

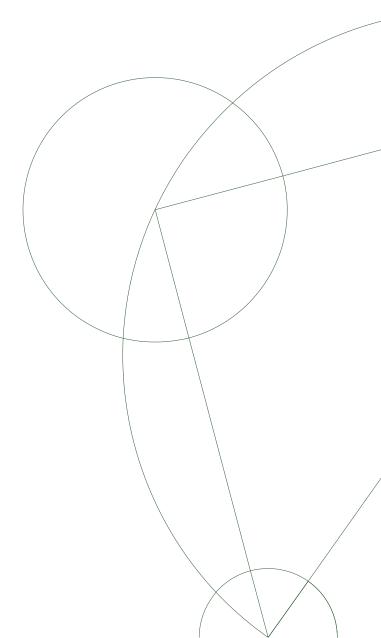
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Analysis of shape spaces

PROJECT OUTSIDE OF THE COURSE SCOPE
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NOVEMBER 9, 2018



	Abstract	
Abstract about analysis of shapes.		

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Introduction

Intro

1. Preliminaries

1.1 Notation

In the following M denotes a smooth manifold and TM is the tangent bundle of M and $\mathcal{T}(M)$ denotes the space of all vector fields on M. For $I \subset \mathbb{R}$, $\gamma : I \to M$ is a curve in M, i.a. a smooth map.

1.2 Connections

To consider the geodesic distance between two points in a manifold, geodesics need to be defined in a coordinate-invariant way such that the distance is independent of the coordinate charts. One property of geodesics in a Euclidean space, straight lines, is that they have acceleration 0. In order to make sense of acceleration of a curve in a manifold, we need to be able to compute "differences" between tangent spaces along the curve. Connections are exactly a way of making computations between tangent spaces possible - they allow us to differentiate vector fields along curves.

Since our use of connections is to define geodesics, we define connections in the tangent bundle of a manifold (instead of defining them generally on smooth sections of vector bundles) following chapter 4 of Lee (1997).

Definition 1.2.1. A connection in TM is a map

$$\Delta : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M),$$

written $(X,Y) \mapsto \Delta_X Y$ satisfying (for $f, g \in C^{\infty}(M)$ and $a, b \in \mathbb{R}$);

- a) $\Delta_{fX_1+gX_2}Y = f\Delta_{X_1}Y + g\Delta_{X_2}Y$ (linearity over $C^{\infty}(M)$ in X)
- b) $\Delta_X(aY_1 + bY_2) = a\Delta_X Y_1 + b\Delta_X Y_2$ (linearity over \mathbb{R} in Y)
- c) $\Delta_X(fY) = f\Delta_X Y + (Xf)Y$ product rule

In accordance with connections allowing "differences" between tangent spaces, $\Delta_X Y$ is called the *covariant derivative of* Y *in the direction of* X. To use connections to derivate along curves, we need the definition of a vector field along a curve, which is a smooth map $V: I \to TM$ such that $V(t) \in T_{\gamma(t}M$ for all $t \in I$. The prime example of a vector field along a curve is its velocity, $\dot{\gamma}(t) \in T_{\gamma(t}M$, which acts on functions, $f \in C^{\infty}(M)$, by

$$\dot{\gamma}(t)f = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t).$$

We denote by $\mathcal{T}(\gamma)$ all vector fields along γ . To define geodesics all we now need is to define what is means to take the covariant derivative of $V \in \mathcal{T}(\gamma)$ along γ . This covariant derivative is noted $D_t V$ and is has the following properties.

Lemma 1.2.2. Let Δ be a linear connection on M. For each $\gamma: I \to M$, Δ determines a unique operator

$$D_t: \mathcal{T}(\gamma) \to \mathcal{T}(\gamma),$$

satisfying (for $f, g \in C^{\infty}(I)$ and $a, b \in \mathbb{R}$);

a)
$$D_t(aV + bW) = aD_tV + bD_tW$$
 (linearity over \mathbb{R}

b)
$$D_t(fV) = \dot{f}V + fD_tV$$
 (product rule)

c) If V is extendible, then for any extension \tilde{V} of V, $D_tV(t) = \Delta_{\dot{\gamma}(t)}\tilde{V}$.

Proof.

Proof of Lemma 4.9 in Lee (1997)

$$\Box$$

V is said to be extendible if it can be constructed by any vector field on M, \tilde{V} by letting $V(t) := \tilde{V}_{\gamma(t)}$. This is not always the case; if V is the velocity of an intersecting curve γ with different covariant derivative at the intersection times. The covariant derivative of the velocity of a curve is now used to define a geodesic.

Definition 1.2.3. Let Δ be a linear connection on M and γ a curve in M. The acceleration of γ is $D_t\dot{\gamma}(t)$. If this vector field is zero, $D_t\dot{\gamma}(t) \equiv 0$, then $\gamma(t)$ is a geodesic with respect to Δ

It follows from Theorem 4.10 in Lee (1997) that for any manifold, M, with a linear connection, for any $p \in M$ and $V \in T_pM$ and $t_0 \in \mathbb{R}$ there exists an un-extendable geodesic, $\gamma_V : I \to M$, with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. The geodesic is called the (maximal) geodesic with initial value p and initial velocity V.

In this construction of geodesics the only necessary structure of M is that it should be a smooth manifold. When M is also equipped with a Riemannian metric, making M a Riemannian manifold, the choice of connection (determining the geodesics) should in some way respect the metric. Geodesics resulting from this specific choice of connection are called $Riemannian\ geodesics$.

Bibliography

Lee, John M (1997). $Riemannian\ Manifolds.$ Vol. 1. Springer.