The manifold of curves

By (some reference) ...

The tangent space of curves

Constructing the space of closed curves. Intuitively, we want to consider the space of all (smooth) closed curves in $\mathbb{R}^2$. This can be seen as the space of all submanifolds in $\mathbb{R}^2$ which are diffeomorphic to the unit circle $\mathbb{S}^1$. If we let $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ denote the space of all *immersion* from the inut circle into the plane, we can define the space we want to consider as

$$B := \left\{ q(\mathbb{S}^1) \mid q \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2) \right\}. \tag{1}$$

Here we simply think of $q(\mathbb{S}^1) \subset \mathbb{R}^2$ as a subspace and forget about the actual map q. (Keeping this mapping in mind we could define the space in another way; but this is not so important right now.)

The tangent space of B. For ordinary finite dimensional manifolds B, Lee defines the tangent space at a point $p \in B$ through the notation of *derivations*; this is a rather abstract construction, but is nice to work with. Using this, one can define the notion of a tanget vector to a path in B passing through the point p. Then, one can define an equivalence relation on the space of such paths and obtain an equivalent definition of the tanget space, which is more intuitive. One can also work the other way around and start by defining the notion of tanget vector to paths in B (as is done on Wiki). We briefly do that here:

For a neighbourhood $U \subset B$ containing p we have a smooth coordinate chart $\varphi\colon U \to \mathbb{R}^n$. For a path $\gamma\colon [-\varepsilon, \varepsilon] \to B$ with $\varphi(0) = p$, it makes perfect sense to consider differentiability of the map $\varphi \circ \gamma\colon [0,1] \to \mathbb{R}^n$. Now, the relation

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0),$$

defines an equivalence relation on the collection of all such paths γ. An equivalence class of such paths, denoted by $[\gamma'_p]$ (or simply γ'_p), is called a *tangent vector* at the point p. The collection of all tanget vectors make up the tanget space T_pB at p.

Returning to B being as defined in (1), we now want to determine what the tangent spaces look like at a point $q \in B$ by following the construction above (taking for granted that B actually is a (Fréchet) manifold, and that the following definitions/constructions make sense). Firstly, a path in B is now a map $\gamma\colon [-\varepsilon, \varepsilon] \to B$ such that $\gamma(t)$ is a curve $q_t \in B$ for all $t \in [-\varepsilon, \varepsilon]$, with $\gamma(0) = q$.

For each of the q_t we can think of them as parametrized by $x \mapsto q_t(x)$, $x \in \mathbb{S}^1$. Then, if the curve γ is such that for each fixed $x \in \mathbb{S}^1$ the map $t \mapsto q_t(x) \in \mathbb{R}^2$ is differentiable, we can define γ_q' as the mapping

$$\gamma_q' := x \mapsto \frac{\partial}{\partial t}\bigg|_{t=0} q_t(x).$$

Thus each γ_q' is a mapping from the unit circle into $\mathbb{R}^2$, which mean that we can think of a tangent vector to a curve $q \in B$ as a vector field on $\mathbb{S}^1$ (though of as $\subset \mathbb{R}^2$).

Question / considerations / imprecisions. Above we made an intuitively reasonable construction, but ignored some of technicalities, which we list here.

- Equivalence classes: We should define the tangent vectors as equivalence classes of path in B; this should all determine a unique vector field.
- The manifold structure of B: We simply used the intuitive idea to differentiate in "time" for each fixed point on a curve, $q_t(x)$. However, as we have not specified the chart for the manifold B, it is not obvious that this construction corresponds to the one made in the finite dimensional case. Technically we would need a chart $\varphi \colon U \to F$, with F some Fréchet space and then show some sort of Fréchet-differentiability of the composite function $\varphi \circ \gamma$.
- We mentioned that in the finite dimensional case the two definitions (through derivates and tangents to curves, respectively) are equivalent. It is not obvious that this also hold in the infinite dimensional case.
- At the beginning we eliminated the knowledge of the parametrization of a curve $q \in B$ to make the definition of B simpler. However, we actually use a parametrization later, and thus we should make sure that reparametrizations does not matter for the construction of the tangent space. (It does not, as it would just move the vector field around $\mathbb{S}^1$ according to the reparametrization.)
- Are there any smoothness assumptions (or something) about vector field we need to validate? For example, just the fact that the map $t \mapsto q_t(x)$ behaves nice does of course *not* imply that also the derived vector field γ_q' behaves nicely in x – which is what we would need to get a smooth vector field (?).
- Is it correctly formulated that we need to think of $\mathbb{S}^1 \subset \mathbb{R}^2$ to make sense of a vector field on the circle?

Defining a metric on the tangent space of curves 1

This is the sort of argument they (Mumford and Michor) give in the beginning of section 3.2. Not sure how to exactly finish this.

