Math 214 Homework 10

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Q10-15. Let V be a finite(n)-dimensional real vector space, and let $G_k(V)$ be the Grassmannian of $k-$dimensional subspaces of V. Let T be the subset $G_k(V) \times V$ defined by

$$T = \{(S, v) \in G_k(V) \times V \ : \ v \in S\}$$

Show that T is a smooth rank-k subbundle of the product bundle $G_k(V) \times V \to G_k(V)$, and is thus a smooth rank-k vector bundle over $G_k(V)$.

Proof:
We have a product bundle $G_k(V) \times V \to G_k(V)$ and want to show that $T = \{(S, v) \in G_k(V) \times V \ : \ v \in S\}$ is a subbundle.

Notice that for each $k-$dimensional subspace of $x \in G_k(V)$, we have a linear subspace

$$T_x = \{(x, v) \ : \ x \in G_k(V), v \in x\} \subseteq G_k(V) \times V$$

Lemma 10.32 tells us that $T = \bigcup_{x \in G_k(V)} T_x$ is a subbundle of $G_k(V) \times V$ if and only if each point of $G_k(V)$ has a neighborhood U on which there exist smooth sections $\sigma_1, \cdots, \sigma_{\dim(G_k(V))} : U \to G_k(V) \times V$ with the property that $\sigma_1, \cdots, \sigma_{k(n-k)}$ form a basis for T_x at each $x \in U$.

Recall that we can cover $G_k(V)$ with charts that look like $\phi_I : \mathrm{GL}(V_I, V_J) \to \mathrm{Gr}_k(V)$ defined by

$$\phi_I(L) = \mathrm{graph}(L) = \{v + L(v) \ : \ v \in V_I\} \subseteq V$$

where $I \subseteq \{1, \cdots, n\}$, $J = \{1, \cdots, n\} \setminus I$, and $V_I = \mathrm{span}(e_i)_{i \in I}$. Also, note that $\dim \mathrm{GL}(V_I, V_J) = k(n-k)$.

Each $x \in G_k(V)$ is a $k-$dimensional linear space whose elements are $n-$dimensional vectors. So, for an open set U containing $x = \mathrm{span}\{v_1, \cdots, v_k\} \in G_k(V)$ we can define the section $\sigma_i : U \to G_k(V) \times V$ by

$$x \mapsto (x, x^i)$$

where x^i is the i^{th} coordinate function in the coordinate $\phi_I(L)$ for $1 \leq i \leq k(n-k)$. Then, certainly, the sections $\sigma_1, \cdots, \sigma_{k(n-k)}$ form a basis for T_x. So, Lemma 10.32 gives us the desired result.

Q10-17. Suppose $M \subseteq \mathbb{R}^n$ is an immersed submanifold. Prove that the ambient tangent bundle $T\mathbb{R}^n\big|_M$ is isomorphic to the Whitney sum $TM \oplus NM$, where $NM \to M$ is the normal bundle.

Proof:
Given an immersed submanifold $M \subseteq \mathbb{R}^n$, we define the ambient tangent bundle $T\mathbb{R}^n\big|_M$ to be the set

$$T\mathbb{R}^n\big|_M = \bigcup_{p \in M} E_p = \bigcup_{p \in M} T_p\mathbb{R}^n$$

with the projection $\pi_M : T\mathbb{R}^n\big|_M \to M$ obtained by restricting π.

Recall that the normal space to a manifold M at a point $x \in M$ is the $(n-m)$ dimensional subspace $N_xM \subseteq T_x\mathbb{R}^n$ consisting of all vectors orthogonal to T_xM and the normal bundle to manifold M is the subset of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$NM = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ x \in M, v \in N_xM\}$$

To show that $T\mathbb{R}^n\big|_M \cong TM \oplus NM$, let's construct an isomorphism between the two. For any $p \in M$,since N_pM and T_pM are orthogonal compliments in $T_p\mathbb{R}^n$ we have a natural isomorphism $N_pM \oplus T_pM \cong T_p\mathbb{R}^n$. Then, we have

$$\bigcup_{p\in M} N_pM \oplus T_pM \cong \bigcup_{p\in M} T_p\mathbb{R}^n$$

We want to show that these local isomorphisms can be "combined" to get isomorphism between the whitney sum bundle and the ambient bundle.

