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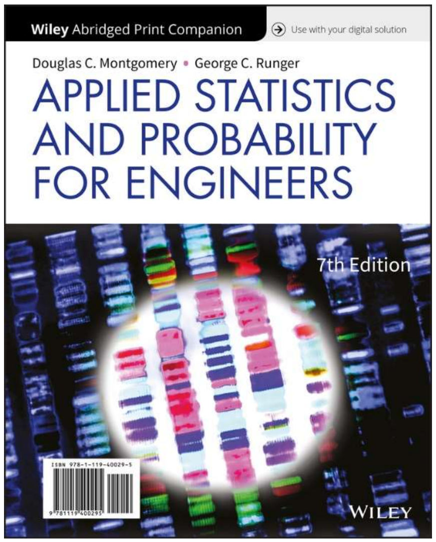

Applied Statistics and Probability for Engineers

Seventh Edition

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Chapter 2

Probability

Chapter 2 Title Slide

2 Probability

CHAPTER OUTLINE

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Learning Objectives for Chapter 2

After careful study of this chapter, you should be able to do the following:

1. Understand and describe sample spaces and events for random experiences with graphs, tables, lists, or tree diagrams
2. Interpret probabilities and use the probabilities of outcomes to calculate probabilities of events in discrete sample spaces
3. Use permutations and combinations to count the number of outcomes in both an event and the sample space
4. Calculate the probabilities of joint events such as unions and intersections from the probabilities of individual events
5. Interpret and calculate conditional probabilities of events
6. Determine independence and use independence to calculate probabilities
7. Use Bayes' theorem to calculate conditional probabilities
8. Describe random variables and the difference between continuous and discrete random variables

Random Experiment

- An experiment is a procedure that is
  - carried out under controlled conditions, and
  - executed to discover an unknown result

> **Random Experiment**
>
> An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.

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Sample Spaces

The set of all possible outcomes of a random experiment is called the sample space, S.

S is **discrete** if it consists of a finite or countable infinite set of outcomes.
S is **continuous** if it contains an interval (either finite or infinite) of real numbers.

Example 2.1 | Camera Flash

- Randomly select a camera and record the recycle time of a flash. $S = R^+ = \{x \mid x > 0\}$, the positive real numbers.
- Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then $S = \{x \mid 1.5 < x < 5\}$ is continuous.
- It is known that the recycle time has only three values (low, medium or high). Then $S = \{low, medium, high\}$ is discrete.
- Does the camera conform to minimum recycle time specifications?

  $S = \{yes, no\}$ is discrete.

Example 2.3 | Message Delays

Messages are classified as on-time or late within the time specified by the system design. Use a tree diagram to represent the sample space of possible outcomes.

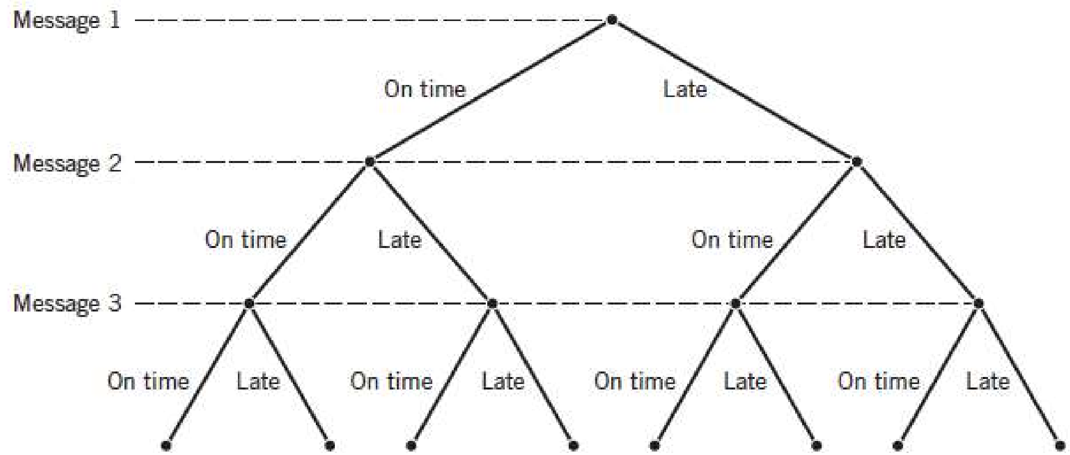

FIGURE 2.5

Tree diagram for three messages.

