Problem 1

(1) $p(x,y) = p(y|x)p(y)$

$$= \frac{1}{\sqrt{2\pi}^m \cdot |\Sigma|^{\frac12}} \exp\left(-\frac12 (x-\mu)^T \Sigma^{-1}(x-\mu)\right) \cdot \frac{1}{\sqrt{2\pi}^D \cdot |S|^{\frac12}} \exp\left(-\frac12 (y-(Ax+b))^T S^{-1}(y-(Ax+b))\right)$$

$$= \frac{1}{\sqrt{2\pi}^m |\Sigma|^{\frac12}} \cdot \frac{1}{\sqrt{2\pi}^D |S|^{\frac12}} \exp\left(-\frac12\{(x-\mu)^T\Sigma^{-1}(x-\mu) + (y-(Ax+b))^T S^{-1}(y-(Ax+b))\}\right)$$

$$= C\cdot \exp\left(-\frac12\{x^T\Sigma^{-1}x - 2\mu^T\cancel{\Sigma^{-1}\mu} + \cancel{\mu^T\Sigma^{-1}\mu} + y^TS^{-1}y - 2(Ax+\cancel{b})^TS^{-1}y + (Ax+b)^TS^{-1}(Ax+b)\}\right.$$

$$= C\cdot \exp\left(-\frac12\{x^T(\Sigma^{-1}+A^TS^{-1}A)x + y^TS^{-1}y - 2x^TA^TS^{-1}y\}\right)$$ ← Terms containing x, y

↳ $\begin{bmatrix}x\\y\end{bmatrix} \sim N(\mu[\ ], \Sigma[\ ])$ 형태로 만든다

$$S_{new}^{-1} = \begin{bmatrix}\overset{M}{\Sigma^{-1}+A^TS^{-1}A} & -A^TS^{-1}\\ -S^{-1}A & S^{-1}\end{bmatrix} = \begin{bmatrix}\Sigma & \Sigma A^T\\ A\Sigma & A\Sigma A^T+S\end{bmatrix}, \quad \mu_{new} = \begin{bmatrix}\mu\\ A\mu+b\end{bmatrix}$$

$$\Rightarrow p(x,y) \sim N\left(\begin{bmatrix}\mu\\ A\mu+b\end{bmatrix}, \begin{bmatrix}\Sigma & \Sigma A^T\\ A\Sigma & A\Sigma A^T+S\end{bmatrix}\right)$$

$x^Tax - 2x^Tb$
$= (x-a^{-1}b)^Ta(x-a^{-1}b) + C$

(2) $p(y) = \int p(x,y)\,dx$

$$= \int C\cdot\exp\left(-\frac12\{x^T\underbrace{(\Sigma^{-1}+A^TS^{-1}A)}_{a}x + y^TS^{-1}y - 2x^T\underbrace{A^TS^{-1}y}_{b}\}\right)dx$$

$$= C'\cdot\exp(y^TS^{-1}y)\cdot\int\exp\left[-\frac12\{(x-(\Sigma^{-1}+A^TS^{-1}A)A^TS^{-1}y)\}^T(\Sigma^{-1}+A^TS^{-1}A)^{-1}(x-(\Sigma^{-1}+A^TS^{-1}A)A^TS^{-1}y)\right]dx$$

$$= C''\cdot\exp(y^TS^{-1}y)\cdot\exp\{(A^TS^{-1}y)^T(\Sigma^{-1}+A^TS^{-1}A)^T(\Sigma^{-1}+A^TS^{-1}A)^{-1}(\Sigma^{-1}+A^TS^{-1}A)A^TS^{-1}y)$$

→ $S^{-1} = S^{-1T}$ (symmetric)

$$= C''\cdot\exp\{y^TS^{-1}y + y^TS^{-1}A(\Sigma^{-1}+A^TS^{-1}A)^TA^TS^{-1}y\}$$

$$= C''\cdot\exp\{y^T(\underbrace{S^{-1} - S^{-1}A(\Sigma^{-1}+A^TS^{-1}A)^TA^TS^{-1}}_{}\,)y\}$$

역행렬 → $A\Sigma A^T+S$

$$p(y) \sim N(A\mu+b,\ A\Sigma A^T+S)$$

3) $Y=X+Z \Rightarrow Z=Y-X$, $Z\sim N(z;\mu_z,\Sigma_z)$ and Consider $p(y|x)$ (fixed x)

