Getting Started

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Contents

- Sorting problem
- 2 sorting algorithms
  - Insertion sort
  - Merge sort

Sorting problem

keys

- **Input**

  each number: item, key

  - A sequence of n number $<\underline{a_1, a_2, . . ., a_n}>$.

  reordering (정렬)

- **Output**
  - A permutation (reordering) $<a'_1, a'_2, . . ., a'_n>$ of the input sequence such that $\underline{a'_1 \le a'_2 \le \ldots \le a'_n}$.

- Ex>
  - Input: < 5, 2, 4, 6, 1, 3>
  - Output: < 1, 2, 3, 4, 5, 6>

Insertion sort

- **Insertion sort**
  - Description
  - Correctness
  - Performance

Description

- What is insertion sort?
  - A sorting algorithm using insertion.

- What is insertion?
  - Given a key and a sorted list of keys, insert the key into the sorted list preserving the sorted order.
  - ex> Insert 3 into <1, 2, 4, 5, 6>

Description

- Insertion sort uses insertion incrementally.
  - Let $A[1..n]$ denote the array storing keys.
  - Insert $A[2]$ into $A[1]$.
  - Insert $A[3]$ into $A[1..2]$.
  - Insert $A[4]$ into $A[1..3]$.

    .

    .

    .
  - Insert $A[n]$ into $A[1..n\text{-}1]$.

Description: example

A[1] A[2]
- 5 2 4 6 1 3
- 5 2 4 6 1 3

A[1-2] A[3]
- 2 5 4 6 1 3

A[1-3] A[4]
- 2 4 5 6 1 3

A[1-4] A[5]
- 2 4 5 6 1 3

A[1-5] A[6]
- 1 2 4 5 6 3
- 1 2 3 4 5 6

Description: pseudo code

INSERTION-SORT(*A*)

```
for j = 2 to A.length
        key = A[j]
        // Insert A[j] into the sorted        이미 정렬
                sequence A[1..j - 1].
        i = j - 1
        while(i > 0 and A[i] > key)
                A[i + 1] = A[i]  → 한 칸씩 뒤로 미루기
                i = i - 1
        A[i + 1] = key
```

Pseudocode conventions are given in p. 19 - 20 of the textbook.

1. identation : block Structure 사용
2. Looping Constmarmcts : 반복문
   repeat-until
   : if else

n: A.length

n-1 iterations of insertion.

Insert $A[j]$ into $A[1..j - 1]$.

Find a place to put $A[j]$.

Put $A[j]$.

Insertion sort

- Insertion sort
  - Description
  - Correctness
  - Performance
    - Running time
    - Space consumption

Running time

- How to analyze the running time of an algorithm?
  - Consider running the algorithm on a specific machine and measure the running time.
    - We cannot compare the running time of an algorithm on a machine with the running time of another algorithm on another machine.
    - So, we have to measure the running time of every algorithm on a specific machine, which is impossible.
  - Hence, we count the number of instructions used by the algorithm.

알고리즘이 수행하는 구체적인 작업이나 단계

Instructions

- Arithmetic
  - Add, Subtract, Multiply, Divide, remainder, floor, ceiling
- Data movement

  값할당, 메모리에서 값 읽기

  - Load, store, copy   리스트 등에서 데이터 복사
- Control
  - Conditional branch   조건
  - Unconditional branch
  - Subroutine call and return

    [illegible]함수 등..

Running time

- The running time of an algorithm grows with the input size, which is the number of items in the input.
- For example, sorting 10 keys is faster than sorting 100 keys.
- So the running time of an algorithm is described as a function of input size n, for example, $T(n)$.

Running time of insertion sort

running time of each line

n : A.length

INSERTION-SORT(*A*)	*cost*	*times*	
1 **for** $j = 2$ **to** *A.length*	c_1	n	$n-1 \underline{+1}$ 조건체크
2 $key = A[j]$	c_2	$n - 1$	
3 // Insert $A[j]$ into the sorted sequence $A[1..j - 1]$.	0	$n - 1$	
4 $i = j - 1$	c_4	$n - 1$	
5 **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$	t_i : j가 n일 때 each line's running time
6 $A[i + 1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$	
7 $i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$	
8 $A[i + 1] = key$	c_8	$n - 1$	

- ***T*(*n*)**: The sum of product of *cost* and *times* of each line.

Running time of insertion sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j-1) + c_7\sum_{j=2}^{n}(t_j-1) + c_8(n-1)$$

cost	*times*
c_1	n
c_2	$n - 1$
c_4	$n - 1$
c_5	$\sum_{j=2}^{n} t_j$
c_6	$\sum_{j=2}^{n} (t_j - 1)$
c_7	$\sum_{j=2}^{n} (t_j - 1)$
c_8	$n - 1$

- ***T*(*n*)**: The sum of product of *cost* and *times* of each line.

