NMM2270A Fall 2024 Assignment 4

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Available: 00:01 on Nov. 11, 2024; Due: 11:59PM on Dec. 01, 2024
Total Points: 36.

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Exam seating: Room –   Row: –   Seat: –

This space is for you to flag potential issues with any questions.

Q12 has a few typos. Commas are shown by ink. It is assumed that's where they were needed.

How Your Assignment will be Graded!

1. Grading short questions (MCQs, T/F, Choice, or simply short questions) will be as follows:
   (a) 2 points for a correct answer and a correct justification.
   (b) 1 point for a correct answer and a wrong justification.
   (c) 1 point for a wrong answer and a correct justification.
   (d) 0 points for a wrong answer and a wrong justification.
   (e) 0.5 a point for reasonable effort in case the score is zero.
   (f) Zero in case cheating is strongly suspected.
2. Grading of long solution problems:
   (a) Full score for a perfect solution.
   (b) A score of zero in case cheating is strongly suspected!
   (c) A score of 60% for a reasonable effort even if the answers are not correct.
   (d) A score of 30% for any relevant attempt

Solve each of the following questions in full detail. Create one PDF file and upload it. Your last upload will be the only file graded. You have two weeks to finish the assignment.

1. (2 points) Find the Laplace transform of $f(t) = t^4 e^{7t}$.

2. (2 points) Use Laplace transforms to solve the initial-value problem $y' - y = e^{-t}, y(0) = 1$.
   Justify your answer in full detail.

3. (2 points) Use Laplace transforms to solve the initial-value problem $y'' - y' - 2y = 0, y(0) = -2, y'(0) = 5$.
   Justify your answer in full detail.

4. (2 points) Use Laplace transforms to solve the initial-value problem $y'' + 6y' + 5y = 0, y(0) = -1, y'(0) = 7$.
   Justify your answer in full detail.

5. (2 points) Find the inverse Laplace transform of $F(s) = \dfrac{\left(1-e^{3s}\right)^2}{s-4}$.
   Justify your answer in full detail.

6. (2 points) True or False? The inverse Laplace transform of $F(s) = \dfrac{2}{3s^2 - 3s - 18}$ is $f(t) = \dfrac{2}{15}e^{3t} - \dfrac{2}{15}e^{-2t}$.
   Justify your answer in full detail.

7. (2 points) True or False? The Laplace transform of the function

$$f(t) = \begin{cases} t^2, 0 \leq t < 1 \\ 0, t \geq 1 \end{cases}$$

   is given by $F(s) = \dfrac{2}{s^3} - \dfrac{(s^2 + 2s + 2)e^{-s}}{s^3}$.
   Justify your answer in full detail.

8. (2 points) True or False? The Laplace transform of $f(t) = t^2 e^{3t} \cos 8t$ is $\dfrac{2(s-3)\left(s^2 - 6s - 183\right)}{\left(s^2 - 6s + 73\right)^3}$
   Justify your answer in full detail.

1)

$$L[f(t)] = L[t^4 e^{7t}]$$

$$L[t^4]_{s \to (s-7)}$$

$$= \left.\frac{4!}{s^5}\right|_{s \to (s-7)}$$

$$= \frac{4!}{(s-7)^5}$$

2) DE: $y' - y = e^{-t}$

$$L(y') - L(y) = L(e^{-t})$$

$$\left[s(Y(s)) - Y(0)\right] - Y(s) = \frac{1}{s+1}$$

$$(s-1)Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{s+2}{(s-1)(s+1)}$$

Turn it into Partial Fractions:

$$\frac{s+2}{(s-1)(s+1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$s+2 = A(s-1) + B(s+1)$$

$3 = B(2)$ $\qquad$ $1 = -2A$

$B = \frac{3}{2}$ $\qquad$ $A = -\frac{1}{2}$

$$Y(s) = \frac{3}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right)$$

To go back to the original formula: Inverse Laplace

$$L^{-1}(Y(s)) = \frac{3}{2} L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right)$$

$$y(t) = \frac{3}{2} e^{t} - \frac{1}{2} e^{-t}$$

3 Laplace Transform on the DE ($y'' - y' - 2y = 0$)

$$L(y'') - L(y') - 2L(y) = 0$$

$$[s^2 Y(s) - sY(0) - sY'(0)] - [sY(s) - sY(0)] - 2Y(s) = 0$$

$$Y(s)(s^2 - s - 2) + 2s - 5 - 2 = 0$$

$$Y(s) = \frac{-2s+7}{s^2-s-2}$$

$$Y(s) = \frac{-2s+7}{(s-2)(s+1)}$$

$$\frac{-2s+7}{(s-2)(s+1)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$-2s+7 = A(s-2) + B(s+1)$$

$$3 = 3B \qquad\qquad 9 = -3A$$
$$B = 1 \qquad\qquad A = -3$$

$$Y(s) = \frac{1}{s-2} - \frac{3}{s+1}$$

Inverse Laplace them:

