Problem 2

(a) Describe term premium. If the term premium is positive, do you expect the risk-neutral or real-world expected path of the risk-free rate to be higher?

The term premium refers to the enhanced yield demanded by investors to invest in fixed income securities for longer periods of time. If the term premium is positive, I would expect the risk-neutral path of the risk-free rate to be higher because it would reflect the positive term premium.

(b) How do term premiums typically vary with term to maturity?

Term premiums usually increase with term to maturity.

(c) Analyze the following: "Risk-neutral scenarios are always market consistent, whereas real-world scenarios are not."

The statement is false. While it is true that real-world scenarios may not be market consistent, they can be.

(d) Recommend whether risk-neutral or real-world scenarios are most appropriate for each of the following situations.

i. Management wants to answer "what-if" questions and look at how bad things could get.
Real-World

ii. Valuing an item with uncertain payoff in a market-consistent way.
Risk-Neutral

iii. Understanding simulation across time, where the simulation involves making investment purchases in the future.
Real-World

iv. Risk Management applications and understanding black swan events.
Real-World

v. Management wants to know how much capital is needed to absorb the potential earnings loss in the current low interest rate environment.
Real-World

vi. To calculate the hedging cost when pricing a new Universal Life product with a minimum crediting rate guarantee.
Risk-Neutral

(e) Suppose you are given the folllowing about a Vasicek interest rate model:

$$dr = \kappa(\theta - r)dt + \sigma d\overline{W}$$

$$d\overline{W} = dW + \lambda dt$$

Show how to compute the risk neutral parameters $\hat{\theta}$, κ, θ, σ and λ.

$$dr = \kappa(\theta - r)dt + \sigma d\overline{W}$$

$$dr = \kappa(\theta - r)dt + \sigma(dW + \lambda dt)$$

$$dr = \kappa(\theta + \frac{\sigma\lambda}{\kappa} - r)dt + \sigma(dW + \lambda dt)$$

So $\hat{\theta} = \theta + \frac{\sigma\lambda}{\kappa}$

(f) Suppose you are given the following about the Cox-Ingersoll-Ross (CIR) interest rate model:

$$dr = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}d\overline{W}$$

$$d\overline{W} = dW - \frac{\lambda\sqrt{r_t}}{\sigma}dt$$

Show how to compute the risk-neutral parameters $\overline{\kappa}$ and $\overline{\theta}$ given κ, θ, σ and λ.

$$dr = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}d\overline{W}$$

$$dr = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}(dW - \frac{\lambda\sqrt{r_t}}{\sigma}dt)$$

$$dr = \kappa(\theta - r_t - \frac{\lambda\sqrt{r_t}}{\sigma})dt + \sigma\sqrt{r_t}dW$$

$$dr = \kappa(\theta - r_t - \frac{\lambda r_t}{\kappa})dt + \sigma\sqrt{r_t}dW$$

$$dr = (\theta\kappa - r_t\kappa - \lambda r_t)dt + \sigma\sqrt{r_t}dW$$

$$dr = (\theta\kappa - (\kappa + \lambda)r_t)dt + \sigma\sqrt{r_t}dW$$

$$dr = (\kappa + \lambda)(\frac{\theta\kappa}{\kappa + \lambda} - r_t)dt + \sigma\sqrt{r_t}dW$$

So,

$$\overline{\kappa} = \kappa + \lambda$$

$$\overline{\theta} = \frac{\theta\kappa}{\kappa + \lambda}$$

(g) Suppose the risk-neutral and real-world instantaneous forward rate curves are reviewed for reasonableness. You are told that the risk-netutral curve is higher than the real-world curve, and the difference between the two curves is roughly constant across all maturities. Assess whether this is reasonable.
It is reasonable that the risk-netural curve is higher than the real-world curve. However, it is not reasonable that that the difference is constant across all maturities. The risk-neutral curve should have an embedded term premium baked in. So the difference between the two curves should rise rapidly with term to maturity and then level off.