Physics 198: Differential Geometry and Lie Groups

Notes

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These are a compilation of notes for the UC Berkeley DeCal 'Physics 198: Differential Geometry and Lie Groups for Physics Students'. These notes are largely based on the primary reference to the class, namely "*Differential Geometry and Lie Groups for Physicists*" by Marián Fecko, and cover topics in roughly the same order as in the book.

This template is based heavily off of the one produced by Kevin Zhou.

Contents

1 Lie Groups

In physics, we can learn a great deal from studying the symmetries of *continuous* systems.

2 Representations of Lie Groups and Lie Algebras

2.1 What is a Representation?

- To understand what a group really *is*, it can be very enlightening to study what the group *does* i.e. to study ***group actions***. It's particularly easy to extract information about a group from its linear actions on linear spaces.
- How exactly do we do this? We can use some sort of map to assign a *linear operator* over that vector space to each group element to describe (or represent) the action of each group element on the vector space elements. The map that we use is a **representation** of the group.

Recall that the space of all linear operators $\rho : V \to V$ is denoted $\mathrm{End}\,(V)$. The subset of these operators which are invertible (isomorphisms on V) is denoted $\mathrm{Aut}\,(V)$.

Notably, $\mathrm{Aut}\,(V)$ has a group structure! **(Check this!)** On the other hand, $\mathrm{End}\,(V)$ becomes an (Associative, and later Lie) Algebra if we define the commutator for $A, B \in \mathrm{End}\,(V)$ as

$$[A, B] = AB - BA$$

Now the formal definition.

Group Representation

Given a group G and vector space V, a group homomorphism

$$\rho : G \to \mathrm{Aut}(V)$$

is called a **representation** of G in V.

Example. *Complete this later*

We can use the same idea to define the representation of an algebra, but this time with $\mathrm{End}\,(V)$.

Lie Algebra Representation

Given a Lie algebra $\mathcal{G}$ and vector space V, an algebra homomorphism

$$\rho' : \mathcal{G} \to \mathrm{End}\,(V)$$

is called a representation of the Lie algebra $\mathcal{G}$ over V.

Note. *The representations ρ and ρ' of a lie group and its lie algebra are related! so ρ' is called the **derived representation**.*

Fecko, Exercise 12.1.4

Consider a Lie algebra $\mathcal{G}$ whose basis elements $\{E_i\}$ satisfy the commutation relations

$$[E_i, E_j] = c^k_{ij} E_k$$

and a representation $f : \mathcal{G} \to \mathrm{End}\,(V)$. Then, define $\mathcal{E}_i \equiv f(E_i)$. Since f is a homomorphisms between algebra, it is linear and respects the commutator i.e. for $A, B \in \mathcal{G}$

$$f\left([A, B]\right) = [f(A), f(B)]$$

Thus,

$$\begin{aligned}
[\mathcal{E}_i, \mathcal{E}_j] &= [f(E_i), f(E_j)] \\
&= f\left([E_i, E_j]\right) \\
&= f\left(c^k_{ij} E_k\right) \\
&= c^k_{ij} f(E_k) \\
&= c^k_{ij} \mathcal{E}_k
\end{aligned}$$

(The basis elements of the representation satisfy the same commutation relation as those of the Lie Algebra!)

Fecko, Exercise 12.1.5

Do this one later

- The assignment from Lie Group to Lie Algebra $G \mapsto \mathcal{G}$ is nice and unique, but the other way around can get messy.
- Similarly, given a Lie group representation ρ there is a unique Lie algebra represenation ρ', but not necessarily the other way around.

Fecko, Exercise 12.1.6

(i) Consider the Lie Group $H = \mathrm{Aut}\,(V) \equiv \mathrm{GL}(V)$. Recall that the Lie Algebra of H is

Write about ρ-invariant inner products.