# Chapter 6: Techniques of Integration

**(Base on Adam and Essex book)**
**Sections covered 6.1-6.2)**

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## 6.2 Integrals of rational functions

**Rational Function:** A quotient P(x)/Q(x) of two polynomials is called a **rational function.**

$$F(x)=\frac{P(x)}{Q(x)}.$$

where P and Q are polynomials.

A **polynomial** of function is defined as follows

$$Q(x)=a_0+a_1x+a_2x^2+......+a_{n-1}x^{n-1}+a_nx^n$$

where $a_0,..,a_n$ are a constants and n is a nonnegative integer.

The Q(x) is called the n degree polynomial.

Let us integrate rational functions

$$\int F(x)dx=\int\frac{P(x)}{Q(x)}dx.$$

## Important Note

- The degree of P should be less than that of Q.
- When the degree of P is equal or greater the degree of Q, then we can use division to express the fraction P(x)/ Q(x). Then we get a polynomial plus another fraction R(x)/Q(x), where R, the remainder in the division.
- Note that in R(x)/Q(x), R has degree less than that of Q.

## Example:

Evaluate the following integral

$$I = \int \frac{x^3 + 3x^2}{x^2 + 1} dx.$$

**Solution**

Let us substitute

$$\frac{P(x)}{Q(x)} = \frac{x^3 + 3x}{x^2 + 1} \qquad P(x) = x^3 + 3x^2 \qquad Q(x) = x^2 + 1$$

Note that

$$P(x) > Q(x)$$

In this case, we divide P by Q as follows

$$\begin{array}{r|l} & x+3 \\ \hline x^2+1 & x^3+3x^2 \\ & x^3+x \\ & ------ \\ & 3x^2-x \\ & 3x^2+3 \\ & ------ \\ & -x-3 \end{array}$$

Here $R(x) = -x - 1$.

The rational function becomes

$$\frac{x^3 + 3x^2}{x^2 + 1} = x + 3 - \frac{x+3}{x^2+1} \qquad R < Q$$

$$\frac{P}{Q} = polynomial + \frac{R}{Q} \qquad polynomial = x + 3 \qquad \frac{R}{Q} = \frac{x+3}{x^2+1}$$

The integration becomes

$$I = \int \left[ x + 3 - \frac{x+3}{x^2+1} \right] dx$$

$$I = \int [x+3]dx - \int \frac{x}{x^2+1}dx - \int \frac{3}{x^2+1}dx$$

$$I = I_1 - I_2 - I_3$$

Integrate $I_1$

$$I_1 = \int [x+3]dx = \frac{1}{2}x^2 + 3$$

$$I_1 = \frac{1}{2}x^2 + 3x + c_1$$

Integrate $I_2$

$$I_2 = \int \frac{x}{x^2+1}dx =$$

$$u = x^2 + 1 \qquad du = 2xdx \to xdx = \frac{1}{2}du$$

$$I_2 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c_2 \qquad u = x^2 + 1$$

$$I_2 = \frac{1}{2}\ln(x^2+1) + c_2$$

Integrate $I_3$

$$I_3 = \int \frac{1}{x^2+1}dx = \tan^{-1}x + c_3$$

Collecting all terms

$$I = \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + c \qquad c = c_1 + c_2 + c_3.$$

# Quadratic Denominators (P<Q)

Suppose that Q(x) has degree n=2 and condition P<Q.

$$Q(x) = x^2 + a^2$$

**Example**

Evaluate the following integrals

$$I = \int \frac{x}{x^2+a^2}dx$$

**Solutions**

We consider the first integral

$$I = \int \frac{x}{x^2 + a^2} dx.$$

We simplify

$$I = \frac{1}{a^2} \int \frac{x}{1 + x^2 / a^2} dx.$$

We substitute

$$u = \frac{x^2}{a^2}$$
$$du = \frac{2x}{a^2} dx$$
$$xdx = \frac{a^2}{2} du$$

The integral becomes

$$I = \frac{1}{a^2} \int \frac{xdx}{1 + x^2 / a^2} \qquad u = \frac{x^2}{a^2} \qquad xdx = \frac{a^2}{2} du$$
$$I = \frac{a^2}{2a^2} \int \frac{1}{1+u} du$$

$$I = \frac{1}{2} \ln(1+u) + c \qquad u = \frac{x^2}{a^2}$$
$$I = \frac{1}{2} \ln(1 + x^2 / a^2) + c$$

## Partial Fractions

**Here we express a complicated rational fraction as a sum of simpler fractions**.

$$\frac{P(x)}{Q(x)} \qquad Q(x) = (x - a_1)(x - a_2)...(x - a_n)$$

We factor $Q$ into a product of $n$ *distinct* linear factors.

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Case (i.e., P<Q).

