Induction and Recursion

Math 2151: Discrete Math for Engineering

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The well ordering principle and Induction

We will assume the following self evident axiom.

The well ordering principle

Every nonempty set of $\mathbb{N}$ has a minimum.

Notice that $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ don't satisfy this property. This property is very important since it implies the following.

Theorem (Principle of Mathematical Induction)

Let $S(n)$ denote a mathematical statement depending on the variable n which ranges over $\mathbb{N}$. If

- *$S(0)$ is true, and*
- *For all natural numbers k $S(k) \to S(k+1)$,*

then for all natural numbers n, $S(n)$ is true.

Proof of the principle of mathematical induction.

Proof.

Let $S(n)$ a mathematical statement satisfying the two conditions in the theorem. We will use the well ordering principle. Consider the set

$$T = \{n \in \mathbb{N} : \neg S(n)\}.$$

(that is, $n \in T \leftrightarrow n \in \mathbb{N} \wedge \neg S(n)$). We want to prove that $T = \emptyset$. Suppose on the contrary that $T \neq \emptyset$. Since $T \subseteq \mathbb{N}$, T has a minimum $k \in T$. Since $k \in T$, $\neg S(k)$. So, $k \neq 1$ by the first hypothesis. This means that $k - 1 \in \mathbb{N}$ and since $k = minT$, $k - 1 \notin T$, i.e., $S(k-1)$. By the second hypothesis, and MP, $S(k)$. This means $k \notin T$, which is impossible. We conclude that $T = \emptyset$ as we wanted to show. □

Principle of mathematical induction

The role of 0, (or 1 in some books) can be played by any natural number $n_0 \in \mathbb{N}$.

Theorem

Mathematical induction Let $S(n)$ denote a mathematical statement depending on the variable n which ranges over $\mathbb{N}$. Let $n_0 \in \mathbb{N}$. If

- *$S(n_0)$ is true, and*
- *For all natural numbers k $S(k) \to S(k+1)$,*

Then, for all $n \geq n_0$, $S(n)$ holds true.

Examples

Let's prove something by induction.
Prove that for all $n \geq 0$,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Here $S(n)$ is the previous line.

$$S(1) : \sum_{i=0}^{1} i = \frac{1(2)}{2} \qquad \text{True}$$

$$S(20) : \sum_{i=0}^{20} i = \frac{20(21)}{2} \qquad ?$$

We will use mathematical induction, so we have to prove that $S(0)$ is true and that for all k, $S(k) \to S(k+1)$ holds true. $S(0)$ is called the Basis step and $S(k) \to S(k+1)$ is called the inductive step. One usually uses direct proof and supposes $S(k)$ to prove $S(k+1)$ (Here, $S(k)$ is refered to as the inductive hypothesis).

Proof

Proof.

Basis step: $S(0)$ holds since $0 = \sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0$*Inductive step:* Suppose that $S(k)$ is true, that is, $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.We have to prove $S(k+1)$. Now,

$$\sum_{i=0}^{n+1} k = 1 + 2 + \cdots + k + k + 1$$

$$= \left(\sum_{k=0}^{n} k \right) + (k+1)$$

$$= \frac{k(k+1)}{2} + k + 1 = (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

(Notice the use of the inductive hypothesis in the 3rd equality.) We are done by induction. $\square$

Example

Let's prove now that for all $n \geq 1$,
$\sum_{i=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
Basis step: For $n = 1$ this is just $1^2 = \frac{1(2)(3)}{6}$.
Inductive step: Suppose that $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$, then

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right) \\
&= (k+1)\frac{(k+2)(2k+3)}{6} = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}
\end{aligned}$$

We are done by induction.

Example

Let's prove that for all $n \geq 0$, $4|9^n + 3$.

Proof.

Basis step: For $n = 0$ the statement is $4|9^0 + 3$. This is evident since $4|4$. *Inductive step:* Suppose that $4|(9^k + 3)$. This means that $9^k + 3 = 4l$, for some integer l. So, $9^k = 4l - 3$. We need to prove that $4|(9^{k+1} + 3)$.

$$\begin{aligned} 9^{k+1} + 3 &= 9^k 9 + 3 \\ &= (4l - 3)9 + 3 \\ &= 36l - 27 + 3 = 36l - 24 = 4(9l - 6) \end{aligned}$$

So, $4|(9^{k+1} + 3)$. We are done by induction. $\square$

Inequality

Let's prove that for $n \geq 3$, $n^2 > 2n$ by induction.

