Class notes

Math 2151

Fall 2024

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(from Colombia)

- Week 1

- Monday

we will start with counting techniques.

The following two principles combined are the building blocks of counting techniques.

The rule of sum:

If a task can be performed in m ways while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in $m+n$ different ways.

Examples:

- In how many ways can one draw a card from a standard deck (52 cards, 2 colors, 4 suits, 13 ranks or kinds) that is either red or a spade?

Picking out a red card $\underline{26}$

picking out a spade $\underline{13}$

total number of possibilities $26+13=39$

- Maybe there are more than 2 tasks. The sum rule still applies as long as no two tasks can be performed at the same time.

My hand in a card game includes 3 fives, 4 Jacks, 2 Kings and a Queen. How many options do I have to play my next card?

$$\underbrace{\underline{3}}_{\text{options for a 5}} + \underbrace{\underline{4}}_{\text{options for 4 Jack}} + \underline{2} + \underline{1} = 10$$

...

This rule doesn't seem very enligthening but we will use it often in tandem with the following rule.

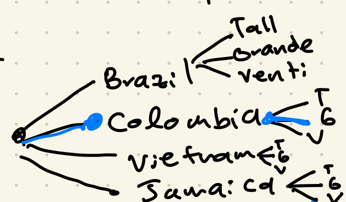
The rule of product:

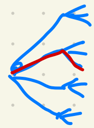
If a procedure can be broken down into first and second stages and if there are m possible outcomes for the first stage and n possible outcomes for the second stage, then the total procedure can be carried out in $m \cdot n$ ways.

- If I want to order a black coffee at a coffee shop and they sell coffees from 4 countries at 3 different sizes, in how many ways can I place my order?

$$\underbrace{4}_{\text{options for variety of coffee}} \cdot \underbrace{3}_{\text{options for size}} = \underbrace{12}_{\text{Total number of options.}}$$

Some people likes to represent this as a tree. If the countries are Brazil, colombia, Vietnam and Jamaica and the sizes are Tall, Grande and Venti, we get the tree





Branches represent possible choices

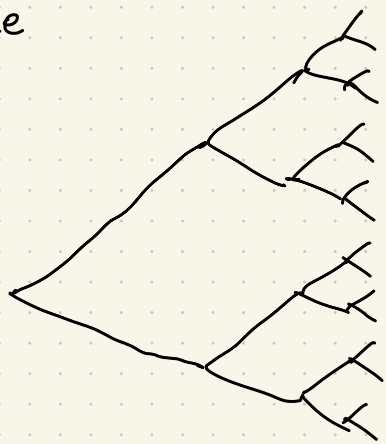
Again, the number of stages being 2 is irrelevant.

A bit string is a sequence of symbols chosen between the symbols 0 and 1. The length of a bit string is the number of symbols.

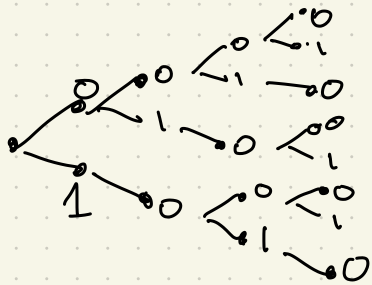
How many bit strings of length 4 are there?

$$\underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 16$$

Tree



How many of these bit strings do not have consecutive ones?



Answer 8 : 0000
0001
0010
0100
0101
1000
1001
1010

- **License plates**: How many licence plates are possible in ON if a plate consist of 4 letters followed by 3 numbers?

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \Rightarrow 26^4 \cdot 10^3$

- what if numbers can't be repeated?

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8} \Rightarrow 26^4 \cdot 10 \cdot 9 \cdot 8$

- what if numbers and letters can't be repeated?

$\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8}$ -

- How many plates start with an O and end with a 0 if no repetitions are allowed?

