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# Math 215A: Algebraic Topology

Homework 2

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**Question 1:** For the manifold of *complete* flags $V^1 \subset V^2 \subset \mathbb{C}^3$ in (say, complex) $3-$space describe explicitly which flags belong to which Bruhat cell.

**Solution:**

The set $\mathbb{C}F(n; 1, 2, \cdots, n-1)$ is the set of complete flags i.e.

$$\mathbb{C}F(n; 1, 2, \cdots, n-1) = \{\text{chains } (V_1 \subset V_2 \cdots \subset V_n) \mid \dim_{\mathbb{C}}(V_i) = i\}$$

In the Bruhat Cell Decomposition of a flag manifold, the Bruhat cells are characterized by the dimensions of intersections $d_{ij} = \dim\left(V_i \cup \mathbb{C}^j\right)$.

In our case, we're considering the manifold of complete flags $\mathbb{C}F(3; 1, 2)$ so we have a total of 6 Bruhat cells, corresponding to the six permutations of the sequence $(1, 2, 3)$:

$$\begin{aligned} &(1,2,3), \quad (3,1,2), \quad (2,3,1) \\ &(1,3,2), \quad (2,1,3), \quad (3,2,1) \end{aligned}$$

The dimension of the cell $e[m_1, \cdots, m_n]$ is equal to the number of pairs $(i, j)$ for which $i < j, m_i > m_j$ So,

$$\begin{aligned} &e[1,2,3] \text{ has dimension } 0 \\ &e[1,3,2] \text{ and } e[2,1,3] \text{ have dimension } 1 \\ &e[3,1,2] \text{ and } e[2,3,1] \text{ have dimension } 2 \\ &e[3,2,1] \text{ has dimension } 3 \end{aligned}$$

The flags in the cells are specified as below ($V_3 = \mathbb{C}^3$ in all cases below) :

(a) The flag contained in the 0-dimensional cell $e[1, 2, 3]$ is $V_1 = \mathbb{C}^1, V_2 = \mathbb{C}^2, V_3 = \mathbb{C}^3$.

(b) The two flags in the 1-dimensional cells are

- $V_1 = \mathbb{C}^1$ and $V_2 =$ any 2-d complex plane containing $\mathbb{C}^1$ other than the standard copy of $\mathbb{C}^2$, $V_3 = \mathbb{C}^3$
- $V_2 = \mathbb{C}^2$ and $V_1 =$ any 1-d complex line contained in $\mathbb{C}^2$ other than the standard copy of $\mathbb{C}^1$

(c) The two flags in the 2-dimensional cells are:

- $V_1$ being any line in $\mathbb{C}^2$ and $V_2$ being any 2d plane containing $V_1$ other than $\mathbb{C}^2$
- $V^2 \neq \mathbb{C}^2$ and $V_1$ being any line contained in $V^2$ which is not $\mathbb{C}^1$

(d) The one 3-dimensional flag is $V_1 =$ any 1-d complex line other than $\mathbb{C}^1$, $V_2 =$ any 2-d complex plane other than $\mathbb{C}^2$

**Question 2:** Let $c_k$ be the number of $k-$dimensional (in complex units) Bruhat cells in the manifold of complete flags in an $n-$dimensional complex space. Show that the generating function $c_0 + c_1 q + c_2 q^2 + \cdots$ of this sequence is equal to the "$q-$factorial" : the product of $\frac{(1-q^k)}{(1-q)}$ over $k = 1, \cdots, n$ and check this for your answer in $(a)$.

**Solution:** (Collaborated with Finn Fraser Grathwol for this question)

Let's consider a finite field. The complete $n-$th flag manifold over finite field $\mathbb{F}_q = \{1, \cdots, q\}$ is $\mathbb{F}F(n; 1, \cdots, n-1) = {}^{\mathrm{GL}_n(\mathbb{F})}/_{B}$ where $B$ is the subgroup of $\mathrm{GL}_n(\mathbb{F})$ formed by upper triangular matrices.