Proof.

Basis step: For $n = 3$, $9 = 3^2 > 2(3) = 6$.
Inductive step: Suppose $k \geq 3$, and $k^2 > 2k$. We have to prove that $(k+1)^2 > 2(k+1)$. Since $k > 3$, $2k > 6$, and $2k + 1 > 7$

$$\begin{aligned}(k+1)^2 = k^2 + 2k + 1 &> 2k + 2k + 1 \\ &> 2k + 7 \\ &> 2k + 2 = 2(k+1)\end{aligned}$$

We are done by induction. $\square$

Recursive definitions

We will use the strong form of the induction principle mainly when working with recursive definitions.
Some sequence are given by formulas, e.g.,

$$a_n = 2^{n+1} - 1$$

is a sequence of integers. Some other sequences are defined by recursion. The first terms of the sequence are defined one by one, and the values of later terms in the sequence are defined by making reference to the terms already defined. Let's see some examples.

Recursive definitions

The sequence a_n is defined by recursion as follows:

$$a_0 = 0, a_1 = 1, \text{ and } a_2 = 2,$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \qquad \text{for } n \geq 3$$

For example:

$$a_3 = a_2 + a_1 + a_0 = 2 + 1 + 0 = 3$$
$$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$$
$$a_5 = 6 + 3 + 2 = 11$$
$$a_6 = 11 + 6 + 3 = 20$$

and so on. Induction and recursion go hand in hand. When we want to prove something about a sequence that is defined recursively we usually use induction.

For example, in the previous examples we have used the fact that when $(a_n)_{n\geq 0}$ is a sequence. We can define the sum sequence $S_n = \sum_{k=0}^{n} a_k$ as

$$\begin{aligned} S_0 &= a_0 \\ S_n &= a_n + S_{n-1}. \end{aligned}$$

We also used that the sequence $a_n = 9^n$ can be defined as

$$\begin{aligned} a_0 &= 1 \\ a_n &= 9(a_{n-1}) \end{aligned}$$

Examples

The Fibonacci numbers are defined by recursion as follows:

$$F_0 = 0, F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 1$$

Prove that

$$\sum_{i=0}^{n} F_i^2 = F_{n+1}F_n.$$

Example with Fibonacci numbers

Proof.

Basis step: For $n = 1$, $1 = 10 + 1 = F_0^2 + F_1^2 \sum_{k=0}^{1} F_i^2 = F_2 F_1 = 1$.
Inductive step: Suppose that $\sum_{i=0}^{k} F_i^2 = F_{k+1}F_k$, then

$$\begin{aligned}
\sum_{i=0}^{k+1} F_i^2 &= \left(\sum_{i=0}^{k} F_i\right) + F_{k+1}^2 \\
&= F_{k+1}F_k + F_{k+1}^2 \\
&= F_{k+1}(F_k + F_{k+1}) \\
&= F_{k+1}F_{k+2} \\
&= F_{k+2}F_{k+1},
\end{aligned}$$

as we wanted to show. □

Another example from Grimaldi's book. The Lucas numbers are defined by recursion as

$$L_0 = 2, L_1 = 1$$
$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2.$$

Prove by induction that $L_n = F_{n-1} + F_{n+1}$
For proving this we will use a variation of the same principle of induction.

Yet another induction principle

Theorem

Alternate form of the induction principle Let $S(n)$ be a mathematical statement depending on the variable n. Let $n_0 \leq n_1 \in \mathbb{N}$. If

- $S(n_0), S(n_0+1), \ldots, S(n_1)$ *is true, and*
- *Whenever* $k \geq n_1$, *and* $S(n_0), S(n_0+1), \ldots, S(k)$ *are, then* $S(k+1)$ *is true.*

Then, $S(n)$ *is true for all* $n \geq n_0$

The difference with the usual principle of induction is that the inductive hypothesis is different. We are not only allowed to use $S(k)$ as our inductive hypothesis. Instead we can use $S(n_0), S(n_0+1), \ldots, S(k)$.

Lucas numbers, strong induction example

Now, the proof of $L_n = F_{n-1} + F_{n+1}$ for all $n \geq 1$.

Proof.

