Phys 2920, Spring 2009
Exam #3

1. For the vector field

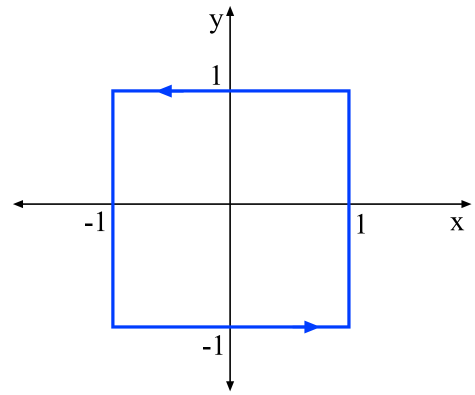
$$\mathbf{a} = -y^3\mathbf{i} + x^3\mathbf{j}$$

consider the path C in the xy plane shown at the right (a square of side 2 with its center at the origin) and the flat square surface S bounded by this curve.

Evaluate both

$$\oint_C \mathbf{a}\cdot d\mathbf{r} \qquad \text{and} \qquad \int_S (\nabla\times\mathbf{a})\cdot d\mathbf{S}$$

Did you get what you expected?



Do the line integral first; divide the path into four parts, starting with the right side, where

$$x = 1,\ dx = 0,\ y: -1 \to 1 \quad \text{so} \quad \int_{(1)} \mathbf{a}\cdot d\mathbf{r} = \int_{-1}^{1} 1^3\,dy = 2$$

On the next part,

$$y = 1,\ dy = 0,\ x: 1 \to -1 \quad \text{so} \quad \int_{(2)} \mathbf{a}\cdot d\mathbf{r} = \int_{1}^{-1} -(1)^3\,dx = 2$$

And on the next part,

$$x = -1,\ dy = 0,\ y: 1 \to -1 \quad \text{so} \quad \int_{(3)} \mathbf{a}\cdot d\mathbf{r} = \int_{1}^{-1} (-1)^3\,dy = 2$$

And finally

$$y = -1,\ dy = 0,\ x: 1 \to -1 \quad \text{so} \quad \int_{(4)} \mathbf{a}\cdot d\mathbf{r} = \int_{-1}^{1} -(-1)^3\,dx = 2$$

So the total line integral is 8.

Going on to the surface integral, take the curl of $\mathbf{a}$:

$$\nabla\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y^3 & x^3 & 0 \end{vmatrix} = (3x^2+3y^2)\mathbf{k} = 3(x^2+y^2)\mathbf{k}$$

And we want to integrate this functions over the square in the xy plane with $x: -1 \to 1$ and $y: -1 \to 1$:

$$\int_S 3(x^2+y^2)dS = 3\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dy\,dx$$

Do the y integral:

$$\rightarrow = 3\int_{-1}^{1}\left(x^2y+\frac{y^3}{3}\right)\Big|_{-1}^{1}\,dx = 3\int_{-1}^{1}\left(2x^2+\frac{2}{3}\right)\,dx$$

Do the x integral:

$$\rightarrow = \left(\frac{6}{3}x^3+2x\right)\Big|_{-1}^{1} = (2\cdot 2+2\cdot 2) = 8$$

These two integrals *should* be the same since this is just an application of Stokes' theorem with a planar surface bounded by the closed curve C.

2. If $z_1 = 2-i$ and $z_2 = -3+4i$, find

$$\text{a)}\quad \left|\frac{z_1+z_2}{z_1-z_2}\right| \qquad\qquad \textbf{b)}\quad \frac{z_2}{z_1}$$

Express the second one in $x+iy$ form.

(a) Since $z_1+z_2 = -1+3i$ and $z_1-z_2 = 5-5i$ then

$$\left|\frac{z_1+z_2}{z_1-z_2}\right| = \left|\frac{-1+3i}{5-5i}\right| = \frac{|-1+3i|}{|5-5i|} = \frac{\sqrt{10}}{\sqrt{50}} = \frac{1}{\sqrt{5}}$$

(b)

$$\frac{z_2}{z_1} = \frac{-3+4i}{2-i}\cdot\frac{2+i}{2+i} = \frac{-10+5i}{5} = -2+i$$

3. Show that for points on the circle $z = Re^{i\theta}$ (with $\theta : 0 \rightarrow 2\pi$), $|e^{iz}| = e^{-R\sin\theta}$.

