Math 214 Homework 11

Keshav Balwant Deoskar

April 25, 2024

Q11-14. Consider the following two covector fields on $\mathbb{R}^3$:

$$\omega = -\frac{4zdx}{(x^2+1)^2} + \frac{2ydy}{y^2+1} + \frac{2xdz}{x^2+1}$$
$$\eta = -\frac{4xzdx}{(x^2+1)^2} + \frac{2ydy}{y^2+1} + \frac{2dz}{x^2+1}$$

(a) Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.

(b) Determine whether either of these covector fields is exact.

(c) For each one that is exact, find a potential function and use it to recompute the line integral.

Proof:

(a) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the following parametrization of the straight line from $(0,0,0)$ to $(1,1,1)$:

$$\gamma(t) = (t,t,t), \quad t \in [0,1]$$

So,

$$\begin{aligned}
\int_\gamma \omega &= \int_0^1 -\frac{4tdt}{(t^2+1)^2} + \frac{2tdt}{t^2+1} + \frac{2tdt}{t^2+1} \\
&= -\int_0^1 \frac{4tdt}{(t^2+1)^2} + \int_0^1 \frac{2tdt}{t^2+1} + \int_0^1 \frac{2tdt}{t^2+1}
\end{aligned}$$

We can solve each of these as in ordinary calculus to get

$$\begin{aligned}
\int_\gamma \omega &= -\int_1^2 \frac{2du}{u^2} + 2 \times \int_1^2 \frac{du}{u} \\
&= -2\left[\frac{-1}{u}\right]_1^2 + 2\left[\ln(u)\right]_1^2 \\
&= 2\left[\frac{1}{2} - \frac{1}{1}\right] + 2\left[\ln(2) - \ln(1)\right] \\
\implies \int_\gamma \omega &= 2\ln(2) - 1
\end{aligned}$$

We can similarly calculate the integral of η:

$$
\begin{aligned}
\int_\gamma \eta &= \int_0^1 \frac{-4t^2dt}{(t^2+1)^2} + \frac{2tdt}{t^2+1} + \frac{2dt}{t^2+1} \\
&= \int_0^1 \left(\frac{-4t^2}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2}{t^2+1} \right) dt \\
&= \int_0^1 \frac{2\left(t^3-t^2+t+1\right)}{(t^2+1)^2} dt \\
&= \ln(2) + 1
\end{aligned}
$$

(b) We can check that ω is closed by verifying all of the mixed second derivatives are equal:

$$
\begin{aligned}
\frac{\partial^2\omega}{\partial y \partial x} &= \frac{\partial\left(\frac{4z}{(x^2+1)^2}\right)}{\partial y} = 0 = \frac{\partial\left(\frac{2y}{y^2+1}\right)}{\partial x} = \frac{\partial^2\omega}{\partial x \partial y} \\
\frac{\partial^2\omega}{\partial z \partial x} &= \frac{\partial\left(\frac{4z}{(x^2+1)^2}\right)}{\partial z} = -\frac{4}{(x^2+1)^2} = \frac{\partial\left(\frac{2x}{x^2+1}\right)}{\partial x} = \frac{\partial^2\omega}{\partial x \partial z} \\
\frac{\partial^2\omega}{\partial z \partial y} &= \frac{\partial\left(\frac{2y}{y^2+1}\right)}{\partial z} = 0 = \frac{\partial\left(\frac{2x}{x^2+1}\right)}{\partial y} = \frac{\partial^2\omega}{\partial y \partial z}
\end{aligned}
$$

Then, since the line segment from $(0,0,0)$ to $(1,1,1)$ is a star-shaped subset of $\mathbb{R}^3$, the Poincare Lemma (Theorem 11.49) tells us that ω is closed $\implies$ ω is exact.

On the other hand, for η we see that

$$
\begin{aligned}
\frac{\partial^2\eta}{\partial z \partial x} &= \frac{\partial\left(\frac{4xz}{(x^2+1)^2}\right)}{\partial z} = \frac{4x}{(x^2+1)^2} \\
\frac{\partial^2\eta}{\partial x \partial z} &= \frac{\partial\left(\frac{2}{x^2+1}\right)}{\partial x} = -\frac{4}{(x^2+1)^2} \\
\implies \frac{\partial^2\eta}{\partial z \partial x} &\neq \frac{\partial^2\eta}{\partial x \partial z}
\end{aligned}
$$

Thus η is not closed, but every exact form must be closed so η is not exact.

