1 The manifold of curves

A 2-dimensional shape can be thought of as a closed (smooth) curve in $\mathbb{R}^2$. Thus, we want to define a manifold structure on the space of these curves. Doing this mathematically correct is rather technical because we need to consider quotient spaces of infinite dimensional manifolds. In our exposition we shall to a large extend "define our way out of this" by, for example, defining tangent vectors of curves instead of deducing how these look like from the formal definition of the underlying manifold. We shall motivate our definitons geometrically and then deduce some properties from these definitons – properties, which can also be deduced from the formal definitions. At the end of this section we briefly address what we miss with our more informal treatment.

A shape in $\mathbb{R}^2$ can either be thought of as a parametrized object, e.g., as a function $\mathbb{S}^1 \ni \theta \mapsto c(\theta) \in \mathbb{R}^2$; but it can also be thought of as an unparametrized object, e.g., the *image* of such a function $\mathrm{Im}(c) \subset \mathbb{R}^2$. We impose some smoothness structure on the curves and define the spaces we want to consider formally as

$$\mathrm{Imm} := \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2) := \{...?\},$$
$$\mathcal{I} := \mathcal{I}(\mathbb{S}^1, \mathbb{R}^2) := \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)/\mathrm{Diff}(\mathbb{S}^1).$$

Imm is the space of *immersion* of the unit circle into $\mathbb{R}^2$, and $\mathcal{I}$ is then this space modulo reparametrization, i.e., we identify two object $q, p \in \mathrm{Imm}$ in $\mathcal{I}$ if $q = p \circ \varphi$, where $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$ is a diffeomorphism on the unit circle.

When our interest is upon the *shape*, we are not really interested in the underlying parametrization of this shape, and so we are mostly interested in the space $\mathcal{I}$. However, it is easiest to construct a manifold structure on Imm and then deduce one on $\mathcal{I}$, so we start by considering the parametrized space of immersions.

The manifold of parametrized curves – $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)$

... Intuitively, we want to consider the space of all (smooth) closed curves in $\mathbb{R}^2$. This can be seen as the space of all submanifolds in $\mathbb{R}^2$ which are diffeomorphic to the unit circle $\mathbb{S}^1$. If we let $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ denote the space of all *immersion* from the inut circle into the plane, we can define the space we want to consider as

$$B := \left\{ q(\mathbb{S}^1) \mid q \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2) \right\}. \tag{1}$$

Here we simply think of $q(\mathbb{S}^1) \subset \mathbb{R}^2$ as a subspace and forget about the actual map q. (Keeping this mapping in mind we could define the space in another way; but this is not so important right now.)

The tangent space of B. For ordinary finite dimensional manifolds B, Lee defines the tangent space at a point $p \in B$ through the notation of *derivations*; this is a rather abstract construction, but is nice to work with. Using this, one can define the notion of a tanget vector to a path in B passing through the point p. Then, one can define an equivalence relation on the space of such paths and obtain an equivalent definition of the tanget space, which is more intuitive. One can also work the other way around and start by defining the notion of tanget vector to paths in B (as is done on Wiki). We briefly do that here:

For a neighbourhood $U \subset B$ containing p we have a smooth coordinate chart $\varphi\colon U \to \mathbb{R}^n$. For a path $\gamma\colon [-\varepsilon, \varepsilon] \to B$ with $\varphi(0) = p$, it makes perfect sense to consider differentiability of the map $\varphi \circ \gamma\colon [0,1] \to \mathbb{R}^n$. Now, the relation

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0),$$

defines an equivalence relation on the collection of all such paths γ. An equivalence class of such paths, denoted by $[\gamma_p']$ (or simply γ_p'), is called a *tangent vector* at the point p. The collection of all tanget vectors make up the tanget space T_pB at p.

