Phys 3810, Spring 2011
Exam #2

1. Give short definitions of the terms (as used by Griffiths)

a) A *complete* set of eigenfunctions.

A set of functions --which are eigenfunctions of a certain operator-- which can be used for an expansion of any proper wave function for the system, i.e. satisfying the boundary conditions.

b) Determinate state.

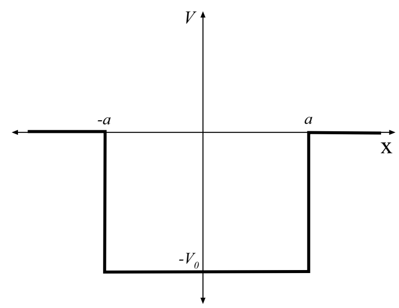
A quantum state for which a measurement of some observable Q will always give the same value (i.e. the uncertainty in Q is zero). Such a state is also an eigenfunction of the operator $\hat{Q}$.

c) The *spectrum* of an operator.

The spectrum of an operator is the set of its eigenvalues, which can be be discrete or continuous or some of both!

2. Recall solving for the states of the odd wave functions in the "finite square well" of value $-V_0$ and width $2a$.

Write down the limiting value for the lowest energy eigenvalues in the limit of large V_0 and large a. You don't need to re-derive it, just explain why you can just write down the answer from a reasonable expectation.



For large depth and width the square well is physically similar to the infinite well for its lowest states, and from the formula for *all* the energy eigenstates for the infinite well of width a:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

we substitute $a \to 2a$ for the new width and for E substitute the amount by which the energy is greater than the bottom of the well,

$$E \to E - (-V_0) = E + V_0$$

and for the states *odd* symmetry, just consider *even* indexes n, so substitute $n \to 2m$. Then we expect for the odd states indexed by m,

$$E \approx \frac{(2m)^2\pi^2(E+V_0)}{2m(2a)^2} = \frac{(m)^2\pi^2(E+V_0)}{2ma^2}$$

3. The momentum space eigenfunction we have been using is (in one way of writing it)

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

which satisfies the "delta function normalization"

$$\langle f_{p'} | f_p \rangle = \delta(p - p')$$

Show how the second of these *follows* from the first one. Hint: Use the symbolic expansion

$$\langle f_{p'} | f_p \rangle = \langle f_{p'} | x \rangle \langle x | f_p \rangle$$

and stuff we did on the problem sets.

$$\begin{aligned} \langle f_{p'} | f_p \rangle &= \langle f_{p'} | x \rangle \langle x | f_p \rangle = \frac{1}{2\pi\hbar} \int e^{-ip'x/\hbar} e^{ipx/\hbar} \, dx \\ &= \frac{1}{2\pi\hbar} \int e^{ix(p-p')/\hbar} \, dx = \frac{1}{2\pi\hbar} (2\pi) \delta\left(\frac{p - p'}{\hbar} \right) \\ &= \frac{1}{\hbar} \hbar \delta(p - p') = \delta(p - p') \end{aligned}$$

4. An operator $\hat{A}$ (representing observable A) has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$ has two normalized eigenstates ϕ_1 and ϕ_2 with eigenvalues b_1 and b_2 respectively. The eigenstates are related by

$$\psi_1 = (-5\phi_1 + 12\phi_2)/13 \qquad \psi_2 = (12\phi_1 + 5\phi_2)/13$$

a) Observable A is measured and the value a_2 obtained. What is the state of the system (immediately) after this measurement? (Easy.)

If the result of a measurement of A the result is a_2, the system has been put ("collapsed") into the state ψ_2.

b) If B is now measured, what are the possible results and what are their probabilities? (Also not too hard.)

A measurement of B can only give the values b_1 and b_2. As the system is now in state ψ_2, the probabilities of getting these values (which go with the states ϕ_1 and ϕ_2) are the coefficients of the expansion of ψ_2 in terms of the ϕ's. Thus they will occur with probabilities

$$b_1 : \quad \frac{(12)^2}{(13)^2} = \frac{144}{169} \qquad b_2 : \quad \frac{(5)^2}{(13)^2} = \frac{25}{169}$$

5. a) Give the definition of a hermitian operator.

