Phys 2920, Spring 2009
Exam #2

1. In what direction from the point $(1,1,1)$ is the directional derivative of $\phi = 4yz - 3x^2$ a maximum? (And what is the magnitude of this maximum?)

Find the gradient of ϕ, evaluate at $(1,1,1)$::

$$\nabla\phi = -6x\,\mathbf{i} + 4z\,\mathbf{j} + 4y\,\mathbf{k}\Big|_{(1,1,1)} = -6\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$$

The direction $\hat{\mathbf{a}}$ in which ϕ has the maximal change per unit length is the direction of the gradient, namely

$$\hat{\mathbf{a}} = \frac{-6\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{36+16+16}} = \frac{-6\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{2\sqrt{17}} = \frac{-3\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}}{\sqrt{17}}$$

and the magnitude of this maximal rate of change is the magnitude of the gradient,

$$\frac{d\phi}{ds}\Big|_{\max} = \sqrt{68} = 2\sqrt{17}$$

2. Give an equation or condition that describes

a) The yz plane, in spherical coordinates.

For the yz plane, ϕ has the values $\pi/2$ and $3\pi/2$ and r and θ have all permitted values. So this plane is given by $\phi = \frac{\pi}{2},\ \frac{3\pi}{2}$.

b) The plane $x = 2$, in cylindrical coordinates.

Since $x = \rho\cos\phi$, this plane would be given by

$$\rho\cos\phi = 2$$

c) The cylinder $\rho = 3$ in cylindrical coordinates, expressed in spherical coordinates.

The relation between the cylindrical ρ and the spherical coordinates is $\rho = r\sin\theta$. Then the relation in spherical coordinates is

$$r\sin\theta = 3$$

3. Find the divergence of the vector field

$$\mathbf{a} = 3r^2\cos\theta\,\hat{\mathbf{r}} + 2r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}$$

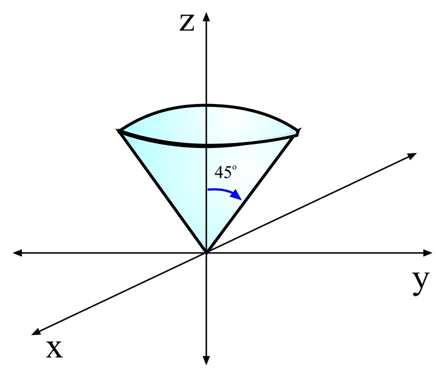
Here we have

$$a_r = 3r^2\cos\theta \qquad a_\theta = 2r^2\cos\phi \qquad a_\phi = -r^2\cos\theta\sin\phi$$

Use the formula for the divergence in spherical coordinates,

$$
\begin{aligned}
\nabla \cdot \mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}3r^4\cos\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}2r^2\cos\phi\sin\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) \\
&= 12r\cos\theta + 2r\cot\theta\cos\phi - r\cot\theta\cos\phi \\
&= 12r\cos\theta + r\cot\theta\cos\phi
\end{aligned}
$$

4. For the 45° "ice–cream cone" shown here (that is, a sector of a sphere of radius R, with $0 < r < R$ and $0 < \theta < \pi/4$) find the solid angle subtended by the cone (at the origin).



Do the angular integration but with the limits: $\phi : 0 \to 2\pi$ and $\theta : 0 \to \pi/4$. Get:

$$
\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/4} \sin\theta\, d\theta\, d\phi &= (2\pi)(-\cos\theta)\Big|_0^{\pi/4} = (2\pi)\left(-\frac{1}{\sqrt{2}}+1\right) = (2\pi)\frac{\sqrt{2}-1}{\sqrt{2}} \\
&= \pi(2-\sqrt{2}) \approx 1.840
\end{aligned}
$$

5. A vector field is given by

$$\mathbf{F}(\mathbf{r}) = 2x\sin y\,\mathbf{i} + (x^2\cos y + 4y\sin z)\,\mathbf{j} + 2y^2\cos z\,\mathbf{k}$$

a) By finding $\nabla \times \mathbf{F}$, show that $\mathbf{F}$ is a conservative field.

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2x\sin y & (x^2\cos y + 4y\sin z) & 2y^2\cos z \end{vmatrix} \\
&= \mathbf{i}(4y\cos z - 4y\cos z) + \mathbf{j}(0-0) + \mathbf{k}(2x\cos y - 2x\cos y) = \mathbf{0}
\end{aligned}
$$

b) Find the scalar field ϕ for which $\mathbf{F} = \nabla\phi$.

If $\nabla\phi = \mathbf{F}$ then

$$\frac{\partial\phi}{\partial x} = F_x = 2x\sin y \qquad \Longrightarrow \qquad \phi = x^2\sin y + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = F_y = (x^2\cos y + 4y\sin z) \qquad \Longrightarrow \qquad \phi = x^2\sin y + 2y^2\sin z + f(x,z)$$

$$\frac{\partial \phi}{\partial z} = F_z = 2y^2 \cos z \qquad \Longrightarrow \qquad \phi = 2y^2 \sin z + f(x, y)$$

These conditions can be fulfilled by

$$\phi = x^2 \sin y + 2y^2 \sin z + C$$

c) For this field, find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where the path C goes from the origin to the point $(2, 2, 3)$.

