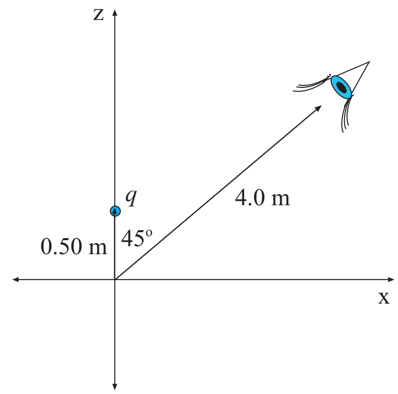
Phys 4610, Fall 2005
Exam #1

1. Jan, the observer is located 4.0 m away from the origin at an angle of 45° from the z axis. A point charge is located on the z axis at $z = 0.50$ m.

Calculate $\mathscr{r}$ for this observer and source.



The location of the observer Jan is (assume she is in the xz plane):

$$
\begin{aligned}
\mathbf{r} &= (4.0 \text{ m} \cos 45°)\hat{\mathbf{x}} + (4.0 \text{ m} \sin 45°)\hat{\mathbf{y}} \\
&= = (2.8 \text{ m})\hat{\mathbf{x}} + (2.8 \text{ m})\hat{\mathbf{z}}
\end{aligned}
$$

The location of the source is

$$\mathbf{r}' = (0.50 \text{ m})\hat{\mathbf{z}}$$

so then the magnitude of the separation vector $\mathscr{r}$ is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'| = \sqrt{(2.8 \text{ m})^2 + (2.8 \text{ m} - 0.50 \text{ m})^2} = 3.7 \text{ m}$$

2. Show that if

$$T = r^2(3\cos^2\theta - 1)$$

then $\nabla^2 T = 0$

You can do this any way you want, but of course you need to explain what you did.

Using the formula for the Laplacian in spherical coordinates,

$$
\begin{aligned}
\nabla^2 T &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(2r^3)(3\cos^2\theta - 1) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(r^2)(-6\cos\theta\sin\theta) \\
&= 6(3\cos^2\theta - 1) + \frac{(-6)}{\sin\theta}(-\sin^3\theta + 2\sin\theta\cos^2\theta) \\
&= 6(3\cos^2\theta - 1) - 6(-\sin^2\theta + 2\cos^2\theta)
\end{aligned}
$$

Use

$$-\sin^2\theta + 2\cos^2\theta = -(1 - \cos^2\theta) + 2\cos^2\theta) = -1 + 3\cos^2\theta \text{ , then}$$

$$\nabla^2 T = 6(3\cos^2\theta - 1) - 6(-1 + 3\cos^2\theta) = 0$$

One can also use $z = r\cos\theta$ and $r^2 = x^2 + y^2 + z^2$ to write

$$T = 3r^2\cos^2\theta - r^2 = 3z^2 - (x^2 + y^2 + z^2) = 2z^2 - x^2 - y^2$$

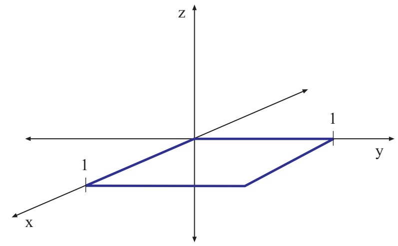
then

$$\nabla^2 T = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T = 4 - 2 - 2 = 0$$

3. Suppose

$$\mathbf{v} = (2x^2 - y)\hat{\mathbf{y}} + 4y\hat{\mathbf{z}}$$

Verify that Stokes' theorem works for the square surface shown at the right.



