Math H185 Notes

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These are notes taken from lectures on Complex Analysis delivered by Professor Tony Feng for UC Berekley's Math H185 class in the Sprng 2024 semester.

Contents

1 January 17 - Introduction to Complex Numbers

1.1 Real Numbers

Before jumping into Complex Numbers, let's recall a property of Real Numbers - the set containing which is denoted $\mathbb{R}$.

Note: If $a \in \mathbb{R}$ then $a^2 \geq 0$. So, in this number system negative real numbers do not have square roots in $\mathbb{R}$.

This is a limitation of $\mathbb{R}$, which we can fix by enlargening our field. (Similar to how the set of rationals was enlargened to the set of reals in Real Analysis).

1.2 Imaginary Numbers

We can introduce a new kind of object called an "Imaginary number" such that imaginary numbers square to negatve (≤ 0) real numbers.

We write $i = \sqrt{-1}$.

Proposition: Any imaginary number can be expressed as bi, $b \in \mathbb{R}$.

Proof:

1.3 Complex Numbers

Complex Numbers

- A complex number is an expression $z = a + bi$ where $a, b \in \mathbb{R}$
- The set of complex numbers is denoted $\mathbb{C}$

Remark: $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

In a sense, this is saying that there are no more "deficiencies" - Unlike polynomials in the reals, *every* complex polynomials is guaranteed to have some complex roots. We will return to this statement later in the course when studying the Fundamental Theorem of Algebra.

Let $z = a + bi$ be a complex number. Then,

- The *real part* of z is $Re(z) = a \in \mathbb{R}$ and the *imaginary part* of z is $Im(z) = b \in \mathbb{R}$.
- The *complex conjugate* of z is $\bar{z} = a - bi$

1.4 Operations on Complex Numbers

"Addition is componentwise"

$$\begin{aligned} \text{Addtion: } z &= a + bi \\ +w &= c + di \\ z + w &= (a + c) + (b + d)i \end{aligned}$$

"Multiplication distributes"

For $z = (a + bi)$, $w = (c + di)$ we have

$$\begin{aligned} z \cdot w &= (a + bi) \cdot (c + di) \\ &= a \cdot (c + di) + bi \cdot (c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Addition and Multiplication satisfy the usual commutativity, associativity, and distributivity. However, Division is a bit more complicated.

Division: If $z \in \mathbb{C}$, $w \in \mathbb{C}\backslash\{0\}$, then $z/w \in \mathbb{C}$ is the <u>unique</u> complex number such that $w{\cdot}(z/w) = z$.

Examples: Write the following complex numbers as $a + bi$ where $a, b \in \mathbb{R}$

1. $(9 - 12i) + (12i - 16) = (9 - 16) + (-12i + 12i) = -7$
2. $(3 + 4i) \cdot (3 - 4i) = 9 - 12i + 12i - 16i^2 = 25$
3. $\frac{50+50i}{3-4i} = \frac{50+50i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{150+200i+150i+200i^2}{25} = \frac{-50+350i}{25} = -2 + 14i$

2 January 19 - The Complex Plane

2.1 What does $\mathbb{C}$ look like geomertrically?

[Insert Diagram of $\mathbb{C}$ with Real and Imaginary Axes]

<u>**Def:**</u> The **Modulus** or **absolute value** of $z = a + bi$ is $|z| = a^2 + b^2$. It is the distance to 0.

There is a correspondence between the structure of $\mathbb{C}$ and the vector structure of $\mathbb{R}^2$.

$$\text{Addition in } \mathbb{C} \iff \text{Vector Addition in } \mathbb{R}^2$$

<u>**Exercise:**</u> Draw the subset of $z \in \mathbb{C}$ defined by

1. $\{|z| = 1\}$
2. $\{|z - 6| = 1\}$
3. $\{|z - 6i| \leq 1\}$
4. $\{\text{Re}(z) \leq \text{Im}(z)\}$
5. $\{|z| = \text{Re}(z) + 1\}$

<u>**Answers:**</u> [Insert figures later.]

1. Circle with radius 1 centered at point $z = 0 + 0i$.
2. Circle with radius 1 centered at point $z = 6 + 0i$.
3. Disk with radius 1 centered at point $z = 0 + 6i$.
4. Everything above the line making 45-degree angle with the real axis.
5. (Horizontal) Parabola with vertex at $z = 0 + 1i$ since

$$\begin{aligned}
\sqrt{x^2 + y^2} &= x + 1 \\
\implies x^2 + y^2 &= x^2 + 2x + 1 \\
\implies y^2 &= 2x + 1 \\
\implies x &= \frac{y^2 - 1}{2}
\end{aligned}$$

So, we get a horizontal parabola.

