NMM2270 Fall 2024 Assignment 2

Najeh M Zarir

Available: 00:01 on Oct. 1, 2024; Due: 11:59PM on Oct. 15, 2024

Print Name: Parth M. Valand   Student Number: 251393822

Signature: Parthy   UWO email/ user ID: pvaland@uwo.ca

Exam seating: Room –   Row: –   Seat: –

This space is for you to flag potential issues with any questions.

None!

Solve each of the following questions in full detail. Create one PDF file and upload it. Your last upload will be the only file graded. You have two weeks to finish the assignment.

1. (4 points) Fill the table below: For each equation indicate if it is linear or nonlinear, and provide enough justification to support your choice. Use extra space below the table if you need.

Equation	Linear	Nonlinear	Justify your answer
$(x^2+1)\frac{dy}{dx} = \sqrt{x+y}$			
$x\tan^{-1}(y)\frac{dy}{dx} = e^{x+y}$			
$y' = \sin(xy)$			
$y' = \frac{1}{x+y}$			

2. (2 points) Rewrite the linear differential equation in the standard form of a linear differential equation.

$$\cos x dy + (\sin x) y dx = dx$$

3. (2 points) Find the general solution of the following differential equation:

$$\frac{dy}{dx} + ky = -kA$$

4. (2 points) Which of the following differential equations is a Bernoulli equation? Justify your answer then solve the equation if you can.
$(x^2+y^2)dy–2xydx = 0$
$3dy = \sin(x)(y^2 - y)dx$
$e^y dx = 5 - \sin(y)dy$
$\frac{dy}{dx} = \ln(y)$

5. (2 points) Which of the following is an example of a separable differential equation? Justify your answer and setup the equation and solve it if you can.
$y' = \sin(xy)$
$(x^2+1)\frac{dy}{dx} = \sqrt{x+y}$
$x\tan^{-1}(y)\frac{dy}{dx} = e^{x+y}$
$y' = \frac{1}{x+y}$

6. (2 points) True or False? The general solution of the differential equation

$$2\frac{dy}{dx} = -4y + 5 \text{ is } y = Cx^{-2} + \frac{5}{4}$$

Justify your answer in detail.

7. (2 points) True or False? The integrating factor of the differential equation

$$-7(y + x^5)dx + x^2 dy = 0 \text{ is } \frac{-7}{x^2}$$

Justify your answer in detail.

1. (4 points) Fill the table below: For each equation indicate if it is linear or nonlinear, and provide enough justification to support your choice. Use extra space below the table if you need.

Equation	Linear	Nonlinear	Justify your answer
$(x^2+1)\frac{dy}{dx} = \sqrt{x+y}$		✓	a
$x\tan^{-1}(y)\frac{dy}{dx} = e^{x+y}$		✓	b
$y' = \sin(xy)$		✓	c
$y' = \frac{1}{x+y}$		✓	d

a) y has a power of $\frac{1}{2}$ which makes it non-linear

b) the coefficient of y is $\tan^{-1}(y)$ which is a function of y, hence non-linear.

c) y is in the function of $\sin(xy)$ so it is non-linear

d) when re-written as $y'(x+y)=1$, the coefficient of y' is $(x+y)$ which is a function, hence non-linear.

2)

2. (2 points) Rewrite the linear differential equation in the standard form of a linear differential equation.

$$\cos x dy + (\sin x) y dx = dx$$

$\rightarrow \quad \cos x \frac{dy}{dx} + (\sin x) y = 1$

$$\frac{dy}{dx} + (\tan x)\, y = \sec x$$

this is the standard linear differential equation.

3)

3. (2 points) Find the general solution of the following differential equation:

$$\frac{dy}{dx} + ky = -kA$$

$k = p(x)$ $\qquad -kA = q(x)$

$$I.F = e^{\int k\, dx}$$
$$= e^{kx\, dx}$$

$$\int e^{kx} \frac{dy}{dx} + \int e^{kx} ky = \int -kAe^{kx}$$

$$e^{kx} y = \frac{-kAe^{kx}}{k} + c$$

$y = -A + ce^{-kx}$ is the general solution.

