Counting

Math 2151: Discrete Math for Engineering

University of Western Ontario
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Outline

The following is a brief outline of the contents.

- Counting.
- Logic and proofs.
- Sets and relations.
- Functions.
- Induction and recursion.
- Arithmetic.
- Algorithms and complexity.
- Graphs.

Books

- Required: Discrete Math with Graph Theory 3rd Edition by Goodaire and Parmenter.
- Suggested: Discrete and Combinatorial Mathematics, 5th Edition, by Ralph Grimaldi. I'll post some pages from time to time on OWL.

Evaluation

The evaluation will consists of:

- Quizzes and assignments (15%): roughly one quiz per week on Wednesdays (except on the first and on exam weeks). Two or three written assignments before the exams.
- Midterm tests (20% each): One on Wednesday Oct 9, and one on Thursday, November 14 7pm-8:30pm. Location TBA.
- Final test (45%): Date and location TBA.

Office hours

Office hours will be held in my office MC 134 at the following times:

- Monday 4:30pm-5:30pm
- Tuesday: 2pm-3pm
- Wednesday: 2:30pm-3:30pm
- Or by appointment through Zoom.

Counting techniques

The rule of sum

If a task can be performed in m ways while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in m+n different ways.

If there are more than 2 tasks the sum rule still applies as long as no two of the tasks can be performed at the same time.

Examples, rule of sum

- In how many ways can one draw a card from a standard deck that is either red or a spade?
- My hand in a card game includes 3 fives, 4 Jacks, 2 Kings and a Queen. I have to play one card. How many options do I have?

Rule of product

The rule of product

The total number of possible outcomes of a series of decisions is found by multiplying the number of choices for each decision.

Examples, product rule

- I want to order a black coffee at a coffee shop. They sell coffee from Brazil, Colombia, Vietnam and Jamaica and the sizes that you can get are Tall, Grande and Venti. How many options do I have for ordering my coffee?
- A bit string is a sequence of symbols chosen between the symbols 0 and 1. The length of a bit string is the number of symbols in the sequence. For example 01101 is a bit string of length 5. How many bit strings of length 4 are there? How many bit strings do not have consecutive 1's?
- How many license plate are possible in Ontario if a plate consists of 4 letters followed by 3 numbers? What if the numbers can't be repeated? What if numbers and letters cannot be repeated? How many plates start with an O and end with a 0 if repetitions of symbols are not allowed?
- In how many ways can we arrange the letters ABCDE?

- In the previous example we saw that the number of ways of arranging five different letters (without repetition) is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. In general, the number of ways of arranging n different symbols is $n! = n(n-1)(n-2)\cdots 1$, with $0! = 1$ by definition. $n!$ is read as n factorial.
- These words without repetition are called arrangements in the book, meaning that the words DACE and CEAD are different, that is, the order in the arrangement of the symbols matters!. There will be situations where the order doesn't matter.

Definition: Permutations

Given a collection of n objects, any ordered arrangement of some of the n objects is called a permutation. If r is an integer with $1 \leq r \leq n$, the number of permutations of size r of the n objects can be written as

$$P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}$$

Examples, permutations

- 10 students enter a competition. How many different possible podiums are there?
- From a team of 10 students you have to pick a president a vicepresident and a secretary. In how many ways can you do this?

Trick: overcounting

Overcounting

If, when counting the number of elements of a group, we overcount by counting each element k times, to get a total of n. Then, the number of elements of the group is $\frac{n}{k}$.

- How many different words can be formed with the letters CAAATV? Hint: imagine that the 3 A's are different and overcount.
- In how many different ways can we arrange the letters in MISSISSAUGA? Hint: several overcounts.
- Determine the number of staircase paths in the xy-plane from (0,0) to (5,6) where each path is made of individual steps going one unit to the right and one unit up.

Examples, overcounting

Arrangements of objects with repetitions

In general, if there are n objects composed of n_1 indistinguishable objects, n_2 indistinguishable objects, ..., and n_r indistigiuishable objects where $n = n_1 + n_2 + \cdots + n_r$, then the number of possible arrangements of the n objects is $\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$

- 5 people seat at a circular table with 5 seats. Suppose that two seating arrangements are equal if one can be obtained from the other by rotation. How many different seating arrangements are there? Hint: forget about the rotation bit and overcount.

Combinations

- How many ways are there of drawing 4 cards out of a standard deck without replacement?What if the order of selection doesn't matter?

In general, we have the following.

Combinations

Suppose we have n objects. The number of ways of selecting r objects out of the n objects, with no reference to the order of selection is n choose r, $\binom{n}{r}$ where

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

An unordered selection of objects is also called a combination. So, $\binom{n}{r}$ is the number of combinations of size r chosen from a group of n objects.

Examples, Combinations

- In how many ways can a comittee of 6 people be formed out of a group of 17 people?What if the group must have a president? What if the group must have a president, a secretary and a treasurer?
- Suppose that out of the 17 people 8 are right wing and 9 are left wing. If the comittee must have 3 left wing and 3 right wing people. How many different commitees can be formed?
- If the Commitee must have at least 3 left wing people, how many different committees can be formed?
- A teacher must make 4 debate teams out of a group of eight people. In how many ways can she choose the teams? Do it two ways! Using product rule and combinations and using permutations.
- How many arrangements are there of the letters in MASSACHUSETTS? How many of these arrangements have no adjacent S's?What if the S's have to be all together?

Examples, Combiations 2

- Out of a standard deck of cards. How many different 5 card hands can be dealt?
- How many of these hands have four kings and a non king?
- How many 5 card hands have exactly 3 jacks?
- How many of these hands have 3 of the same kind?

Sigma notation

Sigma notation is a way of abbreviating sums that works in the same way as cycles or while instructions in computer programming. Instead of writting

$$a_1 + a_2 + \cdots + a_m$$

mathematicians write

$$\sum_{i=1}^{m} a_i.$$

This is read as sum from $i = 1$ to $i = m$ of the a_i's. The variable i takes each of the values between 1 and m and we add all of the a_i's. Of course, the variable i doesn't have to start ranging from one, so

$$\sum_{i=k}^{k+h} a_i = a_k + a_{k+1} + \cdots + a_{k+h}$$

also holds true. Let's work out some examples.

- $\sum_{i=0}^{4} 2i$
- $\sum_{i=1}^{n} 2$

Examples, Sigma notation

- $\sum_{i=3}^{7} \frac{1}{3+i}$
- $\sum_{i=2}^{n-2} a_{i-2} = \sum_{i=1}^{n} a_{i-1}$
- $\sum_{i=2}^{4} \frac{3i}{2i^2-1}$

Binomial Theorem

Binomial Theorem

For $n \geq 2$, and x, y numbers (or polynomial variables):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Write out what this means for $n = 2$ and $n = 3$. Does it coincide with what you think?
- Let's prove the general case.

Examples, Binomial Theorem

- What's the coefficient of x^6y^4 in $(x+y)^10$?
- What's the coefficient of x^6y^4 in $(2x+3y)^10$?
- Calculate
$$\sum_{k=0}^{n} \binom{n}{k}$$

Multinomial Theorem

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et x, n be positive integers. Then for numbers (or polynomial variables) $x_1, \ldots, x_s$:

$$(x_1 + \cdots + x_s)^n = \sum_{n_1 + \cdots + n_s = n} \frac{n! x_1^{n_1} \cdots x_s^{n_s}}{n_1! n_2! \cdots n_s!}$$

Where the sum runs through all of the ways of writing n as a sum of non-negative numbers $n_1, \ldots, n_s$

- Why are the terms of the sum finite? We we solve this in the next class by counting them.
- Why is it true?
- Find the sum of all coefficients in the expansion of $(x + y + z)^3$.

Pascal's triangle

The following is Pascal's triangle. Starting with the 1's in the two non-horizontal sides of the triangle one can get any other number by adding the numbers at its top left and top right.

$n = 0$							1							
$n = 1$						1		1						
$n = 2$					1		2		1					
$n = 3$				1		3		3		1				
$n = 4$			1		4		6		4		1			
$n = 5$		1		5		10		10		5		1		
$n = 6$	1		6		15		20		15		6		1	
$n = 7$							$\ldots$							

Pascal's triangle

Pascal's triangle agrees with the following:

$$\begin{array}{lccccccccc}
n=0 & & & & & \binom{0}{0} & & & & \\
n=1 & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n=2 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n=3 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n=4 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}. \\
n=5 & & & & & \ldots & & & &
\end{array}$$

This is because of the following identities:

- $\binom{n}{0} = 1$, $\binom{n}{n} = 1$
- $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

Examples, Bars and stars

- Warm up: how many words of length 9 can be made with the symbols $\star$ and $|$ that have exactly four bars and 5 stars?
- Find the number of ways of distributing 5 bananas among 5 kids.
  Each way of distributing can be codified by one of the words in the previous example. We have 5 stars correspoinding to the 5 bananas. We have 4 bars that separate the stars. For example, the expression

  $$\star || \star \star | \star \star |$$

  corresponds to the first kid gets 1 banana, 2nd kid=0 bananas, 3rd kid=2 bananas, 4th kid=2 bananas, 5th kid= 0 bananas :(.
- How many integer solutions are there of the equation $x_1 + \cdots + x_5 = 5$ if $x_i \geq 0$? Hint: this is the same problem as the previous one.

Examples, Bars and Stars 2

- How many integer solutions are there of the equation $x_1 + \cdots + x_5 = 8$ if $x_i \geq 1$? This is like giving 8 bananas to 5 kids, but each kid must get at least 1 banana. So, we are really giving away 3 bananas to 5 kids.
- 7 students stop at a restaurant, where each of them has to pick one of the following: a cheeseburger, a hot dog, a taco or a sandwich. How many combinations are possible from the point of view of the guy writing down their order? Example: 3 cheese burgers, 3 tacos and a sandwich is one possible option. The book calls this counting with repetition.
- Count the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

such that $x_i \geq 2$.

Examples, Bars and Stars 3

- Find the number of ways to distribute 5 bananas and 6 oranges among 4 kids if each of the kids has to get at least 1 orange. Hint: product rule.
- Count the number of summands in the expansion of

$$(x_1 + \cdots + x_5)^7.$$

  Hint: what does the multinomial theorem say? Notice that this proves that we are summing over something finite in the multinomial theorem.
- Classic example: count the number of compositions an integer $n \geq 1$. These are the ways of writing n as the sum of smaller integers greater than one, where the order of sum matters.

Bars and stars: cheat sheet

Bars and stars

The number $\binom{n-1+r}{r}$ (corresponding to $n-1$ bars and r stars) counts each of the following:

- The number of integer solutions of
$$x_1 + \cdots + x_n = r$$
with $x_i \geq 0$.
- The number of possible selections, with repetition, of size r out of n types of objects.
- The number of ways r identical objects can be distributed in n distinct containers.