Running time of insertion sort

- t_i: The number of times the **while** loop test is executed for j.

- Note that **for**, **while** loop test is executed one time more than the loop body.

Running time of insertion sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1)$$

$$+ c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- Although the size of the input is the same, we have
  - best case → Input size가 같다는 가정!
  - average case, and
  - worst case.

Running time of insertion sort

- Best case
  - If $A[1..n]$ is already sorted, $t_j = 1$ for $j = 2, 3, \ldots, n$.

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \underbrace{\cancel{\sum_{j=2}^{n}(t_j - 1)}}_{0}$$
$$+ c_7 \underbrace{\cancel{\sum_{j=2}^{n}(t_j - 1)}}_{0} + c_8(n-1)$$
$$= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$
$$= \underbrace{(c_1 + c_2 + c_4 + c_5 + c_8)}_{a} n - \underbrace{(c_2 + c_4 + c_5 + c_8)}_{b}$$

- This running time can be expressed as $an+b$ for *constants a* and b; it is thus a ***linear function*** of n.

Running time of insertion sort

- Worst case
  - → 모든 item이 refresh되어야 함
    → $t_j = j$
  - If $A[1..n]$ is sorted in reverse order, $t_j = j$ for $j = 2, 3, \ldots, n$.

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \quad \text{and} \quad \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right)$$

$$+ c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

$$= \underbrace{\left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)}_{a} n^2 + \underbrace{\left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)}_{b} n - \underbrace{(c_2 + c_4 + c_5 + c_8)}_{c}$$

- This running time can be expressed as $an^2 + bn + c$ for constants a, b, and c; it is thus a ***quadratic function*** of n.

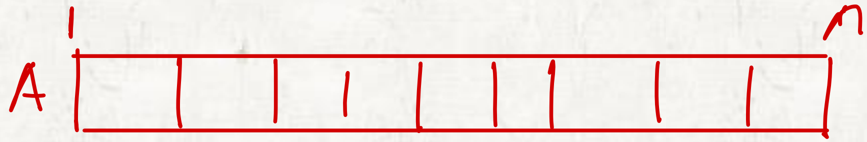
Running time of insertion sort

- **Only the degree of leading term is important.**
  - Because we are only interested in the ***rate of growth*** or ***order of growth.***
  - For example, a quadratic function grows faster than any linear function.

- The degree of leading term is expressed as Θ–notation.
  - The worst-case running time of insertion sort is $\Theta(n^2)$.

Space consumption of insertion sort

- $\Theta(n)$ space.



배열 저장하기 위한 n

- Moreover, the input numbers are ***sorted in place.***
  - $n + c$ space for some constant c.

Self-study on Insertion Sort

- **Exercise 2.1-1**
- **Exercise 2.1-2**

Content

- Sorting problem

$\log n = \log_2 n$

- Sorting algorithms

worst case | best case | sorted in place

- Insertion sort - $\Theta(n^2)$. $\Theta(n)$ (n+c) space
- Merge sort - $\Theta(n\lg n)$. $\Theta(n\log n)$ X (additional space)

Heap sort $\Theta(n\log n)$ $\Theta(n)$ O

Merge

- What is merge sort?
  - A sorting algorithm using merge.

- What is merge?
  - Given two sorted lists of keys, generate a sorted list of the keys in the given sorted lists.
  - <1, 5, 6, 8> < 2, 4, 7, 9> → < 1, 2, 4, 5, 6, 7, 8, 9>

Merge

- Merging example
  - <1, 5, 6, 8> <2, 4, 7, 9> → < 1 >
  - < 5, 6, 8> <2, 4, 7, 9> → < 1, 2 >
  - < 5, 6, 8> < 4, 7, 9> → < 1, 2, 4 >
  - < 5, 6, 8> < 7, 9> → < 1, 2, 4, 5 >
  - < 6, 8> < 7, 9> → < 1, 2, 4, 5, 6 >
  - < 8> < 7, 9> → < 1, 2, 4, 5, 6, 7 >
  - < 8> < 9> → < 1, 2, 4, 5, 6, 7, 8 >
  - < > < 9> → < 1, 2, 4, 5, 6, 7, 8, 9>

