The L2 metric

We have seen (not yet) that the tanget vector to a shape in B can be thought of as smooth functions $a\colon \mathbb{S}^1 \to \mathbb{R}$.

Definition 0.1 (The L2 metric)**.** The *L2 metric* G^2_q at the point $q \in B$ is defined as

$$G^2_q(a,b) := \langle a, b\rangle_{L^2} = \int_{\mathbb{S}^1} a(\theta)b(\theta)\,\mathrm{d}\theta,$$

$a, b \in T_qB = C^\infty(\mathbb{S}^1, \mathbb{R})$.

(Not clear that this definition is flexible enough to talk about it varying smoothly with the point q?)

From this we can then define the length of a path in B. (Can also do this more generally for any metric?)

Definition 0.2. (Length of path) For a path $c : [0,1] \to B$ we define the *length of the path* as

$$L(c) := \int_0^1 \sqrt{G^2_{c(t)}(c_t, c_t)}\,\mathrm{d}t.$$

Definition 0.3. For two curves $q_0, q_1 \in B$ we define the *geodesic distance* between these as

$$d(q_0, q_1) := \inf_{c \in \mathcal{C}} \{L(c)\},$$

where $\mathcal{C}$ denotes all paths c, such that $c(0) = q_0$ and $c(1) = q_1$.

It turns out that this metric vanishes on all of B, i.e., for any two curves $q_0, q_1 \in B$, we can find a path c connecting these such that $L(c)$ can be made arbitrarily small, so $d(q_0, q_1) = 0$ for all curves. We show this in a special case which illustrates the central idea.

We show that we can find an arbitrarily short path between the circle and a shape obtained by deforming the circle along a normal vector field as illustrated in figure 1.

Let $q_0(\theta)$ denote the standard parametrization of the circle. As argued (ref to some previous result of ours) the vector field we move the circle along can be identified with a smooth scalar function $a \in C^\infty(\mathbb{S}^1, \mathbb{R})$, and thus we can identify the final shape q_1 as

$$q_1(\theta) = a(\theta)n_{q_0}(\theta),$$

with $n_c(\theta)$ denoting the unit normal to the curve c at the point θ. Here, $q_0 = \mathbb{S}^1$. From this we see that our assumption essentially simplifies the question of finding a path from q_0 to q_1 to the problem of moving the zero function to the function a. The figure also illustrates one possible path, namely moving each point θ according to the function $t \mapsto ta(\theta)$, $t \in [0,1]$. Figure 2 illustrates another path: Here we first move the circle according to the dark gray pattern and then we move the last part according to the light gray pattern. We can formalize this as moving along the path given as

$$\varphi_n(t,\theta) = \begin{cases} \varphi_n^1(t,\theta) = a(\theta)(1-\cos(\theta n))t & : \quad t < \frac{1}{2} \\ \varphi_n^2(t,\theta) = a(\theta)(t-\cos(\theta n)(1-t)) & : \quad t \geq \frac{1}{2} \end{cases}$$

We see that this is a continous function as the two cases agree in $t = 1/2$. However, it's not smooth, as differentiability fails at $(\theta, t) = (0, 1/2)$, for example. We can mend this by smoothing out the function above by replacing t (and $1-t$) with, e.g.,

$$th_\varepsilon\left(t, \frac{1-\cos(\theta)}{2}\right),$$

where $h_\varepsilon \colon [0,1]^2 \to \mathbb{R}$ is such that

$$\begin{aligned} &h_\varepsilon(t,x) = 0, \text{ for } (t,x) \in [0,1/2] \times \{0\} \cup [1/2,1] \times \{1\}, \text{ and} \\ &h_\varepsilon(t,x) = 1, \text{ for } \varepsilon < x < 1-\varepsilon. \end{aligned}$$

From ref we know that h_ε can be chosen to be smooth. To ease the calculations and notation we refrain from doing this; the following results will thus be approximate, which is enough in our case, because C^∞ is dense in L^2 (good enough?).

[NB: All these smoothness concerns should be eliminated by allowing piece-wise smooth functions in time...!]

