Potential for a Moving Point Charge

In Chapter 10 Griffiths uses the the formulae for the retarded potentials to get the potentials from a moving point charge. From the singular nature of this charge distribution there are mathematical subtleties in working out the integral; in particular there is an extra factor which corrects Coulomb's law. One might miss this factor if one is careless.

Griffiths prefers to justify this factor with a geometric argument which is clear when it applies to a finite charge distribution; he says that it must apply to a point charge as well since we have to be able to consider a point charge as the tiny limit of a finite charge distribution. While this is true and his discussion helps to see that the factor does *not* come from relativity, it *does* follow purely from the mathematics as long as the math is done *carefully*.

We begin with the retarded potential of a charge distribution:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'$$

where

$$t_r = t - \frac{\mathscr{r}}{c} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

At any time, our point charge has a location in space given by $\mathbf{w}(t)$. From that the charge density of the "distribution" at location $\mathbf{r}'$ and time t' is

$$\rho(\mathbf{r}',t') = q\delta^3(\mathbf{r}'-\mathbf{w}(t'))$$

Putting all of these equations together we have

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\,\delta^3\left(\mathbf{r}'-\mathbf{w}\left[t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{\mathscr{r}}\,d\tau' \tag{1}$$

The first thing to note about 1 is that the delta function can *not* be eliminated so quickly because the argument is not a simple function of $\mathbf{r}'$. Furthermore it's a three-dimensional delta function with a messy vector argument! Yikes!

The general method for working out such a complicated delta function is to change the variable of integration to give the delta function a simple argument, and we proceed to do this, by use of a little trick. First, replace the argument of the $\mathbf{w}$ function in 1 by t' with the stipulation that t' be replaced by $t - |\mathbf{r}-\mathbf{r}'|/c$. This proviso can be enforced by including a second delta function and integrating over all t', thus:

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int\int \frac{\delta^3\left(\mathbf{r}'-\mathbf{w}[t']\right)\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{\mathscr{r}}\,d\tau' dt' \tag{2}$$

Now since the first delta function demands that $\mathbf{r}'$ be the same as $\mathbf{w}(t')$ we can replace $\mathbf{r}'$ where it appears in the argument of the second delta function by $\mathbf{w}(t')$:

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int\int \frac{\delta^3\left(\mathbf{r}'-\mathbf{w}[t']\right)\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right)}{\mathscr{r}}\,d\tau' dt' \tag{3}$$

Now the second delta function *only* contains the variable t'; furthermore it is a one-dimensional delta function so it is easier to work on. The roots of the argument can be found by solving the equation

$$t' - t + \frac{|\mathbf{r} - \mathbf{w}(t')|}{c} = 0$$

For a given $\mathbf{r}$ and t there will be *one* root, t_0. (It is true that this is the "retarded time" t_r for $(\mathbf{r}, t)$ but I prefer the former notation because there is only *one* retarded time to be considered here.) Now if we consider the function of t'

$$f(t') = t' - t + \frac{|\mathbf{r} - \mathbf{w}(t')|}{c} \tag{4}$$

then a *relatively* well-known[1] fact about delta functions is:

$$\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|} \delta(x - x_i) \tag{5}$$

where i indexes the roots of $f(x)$. So the only hard thing remaining is to take d/dt' of $f(t')$ in 4.

Recall that $\nabla \mathscr{r} = \boldsymbol{\mathscr{r}}/\mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$. Adapting this to the current problem (we will need to use the chain rule on 4), we find:

$$\begin{aligned} \frac{d}{dt'} f(t') &= 1 + \frac{1}{c} \nabla_{\mathbf{w}} |\mathbf{r} - \mathbf{w}| \cdot \frac{d}{dt'} \mathbf{w}(t') \\ &= 1 - \frac{1}{c} \frac{\mathbf{r} - \mathbf{w}}{|\mathbf{r} - \mathbf{w}|} \cdot \mathbf{w}'(t') \end{aligned}$$

This expression is to be put into the denominator in 5 and then the delta function in 3 is replaced. *Then* the integration on t' in 3 can be done and then the $\delta(t' - t_0)$ just replaces t' by t_0 everywhere. The resulst is

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t_0))\, d\tau'}{|\mathbf{r} - \mathbf{r}'| \left(1 - \frac{\mathbf{r} - \mathbf{w}(t_0)}{c|\mathbf{r} - \mathbf{w}(t_0)|} \cdot \mathbf{w}'(t_0)\right)} \tag{6}$$

Now the three-dimensional delta function on $\mathbf{r}'$ is simple and the result is to replace $\mathbf{r}'$ by $\mathbf{w}(t_0)$ everywhere. In the result we will be compact and keep using $\mathbf{r}'$ where we *really* mean $\mathbf{w}(t_0)$ where t_0 is the retarded time. Also, $\mathbf{w}'(t_0)$ is the velocity of the charge at the retarded time, which we'll just call $\mathbf{v}$. With that understanding (and the usual abbreviation $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$) we can write:

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}\left(1 - \frac{\boldsymbol{\mathscr{r}}}{c\mathscr{r}} \cdot \mathbf{v}\right)} \tag{7}$$

Then, multiplying top and bottom by c gives

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} \tag{8}$$

[1] That is, it's in Jackson.