Phys 2920, Spring 2013
Exam #2

Do all matrix calculations by hand unless otherwise indicated. So you need to show your work.

1. The operator $\mathcal{A}$ is given by

$$\mathsf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

as written in the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ basis.

a) Suppose we want to re-express our vectors and operators in the new basis of the unit (orthonormal!) unit vectors

$$\hat{\mathbf{e}}'_1 = \frac{1}{2}(\hat{\mathbf{i}} + \sqrt{3}\,\hat{\mathbf{j}}) \qquad \hat{\mathbf{e}}'_2 = \frac{1}{2}(-\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

What is the transformation matrix S and its inverse S^{-1}? (Hint: S is an orthogonal matrix!)

The S matrix one constructs from this transformation is

$$\mathsf{S} = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

We note that as the new basis vectors are also orthonormal, S is an orthogonal matrix, so its inverse is its transpose;

$$\mathsf{S}^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

b) Express the operator A in the new basis; note, it won't (necessarily) be diagonal. Express the vector

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

in the new basis. *Check* your answer for $\mathbf{x}$.

Now do the grunt work of transforming. We find:

$$\begin{aligned} \mathsf{A}' &= \mathsf{S}^{-1}\mathsf{A}\mathsf{S} \\ &= \tfrac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \end{aligned}$$

Here the arithmetic is a little tedious, but my answer is

$$\mathsf{A}' = \tfrac{1}{4}\begin{pmatrix} -7+2\sqrt{3} & -2-5\sqrt{3} \\ -2-5\sqrt{3} & 3-2\sqrt{3} \end{pmatrix}$$

For $\mathbf{x}'$ we get

$$\mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2-\sqrt{3} \\ -1-2\sqrt{3} \end{pmatrix}$$

To check this, we directly compute $\mathbf{x}'$ in terms of the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ basis:

$$\begin{aligned} \mathbf{x}' &= \tfrac{1}{2}(2-\sqrt{3})\hat{\mathbf{e}}'_1 + \tfrac{1}{2}(-1-2\sqrt{3})\hat{\mathbf{e}}'_2 \\ &= \tfrac{1}{4}(2-\sqrt{3})(\hat{\mathbf{i}}+\sqrt{3}\hat{\mathbf{j}}] + \tfrac{1}{4}(-1-2\sqrt{3})(\sqrt{3}\hat{\mathbf{i}}+\hat{\mathbf{j}}) \\ &= \tfrac{1}{4}(2-\sqrt{3}+\sqrt{3}+6)\,\hat{\mathbf{i}} + \tfrac{1}{4}(2\sqrt{3}-3-1-2\sqrt{3})\,\hat{\mathbf{j}} \\ &= 2\hat{\mathbf{i}} - \hat{\mathbf{j}} = \mathbf{x} \end{aligned}$$

c) Explain what I would do if I wanted to find a basis in which A is diagonal.

Find the eigenvectors of A. Use these as new basis vectors. (When transformed the diagonal elements of A' will be the eigenvalues of A (and, trivially, A').

2. a) Consider the point given by the cylindrical coordinates $(2, \frac{\pi}{3}, -1))$. What are the spherical coordinates of this point?

$$x = 2\cos(\pi/3) = 1 \qquad y = 2\sin(\pi/3) = 2\cdot(\sqrt{3}/2) = \sqrt{3} \qquad z = -1$$

so

$$r = \sqrt{1+3+1} = \sqrt{5} \qquad \theta = \cos^{-1}(z/r) = \cos^{-1}(-1/\sqrt{5}) = 2.034 \qquad \phi = \phi = \frac{\pi}{3}$$

So the point is

$$(\sqrt{5},\ 2.034,\ \pi/3)$$

b) Consider the point given by the Cartesian coordinates $(1, -2, -3)$. What are the spherical coordinates of this point?

$$r = \sqrt{1+4+9} = \sqrt{14} \qquad \theta = \cos^{-1}(z/r) = 2.50 \qquad \phi = \tan^{-1}_{+\mathrm{brains}}(y/x) = 5.18$$

3. For the scalar field

$$\Phi = -4x^2z^2 + xy^2z$$

a) Find the rate of change of Φ at the point $(1, 1, 1)$ in the direction *parallel to* the vector $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$).