Events

Event
An **event** is a subset of the sample space of a random experiment.

Event combinations

- The union of two events consists of all outcomes that are contained in either of the two events, denoted as $E_1 \cup E_2$.
- The intersection of two events is the event that consists of all outcomes that are contained in both of the two events, denoted as $E_1 \cap E_2$.
- The complement of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the complement of the event E as E'.

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Example 2.4 | Events

- Suppose that the recycle times of two cameras are recorded. Consider only whether or not the cameras conform to the manufacturing specifications. We abbreviate *yes* and *no* as *y* and *n*. Consider the sample space $S = \{yy, yn, ny, nn\}$.
- Suppose that the subset of outcomes for which at least one camera conforms is denoted as E_1. Then, $E_1 = \{yy, yn, ny\}$.
- Suppose that the subset of outcomes for which both cameras do not conform, denoted as E_2, contains only the single outcome, $E_2 = \{nn\}$.
- Other examples of events are $E_3 = \emptyset$, the null set, and $E_4 = S$, the sample space.
- If $E_5 = \{yn, ny, nn\}$, $E_1 \cup E_5 = S$, $E_1 \cap E_5 = \{yn, ny\}$, $E_1' = \{nn\}$

Practical Interpretation: Events are used to define outcomes of interest from a random experiment. One is often interested in the probabilities of specified events.

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Example 2.5 | Camera Recycle Time

Camera recycle times might use the sample space $S = R^+$.

Let $E_1 = \{x \mid 10 \le x < 12\}$ and $E_2 = \{x \mid 11 < x < 15\}$

Then,

$E_1 \cup E_2 = \{x \mid 10 \le x < 15\}$ and $E_1 \cap E_2 = \{x \mid 11 < x < 12\}$

Also

$$E_1' = \{x \mid x < 10 \quad \text{or} \quad 12 \le x\}$$

And

$$E_1' \cap E_2 = \{x \mid 12 \le x < 15\}$$

Venn Diagrams

We can use **Venn diagrams** to represent a sample space and events in a sample space.

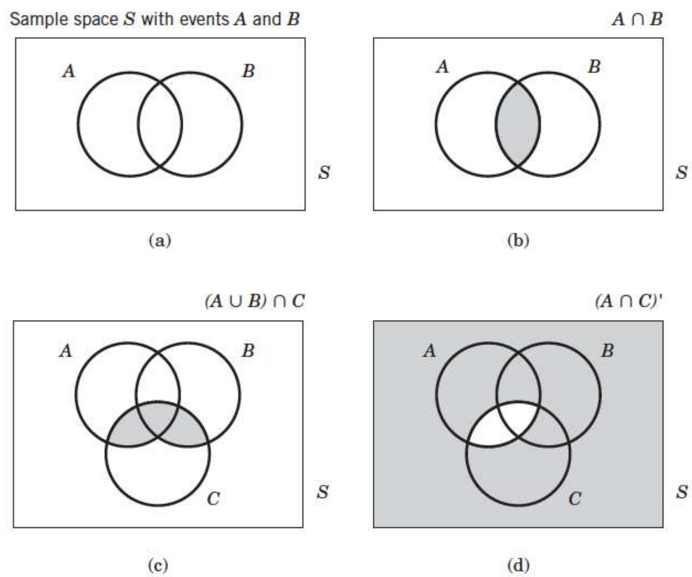

FIGURE 2.6
Venn diagrams.