$\underline{1} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} \cdot \underline{9} \cdot \underline{8} \cdot \underline{1}$

- How many words of four letters can be formed with the letters A B C D E F G H
  1 2 3 4 5 6 7 8

  $\underline{8}\ \underline{8}\ \underline{8}\ \underline{8} = 8^4$

- what if no repetitions are allowed?

  $\underline{8}\ \underline{7}\ \underline{6}\ \underline{5}$

These "words" without repetition are called "arrangements" in the book, meaning that the words DACE and CEAD are different even if they are made with the same letters, so the order matters.

They are also called permutations.

Def: Given a collection of n objects, any ordered arrangement of these objects is called a <u>permutation</u>.

If there are n distinct objects and r is an integer $(1 \leq r \leq n)$, the <u>number of permutations of size r for the n objects is</u>:

(we are counting ordered selections of size r from a pool of n objects)

$$P(n,r) = \underset{1^{st}\text{ element}}{\underline{n}} \cdot \underset{2^{nd}\text{ element}}{\underline{n-1}} \cdot \underline{(n-2)} \cdots \underset{r^{th}\text{ element}}{\underline{(n-(r-1))}}$$

$$= n \cdot (n-1) \cdot \cdots \cdot [n-(r-1)] \cdot \frac{(n-r) \cdots 1}{(n-r) \cdot \cdots \cdot 1}$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

cheat sheet (write what it counts, not just the number)

10 students enter a competition. How many different possible podiums are there?

$$\underline{10} \cdot \underline{9} \cdot \underline{8} = P(10,3) = \frac{10!}{(10-3)!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$$

From a team of 10 students you have to pick a president a vicepresident and a secretary. In how many ways can you do this?

$$10 \cdot 9 \cdot 8 = P(10,3)$$

In how many different words can we arrange the letters WOW? (= How many different word can we form with the letters WOW)

Imagine that the first w is different from the second w

$W_1 O W_2$ the number of words that i can form is:

$$\underline{3}\cdot\underline{2}\cdot\underline{1} = 3! \quad \begin{cases} W_1 O W_2,\ O W_1 W_2,\ W_1 W_2 O, \\ W_2 O W_1,\ O W_2 W_1,\ W_2 W_1 O \end{cases}$$

Each word contains two w's which can be arranged in 2! ways so we have to divide by 2!.

Answer: $\dfrac{3!}{2!} = \dfrac{3\cdot\cancel{2\cdot1}}{\cancel{2\cdot1}} = 3 \quad \{WOW, OWW, WWO\}$

- In how many ways can we arrange the letters in CAAATV

6 letters

1 letter repeated 3 times

$$\frac{\overset{\text{words with six different letters}}{6!}}{\underset{\text{ways of arranging the 3 equal A's}}{3!}} = \frac{6\cdot5\cdot4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2\cdot1}} = 6\cdot5\cdot4$$

for example if $A_1 A_2 A_3$ are different letters

$C A_1 A_2 A_3 T V \quad C A_1 A_3 A_2 T V$
$C A_2 A_1 A_3 T V \quad C A_2 A_3 A_1 T V$
$C A_3 A_1 A_2 T V \quad C A_3 A_2 A_1 T V$

correspond to CAAATV, the same word, that's why we divide by $6 = 3 \cdot 2 \cdot 1$

- In how many ways can we arrange the letters in CCATVV?

$$\frac{6!}{2! \cdot 2!}$$

$6! \rightarrow$ r letters

$2! \rightarrow$ one letter repeated twice

$2! \rightarrow$ another letter repeated twice

$C_1 C_2 A T V_1 V_2$
$C_2 C_1 A T V_1 V_2$
$C_1 C_2 A T V_2 V_1$
$C_2 C_1 A T V_2 V_1$

In how many different ways can we arrange the letters in MASSACHUSETTS?

- 13 letters
- Letter A appears 2 times
- " S " 4 times
- " T " 2 times

$$\frac{13!}{2!\,4!\,2!}$$

- How many of these arrangements have all the S's together?

Hint: consider all the S's as one letter we would form words with the letters

M, A, A, C, H, U, T, T, S S S S, so,

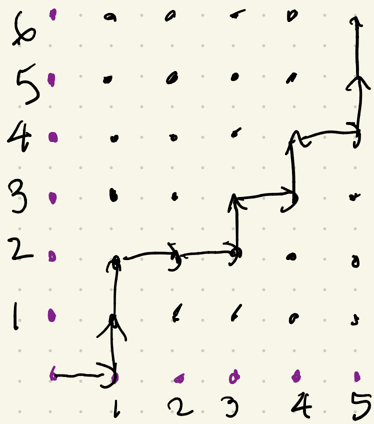
$$\frac{10!}{2!2!}$$

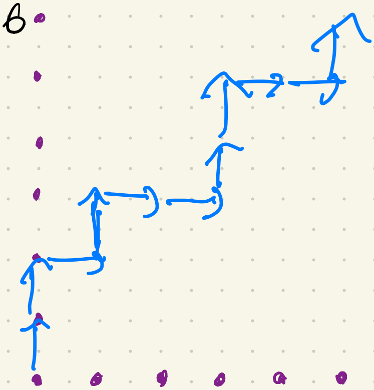
In general, if there are n objects composed of n_1 indistenguishable objects, ..., and n_r indistinguishable objects with $n_1 + \cdots + n_r = n$, the number of possible arrangements of the n objects is: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$

classical important example: ← I skipped this in class. I'll do it later

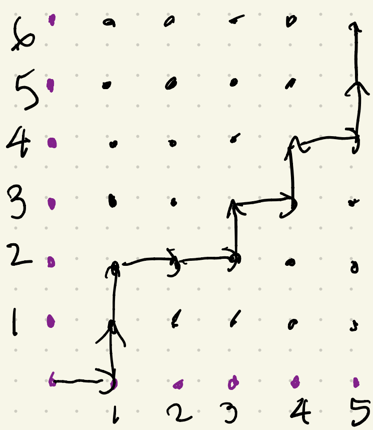
- Determine the number of paths in the xy-plane from $(0,0)$ to $(5,6)$ where each path is made of individual steps going to the right or up.

Let's draw some examples of paths



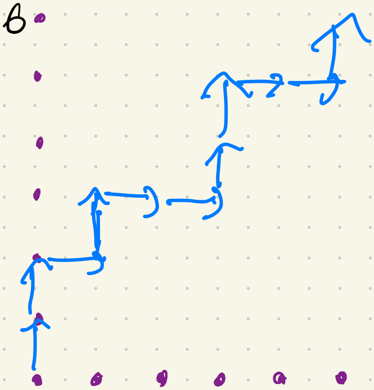


We can identify each of these paths with a sequence of the symbols R and U, where the symbol R appears 5 times and the symbol U appears 6 times



corresponds to

RUURRURURUU



Corresponds to

UURURRUURRU

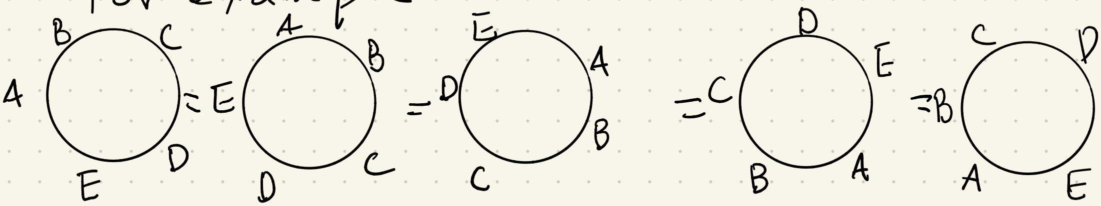
So, the number of paths is

$$\frac{11!}{5!\,6!}$$

Circular table examples

- 5 people seat at a circular table with 5 seats. Suppose that two seating arrangements are equal if one can be obtained from the other by a rotation.

For example



How many different seating arrangements are there?

