Phys 3810, Spring 2011
Exam #3

1. What is meant by:

a) Gyromagnetic ratio (for a particle).

The (constant) ratio of the magnetic moment to the angular momentum for a particle.

b) Larmor precession. *Hint: This arose from considering a spin in a uniform B field...*

A magnetic momentum placed in a uniform B field $\mathbf{B} = B_0\,\hat{\mathbf{k}}$ in a *particular* spin state will maintain its expectation values of S_z but the values for S_x and S_y will undergo oscillations, as if the spin vector were a definite vector precessing about the z axis.

c) Degeneracy pressure.

A pressure (change of E with V) which arises in a multi-fermion system *not* because of a repulsive potential between the particles but because the fermions cannot occupy the same quantum states by antisymmetry (i.e. the Pauli principle). In effect their wave functions avoid one another so as not to overlap.

d) The (so-called) exchange force. *Hint: This arises in a discussion of states of a small number of identical particles.*

A related idea; in a system of bosons or fermions the symmetry/antisymmetry of the wave functions give an effective attraction or repulsion between the particles.

2. Although we can work with a coordinate representation of the spin–1 wave functions (for orbital angular momentum) we can also discuss them in the matrix (intrinsic) form.

a) Write down the matrices for S^2 and S_z (corresponding to the S_z eigenstates).

The eigenvalue of S^2 is $\hbar^2 \cdot 1 \cdot (1+1) = 2\hbar^2$ for all the sub-states, and the eigenvalues of S_z are $\hbar, 0, -\hbar$. Thus:

$$S^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S_z = 2\hbar^2 \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

b) Find the matrices for S_- and S_+.

The action of S_+ and S_- on the states is

$$S_+\,|1\ -1\rangle = \sqrt{2}\hbar|1\ 0\rangle \qquad S_+\,|1\ 0\rangle = \sqrt{2}\hbar|1\ 1\rangle \qquad S_+\,|1\ 1\rangle = 0$$

$$S_-\,|1\ -1\rangle = 0 \qquad S_-\,|1\ 0\rangle = \sqrt{2}\hbar|1\ -1\rangle \qquad S_+\,|1\ 1\rangle = \sqrt{2}\hbar|1\ 0\rangle$$

Then the matrices which perform these actions are

$$S- = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad S_+ = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

c) Find the matrix for S_x.

From its definition $S_\pm = S_x \pm iS_y$ we have

$$S_x = \tfrac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

d) Find the eigenvalues of S_x. (It should be clear what they should be, but show all your work on all parts.)

Before even starting we expect that the eigenvalues of S_x will be the same asthose of S_z, namely $-\hbar, 0 + \hbar$. Let's hope we get that! Solve for λ in

$$|S_x - \lambda \mathbf{1}| = \begin{vmatrix} -\lambda & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\lambda \end{vmatrix} = 0$$

This gives the equation

$$0 = -\lambda^3 + \frac{\hbar^2}{2}\lambda + \frac{\hbar^2}{2}\lambda = -\lambda^3 + \hbar^2\lambda \qquad \Longrightarrow \qquad \lambda(\lambda^2 - \hbar^2) = 0$$

which does have roots $-\hbar, 0, +\hbar$.

3. An electron is in the spin state (for the z basis)

$$\chi = A \begin{pmatrix} 2 \\ 5i \end{pmatrix}$$

a) Find the normalization constant A

We must have $\chi^\dagger \chi = 1$, so

$$\chi^\dagger \chi = |A|^2 \begin{pmatrix} 2 & -5i \end{pmatrix} \begin{pmatrix} 2 \\ 5i \end{pmatrix} = |A|^2(29) = 1 \qquad \Longrightarrow \qquad A = \frac{1}{\sqrt{29}}$$

b) Find the expectation values of S_z and S_y for this state.

$$\langle S_z\rangle = \frac{1}{29}\begin{pmatrix}2 & -5i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}2\\ -5i\end{pmatrix} = \frac{\hbar}{58}\begin{pmatrix}2 & -5i\end{pmatrix}\begin{pmatrix}2\\ -5i\end{pmatrix} = -\frac{\hbar}{58}(4-25) = -\frac{21\,\hbar}{58}$$

$$\langle S_y\rangle = \frac{1}{29}\begin{pmatrix}2 & -5i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}2\\ 5i\end{pmatrix} = \frac{\hbar}{58}\begin{pmatrix}2 & -5i\end{pmatrix}\begin{pmatrix}5\\ 2i\end{pmatrix} = \frac{\hbar}{58}(10+10) = \frac{10\,\hbar}{29}$$

4. Consider a quantum state where an angular momentum $s_1 = 2$ combines with an angular momentum $s_2 = \frac{3}{2}$ to give a state with total angular momentum $s = \frac{5}{2}$ and z component $\frac{3}{2}$.

