Phys 4610, Fall 2007
Exam #1

1. The vector fields $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = 3xz\hat{\mathbf{x}} + y^2\hat{\mathbf{y}} + 4xz\hat{\mathbf{z}} \qquad \text{and} \qquad \mathbf{B} = 2x^2\hat{\mathbf{x}} + xy^2z\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$$

evaluate $(\mathbf{B}\cdot\nabla)\mathbf{A}$.

The operator $(\mathbf{B}\cdot\nabla)$ is given by:

$$(\mathbf{B}\cdot\nabla) = 2x^2\frac{\partial}{\partial x} + xy^2z\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$$

Then

$$\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \left(2x^2\frac{\partial}{\partial x} + xy^2z\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right)(3xz\hat{\mathbf{x}} + y^2\hat{\mathbf{y}} + 4xz\hat{\mathbf{z}}) \\
&= 2x^2(3z\hat{\mathbf{x}} + 4z\hat{\mathbf{z}}) + xy^2z(2y\hat{\mathbf{y}}) + z^2(3x\hat{\mathbf{x}} + 4x\hat{\mathbf{z}}) \\
&= (6x^2z + 3xz^2)\hat{\mathbf{x}} + 2xy^3z\hat{\mathbf{y}} + (8x^2z + 4xz^2)\hat{\mathbf{z}}
\end{aligned}$$

2. A vector field is given in cylindrical coordinates by

$$\mathbf{v} = sz\hat{\mathbf{s}} + s\cos\phi\hat{\boldsymbol{\phi}} + 2sz^2\hat{\mathbf{z}}$$

Evaluate $\nabla\cdot\mathbf{v}$ and verify the divergence theorem for a circular cylinder of radius 2 and length 1, coaxial with the z axis with its ends at $z=0$ and $z=1$.

Using the $\nabla\cdot$ operator in cylindrical coordinates,

$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(s^2z) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\cos\phi) + \frac{\partial}{\partial z}(2sz^2) \\
&= 2z - \sin\phi + 4sz
\end{aligned}$$

Integrate $\nabla\cdot\mathbf{v}$ over the given volume:

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{v}\,d\tau = \int_0^1 dz\int_0^{2\pi} d\phi\int_0^2 s\,ds[2z - \sin\phi + 4sz]$$

Note, the integral on ϕ for the $\sin\phi$ term gives zero, and there are no ϕ's in the other terms, giving a factor of 2π for the ϕ integral. We get:

$$\begin{aligned}
\int_{\mathcal{V}} \nabla\cdot\mathbf{v}\,d\tau &= 2\pi\int_0^1 dz\int_0^2[2sz + 4s^2z]ds = 2\pi\int_0^1 dz[s^2z + \tfrac{4}{3}s^3z]\Big|_0^2 \\
&= 2\pi\int_0^1 dz[4z + \tfrac{4}{3}8z] = 4\pi\int_0^1 dz[2z + \tfrac{16}{3}z] = 4\pi\left(1 + \tfrac{8}{3}(1)\right) \\
&= 4\pi\frac{11}{3} = \frac{44}{3}\pi
\end{aligned}$$

Now integrate $\mathbf{v} \cdot d\mathbf{a}$ over the surface.

On the bottom surface, $d\mathbf{a} = s\,ds\,d\phi(-\hat{\mathbf{z}})$. But note, with $z = 0$ on that surface, $\mathbf{v} \cdot d\mathbf{a}$ gives zero.

On the top surface, $d\mathbf{a} = s\,ds\,d\phi(\hat{\mathbf{z}})$, with $z = 1$. We get:

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi} d\phi \int_0^2 s\,ds[2sz^2]\Big|_{z=1} = 2\pi \int_0^2 2s^2 = 4\pi\tfrac{8}{3}\ .$$

On the side, $s = 2$, and $d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}}$, with $s = 2$. Then

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi} d\phi \int_0^1 2dz(2z) = 4\pi(1) = 4\pi$$

The total is

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = 4\pi(1 + \tfrac{8}{3}) = \frac{44}{3}\pi$$

3. Do the integrals

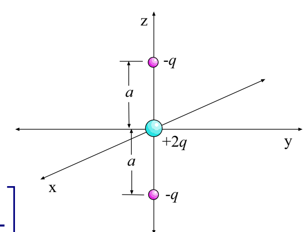
a) $\int_0^{50} x^2 \delta(x+3)dx$

The argument of the δ function is zero at $x = -3$, but the integration range does not include -3. So the integral is $\boxed{\text{zero}}$.

b) $\int \alpha \frac{\delta(r-R)}{2\pi r^3} d\tau$. where α and R are positive constants. The integral is over all space. Do this in spherical coordinates.

$$\int \alpha \frac{\delta(r-R)}{2\pi r^3}\,d\tau = 4\pi \int_0^\infty \frac{\alpha}{2\pi}\delta(r-R)\frac{r^2\,dr}{r^3} = 2\alpha\frac{R^2}{R^3} = \frac{2\alpha}{R}$$

4. What is the work required to assemble the system of charges shown here?

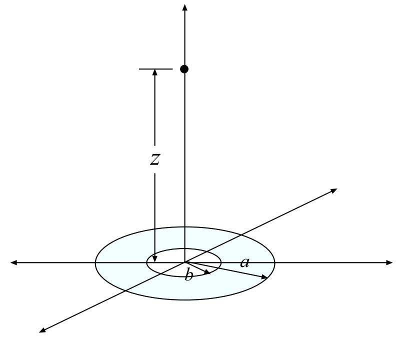


The work is a sum over the pairs of the point charges,

$$\begin{aligned} W &= \sum_{\text{pairs}} \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{4\pi\epsilon_0}\left[\frac{-2q^2}{a} + \frac{-2q^2}{a} + \frac{q^2}{2a}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\left[-4 + \tfrac{1}{2}\right] = -\frac{1}{4\pi\epsilon_0}\frac{7}{2}\frac{q^2}{a} \end{aligned}$$

5. A flat (circular) annulus in the xy plane has charge density σ; it has outer radius a and inner radius b.

Find the electric field and the electric potential at a point z on the z axis.



An element of charge at (s, ϕ) on the annulus has charge $dq = \sigma s\, ds\, d\phi$ and is at a distance $\sqrt{s^2 + z^2}$ from P. The vector $\mathscr{r}$ from the element to P makes an angle θ with the z axis, with

$$\cos\theta = \frac{z}{\sqrt{s^2 + z^2}}$$

The contribution of the charge element to E_z at P is

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\sigma s\, ds\, d\phi\, z}{(s^2+z^2)(s^2+z^2)^{3/2}} = \frac{\sigma z}{4\pi\epsilon_0}\frac{s\, ds\, d\phi}{(s^2+z^2)^{3/2}}$$

Integrate, with $s : b \to a$ and $\phi : 0 \to 2\pi$, then

$$\begin{aligned} E_z &= \frac{2\pi\sigma z}{4\pi\epsilon_0}\int_b^a \frac{s\, ds}{(s^2+z^2)^{3/2}} = \frac{\sigma z}{\epsilon_0}(-2)(-\tfrac{1}{2})(s^2+z^2)^{-1/2}\Big|_b^a \\ &= -\frac{\sigma z}{2\epsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right] = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{\sqrt{b^2+z^2}} - \frac{1}{\sqrt{a^2+z^2}}\right] \end{aligned}$$

To get the potential, integrate $\frac{dq}{4\pi\epsilon_0 \mathscr{r}}$, with no cosine factor!

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_b^a \frac{\sigma s\, ds}{\sqrt{s^2+z^2}} \\ &= \frac{2\pi\sigma}{4\pi\epsilon_0}(2)(\tfrac{1}{2})\sqrt{s^2+z^2}\Big|_b^a = \frac{\sigma}{2\epsilon_0}(\sqrt{a^2+z^2} - \sqrt{b^2+z^2}) \end{aligned}$$

6. Two parallel conducting plates have areas of 50.0 cm^2 and are separated by 2.00 mm. (You can thus approximate the plates as being infinite compared to their separation.) Charges of $\pm 5.0\,\mu\mathrm{C}$ are placed on the plates.

What is the magnitude of the force on an electron which is between the plates?

We showed that the E field between oppositely charged plates has magnitude $E = \sigma/\epsilon_0$ and so the force on an electron would have magnitude $F = qE = e\sigma/\epsilon_0$. With $\sigma = Q/A$ for the plates, we get

$$\begin{aligned} F = e\frac{Q/A}{\epsilon_0} &= (1.602\times 10^{-19}\ \mathrm{C})\frac{(5.0\times 10^{-6}\ \mathrm{C})}{(50\times 10^{-4}\ \mathrm{m}^2)(8.854\times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})} \\ &= 1.8\times 10^{-11}\ \mathrm{N} \end{aligned}$$

7. A solid sphere of radius R carries a charge density given by

$$\rho = \frac{k}{(r+a)^2}$$

a) Find the relation between k, R and the total charge Q of the sphere.

A solid sphere of radius R, with $\rho = \frac{k}{(r+a)^2}$. The total charge of the sphere is

$$Q = \int_{\mathcal{V}} \rho \, d\tau = 4\pi \int_0^R r^2 \rho \, dr = 4\pi k \int_0^R \frac{r^2}{(r+a)^2} dr$$

This integral is a bit messy. Tables give:

$$\begin{aligned} Q &= 4\pi k \left[a + r - 2a \log(a+r) - \frac{a^2}{a+r} \right] \Big|_0^R \\ &= 4\pi k \left[R - 2a \log\left(\frac{a+R}{a} \right) - \frac{a^2}{a+R} + a \right] \end{aligned}$$

This gives Q in terms of k, a, and R.

b) Find the electric field for points inside and outside the sphere.

Inside the sphere, make a spherical Gaussian surface of radius $r < R$. The charge enclosed is

$$Q_{\text{encl}} = 4\pi k \int_0^r \frac{r'^2 \, dr'}{(r'+a)^2} = 4\pi k \left[r - 2a \log\left(\frac{a+r}{a} \right) - \frac{ar}{a+r} \right]$$

which is messy. Since the E field must be radial, we have

$$\oint \mathbf{E} \cdot d\mathbf{a} = 4\pi r^2 E_r$$

on this surface, so by Gauss' law,

$$E_r = \frac{1}{4\pi\epsilon_0 r^2} Q_{\text{encl}} = \frac{k}{r^2 \epsilon_0} \left[r - 2a \log\left(\frac{a+r}{a} \right) - \frac{ar}{a+r} \right]$$

For $r > R$ the field is the same as a point charge at the origin, so

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \, ,$$

Q being the total charge of the sphere.

c) Discuss how you would calculate $V(r)$ for all points in space. (The math may be a little tedious to work out at this point, but just tell me how you'd do the calculation.)

To get the potential at points outside the sphere, do

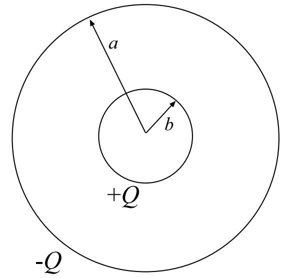
$$V(r) = -\int_\infty^r E_{r,\,out}(r') \, dr' = \frac{Q}{4\pi\epsilon_0 r}$$

and for points inside the sphere,

$$V(r) = -\int_\infty^R E_{r,\,out}(r') \, dr' - \int_R^r E_{r,\,in}(r') \, dr'$$

The last integral is probably messy, and we'll leave it in that form.

8. Two concentric conducting spherical shells of radii a and b contain total charges $-Q$ and Q, respectively.



a) What is the electric field for the regions $r < b$, $b < r < a$ and $r > a$?

There is no electric field for $r < b$ and $r > a$, by application of Gauss' law. Between the spheres, applying Gauss' law gives

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

b) What is the value of the energy density of the E field for these regions?

$$u = \frac{\epsilon_0}{2}E^2 = \frac{Q^2}{16\pi^2\epsilon_0 r^4} \qquad \text{for} \quad b < r < a\ .$$

$u = 0$ elsewhere.

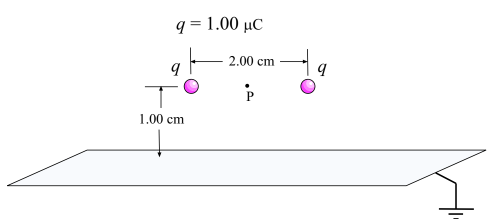
c) Find the total energy contained in the electric field.

Integrate the results of (b) over the volume between the spheres:

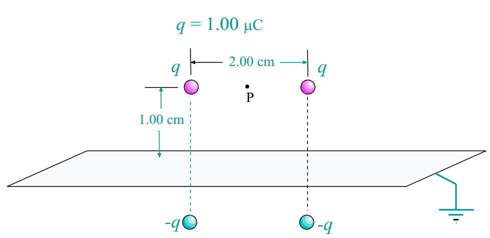
$$\begin{aligned} W &= \int_{\mathcal{V}} u\,d\tau = 4\pi \frac{Q^2}{16\pi^2\epsilon_0}\int_b^a \frac{r^2\,dr}{r^4} \\ &= \frac{q^2}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2} = -\frac{Q^2}{4\pi\epsilon_0}\Big|_b^a = \frac{Q^2}{4\pi\epsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right) \\ &= \frac{Q^2(a-b)}{4\pi\epsilon_0 ab} \end{aligned}$$

9. Two positive $1.0\,\mu$C charges are separated by 2.0 cm and both are 1.0 cm above an infinite conducting plane held at zero potential.

Find a numerical value for the value of the electric potential at the point which is half-way between them.



Solution to the problem uses two image charges below the plane. (See fig) Point P is 1 cm from the 2 $1.0\,\mu$C charges and



$$\sqrt{(1\ \text{cm})^2(2\ \text{cm})^2} = 2.24\ \text{cm}$$

from the $-1.0\,\mu$C. The potential at P is

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\sum\frac{q_i}{\mathscr{r}_i} = \frac{1}{4\pi\epsilon_0}\left[2\frac{(1.0\times10^{-6})}{(1.0\times10^{-2}\ \text{m})} - 2\frac{(1.0\times10^{-6})}{(2.24\times10^{-2}\ \text{m})}\right] \\ &= \frac{2(1.0\times10^{-6}\ \text{C})}{4\pi(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2})(1.0\times10^{-2}\ \text{m})}\left[1-\frac{1}{\sqrt{5}}\right] = 9.9\times10^5\ \text{V} \end{aligned}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + sd\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

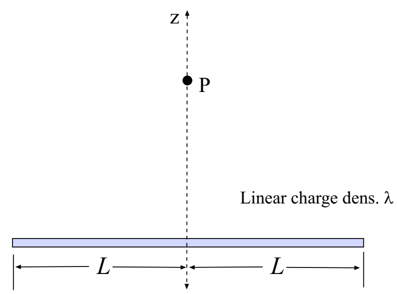
$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V$$

$$V = -\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' \qquad W = \frac{1}{2}\sum_{i=1}^{n}q_iV(\mathbf{r}_i) \qquad W = \frac{\epsilon_0}{2}\int_{\text{all space}}E^2\,d\tau$$

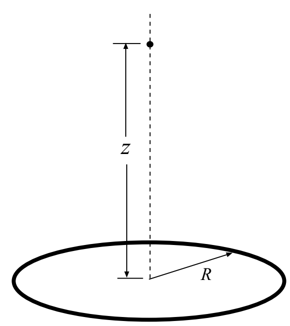
$$e = 1.602\times10^{-19}\ \text{C} \qquad \epsilon_0 = 8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2} \qquad \mu_0 = 4\pi\times10^{-7}\ \frac{\text{T}\cdot\text{m}}{\text{A}} \qquad c = 2.998\times10^{8}\ \frac{\text{m}}{\text{s}}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$E_z = \frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}$$



$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{Q}{2\pi R^2\epsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right] \end{aligned}$$