An operator $\hat{\mathcal{O}}$ is Hermitian if for all functions in the Hilbert space itis true that

$$\int f^*(x) \hat{\mathcal{O}} g(x) \, dx = \int \left[\hat{\mathcal{O}} f(x) \right]^* g(x) \, dx$$

b) Show that $\hat{p}$ (momentum operator for one dimension, x) is a hermitian operator. *Carefully explain* all the steps!!!

$$\begin{aligned}\int f(x)^*\hat{p}g(x)\,dx &= \int f(x)^*\frac{\hbar}{i}\frac{d}{dx}g(x)\,dx \\ &= \left(\frac{\hbar}{i}\right)\left(-\int\left[\frac{d}{dx}f(x)^*\right]g(x)\,dx\right) = \int\left[\frac{\hbar}{i}f(x)\right]^*\hat{p}g(x)\,dx \\ &= \int[\hat{p}f(x)]^*g(x)\,dx\end{aligned}$$

Here, after substituting for $\hat{p}$ we did an integration by parts where we omitted the boundary term since $f(x)$ and $g(x)$ and their derivatives all vanish at ∞; then a minus sign goes with the switched integral.

In the next step we can erase the minus when we put i inside the bracket which is complex-conjugated. Finally, we recognize that we have the $\hat{p}$ operator back again and make the substitution.

6. Give a brief but *correct* summary of the meaning (i.e. the proper usage) of the energy–time uncertainty relation $\Delta t\,\Delta E \geq \frac{\hbar}{2}$.

We first specify some observable Q and then -with the system *not being in a stationary state*, so that there is some uncertainty in the energy, ΔE- we find the time for which the expectation value of Q will change substantially and call that Δt. Then it will be true that $\Delta E\,\Delta t \geq \frac{\hbar}{2}$.

7. Show that

$$[x^n, p] = i\hbar n x^{n-1}$$

Consider a "test function" $f(x)$, then the action of the *operator* given by $[x^n, p]$ is

$$\begin{aligned}[x^n, p]f(x) &= \frac{\hbar}{i}x^n\frac{d}{dx}f(x) - \frac{\hbar}{i}\frac{d}{dx}\left(x^n f(x)\right) \\ &= \frac{\hbar}{i}\left[x^n f'(x) - nx^{n-1}f(x) - x^n f'(x)\right] = \frac{\hbar}{i}\left(-nx^{n-1}\right)f(x)\end{aligned}$$

So the action of the given operator is to multiply $f(x)$ by a factor; thus:

$$[x^n, p] = i\hbar\, n x^{n-1}$$

8. The Hamiltonian for a certain two-level system is given by

$$H = \begin{pmatrix} a & \frac{a}{2} \\ \frac{a}{2} & a \end{pmatrix}$$

If the system starts out in state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

what is $|\mathcal{S}(t)\rangle$?

Find the eigenvalues and eigenvectors for this Hamiltonian. Eigenvalues:

$$\begin{vmatrix} a-E & a/2 \\ a/2 & a-E \end{vmatrix} = 0 = (a-E)^2 - \frac{a^2}{4} = e^2 - 2aE + \frac{3a^2}{4}$$

Solve this either by factoring or the quadratic equation:

$$E = \frac{2a \pm \sqrt{4a^2 - 3a^2}}{2} = \tfrac{1}{2}(2a \pm a)$$

so that the eigenvalues are

$$E_1 = \tfrac{1}{2}a, \; E_2 = \tfrac{3}{2}a$$

Do some algebra (or just judicious guessing) and find that the corresponding eigenvectors are

$$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The state of the system at $t = 0$ was not an eigen state but can be written as a linear combination of them:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

to which we can attach the exponential time factor to get the full time dependence for the state $\mathcal{S}(t)$:

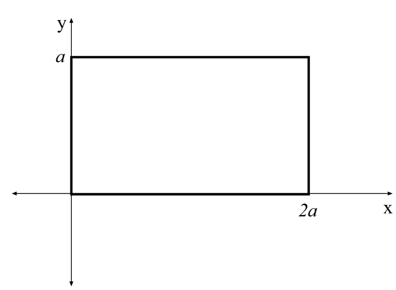
$$\mathcal{S}(t) = \psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}$$

Written out in full this is

$$\mathcal{S}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp(-iat/(2\hbar) + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(-i3at/(2\hbar)$$

One can write this as a single column vector and combine terms, but I will leave it at this.

9. A particle of mass m is confined to move in two dimensions. It is confined to move inside a rectangle of sides $2a$ and a (see figure).



What are the energy eigenvalues for the particle?