$p(y|x) = N(\mu_z+x, \Sigma_z)$, $p(x) = N(\mu_x,\Sigma_x)$

$p(y) = \int N(y;x+\mu_z,\Sigma_z)\cdot N(x;\mu_x,\Sigma_x)\,dx$

$$=\int C\cdot\exp\left(-\frac12(y-x-\mu_z)\Sigma_z^{-1}(y-x-\mu_z) - \frac12(x-\mu_x)^T\Sigma_x^{-1}(x-\mu_x)\right)dx$$

$$=\int\exp\left(-\frac12[x^T\Sigma_z^{-1}x + x^T\Sigma_x^{-1}x - 2x^T\Sigma_z^{-1}(y-\mu_z) - 2x^T\Sigma_x^{-1}\mu_x] + C\right)$$

$$=\int\exp\left(-\frac12\{x^T(\Sigma_z^{-1}+\Sigma_x^{-1})x - 2x^T\Sigma_z^{-1}(y-\mu_z) + \Sigma_x^{-1}\mu_x\} + C\right)$$

$$=\int\exp\left(-\frac12(x-(\Sigma_z^{-1}+\Sigma_x^{-1})^{-1}(\Sigma_z^{-1}(y-\mu_z)+\Sigma_x^{-1}\mu_x))(\Sigma_z^{-1}+\Sigma_x^{-1})(x-(\Sigma_z^{-1}+\Sigma_x^{-1})^{-1}(\Sigma_z^{-1}(y-\mu_z)+\Sigma_x^{-1}\mu_x))\right)$$

$p(x+z) = N(x+z;\mu_x+\mu_z,\Sigma_x+\Sigma_z)$

$p(y|x+z) = N(y;A(x+z)+b, S)$

$$p((x+z),y) = N\left(\begin{bmatrix}\mu_x+\mu_z\\A(\mu_x+\mu_z)+b\end{bmatrix}, \begin{bmatrix}\Sigma_x+\Sigma_z & (\Sigma_x+\Sigma_z)A^T\\ A(\Sigma_x+\Sigma_z) & A(\Sigma_x+\Sigma_z)A^T+S\end{bmatrix}\right)$$

$$p(y) = N(A(\mu_x+\mu_z)+b,\ A(\Sigma_x+\Sigma_z)A^T+S)$$

Problem 2

(1) $p(\tau|X) = p(X|\tau)\cdot p(\tau)$

$p(X,\tau)$ with fixed $\mu \Rightarrow$

$$p(\tau|X) = \prod_{i=1}^{N}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{\tau}{2}(x_i-\mu)^2\right)\cdot\frac{1}{\Gamma(a)}b^a\tau^{a-1}\exp(-b\tau)$$

$$\propto \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\exp\left(-\frac{\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)\cdot\frac{1}{\Gamma(a)}b^a\tau^{a-1}\exp(-b\tau)$$

$$\Rightarrow (2\pi)^{\frac{N}{2}}\cdot\tau^{\frac{N}{2}+a-1}\cdot\exp\left(-\tau\left(\frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2+b\right)\right) + \sim$$

$$\hat{a}_N = \frac{N}{2}+a,\quad b_N = b+\frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2$$

(2)

$$p(X|\mu,\tau) = \prod_{i=1}^{N}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{\tau}{2}(x_i-\mu)^2\right) = \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\exp\left(-\frac{\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

$$p(\mu,\tau) = p(\mu|\tau)\cdot p(\tau) = N(\mu|\mu_0,(\beta\tau)^{-1})\cdot Gam(\tau|a,b)$$

$$= \sqrt{\frac{\beta\tau}{2\pi}}\exp\left(-\frac{\beta\tau}{2}(\mu-\mu_0)^2\right)\cdot\frac{1}{\Gamma(a)}b^a\tau^{a-1}\exp(-b\tau)$$

$$p(\mu,\tau|X) = p(X|\mu,\tau)\cdot p(\mu,\tau)$$

$$= \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\cdot\exp\left(-\frac{\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)\cdot\left(\frac{\beta\tau}{2\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{\beta\tau}{2}(\mu-\mu_0)^2\right)\cdot\frac{1}{\Gamma(a)}b^a\tau^{a-1}\exp(-b\tau)$$

$$= \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\cdot\left(\frac{\beta\tau}{2\pi}\right)^{\frac{1}{2}}\cdot\exp\left(-\frac{\tau}{2}\left(\sum_{i=1}^{N}(x_i-\mu)^2+\beta(\mu-\mu_0)^2\right)\right)\cdot\frac{1}{\Gamma(a)}b^a\tau^{a-1}\cdot\exp(-b\tau)$$

μ에 대한 완전제곱

$\sum_{i=1}^{N}x_i^2 - 2\sum_{i=1}^{N}x_i\mu + N\mu^2 + \beta\mu^2 - 2\beta\mu\mu_0 + \beta\mu_0^2$