(c)

Q11-17. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$ denote the n-torus. For each $j = 1, \cdots, n$, let $\gamma_j : [0,1] \to \mathbb{T}^n$ be the curve segment

$$
\gamma_j(t) = \left(1, \cdots, e^{2\pi i t}, \cdots, 1\right) \quad \text{(with } e^{2\pi i t} \text{ in the } j^{th} \text{ place)}
$$

Show that a closed covector field ω on $\mathbb{T}^n$ is exact if and only if $\int_{\gamma_j} \omega = 0$ for $j = 1, \cdots, n$

Proof:
" $\implies$ ": Suppose closed covector field $\omega \in \mathfrak{X}^*\left(\mathbb{T}^2\right)$ is exact. Then, there exists a potential function

$f : M \to \mathbb{R}$ such that

$$\begin{aligned}\int_{\gamma_j} \omega &= \int_{\gamma_j} df \\ &= f(\gamma_j(1)) - f(\gamma_j(1)) \text{ (By the fundamental theorem of line integrals)} \\ &= f\left(1, \cdots, e^{2\pi i \cdot 1}, \cdots, 1\right) - f\left(1, \cdots, e^{2\pi i \cdot 0}, \cdots, 1\right) \\ &= f\left(1, \cdots, 1, \cdots, 1\right) - f\left(1, \ \cdots, 1, \cdots, 1\right) \\ &= 0\end{aligned}$$

" $\Longleftarrow$ ":

For the reverse direction, suppose that for each $j = 1, 2, \cdots, n$ we have

$$\int_{\gamma_j} \omega = 0$$

i.e. the integral of ω along each circle in the decomposition $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is zero. If we can somehow show that $\int_\gamma \omega$ for *any* piece-wise smooth closed curve, then ω will be conservative and thus exact.

We have the smooth covering map $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ defined by

$$\left(x^1, \cdots, x^n\right) \mapsto \left(e^{2\pi i x^1}, \cdots, e^{2\pi i x^n}\right)$$

Consider any piece-wise smooth closed curve segment $\gamma : [0,1] \to \mathbb{T}^n$ and let $\tilde{\gamma} : [0,1] \to \mathbb{R}^n$ be a lift of γ such that

$$\gamma = \varepsilon^n \circ \tilde{\gamma}$$

Since ε^n is smooth, surjective and γ is also smooth, exercise 4.10 tells us that $\tilde{\gamma}$ must be smooth as well. WLOG, we can assume that $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = \left(m^1, \cdots, m^n\right)$ for integers $m^1, \cdots, m^n$ (if these are not integers, then $\gamma = \varepsilon^n \circ \tilde{\gamma}$ will not be closed).

Going forward, the idea will be to decompose the integral over γ into integrals over paths along the γ_j's. Define $\alpha_i : [0,1] \to \mathbb{T}^n$ as the line segment from $\left(m^1, \cdots, m^{i-1}, 0, \cdots, 0\right)$ to $\left(m^1, \cdots, m^i, 0, \cdots, 0\right)$ and let $\alpha : [0,1] \to \mathbb{T}^n$ be the concatenation of the α_i's such that $\alpha(0) = 0$ and $\alpha(1) = \left(m^1, \cdots, m^n\right)$. Then,

$$\begin{aligned}\int_\gamma \omega &= \int_{\varepsilon^n \circ \tilde{\gamma}} \omega \\ &= \int_{\tilde{\gamma}} \left(\varepsilon^n\right)^* \omega \\ &= \int_\alpha \left(\varepsilon^n\right)^* \omega \text{ (Prop 11.42; Since } \tilde{\gamma}, \alpha \text{ start and end at the same points)} \\ &= \int_{\alpha_1} \left(\varepsilon^n\right)^* \omega + \cdots + \int_{\alpha_n} \left(\varepsilon^n\right)^* \omega \\ &= \int_{\varepsilon^n \circ \alpha_1} \omega + \cdots + \int_{\varepsilon^n \circ \alpha_n} \omega \\ &= \int_{\varepsilon^n \circ \gamma_1} \omega + \cdots + \int_{\varepsilon^n \circ \gamma_n} \omega \text{ (Again due to Prop 11.42)} \\ &= 0\end{aligned}$$

This shows that ω is a conservative covector field on a smooth manifold with or without boundary. Thus, by Proposition 11.42 it must be exact.

