Preface

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List of Scribes

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Institute Name
CS1234 Course Course

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LECTURE
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Quest for Structure in Counting Problems

In the previous lecture, we saw that the counting problem can be as hard as (or harder than) the decision problem as given an algorithm for counting problem the decision problem reduces to just checking the count to be zero or not. We also saw an easy decision problem CYCLE whose counting version #CYCLE is NP-hard (by reduction from HAMCYCLE) implying that easy decision problems can also have corresponding counting problems hard. We also argued that talking about counting problems still makes sense as the count value, though exponential, can still be represented in polynomial number of bits.

In this lecture, we will study counting problems and understand their structural complexity. We shall also make attempts to develop the theory of complexity classes capturing the counting problems (especially #P). We shall also discuss their basic containments.

4.1 Preliminaries

Firstly, we fix our computation model where we have a Turing machine with an input tape, work tape and an output tape. We are interested in the resources used by the Turing machine - space (considering only the work tape) and time.

We want to capture the notion of counting formally. One such way is to see it as computing a function $f : \Sigma^* \to \mathbb{N}$ where $\Sigma = \{0, 1\}$ which gives an integer value. So, how can we capture the notion of computing a function? There are two possible ways of capturing function computation.

Variant 1 We say that a function f is computable if each bit of the output can be computed in some decision complexity class $\mathcal{C}$.

Variant 2 f is computable if the value of the function computation can be written down within the resource bounds.

Analogous to the decision problems, we define complexity classes for function computation problems. A natural extension of P is FP which is defined as

$$\text{FP} = \{f \mid f : \Sigma^* \to \mathbb{N}, f(x) \text{ for any } x \in \Sigma^* \text{ can be written down in } poly(|x|) \text{ time}\}$$

Now, we shall plugging in the two variants of function computation and see which of them is more appropriate.

4.2 Comparing the variants

We quickly observe that the first and the second variant really coincides when we are talking about deterministic computations. Let us do this analysis by attempting the definition of FP. Following the first variant $f \in \mathsf{FP}$ iff there exists an algorithm that can compute each bit of f in class P. But since the algorithm is deterministic, this is equivalent to saying that $\forall\ i$, the language defined by the i^{th} bit

$$L_{f_i} = \{x : (f(x))_i = 1\} \in \mathsf{P}$$

On the other hand, if each bit can be computed in polynomial time and since the count value can be represented in polynomial number of bits, for evaluation, we just run a polynomial time algorithm polynomial times which is still a polynomial. Hence we have an algorithm that satisfies the second variant.

Hence for deterministic polynomial time computation both variants are equivalent. We can define other classes for deterministic computation like FL(log space bounded function computation), FPSPACE (polynomial space bounded function computation). Containments of these classes are analogous to their decision versions. We leave the proof as an exercise.

LEMMA 4.1. $\mathsf{FL} \subseteq \mathsf{FP} \subseteq \mathsf{FPSPACE}$

Now we turn into the non-deterministic world. Following variant 1 of definition of function computation, we must have each bit computable in class NP. But variant 2 is not useful because an non-deterministic poly time Turing machine by our model is not set to output a value. How can we capture the function computation for a non-deterministic machine for decision problems which works by guess-verify mechanism?

Now, consider the non-deterministic algorithm we had for SAT, which does guessing of an assignment and verifying it. We can observe the following additional property.

OBSERVATION 4.2. *Number of satisfying assignments is exactly equal to the number of accepting paths.*

This leads to the question as to whether this is accidental or is there some hidden structure? It also assigns a function value to the non-deterministic Turing machine. This motivates us to give a new model, for us to call f is computable by a non-deterministic polynomial time Turing machine.

DEFINITION 4.3. f is #P if there exists a non-deterministic Turing machine M running in time $p(n)$ such that $\forall x \in \Sigma^*$, $f(x) = \left|\{y \in \{0,1\}^{p(n)} : M \text{ accepts on path } y\ \}\right|$.