2.2 Polar Coordinates

Rather than using the rectangular (cartesian) coordinates to describe $\mathbb{C}$, we can equvalently use **Polar coordinates** wherein we use the distance from the origin (modulus) and the angle made wtih the real axis (argument).

$$\begin{aligned}
&(x, y) \to (r, \theta) \\
&r = \sqrt{x^2 + y^2} \\
&\theta = \arctan\left(\frac{y}{x}\right)
\end{aligned}$$

Notice however there is an amibiguity in the angle θ since $\theta + 2n\pi$ would describe the same point. Thus, we define the **Principal value branch** as the restriction $\theta \in [0, 2\pi]$.

Recall: The exponential Function is defined as below for $z \in \mathbb{C}$.

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

Now the usual definitions for the cosine and sine as in $\mathbb{R}$ in terms of the unit circle doesn't quite work for complex numbers. However, we can still define them using the exponential!

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4}z^4 + \cdots$$
$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$$

Euler's Theorem: $e^{iz} = \cos(z) + i\sin(z)$, $z \in \mathbb{C}$.

Proof: Expanding the exponential, we have

$$e^{iz} = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

and note that

$$\frac{1}{(2n)!}(iz)^{2n} = \frac{1}{(2n!)}z^{2n}(i^{2n}) = \frac{(-1)^n z^{2n}}{(2n)!}$$

Similarly,

$$\frac{1}{(2n-1)!}(iz)^{2n-1} = \frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$$

Thus,

$$e^{iz} = \underbrace{\sum_{2n} \frac{(-1)^n z^{2n}}{(2n)!}}_{\cos(z)} + \underbrace{\left(\sum_{2n-1} \frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}\right)i}_{\sin(z)}$$

Additionally, we can convert back from Polar coordinates to Cartesian Coordinates as

$$(r, \theta) \to (x, y)$$
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Multiplication: The multiplication of two complex numbers can be thought of in terms of polar coordniates as scaling by r and rotating by θ.

Topology of $\mathbb{C}$

The set $\mathbb{C}$ is a **metric space**, and the metric on $\mathbb{C}$ is $\text{dist}(z, w) = |z - w|$ ($\iff$ Eucldiean metric on $\mathbb{R}^2$).

Notation:

- Given a complex number $z_0 \in \mathbb{C}$ and $r > 0, r \in \mathbb{R}$, the set

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

  is called the **open ball around z_0 of radius r**.

- Similarly, the set $\overline{B_r(z_0)}$ is the **closed ball**:

$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

Open and Closed sets

- A subset $U \subseteq \mathbb{C}$ is **open** if for all $z \in U$, there exists $r > 0$ such that $B_r(z) \subset U$.
- A subset $V \subseteq \mathbb{C}$ is said to be **closed** if its complement V^c is open in $\mathbb{C}$.

Ex: The set $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ is closed. This set will be important when studying the complex logarithm. [Insert figure.]

Note that while the closed ball is not open (since the points on the boundary i.e. points with $|z - z_0| = r$ don't satisfy the requirement), a set *can* be both open and closed. For example, the sets $\mathbb{C}$ and $\emptyset$ are both closed and open.

3 January 22 - Holomorphic functions

3.1 Sequences and Series

Recall that a sequence of complex numbers $\{z_n \in \mathbb{C}\}$ is said to *converge* to $z \in \mathbb{C}$ if for all $\epsilon > 0$ there exists a natural number $N \geq 1$ such that

$$|z_n - z| < \epsilon$$

for all $n \geq N$.

Equivalently,

$$\lim_{n \to N} |z_n - z| = 0$$

In HW1, we show that if $z_n = x_n + iy_n$ and $z = x + iy$ where $x, y, x_n, y_n \in \mathbb{R}$ then

$$\lim_{n\to\infty} |z_n - z| = 0 \iff \begin{cases} \lim_{n\to\infty} |x_n - x| = 0 \\ \lim_{n\to\infty} |y_n - y| = 0 \end{cases}$$

3.2 Complex Dfferentiability

Let $f : U \subseteq_{\text{open}} \mathbb{C} \to \mathbb{C}$.

Holomorphic Functions

f is Holomorphic at $z_0 \in U$ if

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If so, call the limit at $f'(z_0)$.

Note: Keep in mind that h is a complex number.

