Phys 2920, Spring 2011
Exam #1

1. Write $-4.5 + 6.0i$ in polar form.

For the form $z = \rho e^{i\phi}$ we have

$$\rho = \sqrt{(4.5)^2 + (6.0)^2} = 7.5 \qquad \tan\phi = \frac{y}{x} = -1.33 \qquad \Longrightarrow \qquad \phi = \tan^{-1}(-1.33) \stackrel{???}{=} -0.927$$

No, this angle can't be right because ϕ must be in the second quadrant. Add π to get $\phi = 2.21$, giving

$$z = (7.5)e^{i(2.21)}$$

2. Give a definition of **linear independence**.

The vectors $\mathbf{a}_1 + \cdots + \mathbf{a}_n$ are linearly independent if there is no linear combination of them which is zero. Likewise, if we cannot write any one of the vectors as a linear combination of the rest.

For problems 3 – 5 use the vectors

$$\mathbf{a} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} \qquad \mathbf{b} = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}} \qquad \mathbf{c} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{k}} \qquad \mathbf{d} = -2\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

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3. Find the angle between the vectors **a** and **b**.

Since

$$a = \sqrt{10} \qquad \text{and} \qquad b = \sqrt{42} \qquad \text{and} \qquad \mathbf{a}\cdot\mathbf{b} = -5 - 3 = -8 = ab\cos\theta$$

then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{-8}{\sqrt{10}\sqrt{42}} = -0.3903 \qquad \Longrightarrow \qquad \theta = 1.97 = 113^\circ$$

4. Find a unit vector perpendicular to both **b** and **c**.

A vector which is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & 1 & 4 \\ 3 & 0 & -3 \end{vmatrix} = -3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

which has magnitude

$$|\mathbf{b}\times\mathbf{c}| = \sqrt{27} = 3\sqrt{3}$$

Then a unit vector pointing in the same direction (perpendicular to both $\mathbf{b}$ and $\mathbf{c}$) is

$$\hat{\mathbf{v}} = \frac{1}{3\sqrt{3}}[-3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}] = \frac{-1}{\sqrt{3}}[\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}]$$

5. Are the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ linearly independent?

There are several ways to do this; one is consider a vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 0 \\ -5 & 1 & 4 \end{vmatrix} = -12\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 14\hat{\mathbf{k}}$$

and if $\mathbf{d}$ is perpendicular to this then lies in the same plane as $\mathbf{a}$ and $\mathbf{b}$. We find

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d} = 24 + 32 - 46 = 0$$

so $\mathbf{d}$ *is* perpendicular to it and hence must be *some* linear combination of $\mathbf{a}$ and $\mathbf{b}$. So they are not linearly independent.

6. Recall the expansion of (suitable) functions on the interval $[0,1]$ In terms of the functions $f_n(x) = \sqrt{2}\sin(n\pi x)$. (They were orthonormal.) We expanded one function which could be done "by inspection"; the function

$$f(x) = 3\sin^2(\pi x)$$

can also be expanded in the $f_n(x)$'s but we have to do it the hard way.

Write down an expression for how we would get n^{th} coefficient c_n in

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

It will involve an integral, but you don't need to work it out.

Since the functions constitute an orthonormal basis (with respect to the given inner product) the coefficient c_n of the "vector" f_n is the "dot product" of basic vector f_n with the given $f(x)$:

$$c_n = \langle n|f\rangle = \int_0^1 \left(\sqrt{2}\sin(n\pi x)\right)\left(3\sin^2(\pi x)\right)\, dx = 3\sqrt{2}\int_0^1 \sin(n\pi x)\,\sin^2(\pi x)\,dx$$

7. Consider a new orthonormal basis for vectors is 3-space:

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}), \qquad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{k}}), \qquad \hat{\mathbf{e}}_3 =$$

a) Oh noes! I haz spilled Coke on the last vector and can't make it out! Well, give me a third vector which would make an orthonormal basis.

