NMM2270-CH2
1ST ORDER DES

- 2.2 Separable Equations
- 2.3 Linear Equations
- 2.4 Exact Equations
- 2.5 Solutions by Substitutions
- 2.6 A Numerical Method
- 2.7 Linear Models
- 2.8 Nonlinear Models

2.5 SOLUTIONS BY SUBSTITUTIONS

"Often the first step in solving a given differential equation consists of transforming it into another differential equation by means of a substitution."

Formal Argument!

$$\frac{dy}{dx} = f(x,y) \;\; (= f(x, g(x,u))) \;\; , \text{ let } y = g(x,u)$$

divide by dx

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\cdot\frac{du}{dx} \leftarrow \text{Full differential}$$

$$\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$$

$$\underbrace{f(x, g(x,u)) = g_x(x,u) + g_u(x,u)}_{\text{Call this } F(x,u)}\frac{du}{dx}$$

, If solution is $u = \phi(x)$

then $y = g(x, \phi(x))$

$$\frac{dy}{dx} = f(x,y) \quad \rightarrow \quad y = g(x,u)$$

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\cancel{\frac{dx}{dx}} + \frac{\partial g}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$$

$$\frac{du}{dx} = \underbrace{\left[\frac{dy}{dx} - g_x(x,u)\right] \Big/ g_u(x,u)}_{F(x,u)}$$

$$\frac{du}{dx} = F(x,u) \leftarrow \text{hopefully easy to solve}$$

Homogeneous Equations

- If $f(tx,ty) = t^\alpha f(x,y)$, then f is called a *homogeneous function* of degree α.
- Example: $f(x,y) = x^5 + y^5$ is homogeneous of order 5.
  $$f(tx,ty) = (tx)^5 + (ty)^5 = t^5(x^5 + y^5) = t^5 f(x,y)$$
- Example: $f(x,y) = \frac{x-2y}{3x+y} = \frac{tx-2ty}{3tx+ty} = \frac{t(x-2y)}{t(3x+y)} = t^0 f(x,y)$ is homogeneous of order 0.
- Example: $g(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ is HMGs of order -1.
  $$g(tx,ty) = \frac{1}{\sqrt{t^2x^2+t^2y^2}} = \frac{1}{t\sqrt{x^2+y^2}} = t^{-1}g(x,y)$$

$$f(x,y) = x^7 + y^7$$

$$f(tx,ty) = (tx)^7 + (ty)^7 = t^7 x^7 + t^7 y^7$$

$$= t^7 \underbrace{(x^7 + y^7)} = t^7 f(x,y)$$

A HMGs eqn. of the form $M(x,y)dx + N(x,y)dy = 0$

- The equation is HMGs of the same order as the functions $M(x,y)$ and $N(x,y)$.

$$M(tx,ty) = t^{\alpha}\, M(x,y) \text{ and } N(tx,ty) = t^{\alpha}\, N(x,y)$$

$$\text{A linear } 1^{st} \text{ order DE } a_1 y' + a_0 y = g(x) = 0$$

- If M and N are HMGs functions of degree α :

$$M(x,y) = x^{\alpha}\, M(1,u) \text{ and } N(x,y) = x^{\alpha}\, N(1,u) \text{ where } u = y/x,$$

$$M(x,y) = y^{\alpha} M(v,1) \text{ and } N(x,y) = y^{\alpha} N(v,1) \text{ where } v = x/y$$

- This suggests that one uses:

$$y = u\,x \text{ or } x = v\,y$$

Note:

$$x^{\alpha}\, M(1,u)dx + x^{\alpha}\, N(1,u)dy = 0 \text{ or } M(1,u)dx + N(1,u)dy = 0$$

where $u = y/x$ or $y = ux$

- Now replace dy by $\underbrace{u\,dx + x\,du}_{dy}$ to get:

$$M(1,u)dx + N(1,u)\,[u\,dx + x\,du] = 0$$

$$[M(1,u) + u\,N(1,u)]dx + xN(1,u)\,du = 0, \text{ or}$$

$$\frac{dx}{x} + \frac{N(1,u)du}{M(1,u) + uN(1,u)} = 0$$

Formal Solution

$\Rightarrow$ Examples $\Rightarrow$

Example 1: Solving a homogeneous DE

Solve: $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$

$M(x,y) = x^2 + y^2$, $N(x,y) = x^2 - xy$

$M(tx,ty) = t^2(x^2+y^2) = t^2 M(x,y)$

$N(tx,ty) = t^2x^2 - tx\,ty = t^2(x^2 - y^2)$

Let $y = ux$,

$dy = u\,dx + x\,du$

$(x^2 + u^2x^2)\,dx + (x^2 - u x^2)\left[u\,dx + x\,du\right] = 0$

$\Rightarrow dx\left[x^2 + \cancel{u^2x^2} + x^2u - \cancel{x^2u^2}\right]$

$+ du\left[x^3 - u x^3\right] = 0$

$\Rightarrow \cancel{x^2}(\cancel{1+u})\,dx + \cancel{x^3}(1-u)\,du = 0$

$\underbrace{\hphantom{x}}_{\cancel{x^3}(\cancel{1+u})\ x} \qquad \underbrace{\hphantom{x}}_{\cancel{x^3}(1+u)}$