Consider again the function φ_n. We calculate the distance from the circle to the shape obtained on the halfway, that is $d(q_0, \mathrm{Im}(\varphi_n^1(1/2,\cdot)))$; the distance from this shape to the final shape can be calculated likewise. The time derivative of the path φ_n^1 with $t \in [0,1/2]$ is given as

$$\frac{\partial \varphi_n^1}{\partial t} = a(\theta)(1-\cos(\theta n)),$$

so we get that

$$\begin{aligned} L(\varphi_n^1)^2 &= \left(\int_0^1 \|a(\theta)(1-\cos(\theta n))\|_{L^2} \,\mathrm{d}t\right)^2 \\ &= \|a(\theta)(1-\cos(\theta n))\|_{L^2}^2 \\ &= \int_0^{2\pi} \left(a(\theta)(1-\cos(\theta n))\right)^2 \mathrm{d}\theta \\ &\leq \sup_{x\in[0,2\pi]} \{a(\theta)^2\} \int_0^{2\pi} (1-\cos(\theta n))^2 \,\mathrm{d}\theta \end{aligned}$$

We have that

$$\int_0^{2\pi} (1-\cos(\theta n))^2 \,\mathrm{d}\theta =$$

... WRONG ...

For some t consider the curve $\varphi(t,\cdot)$. We attach the vector field $a(\theta)(1-\cos(\theta n))$ to this curve at the points $\varphi(t,\theta)$. To identify this with the unique vector field on the circle, denote the length of $\varphi(t,\cdot)$ by l_t. Then we can reparametrize the vector field in accordance with the curve $\varphi(t,\cdot)$ as $a(l_t\theta)(1-\cos(l_t\theta n))$ [explain better]. Having identified the unique vector field on the circle, we can calculate

$$\begin{aligned}
G^2_{\varphi(t)}(\varphi_t,\varphi_t) &= \int_{\mathbb{S}^1} (a(l_t\theta)(1-\cos(l_t\theta n)))^2 \,\mathrm{d}\theta \\
&= \int_0^{2\pi} (a(\gamma)(1-\cos(\gamma n)))^2 l_t^{-1} \,\mathrm{d}\gamma \\
&\leq l_t^{-1} \sup_{x\in[0,2\pi]} \{a(x)^2\} \int_0^{2\pi} (1-\cos(\gamma n))^2 \,\mathrm{d}\gamma \\
&= l_t^{-1} C.
\end{aligned}$$

Now,

$$l_t \geq \text{"minimal difference above 0"}_{x\in[0,2\pi]}\{a(x)\}\pi n,$$

so we see that $G^2_{\varphi(t)}(\varphi_t,\varphi_t) \to 0$ for $n\to\infty$.

Correct approach

Definition of the length of the path φ directly from the path alone:

$$L(\varphi) = \int_0^1 \int_{\mathbb{S}^1} \frac{\langle \varphi_t, i\varphi_\theta\rangle^2}{|\varphi_\theta|} \,\mathrm{d}\theta \,\mathrm{d}t$$

Rewrite this as

$$\int_0^1 \int_{\mathbb{S}^1} \left(\frac{\langle \varphi_t, i\varphi_\theta\rangle}{|\varphi_t||\varphi_\theta|}\right)^2 |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta \,\mathrm{d}t = \int_0^1 \int_{\mathbb{S}^1} \cos(\alpha(\varphi_t, i\varphi_\theta))^2 |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta \,\mathrm{d}t,$$

with $\alpha(x,y)$ denoting the angle between x and y. When constructing a zigzag-path the angle will be (approximately?) constant in θ and t, and it is given by

$$n = \tan(\alpha).$$

We have that

$$\cos(\arctan(n)) = (1+n^2)^{-1/2} = O(n^{-1}),$$

so we can write

$$L(\varphi) = O(n^{-2}) \int_0^1 \int_{\mathbb{S}^1} |\varphi_t|^2 |\varphi_\theta| \, \mathrm{d}\theta \, \mathrm{d}t,$$

in this case. To show that such a zigzag path has arbitrary small length we just need to show that the integral does not grow faster than n^2.