- This means that for any sequence $h_n \to 0$ <u>or</u> $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$|h| < \delta \implies |\frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0)| < \epsilon$$

- This is the most important definition of the course.

<u>Remark:</u> Although $\mathbb{C}$ is the same as $\mathbb{R}^2$ as a metric space, Holomorphicity is much stronger than differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ because the limits along every path to a point are required to be equal.

<u>Example:</u> Consider the function $f(z) = \overline{z}$.

We observe that

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{\overline{z+h} - \overline{z}}{h} \\ &= \frac{\overline{h}}{h} \end{aligned}$$

Now, if we take the limit as $h \to 0$ along the real line, we get $\frac{\overline{h}}{h} = \frac{h}{h} = 1$, however if we take the limit along the imaginary line we get $\frac{\overline{h}}{h} = \frac{-h}{h} = -1$

On the other hand if we consider the counterpart of this function in $\mathbb{R}^2$ as $f(x, y) = (x, -y)$, this function is *smooth everywhere*. In contrast to this, the complex function $f(z) = \overline{z}$ is *not holomorphic at any* $z_0 \in \mathbb{C}$.

We will see that holomorphic functions have strong rigidity properties not shared by real differentiable functions. For instance,

- If f, g are holomorphic on a connected open set $U \subseteq \mathbb{C}$ and $f = g$ on a line segment in U, then in fact they agree at *all* points in U: $f(z) = g(z) \forall z \in U$. This is the **Principle of Analytic Continuation.**
- Another example of surprising rigidity is that if f is holomorphic on U i.e. it is one differentiable on U, then in fact it is *infinitely* differentiable on U.

Examples:

1. $f(z) = z^n$

   Calculate:

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{(z+h)^n - z^n}{h} \\
&=_{\text{binom thm.}} \left[\frac{1}{h}\left(z^n + nz^{n-1}h + \cdots + nzh^{n-1} + h^n\right) - z^n\right] \\
&= nz^{n-1} + h(\cdots) \\
&\implies \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = nz^{n-1}
\end{aligned}$$

3.3 Stability Properties

While holomorphicity is different from real Dfferentiability, there are a number of properties which are justified by the same $\epsilon - \delta$ proofs as those from $\mathbb{R}$ analysis.

- If $f, g : U \to \mathbb{C}$ are holomorphic at $z_0 \in U$ then
  - $(f + g)$ is holomorphic at z_0 and
  $$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$
  - fg is holomorphic at z_0 and
  $$(fg)'(z_0) = f'g(z_0)g(z_0) + f(z_0)g'(z_0)$$
  - Chain Rule: $(f \circ g)$ is holomorphic at z_0 and
  $$(f \circ g)'(z_0) = f'(z_0) \cdot g'(z_0)$$
  - Division: (f/g) is holomorphic at z_0 if $g(z_0) \neq 0$ and
  $$\left(\frac{f}{g}\right) = \frac{f'(z_0)g(z_0) + f(z_0)g(z_0)}{g(z_0)^2}$$

<u>Polynomials:</u> Finite sum of monomials.

$$f(z) = a_n z^n + \cdots + a_0$$

By linearity (Stability property 1), *all* Polynomials are holomorphic on C.

<u>Rational Functions:</u> Ratios of Polynomials.

$$h(z) = \frac{f(z)}{g(z)}$$

By Stability property 3, all rational functions are holomorphic on $\{z \in \mathbb{C} : g(z) \neq 0\} \subseteq_{\text{open}} \mathbb{C}$.

<u>**Warm-down examples:**</u> Where are the following functions holomorphic, and what are their derivatives in those regions?

1. $f(z) = \frac{1}{z}$
2. $f(z) = z^2 + 3z + \frac{1}{2}$
3. $f(z) = \text{Re}(z)$
4. $f(z) = i \cdot \text{Im}(z)$
5. $f(z) = \text{Re}(z) + i \cdot \text{Im}(z)$

<u>**Answers:**</u>

1. Holomorphic on $\mathbb{C} \setminus \{0\}$, and derivative in the region is
$$\frac{-1}{z^2}$$
2. Holomorphic on $\mathbb{C}$, and derivative in the region is
$$2z + 3$$
3. Not holomorphic *anywhere*, snice limit vertically is always zero but limit horizontally will be non-zero.
4. Not holomorphic *anywhere*, snice limit horizontally is always zero but limit vertically will be non-zero.
5. Holomorphic on $\mathbb{C}$, and derivative in the region is 1 ($f(z) = z$, so $f'(z) = 1$ at all points).