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Mutually Exclusive Events

Mutually Exclusive Events

Two events, denoted as E_1 and E_2, such that

$$E_1 \cap E_2 = \emptyset$$

are said to be **mutually exclusive**.

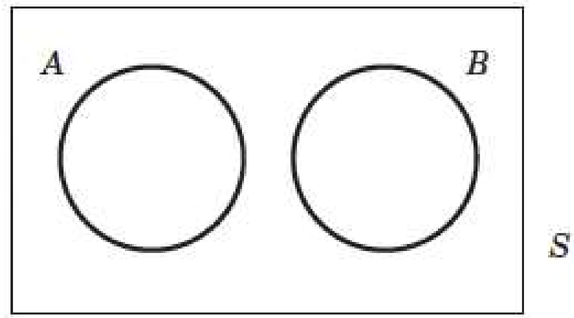

FIGURE 2.7

Mutually exclusive events.

Mutually Exclusive Events - Laws

- The definition of the complement of an event implies that $(E')' = E$

- The distributive law for set operations implies that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \textbf{ and } (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

- DeMorgan's laws imply that

$$(A \cup B)' = A' \cap B' \textbf{ and } (A \cap B)' = A' \cup B'$$

- Also, remember that

$$A \cap B = B \cap A \textbf{ and } A \cup B = B \cup A$$

Counting Techniques

- Determining the outcomes in the sample space (or an event) can be difficult.
- In these cases, counts of the numbers of outcomes in the sample space and various events are used to analyze the random experiments.
- These methods are referred to as **counting techniques**.
  - Multiplication Rule
  - Permutations
  - Combinations

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Counting – Multiplication Rule

Assume an operation can be described as a sequence of k steps, and

- The number of ways to complete step 1 is n_1.
- The number of ways to complete step 2 is n_2 for each way to complete step 1.
- The number of ways to complete step 3 is n_3 for each way to complete step 2, and so on.

The total number of ways to complete the operation is $n_1 \times n_2 \times \cdots \times n_k$

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Example 2.7 | Web Site Design

The design for a Website is to consist of four colors, three fonts, and three positions for an image.

From the multiplication rule, 4 x 4 x 3 = 36 different designs are possible

Practical Interpretation: The use of the multiplication rule and other counting techniques enables one to easily determine the number of outcomes in a sample space or event and this, in turn, allows probabilities of events to be determined.

Counting – Permutations

A **permutation** of the elements is an ordered sequence of the elements.

Note. Ordered means that, for example, (a,b) and (b,a) are different.

Example. Consider a set of elements, such as $S = \{a, b, c\}$.
abc, acb, bac, bca, cab, cba are all the permutations of the elements of S.

The number of **permutations** of n different elements is $n!$ where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \tag{2.1}$$

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Counting – Permutations

Permutations of Subsets

The number of permutations of subsets of *r* elements selected from a set of *n* different elements is

$$P_r^n = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 2.8 | Printed Circuit Board

A printed circuit board has **eight** different locations in which a component can be placed. If **four** different components are to be placed on the board, how many different designs are possible?

Answer: $P_4^8 = \frac{8!}{(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1{,}680$ different designs are possible

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Counting – Permutations

Permutations of Similar Objects

The number of permutations of $n = n_1 + n_2 + \dots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, … , and n_r are of an r^{th} type is

$$\frac{n!}{n_1!\,n_2!\,\dots\,n_r!}$$

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Example 2.9 | Hospital Schedule

A hospital operating room needs to schedule

three knee surgeries and two hip surgeries

in one day. We denote a knee surgery as *k*, and hip surgery as *h*. The number of possible sequences of three knee and two hip surgeries is

$$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!} = 10$$

The 10 sequences are easily summarized as

$$\{kkkhh, kkhkh, kkhhk, khkkh, khkhk, khhkk, hkkkh, hkkhk, hkhkk, hhkkk\}$$

Counting – Combinations

Combinations

The number of combinations, subsets of r elements that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_r^n and