If we forget about the rotations, there are $\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5!$ arrangements.

For each of these arrangements there are 5 arrangements that are equal, for example.

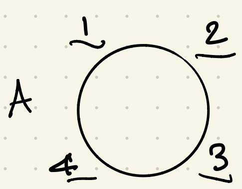
(ABCDE = EABCD = DEABC = CDEAB = BCDEA)

So the total number of arrangements is

$$\frac{5!}{5} = 4!.$$

Another way to think about this:

Since arrangements are equal up to rotation, fix who seats at the leftmost seat. Say, suppose A seats at the leftmost seat so that in the other seats any one can seat and there are no repetitions



$4.3.2.1 = 4!$

Combinations

In how many ways can a committee of 6 people be formed out of a group of 17 people?

(notice that the order of selection doesn't matter!)

If the order mattered there would be $P(17,6) = \frac{17!}{11!}$ possibilities.

However, since the order doesn't matter we are counting each committee 6! times! So, the answer is

$$\frac{\frac{17!}{11!}}{6!} = \frac{17!}{6!\,11!}$$

In general, we have the following:

Suppose n distinct objects are given. The number of ways of selecting r objects out of the n objects (with no reference to the order of selection) is

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

read n choose r

number of ways of selecting r objects if order mattered

number of ways of arranging r objects.

Examples:

In how many ways can we draw 4 cards out of a standard deck without replacement?

$$\binom{52}{4}$$

- Suppose that out of the 17 people 9 are Democrats and 8 are Republicans. If the comittee must have 3 Democrats and 3 Republicans, how many different comittees are there?

We use the product rule. Have to chose 3 Democrats and 3 republicans:

Answer $\binom{9}{3} \cdot \binom{8}{3}$

- What if the comittee must have at least 3 Democrats? Using the sum rule

The comittee can have 3, 4, 5, or 6 Democrats

3 democrats: $\binom{9}{3}\binom{8}{3}$

4 democrats; $\binom{9}{4}\binom{8}{2}$

5 Democrats: $\binom{9}{5} \cdot 8$

6 Democrats: $\binom{9}{6}$

Answer: $\binom{9}{3} \cdot \binom{8}{3} + \binom{9}{4}\binom{8}{2} + \binom{9}{5} \cdot 8 + \binom{9}{6}$

Another way of doing this:

$$\underbrace{\binom{9}{3}}_{\text{Choose 3 Dems}} \cdot \underbrace{\binom{8+6}{3}}_{\text{choose 3 among the rest}}$$

These two are equal because they count the same thing.

*) A teacher must make 4 debate teams out of a group of eight people. In how many ways can she choose the teams? (Debate teams have 2 members)

↓ It's in the book

Using combinations: Using the product rule

$$\underset{\text{1st team}}{\binom{8}{2}} \underset{\text{2nd team}}{\binom{6}{2}} \underset{\text{3rd team}}{\binom{4}{2}} \cdot \underset{\text{4th team}}{1} = \frac{8!}{2!\cancel{6!}} \cdot \frac{\cancel{6!}}{2!\cancel{4!}} \cdot \frac{\cancel{4!}}{2!2!}$$

$$= \frac{8!}{2^4}$$

Using permutations: $\underset{1^{rst}}{A}$ $\underset{2^{nd}}{B}$ $\underset{3^{rd}}{A}$ $\underset{4^{th}}{C}$ $\underset{5^{th}}{B}$ $\underset{6^{th}}{C}$ $\underset{7}{D}$ $\underset{8}{D}$

People ← 1rst

Four teams = 4 letters $\frac{8!}{2!\,2!\,2!\,2!} = \frac{8!}{2^4}$

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$$= \frac{8!}{2^4}$$

Using permutations: $\underset{1^{rst}}{\underline{A}}$ $\underset{2^{nd}}{\underline{B}}$ $\underset{3^{rd}}{\underline{A}}$ $\underset{4^{th}}{\underline{C}}$ $\underset{5^{th}}{\underline{B}}$ $\underset{6^{th}}{\underline{C}}$ $\underset{7}{\underline{D}}$ $\underset{8}{\underline{D}}$