$\Rightarrow \dfrac{dx}{x} + \left(\dfrac{1-u}{1+u}\right)du = 0$

$\dfrac{1-u}{1+u} = \dfrac{1-u+1-1}{1+u} = \dfrac{2-(1+u)}{1+u} = -1 + \dfrac{2}{1+u}$

$\boxed{\left[-1 + \dfrac{2}{1+u}\right]du + \dfrac{dx}{x} = 0}$

$\Rightarrow$ continue

Example 1 cont'd

$$\int \left[-1 + \frac{2}{1+u}\right] du + \int \frac{dx}{x} = C'$$

$$-u + 2\ln|1+u| + \ln|x| = \ln|C|$$

$$-\frac{y}{x} + 2\ln\left|\frac{x+y}{x}\right| + \ln|x| = \ln|C|$$

$$\frac{y}{x} = \ln\left[\frac{(x+y)^2}{x^2} \cdot \frac{x}{C}\right] \Rightarrow \frac{(x+y)^2}{Cx} = e^{y/x}$$

$$(x+y)^2 = Cx\, e^{y/x}$$

$y = ux$
$u = \frac{y}{x}$

Practically: Use $x = vy$ whenever $M(x,y)$ is simpler than $N(x,y)$

Bernoulli's Equation

- $\frac{dy}{dx} + P(x)\,y = f(x)\boxed{y^n}$ where n is a real number.
- $n = 0$ makes the equation linear!
- $n = 1$ linear and separable.
- Leibniz (1646-1716) showed that for $n \neq 0$ and $n \neq 1$, the equation can be reduced to a linear eqn. by the substitution $\boxed{u = y^{1-n}.}$

$$n=0,\quad \frac{dy}{dx} + \underline{P(x)}\,y = f(x)\,y^0 = \underline{f(x)}$$

$$n=1,\quad \frac{dy}{dx} + P(x)\,y = f(x)\,y$$

$$\frac{dy}{dx} + [\overbrace{\underline{P(x) - f(x)}}^{P'(x)}]\,y = 0$$

Jacob Bernoulli (1654--1705)

Leibniz (1646-1716)

Example 2: Solving a Bernoulli Equation

- Solving $x\frac{dy}{dx} + y = x^2 y^2$.
- Divide by $x \Rightarrow \frac{dy}{dx} + \frac{1}{x}y = xy^2$, $n = 2, u = y^{1-2} = y^{-1}$.
- Substitute: $y = u^{-1} \Rightarrow \frac{dy}{dx} = -u^{-2}\frac{du}{dx}$
- The eqn. becomes: $\frac{du}{dx} - \frac{1}{x}u = -x$.

$$\Downarrow$$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx} y\right] = e^{\int P(x)\,dx} f(x)$$

$$\boxed{x\left(-u^{-2}\frac{du}{dx}\right) + u^{-1} = x^2 u^{-2} \longrightarrow \frac{du}{dx} - \frac{1}{x}u = -x}$$

$\Rightarrow$ Linear: $P(x) = -\frac{1}{x} \Rightarrow e^{\int P(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

$$\Rightarrow \frac{d}{dx}\left[-\frac{1}{x}u\right] = (-x)\frac{1}{x} = -1$$

$$u(x) = -x\int(-1)dx = -x^2 + cx$$

$$\Rightarrow y = \frac{1}{u} = \frac{1}{(-x^2+cx)} = (-x^2+cx)^{-1}$$

Verify: $\frac{dy}{dx} = -(-x^2+cx)^{-2}(-2x+c) = \frac{-(-2x+c)}{(-x^2+cx)^2}$

Left: $\frac{-(-2x^2+cx)}{(-x^2+cx)^2} + \frac{1}{(-x^2+cx)} = \frac{2x^2 \cancel{-cx} - x^2 + \cancel{cx}}{(-x^2+cx)^2}$

$$= x^2y^2 = RHS$$

Reduction to Separation of Variables

- A DE of the form $\frac{dy}{dx} = f(Ax + By + C)$ is always separable with the substitution $u = Ax + By + C, B \neq 0$.

Example 3: An Initial-value Problem

- Solve: $\frac{dy}{dx} = (-2x + y)^2 - 7, y(0) = 0$.