Returning to B being as defined in (1), we now want to determine what the tangent spaces look like at a point $q \in B$ by following the construction above (taking for granted that B actually is a (Fréchet) manifold, and that the following definitions/constructions make sense). Firstly, a path in B is now a map $\gamma\colon [-\varepsilon, \varepsilon] \to B$ such that $\gamma(t)$ is a curve $q_t \in B$ for all $t \in [-\varepsilon, \varepsilon]$, with $\gamma(0) = q$. For each of the q_t we can think of them as parametrized by $x \mapsto q_t(x)$, $x \in \mathbb{S}^1$. Then, if the curve γ is such that for each fixed $x \in \mathbb{S}^1$ the map $t \mapsto q_t(x) \in \mathbb{R}^2$ is differentiable, we can define γ_q' as the mapping

$$\gamma_q' := x \mapsto \frac{\partial}{\partial t}\bigg|_{t=0} q_t(x).$$

Thus each γ_q' is a mapping from the unit circle into $\mathbb{R}^2$, which mean that we can think of a tangent vector to a curve $q \in B$ as a vector field on $\mathbb{S}^1$ (though of as $\subset \mathbb{R}^2$).

Defining a metric on the tangent space of curves 2

Definition 1.1 (The L2 metric on Imm)**.** The *L2 metric* G_c^2 at the point $c \in \mathrm{Imm}$ is defined as

$$G_c^2(h,k) := \int_{\mathbb{S}^1} \langle h(\theta), k(\theta) \rangle \, |c_\theta| \, \mathrm{d}\theta,$$

$h, k \in T_cB = C^\infty(\mathbb{S}^1, \mathbb{R}^2)$.

Because this metric is invariant under reparametrizations it also induces a metric on the quotient space $B = \mathrm{Imm}/\mathrm{Diff}(\mathbb{S}^1)$ and thus determines a length measure on the paths in B [define this]. We want to find an expression for this, using parametrized representatives from Imm. For a parametrized curve $c \in \mathrm{Imm}$ we write $C = \mathrm{Im}(c) =: \pi(c) \in B$ for the projection onto the quotient space. For any path $t \mapsto q(t, \cdot) \in \mathrm{Imm}$ running through the space of parametrized curves, this induces a path in B by simply projecting the path down to $t \mapsto \pi(q(t, \cdot))$.

In ... it is shown that ... We here give some intuitive, geometric arguments for why this should be true.

We know that a tangent vector h to a curve $c \in \mathrm{Imm}$ can be visualized as a vector field on the circle. By definition, this vector field is derived from a path $t \mapsto q(t, \cdot)$ in Imm as $h = q_t(0, \cdot)$. At each of the timepoints, t, we get a particular parametrized curve $\theta \mapsto q(t, \theta)$. We cannot directly use such a path to define tangent vectors of the *quotient space* B because this path is sensitive to reparametrizations: Consider a time-dependent reparametrization $\varphi(t, \theta)$, or, equivalently, a path $t \mapsto \varphi(t, \cdot)$ in $\mathrm{Diff}(\mathbb{S}^1)$; then the two paths $t \mapsto \pi(q(t, \cdot))$ and $t \mapsto \pi(q(t, \varphi(t, \cdot)))$ are identical in B but give rise to two different vector fields.

To make better sense of the tangents vectors of the quotient space, we use the following result.

Proposition 1.2. *For every path $t \mapsto q(t, \cdot)$ in Imm there exists a time-dependent reparametrization $t \mapsto \varphi(t, \cdot) \in \mathrm{Diff}(\mathbb{S}^1)$ such that the path*

$$t \mapsto r(t, \theta) := q(t, \varphi(t, \theta))$$

fulfills $\langle e_t, e_\theta \rangle = 0$ for all $(t, \theta) \in [0, 1] \times \mathbb{S}^1$, and such that $\varphi(0, \theta) = \theta$. Furthermore, it holds that

$$\varphi_t = a \circ \varphi = -\frac{\langle q_t \circ \varphi, q_\theta \circ \varphi \rangle}{|q_\theta \circ \varphi|^2}, \quad a := -\frac{\langle q_t, q_\theta \rangle}{|q_\theta|^2}. \tag{2}$$