Merge

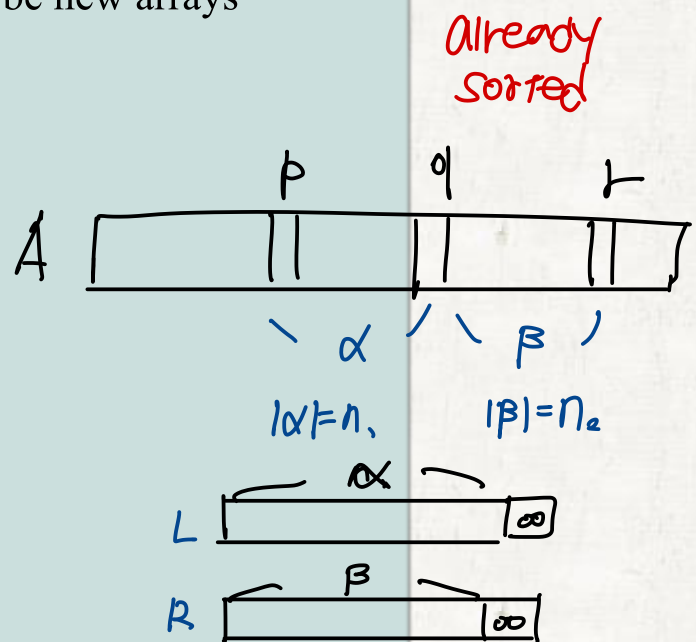
MERGE(*A*, *p*, *q*, *r*)

```
n1 = q − p + 1
n2 = r − q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1  to n1
        L[i]  = A [p + i - 1]
for j = 1  to n2
        R[j]  = A [q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p  to r
        if L[i] ≤ R[j]
            A[k] = L[i]
            i = i  + 1
        else A[k] = R[j]
            j = j  + 1
```

Compare

move here

merge!



$L[n_1+1] = R[n_2+1] = \infty$ 이 있기 때문에 $i \le n_1$, $j \le n_2$ 와 같은 조건 추가할 필요가 없다.