A vector which is perpendicular to the first two is

$$\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

which is normalized to give a suitable third (new) basis vector

$$\hat{\mathbf{e}}_3 = \frac{1}{\sqrt{6}}(-\hat{\mathbf{\imath}} + 2\hat{\mathbf{\jmath}} - \hat{\mathbf{k}})$$

b) Find the matrix S which transforms the basis via

$$\hat{\mathbf{e}}'_j = \sum_{i=1}^{N} \mathsf{S}_{ij}\hat{\mathbf{e}}_i$$

The matrix S is formed from the components of the $\hat{\mathbf{e}}$'s written in columns:

$$\mathsf{S} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \qquad \Longrightarrow \qquad \mathsf{S}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

where we got S^{-1} by taking the transpose of an orthogonal matrix.

c) Note that you should have an *orthogonal* matrix in (b). Using S, transform the vector

$$\mathsf{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

to the new basis.

We get the new representation of $\mathbf{x}$ from

$$\mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{5}{\sqrt{6}} \end{pmatrix}$$

One can calculate the magnitudes of the vectors, and we find that $|\mathbf{x}| = 13$ and $|\mathbf{x}'| = 13$. Indeed, they *must* be equal as an orthogonal transformation preserves the norm.

8. One theorem we used for determinants is that the value is not changed under a *row operation* on the matrix. Explain what is meant by a row operation.

A row operation is one where we take a constant multiple of one row of a matrix and and add it term-by-term to another row. (In doing so we don't change the values of the first row.)

9. If matrices A, B and C are given by

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & -1 & 3 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 5 & -8 & 0 \\ -1 & 3 & 0 \end{pmatrix} \qquad \mathsf{C} = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$$

Find:

a) AB

$$\mathsf{AB} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 5 & -8 & 0 \\ -1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 10 & -15 & 1 \\ -7 & 25 & 4 \\ -8 & 18 & 1 \end{pmatrix}$$

where the entries of the product were gotten by taking the appropriate row from A and multiplying/adding with the respective entries from B.

b) Det(A)

Use the products-along-the-diagonal method to get

$$|\mathsf{A}| = 0 + 14 + 0 + 7 - 24 - 0 = -3$$

c) $\mathrm{Det}(\mathsf{B}^{-1})$

Using the property of the determinant of an inverse, and since $|\mathsf{B}| = 15 - 8 = 7$ we get:

$$|\mathsf{B}^{-1}| = |\mathsf{B}|^{-1} = \frac{1}{7}$$

d) Trace(C)

$$\mathrm{Trace}(\mathsf{C}) = \sum_{i=1}^{N} C_{ii} = 6$$

e) $\mathrm{Det}(\mathsf{C}^{37})$

Whoa! Gotta multiply da matrix C 37 times! This gonna take a while.
Just kidding.
Since $\mathrm{Det}(\mathsf{C}) = 9 - 10 = -1$, the multiplicative property of the determinant gives

$$\mathrm{Det}(\mathsf{C}^{37}) = |\mathsf{C}^{37}| = |\mathsf{C}|^{37} = (-1)^{37} = -1$$

10. If

$$\mathsf{ABx} = \mathsf{Cy}$$

(where x and y are vectors and the others square non-singular matrices) find an expression for x in terms of the other quantities and their inverses. (This has a fairly short answer, but get it right.)

By left-multiplying by the inverses, we get

$$\mathsf{Bx} = \mathsf{A}^{-1}\mathsf{Cy} \qquad \Longrightarrow \qquad \mathsf{x} = \mathsf{B}^{-1}\mathsf{A}^{-1}\mathsf{Cy}$$

11. Find the eigenvalues and (unit) eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Evaluate the determinant

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) + 6 = \lambda^2 - 3\lambda - 4 = 0$$

which can be solved as

$$(\lambda+1)(\lambda-4) = 0 \qquad \Longrightarrow \qquad \lambda = -1 \quad \text{or} \quad 4$$

For the eigenvalue -1, solve

$$\mathsf{A}\mathbf{v} = -\mathbf{v} \qquad \Longrightarrow \qquad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

an equation which gives

$$x + 2y = -x \qquad \Longrightarrow \qquad x = -y \qquad \Longrightarrow \qquad \mathbf{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where we have also produced a *unit* eigenvector.
Likewise, for the eigenvalue 4, solve

$$\mathsf{A}\mathbf{v} = 4\mathbf{v} \qquad \Longrightarrow \qquad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$$

an equation which gives

$$x + 2y = 4x \qquad \Longrightarrow \qquad 3x = 2y \qquad \Longrightarrow \qquad \mathbf{v} = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

So them's the (unit) eigenvectors.

