The L2 metric vanishes

Some introductory text...

Definition of the length of the path φ directly from the path alone:

$$L(\varphi) = \int_0^1 \int_{\mathbb{S}^1} \frac{\langle \varphi_t, i\varphi_\theta \rangle^2}{|\varphi_\theta|} \,\mathrm{d}\theta \,\mathrm{d}t$$

Rewrite this as

$$\int_0^1 \int_{\mathbb{S}^1} \left(\frac{\langle \varphi_t, i\varphi_\theta \rangle}{|\varphi_t||\varphi_\theta|} \right)^2 |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta \,\mathrm{d}t = \int_0^1 \int_{\mathbb{S}^1} \cos(\alpha(\varphi_t, i\varphi_\theta))^2 |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta \,\mathrm{d}t,$$

with $\alpha(x, y)$ denoting the angle between x and y. When constructing a zigzag-path the angle will be (approximately?) constant in θ and t, and it is given by

$$n = \tan(\alpha).$$

We have that

$$\cos(\arctan(n)) = (1 + n^2)^{-1/2} = O(n^{-1}),$$

so we can write

$$L(\varphi) = O(n^{-2}) \int_0^1 \int_{\mathbb{S}^1} |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta \,\mathrm{d}t,$$

in this case. To show that such a zigzag path has arbitrary small length we just need to show that the integral does not grow faster than n^2.

As an example, take the simply case where we expand the circle $e^{i\pi\theta}$ to $2e^{i\pi\theta}$. The zigzag path in then concretely given as

$$\varphi(t, \theta) = e^{i\pi\theta} \sum_{k=0}^{n-1} h^{n,k}(t, \theta) + g^{n,k}(t, \theta)$$

where

$$\begin{aligned} h^{n,k}(t, \theta) &:= 1_{[\frac{k}{n}, \frac{k}{n} + \frac{1}{2n})}(\theta) \left(1 + 2t(n\theta - k)\right), \\ g^{n,k}(t, \theta) &:= 1_{[\frac{k}{n} + \frac{1}{2n}, \frac{k+1}{n})}(\theta) \left(1 + 2t(1 - n\theta - k)\right). \end{aligned}$$

We have that

$$\begin{aligned} |\varphi_t| &= \sum_{k=0}^{n-1} |h_t^{n,k}| + \sum_{k=0}^{n-1} |g_t^{n,k}|, \\ |\varphi_\theta| &= \sum_{k=0}^{n-1} |h_\theta^{n,k} + h^{n,k}| + \sum_{k=0}^{n-1} |g_\theta^{n,k} + g^{n,k}|, \end{aligned}$$

so by symmetry

$$\begin{aligned}
\int_0^1 |\varphi_t|^2 |\varphi_\theta| \,\mathrm{d}\theta &= 2n \int_0^{\frac{1}{2n}} |h_t^{n,0}|^2 |h_\theta^{n,0} + h^{n,0}| \,\mathrm{d}\theta \\
&= 2n \int_0^{\frac{1}{2n}} (2n\theta)^2 (2tn + 1 + t2n\theta) \,\mathrm{d}\theta \\
&= \int_0^1 u^2 (2tn + 1 + tu) \,\mathrm{d}\theta \\
&= O(n),
\end{aligned}$$

for $t \in [0,1]$ which gives the result.