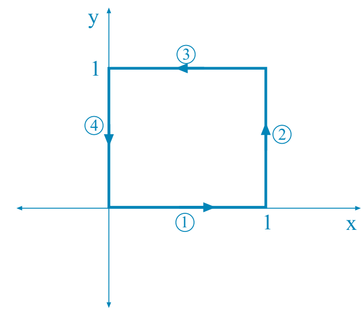
Find $\nabla \times \mathbf{v}$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 0 & (2x^2 - y) & 4y \end{vmatrix} = 4\hat{\mathbf{x}} + 4x\hat{\mathbf{z}}$$

Stokes' theorem sez $\int_{\mathcal{S}} \nabla \times \mathbf{v} \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$. On the lhs we get (with $d\mathbf{a} = \hat{\mathbf{z}}da$)

$$\int_{\mathcal{S}} \nabla \times \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 dy(4x) = (4)(1) \int_0^1 x\, dx = 2$$

On the rhs we do an integral on the four parts of the path.
On 1, $d\mathbf{l} = dx\hat{\mathbf{x}}$, so $\mathbf{v} \cdot d\mathbf{l} = 0$.
On 2, $d\mathbf{l} = dy\hat{\mathbf{y}}$ and $x = a$, so



$$\int_2 \mathbf{v} \cdot d\mathbf{l} = \int_0^1 (2 - y)dy = 2 - \int_0^1 y\, dy = \frac{3}{2}$$

On 3, $d\mathbf{l} = dx\hat{\mathbf{x}}$, so $\mathbf{v} \cdot d\mathbf{l} = 0$.
On 4, $d\mathbf{l} = dy\hat{\mathbf{y}}$ and $x = 0$ so

$$\int_4 \mathbf{v} \cdot d\mathbf{l} = \int_1^0 (0 - y)\, dy = \int_0^1 y\, dy = \frac{1}{2}$$

Adding up the parts,

$$\oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l} = 2$$

so Stokes' theorem checks out.

4. Find the divergence of the function

$$\mathbf{v} = s(2 + \sin 2\phi)\, \hat{\mathbf{s}} + s \sin\phi \cos\phi\, \hat{\boldsymbol{\phi}} + 3z\, \hat{\mathbf{z}}$$

Use the formula for the divergence in cylindrical coordinates. Get:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}[s^2(2 + \sin 2\phi)] + \frac{1}{s}\frac{\partial}{\partial \phi}[s \sin\phi \cos\phi] + \frac{\partial}{\partial z}(3z) \\ &= 2(2 + 2\sin\phi) + (\cos^2\phi - \sin^2\phi) + 3 \\ &= 7 + 2\sin 2\phi + \cos 2\phi \end{aligned}$$

5. Evaluate

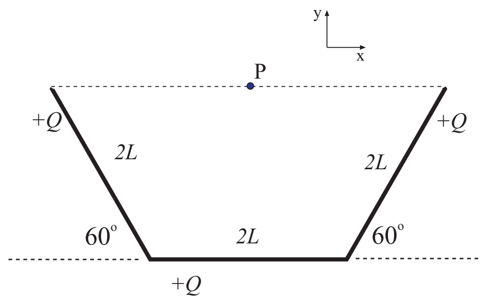
$$\int_0^\infty \left[x^3\delta(x+5) - (x^2+5)\delta(x-4)\right]\,dx$$

The first term can only contribute if the integration range include $x = -5$, which it doesn't. It does include $x = 4$ so the second term does contribute, giving:

$$\implies \;\; = -(4^2+5) = -21$$

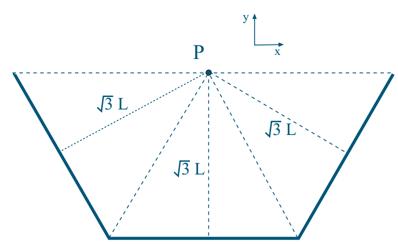
6. Three thin rods each of length $2L$ and uniformly charged with a total charge $+Q$ (each) are arranged in a plane as shown.

Find the magnitude and direction of the electric field at the point P.



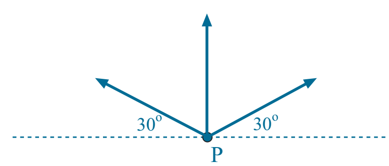
By geometry, the point P is a distance $\sqrt{3}L$ from the midpoint of each segment.

