Almost local metrics

The problem at hand is the following: we wish to construct a notion of distance between two points in $B_e(S^1, \mathbb{R}^2)$ by defining a metric, such that the distance between two points is the length of geodesics between the points. As we have seen, the distance induced by the L^2-metric vanishes on $B_e(S^1, \mathbb{R}^2)$, so we seek to define metrics, which do not vanish. One type of such metrics is *almost local metrics*, which, given $f \in B_e(S^1, \mathbb{R}^2)$, are metrics of the form

$$G_f^\Phi(h,k) = \int_{S^1} \Phi(\mathrm{Vol}(f), H_f, K_f)\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}_{>0}$ is smooth, $\mathrm{Vol}(f) = \int_{S^1} \mathrm{vol}(f^*\bar{g})$ is the total volume of $f(S^1)$, H_f is the mean curvature of f and K_f is the Gauss curvature of f. Both H_f and K_f are local invariant properties with respect to the Riemanninan metric, defined to be the trce and the determinant of the Weingarten mapping, respectively, and so Φ is often chosen to only depend on one of the two curvatures. Φ can also be seen as map from $\mathrm{Imm}(S^1, \mathbb{R}^2)$ to $C^\infty(S^1, \mathbb{R}_{>0}$. When viewed as such, in order for the metric to be invariant under reparametrizations, Φ must also be equivariant with respect to the action of the diffeomorphism group, $\mathrm{Diff}(S^1)$ - i.e. $\Phi(f \circ \varphi) = \Phi(f) \circ \varphi$ for $\varphi \in \mathrm{Diff}(S^1)$.

The total volume of f, $\mathrm{Vol}(f)$, is defined via the volume form induced by the pullback metric, $f^*\bar{g}$, so this definition of almost-local metrics only applies to manifolds which posses a volume form. All compact, oriented manifolds do this (Reference?), and almost local metrics are often defined for this class of manifolds. In the case of $f \in B_e(S^1, \mathbb{R}^2)$, the volume form on S^1 induced by f, is given by $\mathrm{vol}(f^*\bar{g}) = |f_\theta|\, d\theta$. (Reference to Riemannian Geometries on Spaces of Plane Curves 2.2)

$\mathrm{Vol}(f)$ is a non-local property of f, and thus the metrics are not only dependent on the local properties, K_f, H_f, but must be *almost* local metrics.

Note that if Φ depends only on f through $\mathrm{Vol}(f)$ then $G_f^\Phi(h,k)$ is equal to the L^2-metric (up to a constant) (is this obvious from our definition of the L^2 metric ?). But if Φ actually depends on either curvature and the total volume, then point-separation is achieved under certain conditions imposed on Φ;

Theorem 0.1. *If $\Phi(Vol(f), H_f, K_f) \geq AH_f$ for some $A > 0$, then G_f^Φ induces a point-separating metric on $B_e(S^1, \mathbb{R}^2)$.*

Proof.
Reference - perhaps explain heuristically? □

No matter the choice of Φ, an almost local metric is never point-separating on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ - the shape space without quotienting out reparametrizations. To see this let $f \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ and take $\tilde{f}$ to be in the orbit of f of the $\mathrm{Diff}(S^1)$ action - i.e. $\tilde{f} = \varphi \circ f$ for some $\varphi \in \mathrm{Diff}(S^1)$. Since Φ is equivariant w.r.t. the action of $\mathrm{Diff}(S^1)$,

$$G^{\Phi}_{\tilde{f}}(h,k) = \int_{S^1} \Phi(\tilde{f})\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}) = \int_{S^1} \Phi(f) \circ \varphi \bar{g}(h,k)\mathrm{vol}(f^*\bar{g}),$$

the almost local metric restricted to the orbit of f can be viewed as a weighted L^2-type metric with weights represented by $\Phi(f) \circ \varphi$. As the geodesic distance function induced by weighted L^2 metrics vanishes (Reference or follows easily from proof?), the almost local metric vanishes for point in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ which are in the same orbit of the $\mathrm{Diff}(S^1)$-action.

In general, existence and uniqueness of geodesics w.r.t. almost local metrics are not ensured and thus the length of a path in $B_e(S^1, \mathbb{R}^2)$ cannot be determined by constructing a geodesic and computing its length. In certain cases however, the length of a path is exactly the lower bound used in 0.1 Reference to theorem 3.1 in H0 type Riemannian metrics on the space of planar curves.

Example 0.2
Define an almost local metric on $B_e(S^1, \mathbb{R}^2)$ as above with $\Phi(f) = \ell(f)$ where $\ell(f)$ is the ordinary curve length of f (which implicit is a function of the curvatures of f). Let $q_0, q_1 \in B_e(S^1, \mathbb{R}^2)$ be shapes and let $c : [0,1] \to B_e(S^1, \mathbb{R}^2)$ be a path from q_0 to q_1 such that $c(0) = q_0$ and $c(1) = q_1$. The length of the path c is then the area swept out by c in $\mathbb{R}^2$,

$$L_{G^\Phi}(c) = \int_{S^1} \int_{[0,1]} |\det dc(t,\theta)| \, dt d\theta,$$

and the distance between q_0 and q_1 is then the infimum over all paths in $B_e(S^1, \mathbb{R}^2)$ which start in q_0 and end in q_1:

$$d_{G^\Phi}(q_0, q_1) = \inf_{c \in \mathcal{C}} \int_{S^1} \int_{[0,1]} |\det dc(t,\theta)| \, dt d\theta,$$

where $\mathcal{C}$ denotes all paths c, such that $c(0) = q_0$ and $c(1) = q_1$.

Example 0.3
If Φ is a more general function of the curve length, $\Phi = e^{A\ell(f)}$, for some constant $A > 0$, then the distance between two shapes, q_0 and q_1, is bounded by

$$\inf_{c \in \mathcal{C}} \sqrt{Ae} \int_{S^1} \int_{[0,1]} |\det dc(t,\theta)| \, dt d\theta \leq d_{G^\Phi}(q_0, q_1) \leq \inf_{c \in \mathcal{C}} \sqrt{Ae} e^{A\ell_{max}(c)/2} \int_{S^1} \int_{[0,1]} |\det dc(t,\theta)| \, dt d\theta,$$

where $\ell_{max}(c) = \max_{t \in [0,1]} \ell(c(t, \cdot))$ is the maximum length of any immersion on the path from q_0 to q_1. In particular, if $q_0 \neq q_1$, such that there exists no trivial path between the two shapes, then the distance is positive, since the area swept out in $\mathbb{R}^2$ by any path is positive.