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{1}{s-2}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$$

$$y(t) = e^{2t} - 3e^{-t} \quad //$$

4 $DE = y'' + 6y' + 5y = 0$

$$L(y'') + 6L(y') + 5L(y) = 0$$

$$\left[s^2 Y(s) + Y(0) + Y'(0)\right] + 6\left[s(Y(0)) - Y(0)\right] + 5Y(s) = 0$$

$$Y(s) = \frac{-s+1}{s^2+6s+5}$$

$$Y(s) = \frac{-s+1}{(s+5)(s+1)}$$

$$\frac{-s+1}{(s+5)(s+1)} = \frac{A}{s+1} + \frac{B}{s+5}$$

$$-s+1 = A(s+5) + B(s+1)$$

$$6 = B(-4) \qquad 2 = 4A$$

$$B = -\frac{6}{4} = -\frac{3}{2} \qquad A = \frac{1}{2}$$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s+1}\right) - \frac{3}{2}\left(\frac{1}{s+5}\right)$$

Inverse Laplace

$$\mathcal{L}^{-1}(Y(s)) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)$$

$$y(t) = \frac{1}{2}e^{-t} - \frac{3}{2}e^{-5t} \quad //$$

5

$$F(s) = \frac{(1-e^{3s})^2}{s-4}$$

$$= \frac{1-2e^{3s}+6e^{6s}}{s-4}$$

Inverse Laplace:

$$\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) = e^{4t}$$

$$\mathcal{L}^{-1}\left(\frac{e^{3s}}{s-4}\right) = e^{4(t-3)}\mathcal{U}(t-3)$$

$$\mathcal{L}^{-1}\left(\frac{e^{6s}}{s-4}\right) = e^{4(t-6)}\mathcal{U}(t-6)$$

$$\mathcal{L}^{-1}(F(s)) = e^{4t} + e^{4(t-3)}\mathcal{U}(t-3) + e^{4(t-6)}\mathcal{U}(t-6) \quad //$$

6

$$f(s) = \frac{2}{3s^2 - 3s - 18}$$

$$f(s) = \frac{2}{3}\left(\frac{1}{s^2 - s - 6}\right)$$

$$f(s) = \frac{2}{3}\left(\frac{1}{(s-3)(s+2)}\right)$$

Decompose $f(s)$

$$\frac{\frac{2}{3}}{(s-3)(s+2)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$\frac{2}{3} = A(s-3) + B(s+2)$$

$$\frac{2}{3} = 5B \qquad \frac{2}{3} = -5A$$

$$B = \frac{2}{15} \qquad A = -\frac{2}{15}$$

$$f(s) = \frac{\frac{2}{15}}{s-3} - \frac{\frac{2}{15}}{s+2}$$

Inverse Laplace :

$$L^{-1}(f(s)) = \frac{2}{15}L^{-1}\left(\frac{1}{s-3}\right) - \frac{2}{15}L^{-1}\left(\frac{1}{s+2}\right)$$

$$L^{-1}(f(s)) = \frac{2}{15}e^{3t} - \frac{2}{15}e^{-2}$$ $\Rightarrow$ The statement is true //

7 Write $f(t)$ as

$$f(t) = t^2(1-\mathcal{U}(t-1))$$

$$f(t) = t^2 - t^2\mathcal{U}(t-1)$$

$$(t-1)^2\mathcal{U}(t-1) = (t^2-2t+1)\mathcal{U}(t-1)$$

$$= [t^2-2(t-1)-1]\mathcal{U}(t-1)$$

$$t^2\mathcal{U}(t-1) = (t-1)^2\mathcal{U}(t-1) + 2(t-1)\mathcal{U}(t-1) + \mathcal{U}(t-1)$$

Subbing $t^2\mathcal{U}(t-1)$ in $f(t)$.

$$f(t) = t^2 - [(t-1)\mathcal{U}(t-1) + 2(t-1)\mathcal{U}(t-1) + \mathcal{U}(t-1)]$$