We express P/Q as a **partial fraction decomposition**

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)...(x-a_n))}$$
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + .... + \frac{A_n}{(x-a_n)}$$
.

**Determine of** $A_1, A_2 ... A_n$ **constantans**.

**Method 1**:

- Constants are determined by solving the *n* linear equations.
- We can find these constants equating the coefficients of like powers of *x*.

**Method 2**:

We multiply the partial fraction decomposition by $(x-a_k)$ to find $A_k$ as follows

$$(x-a_k)\frac{P(x)}{Q(x)} = \frac{A_1(x-a_k)}{(x-a_1)} + ... + \frac{A_k(x-a_k)}{(x-a_k)} + .... \frac{A_n(x-a_k)}{(x-a_n)}$$
$$(x-a_k)\frac{P(x)}{Q(x)} = \frac{A_1(x-a_k)}{(x-a_1)} + ... + A_k + .... \frac{A_n(x-a_k)}{(x-a_n)}$$

We put $x=a_k$ on the right side.

$$\lim_{x\to a_k}(x-a_k)\frac{P(x)}{Q(x)} = A_k$$

## Example:

Evaluate the following integral

$$I = \int \frac{x+4}{x^2-5x+6}dx \,.$$

**Solution**

Expressing P/Q as a **partial fraction decomposition**

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We write the integral as

$$I = \int \frac{P}{Q} dx = \int \frac{x+4}{x^2-5x+6} dx$$

where

$P = x+4 \qquad Q = x^2-5x+6$.

We factor $Q$ into a product of *n distinct* linear factors as below

$$\begin{aligned} Q(x) &= x^2-5x+6 = x^2-2x-3x+6 \\ Q(x) &= x\ (x-2)-3(x-2) \\ Q(x) &= (x-2)(x-3) \end{aligned}.$$

The integrands becomes

$$I = \int dx \frac{x+4}{x^2-5x+6} = \int dx \frac{x+4}{(x-2)(x-3)}$$

**First Method to calculate constants A1 and A2:**

$$I = \int dx \frac{x+4}{(x-2)(x-3)} = \int dx \left( \frac{A_1}{(x-2)} + \frac{A_2}{(x-3)} \right)$$

Calculation of $A_1, A_2$

$$\frac{x+4}{(x-2)(x-3)} = \frac{A_1(x-3)+A_2(x-2)}{(x-2)(x-3)}$$

$$\frac{x+4}{(x-2)(x-3)} = \frac{A_1x-3A_1+A_2x-2A_2}{(x-2)(x-3)}$$

$$\frac{x+4}{(x-2)(x-3)} = \frac{\left(A_1+A_2\right)x-3A_1-2A_2}{(x-2)(x-3)}$$

Comparing left side and right side

$$\begin{aligned} A_1 + A_2 &= 1 \\ 3A_1 + 2A_2 &= -4 \end{aligned}$$

$$A_1 = -6, A_2 = 7$$

**Second Method for A1 and A2:**

$$A_k = \lim_{x \to a_k} (x - a_k) \frac{P(x)}{Q(x)}$$

$$A_1 = \lim_{x \to 2} (x-2) \frac{x+4}{(x-2)(x-3)} = \lim_{x \to 2} \frac{x+4}{(x-3)} = \frac{2+4}{(2-3)} = -6$$

$$A_2 = \lim_{x \to 3} (x-3) \frac{x+4}{(x-2)(x-3)} = \lim_{x \to 3} \frac{x+4}{(x-2)} = \frac{3+4}{(3-2)} = 7$$

Finally, we evaluate the integral

$$I = \int \frac{A_1}{(x-2)} dx + \int \frac{A_2}{(x-3)} dx = \int \frac{-6}{(x-2)} dx + \int \frac{7}{(x-3)} dx$$

$$I = -6 \ln|x-2| + c_1 + 7 \ln|x-3| + c_2$$

$$I = -6 \ln|x-2| + 7 \ln|x-3| + c \qquad c = c_1 + c_2$$

## Completing the Square Method

We can express Q(x) = $Ax^2 + Bx +$ C can be express as product two linear terms.

Aim

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right)$$

$$Ax^2 + Bx + C = A\left(x^2 + 2\frac{B}{2A}x + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} + \frac{C}{A}\right)$$

$$Ax^2 + Bx + C = A\left(x + \frac{B}{2A}\right)^2 - A\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)$$

$$Ax^2 + Bx + C = A\left(x + \frac{B}{2A}\right)^2 - \left(\frac{B^2 - 4AC}{4}\right)$$