REMARK 4.4. The RHS is also the number of accepting paths of M on x if the lengths of all paths are equal to $p(n)$. We remark that this can be achieved without loss of generality. That is, from an arbitrary TM M, we can get to a new TM M' which has the same number of accepting paths, such that the number of accepting paths on any input x remains the same. We recall our observation that for length of all non-deterministic paths of an NP machine on any input can be made equal without changing the accepted language[1] . But this construction makes the language accepted the same, and need not keep the number of accepting

[1]In particular, we showed that a language $A \in \mathsf{NP}$ if and only if there is a language $B \in \mathsf{P}$ and a polynomial $p(n)$ such that $x \in A \iff \exists y \in \{0,1\}^{p(n)} : (x, y) \in B$

paths the same. We modify it slightly to achieve our goal. Indeed, if a path is shorter than $p(n)$ bits and decided A/R, we extend it to the required length using a binary tree of paths rooted at that node and make the left most path (in this binary tree) report A/R respectively and make all other paths reject. The number of accepting paths does not change due to this construction.

REMARK 4.5. Try this as an exercise. Initiate the thought process on : how does this definition compare with variant 1? What does computing/testing each bit to be 0/1 mean?

Counting version of SAT denoted as #SAT can be defined as

$$\#\mathsf{SAT}(\phi) = |\{\sigma | \phi(\sigma) = 1, \sigma \text{ is a boolean assignment to variables in } \phi\}|$$

It follows from our observation that #SAT $\in$ #P, since we can give a non-deterministic machine (i,e. a machine for SAT) where number of accepting paths equals to the number of satisfying truth assignments. It can also be shown that #CYCLE $\in$ #P.

CLAIM 4.6. #CYCLE $\in$ #P

Proof. Following is a non-deterministic Turing machine N, such that number of cycles equals number of accepting paths.

N = " On input G,

1. Guess subsets $V' \subseteq V(G)$ and $E' \subseteq E(G)$ non-deterministically.
2. Accept iff V', E' form a simple cycle. "

Now it follows that #CYCLE $(G) = |\{\#$ of accepting paths of N on $G\}|$ since any cycle can uniquely be characterised by an edge set and a vertex set. $\square$

4.3 Basic Containments

In the functional world, we have the following scenario.

$$\begin{tikzcd}
\mathsf{FPSPACE} \arrow[d, dash] \\
\mathsf{FP} \arrow[d, dash] \\
\mathsf{FL}
\end{tikzcd}$$

So where does set of functions, #P lie? We will argue that #P lies between FPand FPSPACE thus replicating the picture in the decision world.

LEMMA 4.7. #P $\subseteq$ FPSPACE

Proof. Given a non-deterministic poly time Turing machine M computing function f, we just need to do a simulation in deterministic poly space. This can be done by simulating M over all non deterministic paths while reusing space across the paths. Since length of any path is polynomially bounded, space used will also be polynomial. In the process we need to keep the count of accepting paths which can be $\leq 2^{p(n)}$ but still representable with $p(n)$ bits in binary. Hence space requirement is only polynomial in input length. $\square$

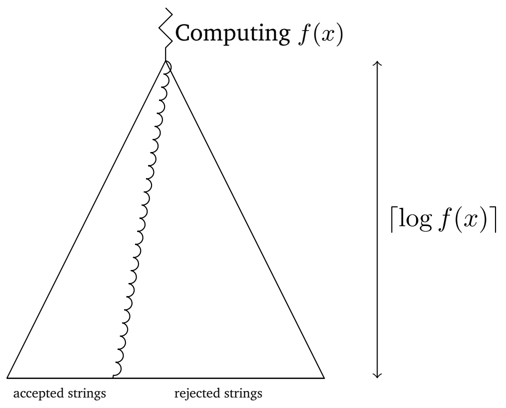
LEMMA 4.8. $\mathsf{FP} \subseteq \#\mathsf{P}$

Proof. Given an $\mathcal{L} \in \mathsf{FP}$, there exists a deterministic Turing machine M which $\forall x \in \mathcal{L}$ writes $f(x)$ in $p(n)$ time where p is a polynomial and $n = |x|$. To show that $\mathcal{L} \in \#\mathsf{P}$, we need to construct a non-deterministic such that

$$\text{No of accepting paths} = f(x).$$

N can compute $f(x)$ by simulating M. Let the value obtained be k. Now, N must have exactly k accepting paths. This can be ensured by guessing $\lceil \log k \rceil$ bits and accepting all the paths whose address have binary representations $\leq k$ and rejecting the remaining paths.