$\Rightarrow (N+\beta)\mu^2 - 2\left(\sum_{i=1}^{N}x_i+\beta\mu_0\right)\mu + C$

$$= \left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\cdot\left(\frac{\beta\tau}{2\pi}\right)^{\frac{1}{2}}\cdot\exp\left(-\frac{\tau}{2}\left((N+\beta)\mu^2 - 2\left(\sum_{i=1}^{N}x_i+\beta\mu_0\right)\mu + c\right)\right)$$

$$\propto \exp\left(-\frac{\tau}{2}(N+\beta)\left\{\mu^2 - \frac{2\left(\sum_{i=1}^{N}x_i+\beta\mu_0\right)\mu}{(N+\beta)} + c'\right\}\right)$$

$$\Rightarrow \mu_N = \frac{\sum_{i=1}^{N}x_i+\beta\mu_0}{N+\beta},\quad \beta_N = N+\beta$$

Remaining term

$$\exp\left\{-\frac{\tau}{2}\left(-\frac{\left(\sum_{i=1}^{N}x_i+\beta\mu_0\right)^2}{N+\beta} + \sum_{i=1}^{N}x_i^2 + \beta\mu_0^2\right) - b\tau\right\}$$

$$= \sum_{i=1}^{N}x_i^2 + \beta\mu_0^2 - \frac{1}{N+\beta}\left(N^2\bar{x}^2 + 2N\beta\mu_0\bar{x} + \beta^2\mu_0^2\right)$$

$$\Rightarrow \exp\left[-\tau\left[b+\frac{1}{2}\left(\sum_{i=1}^{N}(x_i-\bar{x})^2 + \frac{N\beta}{N+\beta}(\bar{x}-\mu_0)^2\right)\right]\right]$$

$$\left(\frac{\tau}{2\pi}\right)^{\frac{N}{2}}\cdot\tau^{a-1} = \tau^{\frac{N}{2}+a-1} \Rightarrow \hat{a} = \frac{N}{2}+a$$

$$b_N = b+\frac{1}{2}\left(\sum_{i=1}^{N}(x_i-\bar{x})^2 + \frac{N\beta}{N+\beta}(\bar{x}-\mu_0)^2\right)$$

Problem 3

(1) $P(\theta|D) = \dfrac{P(D|\theta)\cdot P(\theta)}{P(D)} \propto P(D|\theta)\cdot P(\theta)$

$$P(D|\theta)\cdot P(\theta) = \theta^m (1-\theta)^{N-m} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{m+\alpha-1}(1-\theta)^{N-m+\beta-1}$$

$\Rightarrow$ Beta distribution

(2) $\hat{\alpha} = \alpha + m$, $\hat{\beta} = \beta + N - m$

(3) $$P(\theta|D) = \frac{\Gamma(14)}{\Gamma(9)\Gamma(5)} \cdot \theta^{9}(1-\theta)^{5}$$

$$= \mathrm{Beta}(\theta|9,5)$$

problem 4

$p(x|\mathbf{X}) = \int p(x|\mu)p(\mu|\mathbf{X})\,d\mu$

$p(x|\mu)\cdot p(\mu|\mathbf{X})$

$\propto \exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) - \frac{1}{2}(\mu-\mu_N)^T\Sigma_N^{-1}(\mu-\mu_N))$

$= \exp(-\frac{1}{2}(\mu^T\Sigma^{-1}\mu - 2\mu^T\Sigma^{-1}x + x^T\Sigma^{-1}x + \mu^T\Sigma_N^{-1}\mu - 2\mu^T\Sigma_N^{-1}\mu_N + \mu_N^T\Sigma_N^{-1}\mu_N)$

$= \exp(-\frac{1}{2}(\mu^T(\Sigma^{-1}+\Sigma_N^{-1})\mu - 2(\Sigma^{-1}x+\Sigma_N^{-1}\mu_N)\mu + x^T\Sigma^{-1}x + \mu_N^T\Sigma_N^{-1}\mu_N))$

$= \exp(-\frac{1}{2}(\Sigma^{-1}+\Sigma_N^{-1})\{\mu^T\mu - 2(\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-1}x+\Sigma_N^{-1}\mu_N)\mu) + x^T\Sigma^{-1}x + \mu_N^T\Sigma_N^{-1}\mu_N))$

$= \exp(-\frac{1}{2}(\Sigma^{-1}+\Sigma_N^{-1})\{\mu-(\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-1}x+\Sigma_N^{-1}\mu_N)\}^2 - (\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-1}x+\Sigma_N^{-1}\mu_N)^2$

