Phys 3610, Fall 2008
Exam #3

1. Define:

a) Cyclic (ignorable) coordinate

The generalized coordinate q_i is cyclic if the lagrangian $\mathcal{L}$ does not depend on it. In that case, $\partial\mathcal{L}/\partial\dot{q}_i$ is constant which is also expressed by saying that the momentum conjugate to q_i is conserved.

b) Reduced mass

For a two-particle system, the reduced mass μ is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

In the equivalent one--particle problem μ plays the role of the (single) particle's mass.

c) Coriolis force

This is a velocity-dependent (phony) force used in a rotating frame when we express Newton's 2nd law in that frame. It is given by $2m\mathbf{\Omega} \times \dot{\mathbf{r}}$

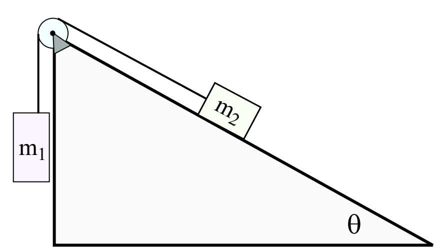
d) Foucault pendulum.

A fine physics demo found a quality science museums everywhere. It is a long simple pendulum which is set to swing in a plane but it is found that over time the plane of motion slowly rotates. The rate of this rotation depends on the latitude of the location. This drift is caused by the Coriolis force.

e) Principal axes.

This are axes associated with a rigid body for which the total angular momentum $\mathbf{L}$ is parallel to ω if $\boldsymbol{\omega}$ lies along one of these axes.

2. Mass m_1 is suspended from a string which passes over an ideal massless pulley; the other end of the string is attached to a mass m_2 which slides on a frictionless inclined surface, inclined at angle θ. (The string pulls parallel to the slope.) You can assume $m_1 > m_2$.



Find the acceleration of the masses using the Lagrange approach. Note: There is one degree of freedom; the total length of the string is constant. You only need to write down an expression for the potential energy up to a constant term.

(Does the answer match what you know from elementary physics?)

If we let x be the distance of m_1 below the pulley (that is, the negative of its y coordinate) then if the length of the string is L, then m_2 is a distance $L - x$ downward along the slope from the pulley. (This ignores the part of string that goes around the pulley, but that just adds a constant to the coordinates.) Then the kinetic energy of the system is

$$T = \tfrac{1}{2}m_1\left(\frac{d}{dt}x\right)^2 + \tfrac{1}{2}m_2\left(\frac{d}{dt}(L-x)\right)^2 = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 = \tfrac{1}{2}(m_2+m_2)\dot{x}^2$$

Of course, both masses have the same *speed*.

The potential energy of the system is

$$U = -m_1gx - m_2g(L-x)\sin\theta = -m_1gx + m_2gx\sin\theta = gx(-m_1 + m_2\sin\theta)$$

so the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + gx(m_1 - m_2\sin\theta)$$

Set up the Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial x} = g(m_1 - m_2\sin\theta) \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1+m_2)\dot{x}$$

Then the Lagrange equation gives us

$$g(m_1 - m_2\sin\theta) - \frac{d}{dt}(m_1+m_2)\dot{x} = (m_1+m_2)\ddot{x}$$

which then gives

$$\ddot{x} = \frac{g(m_1 - m_2\sin\theta)}{(m_1+m_2)}$$

This is what we get from Phys 2110 physics; if the string has tension T then the free-body diagrams give the equations

$$m_1g - T = m_2\ddot{x} \qquad T - m_2g\sin\theta = m_2\ddot{x}$$

which, on eliminating T (add the two equations) gives

$$m_1g - m_2g\sin\theta = (m_1+m_2)\ddot{x} \qquad \Longrightarrow \qquad \ddot{x} = \frac{g(m_1 - m_2\sin\theta)}{(m_1+m_2)}$$

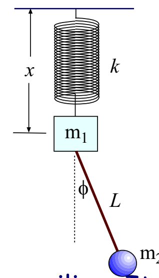
as before.

I've decided to make #3 extra-credit! Finish the rest first and do as much as you have time for.