Consider the following metric on Imm.

Definition 0.1 (The L2 metric on Imm)**.** The *L2 metric* G_c^2 at the point $c \in \mathrm{Imm}$ is defined as

$$G_c^2(h,k) := \int_{\mathbb{S}^1} \langle h(\theta), k(\theta) \rangle \, |c_\theta| \, \mathrm{d}\theta,$$

$h, k \in T_c B = C^\infty(\mathbb{S}^1, \mathbb{R}^2)$.

Adding the parametrization derivative c_θ makes this invariant under (orientation preserving?) reparametrizations. Because of that, this metric also induces a metric on the quotient space B, which we can identify as follows. Consider first the tangent space at c to the *orbit* of the diffeomorphism group

$$T_c(c \circ \mathrm{Diff}(\mathbb{S}^1)), \quad c \circ \mathrm{Diff}(\mathbb{S}^1) := \left\{ c \circ \varphi \mid \varphi \in \mathrm{Diff}(\mathbb{S}^1) \right\}.$$

A path in the orbit $c \circ \mathrm{Diff}(\mathbb{S}^1)$ reduces to

$$t \mapsto (\theta \mapsto (c \circ \varphi_t)(\theta)) .$$

Differentiating this with respect to t shows that we can identify

$$T_c(c \circ \mathrm{Diff}(\mathbb{S}^1)) = \left\{ g c_\theta \mid g \in C^\infty(\mathbb{S}^1, \mathbb{R}) \right\}. \tag{2}$$

The *normal space* $\mathcal{N}_c \subset T_c(\mathrm{Imm})$ consists of tangent vectors that are orthogonal to $T_c(c \circ \mathrm{Diff}(\mathbb{S}^1))$ with respect to G_c^2. As the projection

$$\pi \colon \mathrm{Imm} \to B = \mathrm{Imm}/\mathrm{Diff}(\mathbb{S}^1)$$

sends the whole orbit to a single point it follows that tangents vectors to the projected element $\pi(c) \in B$ can essentially be identified with elements of the normal space $\mathcal{N}_c$. From (2) we see that

$$\mathcal{N}_c = \left\{ g i c_\theta \mid g \in C^\infty(\mathbb{S}^1, \mathbb{R}) \right\},$$

with $i c_\theta = |c_\theta| n_c$, n_c being the unit normal vector field along c. Finally, for a given $h \in T_c(\mathrm{Imm})$ we can explicitly find the decomposition of h into the two orthogonal subspaces of the tangent space by projecting h onto c_θ and $i c_\theta$, which gives

$$\begin{aligned} h &= p_{c_\theta}(h) + p_{i c_\theta}(h) \in T_c(c \circ \mathrm{Diff}(\mathbb{S}^1)) \oplus \mathcal{N}_c, \\ p_{c_\theta}(h) &= \frac{\langle h, c_\theta \rangle}{|c_\theta|^2} c_\theta \in T_c(c \circ \mathrm{Diff}(\mathbb{S}^1)), \\ p_{i c_\theta}(h) &= \frac{\langle h, i c_\theta \rangle}{|c_\theta|^2} i c_\theta \in \mathcal{N}_c, \end{aligned}$$

with $p_v(u)$ denoting the standard orthogonal projection in $\mathbb{R}^2$.

This allows us to calculate a formula for the induced metric on B. In particular, it allows us to explicitly calculate the length of a projected path $t \mapsto \pi(q(t,\cdot)) \in B$ where $t \mapsto q(t,\cdot) \in \mathrm{Imm}$ is a path in Imm. In this setting, at each time point t the tangent vector is q_t so by the above arguments we get that the induced metric is

$$\begin{aligned} L^2(\pi(q)) &= \int_0^1 G^2_{q(t,\cdot)}(p_{iq_\theta}(q_t), p_{iq_\theta}(q_t))\,\mathrm{d}t \\ &= \int_0^1 \int_{\mathbb{S}^1} \left\langle \frac{\langle h, iq_\theta\rangle}{|q_\theta|^2} iq_\theta, \frac{\langle h, iq_\theta\rangle}{|q_\theta|^2} iq_\theta \right\rangle |q_\theta|\,\mathrm{d}\theta\,\mathrm{d}t. \end{aligned}$$

Defining a metric on the tangent space of curves 2

Definition 0.2 (The L2 metric on Imm)**.** The *L2 metric* G^2_c at the point $c \in \mathrm{Imm}$ is defined as

$$G^2_c(h,k) := \int_{\mathbb{S}^1} \langle h(\theta), k(\theta)\rangle\,|c_\theta|\,\mathrm{d}\theta,$$

$h, k \in T_cB = C^\infty(\mathbb{S}^1, \mathbb{R}^2)$.