With $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$, then

$$e^{iz} = e^{iR(\cos\theta+i\sin\theta)} = e^{-R\sin\theta+iR\cos\theta} = e^{-R\sin\theta}e^{iR\cos\theta}$$

But since $R\cos\theta$ is real, $e^{iR\cos\theta}$ has unit magnitude, so

$$|e^{iz}| = |e^{-R\sin\theta}e^{iR\cos\theta}| = |e^{-R\sin\theta}|\,|e^{iR\cos\theta}| = e^{-R\sin\theta}$$

4. Using the \$10 calculator method, evaluate

$$\text{a)}\quad \mathrm{Ln}\,(4+7i) \qquad \text{b)}\quad \sin(2+6i)$$

For the first one, recall that "Ln" means the principal (i.e. most obvious) value of the multi-valued log function.

(a) Write $z = 4+7i$ in simple polar form:

$$|z| = \rho = \sqrt{16+49} = \sqrt{65} \quad \text{and} \quad \phi = \tan^{-1}\frac{7}{4} \approx 1.05$$

Then

$$z = \sqrt{65}e^{i1.05} \qquad \Longrightarrow \qquad \mathrm{Ln}z = \ln(\sqrt{65}) + i1.05 = 2.09 + i1.05$$

(b) Use angle addition formulae and relations for $\sin(iz)$, etc., get

$$\sin(2+6i) = \sin 2 \cos(6i) + \sin(6i)\cos(2) = \sin 2 \cosh 6 + i \sinh 6 \cos 2$$

Then the calculator gives:

$$\Longrightarrow \qquad = 183.4 - i83.9$$

5. Explain the terms **branch point** and **branch cut**. How would you make a branch cut for the function

$$f(z) = (z-3)^{1/2} \quad ?$$

The "square root" function is badly behaved around any point in the complex plane where its argument is zero. If we "walk around" the point and take continuous values of $z^{1/2}$ we find we get a different value when we get back to the starting point. To remind us not to do this, we put in a barrier to block trips around the point; this can be any line which extends from the point off to infinity. The point and line are the branch point and branch cut.

Here the singularity is at $z = 3$. Such a function needs some sort of line running from $z = 3$ out to ∞.

6. If a function $f(z)$ has a complex derivative in a region it is said to be analytic. Give two further consequences of a function being analytic.

Other remarkable properties are: Obeying the Cauchy--Riemann relations; the existence of second, third and all further derivatives; the fact an integral of $f(z)$ around any closed path in the region gives zero. The existence of a Taylor series for $f(z)$.

7. For the *complex* function $f(z) = \cos z$, verify that the real and imaginary parts satisfy the Cauchy–Riemann equations.

With $z = x + iy$, we have

$$z = \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(x) - i\sin(x)\sinh(y)$$

So that

$$u(x,y) = \cos(x)\cosh(y) \qquad \text{and} \qquad v(x,y) = -\sin(x)\sinh(y)$$

These give

$$\frac{\partial u}{\partial x} = -\sin(x)\cosh(y) \qquad \frac{\partial v}{\partial y} = -\sin(x)\cosh(y) = \frac{\partial u}{\partial x}$$

and

$$\frac{\partial u}{\partial y} = \cos(x)\sinh(y) \qquad -\frac{\partial v}{\partial x} = (-1)(-\cos(x)\sinh(y)) = \cos(x)\sinh(y) = \frac{\partial u}{\partial y}$$

so the C-R conditions check out.

8. Evaluate

$$\lim_{z\to i} \frac{z-i}{z^4+10z^2+9}$$

We note that simple substitution gives an expression with 0 on top and bottom, so apply the l'Hopital rule and get:

$$\lim_{z\to i} \frac{z-i}{z^4+10z^2+9} = \lim_{z\to i} \frac{1}{4z^3+20z} = \frac{1}{-4i+20i} = \frac{1}{16i} = -\frac{i}{16}$$

9. What are the locations and orders of the poles of the function

$$f(z) = \frac{5z}{(z+17)(z^2+8)^2}$$

The denominator is zero at $z=-17$ and at $z=\pm\sqrt{8}i = \pm i2\sqrt{2}$ The latter is from the factor z^2+8 which factors as

$$z^2+8 = (z+i\sqrt{8})(z-i\sqrt{8})$$

and which appears *twice* in the denominator. So the poles at $\pm\, i\, 2\sqrt{2}$ are poles of order 2. The pole at $z=-17$ is a pole of order 1 (a simple pole).