Using the result from (b), we have

$$\int_{(0,0,0)}^{(2,2,3)} \mathbf{F} \cdot d\mathbf{r} = \phi\Big|_{(0,0,0)}^{(2,2,3)} = (2)^2 \sin(2) + 2(2)^2 \sin(3) - 0 - 0 = 4\sin(2) + 8\sin(3)$$

6. For the vector field

$$\mathbf{a} = (x + y)\,\mathbf{i} + (y - x)\,\mathbf{j}$$

find the line integral $\int_C \mathbf{a} \cdot d\mathbf{r}$ from the point $(0, -1)$ to $(2, 3)$ along two paths:

a) The parabola $y = x^2 - 1$

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \int_C [(x + y)dx + (y - x)dy]$$

One the given path, $y = x^2 - 1$, $dy = 2x\,dx$ and $x : 0 \to 2$. Then:

$$\begin{aligned} \int_C \mathbf{a} \cdot d\mathbf{r} &= \int_C [(x + x^2 - 1)\,dx + (x^2 - 1 - x)(2x)\,dx] = \int_0^2 (2x^3 - x^2 - x - 1)\,dx \\ &= \left(\tfrac{1}{2}x^4 - \frac{x^3}{3} - \frac{x^2}{2} - x\right)\Big|_0^2 = \left(\tfrac{1}{2}16 - \frac{8}{3} - 2 - 2\right) = 4 - \frac{8}{3} = \frac{4}{3} \end{aligned}$$

b) The straight line from $(0, -1)$ to $(2, -1)$ and then the straight line from $(2, -1)$ to $(2, 3)$.

On the first part, $dy = 0$ and $y = -1$. This gives:

$$\int_{C_1} \mathbf{a} \cdot d\mathbf{r} = \int_0^2 (x - 1)\,dx = \frac{x^2}{2} - x\Big|_0^2 = 2 - 2 = 0$$

On the second part, $dx = 0$ and $x = 2$. This gives:

$$\int_{C_2} \mathbf{a} \cdot d\mathbf{r} = \int_{-1}^3 (y - 2)\,dy = \frac{y^2}{2} - 2y\Big|_{-1}^3 = \frac{9}{2} - 6 - \frac{1}{2} - 2 = -4$$

The total is -4, so $\int_C \mathbf{a} \cdot d\mathbf{r} = -4$.

7. Perform the integral $\oint_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a}$ is the vector field

$$\mathbf{a} = r^3 \sin\theta\,\hat{\mathbf{e}}_r + 4r^2 \cos^2\phi\,\hat{\mathbf{e}}_\theta + 4r^2 \tan\theta\hat{\mathbf{e}}_\phi$$

and the *closed* surface S is that of the upper hemisphere of the sphere of radius 2 centered at the origin.

On the round part of the (closed) hemispherical surface (constant r, $r = 2$), we use $da_r\,\hat{\mathbf{e}}_r = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{e}}_r$, where θ goes from 0 to $\pi/2$ and ϕ goes from 0 to 2π.

$$\begin{aligned}
\int_{S_1} \mathbf{a}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} (r^3\sin\theta) r^2 \sin\theta\, d\theta\, d\phi\big|_{r=2} \\
&= 32(2\pi)\int_0^{\pi/2} \sin^2\theta\, d\theta = 64\pi\left(\frac{\theta}{2} - \tfrac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = 64\pi\frac{\pi}{4} = 16\pi^2
\end{aligned}$$

The flat part is a surface of constant θ, with $\theta = \pi/2$. The area element is $da_\theta\,\hat{\mathbf{e}}_\theta = r\,dr\,d\phi\hat{\mathbf{e}}_\theta$, where r goes from 0 to 2 and ϕ goes from 0 to 2π.

$$\begin{aligned}
\int_{S_2} \mathbf{a}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2} (4r^2\cos^2\phi) r\, dr\, d\phi = 4\int_0^{2\pi}\cos^2\phi\, d\phi \int_0^2 r^3\, dr \\
&= 4\left(\frac{r^4}{4}\right)\Big|_0^2 \left(\frac{\phi}{2} + \tfrac{1}{4}\sin 2\phi\right)\Big|_0^{2\pi} = 4\left(\frac{16}{4}\right)\pi = 16\pi
\end{aligned}$$

The sum of the two parts gives

$$\int_S \mathbf{a}\cdot d\mathbf{S} = 16\pi^2 + 16\pi = 16\pi(\pi+1)$$

The integral can also be worked using the divergence theorem (although that was not the intention for this exam). The using the formula for $\nabla\cdot\mathbf{a}$ in spherical coordinates, the divergence of the field $\mathbf{a}$ is

$$\begin{aligned}
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^5\sin\theta) + \frac{1}{r\sin\theta}(\cos\theta\cos^2\phi) \\
&= 5r^2\sin\theta + 4r\cos^2\phi\cot\theta
\end{aligned}$$

Since $\oint_S \mathbf{a}\cdot d\mathbf{S} = \int_V (\nabla\cdot\mathbf{a})\, dV$, integrate over the volume:

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (5r^2\sin\theta + 4r\cos^2\phi\cot\theta) r^2\,\sin\theta\, dr\, d\theta\, d\phi \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (5r^4\sin^2\theta + 4r^3\cos^2\phi\cos\theta)\, dr\, d\theta\, d\phi
\end{aligned}$$

Do the terms separately and carefully; the first term is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 5r^4\sin^2\theta\, dr\, d\theta\, d\phi = 2^4(2\pi)\int_0^{\pi/2}\sin^2\theta\, d\theta = 64\pi\frac{\pi}{4} = 16\pi^2$$

The second term is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 4r^3\cos^2\phi\cos\theta\, dr\, d\theta\, d\phi = 2^4\pi\int_0^{\pi/2}\sin\theta\, d\theta = 16\pi$$

which again gives a total of $16\pi(\pi+1)$.

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \qquad \text{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\text{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad \hat{\mathbf{z}} = \mathbf{k} \tag{2}$$

$$\mathbf{i} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{j} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{k} = \hat{\mathbf{e}}_z \tag{3}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{4}$$

$$
\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{i} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}
$$

$$
d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi
$$

$$
da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta
$$

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(ra_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(ra_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
$$

$$
\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy
$$