All-Pairs Shortest Paths

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Contents

- Using SSSP (single source shortest path) algorithms
- Floyd-Warshall algorithm
- Transitive closure of a directed graph

Using SSSP algorithms

- We can solve an all-pairs shortest-paths problem by running a ***single-source shortest-paths algorithm* $\boxed{\underline{|V|\ \textit{times}}}$,** once for each vertex as the source.

- **Nonnegative-weight** edges
  - Dijkstra's algorithm
    - The linear-array implementation
      - $O(V \cdot V^2) = O(V^3)$.

      If $E = \Theta(V)$ $= O(VE \log V)$

      → Floyd-warshall algorithm 보다 빠르다.
    - The binary min-heap implementation
      - $O(V \cdot (V \lg V + E \lg V)) = O(V^2 \lg V + \boxed{V E \lg V})$

Using SSSP algorithms

- ***Negative-weight edges***
  - Bellman-Ford algorithm
    - $O(V \cdot VE) = O(V^2E)$
    - $O(V^4)$ on a dense graph

$E = V^2$

$5+6+7-(8+9)-(8+7+6)$

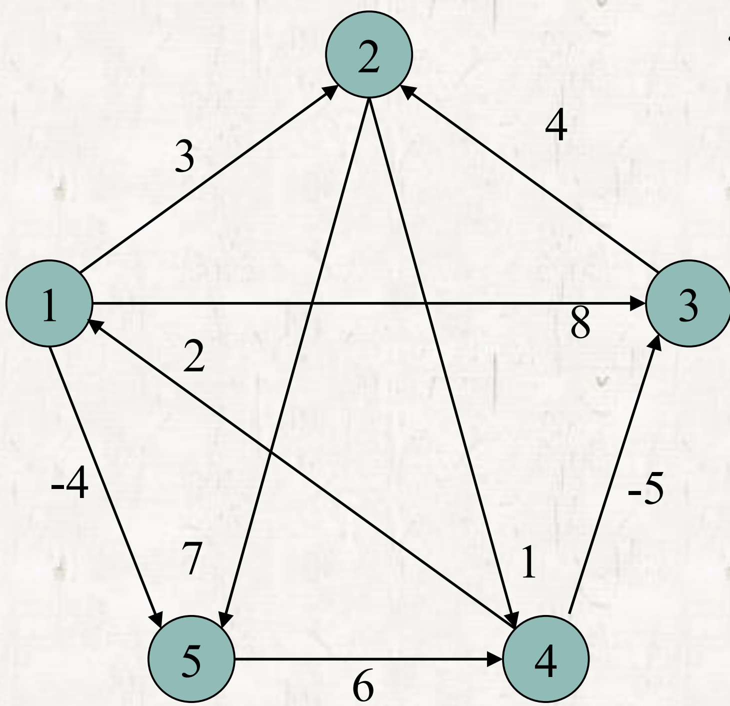
$a+b$

Contents

- *Using SSSP (single source shortest path) algorithms*
- Floyd-Warshall algorithm
  - $\Theta(V^3)$-time
- Transitive closure of a directed graph

Definition

- Adjacency Matrix W
  - $w_{ij} = w(i,j)$



$$\begin{array}{cc} \text{from}\backslash\text{to} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \end{array}$$