People ⟵

Four teams = 4 letters $\dfrac{8!}{2!\,2!\,2!\,2!} = \dfrac{8!}{2^4}$

(We imagine that people are numbered from 1 to eigth and we are forming a word with letters AA ~~D~~B CC D D

This word would correspond to

Team A: 1, 3

Team B: 2, 5

Team C: 4, 6

Team D: 7, 8

The number of words with letters AABBCCDD is $\dfrac{8!}{2!2!2!2!}$ so, we win!

A classic:

The number of arrangements of the letters in MASSACHUSETTS is

$$\frac{13!}{2!\,4!\,2!}$$

How many of these arrangements have no adjacent S's?

Consider the arrangements of the letters in MASSACHUSETTS which are not S's

A classic:

The number of arrangements of the letters in $\underline{M}\,\underline{A}\,\underline{SS}\,\underline{A}\,\underline{CH}\,\underline{U}\,\underline{SETTS}$ is

$$\frac{13!}{2!\,4!\,2!}$$

How many of these arrangements have no adjacent S's?

Consider the arrangements of the letters in MASSACHUSETTS which are not S's

Another one using the sum rule and combinations.

Out of a set of 52 poker cards:

- How many different 5 cards hands can be dealt? (4 suits, 13 cards each)
  - $\binom{52}{5}$

- How many of these hands have four kings and one non-king?

$\underline{1} \cdot \underline{48} = \underline{\binom{4}{4}} \cdot \underline{\binom{52-4}{1}} = 1 \cdot 48 = 48$

options for 4 kings

options for 1- non-king

- How many of these hands have 4 of a Kind?

Sum Rule

$4\ A's = \binom{48}{1} = 48$

$4\ 1's = 48$

$4\ 2's = 48$

$\vdots$

Answer: $48 \cdot 13$

Sigma Notation

Σ (sigma) is an uppercase Greek letter that is used to abbreviate sums.

Instead of writing

$$a_1 + a_2 + \cdots + a_m$$

we write $\sum_{i=1}^{m} a_i = a_1 + \cdots + a_m$

Examples

- $\sum_{i=0}^{4} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = \sum_{i=1}^{5} 2^{i-1}$
- $\sum_{i=1}^{n} a = \underbrace{a + a + a \cdots + a}_{n \text{ times}} = n \cdot a$
- $\sum_{i=0}^{n} a_i = \sum_{i=1}^{n-1} a_{i-1} = \sum_{i=2}^{n-2} a_{i-2}$

$= a_0 + \cdots + a_n$

- $a_m + a_{m+1} + \cdots + a_{m+n} = \sum_{i=0}^{n} a_{m+i}$

- $\sum_{i=0}^{6} \frac{1}{2^i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$

- $\sum_{i=3}^{7} \frac{1}{3+i} = \frac{1}{3+3} + \frac{1}{3+4} + \frac{1}{3+5} + \frac{1}{3+6} + \frac{1}{3+7}$

- $a_3 + a_4 + a_5 + a_6 = \sum_{i=0}^{3} a_{3+i}$

$$= \sum_{i=1}^{4} a_{2+i}$$

$$= \sum_{i=2}^{5} a_{1+i}$$

$$= \sum_{i=3}^{6} a_i$$

Binomial theorem

For $n \geq 2$, $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ holds for x, y numbers or polynomial variables.

For $n=2$ this means

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

For $n=3$

$$(x+y)^3 = (x+y)^2\,(x+y)$$
$$= (x^2+2xy+y^2)(x+y)$$
$$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$$
$$= 1x^3 + 3x^2y + 3xy^2 + y^3$$
$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

How would one go about proving this

In $\underbrace{(x+y)}_{1^{st}\text{ factor}}\underbrace{(x+y)}_{2^{nd}\text{ factor}}\underbrace{(x+y)}_{3^{rd}\text{ factor}}\cdots\underbrace{(x+y)}_{n^{th}\text{ factor}} = (x+y)^n$

The coefficient of $x^r y^{n-r}$ is the number of ways of choosing the x in r factors and, consequently, the y in the rest of the factors.