Q12-6.

(a) Let α be a covariant $k-$tensor on a finite dimensional real vector space V. Show that $\text{Sym}\alpha$ is the unique symmetric $k-$tensor satisfying

$$(\text{Sym}\alpha)\,(v,\cdots,v) = \alpha\,(v,\cdots,v)$$

for all $v \in V$.

(b) Show that the symmetric product is associative: for all symmetric tensors α, β, γ,

$$(\alpha\beta)\,\gamma = \alpha\,(\gamma\beta)$$

(c) Let $\omega^1, \cdots, \omega^k$ be covectors on a finite-dimensional vector space. Show that their symmetric product satisfies

$$\omega^1, \cdots, \omega^k = \frac{1}{k!}\sum_{\sigma\in S_k} \omega^{\sigma(1)} \otimes \cdots \otimes \omega^{\sigma(k)}$$

Proof:

(a) Suppose there exists some other covariant $k-$tensor $(\text{Sym})' \in \Sigma_k(V^*)$ such that

$$(\text{Sym}\ \alpha)'\,(v,\cdots,v) = (v,\cdots,v)$$

Define $\beta \equiv (\text{Sym}) - (\text{Sym})' \in \Sigma_k(V^*)$. Now, fix $v, w_1 \in V$ and for $\epsilon > 0$ let $\gamma_v : (-\epsilon,\epsilon) \to V$ be the map

$$t \mapsto v + tw$$

Then, for all $t_0 \in (-\epsilon,\epsilon)$ we have $\beta(\gamma_v(t_0),\cdots,\gamma_v(t_0)) = 0$ so

$$\left.\frac{d}{dt}\right|_{t_0} \beta(\gamma_v(t_0),\cdots,\gamma_v(t_0)) = 0$$

So, in particular, at $t = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \beta(\gamma_v(t_0),\cdots,\gamma_v(t_0)) = \beta(w_1,v,\cdots,v) + \beta(v,w_1,\cdots,v) + \cdots + \beta(v,\cdots,v,w_1) = 0$$

But since β is symmetric, this means

$$\begin{aligned} &k\cdot\beta(w_1,v,\cdots,v) = 0\\ \implies &\beta(w_1,v,\cdots,v) = 0\\ \implies &(\text{Sym}\alpha)(w_1,v,\cdots,v) = (\text{Sym}'\alpha)(w_1,v,\cdots,v)\end{aligned}$$

We can bascially the same arguement for the remaining $k-1$ entries so we finally obtain

$$(\text{Sym}\alpha)(w_1,w_2,\cdots,w_k) = (\text{Sym}'\alpha)(w_1,w_2,\cdots,w_k)$$

Thus, Sym α is unique.

(b) Suppose we have symmetric covariant tensors α,β,γ of rank k,l,m on a vector space V.

Now, $\beta\gamma = \text{Sym}(\beta\otimes\gamma)$ is an $(l+m)-$rank covector on V, so $\alpha(\beta\gamma) = \text{Sym}\,(\alpha\otimes\text{Sym}(\beta\otimes\gamma))$ is a rank $(k+l+m)$ covariant tensor on V.

For any $v \in V$,

$$\begin{aligned}(\alpha\beta)\,\gamma &= (\mathrm{Sym}\,(\mathrm{Sym}(\alpha\otimes\beta)\otimes\gamma))\,(v,\cdots,v)\\ &= (\mathrm{Sym}(\alpha\otimes\beta))\,(v,\cdots,v)\gamma\,(v,\cdots,v)\\ &= \alpha(v,\cdots,v)\beta(v\cdots,v)\gamma(v,\cdots,v)\\ &= \alpha(v,\cdots,v)\,(\mathrm{Sym}(\beta\otimes\gamma))\,(v,\cdots,v)\\ &= (\mathrm{Sym}\alpha\otimes(\mathrm{Sym}(\beta\otimes\gamma)))\,(v,\cdots,v)\\ &= \alpha(\beta\gamma)\end{aligned}$$

Therefore, the symmeteric product is associative.

$$\alpha\,(\beta\gamma) = (\alpha\beta)\,\gamma = \alpha\beta\gamma$$

(c) Now let $\omega^1, \cdots, \omega^k$ be covectors on a finte-dimensional vector space.

