Phys 2920, Spring 2011
Exam #2

1. The operator

$$\mathsf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$$

(as written in the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ basis) is not simple!

a) Find a basis in which A is diagonal. Give the new (unit) basis vectors and state what the matrix A is in the new basis.

This matrix will be diagonal when expressed in a basis made of its *eigenvectors*. Find the eigenvalues of A; solve:

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1) = 0$$

So the eigenvalues are $\lambda = -1, 4$.

The eigenvector for $\lambda_1 = -1$ is

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

which gives

$$3x + 2y = -x \qquad \Longrightarrow \qquad y = -2x \qquad \Longrightarrow \qquad \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The eigenvector for $\lambda_2 = 4$ is

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

which gives

$$3x + 2y = 4x \qquad \Longrightarrow \qquad x = 2y \qquad \Longrightarrow \qquad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Putting the vectors into columns, the matrix S which gives the transformation is

$$\mathsf{S} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Since the vectors forming S are orthonormal, its inverse is the traspose,

$$\mathsf{S}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

By construction, the matrix A in the new basis *must* be diagonal with elements being the eigenvalues, but to check this:

$$
\begin{aligned}
\mathsf{A} &= \mathsf{S}^{-1}\mathsf{A}\mathsf{S} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\
&= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 8 \\ 2 & 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}
\end{aligned}
$$

as we ex[ected!

b) A vector expressed in the original $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ basis is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is it when expressed in the new basis? (Find $\mathbf{v}'$).
Hint: Are the new basis vectors orthogonal?)

We have:

$$\mathbf{v}' = \mathsf{S}^{-1}\mathbf{v} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The magnitude of this vector is

$$|\mathbf{v}'| = \frac{\sqrt{1+9}}{\sqrt{5}} = \sqrt{\frac{10}{5}} = \sqrt{2}$$

the same as the original vector; an orthogonal transformation preserves the norm!

2. Consider the point given by the Cartesian (rectangular) coordinates (-3, 0, 0).
a) What are the spherical coordinates of this point?

Being in the $z = 0$ plane, we have $\theta = \pi/2$. It is a distance 3 from the origin so $r = 3$. Since x has a negative value and y is zero, $\phi = \pi$. So

$$P = (3,\ \pi/2,\ \pi)$$

b) Express the unit vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$ for this point in terms of the Cartesian unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.

In the xy plane, the direction of increasing θ is "down", so $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{k}}$. The radial direction is the $-x$ direction, so $\hat{\mathbf{e}}_r = -\hat{\mathbf{i}}$. Finally $\hat{\mathbf{e}}_\phi$ points in the counterclockwise direction at P and that is the $-\hat{\mathbf{j}}$ direction, so $\hat{\mathbf{e}}_\phi = -\hat{\mathbf{j}}$. So:

$$\hat{\mathbf{e}}_r = -\hat{\mathbf{i}} \qquad \hat{\mathbf{e}}_\theta = -\hat{\mathbf{k}} \qquad \hat{\mathbf{e}}_\phi = -\hat{\mathbf{j}}$$

3. For the scalar field

$$\Phi = 2x^2y - 3xyz^2$$

a) Find the directional derivative of Φ at the point $P = (1, 3, 1)$ in the (vector) *direction* given by $2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$.

The gradient of Φ is

$$\nabla\Phi = (4xy - 3yz^2)\,\hat{\mathbf{i}} + (2x^2 - 3xz^2)\,\hat{\mathbf{j}} - 6xyz\,\hat{\mathbf{k}}$$

Evaluated at $(1, 3, 1)$ it is

$$\nabla\Phi\Big|_P = 3\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 18\,\hat{\mathbf{k}}$$

The *unit* vector in the given direction is

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{9}}(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}) = \frac{1}{3}(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}})$$

so the rate of change of Φ in that direction is

$$\frac{d\Phi}{ds} = \nabla\Phi\Big|_P \cdot \hat{\mathbf{a}} = \frac{1}{3}(6 + 1 + 36) = \frac{1}{3}(43) = \frac{43}{3}$$

b) In what direction from the point $P = (1, 3, 1)$ is the directional derivative a maximum?