Take the gradient of Φ:

$$\nabla\Phi = (-8xz^2 + y^2z)\,\hat{\mathbf{i}} + (2xyz)\,\hat{\mathbf{j}} + (-8x^2z + xy^2)\,\hat{\mathbf{k}}$$

which when evaluated at the (easy) point $(1,1,1)$ gives

$$\nabla\Phi\Big|_{(1,1,1)} = -7\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 7\,\hat{\mathbf{k}}$$

The *unit* vector in the given direction is

$$\hat{\mathbf{a}} = \tfrac{1}{\sqrt{6}}(\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

so the rate of change of Φ in this direction is

$$\frac{d\Phi}{ds} = \nabla\Phi\cdot\hat{\mathbf{a}} = \tfrac{1}{\sqrt{6}}(-7-4-7) = -\frac{18}{\sqrt{6}} = -3\sqrt{6}$$

b) In what direction from the point $P = (1,1,1)$ is the directional derivative a maximum?

That is in the direction of the gradient itself, which as a unit vector is

$$\frac{\nabla\Phi}{|\nabla\Phi|} = \frac{1}{\sqrt{102}}(-7\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 7\,\hat{\mathbf{k}})$$

4. In spherical coordinates, specify

a) The yz plane.

Since we only have positive r we have to specify

$$\phi = \frac{\pi}{2} \qquad \text{or} \qquad \phi = \frac{3\pi}{2}$$

b) The cone $5z^2 = x^2 + y^2$.

This gives

$$5r^2\cos^2\theta = r^2\sin^2\theta \qquad\Longrightarrow\qquad \tan^2\theta = 5 \qquad\Longrightarrow\qquad \tan\theta = \pm\sqrt{5}$$

This gives

$$\theta = 1.15 \qquad \text{or} \qquad \theta = 1.99$$

5. Find the Laplacian (∇^2) of the scalar field

$$\Phi = 3\rho^3\,\cos 2\phi\,z^4$$

Follow the formula and get

$$\begin{aligned}
\nabla^2\Phi &= \cos 2\phi\,z^4\frac{1}{\rho}\frac{\partial}{\partial\rho}(9\rho^3) + \frac{1}{\rho^2}(-12)\rho^3\cos 2\phi + 36\rho^3\cos 2\phi z^4 \\
&= 27\rho z^4\cos 2\phi - 12\rho z^4\cos 2\phi + 36\rho^3\cos 2\phi z^4 \\
&= 15\rho z^4\cos 2\phi + 36\rho^3\cos 2\phi z^4
\end{aligned}$$

6. Find the divergence of the vector field

$$\mathbf{a} = 3r^2\ \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + 2r^2\ \cos^2\theta\,\hat{\mathbf{e}}_\theta - \cos^2\theta\,\hat{\mathbf{e}}_\phi$$

at the spherical point $(2, \frac{\pi}{2}, \pi)$.

$$\begin{aligned}\nabla\cdot\mathbf{a} &= \frac{1}{r^2}12r^3\sin\theta\cos\phi + \frac{2r^2}{r\sin\theta}[\cos^3\theta - 2\sin^2\theta\cos\theta] \\ &= 12r\sin\theta\cos\phi + 2r[\cot\theta\cos^2\theta - 2\sin\theta\cos\theta]\end{aligned}$$

The the spherical point $(2, \frac{\pi}{2}, \pi)$, this is

$$\nabla\cdot\mathbf{a} = 12\cdot 2\cdot 1\cdot(-1) - 4\cdot 00 - 0) = -24$$

7. Find the curl of the vector field

$$\mathbf{a} = r^2\sin\theta\,\hat{\mathbf{e}}_r + 2r^2\sin\phi\hat{\mathbf{e}}_\theta + b^2\cos^2\phi\,\hat{\mathbf{e}}_\phi$$

where b is some constant.

$$\begin{aligned}\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}[b^2\cos^2\phi\cos\theta - 2r^2\cos\phi]\hat{\mathbf{e}}_r + \frac{1}{r}[0 - b^2\cos^2\phi\cdot 1]\hat{\mathbf{e}}_\theta + \frac{1}{r}[2\sin\phi\cdot 3r^2 - r^2\cos\theta]\hat{\mathbf{e}}_\phi \\ &= \frac{1}{r\sin\theta}[b^2\cos^2\phi\cos\theta - 2r^2\cos\phi]\hat{\mathbf{e}}_r - \frac{b^2}{r}\cos^2\phi\,\hat{\mathbf{e}}_\theta + [6r\sin\phi - r\cos\theta]\,\hat{\mathbf{e}}_\phi\end{aligned}$$

8. Do the line integral $\int_A^B \mathbf{a}\cdot d\mathbf{r}$ where

$$\mathbf{a} = xy\,\hat{\mathbf{i}} - 2y^2\,\hat{\mathbf{j}}$$

where $A = (0,0)$ and $B = (2,3)$ and where the path from A to B is:
a) The line from $(0,0)$ to $(2,0)$ then from $(2,0)$ to $(2,3)$.

On the first part of the path, $d\mathbf{r} = dx\ \hat{\mathbf{i}}$, $y = 0$ and

$$\int_1 \mathbf{a}\cdot d\mathbf{r} = \int_0^2 x\cdot 0dx = 0$$

One the second part, $d\mathbf{r} = dy\ \hat{\mathbf{j}}$, $x = 2$ so

$$\int_2 \mathbf{a}\cdot d\mathbf{r} = \int_0^3 (-2y^2)\,dy = -2\frac{y^3}{3}\Big|_0^3 = -2\cdot 9 = -18$$

So the the total integral is -18.

b) The straight line from $(0,0)$ to $(2,3)$.

Parametrize the path with

$$x = 2t \qquad y = 3t \qquad t : 0 \to 1 \qquad d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} = 2\,dt\,\hat{\mathbf{i}} + 3\,dt\,\hat{\mathbf{j}}$$

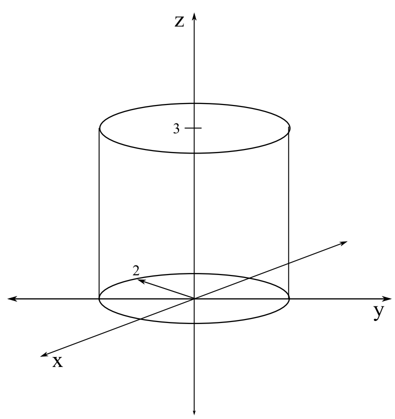
Then the integral is

$$\begin{aligned} \text{Int} &= \int_0^1 (2t)(3t)(2\,dt) - 2\int_0^1 (3t)^2\,3\,dt \\ &= 12\int_0^1 t^2\,dt - 54\int_0^1 t^2\,dt = -42\int_0^1 t^2\,dt \\ &= -42 \cdot \tfrac{1}{3} = -14 \end{aligned}$$

9. Find $\int_V \Phi(\mathbf{r})\,dV$ where Φ is the scalar field

$$\Phi(\rho, \phi, z) = 3\rho^2 z^2 \sin^2\phi$$

and the volume V is the circular cylinder of radius 2 and height 3, concentric with the z axis and whose base is in the xy plane.



The integral is

$$\begin{aligned} \text{Int} &= \int_0^3 dz \int_0^{2\pi} d\phi \int_0^2 \rho\,d\rho\,3\rho^2 z^2 \sin^2\phi \\ &= \int_0^2 3\rho^3\,d\rho \int_0^3 z^2\,dz \int_0^{2\pi} \sin^2\phi\,d\phi = \tfrac{3}{4}\,2^4 \cdot \tfrac{1}{3}\,3^3 \cdot \left.\frac{\phi}{2}\right|_0^{2\pi} \\ &= 12 \cdot 9 \cdot \pi = 108\pi \end{aligned}$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{4}$$

$$
\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{i}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{j}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}
$$

$$
d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi
$$

$$
da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta
$$

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
$$

$$
\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy
$$

$$
\int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}
$$

$$
\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{x}{2} \qquad \int \cos^2 x\,dx = +\frac{1}{4}\sin 2x + \frac{x}{2}
$$

Other integrals furnished upon request.