N runs in poly time since $f(x)$ computation (simulation of M) takes only polynomial time. Even though $f(x)$ is exponential, number of bits guessed is $\lceil \log f(x) \rceil$ which will be polynomial in n. Hence depth is polynomially bounded. Also number of accepting paths equals $f(x)$ by construction. Thus N' is a #Pmachine accepting $\mathcal{L}$. Hence $\mathcal{L} = L(N') \in \#\mathsf{P}$. ∎



It can be observed that if function computation can be done in polynomial time then $\mathsf{P} = \mathsf{NP}$. This is because solving decision problem amounts to checking if the corresponding counting function gives a non zero value or not hence making decision problem easy if function computation is in P.

LEMMA 4.9. $\#\mathsf{P} = \mathsf{FP} \implies \mathsf{P} = \mathsf{NP}$

An interesting question would be to ask if the converse it true? That is

$$\text{Does } \mathsf{P} = \mathsf{NP} \text{ imply } \#\mathsf{P} = \mathsf{FP}?$$

We will address this and show a weaker implication (that is, based on slightly stronger LHS) in the next lecture.

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LECTURE
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FP **vs** #P **question**

We know $(\#\mathsf{P} = \mathsf{FP}) \implies (\mathsf{P} = \mathsf{NP})$. In the last lecture we left the question about the converse. i.e., is it true that, $(\mathsf{P} = \mathsf{NP}) \implies (\#\mathsf{P} = \mathsf{FP})$?

In this lecture, we will show a weaker version of the above containment. We will define a new class of languages PP (probabilistic polynomial time) in decision world, that contains NP, and will show that if we make a stronger assumption PP = P (seemingly stronger than NP = P) we get the required implication. We will show in particular that $\mathsf{PP} = \mathsf{P} \iff \#\mathsf{P} = \mathsf{FP}$.

5.1 Complexity Class PP

For exploring the complexity class, we revisit the definition of NP. With respect to a "branching" Turing machine(NTM) (which can branch at each computation by using a guess bit), we have two different classes of languages as follows, which differs in their acceptance conditions. We denote for a machine M, by $\#acc_M(x)$ as the number of accepting paths. Let the machine run for $p(n)$ time on each path.

- NP = {L : ∃ branching TM M_1, x ∈ L $\iff \#acc_{M_1}(x) \geq 1\}$.
- coNP = {L : ∃ branching TM M_2, x ∈ L $\iff \#acc_{M_2}(x) \geq 2^{p(n)}\}$.

In terms of number of accepting paths on input x, the above classes represent the extremes. On one side, in NP we talk of atleast one accepting path, and on the other side, in coNP we talk of all accepting paths. To understand the structure of these classes we can ask the variant of this as : what different class of languages do we get if we set the accepting condition as the number of accepting paths being more than a fraction of the total number of paths on input *x*. A simpler situation is to consider the fraction to be half and the corresponding class of languages is called PP(probabilistic polynomial) which is defined as follows. We will come across other variants in the later lectures.

DEFINITION 5.1. PP ={L : ∃ NTM M, x ∈ L $\iff \#acc_{M_1}(x) > 2^{p(n)-1}\}$

We first understand this complexity class with respect to the other ones we have already seen. We start with the following proposition.

PROPOSITION 5.2. NP $\subseteq$ PP.

Proof. Let $L \in$ NP via a nondeterministic turing machine M, $x \in L \iff M$ has atleast one accepting path on x. Our aim is to give another turing machine M' for the same language L such that $x \in L \iff M'$ has more than half of the total number of paths as accepting paths on x.
Description of M' (on input x):

1. Simulate M on x.
2. If M accepts then choose one bit nondeterministically and accept in both branches.
3. If M rejects then choose one bit nondeterministically,accept in one branch and reject in other.

As we remarked in the last lecture, we can assume without loss of generality that the height of computation tree of M on x is exactly $p(n)$ for some polynomial p. Since the total number of paths for M' on input x is $2^{p(n)+1}$, it suffices to prove that $x \in L \iff \#acc_M(x) > 2^{p(n)}$. Let $x \in L$. Since M has at least one accepting path on x. Since M', by construction, creates an imbalance (in count) between the number of accepting and rejecting paths precisely when M accepts, the number of accepting paths in M' will be more than the number of rejecting paths. Thus $\#acc_M(x) > 2^{p(n)}$.
If $x \notin L$, all paths reject in M on input x, and thus number of accepting and rejecting paths in M' on input x are exactly equal and each of themsame is equal to $2^{p(n)}$. □

A similar proof will also show that coNP is contained in PP. Thus we have the following containment relationship among different classes of languages

$$\begin{tikzcd}
\text{PSPACE} \arrow[d, dash] \\
\text{PP} \arrow[d, dash] \\
\text{NP} \arrow[d, dash] \\
\text{P}
\end{tikzcd}$$

Now we will prove the main theorem of the lecture which characterizes the FP vs #P question in the decision world.