To do so, note that the map $\phi : TM \oplus NM \to T\mathbb{R}^n\big|_M$ defined by $\phi(p, (v, w)) = (p, \phi_p(v, w))$ is smooth since the local isomorphisms ϕ_p are linear and bundle structures are smooth. Also, ϕ restricts to a linear isomorphism on each fiber, since ϕ_p is a linear isomorphism for each $p \in M$.

To show that ϕ is an isomorphism, we need to construct the inverse. Define a map $\psi : T\mathbb{R}^n\big|_M \to TM \oplus NM$ by $\psi(p, x) = (p, (x_p, y_p))$ where x_p is the orthogonal projection of x onto T_pM and $y_p = x - x_p$ is the orthogonal projection of x onto N_xM. Note that ψ is smooth, since the orthogonal prjections are smooth linear maps.

Both maps are one-to-one and onto, thus we have that ϕ is an isomorphism. So,

$$\boxed{T\mathbb{R}^n\big|_M \cong TM \oplus NM}$$

Q11-5. For any smooth manifold M, show that T^*M is a trivial vector bundle if and only if TM is trivial.

Proof:
Suppose TM is trivial. Then, M is parallelizable and admits a global frame $(E_1, \cdots, E_n)$ which serves as a basis of T_pM, for all $p \in M$. Take some open cover $\{U_\alpha\}_{\alpha\in A}$ of M. Then, $(E_1, \cdots, E_n)\big|_{U_\alpha}$ is a smooth local frame for all $\alpha \in A$.

Then, as in Example 11.13 in LeeSM, every smooth local frame has a dual smooth local coframe $\left(\epsilon^1, \cdots, \epsilon^n\right)\big|_{U_\alpha}$ such that $\epsilon^j\big|_{U_\alpha} E_i\big|_{U_\alpha} = \delta_i^j$. THen, by the gluing lemma for smooth manifolds applied to $\epsilon^j : M \to T^*M$, we can extend to a global section of the tangent bundle to obtaina global coframe for T^*M. Thus T^*M is also trivial.

The converse follows exactly the same procedure.

Q11-6. Suppose M is a smooth $n-$manifold, $p \in M$, and $y^1, \cdots, y^k$ are smooth real-valued functions defined on a neighborhood of p in M. Prove the following statements.

(a) If $k = n$ and $\left(dy^1\big|_p, \cdots, dy^n\big|_p\right)$ is a basis for T_p^*M, then $\left(y^1, \cdots, y^n\right)$ are smooth coordinates for M in some neighborhood of p.

(b) If $\left(dy^1\big|_p, \cdots, dy^k\big|_p\right)$ is a linearly independent $k-$tuple of covectors and $k < n$, then there are smooth functions $y^{k+1}, \cdots, y^n$ such that $\left(y^1, \cdots, y^n\right)$ are smooth coordinates for M in a neighborhood of p.

(c) If $\left(dy^1\big|_p, \cdots, dy^k\big|_p\right)$ span T_p^*M, there are indices $i_1, \cdots, i_n$ such that $\left(y^{i_1}, \cdots, y^{i_n}\right)$ are smooth coordinates for M in a neighborhood of p.

Proof:

(a) Let $\phi = \left(y^1, \cdots, y^n\right) : U \to M$ for some open subset $p \in U \subseteq_{open} M$. To show that $y^1, \cdots, y^n$ form smooth coordinates for M in some neighborhood of p we need to show that ϕ is a local diffeomorphism.

This can be achieved by showing that $d\phi$ is invertible (since then the Inverse Function Theorem will imply that ϕ is a local diffeomorphism).