4) Bernoulli's differential equation follows the form:

$$y' + p(x)\,y = q(x)\,y^n$$

i) $(x^2 + y^2)\,dy - 2xy\,dx = 0$

$$(x^2 + y^2)\frac{dy}{dx} - 2xy = 0$$

$$\frac{dy}{dx} - \frac{2xy}{(x^2+y^2)} = 0$$

↳ the coefficient includes a function of y so it does not match bernoulli's format

ii) $3\,dy = (\sin x)(y^2 - y)\,dx$

$$3\frac{dy}{dx} = y^2\sin x - y\sin x$$

$$3\frac{dy}{dx} + y\sin x = y^2\sin x$$

$$\frac{dy}{dx} + \frac{y\sin x}{3} = \frac{y^2\sin x}{3}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{\sin x}{3y} = \frac{\sin x}{3}$$

Using $a = \frac{1}{y}$ & $da = -\frac{1}{y^2}\,dy$

$$-\frac{da}{dx} + \frac{a\sin x}{3} = \frac{\sin x}{3}$$

$$\frac{da}{dx} - \frac{a\sin x}{3} = \frac{-\sin x}{3}$$

Integrating Factor : $e^{\int \frac{-\sin x}{3}\,dx} = e^{\frac{\cos x}{3}}$

$$\int e^{\frac{\cos x}{3}} \frac{da}{dx} - \frac{\sin x}{3} e^{\frac{\cos x}{3}} a = \int e^{\frac{\cos x}{3}} \frac{\sin x}{3}$$

$$e^{\frac{\cos x}{3}} a = e^{\frac{\cos x}{3}} + c$$

$$a = 1 + ce^{\frac{-\cos x}{3}}$$

Now to sub back a:

$$\frac{1}{y} = 1 + ce^{\frac{-\cos x}{3}}$$

$$\boxed{y = \frac{1}{1 + ce^{\frac{-\cos x}{3}}}}$$

Is the general solution.

iii) $e^y \, dx = 5 - \sin y \, dy$

$$\sin y \frac{dy}{dx} + e^y = \frac{5}{dx}$$

$$\frac{dy}{dx} + \frac{e^y}{\sin y} = \frac{5}{\sin y \, dx}$$

this isn't bernoulli's layout because sin y is a function of y and there is no y multiplied by it

iv) $\frac{dy}{dx} = \ln(y)$

this would never fit because ln is a function of y and is not linear.

5) i) $\frac{dy}{dx} = \sin(xy)$

this is not a separable equation as x and y cannot be separated. Hence it's not a separable D.E.

ii) $(x^2+1)\dfrac{dy}{dx} = \sqrt{x+y}$

Because of $\sqrt{x+y}$, x & y can't be separated and hence it isn't a separable D.E.

iii) $x\tan^{-1}(y)\dfrac{dy}{dx} = e^{x+y}$

$$\frac{\tan^{-1}(y)}{e^y}\,dy = e^x\,dx$$

This is separable D.E as shown above. Now to solve it:

$$\int \frac{\tan^{-1}(y)}{e^y}\,dy = \int e^x\,dx + c$$

iv) $y' = \dfrac{1}{x+y}$

Due to $\dfrac{1}{x+y}$, this can't be separated and hence isn't a separable D.E

6) The D.E is:

$$\frac{dy}{dx} = -2y + \frac{5}{2} \rightarrow \frac{dy}{dx} + 2y = \frac{5}{2}$$

IF $e^{\int 2\,dx} = e^{2x}$

$$\frac{dy}{dx}(e^{2x}) + 2ye^{2x} = \frac{5}{2}e^{2x}$$

Integrate everything:

$$e^{2x}y = \int \frac{5}{2}e^{2x}$$

$$e^{2x}y = \frac{5}{4}e^{2x} + c$$

$$y = \frac{5}{4} + ce^{-2x}$$ is the general solution.

Compared to $y = cx^{-2} + \frac{5}{4}$, it is not the same so the answer is false.

