Anders Munch and Mads Møller Pedersen

Analysis of shape spaces

Project outside of the course scope

Department of Mathematical Sciences
University of Copenhagen

Advisor

Stefan Sommer

November 9, 2018

Abstract

Abstract about analysis of shapes.

Contents

Introduction

Intro

1. Preliminaries

1.1 Notation

In the following M denotes a smooth manifold and TM is the tangent bundle of M and $\mathcal{T}(M)$ denotes the space of all vector fields on M. For $I \subset \mathbb{R}$, $\gamma : I \to M$ is a curve in M, i.a. a smooth map.

1.2 Connections

To consider the geodesic distance between two points in a manifold, geodesics need to be defined in a coordinate-invariant way such that the distance is independent of the coordinate charts. One property of geodesics in a Euclidean space, straight lines, is that they have acceleration 0. In order to make sense of acceleration of a curve in a manifold, we need to be able to compute "differences" between tangent spaces along the curve. *Connections* are exactly a way of making computations between tangent spaces possible - they allow us to differentiate vector fields along curves.

Since our use of connections is to define geodesics, we define connections in the tangent bundle of a manifold (instead of defining them generally on smooth sections of vector bundles) follwing chapter 4 of Lee (1997).

Definition 1.2.1. *A connection in TM is a map*

$$\Delta : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M),$$

written $(X, Y) \mapsto \Delta_X Y$ satisfying (for f, $g \in C^\infty(M)$ and a, $b \in \mathbb{R}$);

a) $\Delta_{fX_1+gX_2} Y = f\Delta_{X_1} Y + g\Delta_{X_2} Y$ *(linearity over $C^\infty(M)$ in X)*

b) $\Delta_X(aY_1 + bY_2) = a\Delta_X Y_1 + b\Delta_X Y_2$ *(linearity over $\mathbb{R}$ in Y)*

c) $\Delta_X(fY) = f\Delta_X Y + (Xf)Y$ *product rule*

In accordance with connections allowing "differences" between tangent spaces, $\Delta_X Y$ is called the *covariant derivative of Y in the direction of X.* To use connections to derivate along curves, we need the definition of a *vector field along a curve*, which is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t}M$ for all $t \in I$. The prime example of a vector field along a curve is its velocity, $\dot{\gamma}(t) \in T_{\gamma(t}M$, which acts on functions, $f \in C^\infty(M)$, by

$$\dot{\gamma}(t)f = \frac{\mathrm{d}}{\mathrm{dt}}(f \circ \gamma)(t).$$

We denote by $\mathcal{T}(\gamma)$ all vector fields along γ. To define geodesics all we now need is to define what is means to take the covariant derivative of $V \in \mathcal{T}(\gamma)$ along γ. This covariant derivative is noted $D_t V$ and is has the following properties.

Lemma 1.2.2. *Let Δ be a linear connection on M. For each $\gamma : I \to M$, Δ determines a unique operator*

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma),$$

satisfying (for f, $g \in C^\infty(I)$ and a, $b \in \mathbb{R}$);

a) $D_t(aV + bW) = aD_t V + bD_t W$ *(linearity over $\mathbb{R}$*

b) $D_t(fV) = \dot{f}V + fD_t V$ *(product rule)*

c) If V is extendible, then for any extension $\tilde{V}$ of V, $D_t V(t) = \Delta_{\dot{\gamma}(t)}\tilde{V}$.

Proof.
Proof of Lemma 4.9 in Lee (1997) □

V is said to be extendible if it can be constructed by any vector field on M, $\tilde{V}$ by letting $V(t) := \tilde{V}_{\gamma(t)}$. This is not always the case; if V is the velocity of an intersecting curve γ with different covariant derivative at the intersection times. The covariant derivative of the velocity of a curve is now used to define a geodesic.

Definition 1.2.3. *Let Δ be a linear connection on M and γ a curve in M. The acceleration of γ is $D_t\dot{\gamma}(t)$. If this vector field is zero, $D_t\dot{\gamma}(t) \equiv 0$, then $\gamma(t)$ is a geodesic with respect to Δ*

It follows from Theorem 4.10 in Lee (1997) that for any manifold, M, with a linear connection, for any $p \in M$ and $V \in T_pM$ and $t_0 \in \mathbb{R}$ there exists an un-extendable geodesic, $\gamma_V : I \to M$, with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. The geodseic is called the (maximal) geodesic with initial value p and initial velocity V.

In this construction of geodesics the only necessary structure of M is that it should be a smooth manifold. When M is also equipped with a Riemannian metric, making M a Riemannian manifold, the choice of connection (determining the geodesics) should in some way respect the metric. Geodesics resulting from this specific choice of connection are called *Riemannian geodesics.*

Bibliography

Lee, John M (1997). *Riemannian Manifolds*. Vol. 1. Springer.