3. Mass m_1 is suspended from a vertical ideal spring and is constrained so that it can move only vertically; the spring has force constant k and natural length ℓ_0.

Attached below m_1 is a simple pendulum; the mass of the bob is m_2.

Find (set up) the Lagrangian of the system and find the equations of motion. Do as much as you have time for.



Suppose the full length of the spring is x, so the position of m_1 is $y = -x$ from the ceiling. Its energy is easy to express and we move on to mass m_2: For m_2, its coordinates (again measured from suspension point of spring) are

$$X = L\sin\phi \qquad Y = -x - L\cos\phi$$

which gives

$$\dot{X} = \dot{\phi}L\cos\phi \qquad \dot{Y} = -\dot{x} + \dot{\phi}L\sin\phi$$

then the total kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{\phi}^2L^2\cos^2\phi + (-x + \dot{\phi}L\sin\phi)^2) = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{x}^2 + \dot{\phi}^2L^2 - 2\dot{x}\dot{\phi}L\sin\phi) \\ &= \tfrac{1}{2}(m_1+m_2)^2 + \tfrac{1}{2}m_2\dot{\phi}^2L^2 - m_2\dot{x}\dot{\phi}L\sin\phi \end{aligned}$$

and the total potential energy is

$$\begin{aligned} U &= -m_1gx - m_2g(x + L\cos\phi) + \tfrac{1}{2}k(x-\ell_0)^2 \\ &= -(m_1+m_2)gx - m_2gL\cos\phi + \tfrac{1}{2}k(x-\ell_0)^2 \end{aligned}$$

and thus the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \tfrac{1}{2}(m_1+m_2)^2 + \tfrac{1}{2}m_2\dot{\phi}^2L^2 - m_2\dot{x}\dot{\phi}L\sin\phi + (m_1+m_2)gx + m_2gL\cos\phi - \tfrac{1}{2}k(x-\ell_0)^2 \end{aligned}$$

Whoa! what a mess! Well, for these coupled-motion problms things usually do get a bit messy.

Get the Lagrange equation for x. We find:

$$\frac{\partial\mathcal{L}}{\partial x} = +(m_1+m_2)g - k(x-\ell_0) \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1+m_2)\dot{x} - m_2\dot{\phi}L\sin\phi$$

giving the Lagrange equation

$$+(m_1+m_2)g - k(x-\ell_0) = (m_1+m_2)\ddot{x} - m_2(\ddot{\phi}L\sin\phi - \dot{\phi}^2L\cos\phi)$$

It's not clear what to do with this without an approximation for small displacements.

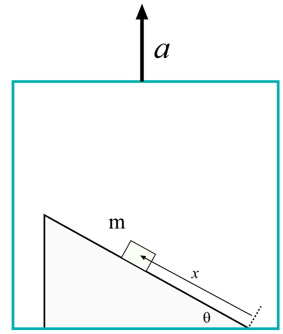
Get the Lagrange equation for ϕ. We find:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m_2gL\sin\phi \qquad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m_2\dot{\phi}L^2 - m_2\dot{x}L\sin\phi$$

giving

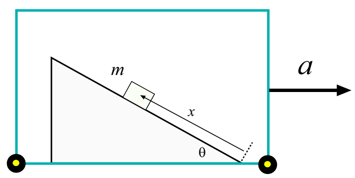
$$-m_2gL\sin\phi = m_2L^2\ddot{\phi} - m_2L(\ddot{x}\sin\phi + \dot{x}\dot{\phi}\cos\phi)$$

4. Take your choice of solving *one* of two similar problems.



Both involve a block sliding (down) a frictionless inclined plane (angle θ). In the first, it all takes place in an elevator car which is accelerating upward with acceleration a. (You can assume that its height is given by $y = \frac{1}{2}at^2$.)

In the other, the inclined plane is inside car accelerating to the right with constant acceleration a (and you can assume its horizontal position is given by $x = \frac{1}{2}at^2$.) In either problem there is one degree of freedom, which you can take to be the distance of the mass from the bottom of the slope (measured along the slope).