Write out the state $|\frac{5}{2}\ \frac{3}{2}\rangle$ as a linear combination of states where the individual z components are determinate, that is, states of the form $|2\ m_1\rangle\,|\frac{3}{2}\ m_2\rangle$.

What can I say? Just read the coefficients off the table of CG coefficients. We get:

$$|\tfrac{5}{2}\ \tfrac{3}{2}\rangle = \sqrt{\tfrac{16}{35}}\,|2\ 2\rangle\,|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{35}}\,|2\ 1\rangle\,|\tfrac{3}{2}\ +\tfrac{1}{2}\rangle - \sqrt{\tfrac{18}{35}}\,|2\ 0\rangle\,|\tfrac{3}{2}\ \tfrac{3}{2}\rangle$$

5. The electron in a hydrogen atom is in an eigenstate of *total* angular momentum from its orbital and spin parts. If we say that the total angular momentum is $\mathbf{J} = \mathbf{L} + \mathbf{S}$, then it is in a state with $j = \frac{5}{2}$, $m_j = \frac{3}{2}$.

If it has the wave function

$$R_{32}\left(a_1 Y_2^1 \chi_+ + a_2 Y_2^2 \chi_-\right)$$

Find the coefficients a_1 and a_2. *Hint: This also has something to do with those C-G coefficients.*

This is also a C-G sum, where we form a state with $j = \frac{5}{2}$ and $m_j = \frac{3}{2}$ (similar to prob 4) but here it is formed from an orbital angular momentum 2 coupled to a spin angular momentum $\frac{1}{2}$.

For the C-G table, find the coefficients to form the $|\frac{5}{2}\ \frac{3}{2}\rangle$ state from the states $|2\ m_l\rangle|\frac{1}{2}\ m_s\rangle$:

$$|\tfrac{5}{2}\ \tfrac{3}{2}\rangle = \sqrt{\tfrac{4}{5}}\,|2\ 1\rangle\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{5}}\,|2\ 2\rangle\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

So the coefficients which would give a state with $j = \frac{5}{2}$ and $m_j = \frac{3}{2}$ are

$$a_1 = \sqrt{\frac{4}{5}} \qquad a_2 = \frac{1}{\sqrt{5}}$$

6. Explain how it comes about that we should use the reduced mass $\mu = \frac{m_1 m_2}{(m_1+m_2)}$ in the Schrödinger equation for the H atom instead of the electron mass m_e.

A mathematical derivation is not required here but you should carefully *describe* the math which goes into identifying μ.

For the two--body Schrödinger equation with a potential that only depends on the distance between the particles (r) one can do a separation of variables between the CM location $\mathbf{R}$ and the separation vector $\mathbf{r} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$. After this is done the $\mathbf{r}$ equation looks just like a single--particle equation but with μ in place of the single--particle mass m (it all comes out of the algebra). The energy value we get from solving the $\mathbf{r}$ equation (with μ) is the energy for relative motion, to which must be added the CM kinetic energy.

7. The NO molecule behaves like two point masses joined by a spring of force constant $k = 1530\,\frac{\mathrm{N}}{\mathrm{m}}$.

a) What is the reduced mass of the system? (The "atomic wts" of N and O are respectively $14.003\frac{\mathrm{g}}{\mathrm{mol}}$ and $15.995\frac{\mathrm{g}}{\mathrm{mol}}$, respectively.)

Find μ in $\frac{\mathrm{g}}{\mathrm{mol}}$, then convert:

$$\mu = \frac{(14.003)(15.995)}{(14.003 + 15.995)}\frac{\mathrm{g}}{\mathrm{mol}} = (7.466\,\frac{\mathrm{g}}{\mathrm{mol}}) \cdot \frac{\mathrm{mol}}{(6.022 \times 10^{23})} = 1.240 \times 10^{-26}\ \mathrm{kg}$$

b) What is the (classical) frequency of oscillation of the system?

Now, for two particles interacting via a potential $\frac{1}{2}k(r - r_0)^2$ the Schrödinger equation for the relative motion has a μ in place of m! The classical frequency we want is then

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{(1530\,\frac{\mathrm{N}}{\mathrm{m}})}{(1.240 \times 10^{-26}\ \mathrm{kg})}} = 3.51 \times 10^{14}\ \mathrm{s}^{-1}$$

(This is angular frequency, of course. The regular frequency ν is this divided by 2π, $f = 5.59 \times 10^{13}\ \mathrm{s}^{-1}$.)

c) What is the difference in energy of the two lowest (quantum) vibrational states?