That would be the same as the direction as $\nabla\Phi$; the unit vector in this direction is

$$\frac{1}{\sqrt{334}}(3\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 18\,\hat{\mathbf{k}})$$

4. Find the Laplacian (∇^2) of the scalar field

$$\Phi = 2\frac{\cos^2\theta\sin\phi}{r^2}$$

Using the formula for ∇^2 in spherical coordinates, we have

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \\
&= \frac{2}{r^2}\frac{\partial}{\partial r}\left(\frac{-2\cos^2\theta\sin\phi}{r}\right) + \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-2\cos\theta\sin^2\theta\sin\phi}{r^2}\right) + \frac{2}{r^2\sin^2\theta}\left(\frac{-\cos^2\theta\sin\phi}{r^2}\right) \\
&= \frac{4}{r^2}\frac{\cos^2\theta\sin\phi}{r^2} + \frac{4\sin\phi}{r^4\sin\theta}\left(\sin^3\theta - 2\sin\theta\cos^2\theta\right) - \frac{2}{r^4}\frac{\cos^2\theta}{\sin^2\theta}\sin\phi
\end{aligned}$$

Combine terms:

$$\begin{aligned}
\nabla^2\Phi &= \frac{2\sin\phi}{r^4}\left[2\cos^2\theta + 2(\sin^2\theta - 2\sin\theta\cos^2\theta) - \frac{\cos^2\theta}{\sin^2\theta}\right] \\
&= \frac{2\sin\phi}{r^4}\left[-2\cos^2\theta + 2\sin^2\theta - \cot^2\theta\right]
\end{aligned}$$

5. Find the divergence of the vector field

$$4x^3y^2\,\hat{\mathbf{i}} + yz^2\,\hat{\mathbf{j}} + 5xy\cos^2 z\,\hat{\mathbf{k}}$$

at the point $(1, 1, \pi)$.

The divergence of this field (call it a) is

$$\nabla \cdot \mathbf{a} = 12x^2y^2 + z^2 - 10xy\cos z \sin z$$

Evaluated at $(1, 1, \pi)$ this is

$$\nabla \cdot \mathbf{a} = 12 + \pi^2 + 0 = 12 + \pi^2$$

6. Find the curl of the vector field

$$\mathbf{a} = \rho \sin^2 \phi \hat{\mathbf{e}}_\rho + 2z\hat{\mathbf{e}}_\phi + \rho^2\hat{\mathbf{e}}_z$$

Use the formula for the curl in cylindrical coordinates. Note, a couple of the partial derivatives give zero. Get:

$$\begin{aligned} \nabla \times \mathbf{a} &= (0-2)\hat{\mathbf{e}}_\rho + (0-2\rho)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}[2z - 2\rho\sin\phi\cos\phi]\hat{\mathbf{e}}_z \\ &= -2\hat{\mathbf{e}}_\rho - 2\rho\hat{\mathbf{e}}_\phi + \left(\frac{2z}{\rho} - \sin 2\phi\right)\hat{\mathbf{e}}_z \end{aligned}$$

7. Do the line integral

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} \qquad \text{for} \qquad \mathbf{a} = (x+y)^2\,\hat{\mathbf{i}} - 5y\,\hat{\mathbf{j}}$$

from $A = (0, 0)$ to $B = (2, 1)$ for the two paths:

a) The line from $(0, 0)$ to $(2, 0)$ then from $(2, 0)$ to $(2, 1)$.

The basic integral is

$$I = \int_C \mathbf{a} \cdot d\mathbf{r} = \int_C [(x+y)^2\,dx - 5y\,dy]\ .$$

On the first part of the path, $dy = 0$ and $y = 0$, with $x : 0 \to 2$. Then

$$I_1 = \int_0^2 (x^2)dx = \frac{8}{3}$$

and on the next part, $dx = 0$ and $x = 2$ with $y : 0 \to 1$. Then

$$I_2 = -\int_0^1 (5y)\,dy = -\frac{5}{2}y^2\Big|_0^1 = -\frac{5}{2}$$

The total is

$$I = I_1 + I_2 = \frac{8}{3} - \frac{5}{2} = \frac{1}{6}$$

b) The straight line from A to B.

The path is given by

$$\mathbf{r} = 2t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} \qquad t : 0 \to 1 \qquad \Longrightarrow \qquad d\mathbf{r} = 2\,dt\,\hat{\mathbf{i}} + dt\,\hat{\mathbf{j}}$$

and this gives

$$\int_0^1 [(2t+t)^2 2\,dt - 5t\,dt] = \int_0^1 (18t^2 - 5t)\,dt = 6t^3 - \frac{5}{2}t^2\Big|_0^1 = 6 - \frac{5}{2} = \frac{7}{2}$$

(The answer is not the same as (a); this vector field is not conservative.)

8. We want to do the line integral $\int_A^B \mathbf{v}\cdot d\mathbf{r}$ for the field

$$\mathbf{v} = (8xy + 2x\cos z)\,\hat{\mathbf{i}} + 4x^2\,\hat{\mathbf{j}} - x^2 \sin z\,\hat{\mathbf{k}}$$

from $A = (0, 0, 0)$ to $B = (2, 2, \pi)$.

a) Show that you will get the same answer *irregardless* of the path from A to B.

The integral is path-independent if $\nabla \times \mathbf{a} = 0$. Test this:

$$\nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 8xy + 2x\cos z & 4x^2 & -x^2\sin z \end{vmatrix} = (0-0)\,\hat{\mathbf{i}} + (-2x\sin z + 2x\sin z)\,\hat{\mathbf{j}} + (8x - 8x)\,\hat{\mathbf{k}} = 0$$

so the field is conservative and the integral $\int_A^B \mathbf{a}\cdot d\mathbf{r}$ will not depend on the path.

b) Find the value of the integral.

We need to find a scalar field Φ of which $\mathbf{a}$ is the gradient. We note

$$a_x = \frac{\partial\Phi}{\partial x} = 8xy + 2x\cos z \qquad \Longrightarrow \qquad \Phi = 4x^2 y + x^2\cos z + f_1(y, z)$$

$$a_y = \frac{\partial\Phi}{\partial y} = 4x^2 \qquad \Longrightarrow \qquad \Phi = 4x^2 y + f_2(x, z)$$

$$a_z = \frac{\partial\Phi}{\partial z} = -x^2\sin z \qquad \Longrightarrow \qquad \Phi = x^2\cos z + f_3(x, y)$$

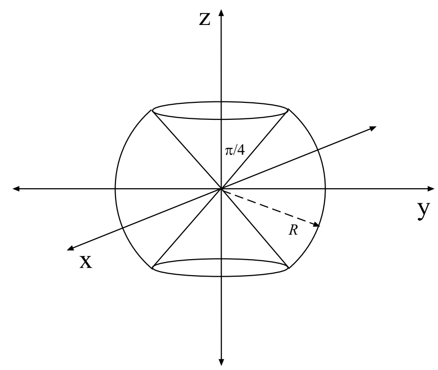
A solution is

$$\Phi = 4x^2 y + x^2\cos z + C$$

and then the value of the integral is

$$\int_A^B \mathbf{a}\cdot d\mathbf{r} = \Phi\Big|_A^B = (32 + 4(-1)) - (0) = 28$$

9. The volume shown at the right was formed by *removing* two "ice-cream cone" shapes from a solid sphere of radius R. The half-angle of the (removed) cones is $\pi/4$.



a) Find the volume of this shape.

The volume extends over the spherical coordintates

$$r : 0 \to R \qquad \theta : \pi/4 \to 3\pi/4 \qquad \phi : 0 \to 2\pi$$

Integrate the volume elements over this range:

$$\begin{aligned} V &= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^R r^2\,dr\sin\theta\,d\theta\,d\phi \\ &= (2\pi)\int_0^R r^2\,dr\int_{\pi/4}^{3\pi/4}\sin\theta\,d\theta = (2\pi)\frac{R^3}{3}\left(-\cos\theta\Big|_{\pi/4}^{3\pi/4}\right) \\ &= \frac{2\pi R^3}{3}\left(+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = \frac{4\pi R^3}{3\sqrt{2}} \end{aligned}$$

b) Write down the integral you would do to find its moment of inertia about the z axis. (You don't need to work the integral.)

The moment of inertia is the sum of mass elements times their squared distance from the z axis; if the object has uniform mass density ρ then the mass of a volume element is $\rho\,dV$ and its squared distance from the axis is $r^2\sin^2\theta$. Then the integral to be done is

$$\rho\int r^2\sin^2\theta\,dV = \rho\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^R r^4\,dr\sin^3\theta\,d\theta\,d\phi$$

Actually, a couple parts of this can be done immediately. Do the ϕ and r integrals to get

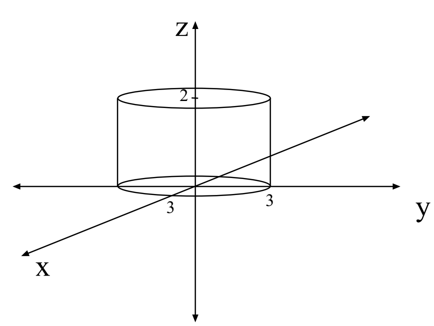
$$\Longrightarrow \qquad = (2\pi\rho)\frac{R^5}{5}\int_{\pi/4}^{3\pi/4}\sin^3\theta\,d\theta$$

and then the θ integral can be done. I'll leave it at this.

10. Find $\oint_S \mathbf{a}\cdot d\mathbf{S}$ where $\mathbf{a}$ is the vector field

$$\mathbf{a} = z\rho\sin^2\phi\hat{\mathbf{e}}_\rho + \rho z\hat{\mathbf{e}}_\phi + \rho^2\hat{\mathbf{e}}_z$$

and S is the surface of the cylinder of radius 3 and height 2 whose axis is the z and whose bottom surface is in the xy plane.



On the top surface we have

$$z = 2 \qquad d\mathbf{S} = da_z\,\hat{\mathbf{e}}_z = \rho\,d\rho\,d\phi\,\hat{\mathbf{e}}_z \qquad \rho : 0 \to 2 \qquad \phi : 0 \to 2\pi$$

so the surface integral picks out the z component of the vector field, giving:

$$\int \mathbf{a} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^3 (\rho^2) \rho \, d\rho \, d\phi = (2\pi) \int_0^3 \rho^3 \, d\rho$$

Then we get:

$$\Longrightarrow \qquad = 2\pi \frac{\rho^4}{4}\Big|_0^3 = 2\pi \frac{81}{4} = \frac{81\pi}{2}$$

However on the bottom surface we have

$$z = 0 \qquad d\mathbf{S} = da_z \, (-\hat{\mathbf{e}}_z) = -\rho \, d\rho \, d\phi \, \hat{\mathbf{e}}_z \qquad \rho : 0 \to 2 \qquad \phi : 0 \to 2\pi$$

so that

$$\int \mathbf{a} \cdot d\mathbf{S} = -\int_0^{2\pi} \int_0^3 (\rho^2) \rho \, d\rho \, d\phi = -(2\pi) \int_0^3 \rho^3 \, d\rho$$

so it cancels the first integral (since the integrand was independent of z). This leaves only the integral on the round part for which we have

$$\rho = 3 \qquad d\mathbf{S} = da_\rho \hat{\mathbf{e}}_\rho = \rho \, d\phi \, dz \hat{\mathbf{e}}_\rho \qquad \phi : 0 \to 2\pi \qquad z : 0 \to 2$$

so that the surface integral picks out the ρ component of $\mathbf{a}$, giving

$$\int \mathbf{a} \cdot d\mathbf{S} = \int_0^2 \int_0^{2\pi} (z\rho \sin^2 \phi) \rho \, d\phi \, dz$$

Separate the factors, get:

$$\Longrightarrow \qquad = (9) \left(\int_0^2 z \, dz \right) \left(\int_0^{2\pi} \sin^2 \phi \, d\phi \right) = 9 \frac{4}{2} \frac{2\pi}{2} = 18\pi$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{4}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{\imath}} + \sin\theta\sin\phi\,\hat{\mathbf{\jmath}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{\imath}} + \cos\theta\sin\phi\,\hat{\mathbf{\jmath}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{\imath}} + \cos\phi\,\hat{\mathbf{\jmath}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{\imath}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{\jmath}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$

$$\int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{x}{2} \qquad \int \cos^2 x\,dx = +\frac{1}{4}\sin 2x + \frac{x}{2}$$

Other integrals furnished upon request.