For either problem: Find the Lagrangian for the system; it will explicitly depend on time, t. Then find the equation of motion for the mass.

From our later work on accelerating reference frames, you can probably make sense of your result.

Car accelerating upward:

Measure x upward from the bottom of the incline and work with that. (Point the x axis to the left in the picture,) Then the coordinates of the mass are

$$X = x\cos\theta \qquad Y = x\sin\theta + \tfrac{1}{2}at^2$$

which give

$$\dot{X} = \dot{x}\cos\theta \qquad \dot{Y} = \dot{x}\sin\theta + at$$

and the kinetic energy of the mass is

$$T = \tfrac{1}{2}m(\dot{X}^2 + \dot{Y}^2) = \tfrac{1}{2}m\left((\dot{x}\cos\theta)^2 + (\dot{x}\sin\theta + at)^2\right)$$

Expand things, use the trig identity and get

$$T = \tfrac{1}{2}m(\dot{x}^2 + 2\dot{x}at\sin\theta + a^2t^2)$$

The potential energy is

$$U = mgY = mg(x\sin\theta + \tfrac{1}{2}at^2)$$

so the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + 2\dot{x}at\sin\theta + a^2t^2) - mg(x\sin\theta + \tfrac{1}{2}at^2)$$

Set up Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial x} = -mg\sin\theta \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} + mat\sin\theta$$

Then the Lagrange equation gives

$$-mg\sin\theta = \frac{d}{dt}(m\dot{x} + mat\sin\theta) = m\ddot{x} + ma\sin\theta$$

so that

$$\ddot{x} = -g\sin\theta - a\sin\theta = -(g+a)\sin\theta$$

We note that in the usual answer of "$g\sin\theta$ down the slope" we have $g+a$ in place of g. This is as we expect; the upward acceleration of the reference frame gives a new effective value of g.

Car accelerating to the right

Use the degree of freedom x in the same way as the case above but have the X axis go to the right. Then the coordinates of the mass are

$$X = -x\cos\theta + \tfrac{1}{2}at^2 \qquad Y = x\sin\theta$$

and the kinetic energy is

$$T = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) = \frac{m}{2}((-\dot{x}\cos\theta + at)^2 + (\dot{x}\sin\theta)^2) = \frac{m}{2}(\dot{x}^2 - 2\dot{x}at\cos\theta + a^2t^2)$$

and the potential energy is

$$U = mgx\sin\theta$$

so

$$\mathcal{L} = T - U = \frac{m}{2}(\dot{x}^2 - 2\dot{x}at\cos\theta + a^2t^2) - mgx\sin\theta$$

Set up the Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial x} = -mg\sin\theta \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{m}{2}(2\dot{x} - 2at\cos\theta) = m(\dot{x} - at\cos\theta)$$

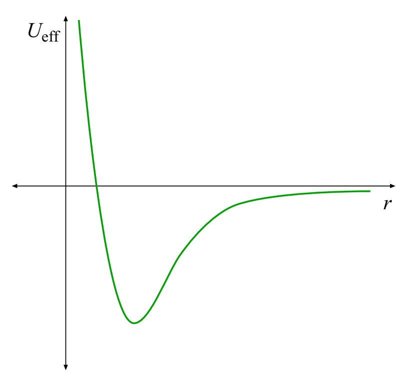
Then the lagrange equation gives

$$-mg\sin\theta = m(\ddot{x} - a\cos\theta) \qquad \Longrightarrow \qquad \ddot{x} = -g\sin\theta + a\cos\theta$$

Remembering that x points *up* the slope (to the left), the effect of gravity is add a component of acceleration $a\cos\theta$ up the slope.

5. What's the deal with the "effective potential" for the Kepler problem? (Graph of $U_{\rm eff}$ shown at right.)

Why does $U_{\rm eff}(r)$ goes to $+\infty$ at $r = 0$? (Doesn't the gravitational potential go to negative infinity there?)



The effective radial potential is a combination of the potential from the *true* central force plus the "centrifugal potential". The latter is proportional to $+1/r^2$ so it blows up at the origin and overpowers the real gravitational potential which only blows up as $-1/r$.

Explain how we would use this graph the find minimum and maximum distance of a Kepler orbit for a given total energy (and ℓ).

At the values of r where a horizontal line at E crosses the curve of $U_{\rm eff}$ the radial velocity $\dot{r}$ is zero, so these are the maximum and minimum radii.

6. Pluto's closest approach to sun is 30.17 AU and its farthest distance from the sun is 48.48 AU. (The "AU" is the average distance of the earth from the sun.) Find the eccentricity ϵ of Pluto's orbit. (Hint: You may want to use the equation for $r(\phi)$ and take ratios for the two distances. Then solve for ϵ.)

Using the formula for a Kepler elliptical orbit ($\epsilon < 1$) relating r and ϕ, the minimum r occurs where $\cos\phi = 1$; this gives

$$r_{\rm min} = 30.17\ \mathrm{AU} = \frac{c}{1+\epsilon}$$

and the maximum is where $\cos\phi = -1$, which gives

$$r_{\rm max} = 48.48\ \mathrm{AU} = \frac{c}{1-\epsilon}$$

Divide the second equation by the first and get

$$\frac{48.48}{30.17} = 1.6069 = \frac{1+\epsilon}{1-\epsilon} \qquad \Longrightarrow \qquad 1+\epsilon = 1.6069(1-\epsilon)$$

Solve for ϵ:

$$2.6069\epsilon = 0.6069 \qquad \Longrightarrow \qquad \epsilon = 0.233$$

7. Suppose the central potential for a mass m moving around a center of force is

$$U(r) = -A\frac{e^{-ar}}{r}$$

a) What is $F(r)$? (I.e. radial component of the central force.)

For the given central potential, $U(r) = -Ae^{-\alpha r}/r$ (which approaches the Kepler problem as $\alpha \to 0$, we have

$$F(r) = -\frac{dU}{dr} = +A\left[\frac{-\alpha e^{-\alpha r}}{r} - \frac{e^{-\alpha r}}{r^2}\right] = -Ae^{-\alpha r}\frac{(\alpha r+1}{r^2}$$

b) Write out the differential equation that would *in principle* allow you to find the trajectory $r(\phi)$ if the mass has angular momentum ℓ. (You don't need to solve it!)

With $u = 1/r$ we have a differential equation for $u(\phi)$:

$$u'' = -u - \frac{\mu}{\ell^2 u^2}F$$

We need to write F as a function of u:

$$F = -Au^2[\alpha/u + 1]e^{-\alpha/u} = -A[\alpha u + u^2]e^{-\alpha/u}$$

so with this we write

$$u''(\phi) = -u(\phi) + A\frac{\mu}{\ell^2 u(\phi)^2}[\alpha u + u^2]e^{-\alpha/u}$$

which is probably very challenging to solve and certainly will not give closed orbits.

8. Recall the equation from the text,

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \mathbf{\Omega} \times \mathbf{Q}$$

a) What does this equation tell us?

The equation relates the rate of change of any vector $\mathbf{Q}$ in the inertial and the rotating frames ($\mathcal{S}_0$ and $\mathcal{S}$, respectively).

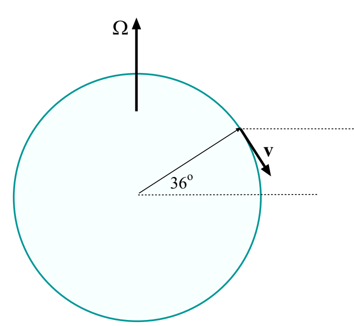
b) Describe how it was used to derive "Newton's 2nd law in a rotating frame".

Newton's 2nd law is first written down for an inertial frame; it gives $\ddot{\mathbf{r}}$ in terms of the *true* forces acting on a particle. This equation involves two d/dt operations acting on a vector $\mathbf{r}$ in the inertial frame; we use the above equation to relate this to two d/dt operating on $\mathbf{r}$ in the rotating frame. The result is that in addition to the true forces we get a couple terms which *act like* forces: The centrifugal "force" and the Coriolis "force".

9. In solving problems showing how the Coriolis force affects free-fall, we grouped the centrifugal force together with gravity and didn't consider its separate effect. Why did we do that?

Since it is independent of the (rotating frame) velocity and proportional to the mass, for a particular location on the earth the centrifugal force is a constant force like gravity. A plumb line points along the direction of the sum of gravitational and centrifugal forces and so we can separate them (experimentally) only with great difficulty. Rather we punt and call this direction the "vertical".

10. Suppose we fire a bullet southward at 36 deg latitude at a speed of 1000 m/s. Find the magnitude and direction of the acceleration due to the Coriolis force. (A picture may help you. . .)



The situation is illustrated at the right. From geometry, the angle between $\mathbf{\Omega}$ and $\mathbf{v}$ is $90^\circ + 54^\circ = 144^\circ$.

The value of Ω is

$$\Omega = \frac{2\pi}{(24)(3600)\ \mathrm{s}} = 7.27 \times 10^{-5}\ \mathrm{s}^{-1}$$

and since the Coriolis force if $2m\dot{\mathbf{r}} \times \mathbf{\Omega}$, its direction is out of the page for the diagram here, which means it points to the West and its magnitude is

$$|\mathbf{F}_{\mathrm{Cor}} = 2m\Omega\dot{r}\sin 144^\circ = m(2)(7.27 \times 10^{-5}\ \mathrm{s}^{-1})(1000\,\tfrac{\mathrm{m}}{\mathrm{s}})\sin 144^\circ = m(8.55 \times 10^{-2}\,\tfrac{\mathrm{m}}{\mathrm{s}^2})$$

(which is all we can get here because I didn't give the mass of the bullet...) but as a fraction of g (or, the force as a fraction of the weight), the Coriolis $acceleration$ is

$$a_{\mathrm{Cor}}/g = 8.7 \times 10^{-3}$$

which is still small, as we expect.

11. What are the Euler angles (ϕ, θ, ψ) used for?

The Euler angles are used to give the orientation of a rotating rigid body. They are angles though which one successively rotates the axes of the rotating system to arrive at their current orientation.

Useful Equations

Math

$$f(x) = (x-x_0)f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!}f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!}f^{(3)}(x_0) + \cdots$$

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Physics:

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{F} = -\nabla U \qquad \mathbf{p} = m\mathbf{v} \qquad T = \tfrac{1}{2}mv^2 \qquad U_{\text{grav}} = mgy$$

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \qquad L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y'(x)^2}\,dx$$

$$\mathcal{L} = T - U \qquad \mathcal{L} = \mathcal{L}(q_1, q_2, \ldots \dot{q}_1, \dot{q}_2, \ldots t) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} \qquad (\text{gen. } \mathbf{p}) = m\mathbf{v} + q\mathbf{A}$$

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 \qquad \mathbf{L}(\text{CM}) = \mathbf{r} \times \mu\dot{\mathbf{r}}$$

$$\dot{\phi} = \frac{\ell}{\mu r^2} \qquad F_{\text{cf}} = \frac{\ell^2}{\mu r^3} \qquad \mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{\text{cf}}(r)] = -\frac{d}{dr}U_{\text{eff}}(r) \qquad U_{\text{eff}} = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$u = \frac{1}{r} \qquad u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F \qquad r(\phi) = \frac{c}{1+\epsilon\cos\phi} \qquad c = \frac{\ell^2}{\gamma\mu} \qquad \gamma = Gm_1m_2$$

$$E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \qquad \tau^2 = \frac{4\pi^2}{GM_s}a^3$$

$$\mathbf{F}_{\text{inertial}} = -m\mathbf{A} \qquad \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}$$

$$m\ddot{\mathbf{r}} = \mathbf{F} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$$

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = M\dot{\mathbf{R}}$$

$$\mathbf{L} = \mathbf{L}(\text{CM}) + \mathbf{L}(\text{rel to CM}) \qquad T = T(\text{CM}) + T(\text{rel to CM})$$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \qquad T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \qquad (\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega}$$