Growth of Functions

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Contents

- **Asymptotic notation**
  - Θ-notation
  - O-notation
  - Ω-notation

Analogy

- $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.

차수

  - $3n^2 + 2n - 1 = \Theta(n^2)$ only highest degree
  - $3n - 1 = \Theta(n)$
  - $3n - 1 \neq \Theta(n^2)$

Analogy

- $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.

  - $3n^2 + 2n - 1 = O(n^2)$
  - $3n - 1 = O(n)$
  - $3n - 1 = O(n^2)$

Analogy

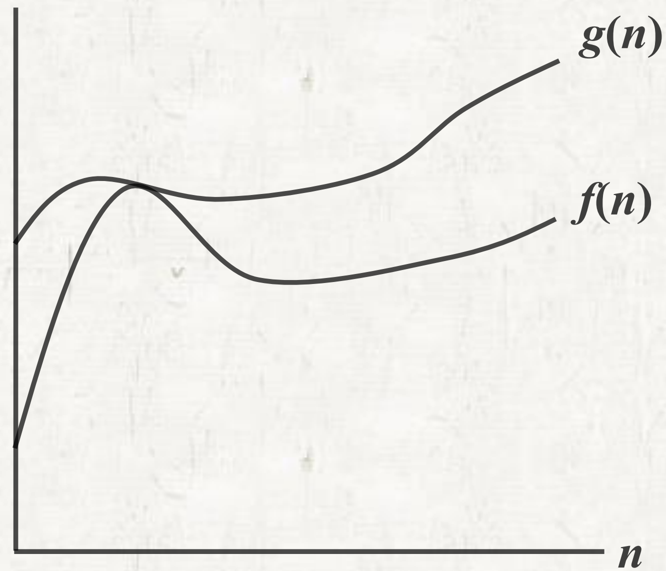
- $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.

  - $3n - 1 = \Omega(n)$
  - $3n^2 - 1 = \Omega(n)$

O-notation

- ***Upper bound***
  - $g(n)$ is an *upper bound* of $f(n)$.

$g(n) \geq f(n)$ for all positive n

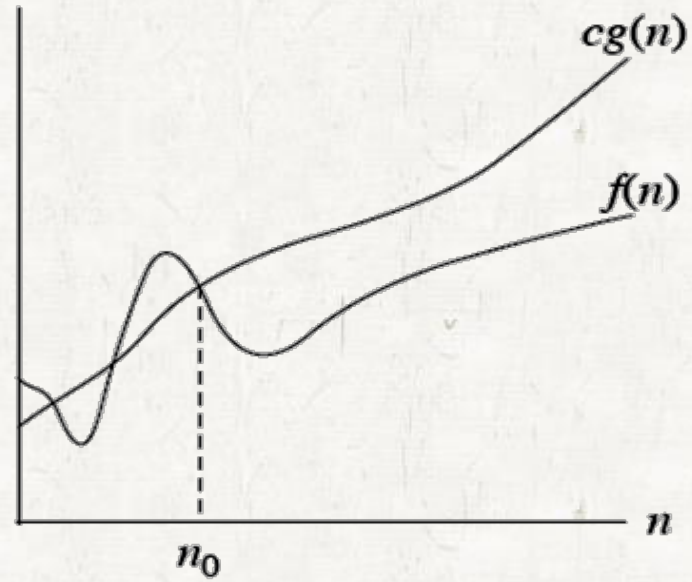


O-notation

- g(n) is an ***asymptotic upper bound*** of $f(n)$.
  - $f(n) = O(g(n))$

There exist positive constants c and n_0 such that
$0 \le f(n) \le \boxed{c}g(n)$ for all $n \ge n_0$.

$n \geq n_0$인 모든 positive n에 대해
$0 \le f(n) \le cg(n)$을 만족시키는 c가 있을때



O-notation

- Example

$$3n + 1 = O(n^2)$$

- Show there are c and n_0 such that $3n + 1 \le cn^2$ for all $n \ge n_0$.
- Dividing by n^2 yields $\underline{\frac{3}{n} + \frac{1}{n^2}} \le c$. maximized of $\frac{3}{n} + \frac{1}{n^2}$