12. a) Check if each matrix has an inverse, and if so find it (any way you can) for:

$$\mathsf{A} = \begin{pmatrix} 2i & 1 \\ 2 & -i \end{pmatrix} \qquad\qquad \mathsf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

We find:

$$|\mathsf{A}| = 2 - 2 = 0$$

so A has no inverse and

$$|\mathsf{B}| = 4 + 2 = 6$$

so B does (necessarily) have an inverse. It can be found by the parallel-row-operation method, or whatever you want to call it; for the fisrt step add the first row to the second row:

$$\left(\begin{array}{cc||cc} 1 & 2 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right) \quad \Longrightarrow \quad \left(\begin{array}{cc||cc} 1 & 2 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{array}\right)$$

Then subtract $\frac{1}{3}$ of the second row from the first row, then multiply the second row by $\frac{1}{6}$:

$$\Longrightarrow \quad \left(\begin{array}{cc||cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 6 & 1 & 1 \end{array}\right) \quad \Longrightarrow \quad \left(\begin{array}{cc||cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \end{array}\right)$$

So then the inverse of B is

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

b) For the computer software *or* calculator that you used on the homework, *explain* how would you find the determinant of

$$\mathsf{A} = \begin{pmatrix} 5 & 3 & 0 \\ 4 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

Obviously, there are many possible answers for this. In Maple, the lines

```
a:=Matrix([[5, 3, 0],[4, 2, -3],[4, -5, 6]]);
|a|;
```

will do it (and you don't have to load the `LinearAlgebra` package first).

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} \qquad \epsilon_{ijk}=\begin{cases}1 & ijk=123\\ \times -1 & \text{switch indices}\\ 0 & \text{any two equal}\end{cases} \qquad \mathbf{c}=\mathbf{a}\times\mathbf{b} \quad\Rightarrow\quad c_k=\sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$P_0(x)=1 \qquad P_1(x)=x \qquad P_2(x)=\tfrac{1}{2}(3x^2-1) \qquad P_3(x)=\tfrac{1}{2}(5x^3-3x)$$

$$P_4(x)=\tfrac{1}{8}(35x^4-30x^2+3) \qquad P_5(x)=\tfrac{1}{8}(63x^5-70x^3+15x)$$

$$(\mathsf{AB})_{ij}=\sum_{k=1}^{N}\mathsf{A}_{ik}\mathsf{B}_{kj} \qquad \mathsf{AA}^{-1}=\mathsf{A}^{-1}\mathsf{A}=\mathcal{I} \qquad \mathrm{Tr}\,\mathsf{A}=\sum_{i=1}^{N}A_{ii} \qquad |\mathsf{A}-\lambda\mathbf{1}|=0$$

$$\hat{\mathbf{e}}'_j=\sum_{i=1}^{N}\mathsf{S}_{ij}\hat{\mathbf{e}}_i \qquad \mathbf{x}'=\mathsf{S}^{-1}\mathbf{x} \qquad \mathsf{A}'=\mathsf{S}^{-1}\mathsf{AS}$$

$$|\mathsf{A}^{\mathrm{T}}|=|\mathsf{A}| \qquad |\mathsf{A}^{\dagger}|=|\mathsf{A}^{*}|=|\mathsf{A}|^{\star} \qquad |\lambda\mathsf{A}|=\lambda^n|\mathsf{A}| \qquad |\mathsf{AB}|=|\mathsf{A}|\cdot|\mathsf{B}|$$

$$(\mathsf{AB})^{-1}=\mathsf{B}^{-1}\mathsf{A}^{-1} \qquad \mathrm{Tr}(\mathsf{ABC})=\mathrm{Tr}(\mathsf{BCA})=\mathrm{Tr}(\mathsf{CAB})$$