Let $u = -2x + y, then$

$$\frac{du}{dx} = -2 + \frac{dy}{dx} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2 = u^2 - 7 \Rightarrow \frac{du}{dx} = u^2 - 9$$

$$\frac{du}{u^2 - 9} = \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right] du = dx$$

$$\ln\left|\frac{u-3}{u+3}\right| = 6x + c' \Rightarrow \frac{u-3}{u+3} = c\, e^{6x}$$

$$y(0) = 0 \Rightarrow -1 = c$$

$u = -2x + y$

$x = 0, y = 0$

$\Leftarrow$ $\Rightarrow u = 0$

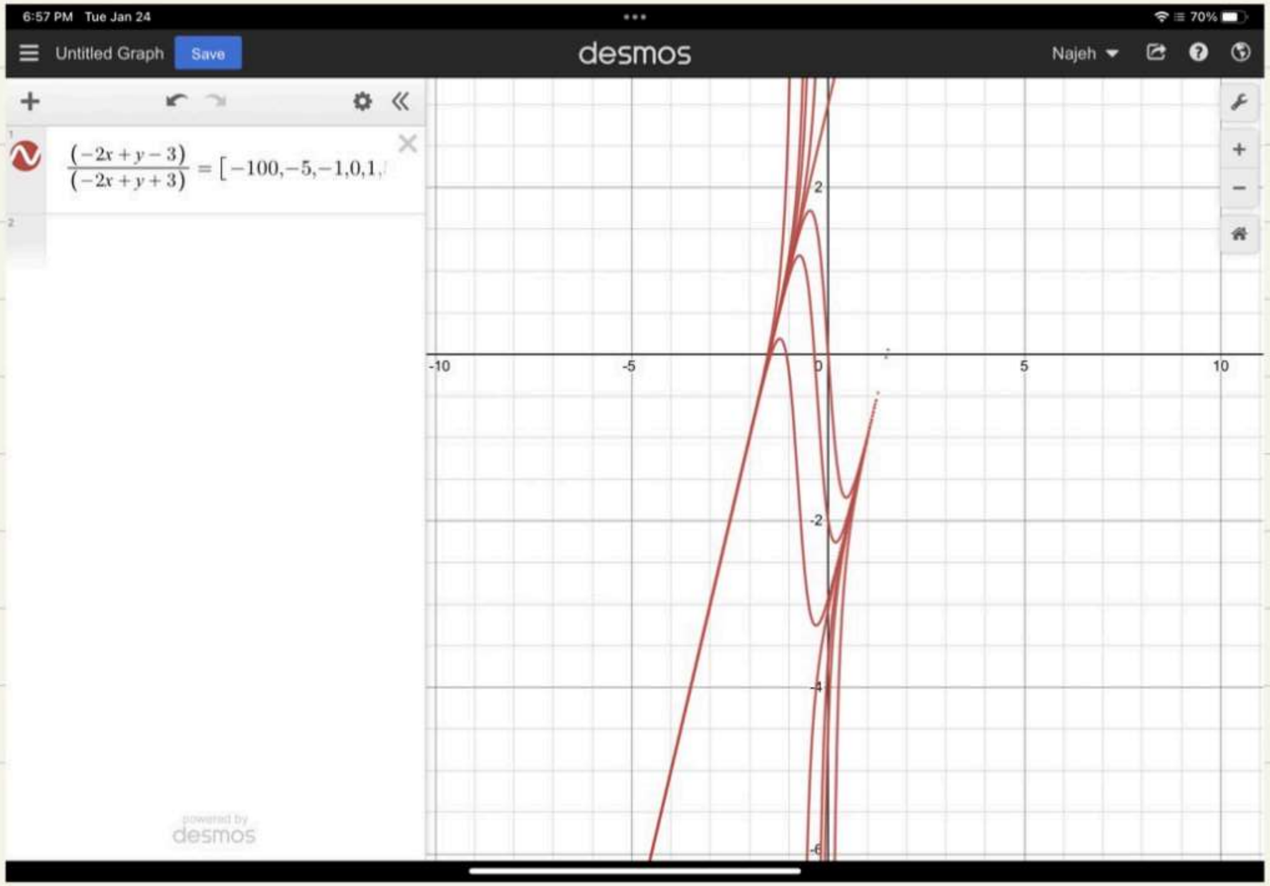
$$u - 3 = (u+3)(-e^{6x})$$

$$\Rightarrow u(1 + e^{6x}) = 3(1 - e^{6x}) \Rightarrow u(x) = 3\,\frac{1 - e^{6x}}{1 + e^{6x}}$$

$$\boxed{y(x) = 2x + 3\,\frac{1 - e^{6x}}{1 + e^{6x}}}$$

$$\boxed{\frac{u-3}{u+3} = c\, e^{6x}} \Rightarrow \boxed{\frac{-2x+y-3}{-2x+y+3} = c\, e^{6x}}$$