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad (2.4)$$

- In combinations, order is not important.
- Example. (a,b) is equal to (b,a)

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Example 2.11a | Sampling without Replacement

- A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. How many samples of size 6 contain 2 defective parts?
- We can select 2 defective from the total of 3 defective in

$$\frac{3!}{2! \cdot 1!} = 3 \text{ different ways}$$

- The number of different ways we can select the remaining 4 from the total of 47 non-defective in

$$\frac{47!}{4! \cdot 43!} = \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 43!} = 178{,}365 \text{ different ways}$$

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Example 2.11c | Sampling without Replacement

Therefore, from the multiplication rule, the number of subsets of size 6 that contain exactly 2 defective parts is

$$3 \cdot 178,365 = 535,095$$

Note. The total number of different subsets of size 6 is

$$\binom{50}{6} = \frac{50!}{6!44!} = 15,890,700$$

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Interpretations and Axioms of Probability

- **Probability** is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur
- The likelihood of an outcome is quantified by assigning a number from the interval [0,1] to the outcome (or a percentage from 0 to 100%)
- 0 indicates an outcome will not occur
- 1 indicates that an outcome will occur with certainty

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Interpretations and Axioms of Probability

- Subjective probability, or *degree of belief*
  - Different individuals will no doubt assign different probabilities to the same outcomes
  - Ex: "The chance of rain today is 30%"
- Relative frequency probability
  - Interpreted as the limiting value of the proportion of times the outcome occurs in n repetitions of the random experiment as n increases beyond all bounds

Example: If we assign probability 0.2 to the outcome that there is a corrupted pulse in a digital signal, we might interpret this assignment as implying that, if we analyze many pulses, approximately 20% of them will be corrupted

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Interpretations and Axioms of Probability

Equally Likely Outcomes

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

For example, suppose that we select 1 laser diode randomly from a batch of 100.
Randomly implies that it is reasonable to assume that each diode in the batch has an equal chance of being selected.
The probability model for this experiment assigns probability of 0.01 to each of the 100 outcomes, because each outcome in the sample space is equally likely

Interpretations and Axioms of Probability

For a discrete sample space, the probability of an event can be defined by the reasoning used in the preceding example

Probability of an Event

For a discrete sample space, the *probability of an event E*, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

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Example 2.13 | Probabilities of Events

A random experiment can result in one of the outcomes {*a, b, c, d*} with probabilities 0.1, 0.3, 0.5, 0.1, respectively.
Let *A* denote the event {*a, b*}, *B* the event {*b, c, d*}, and *C* the event {*d*}

$$P(A) = 0.1 + 0.3 = 0.4$$
$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$
$$P(C) = 0.1$$
$$P(A') = 0.6 \; and \; P(B') = 0.1 \; and \; P(C') = 0.9$$

Because $A \cap B = \{b\}, then\ P(A \cap B) = 0.3$
Because $A \cup B = \{a, b, c, d\}, then\ P(A \cup B) = 1.0$
Because $A \cap C = \{null\}, then\ P(A \cap C) = 0$

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

1. $P(S) = 1$
2. $0 \le P(E) \le 1$
3. For any two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$,
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

The axioms imply that:

- $P(\emptyset) = 0$ and $P(E') = 1 - P(E)$
- If E_1 is contained in E_2, then $P(E_1) \le P(E_2)$.

Unions of Events and Addition Rules

Joint events are generated by applying basic set operations to individual events, specifically:

- Unions of events, $A \cup B$
- Intersections of events, $A \cap B$
- Complements of events, A'

Probabilities of joint events can often be determined from the probabilities of the individual events it comprises

Example 2.15a | Semiconductor Wafers

A wafer is randomly selected from a batch of 940 wafers in a semiconductor manufacturing process

- Let H denote the event that the wafer contains high levels of contamination
  - Then $P(H) = 358/940$.
- Let C denote the event of the wafer is in center of a sputtering tool
  - Then $P(C) = 626/940$.

