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Physics c191: Introduction to Quantum Computing

Homework 3

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Question 1

(a). This cannot be used for instantaneous communication because the measurement process is probabilistic and no information is transferred when Alice takes her measurement.

(b). For operators $A, B \in \mathcal{B}(\mathbb{C}^2)$ and

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

we have

$$\begin{aligned}
\langle\Omega|A \otimes B|\Omega\rangle &= \frac{1}{2}\langle 00|A \otimes B|00\rangle + \frac{1}{2}\langle 00|A \otimes B|11\rangle + \frac{1}{2}\langle 11|A \otimes B|00\rangle + \frac{1}{2}\langle 11|A \otimes B|11\rangle \\
&= \frac{1}{2}(\text{Diagonal elements of } A \otimes B) \\
&= \frac{1}{2}\operatorname{tr}(A \otimes B) \\
&= \frac{1}{2}\operatorname{tr}(A)\operatorname{tr}(B) \\
&= \frac{1}{2}\operatorname{tr}(A^T)\operatorname{tr}(B) \\
&= \frac{1}{2}\operatorname{tr}(A^T B)
\end{aligned}$$

(c). In the case $A = \cos\alpha Z + \sin\alpha X$, $B = \cos\beta Z + \sin\beta X$ we have

$$A = \begin{pmatrix} \cos\alpha & 0 \\ 0 & -\cos\alpha \end{pmatrix} + \begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

and similarly,

$$B = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

We know that $\operatorname{tr}(AB) = (\operatorname{tr} A)(\operatorname{tr} B)$ and $\operatorname{tr} A^T = \operatorname{tr} A$. So,

$$\begin{aligned}
\operatorname{tr}(A^T B) &= \operatorname{tr}\left[\begin{pmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & \sin\beta \\ \sin\beta & -\cos\beta\end{pmatrix}\right] \\
&= \operatorname{tr}\left[\begin{pmatrix}\cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta - \cos\alpha\sin\beta & \sin\alpha\sin\beta + \cos\alpha\cos\beta\end{pmatrix}\right] \\
&= (\cos\alpha\cos\beta + \sin\alpha\sin\beta) + (\sin\alpha\sin\beta + \cos\alpha\cos\beta) \\
&= 2\,(\sin\alpha\sin\beta + \cos\alpha\cos\beta) \\
&= 2\cdot\cos(\alpha-\beta)
\end{aligned}$$

So,

$$\langle\Omega|A\otimes B|\Omega\rangle = \frac{1}{2}\operatorname{tr}(A^T B) = \cos(\alpha-\beta)$$

Now, since A and B are linear combinations of X and Z, which each have eigenvalues ± 1, the same will hold for A and B. **(Will it really? Double check)** . Hence $A\otimes B$ will have eigenvalues ± 1 where the observable value is $+1$ if the measurement outcomes for both Alice and Bob are the same, and -1 otherwise.

Let p denote the probability that their measurements are identical. Then the expected value of $A\otimes B$ upon measurement is $p(1) + (1-p)(-1) = 2p-1$, and this should be exactly $\langle\Omega|A\otimes B|\Omega\rangle$. Thus,

$$\begin{aligned}
\cos(\alpha-\beta) &= 2p-1 \\
\implies p &= \frac{1}{2}\left[1+\cos(\alpha-\beta)\right]
\end{aligned}$$

and using the identity $1+\cos(\theta) = 2\cos^2(\theta/2)$ we have

$$\begin{aligned}
p &= \frac{1}{2}\cdot 2\cos\left(\frac{\alpha-\beta}{2}\right) \\
\implies & \boxed{p = \cos^2\left(\frac{\alpha-\beta}{2}\right)}
\end{aligned}$$

(d). A_1, A_2, B_1, B_2 are defined by the angles $\alpha_1 = \pi/2,\ \alpha_2 = 0, \beta_1 = \pi/4,\ \beta_2 = 3\pi/4$ respectively. The chance of success is then,

$$\begin{aligned}
P_s &= \frac{1}{4}\cdot\underbrace{\cos^2\left(\frac{\alpha_1-\beta_1}{2}\right)}_{A_1=B_1} + \frac{1}{4}\cdot\underbrace{\cos^2\left(\frac{\alpha_1-\beta_2}{2}\right)}_{P(A_1=B_2)} + \frac{1}{4}\cdot\underbrace{\cos^2\left(\frac{\alpha_2-\beta_1}{2}\right)}_{P(A_2=B_1)} + \frac{1}{4}\cdot\underbrace{\left[1-\cos^2\left(\frac{\alpha_2-\beta_2}{2}\right)\right]}_{P(A_2\neq B_2)} \\
&= \frac{1}{4}\cdot\cos^2\left(\frac{\pi}{8}\right) + \frac{1}{4}\cdot\cos^2\left(-\frac{\pi}{8}\right) + \frac{1}{4}\cdot\cos^2\left(-\frac{\pi}{8}\right) + \frac{1}{4}\cdot\left[1-\cos^2\left(-\frac{3\pi}{8}\right)\right] \\
&= \frac{3}{4}\cos^2\left(\frac{\pi}{8}\right) - \frac{1}{4}\cos^2\left(\frac{3\pi}{8}\right) + \frac{1}{4}
\end{aligned}$$