We know that $d\phi_p : T_pU \cong T_pM \to T_{\phi(p)}\mathbb{R}^n$ is a map between tangent spaces of equal dimension. So, to show bijection, it suffices to show $d\phi$ is injective.

Since $\left(dy^1\big|_p, \cdots, dy^n\big|_p\right)$ forms a smooth coframe for the cotangent bundle, we know there must be a smooth frame $\left(\partial/\partial y^1, \cdots, \partial/\partial y^n\right)$ for the tangent bundle dual to the coframe.

Now, if $\left(x^1, \cdots, x^n\right)$ are the coordinate functions on $\mathbb{R}^n$, the coordinate representation of ϕ has components $\hat{\phi}^j = x^j \circ \phi = y^j$. Then, consider any $v = v^i \frac{\partial}{\partial x^i} \in T_pM, v \neq 0$. Then,

$$\begin{aligned}
d\phi_p(v) &= d\phi_p\left(v^i(p)\frac{\partial}{\partial y^i}\right) \\
&= \frac{\partial \hat{\phi}^j}{\partial y^i}(\hat{p})\frac{\partial}{\partial x^j} \\
&= \underbrace{\frac{\partial y^j}{\partial y^i}}_{\delta_i^j}(\hat{p})\frac{\partial}{\partial x^j} \\
&= v^j \frac{\partial}{\partial x^j}
\end{aligned}$$

which is non-zero since at least one of the components v^j is non-zero. This shows that $d\phi$ is injective.

Therefore, $d\phi$ is a bijection i.e. it is invertible, so by the Inverse Function Theorem, ϕ is a local diffeomorphism meaning $\left(y^1, \cdots, y^n\right)$ form coordinate functions on some open neighborhood of p.

Lemma for (b): Let M be a smooth manifold and $p \in M$ with $\lambda \in T_p^*M$. Then, there exists a neighborhood U of p and smooth function $y^j : M \to \mathbb{R}$ such that $dy\big|_p = \lambda_p$.

Proof: Let $\left(U, (x^i)\right)$ be a smooth chart with $p \in U$. Let $\frac{\partial}{\partial x^i}\big|_p$ be the standard basis for T_pM, and $dx^i\big|_p$ be the dual basis i.e. basis for T_p^*M. Then, we can write

$$\lambda = \lambda_i dx^i\big|_p$$

for scalars λ_i. Define $y = \lambda^i x_i$ where $\lambda^i = \lambda_i$. This is smooth since its just a linear combination of the coordinate functions. Then, indeed, we find

$$\begin{aligned}
dy\big|_p &= d\left(\lambda^i x_i\right) \\
&= \lambda_i dx^i \\
&= \lambda
\end{aligned}$$

(b) T_p^*M is an $n-$dimensional vector space, so we can choose $\omega^{k+1}, \cdots, \omega^n \in T_p^*M$ such that $\left(dy^1, \cdots, dy^k, \omega^{k+1}, \cdots, \omega^n\right)$ form a basis for T_p^*M. Then, we know from the Lemma above that there exists an open neighborhood U of p and smooth coordinate maps $y_{k+1}, \cdots, y_n$ such that $d\left(y_{k+1}\right)|_p = \omega^{k+1}, \cdots, d\left(y_n\right)|_p = \omega^n$. Then, by part (a), $y_1, \cdots, y_n$ form smooth coordinates for M in a neighborhood of p.

(c) Assuming $k > n$, the fact that $\left(dy^1\big|_p, \cdots, dy^k\big|_p\right)$ span T_p^*M means that there is some subset of n of these vectors which also span T_p^*M i.e. there exist indices $i_1, \cdots, i_n$ such that $\left(dy^{i_1}\big|_p, \cdots, dy^{i_n}\big|_p\right)$ form a basis for T_p^*M. Then, once again, by part (a) we get the desired result.