Proof. todo or ref. □

Remark 1.3
Note that for every vector field $h \in T_c(\mathrm{Imm})$, determined from the path q, we can make a pointwise decomposition of h onto $q_\theta(0, \cdot)$ and $iq_\theta(0, \cdot)$ by using the pointwise orthogonal projection. Explicitly we have that

$$h = q_t = p_{q_\theta}(q_t) + p_{iq_\theta}(q_t),$$

where p is taken to be the standard *pointwise* $\mathbb{R}^2$ orthogonal projection, which is given as

$$p_v(u) = \frac{\langle v, u \rangle}{|v|^2} v, \quad u, v \in \mathbb{R}^2.$$

More correctly we should thus write

$$h(\theta) = q_t(0,\theta) = p_{q_\theta(0,\theta)}(q_t(0,\theta)) + p_{iq_\theta(0,\theta)}(q_t(0,\theta)).$$

From this we see that the time derivative of the reparametrization in the previous Proposition is the coefficient function for the projection onto the parameter derivative of the original path q; this becomes relevant in a moment.

(We need some sort of parametrization when we talk about tangent vectors of of B because it matters at "what point the arrows are attached".)

We can use this result to define tangent vectors to elements of B in a consistent way:

Definition 1.4. A *tangent vector* h to an element $C = \pi(c) \in B$ is defined as a vector fields obtained from some path $t \mapsto q(t,\cdot) \in \mathrm{Imm}$, with $\pi(q(0,\cdot)) = C$, by

$$h(\theta) = \frac{\partial}{\partial t}\bigg|_{t=0} q(t, \varphi(t,\theta)), \tag{3}$$

where φ is the reparametrization from Proposition 1.2.

First we note that this gives us the following visualization of the tangents spaces of B.

Proposition 1.5. *The tangent space to an element $C = \pi(c) \in B$ consists of orthonormal vector fields on the circle, i.e.,*

$$T_C(B) = \left\{ bic_\theta \mid b \in C^\infty(\mathbb{S}^1, \mathbb{R}) \right\}.$$

If the

Proof. This follows from Definition 1.4 and the property of the reparametrization φ. $\square$

By this construction, we now have a way of transforming a path $t \mapsto q(t,\cdot) \in \mathrm{Imm}$ into a path $q(t, \varphi(t,\cdot))$ such that every tangent vector in Imm along this path will also be a tanget vector in ... [YES, need to define this! – make it an analog construction of the one in Imm] ... In Imm we could just differentiate along a the path q to obtain tangents vectors at each point and then define the length of the path as the integral over the pointwise norm of these tangent vectors; we can not directly do the same when thinking of q as a path in B because we can not be sure that differentiating this path will yield a valid tangent vector in this space in accordance with Defition x. However, we can simply use the same horizontalization trick to make a suitably definition.

From the above defintion we also get an explicit formula for the length of a path in B with the metric $\mathcal{G}^2$ when this path is obtained from a path in Imm.

Proposition 1.6. *$\mathcal{L}$ is well-defined and for any representative $t \mapsto q(t,\cdot) \in$ Imm of the path $t \mapsto \tilde{q}(t) \in B$ the length can be calculated as*

$$\mathcal{L}(\tilde{q}) = \int_0^1 \left(\int_{\mathbb{S}^1} \frac{\langle q_t, iq_\theta \rangle^2}{|q_\theta|} \,\mathrm{d}\theta \right)^{\frac{1}{2}} \mathrm{d}t. \tag{4}$$

Proof. For ease of notation, write $q \circ \varphi$ to mean $q(t, \varphi(t,\theta))$ and so on during this proof. First, we shows that (4) implies that $\mathcal{L}$ is well-defined; so assume (4) holds and let $q(t,\cdot)$ and $p(t,\cdot)$ be two different representatives for $\tilde{q}(t)$. This means that we must have a reparametrization $\psi(t,\theta)$ such that

$$p(t, \psi(t,\theta)) = q(t,\theta).$$

Then

$$p_t = q_t \circ \psi + \psi_t (q_t \circ \psi), \quad p_\theta = \psi_\theta (q_\theta \circ \psi),$$