Now, from the result shown in Chap. 2 and included on the exam, each segment contributes an E field of magnitude



$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{(\sqrt{3}L)\sqrt{3L^2+L^2}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{(\sqrt{3}L)} = \frac{Q}{8\sqrt{3}\pi\epsilon_0 L^2}$$

Two of the contributions are directed at 30° fom the x axis, giving a net y component of

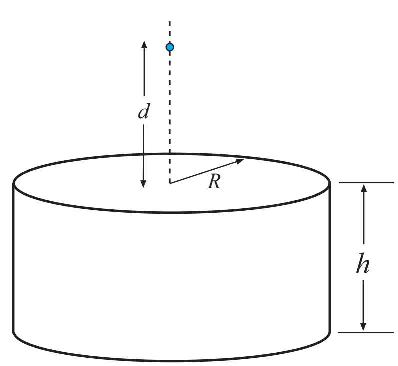


$$E_y = (2\sin 30^\circ + 1)E = 2E = \frac{Q}{4\sqrt{3}\pi\epsilon_o L^2}$$

(The E field points in the y direction.)

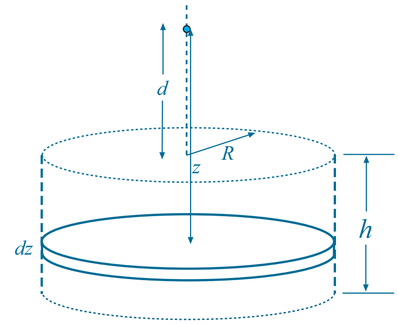
7. Find the electric field on the axis a distance d above a uniformly charged cylinder of radius R and height h. The charge density of the cylinder is ρ.

It will be easiest to use the result for the electric field above a charged disc and set up an integral.



We have the result for the E field at a distance z above the center of a uniformly--charged disk, and we can form the cylinder by adding up a lot of charged disks!

A thin disk of thickness dz at a distance z from the observation point contains a charge $dQ = \rho\pi R^2 dx$ and so it contributes a field



$$E = \frac{dQ}{2\pi R^2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] = \frac{\rho dx}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$

at the observation point. Adding up the disks as z ranges from d to $d+h$, get

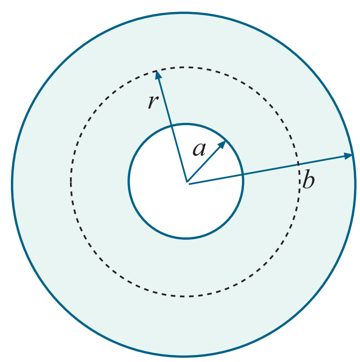
$$\begin{aligned} E &= \int_d^{d+h} \frac{\rho dz}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{\rho}{2\epsilon_0}[z - \sqrt{z^2+R^2}\Big|_d^{d+h} = \frac{\rho}{2\epsilon_0}[h - \sqrt{(d+h)^2+R^2} + \sqrt{d^2+R^2}] \end{aligned}$$

8. A hollow spherical shell carries a charge density

$$\rho = \frac{k}{r}$$

in the region $a < r < b$. Find the electric field in the region $a < r < b$

The electric field at r points in the radial direction. Draw a Gaussian surface of radius r. Since $\rho = k/r$, the charge contained in the surface is



$$\begin{aligned} Q_{\text{enc}} &= 4\pi\int_a^r \left(\frac{k}{r'}\right) r'^2 dr' \\ &= 4\pi k\int_a^r r' dr' = 4\pi k \tfrac{1}{2}(r^2 - a^2) = 2\pi k(r^2 - a^2) \end{aligned}$$

E^r has the same value over the surface, so

$$\oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} = E_r(4\pi r^2)$$

Then Gauss' law gives

$$E_r(4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{2\pi k(r^2-a^2)}{\epsilon_0} \qquad \Longrightarrow \qquad E_r = \frac{k}{2\epsilon_0}\frac{(r^2-a^2)}{r^2}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2 \, \sin\theta \, dr \, d\theta \, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

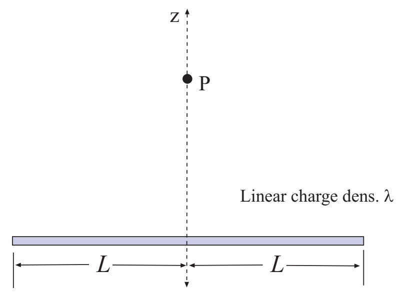
Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{Q}{2\pi R^2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \end{aligned}$$