Professor: Alexander Givental

Math 215A: Algebraic Topology

Homework 7

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Question 1: Show that the tautological embedding of $\mathbb{CP}^\infty$ into $G_+(\infty, 2)$ is a homotopy equivalence.

Solution: (Collaborated with Finn Fraser Grathwol)

An element in $\mathbb{CP}^\infty \cong \mathbb{RP}^{2\infty}$ is a complex line i.e. a copy of $\mathbb{C} \cong \mathbb{R}^2$. So each such element $L \in \mathbb{CP}^\infty$ can be thought of as the span $\{\mathrm{Re}(v), i \cdot \mathrm{Im}(v)\}$ (i is the imaginary unit) for some $v \in \mathbb{C} \cong \mathbb{R}^2$. This defines an embedding $\mathbb{CP}^\infty \hookrightarrow G_+(\infty, 2)$, the elements of which are oriented planes of (real) dimension 2.

The orientation of the $C-$line is given by noting that multiplying by i gives a counterclockwise rotation so we can use, say, a righthand rule to obtain a normal vector to the surface.

Now, the spaces $\mathbb{CP}^\infty, \mathrm{Gr}_2(\infty, 2)$ are the classifying spaces $BU(1)$ and $BSO(2)$ respectively, and it's a well known fact that indeed $U(1) \cong SO(2)$.

Our embeddings induces the universal $U(1)$-bundle from the universal $SO(2)$-bundle when we consider the right-oriented orthonormal bases in L as a euclidean plan of the form $(u, i \cdot u)$ where u is a unit vector in L. Thus, we have a weak homotopy equivalence between the two spaces, and since we're dealing with CW-complexes, this is the same as homotopy equivalence.

Question 2: Prove that a continuous group homomorphism f from G to G' induces a map from BG to BG', which is a weak homotopy equivalence provided that f is.

Solution: (Answer inspired by that of Finn Fraser Grathwol - follow student in Math 215A)

We have a group homomorphism $f : G \to G'$ which is a Weak Homotopy Equivalence (WHE). Now, we can use f to construct a map between associated fiber bundles $\tilde{f} : EG \times_G G \to \times EG \times_G G'$

Each of these are the total spaces obtained from taking the universal principal bundle $EG \xrightarrow{G} BG$ and replacing the fiber G with either G or G' via translations by g and $f(g)$ for $g \in G$.

Since left and right translations commute, G and G' (resp.) act freely on $EG \times_G G$ and $EG \times_G G'$ via right translations. So, we have principal $G-$ and $G'-$ bundle structures over $BG = EG/G$ with the equivariant map $\tilde{f}$ being fiberwise equivalent to f.

Now, f is a WHE, meaning that f_* is an isomorphism between homotopy groups. Applying the 5-lemma to the morphism induced between the exact homotopy sequences of the bundles, and noting that $\pi_n(EG) = 0$ because it is contractible, we see that the G'-bundle over BG

$$G' \hookrightarrow EG \times_G G' \to BG$$

is universal, and so $BG' = BG$.

Question 3: Classify principal $\mathrm{SL}_2(\mathbb{C})$-bundles over $\mathbb{CP}^2$.

__Solution:__ (Answer inspired by Finn Fraser Grathwol)

Recall that, by Milnor's theorem, the isomorphism classes of principal $\mathrm{SL}_2(\mathbb{C})-$bundles over $\mathbb{CP}^2$ are in bijective correspondence with homotopy classes of maps $\mathbb{CP}^2 \to B(\mathrm{SL}_2(\mathbb{C}))$ i.e.

$$\mathcal{P}(\mathbb{CP}^2, \mathrm{SL}_2(\mathbb{C})) \cong [\mathbb{CP}^2, B(\mathrm{SL}_2(\mathbb{C}))]$$

Now, note that $\mathrm{SL}_2(\mathbb{C})$ deformation retracts onto $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. The classifying space for these two is $\mathbb{HP}^\infty$. So, principal $\mathrm{SL}_2(\mathbb{C})-$bundles over $\mathbb{CP}^2$ are classified by homotopy classes of maps $[\mathbb{CP}^2, \mathbb{HP}^\infty] = \pi(\mathbb{CP}^\infty, \mathbb{HP}^\infty)$.

Now, it'd be nice if we could get this down to a homotopy group that we can compute.

Recall that $\mathbb{H} \cong \mathbb{R}^4, \mathbb{C} \cong \mathbb{R}^2$. The CW Complexes $\mathbb{CP}^2$ and $\mathbb{HP}^\infty$ have cells of dimensions $\{0, 4, 8, \cdots\}$ and $\{0, 2\}$ respectively. By the Cell Approximation Theorem,

$$\pi(\mathbb{CP}^\infty, \mathbb{HP}^\infty) = \pi(\mathbb{CP}^2, \mathbb{S}^4)$$

We can assume, by Borsuk's Theorem, that maps $\mathbb{CP}^2 \to \mathbb{S}^4$ factor homotopically through the projection $p : \mathbb{CP}^2 \to \mathbb{CP}^2/\mathbb{CP}^1 = \mathbb{S}^4$. So, we really only need to consider the homotopy classes of maps $\mathbb{S}^4 \to \mathbb{S}^4$ i.e. the bundles are classified by $\pi_4(\mathbb{S}^4) = \mathbb{Z}$.