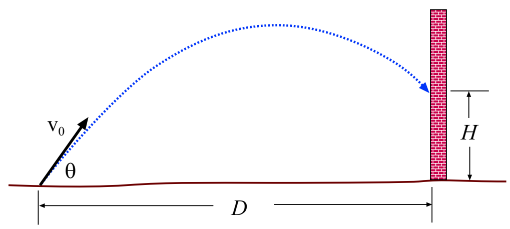
Phys 3610, Fall 2009
Exam #1

1. State Newton's First law (colloquially, "no forces, no accelerations") in a way that is substantive, i.e. not just a trivial consequence of the second law. (Hint: Say something about reference frames.)

More precisely, Newton's First Law postulates that there is a reference frame in which in the absence of *genuine* forces, any particle will have no acceleration (i.e. move at constant velocity. This frame is taken to be the one which is at rest and not rotating with repsect to the "fixed stars". It easily follows that the same will be true for any reference frame moving at constant velocity with respect to this one (an inertial frame).

2. A projectile fired from ground level at speed v_0 and angle θ from the ground level toward a very large wall. The wall is at a distance D from the firing point.

Find an expression for H, the height at which the projectile strikes the wall.



As the x coordinate of the projectile is given by

$$x = v_0 \cos\theta \, t$$

then when $x = D$ (projectile strikes wall) we have

$$D = v_0 \cos\theta t \qquad \Longrightarrow \qquad t = \frac{D}{v_0 \cos\theta}$$

And since the y coordinate is given by

$$y = v_0 \sin\theta t - \tfrac{1}{2} g t^2$$

then at the time of impact we have

$$y = (v_0 \sin\theta) \frac{D}{v_0 \cos\theta} - \tfrac{1}{2} g \left(\frac{D}{v_0 \cos\theta} \right)^2$$

Simplifying, the height H of the impact is

$$H = D \tan\theta - \frac{g D^2}{2 v_0^2 \cos^2\theta}$$

3. a) The examples of motion with quadratic and linear resistive forces (with gravity) each had a quantity defined as the *terminal velocity*.

Give a careful *physical* definition of terminal velocity, suitable for *both* cases.

In both cases this is the speed which is asymptotically attained by a falling object, the asymptotic condition being that where the downward force of gravity equals the upward resistive force (and hence the object moves at constant speed).

It will depend on m and g but the form of $v_{\rm ter}$ depends on whether it is linear or quadratic resistive force.

b) Even though we got a full solution for projectile motion in two dimensions with a linear resistive force, we could not do this with the quadratic resistive force. Explain the problem with quadratic resistive force.

The quadratic resistive force has the form

$$\mathbf{f}_{\rm res} = -cv^2\hat{\mathbf{v}} = -cv\mathbf{v}$$

which has components

$$\mathbf{f}_{\rm res,\,x} == -c\sqrt{v_x^2+v_y^2}\,v_x \qquad \mathbf{f}_{\rm res,\,y} == -c\sqrt{v_x^2+v_y^2}\,v_y$$

The fact that the DE's mix x and y in a complicated way make them soluble only with numerical methods.

4. A mass m moves along the x axis; it has velocity $v_0 > 0$ at $t = 0$ and experiences a drag force given by $f(v) = -F_0e^{v/V}$ where F_0 and V are constants. (This is for "horizontal motion"; there is no gravity force here.)

a) Find v in terms of t and the given parameters. At what time (if any) does it come to rest?