Contamination	Location of Tool		Total
	Center	Edge	
Low	514	68	582
High	112	246	358
Total	626	314	940

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Example 2.15b | Semiconductor Wafers

$P(H \cap C)$ is the probability that the wafer is from the center of the sputtering tool and contains high levels of contamination

$$P(H \cap C) = 112/940$$

The event $(H \cup C)$ is the event that a wafer is from the center of the sputtering tool or contains high levels of contamination (or both)

$$P(H \cup C) = P(H) + P(C) - P(H \cap C)$$
$$= (358 + 626 - 112)/940 = 872/940$$

Contamination	Location of Tool		Total
	Center	Edge	
Low	514	68	582
High	112	246	358
Total	626	314	940

Practical Interpretation: To better understand the sources of contamination, yield from different locations on wafers are routinely aggregated

Addition Rule

Probability of a Union

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2.5}$$

If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B) \tag{2.6}$$

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Addition Rule

Three or More Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad (2.7)$$

Mutually Exclusive Events

A collection of events, $E_1, E_2, \ldots, E_k$, is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \emptyset$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \cdots P(E_k) \quad (2.8)$$

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Conditional Probability

- The probability of an event *B* under the knowledge that the outcome will be in event *A* is called the **conditional probability** of *B* given *A*, denoted as $P(B \mid A)$
- A digital communications channel has an error rate of 1 per 1000 bits transmitted. Errors are rare, occur in bursts. If a single bit is transmitted, we might model the probability of an error as 1/1000. However, if the previous bit was in error because of the bursts, we might believe that the probability that the next bit will be in error is greater than 1/1000.

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Conditional Probability

Conditional Probability

The **conditional probability** of an event B given an event A, denoted as $P(B \mid A)$, is

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad (2.9)$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are N total outcomes,

$P(A)$ = (number of outcomes in A) / N

$P(A \cap B)$ = (number of outcomes in $A \cap B$) / N

$P(B \mid A)$ = (number of outcomes in $A \cap B$) / (number of outcomes in A)

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Example 2.17 | Surface Flaws and Defectives

Table 2.2 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Of the parts with surface flaws (40 parts), the number of defective ones is 10. Therefore,

$$P(D \mid F) = 10/40 = 0.25$$

And of the parts without surface flaws (360 parts), the number of defective ones is 18. Therefore,

$$P(D \mid F') = 18/360 = 0.05$$

Parts Classified			
Defective	Surface Flaws		Total
	Yes (F)	No (F')	
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

Practical Interpretation

The probability of being defective is five times greater for parts with surface flaws. This calculation illustrates how probabilities are adjusted for additional information. The result also suggests that there many be a link between surface flaws and functionally defective parts, which should be investigated

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Intersections of Events and Multiplication and Total Probability Rules

- The conditional probability definition can be rewritten to provide a formula known as the **multiplication rule** for probabilities

Multiplication Rule

$$P(A \cap B) = P(B \mid A)P(A) = P(A \mid B)P(B) \qquad (2.10)$$

- This expression is obtained by interchanging *A* and *B*

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Example 2.19 | Machining Stages

The probability that the 1st stage of a numerically controlled machining operation meets specifications is 0.90. The probability that it meets specifications in the 2nd stage, given that it met specifications in the first stage is 0.95.

What is the probability that both stages meet specifications?