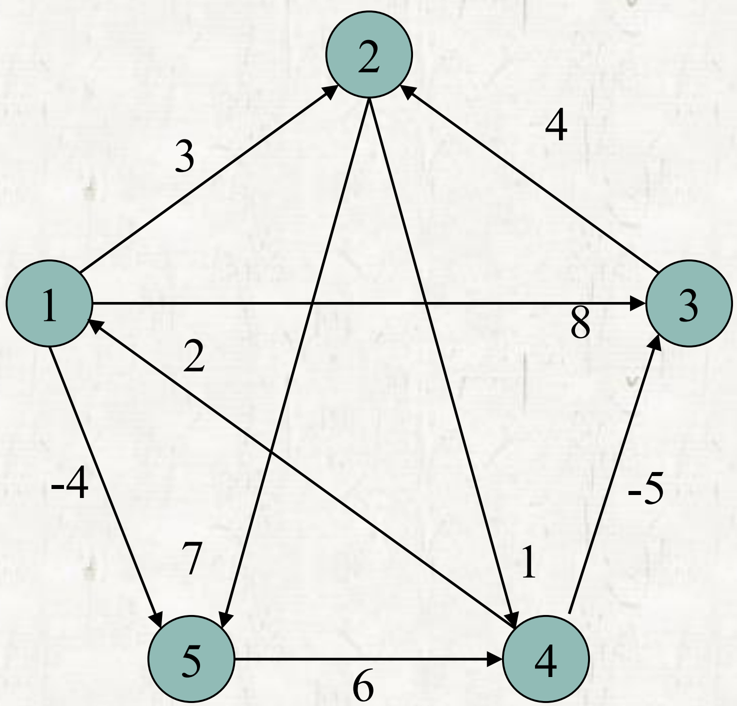
↳ no edge

Definition

- Shortest Distance Matrix *D*
  - $d_{ij} = \delta(i,j)$

길이를 합해 확인

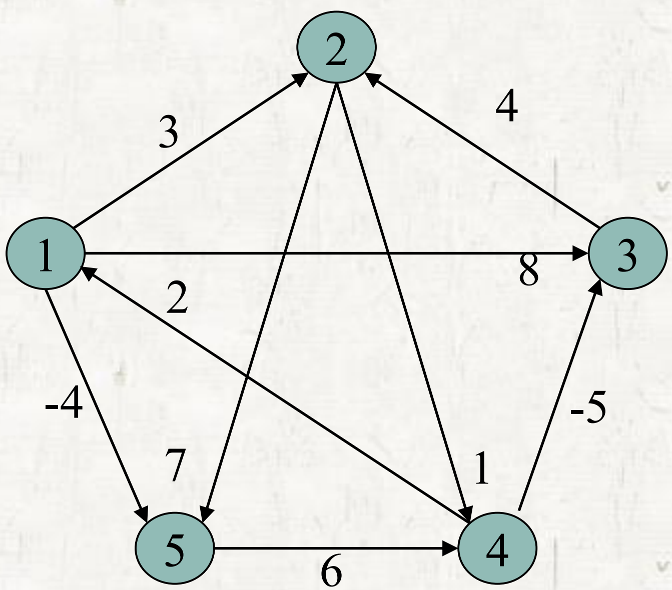
$d_{43} + d_{24}$



$$\begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Definition

- Predecessor Matrix $\boldsymbol{\Pi}$ → Shortest Distance Matrix를 이용해 Path 계산을 위해
  - π_{ij} = NIL if either $i = j$ or there is not a path from i to j.
  - π_{ij} is the predecessor of j on some shortest path from i to j.



$$\begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

Definition

- The following procedure prints a shortest path from i to j due to the optimal substructure of the shortest-paths problem.

PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)

if $i == j$
    print i
elseif π_{ij} == NIL
    print "no path from" i "to" j "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_{ij})   → $i, \ldots, a_{ij}, j$
    print j → print 할 이전에 사용하면 Reversed Order

Floyd-Warshall algorithm

- **Intermediate Vertex**
  - An intermediate vertex of a simple path $p = \langle v_1, v_2, \cdots, v_l \rangle$ is any vertex of p between v_1 and v_l.

