Physics 137B Lecture 4

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These are notes taken from lectures on Quantum Mechanics delivered by Professor Raúl A. Briceño for UC Berekley's Physics 137B class in the Sprng 2024 semester.

Contents

1 January 24 - Introduction to Perturbation Theory

In 137A, we could solve $\hat{H}|n\rangle = E_n|n\rangle$ exactly and obtain the stationary states, then build the general solution as

$$|\psi(t=0)\rangle = \sum_n c_n|n\rangle$$

and

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|n\rangle$$

This was possible because we were studying systems with relatively simple/convenient Hamiltonians, but *most* situations that we want to study aren't so simple.

Where can we apply P.T.?

- Perturbation Theory allows us to study Hamiltonians of the form

  $$\hat{H} = \hat{H}_0 + \lambda\hat{H}'$$

  where $\hat{H}_0$ is a Hamiltonian we can solve

  $$\hat{H}_0|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle$$

  and $\lambda \in (0, 1]$.

- The idea is to parametrize our solutions in terms of λ and find the $\lambda \to 1$ solution.

> Why avoid $\lambda = 0$?
>
> We can think of PT as doing
>
> $$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_1 \\ &= \hat{H}_0 + \lambda\left(\frac{\hat{H}_1}{\lambda}\right) \\ &= \hat{H}_0 + \lambda\hat{H}'\end{aligned}$$
>
> So, $\lambda = 0$ would be a problem.

1.1 Time Independent Perturbation Theory

We want to solve the Hamiltonian

$$\hat{H}|n\rangle = \left(\hat{H}_0 + \lambda\hat{H}'\right)|n\rangle = E_n|n\rangle \quad (1)$$

We assume that our solutions can be parametrized as functions of λ. Then, we taylor expand them as

$$|n\rangle = \sum_{j=0}^{\infty} \lambda^j |n^{(j)}\rangle$$

$$E_n = \sum_{j=0}^{\infty} E_n^{(j)}\lambda^j$$

Now, if we bring the RHS of equation (1) to the left, we have

$$\begin{aligned}
0 &= \left(\hat{H}_0 + \lambda \hat{H}' - E_n\right)|n\rangle \\
&= \left(\hat{H}_0 + \lambda \hat{H}' - \sum_{j=0}^{\infty} E_n^{(j)}\lambda^j\right)\left(\sum_{k=0}^{\infty}\lambda^k |n^{(k)}\rangle\right) \\
&= \left(\hat{H}_0 + \lambda \hat{H}' - (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots)\right)\left(|n\rangle^{(0)} + \lambda |n\rangle^{(1)} + \lambda^2 |n\rangle^{(2)} + \cdots\right)
\end{aligned}$$

Now, let's see what happens when we trunate the resulting sum at different powers of λ:

$$\begin{aligned}
\mathcal{O}(\lambda^0) &: \left(\hat{H}_0 - E_n^0\right)|n^0\rangle = 0 \quad \left[\langle k^{(0)}|n^{(0)}\rangle = \delta_{kn}\right] \\
\mathcal{O}(\lambda^1) &: \lambda\left(\left(\hat{H}' - E_n^{(1)}\right)|n^{(0)}\rangle + \left(\hat{H}_0 - E_n^{(0)}\right)|n^1\rangle\right)
\end{aligned}$$

In the $\mathcal{O}(1)$ seres, we don't know what $E_n^{(1)}$ is. To obtain this correction, we simply **act** $\langle n^{(0)}|$ **on the equation:**

$$\begin{aligned}
0 &= \langle n^{(0)}|\left(\hat{H}' - E_n^{(1)}\right)|n^0\rangle + \underbrace{\langle n^{(0)}|\left(\hat{H}_n^{(0)} - E_n^{(0)}\right)|n^1\rangle}_{=0,\text{ since } E_n^{(0)}\langle n^{(0)}| - E_n^{(0)}\langle n^{(0)}| = 0} \\
&= \langle n^{(0)}|\hat{H}'|n^{(0)}\rangle - E_n^{(1)}\underbrace{\langle n^{(0)}|n^{(0)}\rangle}_{1} \\
&\implies \boxed{E_n^1 = \langle n^{(0)}|\hat{H}'|n^{(0)}\rangle}
\end{aligned}$$

This gives us the leading order Eigenenergy correction! But we still need to find actual states. So, next, we need to solve for $|n^{(1)}\rangle$.

> So far we've been studynig Non-degenerate Perturbation Theory. This only applies for Hamiltonians with no degeneracies i.e. Hamiltonians for which
>
> $$E_n^{(0)} = E_k^{(0)} \iff n = k$$

We can solve for $|n\rangle^{(0)}$ in terms of the non-perturbative stationary states $|k^{(0)}\rangle$ as

$$\begin{aligned}
|n^{(1)}\rangle &= \sum_k c_{nk}^{(1)}|k^{(0)}\rangle \\
&= c_{nn}^{(1)}|n^{(0)}\rangle + \sum_{k\neq n} c_{nk}^{(1)}|k^{(0)}\rangle
\end{aligned}$$

Note that $|n\rangle$ is not yet normalized, so for now we can assume $|n^{(\lambda)}\rangle$ is some arbitrary linear combination of the $\{|n^{(0)}\rangle, |n^{(1)}\rangle, \dots\}$ and worry about the norm *later*.

So,

$$\begin{aligned}
|n^{(\lambda)}\rangle &= |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \cdots \\
&= \underbrace{\left(1 + \lambda c_{nn}^{(0)}\right)}_{A}\left(|n^{(0)}\rangle + \underbrace{\left(\frac{\lambda}{A}\right)}_{\lambda^1}\sum_{k\neq n} c_{nk}^{(1)}|k^{(0)}\rangle + \cdots\right)
\end{aligned}$$

Let

$$\boxed{|n^{(1)}\rangle = \sum_{k\neq n} c_{nk}^{(1)} |k^{(0)}\rangle}$$

Our current goal, then, is to find $c_{nk}^{(1)}$.
Then,

$$\begin{aligned}
0 &= \left(\hat{H}_0 - E_n^{(0)}\right) |n^{(1)}\rangle + \left(\hat{H}' + E_n^{(1)}\right) |n^{(0)}\rangle \\
&= \left(\hat{H}_0 - E_n^{(0)}\right) \sum_{k\neq n} c_{nk}^{(1)} |k\rangle^{(0)} + \left(\hat{H}' - E_n^{(1)}\right) |n^{(0)}\rangle \\
&= \sum_{k\neq n} c_{nk}^{(1)} \left(E_k^{(0)} - E_n^{(0)}\right) |k^{(0)}\rangle + \left(\hat{H}' - E_n^{(1)}\right) |n^{(0)}\rangle
\end{aligned}$$

Now, to extract $|n^{(0)}\rangle$, we act using another stationary state $|l^{(0)}\rangle$ where $l \neq n$.

$$\implies \sum_{k\neq n} c_{nk}^{(1)} \left(E_k^{(0)} - E_n^{(0)}\right) \underbrace{\langle l^{(0)}|k^{(0)}\rangle}_{\delta_{lk}} + \langle l^{(0)}|\hat{H}'|n^{(0)}\rangle - E_n^{(1)} \underbrace{\langle l^{(0)}|n^{(0)}\rangle}_{0} = 0$$

[Lecture ended, so stopped abruptly. Pick up from here in lec 5 notes.]