(e.g. for $n=3$ the coefficient of xy^2 in

$(x+y)^3 = (x+y)(x+y)(x+y) = xy^2 + y^2x + yxy + \cdots$

is 3 corresponding to adding the three xy^2 comming from the multiplication of the blue, red, and green factors. All other multiplications of sumands in the factors give different monomals)

we have to choose r factors out of n and so the coefficient of $x^r y^{n-r}$ is $\binom{n}{r} = \binom{n}{n-r}$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Notice that $\binom{n}{r} = \binom{n}{n-r}$

Examples:

- Coefficient of x^6y^4 in $(x+y)^{10}$: $\binom{10}{4}$
- Coefficient of x^6y^4 in $(2x+3y)^{10}$

$a = 2x \quad b = 3y \quad (= (a+b)^{10})$

a^6b^4 has coefficient: $\binom{10}{4}$

so x^6y^4 has coefficient: $\binom{10}{4}2^6 3^4$

- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \dots + \binom{n}{n} = 2^n$

If we set $x = 1,\ y = 1$ in $(x+y)^n$ we get

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 1^r = \sum_{r=0}^{n} \binom{n}{r}$$

Multinomial theorem

$$(x_1 + \dots + x_s)^n = \sum_{\substack{n_1 + \dots + n_s = n \\ 0 \le n_i \le n}} \frac{n!\ x_1^{n_1} \dots x_s^{n_s}}{n_1!\, n_2! \dots n_t!}$$

- Coefficient of $x^2y^3z^4$ in $(x+y+z)^9$

Answer: $\dfrac{9!}{2!3!4!}$ $\quad (s = 3,\ n_1 = 2,\ n_2 = 3,\ n_3 = 4,\ n = 9)$

- Example: sum of all coefficients in $(x+y+z)^3$

$$(x+y+z)^3 = \sum_{\substack{n_1+\cdots+n_k=3 \\ 0\le n_i\le 3}} \frac{3!}{n_1!\cdots n_k!} x^{n_1} y^{n_2} z^{n_3}$$

The sum of all coefficients is

$$\sum_{\substack{n_1+\cdots+n_k=3 \\ 0\le n_i\le 3}} \frac{3!}{n_1!\cdots n_k!} = \sum_{\substack{n_1+\cdots+n_k=3 \\ 0\le n_i\le 3}} \frac{3!}{n_1!\cdots n_k!} 1^{n_1} 1^{n_2} 1^{n_3} = (1+1+1)^3$$

$$= 3^3 = 27$$

Note: The ten elements in the sum are

n_1	n_2	n_3	$\frac{3!}{n_1!\cdots n_k!}$
0	0	3	6/6 1
0	3	0	6/6 1
3	0	0	6/6 1
0	1	2	6/2 3
0	2	1	6/2 3
1	0	2	6/2 3
2	0	1	6/2 3
1	2	0	6/2 3
2	1	0	6/2 3
1	1	1	6

$1+1+1+3+3+3+3+3+3+6$

$= 7\cdot 3 \quad +6$

$= 27$

- $$\sum_{i=0}^{n} \frac{(-1)^{i+1}}{i!(n-i)!} = \sum_{i=0}^{n} \frac{(-1)^i(-1)\,1^{n-i}}{i!(n-i)!} = -\sum_{i=0}^{n} \frac{(-1)^i\,1^{n-i}}{i!(n-i)!} = (-1+1)^n = 0$$

The triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

where we obtain the entries off the boundary of the triangle by adding the entries to the top right and top left agrees with

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

This holds because

- $\binom{n}{0} = \binom{n}{n} = 1$, and
- $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

proof of the second

the LHS counts the number of selections of $r+1$ elements out of $n+1$ elements. Let's count this in a different way. suppose that, among the $n+1$ elements there is one special element a, and other n.

