Almost local metrics

The problem at hand is the following: we wish to construct a notion of distance between two points in $B_e(S^1, \mathbb{R}^2)$ by defining a metric, such that the distance between two points is the length of geodesics between the points. As we have seen, the distance induced by the L^2-metric vanishes on $B_e(S^1, \mathbb{R}^2)$, so we seek to define metrics, which do not vanish. One type of such metrics is *almost local metrics*, which, given $f \in B_e(S^1, \mathbb{R}^2)$, are metrics of the form

$$G_f^\Phi(h,k) = \int_M \Phi(\mathrm{Vol}(f), H_f, K_f)\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}_{>0}$ is smooth, $\mathrm{Vol}(f) = \int_M \mathrm{vol}(f^*\bar{g})$ is the total volume of $f(M)$, H_f is the mean curvature of f and K_f is the Gauss curvature of f. Both H_f and K_f are local invariant properties with respect to the Riemanninan metric, defined to be the trce and the determinant of the Weingarten mapping, respectively, and so Φ is often chosen to only depend on one of the two curvatures.

The total volume of f, $\mathrm{Vol}(f)$, is defined via the volume form induced by the pullback metric, $f^*\bar{g}$, so this definition of almost-local metrics only applies to manifolds which posses a volume form. All compact, oriented manifolds do this (Reference?), and almost local metrics are often defined for this class of manifolds. In the case of $f \in B_e(S^1, \mathbb{R}^2)$, the volume form on S^1 induced by f, is given by $\mathrm{vol}(f^*\bar{g}) = |f_\theta|\, d\theta$. (Reference to Riemannian Geometries on Spaces of PLane Curves 2.2)

$\mathrm{Vol}(f)$ is a non-local property of f, and thus the metrics are not only dependent on the local properties, K_f, H_f, but must be *almost* local metrics.

Note that if Φ depends only on f through $\mathrm{Vol}(f)$ then $G_f^\Phi(h,k)$ is equal to the L^2-metric (up to a constant) (is this obvious from our definition of the L^2 metric ?). But if Φ actually depends on either curvature and the total volume, then point-separation is achieved under certain conditions imposed on Φ;

Theorem 0.1. *If $\Phi(\mathit{Vol}(f), H_f, K_f) \geq AH_f$ for some $A > 0$, then G_f^Φ induces a point-separating metric on $B_e(S^1, \mathbb{R}^2)$.*

Proof.
Reference - perhaps explain heuristically? ☐