Problem_3

Suppose at time $t = 0$, we are given four zero-coupon bond prices {B_1, B_2, B_3, B_4} that mature at times $t = 1, 2, 3, 4$. This forms the term structure of interest rates.

We also have one-period forward rates {f_0, f_1, f_2, f_3} where each f_i is the rate contracted at time $t = 0$ on a loan that begins at time $t = i$ and ends at time $t = i + 1$. In other words, if a borrower borrowns \$N at time $t = i$, he or she will pay back \$N(1 + f_i) at time $t = i + 1$. The spot rate is denoted by r_i. Clearly, $r_0 = f_o$. The {B_i} and all forward loans are default free.

At each time period there are two possible states of the world, denoted by {u_i,d_i} for \$i = 1, 2, 3, 4 \$.

i	B_i	f_{i-1}
1	.90	.08
2	.87	.09
3	.82	.10
4	.75	.18

(a) At time $i = 0$, how many possible states of the world are there at $i = 3$?
There are $2^3 = 8$ possible states of the world at $i = 3$.

(b) Form an arbitrage portfolio that will guarantee a positive payoff at time i = 0 and non-negative payoff at times $i \geq 1$.

An arbitrage portfolio that guarantees a postive payoff at time i = 0 and a non-negative payoff at time i = 1 would be purchasing the bond that matures at time i = 1 and lending money at f_0. The cashflows from this strategy would be:

i	CF from B_1	CF from Lending	Total
0	-.90	.93	.03
1	1	-1	0

(c) Given a default-free zero-coupon bond, B_n that matures at time $t = n$, and all the forward rates {f_0, ..., f_{n-1}} obtain a formula that expresses B_n as a function of f_i.

$$B_n = \prod_{i=0}^{n-1} \frac{1}{1 + f_i}$$

(d) Consider the system below. Can the B_i be determined independently?

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ B_2^u & B_2^d \\ B_3^u & B_3^d \\ B_4^u & B_4^d \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

No, all the B_i depend on the same underlying term structure.

(e) In the above system, can all the {f_i} be determined independently?

No they cannot. The forward rates are defined recursively. So forward rates with longer time horizons depend on forward rates with shorter time horizons.