- Let *A* and *B* denote the events that the 1st and 2nd stages meet specifications, respectively.
- $P(A \cap B) = P(B \mid A) \cdot P(A) = 0.95 \cdot 0.90 = 0.855$

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Total Probability Rule

- A and A' are mutually exclusive.
- $A \cap B$ and $A' \cap B$ are mutually exclusive
- $B = (A \cap B) \cup (A' \cap B)$

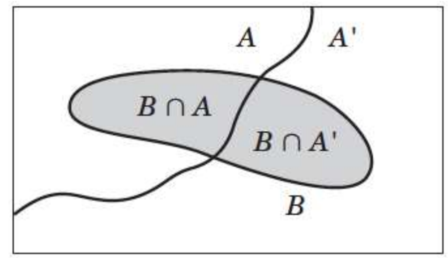

FIGURE 2.13

Total Probability Rule

For any two events A and B

Total Probability Rule (Two Events)

For any events A and B,

$$P(B) = P(B \cap A) + P(B \cap A') = P(B \mid A)P(A) + P(B \mid A')P(A') \qquad (2.11)$$

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Example 2.20 | Semiconductor Contamination

Information about product failure based on chip manufacturing process contamination is given below. Find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not high	0.8

Let F denote the event that the product fails

Let H denote the event that the chip is exposed to high contamination. Then

- $P(F \mid H) = 0.10$ and $P(F \mid H') = 0.005$
- $P(H) = 0.20$ and $P(H') = 0.8$
- $P(F) = 0.10(0.20) + 0.005(0.80) = 0.024$

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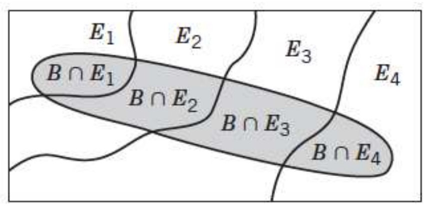
Total Probability Rule

Total Probability Rule (Multiple Events)

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k) \qquad (2.12) \end{aligned}$$



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$

FIGURE 2.14

Partitioning an event into several mutually exclusive subsets.

Independence

Knowledge that the outcome of the experiment is in event *A* does not affect the probability that the outcome is in event *B*

Independence (two events)

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A \mid B) = P(A)$

(2) $P(B \mid A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$ (2.13)

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Independence

Independence (multiple events)

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events

$$P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) = P\left(E_{i_1}\right) \times P\left(E_{i_2}\right) \times \cdots \times P\left(E_{i_k}\right) \tag{2.14}$$

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Bayes' Theorem

- Bayes' theorem states that,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0 \qquad (2.15)$$

Example 2.26

The conditional probability that a high level of contamination was present when a failure occurred is to be determined. The information is summarized here.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not high	0.8

The probability of $P(H \mid F)$ is determined from

$$P(H \mid F) = \frac{P(F \mid H)P(H)}{P(F)} = \frac{0.10(0.20)}{0.024} = 0.83$$

The value of $P(F)$ in the denominator of our solution was found from $P(F) = P(F \mid H)P(H) + P(F \mid H')P(H')$.

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Bayes' Theorem

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k)} \qquad (2.16)$$

for $P(B) > 0$

Note:

Numerator expression is always one of the terms in the sum of the denominator.

Random Variables

- The variable that associates a number with the outcome of a random experiment is referred to as a **random variable**

> **Random Variable**
> A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

- Notation is used to distinguish between a random variable and the real number

> **Notation**
> A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes.

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Discrete and Continuous Random Variables

- A **discrete random variable** is a random variable with a finite (or countably infinite) range
- A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range

Examples of Random Variables

Examples of **continuous** random variables:

electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:

number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error

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Important Terms & Concepts of Chapter 2

- Addition Rule
- Axioms of probability
- Bayes' Theorem
- Combination
- Conditional probability
- Counting techniques
- Equally likely outcomes
- Event
- Independence
- Multiplication rule
- Mutually exclusive events
- Outcome
- Permutation
- Probability
- Random samples
- Random variables – discrete and continuous
- Sample spaces – discrete and continuous
- Total probability rule
- Tree diagram
- Venn diagram
- With or without replacement

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