$v_2 \cdots v_{l-1}$

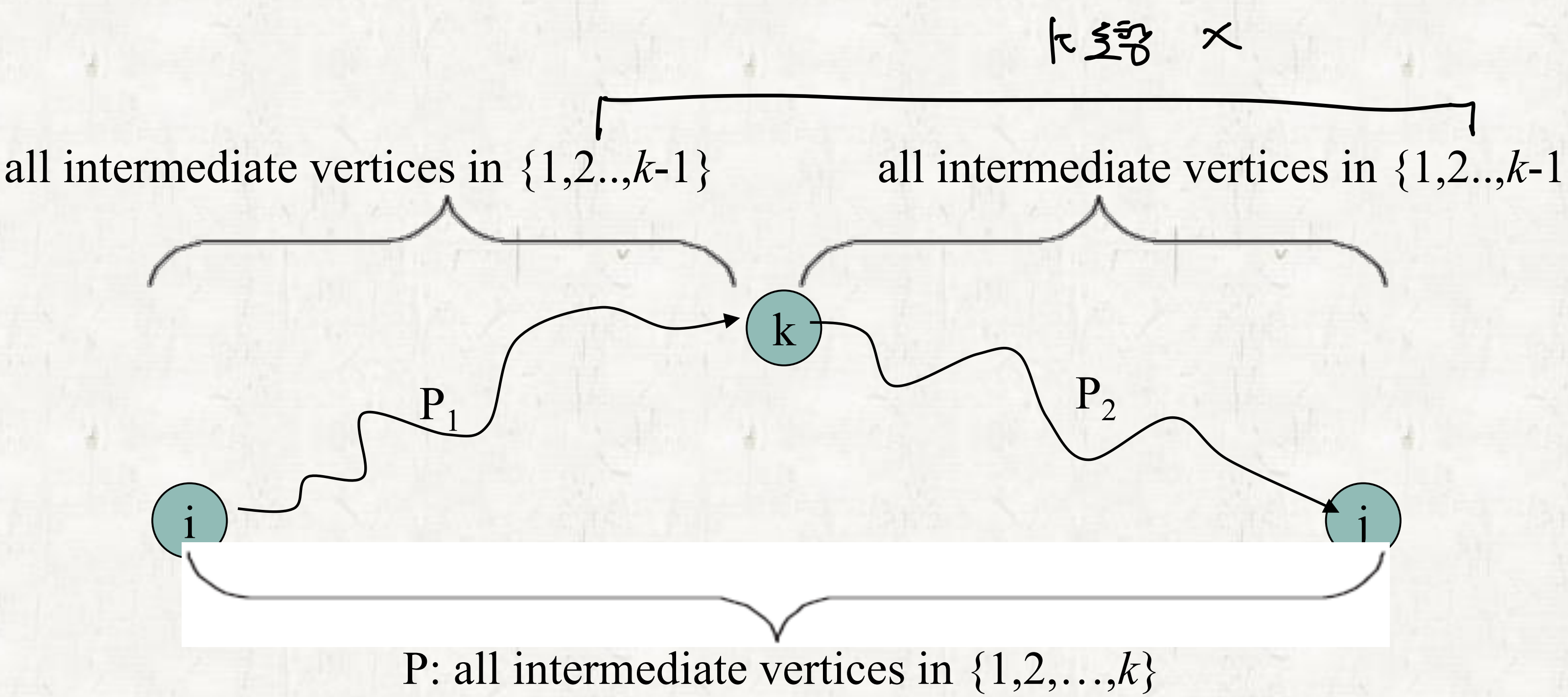
Floyd-Warshall algorithm

- **The structure of a shortest path**
  - Floyd-Warshall algorithm is based on the observation of the intermediate vertices, which costs $\Theta(V^3)$ time.
  - Let $V = \{1, 2, \cdots, n\}$.
  - For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from $\{1, 2, \cdots, k\}$, and let p be a minimum weight path from among them.

Subset : Intermediate vertex들은
1, 2, ···, k로 돌아가는 수 있다.

Floyd-Warshall algorithm

- Case 1 If k is not an intermediate vertex of path p, then all intermediate vertices of p are in $\{1, 2, \cdots, k-1\}$.
- Case 2 If k is an intermediate vertex of path p, then we break p down into $i \overset{p_1}{\leadsto} k \overset{p_2}{\leadsto} j$.



Floyd-Warshall algorithm

- A recursive solution to the all-pairs shortest-paths problem
  - Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex i to vertex j for which all intermediate vertices are in the set $\{1, 2, \cdots, k\}$.
  - We have the following recurrence:

Intermediate vertex $= \emptyset$

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min\left(\underbrace{d_{ij}^{(k\text{-}1)}}_{i \to j}, \underbrace{d_{ik}^{(k\text{-}1)} + d_{kj}^{(k\text{-}1)}}_{i \to k \to j}\right) & \text{if } k \geq 1. \end{cases} \quad (25.5)$$

  - Because for any path, all intermediate vertices are in the set $\{1, 2, \cdots, n\}$, the matrix $D^{(n)} = d_{ij}^{(n)}$ gives the final answer: $d_{ij}^{(n)} = \delta(i, j)$ for all $i, j \in V$.