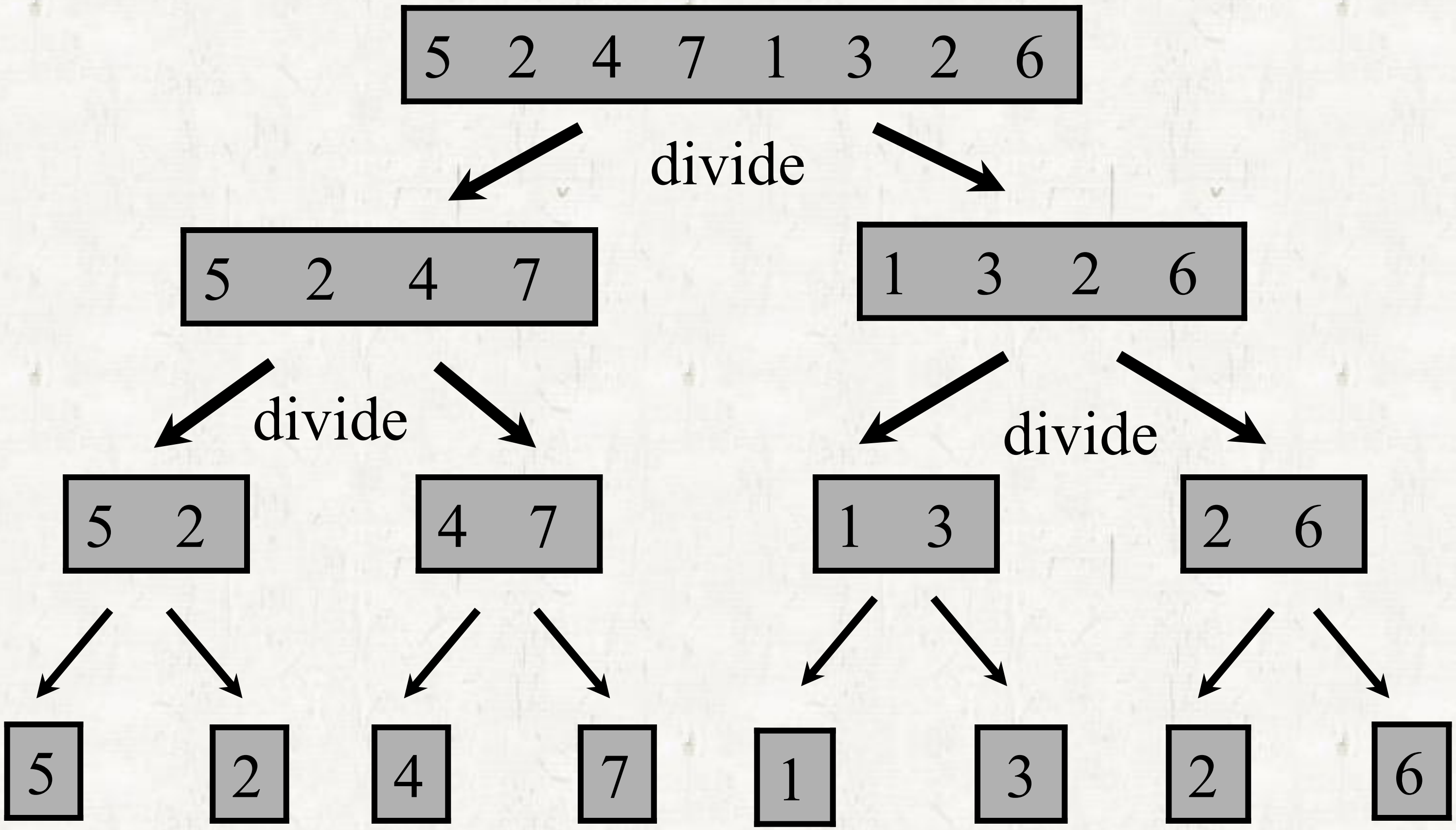
Merge

- Running time of merge
  - Let n_1 and n_2 denote the lengths of two sorted lists.
  - $\Theta(n_1 + n_2)$ time.
    - Main operations: compare and move
    - #comparison ≤ #movement → WHY? Every compare op is followed by a movement op.
    - Obviously, #movement = $n_1 + n_2$
    - So, #comparison ≤ $n_1 + n_2$
      All items copy to & copy back
    - Hence, #comparison + #movement ≤ $2(n_1 + n_2)$
    - which means $\Theta(n_1 + n_2)$.

