Phys 4620, Spring 2008
Exam #1

1. Show how the equation of continuity (that is, the charge conservation equation) follows from the Maxwell equations.

Take the fourth Maxwell equation (Ampere-Maxwell) and take the divergence of both sides. Get:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}$$

The divergence of a curl is zero, and use the fact that space and time derivatives commute. This gives:

$$0 = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial (\nabla \cdot \mathbf{E})}{\partial t}$$

Now use the first Maxwell equation (Gauss) for $\nabla \cdot \mathbf{E}$ and get:

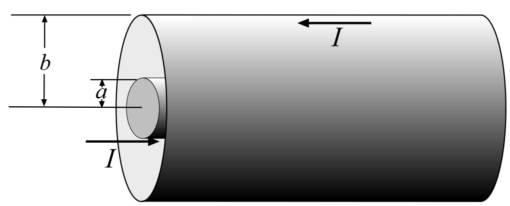
$$\mu_0 \nabla \cdot \mathbf{J} + \frac{\mu_0 \epsilon_0}{\epsilon_0} \frac{\partial \rho}{\partial t} = 0$$

Cancel the μ_0 and then get:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

which is the continuity equation.

2.a) A long coaxial cable carries current I; the current flows down the down the surface of the inner cylinder of radius a and back along the outer cylinder of radius b.



a) Find the magnetic energy stored in a length l.

Using old arguments from Ampere's law, the magnetic field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}$$

where the z axis points to the right. The energy density of the B field is

$$\frac{B^2}{2\mu_0} = \frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi s} \right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$$

Then considering a length l of the cable, integrate over the volume between the cylinders to get the magnetic energy:

$$W_{\text{mag}} = l \int_0^{2\pi} d\phi \int_a^b \frac{\mu_0 I^2}{8\pi^2 s^2} s ds = 2\pi l \frac{\mu_0 I^2}{8\pi^2} \int_a^b \frac{ds}{s} = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{a}{b}\right)$$

b) Find the self–inductance of the cable (per unit length) any way you want.

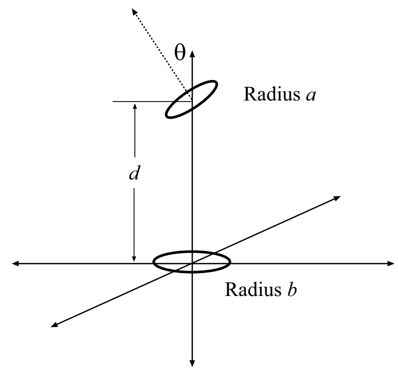
If we set the energy stored (in the length l) to $\frac{1}{2}LI^2$, then

$$\tfrac{1}{2}LI^2 = \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{a}{b}\right) \qquad \Longrightarrow \qquad L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{a}{b}\right)$$

so that the self-inductance *per unit length* is

$$\tilde{L} = \frac{L}{l} = \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right)$$

3. A ring of radius b lies in the xy plane with its center at the origin. Another ring, of radius a has its center on the z axis at $z = d$, with its normal tilted at some fixed angle θ from the z axis. The rings are both "small": $a \ll d$ and $b \ll d$.



a) Find an approximate value for the mutual inductance of the rings. State any approximations being used.

Suppose there is a current I in the bottom loop. With the great distance (compared to its size) of the upper loop, it makes sense to suppose the the B field due to the lower loop is uniform in the vicinity of the upper loop.

To find the value of this field we can use the formula for the B field from a loop on the axis where $z \gg R$:

$$B_z = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \xrightarrow{z \gg R} \frac{\mu_0 I}{2}\frac{R^2}{z^3} = \frac{\mu_0 I}{2}\frac{b^2}{d^3}$$

Another way to get it is to approximate the lower current as a point magnetic dipole $m\hat{\mathbf{z}}$ where $m = \pi b^2$. Then using the formula for the B field at $\mathbf{r} = z\hat{\mathbf{z}}$ from this point dipole,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{z^3}[2m\hat{\mathbf{z}}] = \frac{\mu_0}{\pi}\frac{1}{z^3}[\pi b^2\hat{\mathbf{z}}] = \frac{\mu_0 I}{2}\frac{b^2}{d^3}\hat{\mathbf{z}}$$

With this uniform field in the vicinity of the upper loop, it is clear that the magnetic flux through the upper loop is

$$\Phi = BA_{\rm up}\cos\theta = \frac{\mu_0 I}{2}\frac{b^2}{z^3}(\pi a^2)\cos\theta = \frac{\mu_0 \pi I a^2 b^2}{2d^3}\cos\theta$$

Using the definition $\Phi_2 = MI_1$ we get

$$M = \frac{\mu_0 \pi a^2 b^2}{2d^3}\cos\theta$$

b) For this system, suppose the current in the lower loop depends on time via:

$$I_1(t) = I_0\cos(\omega t)$$

Find an expression for the current induced in the upper loop. (The upper loop has resistance R.)