As the energy levels of an oscillator are $E_n = \hbar\omega(n + \frac{1}{2})$, the energy difference of adjacent levels is just $\hbar\omega$,

$$\Delta E = \hbar\omega = (1.055 \times 10^{-34}\ \mathrm{J\cdot s})(3.51 \times 10^{14}\ \mathrm{s}^{-1}) = 3.70 \times 10^{-20}\ \mathrm{J} = 0.231\ \mathrm{eV}$$

8. a) Write down the term in the Hamiltonian for the He atom which gives the electron–electron repulsion.

Same as the classical expression for the potential energy of two charges $-e$ separated by a distance $|\mathbf{r}_1 - \mathbf{r}_2|$:

$$V(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|}$$

b) Recall that the bone-head wave function for the ground state of He is

$$\psi_{\mathrm{b-h}}(\mathbf{r}_1, \mathbf{r}_2) = \psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2)$$

where the ψ_{100} represent the hydrogenic wave functions for $Z = 2$.

Using this wave function, how could we get a lowest-order estimate for the effect on the energy of e–e repulsion? Explain and set up the integral; you don't need to actually do it.

A good guess is to find the average (expectation value) of the *operator* in part (a) for the state given by $\psi_{\rm b-h}$. (The only reason this would not give the *exact* contribution is that the true state of the system is not $\psi_{\rm b-h}$.) Thus we guess that a first correction to the energy would be given by

$$E^{(1)} = \int \psi^*_{\rm b-h}(\mathbf{r}_1, \mathbf{r}_2) \frac{e^2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|} \psi_{\rm b-h}(\mathbf{r}_1, \mathbf{r}_2)\, d^3r$$

We can substitute for the wave functions and in fact the integral can be done exactly though it's a bit challenging (solution was handed out in class) but I will leave it at this.

c) By the way, $\psi_{\rm b-h}(\mathbf{r}_1, \mathbf{r}_2)$ is *symmetric* for exchange of electron coordinates. How can we even consider such a wave function for the ground state?

Such a *spatial* wave function for electrons is legal if the *spin* part of the wave function is antisymmetric, since the wave function needs to be antisymmetric under exchange of *all* the coordinates.

d) If the first ionization energy of the Lithium $(Z = 3)$*atom* is 5.39 eV and the second ionization energy is 75.6 eV, what is the *total* energy of the neutral Li atom?

The first energy given is that required to make Li into Li^+. The second energy given is that required to make Li^+ into Li^{2+}. The energy required to make Li^{2+} into Li^{3+} is the ionization energy of a hydrogenic atom with $Z = 3$ which by now we know is proportional to Z^2. Thus

$$\mathrm{Li}^{2+} \to \mathrm{Li}^{3+} + e^- \ \ : 9(13.6\ \mathrm{eV} = 122.4\ \mathrm{eV}$$

so the energy of the Li atom is

$$5.39\ \mathrm{eV} + 75.6\ \mathrm{eV} + 122.4\ \mathrm{eV} = 203.4\ \mathrm{eV}$$

9. a) Find the numerical value for the Fermi energy for an electron gas at zero temperature at a fermion density of $8.50 \times 10^{28} /\ \mathrm{m}^3$

The Fermi energy E_F is related to the number density ρ by

$$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$$

Plug in the numbers:

$$E_F = \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{2(9.11 \times 10{-}31\ \mathrm{kg})}(3(8.50 \times 10^{28}\ \mathrm{m}^{-3})\pi^2)^{2/3} = 1.13 \times 10^{-18}\ \mathrm{J} = 7.05\ \mathrm{eV}$$

b) Are we justified in not worrying about relativity in this case?

The rest mass (energy) of an electron is $511\ \mathrm{keV}$ and the energy found in (a) (the maximum kinetic energy of the electrons) is much small than that, so we can certainly ignore relativity.

c) Calculate the "degeneracy pressure" for this Fermi gas.

Degeneracy pressure is related to the number density by

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

Plug in the numbers:

$$P = \frac{(3\pi^2)^{2/3}(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{5(9.11 \times 10^{-31}\ \mathrm{kg})}(8.50 \times 10^{28}\ \mathrm{m}^{-3})^{5/3} = 3.84 \times 10^{10}\ \frac{\mathrm{N}}{\mathrm{m}^2}$$

10. Give the meaning of an "energy band" for single–particle motion in a potential, at least insofar as we used the term in class. Be sure to state in what type potential the term has meaning, and if possible give me the meaning of the state specifier K.

