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Special assignment: Proofs

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These four exercises are due on Friday November 29 on Gradescope.

1. Prove that $16^n + 10n - 1$ is divisible by 25 for all $n \geq 1$.

2. Prove the result known as De Moivre's Theorem: for all integers $n \geq 1$,

$$(cos(\theta) + i\sin(\theta))^n = cos(n\theta) + i\sin(n\theta)$$

Hint: you can find this on the internet but it is a good exercise to do it by yourself. You only need to know that $i^2 = -1$ as well as the formulas for cosine and sine of a sum:

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

3. Recall the sequence of Fibonacci numbers defined in class.

$$F_0 = 0, F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2}, n \geq 2.$$

Prove that for all $n \in \mathbb{N}$,

$$F_{n+2} = 1 + \sum_{i=0}^{n} F_i$$

4. Consider the following sequence defined by recursion.

$$a_1 = 5, a_2 = 10$$
$$a_n = 2a_{n-1} + a_{n-2}, n \geq 3$$

Prove that $a_n \leq 3^n$ for all $n \geq 3$.

1) Basis step: (should work at boundary condition): $n=1$

$$16^1 + 10(1) - 1 = 26 - 1 = 25$$
$$25 \div 25 = 1$$

It works.

Inductive step: if $n=k$, $(16^k + 10k - 1) \div L = 25$

so $16^k + 10k - 1 = 25L$ (k representing any n)

Proving $k+1$ (representing all n) is divisible by 25:

$$16^{k+1} + 10(k+1) - 1$$
$$16^k \times 16 + 10k + 10 - 1 \qquad (16^k = 25L - 10k + 1)$$
$$[(25L - 10k + 1) \times 16] + 10k + 10 - 1$$

$$400L - 150k + 25$$
$$25(16L - 6k + 1)$$

↳ Proves that $16^n + 10n - 1$ is divisible by 25 for all $n \geq 1$.

2) Basis step: $n=1$

$$(\cos\theta + i\sin\theta)^1 = \cos(1)\theta + i\sin(1)\theta$$

It is valid and equal

Inductive Step: $n=k$ (for any n)

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

$n = k+1$ (for all n)

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$$
$$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$$
$$\underbrace{(\cos k\theta\cos\theta - \sin k\theta\sin\theta)} + i\underbrace{(\cos k\theta\sin\theta + \sin k\theta\cos\theta)} = \ldots$$

$$\cos(k\theta+\theta) + i\sin(k\theta+\theta) = \cos(k+1)(\theta) + i\sin(k+1)(\theta)$$
$$\cos(k+1)(\theta) + i\sin(k+1)(\theta) = \cos(k+1)(\theta) + i\sin(k+1)(\theta)$$

Hence, for $n=k+1$, it is validated.

3) Basis step: $n=0$

$$F_{0+2} = 1 + F_0 \quad (\text{where } F_0 = 0)$$
$$= 1$$

↳ it works and is valid.

Inductive step: $n=k$

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i \text{ is true}$$

$n = k+1$

$$F_{(k+1)+2} = 1 + \sum_{i=0}^{k+1} F_i = 1 + \sum_{i=0}^{k} F_i + F_{k+1}$$

$$F_{k+3} = F_{k+2} + F_{k+1}$$

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i$$

Hence validated for $n = k+1$ and valid for all $n \in \mathbb{N}$

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Basis step: $n = 3$

$$a_3 = 2(10) + 5 = 25 \leqslant 3^3 = 27 \iff \text{True}$$

Inductive step: $n = k$

$$a_k \leqslant 3^k$$

$n = k+1$

$$a_{k+1} = 2a_k + a_{k-1} \qquad a_k \leqslant 3^k$$

$$a_{k+1} \leqslant 2 \cdot 3^k + 3^{k-1}$$

$$a_{k+1} \leqslant \left(2 + \tfrac{1}{3}\right) 3^k$$

$$a_{k+1} \leqslant \left(\tfrac{7}{3}\right) 3^k \leqslant 3^{k+1}$$

For $n = k+1$, hence validated.

Hence $a_n \leqslant 3^n$ for all $n \geqslant 3$