}$$

which matches with the result obtained earlier.

Theorem:

1. If $|z| < r$, then $f(z)$ converges absolutely.
2. If $|z| > r$, then it diverges.
3. At $|z| = r$, more care is needed.

Proof Sketch:

1. Consider z such that $|z| < (1-\epsilon)r$ for some $\epsilon > 0$.

$$\begin{aligned} \implies |a_n z^n| &< |a_n|(1-\epsilon)^n r^n \\ &\leq |a_n|(1-\epsilon)^n \left(\frac{1}{|a_n|^{1/n}}\right)^n \quad \text{Assume } a_n \neq 0 \\ &\leq (1-\epsilon)^n \quad (\text{If } a_n = 0, \text{ this inequality is true trivially}) \end{aligned}$$

$$\implies \text{Convergence by Geometric series}$$

Term by term, the series is smaller than the geometric series (which converges), thus it also converges (Dominated Convergence Theorem).

2. If $|z| > r$, then $|z| > r/(1-\epsilon)$ for some $\epsilon > 0$ while

$$|a_n|^{1/n} > \left(\lim_{k\to\infty} \sup |a_k|^{1/k}\right)(1-\epsilon)$$

for infinitely many n.

$$\begin{aligned} &\implies |a_n z^n| > \frac{1}{r^n}(1-\epsilon)^n \cdot \frac{r^n}{(1-\epsilon)^n} > 1 \quad \text{for inf. many } n \\ &\implies \text{the sum diverges} \end{aligned}$$

Example: Consider a polynomial

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

$$\begin{aligned} &\implies \lim_{n\to\infty} \sup |a_n|^{1/n} = 0 \\ &\implies r = \infty \end{aligned}$$

Example: Consider the exponential function

$$\exp(z) = e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

$$\implies r = \lim_{n\to\infty} (n!)^{1/n} =_{\text{claim}} \infty$$

Proof of claim: We wll show that for any b, $b < (n!)^{1/n}$ for all $n >> 0$.

For very large n, we have $n! = \underbrace{1 \cdot 2 \cdots b}_{\geq 1} \cdot \underbrace{(b+1)}_{\geq b} \cdot \underbrace{(b+2)}_{\geq b} \cdots \underbrace{n}_{\geq b}$. So,

$$(n!)^{1/n} > \left(b^{n-m}\right)^{1/n} = b^{1-m/n} = b \cdot b^{-m/n}$$

and

$$\lim_{n \to \infty} b^{-m/n} = 1$$

So, in the $n \to \infty$ limit,

$$(n!)^{1/n} > b$$

Since this holds for *any* constant b, it must be the case that

$$\lim_{n \to \infty} (n!)^{1/n} = \infty \implies \boxed{r = \infty}$$

Note: So far we've only considered series centered at hte origin but we can shift the center of the series from 0 to some point z_0.
So, instead of

$$\sum_{n \geq 0} a_n z^n$$

We have

$$\sum_{n \geq 0} a_n (z - z_0)^n$$

The radius of convergence is given by the same expression.

1.2 Differentiation of Power Series

Knowing convergence properties of a series is also useful for the purposes of *differentiation.*

Theorem: Let r be the radius of convergence of $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$. Then, f is (complex) differentiable on $B_r(z_0)$ and

$$f'(z_0) = \sum_{n \geq 0} n a_n (z - z_0)^{n-1}$$

Proof: The proof follows the same argument as that in real analysis. Write the proof later. One proof can be found here. Otherwise see Stein-Shakarchi Theorem 2.6.

We will break the proof into two parts:

(a) First, we show that if $\sum_{n \geq 0} a_n (z - z_0)^n$ has radius of convergence r, then so does $\sum_{n \geq 0} n a_n (z - z_0)^{n-1}$.

(b) Second, we show that if $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ then the derivative at z_0 is indeed

$$f(z_0) = \sum_{n \geq 0} n a_n (z - z_0)^{n-1}$$