Math H185 Lecture64

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1 January 28 - Integrating over curves

In calculus, we learned how to integrate over intervals. Today, we'll learn how to integrate complex valued functions over curves in the complex plane.

Integral over a curve

We define the integral of a function $f(z)$ over a curve γ

$$\int_\gamma f(z)dz := \int_\gamma \mathrm{Re}(f(z))dz + i\int_\gamma \mathrm{Im}(f(z))dz$$

1.1 What is a curve?

- A ***parameterized curve*** is a continuous function $\gamma : \underbrace{[a,b]}_{\subset\mathbb{R}} \to \mathbb{C}$
- We say γ is ***piece-wise smooth*** if there exist finite subdivisions of $[a,b]$ on which γ is smooth (in the Math 104 sense i.e. the real and imaginary parts are separately infinitely differentiable).

  [include graphs of piece-wise smooth paths]
- Example: $\gamma(t) = z_0 + re^{i\theta}$ where $z_0 \in \mathbb{C}$, $r \in \mathbb{R}_{\geq 0}$, $t \in [0, 2\pi]$. This path traces out a circle of radius r centered around the point z_0.

  [include graph]
- Example: Given a path $\gamma : [a,b] \to \mathbb{C}$. Let $\gamma^- : [a,b] \to \mathbb{C}$ be $\gamma^-(t) = \gamma(a+b-t)$. Then γ^- is the same curve but traversed in reverse orientation.

1.2 So, how do we actually integrate?

Let $\gamma : [a,b] \to \mathbb{C}$ be a "nice" curve, where "nice" means piece-wise smooth parameterized. Then,

Theorem: The integral of $f(z)$ over γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

and these integrals have some nice properties:

1. Linearity: Say $\lambda, \mu \in \mathbb{C}$ Then

$$\int_\gamma \lambda f(z) + \mu g(z)dz = \lambda \in_\gamma f(z)dz + \mu\int_\gamma g(z)dz$$

2. orientation:

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$$

Examples:

(a) $\gamma : [a,b] \to \mathbb{C}$ given by $\gamma(t) = t$:

$$\int_\gamma f(z)dz = \int_a^b f(t)dt$$

(b) $\gamma : [a,b] \to \mathbb{C}$ given by $\gamma(t) = it$:

$$\int_\gamma f(z)dz = \int f(it)idt = i\int f(it)dt$$

Recall that a ***primitive*** (i.e. antiderivative) of f is F such that $F'(z) = f(z)$. Then, the fundamental theorem of calculus is

> ***Fundamental Theorem of Calculus:*** For a "nice" curve $\gamma : [a,b] \to \mathbb{C}$ and function f with primitive F,
>
> $$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

Corollary: If γ is a ***closed*** curve i.e. $\gamma(a) = \gamma(b)$ and f has a primitive on (an open neighborhood of) γ, then

$$\int_\gamma f(z)dz = 0$$

1.3 Most fundamental example: $f(z) = z^n, n \in \mathbb{Z}$

Consider the function $f(z) = z^n, n \in \mathbb{Z}$ and the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = re^{it}$. What is $\int_\gamma f(z)dz$?

> The integral is
>
> $$\begin{aligned}\int_\gamma f(z)dz &= \int_0^{2\pi} re^{int}(rie^{it})dt \\ &= r^{n+1}i\int_0^{2\pi} e^{i(n+1)t}dt\end{aligned}$$
>
> <u>If $n \neq -1$:</u> $e^{i(n+1)t}dt$ has primitive
>
> $$\frac{1}{i(n+1)}e^{i(n+1)t}$$
>
> So, the integral is
>
> $$\int_0^{2\pi} e^{i(n+1)t}dt = 0$$
>
> <u>If $n = -1$:</u> Then, we have
>
> $$\begin{aligned}\int_\gamma f(z)dz &= i\int_0^{2\pi} e^{i(n+1)t}dt \\ &= \int_0^{2\pi} \\ &= 2\pi i\end{aligned}$$

Precisely why does the primitive not work for $n = -1$? The issue lies with the fact that the primitive of $\frac{1}{z}$ is the ***logarithm***.

The complex logarithm isn't defined along a full circle around the origin. We'll revisit this when studying branch cuts soon.

Conclusion

We find that

$$\int_{\partial B_r(0)} z^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$$

Note: A very interesting observation is that this in ***independent of*** r. This is surprising, and foreshadows some incredible results we'll see soon.