An energy band is a group of relatively close-spaced energy levels for an electron moving in a *periodic potential*; between these set of levels there are relatively large "energy gaps" where no energy eigenvalue is possible for the given potential. Legal states are indexed by the number K which gives the change in phase of the wave function if we shift the coordinate x by a, where a is the period length of the potential.

11. From what we learned from doing the compact-star homework problems, what is the (basic) reason that a white dwarf star has a maximum mass? (Explain why it collapses in the mass is too big.)

The lesson from the homework problem was that as the electrons become more relativistic the relation between energy and density for the Fermi gas becomes such that the star is no longer stable against the pull of gravity..

12. In the computer "project" for this semester, explain why:
a) We could arbitrarily use the boundary condition $u'(0) = 1$ for the wave function. (Is the slope really "1" at $r = 0$?)

No, in the correct radial wave function it is undoubtedly not 1. However the value of the slope at $r = 0$ will be *proportional to the normalization* of the wave function and for now we don't care how the wave function is normalized; we only care about the behavior at large r.

b) We only had to look for the energy where the wave function at large r changed sign.

For the exact eigenvalue the wave function will approach zero at large r, but if the trial eigenvalue is slightly incorrect the eventually the wave function will blow up in the plus or minus direction. When we cross from "blowing up" to "blowing down" behavior (or vice versa) then we know that somewhere in between the wave function had to approach zero at large r.

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}} \qquad N_A = 6.022 \times 10^{23}\ \mathrm{mol}^{-1}$$

Physics

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

∞ Square Well: $$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

Harmonic Oscillator: $$V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$E_n = \hbar\omega(n+\tfrac{1}{2}) \qquad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2}$$

Free particle: $$\Psi_k(x) = Ae^{i(kx-\frac{\hbar k^2}{2m})t} \qquad v_{\text{phase}} = \frac{\omega}{k} \qquad v_{\text{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}\,dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,dx$$

Delta Fn Potl: $$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi$$

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)\qquad\frac{d^2\Phi}{d\phi^2}=-m^2\Phi\qquad\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+[\ell(\ell+1)\sin^2\theta-m^2]\Theta=0$$

$$Y_0^0=\sqrt{\frac{1}{4\pi}}\qquad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta\qquad Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)\qquad Y_2^{\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi}\qquad Y_2^{\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm 2i\phi}\text{etc.}$$

$$u(r)\equiv rR(r)\qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$$

$$a=\frac{4\pi\epsilon_0\hbar^2}{me^2}=0.529\times10^{-10}\text{ m}\qquad E_n=-\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}\equiv\frac{E_1}{n^2}\qquad\text{for}\quad n=1,2,3,\ldots$$

where $E_1=-13.6$ eV.

$$R_{10}(r)=2a^{-3/2}e^{-r/a}\qquad R_{20}(r)=\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-r/2a}\qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f=c\qquad E_\gamma=hf\qquad\frac{1}{\lambda}=R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)\qquad\text{where}\qquad R=\frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2=1.097\times10^7\text{ m}^{-1}$$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}\qquad[L_x,L_y]=i\hbar L_z\qquad[L_y,L_z]=i\hbar L_x\qquad[L_z,L_x]=i\hbar L_y$$

$$L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi}\qquad L_\pm=\pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\phi}\right)\qquad L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2f_l^m=\hbar^2l(l+1)f_l^m\qquad L_zf_l^m=\hbar mf_l^m$$

$$[S_x,S_y]=i\hbar S_z\qquad[S_y,S_z]=i\hbar S_x\qquad[S_z,S_x]=i\hbar S_y$$

$$S^2|s\;m\rangle=\hbar^2s(s+1)|s\;m\rangle\qquad S_z|s\;m\rangle=\hbar m|s\;m\rangle\qquad S_\pm|s\;m\rangle=\hbar\sqrt{s(s+1)-m(m\pm1)}\,|s\;m\pm1\rangle$$

$$\chi=\begin{pmatrix}a\\b\end{pmatrix}=a\chi_++b\chi_-\quad\text{where}\quad\chi_+=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad\chi_-=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathsf{S}^2=\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\mathsf{S}_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\,\hat{\mathbf{k}} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$dE = \frac{\hbar^2 k^2}{2m}\frac{V}{\pi^2}k^2\,dk \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2 k_F^5 V}{10\pi^2 m} \qquad P = \frac{2}{3}\frac{\hbar^2 k_F^5}{10\pi^2 m} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

$$V(x+a) = V(x) \qquad \psi(x+a) = e^{iKx}\psi(x) \qquad K = \frac{2\pi n}{Na} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} = \frac{\hbar^2}{2I}n(n+1) \quad n = 0, 1, 2, \dots$$