Problem 1

Consider the following process X_t for $t \in [0,1]$. $X_0 = X_1 = 0$. For $0 < t < 1$, X_t is defined as:

$$X_t = (1-t)\int_0^t \frac{1}{1-u}\,dW_u$$

Note: For parts (a) and (b), find the SDE over $0 < t < 1$ for the process X_t defined for $0 \le t \le 1$.

(a) Determine the stochastic differential equation dX_t. Is X_t a martingale?

$$\begin{aligned}
X_t &= (1-t)\int_0^t \frac{1}{1-u}\,dW_u \\
\frac{X_t}{1-t} &= \int_0^t \frac{1}{1-u}\,dW_u \\
d\left(\frac{X_t}{1-t}\right) &= d\left(\int_0^t \frac{1}{1-u}\,dW_u\right)
\end{aligned}$$

By quotient rule:

$$\begin{aligned}
\frac{(1-t)dX_t - (-1)dtX_t}{(1-t)^2} &= \frac{1}{1-t}dW_t \\
(1-t)dX_t + X_t dt &= (1-t)dW_t \\
(1-t)dX_t &= -X_t dt + (1-t)dW_t \\
dX_t &= -\frac{X_t}{(1-t)}dt + dW_t
\end{aligned}$$

X_t is not a martingale because dX_t has a drift component.

(b) Determine the stochastic differential equation dF_t where $F_t = X_t^2$. Is F_t a martingale?

Use Ito's Lemma:

$$\begin{aligned}
\frac{\partial F_t}{\partial t} &= 0 \\
\frac{\partial F_t}{\partial X_t} &= 2X_t \\
\frac{\partial^2 F_t}{\partial X_t^2} &= 2
\end{aligned}$$

$$\begin{aligned}
dF_t &= 0dt + 2X_t dX_t + \frac{1}{2}2(dX_t)^2 \\
dF_t &= 2X_t\left(-\frac{X_t}{(1-t)}dt + dW_t\right) + dt \\
dF_t &= \left(-\frac{2X_t^2}{(1-t)}dt + 2X_t dW_t\right) + dt
\end{aligned}$$

$$dF_t = (1 - \frac{2X_t^2}{(1-t)})dt + 2X_t dW_t$$

F_t is not a martingale because dF_t has a drift term.

(c) Calculate $E_0(X_t)$

$$E_0(X_t) = E_0((1-t)\int_0^t \frac{1}{1-u}\,dW_u) = 0$$

Ito integral.

(d) Show that:

$$Cov(X_s, X_t) = (1-s)(1-t)Var(\int_0^s \frac{1}{1-u}\,dW_u)$$

for $s \le t$ and $s, t \in [0,1]$

Hint: Use the fact that

$$Cov(\int_0^s \frac{1}{1-u}\,dW_u, Cov(\int_s^t \frac{1}{1-u}\,dW_u = 0$$

$$Cov(X_s, X_t) = Cov((1-s)\int_0^s \frac{1}{1-u}\,dW_u, (1-t)\int_0^t \frac{1}{1-u}\,dW_u)$$

$$Cov(X_s, X_t) = Cov((1-s)\int_0^s \frac{1}{1-u}\,dW_u, (1-t)\int_0^s \frac{1}{1-u}\,dW_u)$$

$$Cov(X_s, X_t) = (1-s)(1-t)Cov(\int_0^s \frac{1}{1-u}\,dW_u, \int_0^s \frac{1}{1-u}\,dW_u)$$

$$Cov(X_s, X_t) = (1-s)(1-t)Var(\int_0^s \frac{1}{1-u}\,dW_u)$$

(e) Calculate the following for $s \le t$ and s,t $s, t \in [0,1]$:

(i) $Cov(X_s,X_t)$
From part (d)

$$Cov(X_s, X_t) = (1-s)(1-t)Var(\int_0^s \frac{1}{1-u}\,dW_u)$$

$$Cov(X_s, X_t) = (1-s)(1-t)(E[(\int_0^s \frac{1}{1-u}\,dW_u)^2] - E[\int_0^s \frac{1}{1-u}\,dW_u])$$

$$Cov(X_s, X_t) = (1-s)(1-t)(E[(\int_0^s \frac{1}{1-u}\,dW_u)^2])$$

By Ito's Isometry

$$Cov(X_s, X_t) = (1-s)(1-t)(E[(\int_0^s \frac{1}{(1-u)^2}\,du)])$$

$$Cov(X_s, X_t) = (1-s)(1-t)(E[\frac{1}{1-u}\Big|_0^s])$$

$$Cov(X_s, X_t) = (1-s)(1-t)(\frac{1}{1-s} - 1)$$

$$Cov(X_s, X_t) = (1-s)(1-t)(\frac{1}{1-s} - 1)$$

$$Cov(X_s, X_t) = s(1-t)$$

(ii) Var(X_t)

$$Var(X_t) = Var((1-t)\int_0^t \frac{1}{1-u}\,dW_u)$$

$$Var(X_t) = E(((1-t)\int_0^t \frac{1}{1-u}\,dW_u)^2) - E((1-t)\int_0^t \frac{1}{1-u}\,dW_u)^2$$

$$Var(X_t) = E(((1-t)\int_0^t \frac{1}{1-u}\,dW_u)^2)$$

By Ito's Isometry

$$Var(X_t) = E((1-t)^2\int_0^t \frac{1}{(1-u)^2}\,du)$$

$$Var(X_t) = (1-t)^2(\left.\frac{1}{1-u}\right|_0^t)$$

$$Var(X_t) = (1-t)^2(\frac{1}{1-t} - 1)$$

$$Var(X_t) = t(1-t)$$

(f) Find $t \in [0,1]$ such that $Var(X_t)$ is :

(i) Minimized
$Var(X_t)$ is minimized when t = 0 or t = 1.

(ii) Maximized
$Var(X_t)$ is maximized when $t = \frac{1}{2}$