- The number of selection of $r+1$ elements out of $n+1$ that contain the special element is $\binom{n}{r}$ → choose r elements out of the non special ones. These, together with the special element give $r+1$ elements
- The number of selections of $r+1$ elements out $n+1$ that don't contain the special element is $\binom{n}{r+1}$ → choose $r+1$ out of the non special ones

So, the total number of selections of $r+1$ elements out of $n+1$ elements is

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Bars and stars

- Number of ways of distributing 5 bananas among 5 kids.

Each distribution can be identified with an expression containing 4 bars and 5 stars

$$* \mid ** \mid\mid ** \mid$$

corresponds to giving

1st kid $\to$ 1 banana

2nd kid $\to$ 2 bananas

3rd kid $\to$ 0 bananas

4th kid $\to$ 2 bananas

5th kid $\to$ 0 bananas

So, the answer is $\dfrac{(5+4)!}{5!\,4!} = \binom{9}{4}$

$$\left[\ \underline{*}\ \underline{\mid}\ \underline{*}\ \underline{*}\ \underline{\mid}\ \underline{\mid}\ \underline{*}\ \underline{*}\ \underline{\mid}\ \to \text{choose 4 spots where the bars go}\ \right]$$

- How many integer solutions are there of the equation

$$x_1 + \cdots + x_5 = 9$$

if $x_i \geq 0$.

$$\binom{9+4}{4} = \binom{13}{4} = \frac{13!}{4!\,9!}$$

(notice this is the same as $\binom{13}{9}$

- What if $x_i > 0$? $x_1 + \cdots + x_5 = 9$

since $x_i \geq 1$ call $a_i = x_i - 1 \geq 0$

we are looking for solutions

$$(a_1+1) + \cdots + (a_5+1) = 9$$

That is

$$a_1 + \cdots + a_5 = 9 - 5 = 4, \; a_i \geq 0$$

So we have 4 bars and 4 stars

$$\frac{8!}{4!\,4!}$$

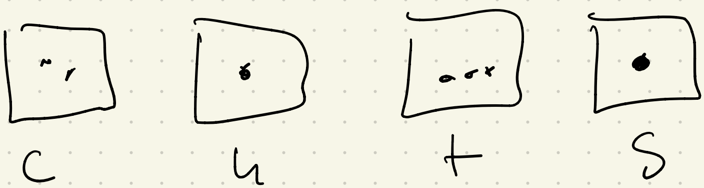
Example:

7 highschool freshmen stop at a restaurant, where each of them has to pick one of the following: a cheeseburger, a hot dog, a taco or a sandwich.

How many combinations are possible from the point of view of the restaurant?

Example: one option would be

c, c, h, t, t, t, s



7 stars and 3 bars

* * | * | * * * | *

Answer $\binom{7+3}{3}$

- In how many ways can we distribute seven bananas and six oranges among four children so that each child gets at least one banana?

Since we have to give each child 1 banana at least, 4 bananas have to be (necessarily) given away.
Let's consider the problem of distributing 6 oranges an 3 bananas amog 4 children which has the same number of solutions.
We use the product rule. First we need to distribute 6 oranges among 4 children (6 stars, 3 bars) which has $\binom{6+3}{3} = \binom{6+3}{6}$ of being done, then we have

- $\binom{4+3}{3}$ ways of distributing 4 bananas among 4 children. So, the answer is

$$\binom{9}{3} \cdot \binom{7}{3}.$$

Fact:

The following numbers are equal to $\binom{n-1+r}{r}$

($n-1$ bars, r stars)

i) The number of integer solutions of

$$x_1 + \cdots + x_n = r \quad x_i \geq 0$$

ii) The number of selections, with repetition, of size r from n types of objects.

iii) The number of ways r identical objects can be distributed in n distinct containers