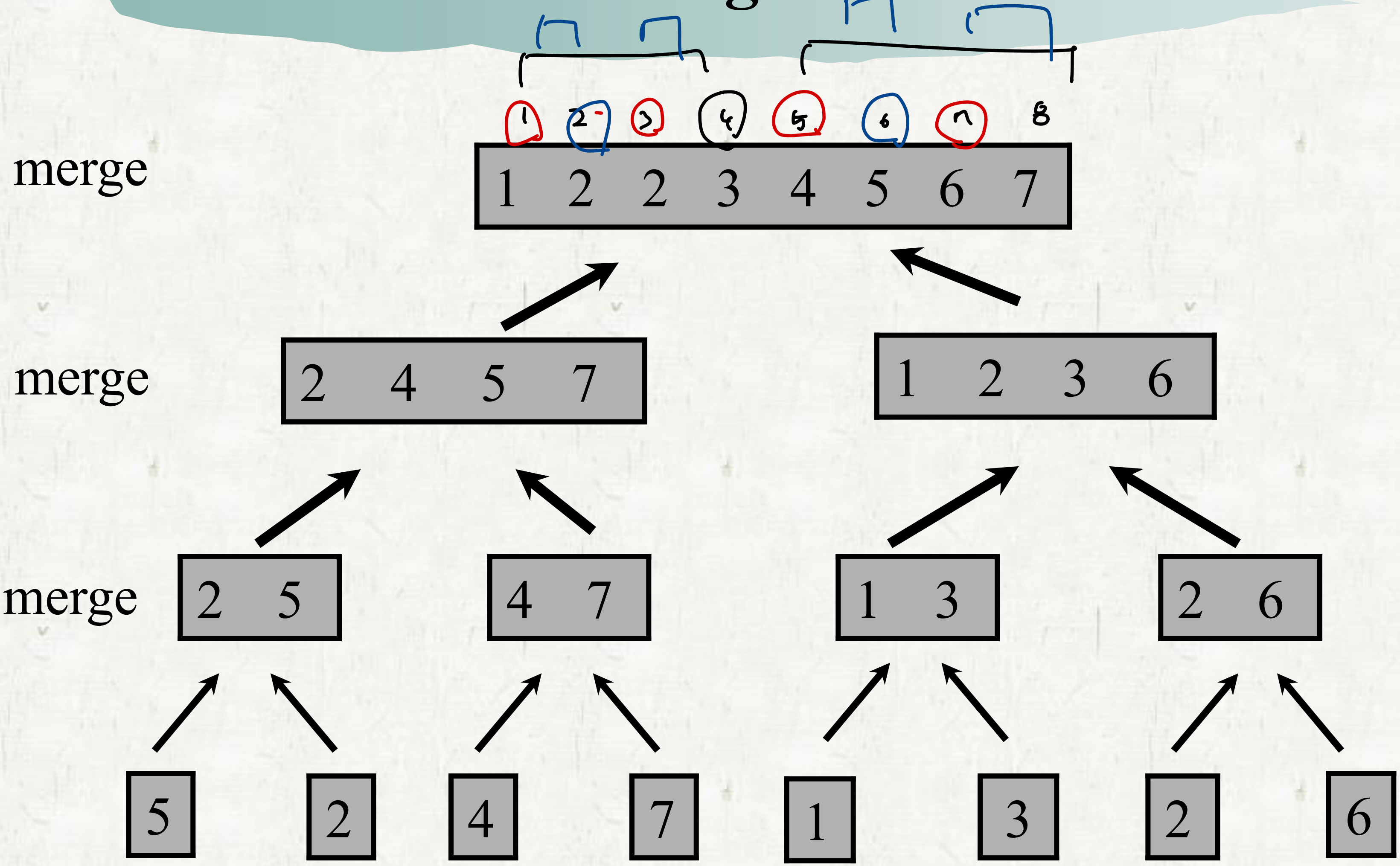
Merge sort

- A divide-and-conquer approach
  - **Divide:** Divide the n keys into two lists of $n/2$ keys.
  - **Conquer:** Sort the two lists recursively using merge sort.
  - **Combine:** Merge the two sorted lists.

Merge sort


5 2 4 7 1 3 2 6
divide
5 2 4 7
1 3 2 6
divide
divide
5 2
4 7
1 3
2 6
5
2
4
7
1
3
2
6

Merge sort



merge
merge
merge
1 2 3 4 5 6 7 8
1 2 2 3 4 5 6 7
2 4 5 7
1 2 3 6
2 5
4 7
1 3
2 6
5
2
4
7
1
3
2
6

Pseudocode

MERGE-SORT(A, p, r)

    if $p < r$

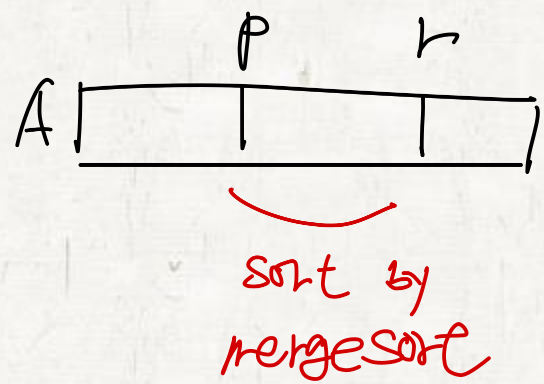
        $q = \lfloor (p + r)/2 \rfloor$ → divide

        MERGE-SORT(A, p, q)

                                                → conquer

        MERGE-SORT$(A, q + 1, r)$

        MERGE(A, p, q, r) → combine



Running time

T(n) → merge sort's running time of merge sort

- **Divide:** $\Theta(1)$
  - The divide step just computes the middle of the subarray, which takes constant time.
- **Conquer:** $2T\,(n/2)$
  - We recursively solve two subproblems, each of size $n/2$.
- **Combine:** $\Theta(n)$
  - We already showed that merging two sorted lists of size $n/2$ takes $\Theta(n)$ time.

무시

$$T(n) = \cancel{\Theta(1)} + 2T\left(\frac{n}{2}\right) + \Theta(n)$$

Running time

- $T(n)$ can be represented as a recurrence.

→ only if $p < r$ 만 동작

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Running time

- where the constant c represents the time required to solve problems of size 1 as well as the time per array element of the divide and combine steps.