so

$$\begin{aligned} \langle p_t, ip_\theta \rangle &= \langle q_t \circ \psi + \psi_t (q_\theta \circ \psi), \psi_\theta (iq_\theta \circ \psi) \rangle \\ &= \langle q_t \circ \psi, \psi_\theta (iq_\theta \circ \psi) \rangle \\ &= (\langle q_t, iq_\theta \rangle \circ \psi) \psi_\theta, \end{aligned}$$

and thus

$$\int_{\mathbb{S}^1} \frac{\langle p_t, ip_\theta \rangle^2}{|p_\theta|} \,\mathrm{d}\theta = \int_{\mathbb{S}^1} \left(\frac{\langle q_t, iq_\theta \rangle^2}{|q_\theta|} \right) \circ \psi |\psi_\theta| \,\mathrm{d}\theta = \int_{\mathbb{S}^1} \frac{\langle q_t, iq_\theta \rangle^2}{|q_\theta|} \,\mathrm{d}\theta,$$

which shows that the length does not depend on the parametrization of the path.

Next, by construction, the tangent vectors along the path $\tilde{q}$ in B is given as

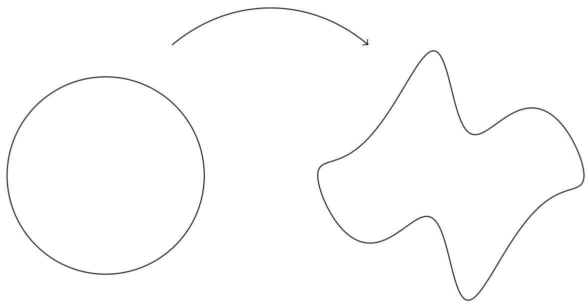
$$\frac{\partial}{\partial t}(q \circ \varphi) = q_t \circ \varphi + \varphi_t (q_\theta \circ \varphi)$$

Now, as in remark 1.3, decompose $q_t \circ \varphi$ by projecting pointwise onto $q_t \circ \varphi$ and $iq_t \circ \varphi$. Then we get

$$q_t \circ \varphi = \left(\frac{\langle q_t, iq_\theta \rangle}{|q_\theta|^2} iq_\theta \right) \circ \varphi + \left(\frac{\langle q_t, q_\theta \rangle}{|q_\theta|^2} q_\theta \right) \circ \varphi,$$

and by Proposition 1.2 we see that the last term cancels with $\varphi_t (q_\theta \circ \varphi)$, so

$$\frac{\partial}{\partial t}(q \circ \varphi) = \left(\frac{\langle q_t, iq_\theta \rangle}{|q_\theta|^2} iq_\theta \right) \circ \varphi.$$



For any fixed $t \in [0,1]$, the reparametrization φ is just an ordinary reparametrization of the curve $\theta \mapsto q(t,\theta)$, so by invariance of the metric we have that

$$\begin{aligned}
&G^2_{q(t,\varphi(t,\cdot))}\left(\frac{\partial}{\partial t}(q\circ\varphi), \frac{\partial}{\partial t}(q\circ\varphi)\right)\\
&\quad= G^2_{q(t,\varphi(t,\cdot))}\left(\left(\frac{\langle q_t, iq_\theta\rangle}{|q_\theta|^2} iq_\theta\right)\circ\varphi, \left(\frac{\langle q_t, iq_\theta\rangle}{|q_\theta|^2} iq_\theta\right)\circ\varphi\right)\\
&\quad= G^2_{q(t,\cdot)}\left(\frac{\langle q_t, iq_\theta\rangle}{|q_\theta|^2} iq_\theta, \frac{\langle q_t, iq_\theta\rangle}{|q_\theta|^2} iq_\theta\right)\\
&\quad= \int_{\mathbb{S}^1}\left\|\frac{\langle q_t, iq_\theta\rangle}{|q_\theta|^2} iq_\theta\right\|^2 |q_\theta|\,\mathrm{d}\theta\\
&\quad= \int_{\mathbb{S}^1}\frac{\langle q_t, iq_\theta\rangle^2}{|q_\theta|}\,\mathrm{d}\theta,
\end{aligned}$$

from which (4) follows immediately by definition. □

Test section 2