We know from proposition 12.15(b) that

$$\omega^1\omega^2 = \frac{1}{2}\left(\omega^1\otimes\omega^2 + \omega^2\otimes\omega^1\right) = \frac{1}{2!}\sum_{\sigma\in S_2}\omega^{\sigma(1)}\otimes\cdots\omega^{\sigma(k)}$$

So the base case $k=2$ is satisfied.

Now, let's assume that the inductive hypothesis holds for $k-1$ i.e.

$$\omega^1\cdots\omega^{k-1} = \frac{1}{(k-1)!}\sum_{\sigma\in S_{k-1}}\omega^1\otimes\cdots\otimes\omega^{k-1}$$

Then,

$$\begin{aligned}\left(\omega^1\cdots\omega^{k-1}\right)\omega^k &= \frac{1}{2}\left[\left(\omega^1\cdots\omega^{k-1}\right)\otimes\omega^k + \omega^k\otimes\left(\omega^1\cdots\omega^{k-1}\right)\right]\\ &= \frac{1}{2}\left[\frac{1}{(k-1)!}\sum_{\sigma\in S_{k-1}}\omega^1\otimes\cdots\otimes\omega^{k-1}\otimes\omega^k + \omega^k\otimes\frac{1}{(k-1)!}\sum_{\sigma\in S_{k-1}}\omega^1\otimes\cdots\otimes\omega^{k-1}\right]\end{aligned}$$

Now, by the associativity of the symmetric product shown in (b), we have

$$(\omega^1,\cdots,\omega^{k-1})\omega^k = \omega^1(\omega^2\cdots,\omega^{k-1}\omega^k) = \left(\omega^1,\cdots,\omega^{i-1}\right)\omega^i\left(\omega^{i+1},\cdots,\omega^k\right)$$

There are k ways to choose i, and all of these terms are equal. So, dividing by k, we find that the ω^i's can be permuted in $k!$ ways. Thus,

$$\begin{aligned}\omega^1,\cdots,\omega^k &= \frac{1}{k}\sum_{i=1}^{k}\left(\omega^1,\cdots,\omega^{i-1}\right)\omega^i\left(\omega^{i+1},\cdots,\omega^k\right)\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\omega^{\sigma(1)}\otimes\omega^{\sigma(k)}\end{aligned}$$

Q12-11. Suppose M is a smooth manifold, A is a smooth covariant tensor field on M, and $V, W \in \mathfrak{X}(M)$. Show that

$$\mathcal{L}_V\mathcal{L}_W A - \mathcal{L}_W\mathcal{L}_V A = \mathcal{L}P_{[V,W]}A$$

Proof:

Q13-2. Suppose F is a smooth vector bundle over a smooth manifold M with or without boundary, and $V \subseteq E$ is an open subset with the property that for each $p \in M$, the intersection of v with the fiber E_p is convex and non-empty. By a "section of V", we mean a (local or global section of E whose image lies in V.)

(a) Show that there exists a smooth global section of V.

(b) Suppose $\sigma : A \to V$ is a smooth section of V defined on a closed subset $A \subseteq M$. Show that there exists a smooth global section $\tilde{\sigma}$ of V whose restriction to A is equal to σ. Show that if V contains the image of the zero section of E, then $\tilde{\sigma}$ can be chosen to be supported in any predetermined neighborhood of A.

Proof:

Q13-13. Let (M, g) be a Riemannian manifold. A smooth vector field V on M is called a **Killing vector field for** g if the flow of V acts by isometries of g.

(a) Show that the set of all Killing vector fields on M constitutes a Lie subalgebra of $\mathfrak{X}(M)$.

(b) Show that a smooth vector field V on M is a Killing vector field if and only if it satisfies the follownig equation in each smooth local coordinate chart:

$$V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial V^k}{\partial x^i} + g_{ik} \frac{\partial V^k}{\partial x^j} = 0$$

Proof:
