Discrete Calculus: The Power of Finite Differences

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This article explores how the familiar rules of calculus can be adapted for discrete sequences, providing a powerful tool for summation. By replacing infinitesimals with finite steps, we uncover a world where "integration" is simply summation, and "differentiation" is the difference operator.

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1 Introduction: From Curves to Sequences

In continuous calculus, the definite integral $\int_a^b f(x)\,dx$ represents the area under a curve. We often approximate this area using a **Riemann sum**: splitting the interval into many thin rectangles of width h and summing their areas.

In a discrete setting, the "smallest" possible width is 1. This suggests a natural analogue: integration becomes simple summation. Given a sequence (a_n), we can regard the sum $\sum_{n=a}^{b} a_n$ as a discrete "definite integral." While continuous calculus works on a continuum, discrete calculus operates on sequences—functions defined on the integers.

2 The Discrete Derivative

The continuous derivative $f'(x)$ measures the instantaneous rate of change. For a sequence (a_n), we cannot take a limit as $h \to 0$ because the smallest gap between points is fixed at 1. Instead, we define the **forward difference operator**, denoted by Δ:

$$\Delta a_n = a_{n+1} - a_n \tag{1}$$

This acts as our discrete derivative. While other versions exist (like backward or centered differences), Δ is the standard choice because it aligns nicely with summation.

3 The Fundamental Theorem of Discrete Calculus

In continuous calculus, the Fundamental Theorem states that $\int_a^b f'(x)\,dx = f(b) - f(a)$. This formalises the notion that integration is the "inverse" of differentiation. We can show this is true for the difference operator and summation using a **telescoping sum**:

$$\sum_{n=a}^{b} \Delta u_n = (u_{a+1} - u_a) + (u_{a+2} - u_{a+1}) + \cdots + (u_{b+1} - u_b) = u_{b+1} - u_a \tag{2}$$

The slight index shift (u_{b+1} vs $f(b)$) arises because our "rectangles" have a non-zero width of 1.

4 Falling Factorials and the Power Rule

We run into a problem when we try to apply the standard power rule. In calculus, $(x^k)' = kx^{k-1}$. However, $\Delta(n^2) = (n+1)^2 - n^2 = 2n + 1$. The "extra" $+1$ prevents a clean analogy. To fix this, we introduce **falling factorials**:

$$n^{\underline{k}} = \overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{\text{k factors}}$$

Using these, we obtain a discrete power rule:

$$\Delta(n^{\underline{k}}) = k \cdot n^{\underline{k-1}} \tag{3}$$

Just as e^x is the function that is its own derivative, the discrete world has an exponential: $\Delta(2^n) = 2^{n+1} - 2^n = 2^n$. In this system, the base 2 plays the role of e. We also have that $H_n = \sum_{k=1}^{n} \frac{1}{k}$, the sequence of harmonic numbers, plays the role of $ln(x)$, as $\Delta H_n = \frac{1}{n+1} = n^{\underline{-1}}$.

5 Example: The Sum of Squares

We can use what we've learned to derive the formula for $\sum_{k=1}^{n} k^2$. First, we rewrite k^2 in terms of falling factorials:

$$k^{\underline{2}} = k(k-1) = k^2 - k \implies k^2 = k^{\underline{2}} + k^{\underline{1}}$$

Now, we find the "anti-difference" of each term and apply the Fundamental Theorem (2):

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) = \left[\frac{1}{3}k^{\underline{3}} + \frac{1}{2}k^{\underline{2}}\right]_{k=1}^{k=n+1}$$

Evaluating at the limits (noting that $1^{\underline{3}}$ and $1^{\underline{2}}$ are 0):

$$= \frac{1}{3}(n+1)(n)(n-1) + \frac{1}{2}(n+1)(n) = (n+1)(n)\left(\frac{n-1}{3} + \frac{1}{2}\right) = \frac{n(n+1)(2n+1)}{6}$$

This recovers the classic formula without proof by induction!

6 Conclusion

Discrete calculus is a huge topic and there exist analogues for advanced topics like Taylor series, PDEs, Laplacians and more. It is an essential tool in combinatorics and algorithm analysis, and it offers fresh intuition for the rules of "regular" calculus we learn in class.

References & Challenges

- *Concrete Mathematics* by Graham, Knuth, and Patashnik.
- *Finite Difference*, Wikipedia.

- **Challenge 1: Summation by Parts.** Show that $\Delta(u_n v_n) = u_n \Delta v_n + v_{n+1} \Delta u_n$. Use this to derive the "Summation by Parts" formula: $\sum u_n \Delta v_n = u_n v_n - \sum v_{n+1} \Delta u_n$.
- **Challenge 2:** Use Summation by Parts to find a closed form for $\sum_{k=1}^{n} k 2^k$.
- **Challenge 3:** Come up with a general method to find $\sum_{k=1}^{n} k^a$. If you can code, try implementing it in python/sympy.

Detailed solutions to these challenges, including a derivation of **Faulhaber's Formula** (the general form for the sum of m-th powers), can be found on my GitHub:
`github.com/kdlegum/Discrete-Calculus-U-Maths`