The tangent space of closed curves in $\mathbb{R}^2$

Constructing the space of closed curves. Intuitively, we want to consider the space of all (smooth) closed curves in $\mathbb{R}^2$. This can be seen as the space of all submanifolds in $\mathbb{R}^2$ which are diffeomorphic to the unit circle $\mathbb{S}^1$. If we let $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ denote the space of all *immersion* from the inut circle into the plane, we can define the space we want to consider as

$$B := \left\{ q(\mathbb{S}^1) \mid q \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2) \right\}. \tag{1}$$

Here we simply think of $q(\mathbb{S}^1) \subset \mathbb{R}^2$ as a subspace and forget about the actual map q. (Keeping this mapping in mind we could define the space in another way; but this is not so important right now.)

The tangent space of B. For ordinary finite dimensional manifolds B, Lee defines the tangent space at a point $p \in B$ through the notation of *derivations*; this is a rather abstract construction, but is nice to work with. Using this, one can define the notion of a tanget vector to a path in B passing through the point p. Then, one can define an equivalence relation on the space of such paths and obtain an equivalent definition of the tanget space, which is more intuitive. One can also work the other way around and start by defining the notion of tanget vector to paths in B (as is done on Wiki). We briefly do that here:

For a neighbourhood $U \subset B$ containing p we have a smooth coordinate chart $\varphi \colon U \to \mathbb{R}^n$. For a path $\gamma \colon [-\varepsilon, \varepsilon] \to B$ with $\varphi(0) = p$, it makes perfect sense to consider differentiability of the map $\varphi \circ \gamma \colon [0, 1] \to \mathbb{R}^n$. Now, the relation

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0),$$

defines an equivalence relation on the collection of all such paths γ. An equivalence class of such paths, denoted by $[\gamma_p']$ (or simply γ_p'), is called a *tangent vector* at the point p. The collection of all tanget vectors make up the tanget space T_pB at p.

Returning to B being as defined in (1), we now want to determine what the tangent spaces look like at a point $q \in B$ by following the construction above (taking for granted that B actually is a (Fréchet) manifold, and that the following definitions/constructions make sense). Firstly, a path in B is now a map $\gamma \colon [-\varepsilon, \varepsilon] \to B$ such that $\gamma(t)$ is a curve $q_t \in B$ for all $t \in [-\varepsilon, \varepsilon]$, with $\gamma(0) = q$. For each of the q_t we can think of them as parametrized by $x \mapsto q_t(x)$, $x \in \mathbb{S}^1$. Then, if the curve γ is such that for each fixed $x \in \mathbb{S}^1$ the map $t \mapsto q_t(x) \in \mathbb{R}^2$ is differentiable, we can define γ_q' as the mapping

$$\gamma_q' := x \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} q_t(x).$$

Thus each γ_q' is a mapping from the unit circle into $\mathbb{R}^2$, which mean that we can think of a tangent vector to a curve $q \in B$ as a vector field on $\mathbb{S}^1$ (though of as $\subset \mathbb{R}^2$).

Question / considerations / imprecisions. Above we made an intuitively reasonable construction, but ignored some of technicalities, which we list here.

- Equivalence classes: We should define the tangent vectors as equivalence classes of path in B; this should all determine a unique vector field.
- The manifold structure of B: We simply used the intuitive idea to differentiate in "time" for each fixed point on a curve, $q_t(x)$. However, as we have not specified the chart for the manifold B, it is not obvious that this construction corresponds to the one made in the finite dimensional case. Technically we would need a chart $\varphi\colon U \to F$, with F some Fréchet space and then show some sort of Fréchet-differentiability of the composite function $\varphi \circ \gamma$.
- We mentioned that in the finite dimensional case the two definitions (through derivates and tangents to curves, respectively) are equivalent. It is not obvious that this also hold in the infinite dimensional case.
- At the beginning we eliminated the knowledge of the parametrization of a curve $q \in B$ to make the definition of B simpler. However, we actually use a parametrization later, and thus we should make sure that reparametrizations does not matter for the construction of the tangent space. (It does not, as it would just move the vector field around $\mathbb{S}^1$ according to the reparametrization.)
- Are there any smoothness assumptions (or something) about vector field we need to validate? For example, just the fact that the map $t \mapsto q_t(x)$ behaves nice does of course *not* imply that also the derived vector field γ_q' behaves nicely in x – which is what we would need to get a smooth vector field (?).
- Is it correctly formulated that we need to think of $\mathbb{S}^1 \subset \mathbb{R}^2$ to make sense of a vector field on the circle?

Test section