Basis step: For $n = 1 : 1 = L_1 = F_0 + F_2 = 1$. We will also need $L_2 = 2 + 1 = 3 = F_1 + F_3 = 1 + 2$. *Inductive step:* Suppose that for $1 \leq i \leq k$ $L_i = F_{i-1} + F_{i+1}$, and let's prove that $L_{k+1} = F_k + F_{K+2}$.

$$
\begin{aligned}
L_{k+1} &= L_k + L_{k-1} \\
&= F_k + F_{k+1} + L_{k-1} && \text{Because } L_k = F_k + F_{k+1} \\
&= F_k + F_{k+1} + F_{k-1} + F_k && \text{Because } L_{k-1} = F_{k-1} + F_k \\
&= F_k + F_{k-1} + F_{k+1} + F_k \\
&= F_{k+1} + F_{k+2}.
\end{aligned}
$$

□

Notice that if $S(n)$ is $L_n = F_{n-1} + F_{n+1}$, we used both $S(k)$ and $S(k-1)$ in the proof of $S(k+1)$ in the inductive step!

Eulerian numbers

The Eulerian numbers $a_{m,k}$ for $m \geq 0$ and $k \geq -1$ integers are defined by recursion as follows.

$$a_{0,0} = 1, a_{m,k} = 0 \text{ for } k \geq m$$
$$a_{m,-1} = 0$$
$$a_{m,k} = (m-k)a_{m-1,k-1} + (k+1)a_{m-1,k} \text{ for } 0 \leq k \leq m-1$$

Prove that for $m \geq 1$,

$$\sum_{k=0}^{m-1} a_{m,k} = m!$$

Eulerian numbers

First, we'll ilustrate why this should be true. You can easily prove by induction that $a_{m,0} = 1$ for all $m \geq 0$. Let's calculate some other values.

$$\begin{aligned}
a_{1,0} &= (1-0)a_{0,-1} + 1a_{0,0} = 1 \\
a_{2,0} &= (2-0)a_{1,-1} + 1a_{1,0} = 1 \\
a_{2,1} &= (2-1)a_{1,0} + 2a_{1,1} = 1 \\
a_{3,0} &= 3a_{2,-1} + 1a_{2,0} = 1 \\
a_{3,1} &= (3-1)a_{2,0} + 2a_{2,1} = 2 + 2 = 4 \\
a_{3,2} &= (3-2)a_{2,1} + 3a_{2,2} = 1 \\
a_{4,0} &= 1 \\
a_{4,1} &= 3a_{3,0} + 2a_{3,1} = 3 + 2(4) = 11 \\
a_{4,2} &= 2a_{3,1} + 3a_{3,2} = 2(4) + 3 = 11 \\
a_{4,3} &= a_{3,2} + 4a_{3,3} = 1
\end{aligned}$$

We could carry on but we are lazy. We can however display the values in a sort of Pascal triangle.

Eulerian numbers

$$
\begin{array}{lccccccccccc}
n=1 & & & & & & 1 & & & & & \\
n=2 & & & & & 1 & & 1 & & & & \\
n=3 & & & & 1 & & 4 & & 1 & & & \\
n=4 & & & 1 & & 11 & & 11 & & 1 & & \\
n=5 & & 1 & & 26 & & 66 & & 26 & & 1 & \\
n=6 & 1 & & 57 & & 302 & & 302 & & 57 & & 1
\end{array}
$$

Notice the sums of the elements of each row form the sequence 1,2,6,24,120,720.

Also: $a_{m,k}$ count something! See the exercises!

Eulerian numbers

Let's now prove that for all $k \geq 1$,

$$\sum_{k=0}^{m-1} a_{m,k} = m!$$

Proof.

Basis step: For 1,

$$\sum_{k=0}^{0} a_{1,k} = a_{1,0} = 1 = 1!$$

Inductive step: Suppose $\sum_{k=0}^{m-1} a_{m,k} = m!$. Then, □

Proof.

$$\sum_{k=0}^{m+1} a_{m+1,k} = \sum_{k=0}^{m} (m-(k+1))a_{m,k-1} + (k+1)a_{m,k}$$

$$= (m+1)a_{m,-1} + a_{m,0} +$$

$$+ ma_{m,0} + 2a_{m,1} +$$

$$+ (m-1)a_{m,1} + 3a_{m,2} +$$

$$+ \cdots +$$

$$+ a_{m,m-1} + (m+1)a_{m,m}$$

$$= (m+1)a_{m,0}a + (m+1)a_{m,1} + \cdots + (m+1)a_{m,m-1}$$

$$= (m+1)\sum_{k=0}^{m-1} a_{m,k} = (m+1)m! = (m+1)!$$

$\square$