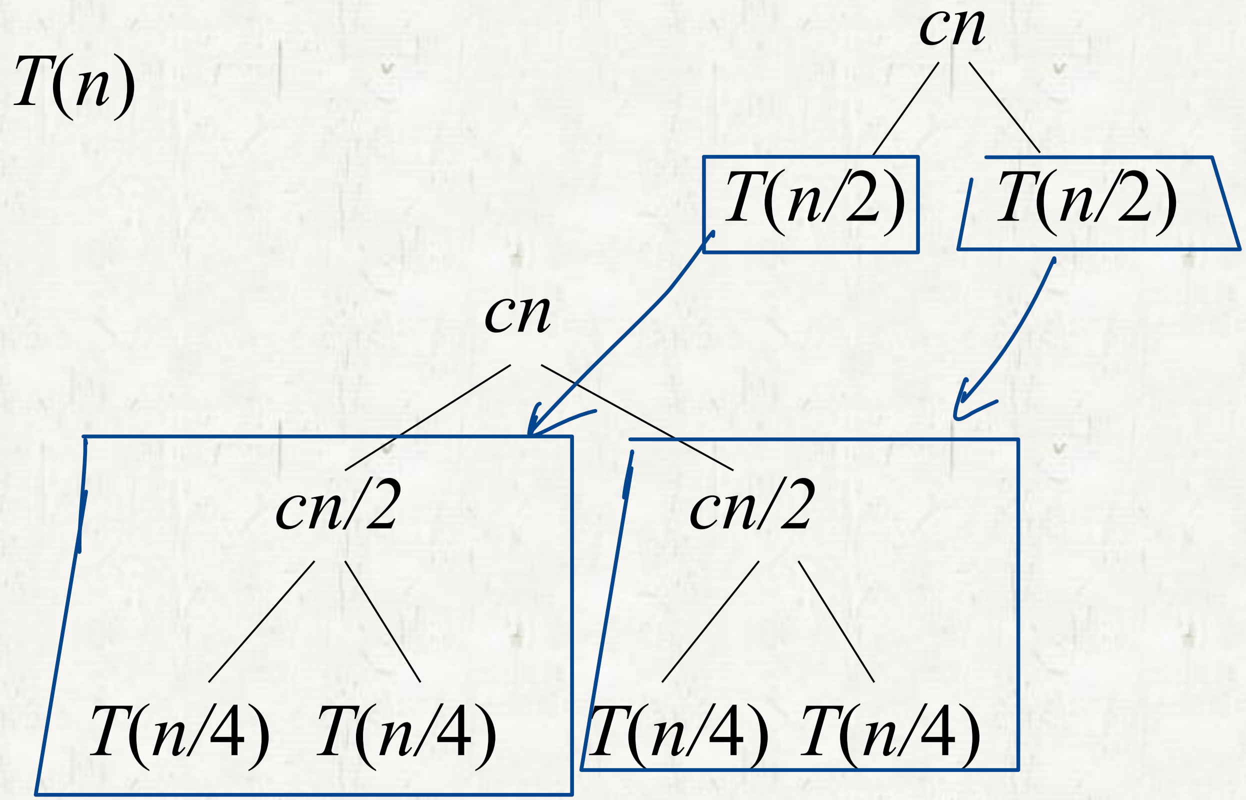
$$T(n) = \begin{cases} \boxed{\Theta(1)} & \text{if } n=1, \\ 2T(n/2) + \Theta(n) & \text{if } n>1 \end{cases}$$

$$\Downarrow$$

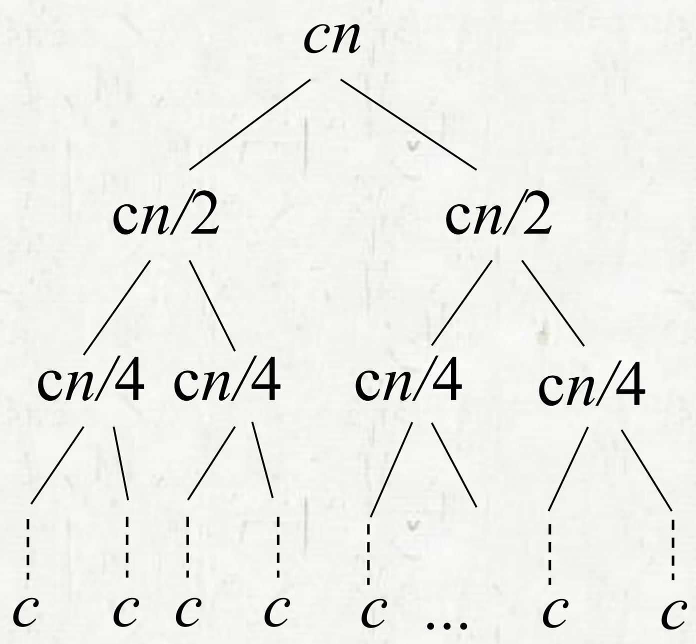
$$T(n) = \begin{cases} c & \text{if } n=1, \\ 2T(n/2) + cn & \text{if } n>1 \end{cases}$$