Inverse Laplace Transform :

$$L^{-1}(f(t)) = L^{-1}(t^2) - [L^{-1}((t-1)\mathcal{U}(t-1)) + 2L^{-1}((t-1)\mathcal{U}(t-1)) + L^{-1}(\mathcal{U}(t-1))]$$

$$= \frac{2!}{s^3} - [e^{-s}(\frac{2!}{s^3}) + 2e^{-s}(\frac{1!}{s^2}) + \frac{e^{-s}}{s}]$$

$$= \frac{2}{3} - \frac{e^{-s}}{s^3}(2+2s+s^2)$$

$$L^{-1}(f(t)) = \frac{2}{s^3} - \frac{(s^2+2s+2)e^{-s}}{s^3}$$

The statement is true //

8 $L(e^{3t}\cos 8t) = L(\cos 8t)_{s\to s-3}$

$$= \frac{s}{s^2+64}$$

$$= \frac{s-3}{(s-3)^2+64}$$

$$= \frac{s-3}{s^2-6s+73}$$

Laplace transform of $f(t)$:

$$= L(t^2 e^{3t}\cos 8t) = \frac{d^2}{ds^2} L(e^{3t}\cos 8t)$$

So

$$\frac{d}{ds} L(e^{3t}\cos 8t) = \frac{d}{ds}\left(\frac{s-3}{s^2-6s+73}\right)$$

$$= \frac{(s^2-6s+73)-(2s-6)(s-3)}{(s^2-6s+73)^2}$$

$$= \frac{-s^2+6s+55}{(s^2-6s+73)}$$

$$= \frac{-1}{s^2-6s+73} + \frac{128}{(s^2-6s+73)^2}$$

Find the second derivative:

$$\frac{d^2}{ds^2} L(e^{3t}\cos 8t) = \frac{2s-6}{(s^2-6s+73)^2} + \frac{128(-2)(2s-6)}{(s^2-6s+73)^3}$$

$$= \frac{2s-6}{(s^2-6s+73)^3}\left(s^2-6s+73-256\right)$$

$$L(f(t)) = \frac{(2s-6)(s^2-6s-183)}{(s^2-6s+73)^3}$$

So, the given statement is true.

Long Solution Problems

Solve the following problems in detail.

9. (5 points) Use the Laplace transform to solve the initial-value problem $y'' + 5y' + 6y = 0, y(0) = 0, y'(0) = 1$.

10. (5 points) Use the Laplace transform to solve the initial-value problem $y'' - 4y' + 12y = 0, y(0) = 0, y'(0) = 1$.

11. (5 points) Use the Laplace transform to solve the integrodifferential equation $y'(t) = 2t\cos t - 3\int_0^t y(\tau)d\tau, y(0) = 0$.

12. (5 points) Use the Laplace transform to solve the system

$$\frac{dx}{dt} + 2x + 2y = 0, x + \frac{dy}{dt} + 3y = 0, x(0) = 0, y(0) = 1.$$

9 DE: $y'' + 5y' + 6y = 0$

$$L(y'') + 5L(y') + 6L(y) = 0$$

$$s^2 Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = 0$$

$$y(s)(s^2 + 5s + 6) - 1 = 0$$

$$y(s) = \frac{1}{s^2+5s+6}$$

$$y(s) = \frac{1}{(s+3)(s+2)}$$

$$y(s) = \frac{(s+3) - (s+2)}{(s+3)(s+2)}$$

$$y(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

Inverse Laplace Transform

$$y(t) = L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{s+3}\right)$$

$$y(t) = e^{-2t} - e^{-3t}$$ //

10 DE: $y'' - 4y' + 12y' = 0$

Laplace Transform:

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 12Y(s) = 0$$

$$Y(s)(s^2 - 4s + 12) - 1 = 0$$

$$Y(s) = \frac{1}{s^2 - 4s + 12}$$

$$= \frac{1}{(s-2)^2 + 8}$$

$$= \frac{1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{(s-2)^2 - (2\sqrt{2})^2}$$

Inverse Laplace Transform

$$y(t) = \frac{1}{2\sqrt{2}} L^{-1}\left(\frac{2\sqrt{2}}{(s-2)^2 - (2\sqrt{2})^2}\right)$$

$$y(t) = \frac{1}{2\sqrt{2}} e^{2t} \sin(2\sqrt{2}t)$$ //

11 DE: $y'(t) = 2t\cos t - 3\int_0^t y(\tau)\,d\tau$

Laplace Transform

$$L(y'(t)) = L(2t\cos t) - 3L\left(\int_0^t y(\tau)\,d\tau\right)$$

$$\to L(2t\cos t) = -2\frac{d}{ds}L(\cos t)$$

$$= -2\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -2\left(\frac{-s^2+1}{(s^2+1)^2}\right)$$