Draw w/ Desmos



End of Section 2.5 Problems

In Problems 1–10, solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$
2. $(x + y)\,dx + x\,dy = 0$
3. $x\,dx + (y - 2x)\,dy = 0$
4. $y\,dx = 2(x + y)\,dy$
5. $(y^2 + yx)\,dx - x^2\,dy = 0$
6. $(y^2 + yx)\,dx + x^2\,dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

$M(x,y)$ is simpler $\Rightarrow$

$x\,dx + (y-2x)\,dy = 0$

3. $M(tx,ty) = tx = t^1 M(x,y)$

$N(tx,ty) = (ty-2tx) = t(y-2x) = t^1 N(x,y)$

Soln: let $x = vy \Rightarrow dx = v\,dy + y\,dv$

$\Rightarrow vy[v\,dy + y\,dv] + [y - 2vy]\,dy = 0$

$v^2\cancel{y}\,dy + vy^{\cancel{2}}\,dv + \cancel{y}\,dy - 2v\cancel{y}\,dy = 0$

$-y\,v\,dv = [v^2 - 2v + 1]\,dy$

$\frac{-v\,dv}{(v-1)^2} = \frac{-v\,dv}{v^2 - v + 1} = \frac{dy}{y} \quad \Longrightarrow$

$$\int \frac{v\,dv}{(v-1)^2} = \int \frac{1+u}{u^2}\,du \qquad \begin{aligned} u &= v-1 \\ du &= dv \end{aligned}$$

$$= \int \frac{du}{u^2} + \int \frac{du}{u} = -\frac{1}{u} + \ln|u|$$

$$-\frac{v\,dv}{(v-1)^2} = \frac{-v\,dv}{v^2-v+1} = \frac{dy}{y} \quad \Rightarrow \quad -\left[\ln|v-1| - \frac{1}{v-1}\right] = \ln|y|$$

v

$$\Rightarrow \ln\left|\frac{x}{y}-1\right| - \frac{1}{\frac{x}{y}-1} + \ln|y| = C$$

$$\ln|x-y| - \frac{y}{x-y} = C \Rightarrow \boxed{(x-y)\ln|x-y| - y = C(x-y)}$$

End of Section 2.5 Problems

In Problems 11–14, solve the given initial-value problem.

11. $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$

12. $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, \quad y(-1) = 1$

13. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \; y(1) = 0$

14. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \; y(1) = e$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20, solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

16. $\dfrac{dy}{dx} - y = e^x y^2$

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

19. $t^2\dfrac{dy}{dt} + y^2 = ty$

20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

11. let $y = ux$, $dy = u\,dx + x\,du$

$$\cancelto{1}{x^3}u^2\,dy = (y^3 - x^3)\,dx = (\cancelto{1}{x^3}u^3 - \cancelto{1}{x^3})\,dx$$

$$u^2\left[u\,dx + x\,du\right] = (u^3 - 1)\,dx \Rightarrow$$

$$\cancel{u^3\,dx} + u^2 x\,du = \cancel{u^3}\,dx - dx$$

$$\Rightarrow u^2\,du = -\frac{dx}{x} \Rightarrow \frac{u^3}{3} = -\ln|x|$$

$$\ln|x| + \frac{y^3}{3x^3} = c \Rightarrow y(1) = 2$$

$$\ln(1) + \frac{8}{3} = c \Rightarrow c = \frac{8}{3}$$

19. $$t^2\frac{dy}{dt} + y^2 = ty$$

$$\frac{dy}{dt} + \frac{1}{t^2}y^2 = \frac{1}{t}y$$

$$\frac{dy}{dt} - \frac{1}{t}y = t^{-2}y^2$$

$$u^2\left(-u^{-2}\frac{du}{dt} - \frac{1}{t}\frac{1}{u} = t^{-2}\frac{1}{u^2}\right)$$

$$\frac{du}{dt} + \frac{1}{t}u = -\frac{1}{t^2}$$

$u = y^{-1}$

$y = u^{-1}$

$\frac{dy}{dt} = -u^{-2}\frac{du}{d\cdot}$

Linear $f(t) = -t$

$P(t) = \frac{1}{t}$

$e^{\int P(t)dt} = e^{\int\frac{dt}{t}} =$

$$\frac{d}{dt}\left[t\,u(t)\right] = -\frac{1}{t} \Rightarrow +u(t) = -\ln|t| + \ln|c|$$

$$u(t) = \frac{1}{y} = \frac{\ln|\frac{c}{t}|}{t} \Rightarrow y(t) = \frac{t}{\ln|\frac{c}{t}|}$$

2.7 LINEAR MODELS

2.8 NONLINEAR MODELS