As the Schrödinger equation separates in the x and y coordinates, the solutions are products of the familiar 1D box wavefunctions, adapted for the lengths given in the problem. The energy eigenfunctions are

$$\begin{aligned} \psi_{n_x,n_y} &= \sqrt{\frac{2}{2a}} \sin\left(\frac{n_x \pi x}{2a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi x}{a}\right) \\ &= \frac{\sqrt{2}}{a} \sin\left(\frac{n_x \pi x}{2a}\right) \sin\left(\frac{n_y \pi x}{a}\right) \end{aligned}$$

and the energy eigenvalues are

$$E_{n_x,n_y} = \frac{\hbar^2}{2m}\left(\frac{n_x^2\pi^2}{4a^2}\right) + \frac{\hbar^2}{2m}\left(\frac{n_y^2\pi^2}{a^2}\right) = \frac{\hbar^2\pi^2}{2ma^2}\left(\frac{n_x^2}{4} + n_y^2\right) \equiv E_0\left(\frac{n_x^2}{4} + n_y^2\right)$$

10. There were actually *two* linearly independent solutions to the radial Schrödinger equation for the zero-potential case. (For example, inside the "Easter egg") The other solutions were called $n_l(x)$. Why were they excluded from the spherical box problem?

The functions n_l were excluded because they blow up at the origin. (If the region for our solution did notinclude the origin we $would$ have to include them.)

11. a) What piece of information did we get about the H radial wave function by considering the behavior as $r \to 0$?

We got a simpler radial differential equation for $u_l(r)$ for which the solution was the power ρ^l. This suggests that we should pull off this factor and work on the remaining part of u_l.

b) What piece of information did we get about the H radial wave function by considering the behavior as $r \to \infty$?

Again, we got a simpler radial differential equation for $u_l(r)$ for which the solution was the exponential $e^{\rho/a}$. This suggests that we should pull off this factor as well.

12. Find $\langle r \rangle$ for the electron in the ground state of the H atom. (Express the answer in terms of the Bohr radius.)

Find the expectation value of r for the ground state; as the angular part just integrates to 1, evaluate the integral:

$$\int_0^\infty r\,|R_{10}(r)|^2 r^2 dr = \frac{2^2}{a^3}\int_0^\infty r^3 e^{-2r/a}\,dr = \frac{4}{a^3}(3!)\frac{a^4}{16} = \frac{3a}{2}$$

One might have expected the Bohr radius a $exactly$ for this, but in fact it's bit higher.

13. Find the energy difference between the $n = 4$ and $n = 3$ levels in the He^+ ion. (Express the answer in eV and as a wavelength for the corresponding photon.)

In the H atom, the energy levels are given by E_1/n^2 with $E_1 = -13.6$ eV. As we saw, for a central charge Ze, the energies are proportional to Z^2, so that for the He^+ ion the energy difference is

$$\Delta E = 4(13.6\ \text{eV})\left(\frac{1}{9} - \frac{1}{16}\right) = 2.64\ \text{eV} = 4.23 \times 10^{-19}\ \text{J}$$

This gives:

$$h\nu = \frac{hc}{\lambda} = E \qquad \Longrightarrow \qquad \lambda = \frac{hc}{E} = \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(2.998 \times 10^8\ \frac{\text{m}}{\text{s}})}{(4.23 \times 10^{-19}\ \text{J})} = 4.70{\times}10^{-7}\ \text{m} = 470\ \text{nm}$$

14. Our solution for the H atom is a first-order answer, good to at least 1%, but some "physics" has been left out. Identify some way in which our H operator is deficient.

Most obviously, it ignores relativity; the Schrödinger equation is non-relativistic; we would guess that we are in error to the degree that the speed of the electron (or its expectation value) is comparable to c.

We have also ignored any possible terms in the Hamiltonian which depend on the spin state of the electron or for that matter the proton.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2} a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

Numbers

$$\hbar = 1.05457\times10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938\times10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262\times10^{-27}\ \mathrm{kg}$$

$$e = 1.60218\times10^{-19}\ \mathrm{C} \qquad c = 2.99792\times10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

∞ Square Well: $$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

Harmonic Oscillator: $$V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

Free particle: $$\Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\, \cos\theta e^{\pm i\phi} \qquad \text{etc.}$$

$$R(r) = A j_l(kr) \qquad k \equiv \sqrt{\frac{2mE}{\hbar}}$$

$$j_1(x) = \frac{\sin x}{x} \qquad j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x$$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10}\ \text{m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7\ \text{m}^{-1}$$