Because this metric is invariant under reparametrizations it also induces a metric on the quotient space $B = \mathrm{Imm}/\mathrm{Diff}(\mathbb{S}^1)$ and thus determines a length measure on the paths in B [define this]. We want to find an expression for this, using parametrized representatives from Imm. For a parametrized curve $c \in \mathrm{Imm}$ we write $C = \mathrm{Im}(c) =: \pi(c) \in B$ for the projection onto the quotient space. For any path $t \mapsto q(t,\cdot) \in \mathrm{Imm}$ running through the space of parametrized curves, this induces a path in B by simply projecting the path down to $t \mapsto \pi(q(t,\cdot))$.

In ... it is shown that ... We here give some intuitive, geometric arguments for why this should be true.

We know that a tangent vector h to a curve $c \in \mathrm{Imm}$ can be visualized as a vector field on the circle. By definition, this vector field is derived from a path $t \mapsto q(t,\cdot)$ in Imm as $h = q_t(0,\cdot)$. At each of the timepoints, t, we get a particular parametrized curve $\theta \mapsto q(t,\theta)$. We cannot directly use such a path to define tangent vectors of the *quotient space* B because this path is sensitive to reparametrizations: Consider a time-dependent reparametrization $\varphi(t,\theta)$, or, equivalently, a path $t \mapsto \varphi(t,\cdot)$ in $\mathrm{Diff}(\mathbb{S}^1)$; then the two paths $t \mapsto \pi(q(t,\cdot))$ and $t \mapsto \pi(q(t,\varphi(t,\cdot)))$ are identical in B but give rise to two different vector fields.

To make better sense of the tangents vectors of the quotient space, we use the following result.

Proposition 0.3. *For every path $t \mapsto q(t,\cdot)$ in* Imm *there exists a time-dependent reparametrization $t \mapsto \varphi(t,\cdot) \in \mathrm{Diff}(\mathbb{S}^1)$ such that the path*

$$t \mapsto r(t,\theta) := q(t,\varphi(t,\theta))$$

fulfills $\langle e_t, e_\theta \rangle = 0$ for all $(t,\theta) \in [0,1] \times \mathbb{S}^1$. For the reparametrization φ it holds that $\varphi(0,\theta) = \theta$ and that

$$\varphi_t = a \circ \varphi, \quad a := -\frac{\langle q_t, q_\theta \rangle}{|q_\theta|^2}. \tag{3}$$

Proof.
todo or ref. □

Remark 0.4
Note that for every vector field $h \in T_c(\mathrm{Imm})$, determined from the path q, we can make a pointwise decomposition of h onto $q_\theta(0,\cdot)$ and $iq_\theta(0,\cdot)$ by using the pointwise orthogonal projection. Explicitly we have that

$$h = q_t = p_{q_\theta}(q_t) + p_{iq_\theta}(q_t),$$

where p is taken to be the standard *pointwise* $\mathbb{R}^2$ orthogonal projection, which is given as

$$p_v(u) = \frac{\langle v, u \rangle}{|v|^2} v, \quad u, v \in \mathbb{R}^2.$$

More correctly we should thus write

$$h(\theta) = q_t(0,\theta) = p_{q_\theta(0,\theta)}(q_t(0,\theta)) + p_{iq_\theta(0,\theta)}(q_t(0,\theta)).$$

From this we see that the time derivative of the reparametrization in the previous Proposition is the coefficient function for the projection onto the parameter derivative of the original path q; this becomes relevant in a moment.

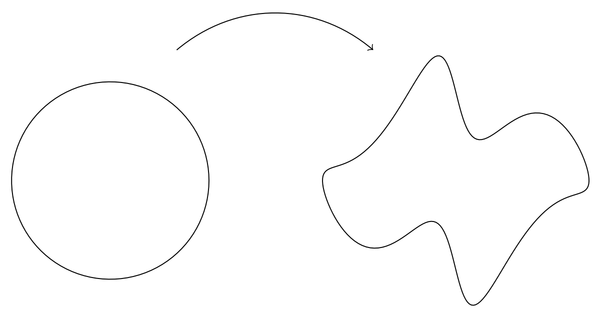
We can use this result to define tangent vectors to elements of B in a consistent way:

Definition 0.5. test

For some path q in Imm, reparametrize this according to Proposition 0.3; then take derivative as in the parametrized case of this new path. This leads us to the following result.

Proposition 0.6. *The tangent space to an element $C = \pi(c) \in B$ consists of orthonormal vector fields on the circle, i.e.,*

$$T_C(B) = \left\{ bic_\theta \mid b \in C^\infty(\mathbb{S}^1,\mathbb{R}) \right\} = C^\infty(\mathbb{S}^1,\mathbb{R}).$$



Proof.
We have justified why this should be the case in the previous discussion, by defining a consistent way to construct the tangent space. $\square$

Test section 2