10. a) What ("closed–form") function is represented by the series

$$f(z) = z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \frac{z^9}{7!} + \cdots \qquad ?$$

This is very similar to the series for $\sin z$ except that for each term the power of z on top is two too big! One can factor out the z^2 to get

$$\begin{aligned} f(z) &= z^2(z^1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots) \\ &= z^2 \sin z \end{aligned}$$

b) Evaluate the sum

$$\frac{1}{2^3} - \frac{1}{3!2^5} + \frac{1}{5!2^7} - \frac{1}{7!2^9} + \cdots$$

We see that this series is the same as the one given in (a) with $\frac{1}{2}$ substituted for z. Then

$$\text{Sum} \;=\; f(\tfrac{1}{2}) = (\tfrac{1}{2})^2 \sin(\tfrac{1}{2}) \approx 0.120$$

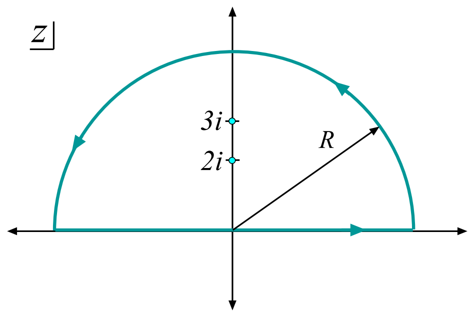
11. Evaluate the following integral, using contour integration:

$$\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}\, dx$$

Be clear about what you are doing! You should discuss (but you don't need to prove rigorously) why any part of your contour gives a zero integral.

We know that we want to extend the integration range to $-\infty \to \infty$

$$\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}\,dx = \tfrac{1}{2}\int_{-\infty}^\infty \frac{1}{(x^2+4)(x^2+9)}\,dx$$



and close the path to form a contour. So we will close it in the upper half-plane with a semi-circle of radius R (as $R \to \infty$) and evaluate

$$\oint_C \frac{dz}{(z^2+4)(z^2+9)}$$

The integrand has poles at $z = \pm 2i$ and $z = \pm 3i$ and both poles are of order 1.

We have added a curvy part to the integral; the integral on this part undoubtedly vanishes, as in the class examples the large power of z in the denominator made the integrand $decrease$ faster than the length of the curvy path $increases$; so in the limit $R \to \infty$ it vanishes.

Evaluate the residues a_{-1} at the poles enclosed by the path, namely those at $z = 2i$ and $z = 3i$. The residue at $2i$ is

$$\lim_{z\to 2i} \frac{(z-2i)}{z^4+13z^2+36} = \lim_{z\to 2i} \frac{1}{4z^3+26z} = \frac{1}{-32i+52i} = \frac{-i}{20}$$

and the residue at $3i$ is

$$\lim_{z\to 3i} \frac{(z-3i)}{z^4+13z^2+36} = \lim_{z\to 3i} \frac{1}{4z^3+26z} = \frac{1}{-108i+78i} = \frac{+i}{30}$$

Then the residue theorem gives

$$\oint_C \frac{dz}{(z^2+4)(z^2+9)} = 2\pi i\left(\frac{-i}{20}+\frac{i}{30}\right) = 2\pi\left(\frac{1}{20}-\frac{1}{30}\right) = \frac{\pi}{30}$$

Since the original integral is half the contour integral, we get

$$\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}\,dx = \tfrac{1}{2}\oint_C \frac{dz}{(z^2+4)(z^2+9)} = \tfrac{1}{2}\frac{\pi}{30} = \frac{\pi}{60}$$

`Maple` gave an awkward (but correct) expression for the answer.

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad \hat{\mathbf{z}} = \mathbf{k} \tag{2}$$

$$\mathbf{i} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{j} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{k} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz \tag{4}$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi \tag{5}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{6}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\mathbf{i} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad \int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$z = x + iy = \rho e^{i\phi} \qquad |z| = \rho = \sqrt{x^2+y^2} \qquad z^* = x - iy \qquad w = \ln z = \ln r + i(\theta + 2k\pi)$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cosh z = \frac{e^z+e^{-z}}{2} \quad \sinh z = \frac{e^z-e^{-z}}{2}$$

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2 \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \mathrm{sech}^2 z \qquad \coth^2 z - 1 = \mathrm{csch}^2 z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin(iz) = i \sinh z \qquad \cos(iz) = \cosh z$$

$$\sin^{-1}(z) = \frac{1}{i} \ln(iz + \sqrt{1 - z^2}) \qquad \cos^{-1}(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}) \qquad \tan^{-1}(z) = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad w = f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

$$(1 + z)^p = 1 + pz + \frac{p(p - 1)}{2!} z^2 + \cdots + \frac{p(p - 1) \cdots (p - n - 1)}{n!} z^n + \cdots$$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{(z - a)} + \frac{a_{-2}}{(z - a)^2} + \frac{a_{-3}}{(z - a)^3} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \qquad a_{-1} = \lim_{z \to a} \left(\frac{1}{(k - 1)!} \frac{d^{k-1}}{dz^{k-1}} (z - a)^k f(z) \right)$$

$$\oint_C f(z) \, dz = 2\pi i \{a_{-1} + b_{-1} + c_{-1} + \ldots\}$$