$+ x^T\Sigma^{-1}x + \mu_N^T\Sigma_N^{-1}\mu_N))$

$\rightarrow$ marginalizing μ

$\exp(-\frac{1}{2}\{x^T\Sigma^{-1}x - (\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-1}x+\Sigma_N^{-1}\mu_N)^2\})$

$= \exp(-\frac{1}{2}\{x^T\Sigma^{-1}x - (\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-2}x^2 + 2\Sigma^{-1}x\Sigma_N^{-1}\mu_N + \Sigma_N^{-2}\mu_N^2)\})$

$= \exp(-\frac{1}{2}\Sigma^{-1}\{x^Tx - (\Sigma^{-1}+\Sigma_N^{-1})^{-1}(\Sigma^{-1}x^2 + 2x\Sigma_N^{-1}\mu_N + C)\})$

$= \exp(-\frac{1}{2}\Sigma^{-1}\{(I-(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\Sigma^{-1})x^2 - 2\Sigma_N^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\mu_N x)$

$= \exp(-\frac{1}{2}(\Sigma^{-1}(I-(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\Sigma^{-1})x^2 - 2\Sigma_N^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\mu_N x)$

$= \exp(-\frac{1}{2}(\underbrace{(\Sigma^{-1}-\Sigma^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\Sigma^{-1})}_{\text{Woodbury}}x^2 - 2\Sigma_N^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\mu_N x)$

$= \exp(-\frac{1}{2}(\Sigma^{-1}-\Sigma^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\Sigma^{-1})\{x - A^{-1}(\Sigma_N^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}\mu_N)\}^2$

$A^{-1} = \Sigma - \Sigma\cdot(-\Sigma^{-1})(\Sigma^{-1}+\Sigma_N^{-1}+\Sigma^{-1}\Sigma\cdot(-\Sigma^{-1}))^{-1}\Sigma$

$= \Sigma + \Sigma_N$ $\qquad (A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$

$= \exp(-\frac{1}{2}(\Sigma+\Sigma_N)^{-1}\{x - (\Sigma+\Sigma_N)\underbrace{(\Sigma_N^{-1}(\Sigma^{-1}+\Sigma_N^{-1})^{-1}}_{\Sigma(\Sigma+\Sigma_N)^{-1}\Sigma_N}\mu_N)\}^2$

$= \exp(-\frac{1}{2}(\Sigma+\Sigma_N)^{-1}\{x - \mu_N\}^2)$

Problem 5

(1) $\log P(w \mid D, \sigma^2, \alpha) = \log\{P(Y \mid w, x, \sigma^2) \cdot P(w \mid \alpha)\}$

$$\propto \; -\frac{\sum_{i=1}^{N}(Y_i - wX_i)^2}{2\sigma^2} - \frac{\alpha}{2}w^2$$

(2)
$$-\frac{\sum_{i=1}^{N}(Y_i - wX_i)^2}{2\sigma^2} - \frac{\alpha}{2}w^2$$

$$= -\frac{\sum_{i=1}^{N} Y_i}{2\sigma^2} + \frac{2w\sum_{i=1}^{N}(X_i Y_i)}{2\sigma^2} - \frac{w^2\sum_{i=1}^{N}(X_i^2)}{2\sigma^2} - \frac{\alpha}{2}w^2$$

$$= \left(-\frac{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}{2\sigma^2}\right)w^2 + \frac{2\sum_{i=1}^{N}(Y_i X_i)}{2\sigma^2}w - \frac{\sum_{i=1}^{N} Y_i}{2\sigma^2}$$

$$= \left(-\frac{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}{2\sigma^2}\right)\left\{w^2 - \frac{2\sum_{i=1}^{N}(Y_i X_i)}{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}w + c\right) - \frac{NY}{2\sigma^2}$$

$$= -\frac{1}{2}\left(\frac{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}{\sigma^2}\right)\left(w - \frac{Y\sum_{i=1}^{N}(X_i)}{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}\right)^2$$

$$\exp\left(-\frac{1}{2}\left(\frac{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}{\sigma^2}\right)\left(w - \frac{\sum_{i=1}^{N}(Y_i X_i)}{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}\right)^2\right) \rightarrow \text{Gaussian}$$

(3) $\sigma_N^2 = \dfrac{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}{\sigma^2}$, $\mu_N = \dfrac{\sum_{i=1}^{N}(X_i Y_i)}{\sum_{i=1}^{N}(X_i^2) + \alpha\sigma^2}$