Faraday's law sez

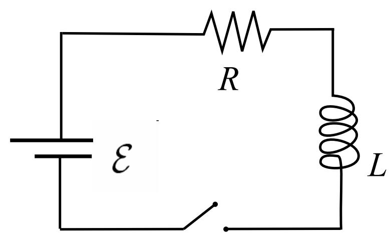
$$\mathcal{E}_2 = -M\frac{dI_1}{dt}$$

so with the given form of $I(t)$, and $I_2 = \mathcal{E}_2/R$, we get

$$\begin{aligned} I_2(t) &= -\frac{M}{R}\frac{dI}{dt} = -\frac{\mu_0\pi a^2b^2\cos\theta}{2d^3R}(-\omega I_0)\sin(\omega t) \\ &= \frac{\mu_0\pi a^2b^2\omega I_0\cos\theta}{2d^3R}\sin(\omega t) \end{aligned}$$

4. In the simple RL circuit shown at the right, the switch is closed at $t = 0$.

Derive the expression for $I(t)$. Remember that across a resistor there is a voltage drop of IR and across an inductor there is a voltage *gain* of $L\frac{dI}{dt}$.



Consider a clockwise path around the loop, and assign a clockwise direction to the current i. Across the battery there is an increase of $\mathcal{E}$; across the resistor there is a potential change of $-IR$ and across the inductor there is a potential increase of $+L\frac{dI}{dt}$. We then have:

$$\mathcal{E} - IR + L\frac{dI}{dt} = 0$$

We just need to solve the DE subject to the condition $I(0) = 0$. Rearrange and get:

$$(\mathcal{E} - IR)dt + LdI = 0 \qquad \Longrightarrow \qquad dt = \frac{LdI}{(\mathcal{E} - IR)}$$

Integrate and get

$$t + C_1 = \frac{L}{(-R)}\ln(\mathcal{E} - IR) \qquad \Longrightarrow \qquad -\frac{R}{L}t + C_2 = \ln(\mathcal{E} - IR)$$

Exponentiate both sides:

$$C_3e^{-Rt/L} = \mathcal{E} - IR$$

More algebra:

$$IR = \mathcal{E} - C_3e^{-Rt/L} \qquad \Longrightarrow \qquad I = \frac{\mathcal{E}}{R} - C_4e^{-Rt/L}$$

The condition $I(0) = 0$ then requires $C_4 = \mathcal{E}/R$. So finally,

$$I = \frac{\mathcal{E}}{R} - \frac{\mathcal{E}}{R}e^{-Rt/L} = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

The current starts off with the value of zero and asymptotically approaches the value of $\mathcal{E}/R$.

5. a) What is the physical meaning of the Poynting vector? (This is, give a couple sentences telling of its significance to physics.)

The Poynting vector ($\mathbf{S}$ gives the flux of electromagnetic energy. This is the rate of energy transport per unit area in the direction of $\mathbf{S}$.

It is also the *density* of the (vector) momentum of the EM field.

b) If $d\mathbf{a}$ is an element of a surface, and $\overleftrightarrow{\mathbf{T}}$ is the Maxwell stress–energy tensor, explain the physical significance of

$$(d\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}})_i \ .$$

This gives the flow of electromagnetic momentum of component i through the surface $d\mathbf{a}$. (Momentum per time.)

c) In electrodynamics we will find that when charges q_1 and q_2 are in motion, the force of q_1 on q_2 is *not* "equal and opposite" to the force of q_2 on q_1. This means that the usual statement of Newton's 3rd law is wrong.

Does it also mean that we must give up the principle of momentum conservation? Why or why not?

If all momentum was *mechanical* (associated with massive particles) it would, but we must also associate momentum with the EM field. When we do this we find that the total momentum of particles and field is conserved.

6. Write the wave function

$$f(z,t) = 8\cos(4z - 2t + \tfrac{\pi}{4})$$

in the form of a complex wave $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$ where

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

If the "4" really means 4.0 m^{-1} and the "2" means 2.0 s^{-1}, what is the speed of this wave?

