Problem_6

2025-03-28

(a) Consider $Z(t) = e^{\sigma W_t - \alpha t}$ For what value of α is Z a martingale?
Recall that,

$$E[e^{\sigma W_t}] = e^{\mu + \frac{1}{2}\sigma^2} = e^{\frac{1}{2}\sigma^2 t}$$

So for $Z(t)$ to be a martingale we need $E_t[Z(T)] = Z(t)$. So we choose $\alpha = \frac{1}{2}\sigma^2$

$$E_t[Z(T)] = Z(t)e^{\frac{1}{2}\sigma^2(T-t)}e^{-\alpha(T-t)}$$
$$= Z(t)e^{\frac{1}{2}\sigma^2(T-t)}e^{-\frac{1}{2}\sigma^2(T-t)} = Z(t)$$

Next, consider under Black-Scholes that a stock price process S_t and a derivative security D_t written on this stock have prices given by:

$$S_t = S_0 e^{2W_t^{\mathbb{P}} + .2t}$$
$$D_t = 2e^{6W_t^{\mathbb{P}} + .39t}$$

$W_t^{\mathbb{P}}$ is a standard Brownian motion under the real world measure $\mathbb{P}$.
$W_t^{\mathbb{Q}}$ is a standard Brownian motion under the risk neutral measure $\mathbb{Q}$.
There is a constant, continuous risk-free rate r.

(b) $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + kt$. Solve for k.
[Hint: Notice that $e^{-rt}S_t$ and $e^{-rt}D_t$ are martingales under $\mathbb{Q}$]

$$e^{-rt}S_t = e^{-rt}S_0 e^{.2W_t^{\mathbb{P}} + .2t}$$
$$= S_0 e^{.2(W_t^{\mathbb{Q}} - kt) - rt + .2t}$$
$$= S_0 e^{.2W_t^{\mathbb{Q}} - (r - .2 + .2k)t}$$

From part (a), we know that in order for $e^{-rt}S_t$ to be a martingale it must be that:

$$\frac{1}{2}(.2)^2 = r - .2 + .2k$$

For the derivative,

$$e^{-rt}D_t = e^{-rt}2e^{.6W_t^{\mathbb{P}} + .39t}$$
$$= 2e^{.6(W_t^{\mathbb{Q}} - kt) - rt + .39t}$$
$$= 2e^{.6W_t^{\mathbb{Q}} - (r - .39 + .6k)t}$$

From part (a), we know that in order for $e^{-rt}D_t$ to be a martingale it must be that:

$$\frac{1}{2}(.6)^2 = r - .39 + .6k$$

Now solve the below system of equations for k.

$$\frac{1}{2}(.2)^2 = r - .2 + .2k$$
$$\frac{1}{2}(.6)^2 = r - .39 + .6k$$
$$.57 - .6k = r$$

$$.02 = r - .2 + .2k$$
$$.22 = .57 - .6k + .2k$$
$$-.35 = -.4k$$
$$k = .875$$

(c) Solve for r.

$$r = .57 - .6k$$
$$= .57 - .6(.875)$$
$$= .045$$

(d) Your coworker is used to applying Girsanov's theorem in the below form:

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t X_u du$$

Help your coworker by selecting the appropriate value for X_u. For the appropriate value of X_u, will this statement mean the same thing as part (b).

Yes, for the appropriate value of X_u this can be stated the same way as part (b).

$$X_u = -k = -.875$$

So,

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + .875 \int_0^t du = W_t^{\mathbb{P}} + .875t$$