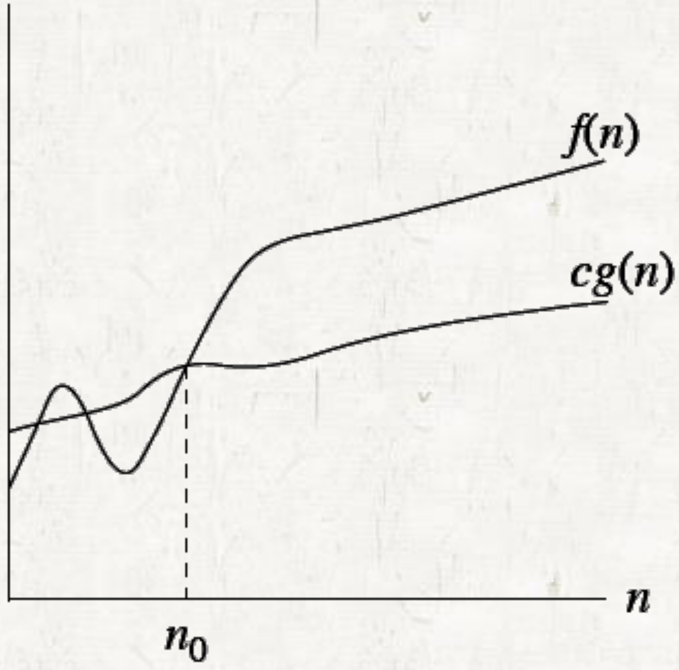
decreasing → '$n=1$'

- The inequality holds for any $n \ge 1$ ($n_0 = 1$) and $c = 4$.

Ω-notation

- **Asymptotic lower bound**
  - $f(n) = \Omega(g(n))$

There exist positive constants c and n_0 such that
$0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.



Ω-notation

- Example

$$3n^2 - 4n + 1 = \Omega(n)$$

- Show there are c and n_0 such that $3n^2 - 4n + 1 \geq cn$ for all $n \geq n_0$.

$n=1 \rightarrow 0$
$n=2 \rightarrow 2.5$
$n=3 \rightarrow 5.xx$
$\rightarrow C=2$
$n_0=2$

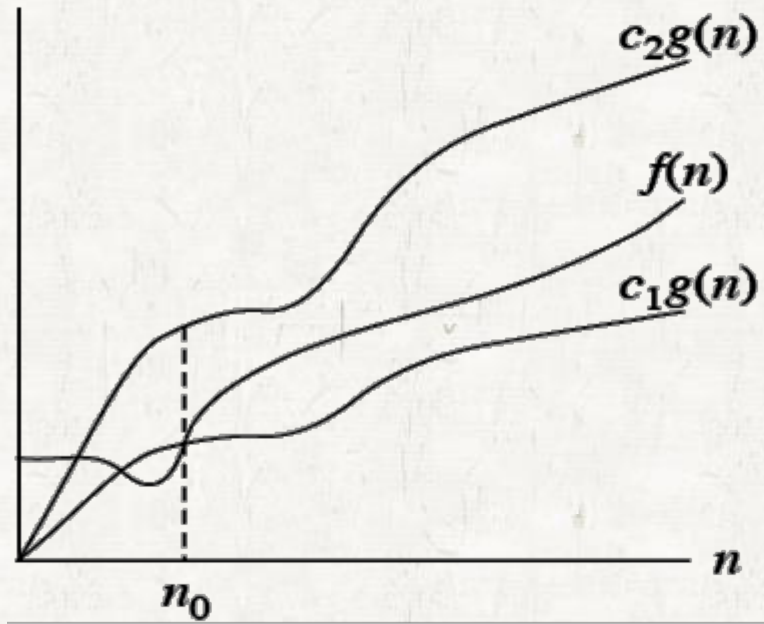
- Dividing by n yields $\boxed{3n - 4 + \frac{1}{n}} \geq c$.

- The inequality holds for any $n \geq 2$ ($n_0 = 2$) and $c = 2$.

Θ-notation

- **Asymptotically tight bound**
  - $f(n) = \Theta(g(n))$

there exist positive constants c_1, c_2 ,and n_0 such that
$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$
for all $n \ge n_0$.



Θ-notation

- Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

To show there exist positive constants c_1, c_2 and n_0 such that

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Dividing by n^2 yields $c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$

Θ-notation

- Example

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$$

max

- The right-hand inequality holds for $n \ge 1$ by choosing $c_2 \ge 1/2$.
- The left-hand inequality holds for $n \ge 7$ by choosing $c_1 \le 1/14$.
- Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$,

  we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

$\frac{1}{2} - \frac{3}{n} > 0$을
만족시키는 가장 작은
$n = 7$

Θ-notation

- Example
  - $f(n) \geq 0$
  - Consider any quadratic function $f(n) = an^2 + bn + c$, where a, b, and c are constants and $a > 0$.
  - Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$.
  - The reader may verify that $0 \leq c_1 n^2 \leq an^2 + bn + c \leq c_2 n^2$ for all $n \geq n_0$. (Self-study)
  - In general, for any polynomial $p(n) = \sum_{i=0}^{d} a_i n^i$ where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$.

