0.1 The L2 metric vanishes

In the previous section we went through some work to construct a measure of length for a path q in the space $\mathcal{I}$. As the tangent spaces were seen to consist of some functions on the circle, it was natural to consider using a version of the L^2-metric (a version invariant to reparametrizations). However, as we illustrate in this section, this does not give rise to a usable notation of length, because the geodesic distance following from this metric becomes 0 for every two curves in the space.

It can seem weird to go through so much trouble to define a useless distance. But firstly, the construction illustrates some of the difficulties in defining a reasonable notion on length on $\mathcal{I}$; secondly, and more importantly, the fact that the distance vanishes is a quite surprising result, which depends crucially on the formula for the length of a path given in Proposition **??**. This gives us some justification for the somewhat unfruitfull work of definiting the L^2-metric.

The "proof" below is mostly heuristic, and we only consider the very simply case of a path transforming the circle into a larger circle, but this captures the main idea.

Theorem 1. *Let $a, b \in \mathcal{I}$. Then for every $\varepsilon > 0$ there exists a path q in $\mathcal{I}$ such that*

$$\mathcal{L}(q) < \varepsilon.$$

Corollary 2. *For all $a, b \in \mathcal{I}$*

$$\mathcal{D}(a, b) = 0.$$

Proof of Theorem 1. Start by rewriting the (**??**) from Proposition **??** as follows.

$$\begin{aligned}
\mathcal{L}(q) &= \int_0^1 \left(\int_{\mathbb{S}^1} \frac{\langle q_t, iq_\theta \rangle^2}{|q_\theta|} \, \mathrm{d} \right)^{1/2} \theta \, \mathrm{d}t \\
&= \int_0^1 \left(\int_{\mathbb{S}^1} \left(\frac{\langle q_t, iq_\theta \rangle}{|q_t||q_\theta|} \right)^2 |q_t|^2 |q_\theta| \, \mathrm{d}\theta \right)^{1/2} \mathrm{d}t \\
&= \int_0^1 \left(\int_{\mathbb{S}^1} \cos(\alpha(q_t, iq_\theta))^2 |q_t|^2 |q_\theta| \, \mathrm{d}\theta \right)^{1/2} \mathrm{d}t,
\end{aligned}$$

with $\alpha(x, y)$ denoting the angle between x and y. When constructing a zigzag-path the angle will be (approximately?) constant in θ and t, and it is given by

$$n = \tan(\alpha).$$

□

We have that

$$\cos(\arctan(n)) = (1+n^2)^{-1/2} = O(n^{-1}),$$

so we can write

$$L(\varphi) = O(n^{-2}) \int_0^1 \int_{\mathbb{S}^1} |\varphi_t|^2 |\varphi_\theta| \, \mathrm{d}\theta \, \mathrm{d}t,$$

in this case. To show that such a zigzag path has arbitrary small length we just need to show that the integral does not grow faster than n^2.

As an example, take the simply case where we expand the circle $e^{i\pi\theta}$ to $2e^{i\pi\theta}$. The zigzag path in then concretely given as

$$\varphi(t,\theta) = e^{i\pi\theta} \sum_{k=0}^{n-1} h^{n,k}(t,\theta) + g^{n,k}(t,\theta)$$

where

$$\begin{aligned} h^{n,k}(t,\theta) &:= 1_{[\frac{k}{n}, \frac{k}{n} + \frac{1}{2n})}(\theta) \left(1 + 2t(n\theta - k)\right), \\ g^{n,k}(t,\theta) &:= 1_{[\frac{k}{n} + \frac{1}{2n}, \frac{k+1}{n})}(\theta) \left(1 + 2t(1 - n\theta - k)\right). \end{aligned}$$

We have that

$$\begin{aligned} |\varphi_t| &= \sum_{k=0}^{n-1} |h_t^{n,k}| + \sum_{k=0}^{n-1} |g_t^{n,k}|, \\ |\varphi_\theta| &= \sum_{k=0}^{n-1} |h_\theta^{n,k} + h^{n,k}| + \sum_{k=0}^{n-1} |g_\theta^{n,k} + g^{n,k}|, \end{aligned}$$

so by symmetry

$$\begin{aligned} \int_0^1 |\varphi_t|^2 |\varphi_\theta| \, \mathrm{d}\theta &= 2n \int_0^{\frac{1}{2n}} |h_t^{n,0}|^2 |h_\theta^{n,0} + h^{n,0}| \, \mathrm{d}\theta \\ &= 2n \int_0^{\frac{1}{2n}} (2n\theta)^2 (2tn + 1 + t2n\theta) \, \mathrm{d}\theta \\ &= \int_0^1 u^2 (2tn + 1 + tu) \, \mathrm{d}\theta \\ &= O(n), \end{aligned}$$

for $t \in [0,1]$ which gives the result.