As an example, take the simply case where we expand the circle $e^{i\pi\theta}$ to $2e^{i\pi\theta}$. The zigzag path in then concretely given as

$$\varphi(t,\theta) = e^{i\pi\theta} \sum_{k=0}^{n-1} h^{n,k}(t,\theta) + g^{n,k}(t,\theta)$$

where

$$\begin{aligned} h^{n,k}(t,\theta) &:= 1_{[\frac{k}{n}, \frac{k}{n} + \frac{1}{2n})}(\theta) \left(1 + 2t(n\theta - k)\right), \\ g^{n,k}(t,\theta) &:= 1_{[\frac{k}{n} + \frac{1}{2n}, \frac{k+1}{n})}(\theta) \left(1 + 2t(1 - n\theta - k)\right). \end{aligned}$$

We have that

$$\begin{aligned} |\varphi_t| &= \sum_{k=0}^{n-1} |h_t^{n,k}| + \sum_{k=0}^{n-1} |g_t^{n,k}|, \\ |\varphi_\theta| &= \sum_{k=0}^{n-1} |h_\theta^{n,k} + h^{n,k}| + \sum_{k=0}^{n-1} |g_\theta^{n,k} + g^{n,k}|, \end{aligned}$$

so by symmetry

$$\begin{aligned} \int_0^1 |\varphi_t|^2 |\varphi_\theta| \, \mathrm{d}\theta &= 2n \int_0^{\frac{1}{2n}} |h_t^{n,0}|^2 |h_\theta^{n,0} + h^{n,0}| \, \mathrm{d}\theta \\ &= 2n \int_0^{\frac{1}{2n}} (2n\theta)^2 (2tn + 1 + t2n\theta) \, \mathrm{d}\theta \\ &= \int_0^1 u^2 (2tn + 1 + tu) \, \mathrm{d}\theta \\ &= O(n), \end{aligned}$$

for $t \in [0,1]$ which gives the result.

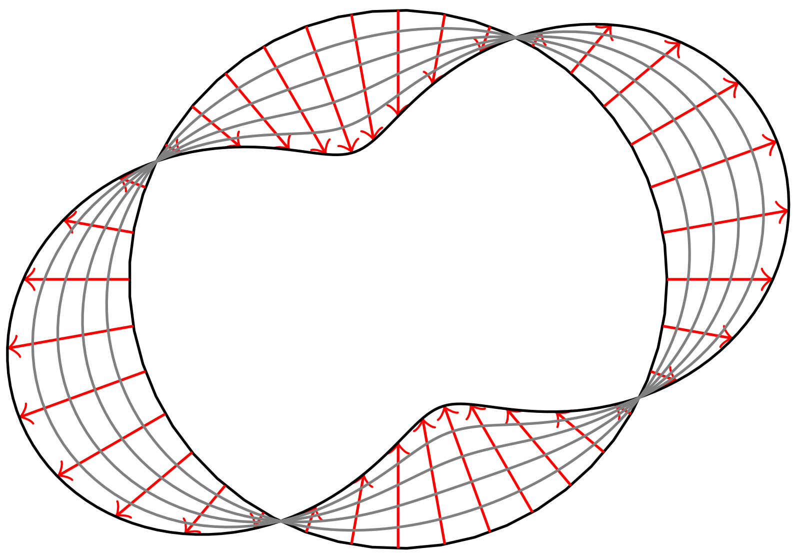

Figure 1: Deformation of the circle along a normal vector field.

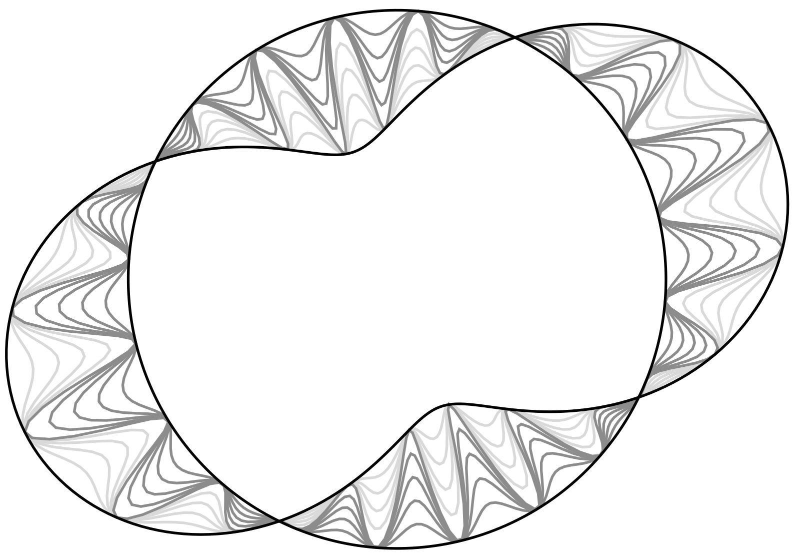

Figure 2: A short path in B.