Using $\cos x = \mathrm{Re}(e^{ix})$, then

$$8\cos(4z - 2t + \tfrac{\pi}{4}) = \mathrm{Re}\left(8e^{i(4z-2t+\pi/4)}\right) = \mathrm{Re}\left(8e^{i\pi/4}e^{i(4z-2t)}\right)$$

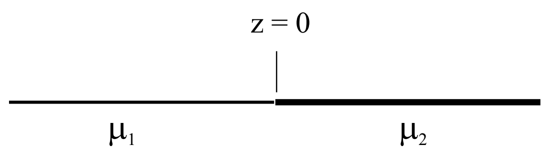
So then in the suggested form of the complex wave we have

$$A = 8e^{i\pi/4} \qquad k = 4 \ \mathrm{m}^{-1} \qquad \omega = 2 \ \mathrm{s}^{-1}$$

This gives a wave speed of

$$v = \frac{\omega}{k} = 0.50 \ \tfrac{\mathrm{m}}{\mathrm{s}}$$

7. Strings of different thickness (mass densitites) are joined at $z = 0$ forming a string where for $z < 0$ the linear mass density is μ_1 and for $z > 0$ it is μ_2 A harmonics wave of frequency ω is incident from the left.



a) Write down forms of the waves for both $z < 0$ and $z > 0$. (Use complex notation for the waves!) Make sure all terms are defined; recall that the speed of wave on a string is $v = \sqrt{\frac{T}{\mu}}$.

The incident wave is

$$\tilde{A}_I e^{i(k_1 z - \omega t)}$$

The reflected wave is

$$\tilde{A}_R e^{i(-k_1 z - \omega t)}$$

The transmitted wave is

$$\tilde{A}_T e^{i(k_2 z - \omega t}$$

where

$$v_1 = \sqrt{T/\mu_1} \qquad v_2 = \sqrt{T/\mu_2} \qquad k_1 = \omega/v_1 \qquad k_2 = \omega/v_2$$

The wave function (before applying boundary conditions) is

$$f(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + A_R e^{-i(k_1 z + \omega t)} & \text{for } z < 0; \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & \text{for } z > 0; \end{cases}$$

b) What are the boundary conditions we apply to solve for the wave in both regions?

The string has no breaks in it so the functions is continuous at $z = 0$ at all times:

$$f(z,t)\Big|_{0^-} = f(z,t)\Big|_{0^+}$$

Also, the forces acting exactly at the junction must cancel (there is no finite mass point at $z = 0$ so that the slope of f must be continous at all times:

$$\frac{\partial f}{\partial z}\Big|_{0^-} = \frac{\partial f}{\partial z}\Big|_{0^+}$$

8. Starting from the Maxwell equations, show how we get the wave equation for the $\mathbf{E}$ field (in vacuum),

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Start with the equation (Faraday)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and take the curl of both sides. Space and time derivatives commute so we get

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

On the left side use a second-derivative identity to get

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

where we have used a relation good in vacuum,

$$\nabla \cdot \mathbf{E} = 0$$

On the right side, use the relation for vacuum,

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

to get

$$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Equate the results for left and right sides and (cancelling the minus sign) get

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

9. Suppose we choose the form of the plane waves for the $\mathbf{E}$ and $\mathbf{B}$ fields as

$$\tilde{\mathbf{E}}(z,t) = Ce^{i(kz-\omega t)} \qquad\qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

where the vectors $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are complex and a constant.

Show how the Maxwell equations imply that the E and B fields are transverse.

Consider the E field wave. Since $\nabla \cdot \mathbf{E} = 0$, we must have

$$\nabla \cdot \left[\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\right]$$

(Actually, it only needs to true for the real part, but that condition has the same result as applying $\nabla\cdot$ to the complex wave. We'll go with this condition!)

Now, since the amplitude (the vector in front of the oscillating exponential) is a constant there is only a dependence on z in the exponential. If the vector $\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$ does indeed have a z component then this gives:

$$ik\tilde{E}_{0z} e^{i(kz-\omega t)} = 0$$

which implies that $\tilde{E}_{0z}$ is zero, i.e. $\tilde{\mathbf{E}}(z,t)$ does not have a z component. Since the assumed direction of porpagation here is along z, the wave only has x and y components; it is transverse.

Similar remarks hold for $\mathbf{B}$ (since $\nabla \cdot \mathbf{B} = 0$ so the B field is also transverse.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2\, \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of gradient)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\hat{\mathscr{r}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times 10^{-7}\,\tfrac{\text{N}}{\text{A}^2} \qquad 1\ \text{T} = 1\,\tfrac{\text{N}}{\text{A}\cdot\text{m}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\text{above}} = B^\perp_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint \mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp - B_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \qquad \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \qquad \tilde{\mathbf{B}} = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}$$

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0R} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \qquad \tan\theta_B = \frac{n_2}{n_1}$$

Specific Results:

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$