Physics 137B Lecture 12

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These are notes taken from lectures on Quantum Mechanics delivered by Professor Raúl A. Briceño for UC Berekley's Physics 137B class in the Sprng 2024 semester.

Contents

1 February 12 - Spin-Orbit Coupling continued

Recap

- Earlier, we discussed the three corrections to the Hydrogen-atom Hamiltonian, the second one being the *spin-orbit coupling correction*

$$\hat{H}_{SO} = \frac{e^2}{8m^2\epsilon_0 c^2}\frac{\vec{S}\cdot\vec{L}}{r^3}$$

- We wrote $2\vec{S}\cdot\vec{L} = \left(J^2 - S^2 - L^2\right)$ and found the first-order energy correction to be

$$E_{SO}^{(1)} = \frac{e^2}{8m^2\epsilon_0 c^2}\langle\frac{1}{r^3}\rangle_{nl}\cdot[j(j+1) - l(l+1) - s(s+1)]$$

  and we can derive

$$\langle\frac{1}{r^3}\rangle_{nl} = \frac{1}{l(l+\frac{1}{2})(l-1)n^3a^3}$$

- Note that if $l = 0$ then we have $j = s$, the energy correction is just zero. So, Spin-Orbit coupling doesn't impact the energies of the zero-angular momentum states.

1.1 Darwin Term

Today, we'll find the correction due to the Darwin Term

$$\hat{H}'_{DAR} = \frac{\pi\hbar^2}{2m^2c^2}\cdot\frac{e^2}{4\pi\epsilon_0}\delta^3(\vec{r})$$

which *only* contributes to the $l = 0$ states.

Why is this? One way to think about it is to recall that when we solved the Schrodinger Equation for the Coulomb potential, we had

$$V_{eff}(r) = V_{coulomb}(r) + \underbrace{\frac{\hbar}{2m}\frac{l(l+1)}{r^2}}_{centrifugal}$$

[Include picture]

At lower r, the centrifugal potential dominates and is a large positive number whereas for large r the coulomb potential dominates and is a negative number close to zero (negative since it's attractive).

As a result, we required our ormalizable states to go to satisfy $\psi_{nlm}(0) \to 0$ for all $l > 0$.
[Explain more clearly; watch recording]

Now, let's calculate the Darwin Correction.

$$\begin{aligned} E_{DAR}^{(1)} &= \frac{\pi\hbar^2}{2m^3c^2}\frac{e^2}{4\pi\epsilon_0}\langle n00|\delta^3(\vec{r})|n00\rangle \\ &= \frac{\pi\hbar^2}{2m^3c^2}\frac{e^2}{4\pi\epsilon_0}\frac{1}{\pi n^3a^3} \end{aligned}$$

Note that

$$E_n^{(0)} = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n}$$
$$= -\frac{mc^2}{2}\underbrace{\left(\frac{e^2}{4\pi\epsilon_0}\frac{1}{c\hbar}\right)^2}_{\alpha^2}\frac{1}{n^2}$$

and since $\alpha \approx \frac{1}{137} << 1$ we have

$$E_n^{(0)} \approx \frac{E_1^{(0)}}{n^2}$$

1.2 Fine Structure corrections

Now that we've calculate each of the corrections individually, the total Fine Structure Corrections are given by

$$E_n^{(1)} = E_{rel}^{(1)} + E_{SO}^{(1)} + E_{DAR}^{(1)}$$
$$= E_n^{(0)}\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)$$

Thus,

$$E^{(0)} \sim \mathcal{O}(\alpha^2)$$
$$E^{(1)} \sim \mathcal{O}(\alpha^4)$$

1.3 Splitting of Energy Levels

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1.4 External Magnetic Fields - Zeeman Effect

Everything we've done earlier has been *internal*. Now, we're going to submerge the entire atom i.e. *both* the electron and proton in an external magnetic field $\vec{B}_{ext}$.

Then, we have a perturbation to the Hamiltonian given by

$$\hat{H}_z = -(\mu_s + \mu_L) \cdot \vec{B}_{ext}$$

where the z-subscript stands for "Zeeman", *not* $z-$axis.

Earlier, we discussed how the magnetic moment of a charged particle is inversely proportional to its mass:

$$\mu_s = \frac{q}{m}\vec{S}, \quad \mu_L = \frac{q}{2m}\vec{L}$$

Because the proton is on the order of 1000 times heavier than the electron, its magnetic moment μ_p is negligible compared to μ_e. Thus, we will neglect the effect due to the proton and only worry about the electron.

$$\implies \hat{H}_z = \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right)\cdot\vec{B}$$

Since we have a uniform magnetic field $\vec{B}(\vec{r}) = \vec{B}_0$. Let's consider a *weak field* $\vec{B}_{ext} = B_0\hat{z}$ where B_0 is small relative to the fine-structure corrections.

All we need to do is use $J = L + S$ since it's a good quantum number.

$$\begin{aligned}
E^{(1)}_{zeeman} &= \frac{e}{2m} B_0 \langle (L_z + 2S_z) \rangle_{rel} \\
&= \frac{e}{2m} B_0 \langle (J_z + S_z) \rangle_{rel} \\
&= \frac{e}{2m} B_0 \left(m_j \hbar + \langle S_z \rangle_{rel} \right)
\end{aligned}$$

We can then evaluate the expectation value using the $|l, m\rangle$-space representation of the stat e $|j = l \pm \frac{1}{2}, m_s\rangle$

$$|j = l \pm \frac{1}{2}, m_s\rangle =$$

We find the expectation value to be

$$\langle S_z \rangle_{rel} = \pm \frac{m_j \hbar}{2l + 1}$$

Thus,

$$\boxed{E^{(1)}_{zeeman} = \frac{eB_0}{2m} m_j \hbar \left(1 \pm \frac{1}{2l + 1} \right)}$$