THEOREM 5.3. $(\#\text{P} = \text{FP}) \iff (\text{PP} = \text{P})$

Proof. ($\Rightarrow$) Assume #P = FP, our aim is to prove PP = P. The reverse containment follows since we know NP $\subseteq$ PP by proposition5.2. Now we show that PP $\subseteq$ P . Let $L \in$ PP via a machine M running in time $p(n)$ for some polynomial p, such that $x \in L \iff \#acc_M(x) > 2^{p(n)-1}$. Define the function, $f(x) = \#acc_M(x)$. By definition $f \in$ #P and hence $f \in$ FP. There is a deterministic polynomial time Turing machine N which on input x outputs the value of $f(x)$. Given an x, to test whether it is in L or not, it suffices to test whether whether the MSB of the binary representation of $f(x)$ is 1 or not. This can be done by simply running the machine N on x and testing the MSB of the output. Hence $L \in P$.

($\Leftarrow$) Assume PP = P, our aim is to prove #P = FP. The reverse direction is easy. FP $\subseteq$ #P as we argued in the last lecture. To show the forward direction, let $f \in$ #P via

M such that $\forall x \in \Sigma^*$, $f(x) = \#accept_M(x)$. Note that the naive approach to compute $f(x)$ is to compute the number of accepting paths in computation tree of height $p(n)$ will take exponential time. But it suffices to find the minimum $0 \leq k \leq 2^{p(n)}$ such that :

$$k + \#acc_M(x) > 2^{p(n)} \tag{5.1}$$

Indeed, the minimum is achieved when $k = k_{min} = 2^{p(n)} - \#acc_M(x) + 1$. Thus $\#acc_M(x) = 2^{p(n)} - k_{min} + 1$. This is an indirect way of finding out $\#acc_M(x)$ and we are moving towards using our assumption that PP = P.

We have search problem in hand; to search for the minimum $k(k_{min})$ satisfying equation (5.1). We do this by binary search over the range $0 \leq k \leq 2^{p(n)}$. We solve the decision problem first. Given x and k, check if $k + \#acc_M(x) > 2^{p(n)}$.

For this, we construct an another Turing machine N such that the number of accepting paths is exactly $k + \#acc_M(x)$ and the total number of paths is $2^{p(n)+1}$. Assume such a construction exists. Define a language $A \subset \Sigma^*$ such that $x \in A \iff k + \#acc_M(x) > 2^{p(n)}$. Thus $x \in A \iff \#acc_N(x) > 2^{p(n)}$. This implies that $A \in PP$ (via the machine N !) and hence $A \in P$ by assumption. Given $x \in \Sigma^*$ we can test if $k + \#acc_M(x) > 2^{p(n)}$ by testing if $x \in A$ or not, which can be done in polynomial time. Now we can do this construction and simulation through a binary search in order to find the minimum value of k that satisfies our inequality.

To complete the proof, we give the construction of N (for a given k). We first construct a machine M_k that runs in time $p(n)$ and has exactly k accepting paths. We slightly modify the idea in the previous lecture to do this. Let $\ell = \lceil \log k \rceil$. The machine M_k guesses a $y \in \{0,1\}^{p(n)}$ and *accepts* if the first ℓ bits of y in binary represents a number less than k and the last $p(n) - \ell$ bits is all-zero (lexicographically first path) and *reject* otherwise.

We combine M_k and N (both using exactly $p(n)$ non-deterministic bits), to get the machine N. The machine N on input x guesses 1 bit and on the 0-branch it simulates M_k on x and on the 1-branch it simulates M on x. Clearly $\#acc_N(x) = \#acc_{M_k}(x) + \#acc_M(x) = k + \#acc_M(x)$. The length of each path is exactly $p(n) + 1$ and hence total number of paths is $2^{p(n)+1}$. $\square$