7) The DE is

$$-7(y+x^5)\,dx + x^2\,dy = 0$$

$$x^2\,dy = 7(y+x^5)\,dx$$

$$\frac{dy}{dx} = \frac{7(y+x^5)}{x^2}$$

$$\frac{dy}{dx} - 7x^{-2}(y+x^5) = 0$$

$$\frac{dy}{dx} - 7x^{-2}y - 7x^3 = 0$$

$$\frac{dy}{dx} - 7x^{-2}y = 7x^3$$

IF:

$$e^{\int -7x^{-2}\,dx} = e^{7x^{-1}} = e^{\left(\frac{7}{x}\right)}$$

$$e^{\left(\frac{7}{x}\right)} \neq \frac{-7}{x^2}$$

So the answer is false, they are not the same.

8. (2 points) True or False? The differential equation

$$\left(2y\sin(x)\cos(x)-y+2y^2e^{xy^2}\right)dx = \left(x-\sin^2(x)-4xye^{xy^2}\right)dy$$

   is exact.

   Justify your answer.

9. (3 points) True or False? The population of a certain town is known to increase at a rate proportional to the current population. If the population doubles every 15 years, and the current population is 32,000, then the population in 11 years will be 53,199.

   Justify your answer.

10. (3 points) Solve the differential equation:

$$e^{2x}\frac{dy}{dx} = e^{-2y} + e^{-2x-2y}$$

   .

11. (3 points) Find an implicit solution of the initial-value problem:

$$y' = \frac{y^2-4}{x^2-9}, y(0) = 3$$

   .

12. (3 points) Find the general solution of the differential equation and give the largest value internal over which the general solution is defined.

$$(x+4)\frac{dy}{dx} + (x+5)y = 5xe^{-2x}$$

13. (3 points) Solve the initial-value problem

$$2\frac{dy}{dx} - 4xy - 3 = 0, y(0) = 4$$

   .

14. (3 points) Initially, 500 milligrams of a radioactive substance were present. After 3 days, the mass had decreased by 145 milligrams. If the rate of decay is proportional to the amount of the substance present at time then find the amount remaining after 10 days (rounded to the nearest milligram).

15. (3 points) A thermometer reading 65° F is placed in an oven heated to a constant temperature. The thermometer reads 125° F after 1 minute and 170°F after 2 minutes. What is the temperature of the oven?

16. (5 points) Find the value of k so that the differential equation is exact

$$\left[8yt^5 + kt^2e^{ty}\right]dy = \left[\frac{2}{t} - 20y^2t^4 - ktye^{ty} - 5e^{ty}\right]dt$$

8) $M(x,y) = 2y \sin x \cos x - y + 2y^2 e^{xy^2}$

$N(x,y) = -x + \sin^2 x + 4xye^{xy^2}$

$$\frac{\partial M}{\partial y} = 2\sin x \cos x \quad -1 + 4ye^{xy^2} + e^{xy^2}(4y^3x)$$

$$\frac{\partial N}{\partial x} = -1 + 2\sin x \cos x + 4ye^{xy^2} + e^{xy^2}(4xy^2)(y)$$

$$= -1 + 2\sin x \cos x + 4ye^{xy^2} + e^{xy^2}(4xy^3)$$

As shown by the highlights, all the terms match up so the DE is exact. True.

9)
$$\frac{d(P(t))}{dt} = k\,dt$$

$$\frac{d(P(t))}{P(t)} = k\,dt$$

$$\ln P(t) = kt + c$$

$$P(t) = e^{kt+c} = e^{kt}e^{c}$$

at $t=0$, $e^{kt} = 1$ so

$$P(0) = e^{c}$$

and from the question: $P(0) = 32\,000$

From the question:

$$P(15) = 2P(0)$$

$$\cancel{P(0)}e^{k(15)} = 2\cancel{P(0)}$$

$$e^{15k} = 2$$

$$15k = \ln(2)$$

$$k = \frac{\ln 2}{15}$$

$$P(11) = 32000\,e^{\frac{\ln 2}{15}(11)}$$

$P(11) = 53'199$ So true, the population after 11 years is 53 199

10) The D.E is: $e^{2x}\frac{dy}{dx} = e^{-2y} + e^{-2x-2y}$

$$e^{2x}\frac{dy}{dx} = e^{-2y}(1 + e^{-2x})$$

$$e^{2y}\,dy = (1+e^{-2x})e^{-2x}\,dx$$

$$e^{2y}\,dy = e^{-2x} + e^{-4x}\,dx$$

Integrate:

$$\int e^{2y}\,dy = \int e^{-2x} + e^{-4x}\,dx$$

$$\tfrac{1}{2}e^{2y} = -\tfrac{1}{2}e^{-2x} - \tfrac{1}{4}e^{-4x} + c$$

$$e^{2y} = -e^{-2x} - \tfrac{1}{2}e^{-4x} + 2c$$

$$2y = \ln\left(-e^{-2x} - \tfrac{1}{2}e^{-4x} + 2c\right)$$

$$y = \tfrac{1}{2}\ln\left(-e^{-2x} - \tfrac{1}{2}e^{-4x} + 2c\right)$$ is the general solution.

11) The D.E is: $\dfrac{dy}{dx} = \dfrac{y^2-4}{x^2-9}$

$$\frac{dy}{y^2-4} = \frac{dx}{x^2-9}$$

$$\int \frac{dy}{y^2-4} = \int \frac{dx}{x^2-9}$$

$$\tfrac{1}{2}\ln\left|\frac{2+y}{2-y}\right| = \tfrac{1}{3}\ln\left|\frac{3+x}{3-x}\right| + c$$

Given $y(0) = 3$

$$\tfrac{1}{2}\ln\left|\frac{2+3}{2-3}\right| = \tfrac{1}{3}\ln\left|\frac{3+0}{3-0}\right| + c$$

$$\frac{1}{2}\ln 5 = 0 + c$$

$$\boxed{c = \frac{1}{2}\ln 5}$$

So the implicit solution is :

$$\frac{1}{2}\ln\left|\frac{2+y}{2-y}\right| = \frac{1}{3}\ln\left|\frac{3+x}{3-x}\right| + c$$

where $c = \frac{1}{2}\ln 5$

So the answer is :

$$\frac{1}{2}\ln\left|\frac{2+y}{2-y}\right| = \frac{1}{3}\ln\left|\frac{3+x}{3-x}\right| + \frac{1}{2}\ln 5 \quad //$$

12) The D.E is :

$$(x+4)\frac{dy}{dx} + (x+5)y = 5xe^{-2x}$$

$$\frac{dy}{dx} + \frac{(x+5)}{(x+4)}y = \frac{5x}{(x+4)e^{2x}}$$

Following the linear format :

IF : $e^{\int \frac{x+5}{x+4}dx} = e^{\int (1+\frac{1}{x+4})dx} = e^{x+\ln(x+4)}$

$$\int \left(e^{x+\ln(x+4)}\right) \frac{dy}{dx} + \frac{(x+5)}{(x+4)} y \left(e^{x+\ln(x+4)}\right) = \int \frac{5x\left(e^{x+\ln(x+4)}\right)}{(x+4)e^{2x}}$$

$$e^{x+\ln(x+4)} y = \int 5xe^{-x}$$

$$e^{x}(x+4)y = 5\int xe^{-x}$$

$$= 5\left(x\frac{d(e^{-x})}{dx} - \int \frac{d(x)}{dx}\left[\int e^{-x}\, dx\right] dx\right)$$

$$= 5\left(xe^{-x} - \int 1(-1)e^{-x}\, dx\right)$$

$$= 5\left(xe^{-x} - e^{-x}\right)$$
$$= -5e^{-x}(1-x)$$

The general solution is

$$e^{x}(x+4)y = -5e^{-x}(1-x) + c$$

$$y = \frac{-5e^{-x}(1-x)}{e^{x}(x+4)} + \frac{c}{e^{x}(x+4)}$$

For this to be valid, all values of x have a y-value except $x = -4$ for $(x+4)$.

So the largest interval is $R - \{-4\}$ for which the general solution is defined.