Q11-7. In the following problems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and $\omega \in \mathfrak{X}^*(N)$. Compute $F^*\omega$ in each case.

(a) $M = N = \mathbb{R}^2$;
$F(s,t) = (st, e^t)$;
$\omega = xdy - ydx$

(b) $M = \mathbb{R}^2, N = \mathbb{R}^3$;
$F(\theta, \phi) = ((\cos\phi + 2)\cos\theta, (\cos\phi + 2)\sin\theta, \sin\phi)$;
$\omega = z^2dx$

(c) $M = \{(s,t) \in \mathbb{R}^2 \ : \ s^2 + t^2 < 1\}, N = \mathbb{R}^3 \setminus \{0\}$;
$F(s,t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$;
$\omega = \left(1 - x^2 - y^2\right) dz$

Solutions:

Using Proposition 11.26, we can compute the pullback of a covector field $\omega \in \mathfrak{X}^*(N)$ under the action of smooth map $F : M \to N$ using the formula

$$F^*\omega = (\omega_j \circ F)\, d\left(y^j \circ F\right)$$

(a) We have

$$\begin{aligned} F^*\omega &= (x \circ F)\, d\,(y \circ F) + (y \circ F)\, d\,(x \circ F) \\ &= st \cdot d\left(e^t\right) + e^t d\,(st) \\ &= st \cdot te^t + st \cdot e^t \\ \implies F^*\omega &= st \cdot (e^t + 1) \end{aligned}$$

(b)

$$\begin{aligned} F^*\omega &= \left(z^2 \circ F\right) d\,(x \circ F) \\ &= \left(\sin^2\phi\right) \cdot d\,((\cos\phi + 2)\cos\theta) \\ &= \sin^2\phi \cdot [-(\cos\phi + 2)\sin\theta - \sin\phi\cos\theta] \\ &= -\sin^2\phi \cdot [(\cos\phi + 2)\sin\theta + \sin\phi\cos\theta] \end{aligned}$$

(c)

$$\begin{aligned} F^*\omega &= \left((1-x^2-y^2)\circ F\right) d\,(z\circ F) \\ &= \left(1-s^2-t^2\right)\cdot d\left(\sqrt{1-s^2-t^2}\right) \\ &= \left(1-s^2-t^2\right)\cdot\left[\frac{-s-t}{\sqrt{1-s^2-t^2}}\right] \\ &= -(s+t)\sqrt{1-s^2-t^2} \end{aligned}$$

Q11-11. Let M be a smooth manifold, and $C \subseteq M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minmium value among points in C. Given a smooth local defining function $\Phi : U \to \mathbb{R}^k$ for C on a neighborhood U of p in M, show that there are real numbers $\lambda_1, \cdots, \lambda_k$ (called ***Lagrange Multipliers***) such that

$$df_p = \lambda_1 d\Phi^1\big|_p + \cdots + \lambda_1 d\Phi^k\big|_p$$

Proof:

The smooth function $f : M \to \mathbb{R}$ attains a local maximum or minmium on $C \subseteq_{\text{embed.}} M$. So, for $p \in C$ and $v \in T_pC$, we have $df_p(v) = 0$. Now, Φ is a local defining function for C, so its differential has full rank k at every $p \in C$. So the component functions of the differential, $d\Phi^i$ for $i = 1, \cdots, k$, are linearly independent.

The component functions $d\Phi^i$ form a basis for a $k-$dimensional subspace of the cotangent space T_p^*M. In fact, this subspace is exactly the annihilator of T_pC i.e. the set of all covectors that vanish on T_pC.

This can be seen as

- For any $v \in T_pC$ we have $d\Phi^i(v) = 0$ because f attains a local extremum on C.

Now, any covector that vanishes on T_pC must be a linear combination of the basis elements, so

$$df_p = \lambda_1 d\Phi^1\big|_p + \cdots + \lambda_k d\Phi^k\big|_p$$

for scalars $\lambda_1, \cdots, \lambda_k$.