$c_1 n^2 \leq an^2 + bn + c$ 를 성립하려면

$c_1 \leq a + b + c$

$an^2 + bn + c \leq c_2 n^2$

$(c_2 - a)n^2 + bn + c \geq 0 \rightarrow c_2$는 a보다 크게

n에 대한 d차 다항식

$\rightarrow p(n) = \Theta(n^d)$

Examples

The running time is $O(n^2)$ & $\Omega(n)$

- **Insertion sort**
  - $O(n^2)$, $\Omega(n)$
    worst best
- **Selection sort**
  - $\Theta(n^2)$
- **Merge sort**
  - $\Theta(n \lg n)$
- **Binary search**
  - $O(\lg n)$, $\Omega(1)$

$* \; O(\lg n) = O\left(\frac{\log n}{\log 2}\right) = O(\log n)$
$= O(\ln n)$

worst case	best case
$\Theta(n^2)$	$\Theta(n)$
$\Theta(n^2)$	$\Theta(n^2)$
$\Theta(n\log n)$	$\Theta(n\log n)$
$\Theta(\log n)$	$\Theta(1)$

Analogy

- **Analogy**
  - $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
  - $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
  - $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
  - $f(n) = o(g(n)) \approx f(n) < g(n)$ in degree.
  - $f(n) = \omega(g(n)) \approx f(n) > g(n)$ in degree.

등호 제거

Comparison of functions

R: relation

- **Comparison of functions**
  - Transitivity : aRb and $bRc \rightarrow aRc$ $\quad a=b$ & $b=c \Rightarrow a=c$
  - Reflexivity : aRa이 성립하는
  - Symmetry : aRb이 성립하면 bRa이 성립할 때
  - Transpose symmetry : $aR_1b \Rightarrow bR_2a$

ex) $a \le b \Rightarrow b \ge a$

$a < b \Rightarrow b > a$

Comparison of functions

- **Comparison of functions**
  - Transitivity ($=$, $\leq$, $\geq$, $<$, $>$)
  - Reflexivity ($=$, $\leq$, $\geq$)
  - Symmetry ($=$)
  - Transpose symmetry ($\leq \leftrightarrow \geq$, $< \leftrightarrow >$)

↳ 등호는 포함 X

Transitivity

- **Transitivity (=, ≤, ≥, <, >)**
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$,
  - $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$,
  - $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$,
  - $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$,
  - $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$.

그래프적 해석이 가능

Reflexivity

- **Reflexivity ($=, \leq, \geq$)**
  - $f(n) = \Theta(f(n))$
  - $f(n) = O(f(n))$
  - $f(n) = \Omega(f(n))$

Symmetry and transpose symmetry

- **Symmetry (=)**
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

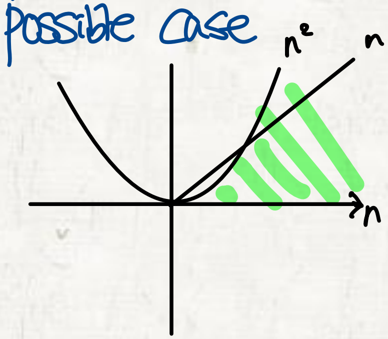
    f(n)과 g(n)는 같은 차수를 가진다.

- **Transpose symmetry** ($\leq \leftrightarrow\ \geq$, $< \leftrightarrow >$)
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$,
  - $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Comparison of functions

- **Trichotomy**
  - For any two real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, $a > b$.
  - That is, any two numbers are comparable.
  - Are any two functions asymptotically comparable?
    - Is it possible $f(n) \neq O(g(n))$ and $f(n) \neq \Omega\,(g(n))$?
    - n and $n^{1+\sin n}$

possible case



$n^0 < n^{1+\sin n} < n^2$

Self-study

- **Exercise 3.1-1**
  - Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$
- **Exercise 3.1-4**
  - Is $2^{n+1} = O(2^n)$? $\rightarrow \Theta(2 \cdot 2^n) = O(2^n)$
  - Is $2^{2n} = O(2^n)$? X
- **Problem 3-2 for O, Θ, and Ω.**
  - Use $\lg(n!) = \Theta(n\lg n)$