and we note that

$$\frac{3\pi}{8} = \frac{\pi}{4} + \frac{\pi}{8}$$

Then using the identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

we have

$$\cos(\frac{3\pi}{8}) = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\right]$$

Doing some more algebra we can find that

$$1 - \cos^2\left(\frac{3\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right)$$

Thus giving us

$$\boxed{P_s = \cos^2\left(\frac{\pi}{8}\right)}$$

(e). If we were dealing with classically preprogrammed and predetermined values of A_1, A_2, B_1, B_2 then the maximal average success probability would be $3/4$ since it's possible to simultaneously satisfy three of the four conditions.

Question 2

(a). We want to show that the n-qubit Hadamard gate acts as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|$$

Recall that the Hadamard Gate acting on a single qubit can be expressed as

$$\begin{aligned} H &= |+\rangle\langle 0| + |-\rangle\langle 1| \\ &= \frac{1}{\sqrt{2}} \left[(|0\rangle + |1\rangle) \langle 0| + (|0\rangle - |1\rangle) \langle 1| \right] \\ &= \frac{1}{\sqrt{2^1}} \sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |x\rangle\langle y| \end{aligned}$$

Now, the hilbert space for an $n-$qubit system is $\mathcal{H} = \mathbb{C}^n$ and has (tensor-product) basis vectors of the form $|x\rangle = |x_1 \cdots x_n\rangle = \otimes_{i=1}^n |x_i\rangle$, for $x \in \{0,1\}^n$ or equivalently $x_i \in \{0,1\}$.

The Hadamard gate acting on $n-$qubits can thus be written as

$$\begin{aligned} \left[H^{\otimes n} \left(\sum_x |x\rangle \right) \right] \left(\sum_y \langle y| \right) &= \sum_y \left\{ \sum_{x = x_1 \cdots x_n} (H|x_1\rangle \otimes \cdots \otimes H|x_n\rangle) \right\} \langle y| \\ &= \sum_y \left[\sum_{x = x_1 \cdots x_n} \frac{1}{\sqrt{2}} (|\pm\rangle_i) \right] \langle y| \\ &= \left(\frac{1}{\sqrt{2}} \right)^n \sum_{y = y_1 \cdots y_n} \sum_{x = x_1 \cdots x_n} (-1)^{x \cdot y} |x\rangle\langle y| \end{aligned}$$

where, in the third line, $|\pm\rangle_i$ is supposed to denote that H maps $|x_i\rangle$ to $|+\rangle$ if $x_1 = 0$ and $|-\rangle$ if $x_i = 1$. We go from the third to the fourth line via the exact same logic as the $1-$qubit case. Thus, we have that

$$\boxed{H = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|}$$

(b). Explicitly calculating the tensor product $H \otimes H$, we have

$$\begin{aligned} H^{\otimes 2} &= H \otimes H \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{aligned}$$

Using the formula from (a), we have

$$H^{\otimes 2} = \frac{1}{\sqrt{2^2}} \sum_{x\in\{0,1\}^2} \sum_{y\in\{0,1\}^2} (-1)^{x\cdot y} |x\rangle\langle y|$$

The only time the (-1) factor survives is when x, y have a different number of ones. Evaluating the formula for each bitstring x, y we have:

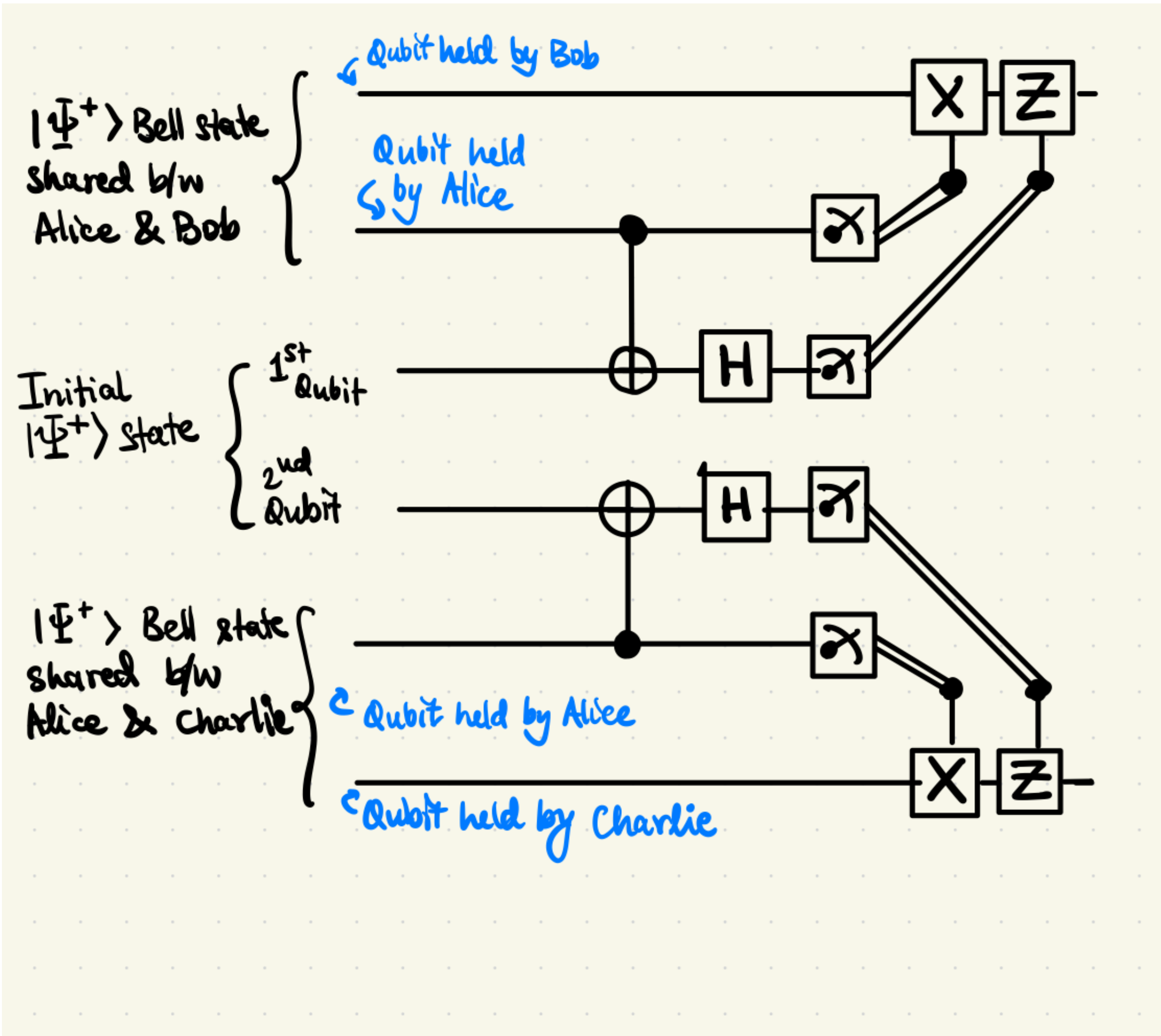
1. $x = 00, y = 00 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|00\rangle\langle 00|$
2. $x = 00, y = 01 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|00\rangle\langle 01|$
3. $x = 00, y = 10 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|00\rangle\langle 10|$
4. $x = 00, y = 11 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|00\rangle\langle 11|$
5. $x = 01, y = 00 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|01\rangle\langle 00|$
6. $x = 01, y = 01 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|01\rangle\langle 01|$
7. $x = 01, y = 10 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|01\rangle\langle 10|$
8. $x = 01, y = 11 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|01\rangle\langle 11|$
9. $x = 10, y = 00 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|10\rangle\langle 00|$
10. $x = 10, y = 01 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|10\rangle\langle 01|$
11. $x = 10, y = 10 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|10\rangle\langle 10|$
12. $x = 10, y = 11 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|10\rangle\langle 11|$
13. $x = 11, y = 00 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|11\rangle\langle 00|$
14. $x = 11, y = 01 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|11\rangle\langle 01|$
15. $x = 11, y = 10 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^0|11\rangle\langle 10|$
16. $x = 11, y = 11 \implies (-1)^{x\cdot y}|x\rangle\langle y| = (-1)^1|11\rangle\langle 11|$

giving us

$$H^{\otimes 2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Question 3

(a) The circuit can be shown as



(b) Yes, because the qubits held by all three individuals just before the classical measurements are entangled. **(I'm not quite sure how to phrase it, but it's almost like they must be entangled by transitivity.)**