13) With the given DE:

$$2\frac{dy}{dx} - 4xy - 3 = 0$$

$$\frac{dy}{dx} - 2xy = \frac{3}{2}$$

I F: $e^{\int -2x\,dx} = e^{-x^2}$

$$\int \frac{dy}{dx} e^{-x^2} - 2xye^{-x^2} = \int \frac{3}{2} e^{-x^2}$$

$$ye^{-x^2} - c = \int \frac{3}{2} e^{-x^2}\,dx$$

$$ye^{-x^2} - y(0) = \int \frac{3}{2} e^{-x^2}\,dx$$

Using the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2}\,dx$

$$ye^{-x^2} - y(0) = \frac{3}{2}\left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right)\int e^{-x^2}\,dx$$

$$ye^{-x^2} - 4 = \frac{3\sqrt{\pi}}{4}\operatorname{erf}(x)$$

$y = e^{x^2}\left(4 + \frac{3\sqrt{\pi}}{4}\operatorname{erf}(x)\right)$ is the solution

14) Given $m(0) = 500$ mg & $m(3) = 500 - 45 = 355$ mg

$$\frac{d(m(t))}{dt} = -k\,m(t)$$

$$\int \frac{d(m(t))}{m(t)} = \int -k \, dt$$

$$\ln(m(t)) = -kt + c$$

$$m(t) = e^{-kt+c}$$

$$m(t) = e^{-kt} e^{c}$$

$$500 = e^{-k(0)} e^{c} \qquad e^{c} = 500 \qquad c = \ln(500)$$

$$m(3) = 500 \, e^{-kt}$$

$$355 = 500 \, e^{-kt}$$

$$\ln\left(\frac{355}{500}\right) = -3k$$

$$-\frac{1}{3} \ln\left(\frac{355}{500}\right) = k$$

Now for $m(10)$:

$$m(10) = 500 \, e^{\frac{10}{3} \ln\left(\frac{355}{500}\right)}$$

$= 160$ mg remains after 10 days.

15) Using Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - T_s) \qquad T_s = \text{oven's temperature}$$

Integrating that:

$$\ln(T - T_s) - \ln(T_0 - T_s) = -kt$$

$$\frac{T - T_s}{T_0 - T_s} = e^{-kt}$$ $T_0 = 65°F$ (initial temp. of thermometer).

After 1 min, $T = 125°F$:

$$\frac{125 - T_s}{65 - T_s} = e^{-k}$$

After 2 min, $T = 170°F$

$$\frac{170 - T_s}{65 - T_s} = e^{-2k}$$

Using these two equasions:

$$(e^{-2k}) = (e^{-k})^2 \longrightarrow \frac{170 - T_s}{65 - T_s} = \left(\frac{125 - T_s}{65 - T_s}\right)^2$$

$$(170 - T_s)(65 - T_s) = (125 - T_s)^2$$

$$11050 - 235T_s + \cancel{T_s^2} = 125^2 - 250T_s + \cancel{T_s^2}$$

$$15T_s = 125^2 - 11050$$

$T_s = 305°F$ is the temperature of the oven.

16) The given D.E follows: $M(t,y)\,dt + N(t,y)\,dy = 0$

where:

$$M(t,y) = -2t^{-1} + 20y^2t^4 + ktye^{ty} + 5e^{ty}$$

$$N(t,y) = 8yt^5 + kt^2e^{ty}$$

$$\frac{\partial M}{\partial y} = 0 + 40t^4y + kte^{ty} + kt^2e^{ty} + 5te^{ty}$$

$$\frac{\partial N}{\partial t} = 40yt^4 + 2kte^{ty} + kt^2ye^{ty}$$

For DE to be exact:

$$\frac{\partial N}{\partial t} = \frac{\partial M}{\partial y}$$

$$\cancel{40yt^4} + \cancel{2}kte^{ty} + \cancel{kt^2ye^{ty}} = \cancel{40t^4y} + \cancel{kte^{ty}} + \cancel{kt^2e^{ty}} + 5te^{ty}$$

$$k\cancel{te^{ty}} = 5\cancel{te^{ty}}$$

$$k = 5 \;//$$