Recursion tree

$T(n) = 2T(\frac{n}{2}) + cn$



Recursion tree



Recursion tree

merge sort

Worst : $\theta(n\log n)$

best : $\theta(n\log n)$

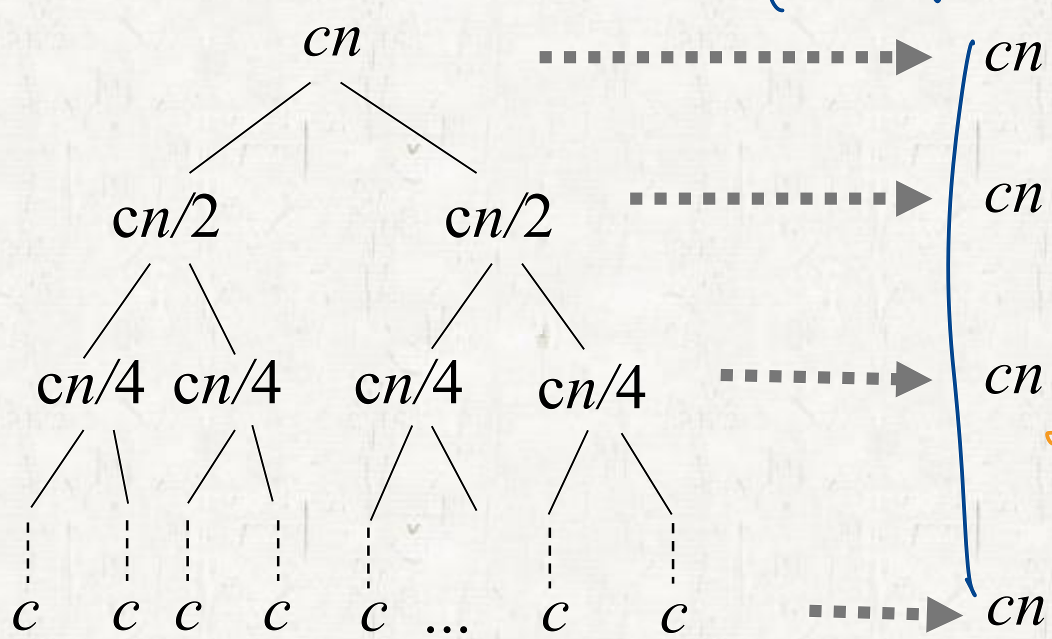
$(\log n+1)$

$cn/2^0$

$cn/2^1$

$cn/2^2$

$c=cn/2^k$



In best case, Insertion sort is better

In worst case, merge sort is better

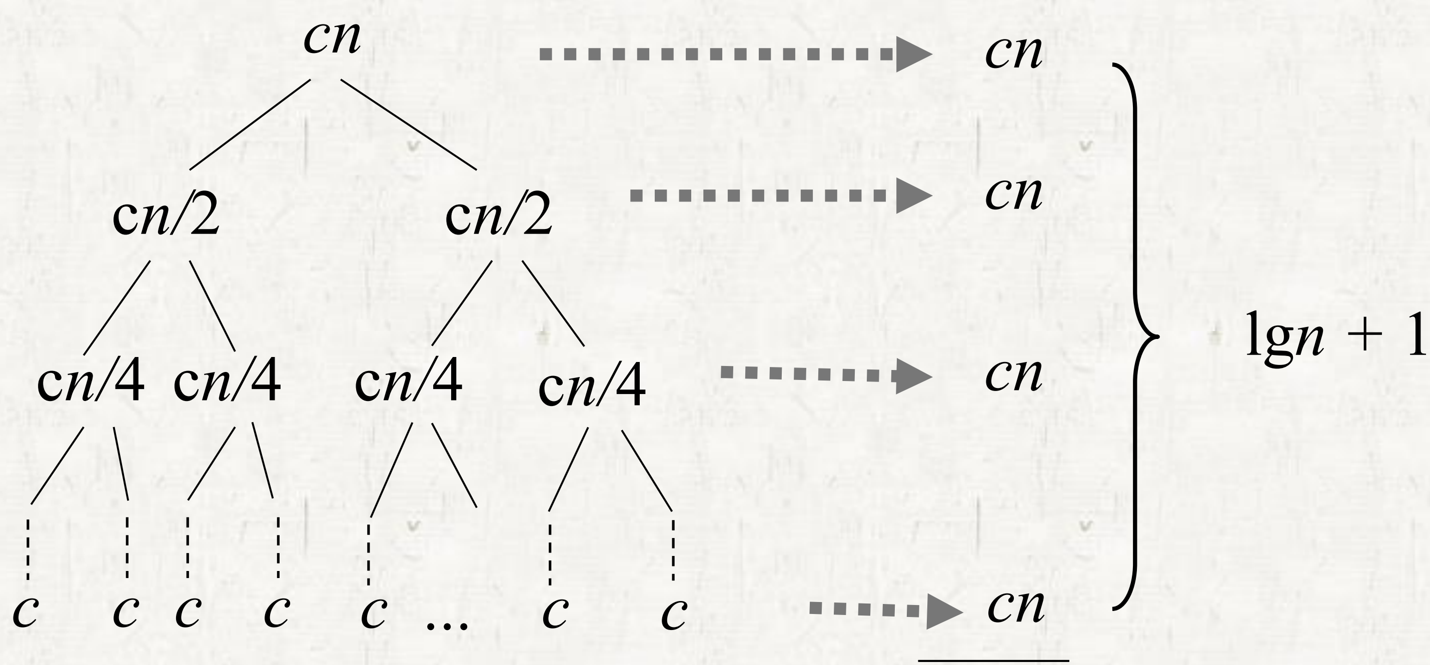
$2^k=n \rightarrow k=\log n$

전체: $cn(\log n+1)$

$= cn\log n + cn$

$= \theta(n\log n)$

Recursion tree



Total : $cn\lg n + cn = \Theta(n\lg n)$

Self-study

- **Merge sort**
  - Exercise 2.3-1
  - Exercise 2.3-2

- **Horner's rule**
  - Problem 2-3 (a) (b)

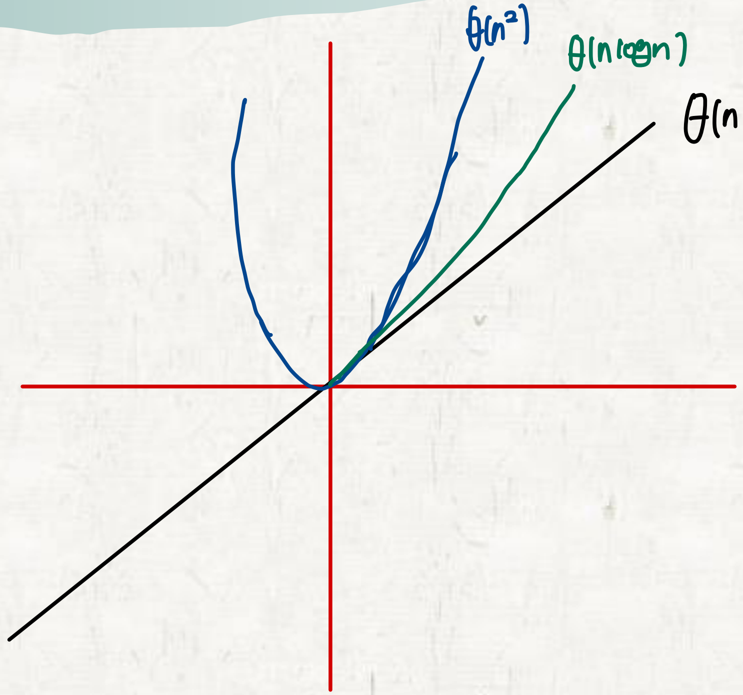
$f(x) = 3x^n + 2x^{n-1} + \cdots$

$\rightarrow \frac{n(n+1)}{2} - 1$ multiple

