1. The Fourier transform of a function $g(t)$ with an arbitrary time shift, τ, is:

$$\int_{-\infty}^{\infty} g(t-\tau)\, e^{-2\pi i f t}\, dt$$

The most straightforward way to proceed is by making the substitution:

$$t_0 = t + \tau$$

$$\Rightarrow \frac{dt_0}{dt} = 1$$

Then the integral above becomes:

$$\int_{-\infty}^{\infty} g(t_0)\, e^{-2\pi i f [t_0 - \tau]}\, dt_0$$

$$= e^{2\pi i f \tau} \underbrace{\int_{-\infty}^{\infty} g(t_0)\, e^{-2\pi i f t_0}\, dt_0}$$

This is the Fourier transform of $g(t)$!

So, we can reduce this expression to:

$$e^{2\pi i f \tau} \hat{f}(g(t))$$

The PSD of this expression is:

$$\left| e^{2\pi i f \tau} \hat{f}(g(t)) \right|^2$$

$$= \underbrace{\left| e^{2\pi i f \tau} \right|^2} \left| \hat{f}(g(t)) \right|^2$$

This equals 1! One way to prove this fact to yourself is by remembering that $e^{2\pi i f \tau} = \cos(2\pi f \tau)$

$+ i\sin(2\pi f\tau)$, and the squared magnitude of this is $\cos^2(2\pi f\tau) + \sin^2(2\pi f\tau) = 1$.

So, we have shown that:

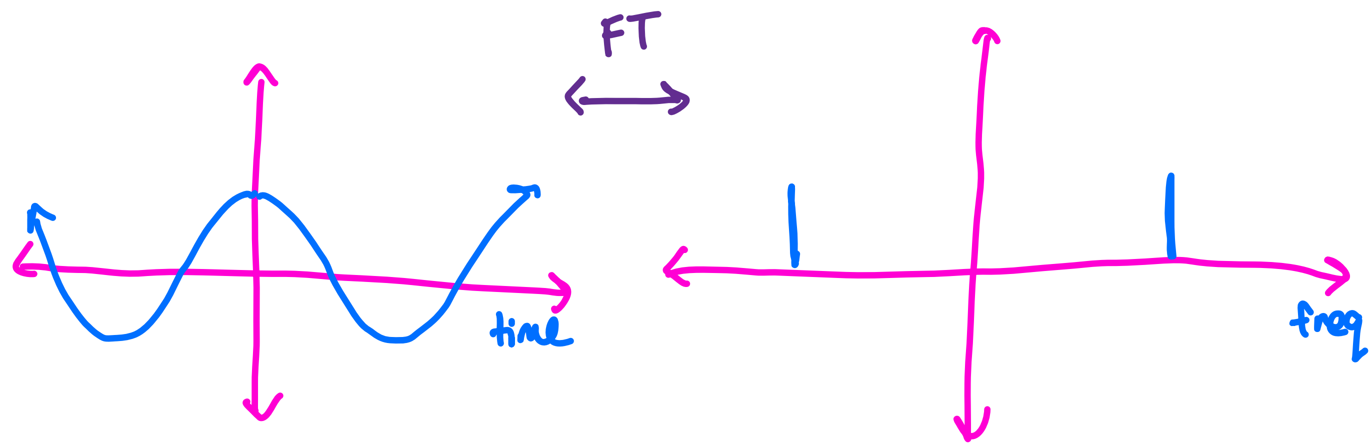
$$\left|\hat{f}(g(t))\right|^2 = \left|\hat{f}(g(t-\tau))\right|^2,$$

or in words that the PSD of an arbitrary function is invariant to time shifts.

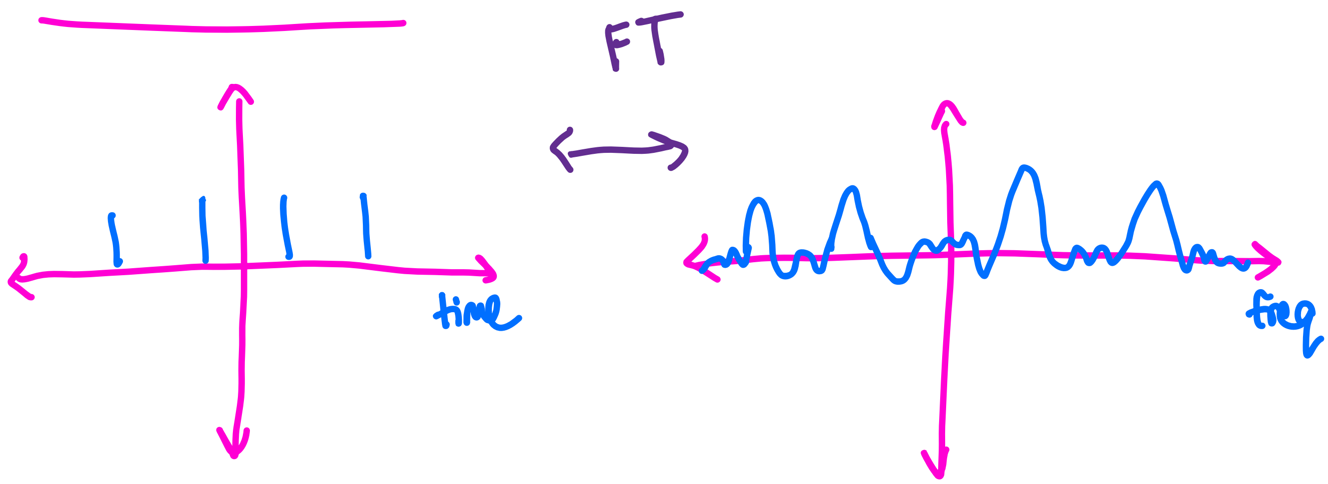
This is important for time series analysis because it shows that the PSD, as an operation, can pick out the periodic component(s) of a signal regardless of its phase. In the eyes of the PSD, cos(t) and sin(t) are the same.

4.

signal:



window function:



Note that the window function I drew above is the pointwise product of a top hat function and an infinite series of delta functions, so in Fourier space, it's the convolution of those two (that's how I was thinking about this problem).

observed signal: