Almost local metrics

The problem at hand is the following: we wish to construct a notion of distance between two points in $B_e(S^1, \mathbb{R}^2)$ by defining a metric, such that the distance between two points is the length of geodesics between the points. As we have seen, the distance induced by the L^2-metric vanishes on $B_e(S^1, \mathbb{R}^2)$, so we seek to define metrics, which do not vanish. One type of such metrics is *almost local metrics*, which, given $f \in B_e(S^1, \mathbb{R}^2)$, are metrics of the form

$$G_f^{\Phi}(h,k) = \int_{S^1} \Phi(\mathrm{Vol}(f), H_f, K_f)\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}_{>0}$ is smooth, $\mathrm{Vol}(f) = \int_{S^1} \mathrm{vol}(f^*\bar{g})$ is the total volume of $f(S^1)$, H_f is the mean curvature of f and K_f is the Gauss curvature of f. Both H_f and K_f are local invariant properties with respect to the Riemanninan metric, defined to be the trace and the determinant of the Weingarten mapping, respectively, and so Φ is often chosen to only depend on one of the two curvatures. In the case of $f \in B_e(S^1, \mathbb{R}^2)$, $H_f(\theta) = \frac{\det(f_\theta, f_{\theta\theta})}{|f_\theta|^3}$, which is just the usual formula for curvature of a plane curve.

Φ can also be seen as map from $\mathrm{Imm}(S^1, \mathbb{R}^2)$ to $C^\infty(S^1, \mathbb{R}_{>0})$. When viewed as such, in order for the metric to be invariant under reparametrizations, Φ must also be equivariant with respect to the action of the diffeomorphism group, $\mathrm{Diff}(S^1)$ - i.e. $\Phi(f \circ \varphi) = \Phi(f) \circ \varphi$ for $\varphi \in \mathrm{Diff}(S^1)$.

The total volume of f, $\mathrm{Vol}(f)$, is defined via the volume form induced by the pullback metric, $f^*\bar{g}$, so this definition of almost-local metrics only applies to manifolds of embeddings from manifolds which posses a volume form. All compact, oriented manifolds do this, such as S^1, (Reference?), and almost local metrics are often defined for embeddings from this class of manifolds to $\mathbb{R}^n$. In the case of $f \in B_e(S^1, \mathbb{R}^2)$, the volume form on S^1 induced by f, is given by $\mathrm{vol}(f^*\bar{g}) = |f_\theta|\, d\theta$. (Reference to Riemannian Geometries on Spaces of Plane Curves 2.2). In our case, almost local metrics therefore take on the form

$$G_f^{\Phi}(h,k) = \int_{S^1} \Phi(\mathrm{Vol}(f), H_f, K_f)\bar{g}(h,k)\,|f_\theta|\, d\theta.$$

$\mathrm{Vol}(f)$ is a non-local property of f, and thus the metrics are not only dependent on the local properties, K_f, H_f, but must be *almost* local metrics.

Remark 0.1
Both curvatures and the volume form of $f \in B_e(S^1, \mathbb{R}^2)$ take on a particular nice form, but expressions can also be found for the general case where $f \in B_e(M, \mathbb{R}^n)$ with M a compact orientable $n-1$ dimensional manifold. This is done by using the Levi-Civita connections of the Riemannian manifolds $(\mathbb{R}^n, \bar{g})$ and (M, G^Φ) to construct the Weingarten mapping. See sections 3.4 and 3.9 of Almost local metrics on shape space of hypersurfaces in n-space

Note that if Φ depends only on f through $\mathrm{Vol}(f)$ then $G_f^{\Phi}(h,k)$ is equal to the L^2-metric (up to a constant) (is this obvious from our definition of the L^2 metric ?). But if Φ actually depends on either curvature and the total volume, then point-separation is achieved under certain conditions imposed on Φ;

Theorem 0.2. *If $\Phi(\mathit{Vol}(f), H_f, K_f) \geq AH_f$ for some $A > 0$, then G_f^{Φ} induces a point-separating metric on $B_e(S^1, \mathbb{R}^2)$.*

Proof.
The proof is found in section 3 in Reference til Riemannian Geometries... with a specific choice of Φ. We sketch a few ideas of this proof but emphasize that the specific choice of Φ is not important (is this actually true? Don't think so. It uses $\Phi \geq 1$. But that is just a scalar condition w.r.t to the L^2-metric???), but merely that $\Phi(f) > AH_f$ for some $A > 0$.

Given a path of un-parametrized shapes, $\pi(c) : [0,1] \times S^1 \to B_e(S^1, \mathbb{R}^2)$, one can choose a path, c, in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ such that $c(0,\cdot)$ is an immersion of constant speed, $\langle c_t, c_\theta \rangle = 0$ for all t and θ, and $c(t,\theta)$ has constant speed. Let c be such a path, and let

$$\Phi(f) = 1 + AH_f^2$$

for some constant $A > 0$ (which implies $\Phi(f) \geq AH_f$). Consider the Hilbert space $L^2(S^1, |c_\theta(t,\theta)|\, d\theta) = L^2(S^1, \mathrm{vol}(c(t)^*\bar{g}))$. The Cauchy-Schwarz inequality yields

$$\int_{S^1} |c_t(t,\theta)|\, |c_\theta(t,\theta)|\, d\theta \leq \left(\int_{S^1} |c_\theta(t,\theta)|\, d\theta \right)^{\frac{1}{2}} \left(\int_{S^1} |c_t(t,\theta)|^2\, |c_\theta(t,\theta)|\, d\theta \right)^{\frac{1}{2}}.$$