### Example

Integrate the following integrate

$$I = \int \frac{x-1}{(x^2 - 2x + 2)} dx$$

**Completing the square method**

$$x^2-2x+2 \qquad A=1 \qquad B=-2 \qquad C=2$$

$$Ax^2+Bx+C=A\left(x+\frac{B}{2A}\right)^2-\left(\frac{B^2-4AC}{4}\right)$$

$$x^2-2x+2=\left(x-1\right)^2+1$$

The integrand becomes

$$I=\int dx\frac{x-1}{(x^2-2x+2)}$$

$$I=\int dx\frac{x-1}{\left(x-1\right)^2+1}.$$

Putting

$$u=\left(x-1\right)^2$$

$$du=2\left(x-1\right)dx$$

$$\left(x-1\right)dx=\frac{1}{2}du$$

The integral reduces to

$$I=I=\int dx\frac{x-1}{\left(x-1\right)^2+1}=\frac{1}{2}\int\frac{du}{u+1}$$

Thus

$$I=\frac{1}{2}\ln\left(u+1\right) \qquad u=x-1$$

$$I=\frac{1}{2}\ln\left(\left(x-1\right)^2+1\right)$$

## Denominators with Repeated Factors

In this case, the partial fraction decomposition of $P$ (x) / $Q$ (x) requires $m$ distinct fractions.

For example, we have

$$\frac{P}{Q}=\frac{P}{(x-1)^2}$$

## Example:

Evaluate the following integral.

$$I = \int \frac{1}{x(x-1)^2}dx$$

**Solution**

$$I = \int \frac{1}{x(x-1)^2}dx = \int \frac{1}{x(x-1)(x-1)}dx$$

Note that (x-1) is repeated in the above integrand.

Let us rewrite the integrand

$$I = \int dx \frac{1}{x(x-1)^2} = \int dx \left( \frac{A_1}{x} + \frac{A_2}{(x-1)} + \frac{A_3}{(x-1)^2} \right)$$

$$\frac{1}{x(x-1)^2} = \frac{A_1}{x} + \frac{A_2}{(x-1)} + \frac{A_3}{(x-1)^2}$$

$$\frac{1}{x(x-1)^2} = \frac{A_1(x-1)^2 + A_2x(x-1) + A_3x}{x(x-1)^2}$$

$$\frac{1}{x(x-1)^2} = \frac{A_1(x^2-2x+1) + A_2(x^2-x) + A_3x}{x(x-1)^2}$$

$$\frac{1}{x(x-1)^2} = \frac{A_1x^2 - 2A_1x + A_1 + A_2x^2 - A_2x + A_3x}{x(x-1)^2}$$

$$\frac{1}{x(x-1)^2} = \frac{(A_1+A_2)x^2 + (-2A_1 - A_2 + A_3)x + A_1}{x(x-1)^2}$$

Comparing powers of x in right and left sides terms, we get

$$A_1 + A_2 = 0$$
$$-2A_1 - A_2 + A_3 = 0.$$
$$A_1 = 1$$

We found

$A_1 = 1, A_2 = -1, A_3 = 1$.

Finally, the integral is written as

$$I=\int\frac{1}{x(x-1)^2}dx=\int\frac{A_1}{x}dx+\int\frac{A_2}{(x-1)}dx+\int\frac{A_3}{(x-1)^2}dx.$$

$$A_1=1, A_2=-1, A_3=1$$

$$\int\frac{1}{x(x-1)^2}dx=\int\frac{1}{x}dx-\int\frac{1}{(x-1)}dx+\int\frac{1}{(x-1)^2}dx$$

Completing the integration

$$I=\ln|x|-\ln|x-1|-\frac{1}{(x-1)}+c$$

$$I=\ln\left|\frac{x}{x-1}\right|-\frac{1}{(x-1)}+c$$

# Extra Example

$$I=\frac{1}{3}\int\frac{x-2}{(x^2-x+1)}dx$$

**Completing the square method**

$$x^2-2x+2 \qquad A=1 \qquad B=1 \qquad C=1$$

$$Ax^2+Bx+C=A\left(x+\frac{B}{2A}\right)^2-\left(\frac{B^2-4AC}{4}\right)$$

$$x^2-x+1=(x-1/2)^2+\frac{3}{4}$$

The integrand becomes

$$I=\frac{1}{3}\int dx\frac{x-2}{(x^2-x+1)}$$

$$I=\frac{1}{3}\int dx\frac{x-2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}.$$

$$I=\frac{1}{3}\int dx\frac{\left(x-\frac{1}{2}\right)-\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$I=\frac{1}{3}\int dx\frac{\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{3}\int dx\frac{\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

Putting

$$u=\left(x-\frac{1}{2}\right)\qquad du=dx$$

The integral reduces to

$$I=\frac{1}{3}\int\ \frac{u}{u^2+\frac{3}{4}}du-\frac{1}{2}\int\ \frac{1}{u^2+\frac{3}{4}}du$$

Thus

$$I=\frac{1}{6}\ln\left(u^2+\frac{3}{4}\right)-\frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+c_2$$

$$I=\frac{1}{6}\ln\left(x^2-x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c_2$$