Floyd-Warshall algorithm

FLOYD-WARSHALL(W) → Adjacency matrix

$n = W.rows$ // number of vertex

$D^{(0)} = W$ $d^{0}_{ij} = W_{ij}$ → $D^{0} = W$

for $k = 1$ **to** n

  let $D^{(k)} = (d_{ij}^{(k)})$ be a new n x n matrix

  for $i = 1$ **to** n

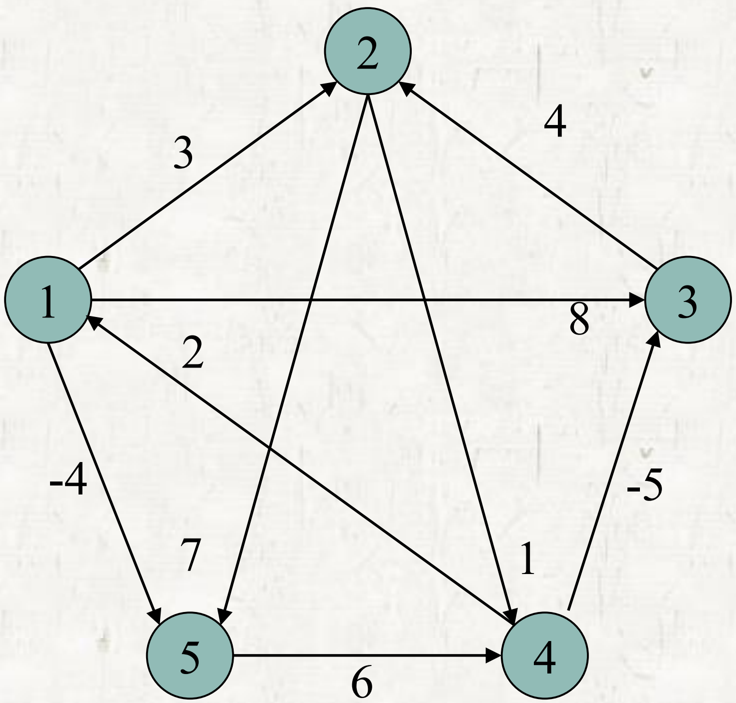
    for $j = 1$ to n

      $O(1)$ ← $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$ → recurrence from 라충

 return $D^{(n)}$

- costs $\Theta(n^3)$ time. $O(1) \times n \times n \times n$ → $\Omega(n^3)$

Floyd-Warshall algorithm



$$W = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-Warshall algorithm

$d_{24}^{(1)} = \min(d_{24}^{(0)}, d_{21}^{(0)} + d_{14}^{(0)})$

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$= W$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

update

$D^{0}_{ij} \neq \infty \rightarrow$ i를 저장

$$\Pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$\Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

4 → 5 = 4일까? 아니다.

$\pi_{25}^{(4)}$: $\pi_{45}^{(3)}$ 이다!!

$2 \to 4 \to 5$

$\pi_{45}^{(3)} = 1$

$d_{25}^{(4)} = \min(d_{25}^{(3)}, d_{24}^{(3)} + d_{45}^{(3)})$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

- Constructing A Shortest Path
  - Let Π_{ij}^{k} be the predecessor of vertex j on a shortest path from vertex i with all intermediate vertices in $\{1, 2, \cdots, k\}$.

$$\Pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty, \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty. \end{cases}$$

$k>0$

$$\Pi_{ij}^{(k)} = \begin{cases} \Pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ \Pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}. \end{cases}$$

Floyd-Warshall algorithm

- Transitive Closure of Graph
  - Given a directed graph $G = (V,E)$ with vertex set $V = \{1, 2, \cdots, n\}$.
  - The transitive closure of G is defined as the graph $G^* = (V,E^*)$, where $E^* = \{(i, j)$: there is a path from vertex i to vertex j in G$\}$.

1 → 2   2 → 3 존재

→ 1 → 3 3 도착