$$L(2t\cos t) = \frac{2(s^2-1)}{(s^2+1)^2}$$

$$\to sY(s) - y(0) = \frac{2(s^2-1)}{(s^2+1)^2} - \frac{3Y(s)}{s}$$

$$Y(s)\left(s+\frac{3}{s}\right) = \frac{2(s^2-1)}{(s^2+1)^2}$$

$$Y(s) = \frac{2s(s^2-1)}{(s^2+3)(s^2+1)^2}$$

Partial Decomposition

$$\frac{2s(s^2-1)}{(s^2+3)(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{s^2+3}$$

$$As+B(s^4+4s^2+3) = As^5 + 4As^3 + 3As + 4Bs^4 + 4Bs^2 + 3B$$
$$Cs+D(s^2+3) = Cs^3 + 3Cs + Ds^2 + 3D$$
$$Es+F(s^4+2s^2+1) = Es^5 + 2Es^3 + Es + Fs^4 + 2Fs^2 + F.$$

$$2s^3 - 2s = (A+E)s^5 + (B+F)s^4 + (4A + 2E + C)s^3$$
$$+ (4B + D + 2F)s^2 + (3A + 3C + E)s +$$
$$(3B + 3D + F)$$

$B + F = 0$ $\quad$ $A + E = 0$ $\quad$ $4A + 2E + C = 2$ $\quad$ $4B + D + 2F = 0$

$B = -F$ $\quad$ $A = -E$ $\quad$ $2A + C = 2$ $\quad$ $2B + D = 0$

$3A + 3C + E = -2$ $\quad$ $3B + 3D + F = 0$

$2A + 3C = -2$ $\quad$ $2B + 3D = 0$

$A = 2$

$B = 0$

$C = -2$

$D = 0$

$E = -2$

$F = 0$

$$Y(s) = \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2} - \frac{2s}{s^2+3}$$

Inverse Laplace

$$y(t) = L^{-1}\left(\frac{2s}{s^2+1}\right) - L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) - L^{-1}\left(\frac{2s}{s^2+3}\right)$$

$$y(t) = 2\cos t - t\sin t - 2\cos\sqrt{3}\,t$$

12 1^{st} DE is $\frac{dx}{dt} + 2x + 2y = 0$

2^{nd} DE is $\frac{dy}{dt} + x + 3y = 0$

Laplace Transform (1^{st} DE & 2^{nd} DE)

1) $sX(s) - x(0) + 2X(s) + 2Y(s) = 0$

$\rightarrow (s+2)X(s) + 2Y(s) = 0$

2) $sY(s) - y(0) + X(s) + 3Y(s) = 0$

$\rightarrow (s+3)Y(s) + X(s) = 1$

$X(s) = 1-(s+3)Y(s) \longrightarrow$ 1)

$$(s+2)(1-(s+3)Y(s)) + 2Y(s) = 0$$
$$(s+2) - (s^2+5s+4)Y(s) = 0$$
$$Y(s) = \frac{s+2}{s^2+5s+4}$$
$$Y(s) = \frac{s+2}{(s+4)(s+1)}$$

Partial Decomposition:

$$\frac{s+2}{(s+4)(s+1)} = \frac{A}{(s+4)} + \frac{B}{(s+1)}$$

$$s+2 = A(s+1) + B(s+4)$$

$1 = 3B \qquad -2 = -3A$

$B = \frac{1}{3} \qquad A = \frac{2}{3}$

$$Y(s) = \frac{\frac{2}{3}}{s+4} + \frac{\frac{1}{3}}{s+1}$$

Inverse Transform

$$y(t) = L^{-1}\left(\frac{\frac{2}{3}}{s+4}\right) + L^{-1}\left(\frac{\frac{1}{3}}{s+1}\right)$$

$$y(t) = \frac{2}{3} e^{-4t} + \frac{1}{3} e^{-t}$$

$$Y(s) = \frac{s+2}{(s+4)(s+1)}$$

$$X(s) = 1 - \frac{(s+3)(s+2)}{(s+4)(s+1)}$$

$$X(s) = \frac{s^2 + 5s + 4 - (s^2 + 5s + 6)}{(s+4)(s+1)}$$

$$X(s) = \frac{-2}{(s+4)(s+1)}$$

$$X(s) = \frac{-2}{3}\left(\frac{(s+4) - (s+1)}{(s+4)(s+1)}\right)$$

$$X(s) = \frac{-2}{3}\left(\frac{1}{s+1} - \frac{1}{s-4}\right)$$

Inverse Transform

$$x(t) = -\frac{2}{3}\left(\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+4}\right)\right)$$

$$\boxed{x(t) = -\frac{2}{3}\left(e^{-t} - e^{4t}\right)}$$

$$x(t) = -\frac{2}{3}\left(e^{-t} - e^{4t}\right) \quad \& \quad y(t) = \frac{2}{3}e^{-4t} + \frac{1}{3}e^{-t}$$