$\mathbb{F}_q^n$ contains $q^n$ vectors, with $q^n - 1$ of them being non-zero. Now, consider some 1-d subspace $V_1 \subseteq \mathbb{F}_q^n$. The $q-1$ non-zero scalar multiples of the $q^n - 1$ nonzero vectors span the same subspaces, so there are

$$\frac{q^n - 1}{q - 1}$$

vectors that could intersect $V_1$. Similarly for a 2d subspace $V_2$, since one dimension is already fixed, there are $\frac{q^{n-1}-1}{q-1}$ choices we can make, and so on for $V_i$ until we hit $V_n$ for which the number of choices is $\frac{q-1}{q-1} = 1$.

Thus, we find that the number of complete flags is

$$\Pi_{k=1}^n \frac{q^k - 1}{q - 1}$$

On the other hand, each Bruhat cell of dimension $k$ is parametrized by $k-$points (from a $k-$dimensional affine space). Thus, if we denote the number of cells as $c_k$ then we have the result.

This matches up with (a) wherein we have $f_3 = \left(q^3 - 1\right)\left(q^2 - 1\right)\left(q^2 - 1\right) / (q-1)^3 = 1 + 2q + 2q^2 + 2q^3$.

**Question 3:** Prove that $\mathbb{S}^\infty$ is contractible.

**Solution:**

Let's denote the natural "equitorial" inclusion $x \mapsto (x, 0)$ as $\iota : \mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$. Consider the following map:

$$\begin{aligned} F : \mathbb{S}^n \times I &\to \mathbb{S}^{n+1} \\ (x,t) &\mapsto \left(\sqrt{1-t^2}x, t\right) \end{aligned}$$

This is a continuous map since its components are continuous, and we notice that $F(x,0) = (x,0) = \iota(x)$ and $F(x,1) = (0,1)$. Thus, $F(x,t)$ is a homotopy between $\iota$ and the contant map $\mathbb{S}^n \ni x \mapsto (0,1) \in \mathbb{S}^{n+1}$. Thus, for every $n \in \mathbb{N}$, $\mathbb{S}^n$ is contractible in $\mathbb{S}^{n+1}$.

This particular homotopy can be visualized as dragging $\mathbb{S}^n$ from the equator to the north pole of $\mathbb{S}^{n+1}$, but an equivalent homotopy would be to imagine one point $x_0$ on the inclusion of $\mathbb{S}^n$ to be fixed and to drag the rest of $\mathbb{S}^n$ over the surface of $\mathbb{S}^{n+1}$, passing the nole pole, and collecting into the fixed point $x_0$.

Now, to make $\mathbb{S}^\infty$ contractible we can extend homotopies between the finite-dimensional spheres to $\mathbb{S}^\infty$ using Borsuk's theorem.

For time-interval $[0, 1/2)$ contract $\mathbb{S}^1$ to a point $x_0 \in \mathbb{S}^2$, and then extend the homotopy $F_1 : \mathbb{S}^1 \times I \to \mathbb{S}^2$ to a homotopy from $S^\infty \times I$ to $\mathbb{S}^2$, for time-interval $[1/2, 3/4)$ contract $\mathbb{S}^2$ to a point $x_0 \in \mathbb{S}^3$ (imagining this to be the same fixed point mentioned earlier) and similarly extend it to a homotopy on $\mathbb{S}^\infty$ using Borsuk's Theorem.

Doing this for all $\mathbb{S}^n$, and taking the composition of all the homotopies we get a map $\mathbb{S}^\infty \times [0,1) \to \mathbb{S}^\infty$ (we never actually hit $t = 1$ in the description above since we keep going for infinitely many $n$), which we can extend to a map $\mathbb{S}^\infty \times [0,1] \to \mathbb{S}^\infty$ such that the map $\mathbb{S}^\infty \times 0 \to \mathbb{S}^\infty$ is just the identity and the map $\mathbb{S}^1 \times 1 \to \mathbb{S}^\infty$ is the constant map to $x_0$.

Thus, we have $\mathbb{S}^\infty$ is contractible to a point.