$\theta(n^2)$

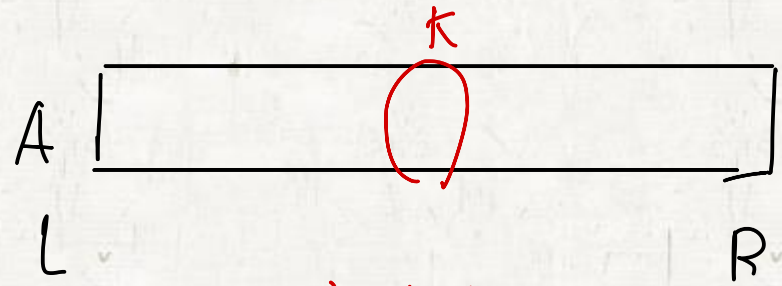
$$\boxed{\begin{aligned} &f(x) = x(3x^{n-1} + 2x^{n-2} + \cdots) \cdots \\ &\Rightarrow x(x(3x^{n-2} + 2x^{n-3} + \cdots) \cdots \\ &\rightarrow \theta(n) \end{aligned}}$$

이 경우가 낫다



More (sorting) algorithms

- **Binary Search**
  - Exercise 2.3-5



$A[\lfloor \frac{L+R}{2} \rfloor]$의 값과 k값 비교

작다 → k-1 왼쪽에 recursive

크다 → k+1 오른쪽에 recursive

같다 → stop

$T(n) = T(\frac{n}{2}) + \theta(1)$

- **Selection sort** → algorithm using maximum/minimum selection
  - Exercise 2.2-2

A | 11 | 7 | 6 | 2 | 3 | 10 | 9 | 2 | 3 |

1. find the index of storing smallest value
2. Change!!

$T(n) = T(n-1) + \underline{\theta(n)}_{cn}$

$= cn + c(n-1) + c(n-2) \cdots = \frac{cn(n+1)}{2} = \theta(n^2)$ best & worst