Problem_5

You are given that:

- $\varepsilon_t = exp[\int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du]$
- γ_u is assumed to be deterministic and continuous in t and the integral $\int_0^t \gamma_u \, dW_u$ is well defined
- For all $t \geq 0$, the following non-explosiveness condition holds:

$$exp(\int_0^t \gamma_u^2 \, du) < \infty$$

- Under $\mathbb{P}$, W_t is a standard Wiener process

(a) Determine the distribution of $\int_0^t \gamma_u \, dW_u$ under $\mathbb{P}$ $\int_0^t \gamma_u \, dW_u$ is an Ito integral. So:

$$\int_0^t \gamma_u \, dW_u \sim \mathcal{N}(0, \int_0^t \gamma_u^2 \, du)$$

(b) Calculate $E^{\mathbb{P}}(\varepsilon_t)$

$$E^{\mathbb{P}}(\varepsilon_t) = E^{\mathbb{P}}(exp[\int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du])$$

From part (a), we know this is a log-normal random variable with parameters $\mu = -\frac{1}{2}\int_0^t \gamma_u^2 \, du$ and $\sigma^2 = \int_0^t \gamma_u^2 \, du$. So, the expected value of ε_t is:

$$E^{\mathbb{P}}(\varepsilon_t) = e^{\mu + \frac{1}{2}\sigma^2}$$

$$E^{\mathbb{P}}(\varepsilon_t) = e^{-\frac{1}{2}\int_0^t \gamma_u^2 \, du + \frac{1}{2}\int_0^t \gamma_u^2 \, du}$$

$$E^{\mathbb{P}}(\varepsilon_t) = 1$$

(c) Determine the stochastic differential equation for $d\varepsilon_t$ under $\mathbb{P}$

$$d\varepsilon_t = d[e^{\int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du}]$$

Let $Y_t = \int_0^t \gamma_u \, dW_u$. Then

$$\varepsilon_t = f(Y_t, t) = e^{Y_t - \frac{1}{2}\int_0^t \gamma_u^2 \, du}$$

First, note that:

$$f_t = -\frac{1}{2}\gamma_t^2 \varepsilon_t$$

$$f_{Y_t} = f_{Y_t Y_t} = \varepsilon_t$$

$$d_{Y_t} = \gamma_t dW_t$$

$$(d_{Y_t})^2 = \gamma_t^2 dt$$

Then using Ito's lemma:

$$d\varepsilon_t = f_t dt + f_{Y_t} dY_t + \frac{1}{2} f_{Y_t Y_t} (dY_t)^2$$

$$= -\frac{1}{2}\gamma_t^2\varepsilon_t dt + \varepsilon_t dY_t + \frac{1}{2}\varepsilon_t (dY_t)^2$$

$$= -\frac{1}{2}\gamma_t^2\varepsilon_t dt + \varepsilon_t \gamma_t dW_t + \frac{1}{2}\varepsilon_t \gamma_t^2 dt$$

$$= \varepsilon_t \gamma_t dW_t$$

Therefore,

$$d\varepsilon_t = \varepsilon_t \gamma_t dW_t$$

(d) Simplify the following quantity: $E^{\mathbb{P}}[exp(\int_0^t (\alpha + \gamma_u) dW_u - \frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du)]$

The quantity is log-normally distributed since $\int_0^t (\alpha+\gamma_u) dW_u$ is an Ito integral. The mean of a log-normal distribution is $e^{\mu + \frac{1}{2}\sigma^2}$

$$\mu = -\frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du$$

$$\sigma^2 = \int_0^t (\alpha+\gamma_u)^2 du$$

$$E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u) dW_u - \frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du)] = exp(-\frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du + \frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du)$$

$$= e^0 = 1$$

Additionally you are given that: $\overline{W_t} = W_t - \int_0^t \gamma_u du$

(e) Prove that:

$$E^{\mathbb{Q}}[exp(\alpha \overline{W_t})] = E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u) dW_u - \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2) du)] = e^{\frac{\alpha^2 t}{2}}$$

$$E^{\mathbb{Q}}[exp(\alpha \overline{W_t})] = E^{\mathbb{Q}}[exp(\alpha(W_t - \int_0^t \gamma_u du))]$$

To convert between $\mathbb{Q}$ and $\mathbb{P}$ one must multiply by the Radon-Nikodym derivative,

$$E^{\mathbb{Q}}[exp(\alpha \overline{W_t})] = E^{\mathbb{P}}[exp(\alpha(W_t - \int_0^t \gamma_u du) + \int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du)]$$

$$= E^{\mathbb{P}}[exp(\alpha W_t - \int_0^t \alpha\gamma_u du + \int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du)]$$

$$= E^{\mathbb{P}}[exp(\int_0^t \alpha dW_u - \int_0^t \alpha\gamma_u du + \int_0^t \gamma_u \, dW_u - \frac{1}{2}\int_0^t \gamma_u^2 \, du)]$$

$$= E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u) dW_u - \int_0^t \alpha\gamma_u du - \frac{1}{2}\int_0^t \gamma_u^2 \, du)]$$

$$= E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u) dW_u - \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2) \, du)]$$

So,

$$E^{\mathbb{Q}}[exp(\alpha\overline{W_t})] = E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u)dW_u - \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du)]$$

Note, $exp(\int_0^t (\alpha+\gamma_u)dW_u - \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du)$ is log-normally distributed. So,

$$\mu = -\frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du$$

$$\sigma^2 = \int_0^t (\alpha+\gamma_u)^2 du$$

$$E^{\mathbb{P}}[exp(\int_0^t (\alpha+\gamma_u)dW_u - \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du)] = e^{\mu+\frac{1}{2}\sigma^2}$$

$$= exp(-\frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du + \frac{1}{2}\int_0^t (\alpha+\gamma_u)^2 du)$$

$$= exp(-\frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du + \frac{1}{2}\int_0^t (\alpha^2 + 2\alpha\gamma_u + \gamma_u^2) du)$$

$$= exp(-\frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2)\ du + \frac{1}{2}\int_0^t (2\alpha\gamma_u + \gamma_u^2) du + \frac{1}{2}\int_0^t \alpha^2 du)$$

$$= exp(\frac{1}{2}\int_0^t \alpha^2 du)$$

$$= e^{\frac{\alpha^2 t}{2}}$$

(f) Identify the distribution of $\overline{W_t}$ under the probability measure $\mathbb{Q}$.
$\overline{W_t}$ is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = t$.

(g) Is the Radon-Nikodym derivative a martingale under $\mathbb{P}$?
Yes, the Radob-Nikodym derivative is a martingale under $\mathbb{P}$. We can see this from part (c). The stochastic differential equation $d\epsilon_t$ has no drift term.

(h) Suppose that $\gamma_t = 0$ for all t. In this case, what is the value of the Radon-Nikodym derivative?
The value of the Radon-Nikodym derivative is

$$\varepsilon_t = e^{\int_0^t \gamma_u\ dW_u - \frac{1}{2}\int_0^t \gamma_u^2\ du}$$

$$= e^0 = 1$$