α: Prior를 얼마나 신뢰할지에 대한 가중치

σ_N^2: 데이터의 정보 + 사전 정보에 대한 총합

μ_N: W추정 × 이전 신뢰도

(4) $\mu_N = \dfrac{14}{10 + 1} = \dfrac{14}{11}$

Problem 6

(1) $k(X,Y) = (x_1y_1 + x_2y_2)^3 = (x_1y_1)^3 + 3(x_1y_1)^2(x_2y_2) + 3(x_1y_1)(x_2y_2)^2 + (x_2y_2)^3$

(2) $k(X,Y) = 1 + X^TAY \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$= 1 + X^TLL^TY$

$\Rightarrow \quad \phi(x) = \begin{pmatrix} 1 \\ L^TX \end{pmatrix} \qquad L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \Rightarrow LL^T = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab+ac \\ ab & b^2+c^2 \end{pmatrix}$

$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$a = \sqrt{a_{11}}, \quad b = \frac{a_{21}}{\sqrt{a_{11}}} \qquad c = \sqrt{a_{22} - \frac{a_{21}^2}{a_{11}}}$

$$\phi(x) = \begin{pmatrix} 1 \\ \begin{pmatrix} \sqrt{a_{11}} & \frac{a_{21}}{\sqrt{a_{11}}} \\ 0 & \sqrt{a_{22} - \frac{a_{21}^2}{a_{11}}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{a_{11}}\, x_1 + \frac{a_{21}}{\sqrt{a_{11}}} x_2 \\ \sqrt{a_{22} - \frac{a_{21}^2}{a_{11}}}\, x_2 \end{pmatrix} \qquad F = 3$$

(3) $k(X,Y) = k_1(X,Y) + k_2(X,Y) = \psi(X)^T\psi(Y) + \varepsilon(X)^T\varepsilon(Y)$

$$\phi(X) = \begin{pmatrix} \psi(x) \\ \varepsilon(x) \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_{F_1}(x) \\ \varepsilon_1(x) \\ \vdots \\ \varepsilon_{F_2}(x) \end{pmatrix} \qquad F = F_1 + F_2$$

problem 17

$$x_t = a x_{t-1} + \varepsilon_{t-1} = a(a x_{t-2} + \varepsilon_{t-2}) + \varepsilon_{t-1}$$

$$= a^t x_0 + \sum_{i=1}^{t-1} a^{t-1-i} \varepsilon_i$$

$$M(t) = E[x_t] = E[a^t x_0 + \sum_{i=1}^{t-1} a^{t-1-i} \varepsilon_i] = a^t + 0 = a^t$$

$$K(t_1, t_2) = Cov[x_{t_1}, x_{t_2}] = E[x_{t_1} x_{t_2}] - E[x_{t_1}] E[x_{t_2}]$$

$$E[X_{t_1}] E[X_{t_2}] = a^{t_1} \times a^{t_2} = a^{t_1+t_2}$$

$$x_{t_1} = a^{t_1} x_0 + \sum_{i=1}^{t_1-1} a^{t_1-1-i} \varepsilon_i$$

$$x_{t_2} = a^{t_2} x_0 + \sum_{j=1}^{t_2-1} a^{t_2-1-j} \varepsilon_j$$

$$x_{t_1} x_{t_2} = a^{t_1+t_2} \cdot x_0^2 + \cancelto{0}{a^{t_1} x_0 \sum_{j=1}^{t_2-1} a^{t_2-1-j} \varepsilon_j} + \cancelto{0}{a^{t_2} x_0 \sum_{i=1}^{t_1-1} a^{t_1-1-i} \varepsilon_i} +$$

$$\sum_{i=1}^{t_1-1} \sum_{j=1}^{t_2-1} a^{t_2-1-j} \varepsilon_j \cdot a^{t_1-1-i} \varepsilon_i$$

$$\sum_{i=1}^{t_1-1} \sum_{j=1}^{t_2-1} a^{t_2-1-j} \varepsilon_j \cdot a^{t_1-1-i} \varepsilon_i$$

$$= \sum_{k=0}^{\min(t_1,t_2)-1} a^{t_1+t_2-2-2k} \varepsilon_k^2$$

$$E[X_{t_1} X_{t_2}] = a^{t_1+t_2} E[x_0^2] + \sum_{k=0}^{\min(t_1,t_2)-1} a^{t_1+t_2-2-2k} E[\varepsilon_k^2]$$

$$\Rightarrow K(t_1, t_2) = a^{t_1+t_2} (x_0^2 + 1) + \sigma^2 \sum_{k=0}^{\min(t_1,t_2)-1} a^{t_1+t_2-2-2k}$$