The length of the path c is then

$$\begin{aligned} L_{G^\Phi}(c) := \int_0^1 \sqrt{G^{\Phi}_{c(t)}(c_t, c_t)} dt &= \int_0^1 \left(\int (1 + AH^2_{c(t)})\, |c_t(t,\theta)|^2\, |c_\theta(t,\theta)|\, d\theta \right)^{\frac{1}{2}} dt \\ &\geq \int_0^1 \left(\int_{S^1} |c_\theta(t,\theta)|\, d\theta \right)^{-\frac{1}{2}} \int_{S^1} |c_t(t,\theta)|\, |c_\theta(t,\theta)|\, d\theta dt. \end{aligned}$$

The mean value theorem for integrals then yields that there exists $t_0 \in [0,1]$ such that

$$L_{G^\Phi}(c) \geq \left(\int_{S^1} |c_\theta(t_0,\theta)|\, d\theta \right)^{-\frac{1}{2}} \int_0^1 \int_{S^1} |c_t(t,\theta)|\, |c_\theta(t,\theta)|\, d\theta dt,$$

where the first factor is the curve length of $c(t_0,\cdot)$ to the power of $-\frac{1}{2}$, and the second factor can be written as

$$\int_0^1 \int_{S^1} |c_t(t,\theta)|\, |c_\theta(t,\theta)|\, d\theta dt = \int_0^1 \int_{S^1} |\det(dc(t,\theta)|\, d\theta dt,$$

which is the area in $\mathbb{R}^2$ swept out by the path c (make a figure). We note that if the shape is not trivially a point in $\mathbb{R}^2$ (such that the length at time t_0 is 0) and if the path is not trivial (such that $c(0,\cdot) = c(1,\cdot)$), then this lower bound is strictly positive. Thus any path from two distinct shapes have length greater than 0, such that the metric induces a point-separating distance function. $\square$

No matter the choice of Φ, an almost local metric is never point-separating on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ - the shape space without quotienting out reparametrizations. To see this let $f \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ and take $\tilde{f}$ to be in the orbit of f of the $\mathrm{Diff}(S^1)$ action - i.e. $\tilde{f} = \varphi \circ f$ for some $\varphi \in \mathrm{Diff}(S^1)$. Since Φ is equivariant w.r.t. the action of $\mathrm{Diff}(S^1)$,

$$G^{\Phi}_{\tilde{f}}(h,k) = \int_{S^1} \Phi(\tilde{f})\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}) = \int_{S^1} \Phi(f) \circ \varphi\bar{g}(h,k)\mathrm{vol}(f^*\bar{g}),$$

the almost local metric restricted to the orbit of f can be viewed as a weighted L^2-type metric with weights represented by $\Phi(f) \circ \varphi$. As the geodesic distance function induced by weighted L^2 metrics vanishes (Reference or follows easily from proof?), the almost local metric vanishes for point in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ which are in the same orbit of the $\mathrm{Diff}(S^1)$-action.

In general, existence and uniqueness of geodesics w.r.t. almost local metrics are not ensured and thus the length of a path in $B_e(S^1, \mathbb{R}^2)$ cannot be determined by constructing a geodesic and computing its length (Reference to first article page 11). In certain cases however, the length of a path is exactly the lower bound used in 0.2 Reference to theorem 3.1 in H0 type Riemannian metrics on the space of planar curves.

Example 0.3
Define an almost local metric on $B_e(S^1, \mathbb{R}^2)$ as above with $\Phi(f) = \ell(f)$ where $\ell(f)$ is the ordinary curve length of f (which implicit is a function of the curvatures of f). Let $q_0, q_1 \in B_e(S^1, \mathbb{R}^2)$ be shapes and let $c : [0,1] \to B_e(S^1, \mathbb{R}^2)$ be a path from q_0 to q_1 such that $c(0) = q_0$ and $c(1) = q_1$. The length of the path c is then the area swept out by c in $\mathbb{R}^2$,

$$L_{G^\Phi}(c) = \int_{[0,1]} \int_{S^1} |\det dc(t,\theta)| \, d\theta dt,$$

and the distance between q_0 and q_1 is then the infimum over all paths in $B_e(S^1, \mathbb{R}^2)$ which start in q_0 and end in q_1:

$$d_{G^\Phi}(q_0, q_1) = \inf_{c \in \mathcal{C}} \int_{[0,1]} \int_{S^1} |\det dc(t,\theta)| \, d\theta dt,$$

where $\mathcal{C}$ denotes all paths c, such that $c(0) = q_0$ and $c(1) = q_1$.

Example 0.4
If Φ is a more general function of the curve length, $\Phi = e^{A\ell(f)}$, for some constant $A > 0$, then the distance between two shapes, q_0 and q_1, is bounded by

$$\inf_{c \in \mathcal{C}} \sqrt{Ae} \int_{[0,1]} \int_{S^1} |\det dc(t,\theta)|\, d\theta dt \le d_{G^\Phi}(q_0, q_1) \le \inf_{c \in \mathcal{C}} \sqrt{Ae} e^{A\ell_{max}(c)/2} \int_{[0,1]} \int_{S^1} |\det dc(t,\theta)|\, d\theta dt,$$

where $\ell_{max}(c) = \max_{t \in [0,1]} \ell(c(t,\cdot))$ is the maximum length of any immersion on the path from q_0 to q_1. In particular, if $q_0 \neq q_1$, such that there exists no trivial path between the two shapes, then the distance is positive, since the area swept out in $\mathbb{R}^2$ by any path is positive.