Newton's law gives us

$$m\frac{dv}{dt} = -F_0e^{v/V} \qquad \Longrightarrow \qquad dt = -\frac{m}{F_0}e^{-v/V}\,dv$$

Integrate from time 0 to a later time t:

$$t \;=\; -\frac{m}{F_0}\int_{v_0}^{v} e^{-v'/V}\,dv \;=\; \frac{mV}{F_0}e^{-v'/V}\Big|_{v_0}^{v}$$

This gives

$$t = \frac{mV}{F_0}\left(e^{-v/V} - e^{-v_0/V}\right) \qquad \Longrightarrow \qquad e^{-v/V} = \frac{F_0}{mV}t + e^{-v_0/V}$$

Isolate v by taking logs, then

$$v = -V\ln\left(\frac{F_0t}{mV} + e^{-v_0/V}\right)$$

The mass is momentarily at rest when $v = 0$, or when the argument of the log in our expression above is 1. This gives

$$1 = \frac{F_0t}{mV} + e^{-v_0/V} \qquad \Longrightarrow \qquad t = \frac{mV}{F_0}(1 - e^{-v_0/V})$$

b) If it starts at $x = 0$, can you get $x(t)$?

Integrate $v(t)$ to get $x(t)$:

$$x(t) = x(0) + \int_0^t v(t')dt' = -V \int_0^t \ln\left(\frac{F_0 t'}{mV} + e^{-v_0/V}\right) dt'$$

which requires us to the integral of a log of a linear function of t'; not hard with a table of integrals, but this is a bit tedious so it's OK if you stop here; for what it's worth, I did this and got

$$x(t) = \frac{mV^2}{F_0}\left(\frac{F_0 t}{mV} + e^{-v_0/V}\right) \ln\left(\frac{F_0 t}{mV} + e^{-v_0/V}\right) + Vt + \frac{mVv_0}{F_0} e^{-v_0/V}$$

5. Show that moment of inertia of a uniform circular disk of radius R and mass M is $\frac{1}{2}MR^2$.

Area element of the plane in polar coordinates is $r\,dr\,d\phi$. If the mass density of the disk is $\rho_{\rm den}$, then the mass of an area element is $dm = \rho_{\rm den} r\,dr\,d\phi$ and since its distance from the axis is $s = r$, the moment of inertia is

$$I = \int s^2 dm = \int_0^{2\pi} \int_0^R r^2 \rho_{\rm den} r\,dr\,d\phi$$

Here, $\rho_{\rm den} = \frac{M}{\pi R^2}$, so that

$$I = \rho_{\rm den}(2\pi) \int_0^R r^3 dr = 2\pi \frac{M}{\pi R^2} \frac{R^4}{4} = \tfrac{1}{2} M R^2$$

6. In solving the classic "rocket problem" we arrived at at a result of the form

$$-F_{\rm thrust} = v \frac{dm}{dt}$$

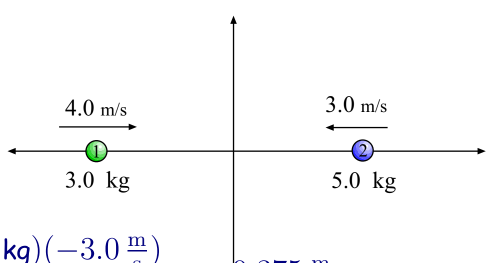
On the right hand side of this equation, what is meaning of v? (What velocity *is* it?)

Here v means $v_{\rm rel}$, the speed of the *ejected mass* relative to that of the remainder of the rocket.

Note that because the "v" and "$\frac{dm}{dt}$" on the right side refer to two different things, the expression could *not* have arisen as an application of the derivative product rule to mv.

7. A 3.0 kg and a 5.0 kg mass approach each other in the *lab* frame with speeds as shown at the right.

a) What is the velocity of the center of mass (as seen in the lab frame)?



$$v_{\rm cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(3.0\ \text{kg})(4.0\ \frac{\rm m}{\rm s}) + (5.0\ \text{kg})(-3.0\ \frac{\rm m}{\rm s})}{8.0\ \text{kg}} = -0.375\ \frac{\rm m}{\rm s}$$

b) What are the initial velocities of the masses as seen in the CM frame?

To get the velocities in the CM frame, *subtract* the CM velocity from the lab velocities. Thus the initial $x-$velocities are respectively

$$v_{1,\,\mathrm{cm}} = 4.0\,\tfrac{\mathrm{m}}{\mathrm{s}} - (-0.375\,\tfrac{\mathrm{m}}{\mathrm{s}}) = +4.8\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad v_{2,\,\mathrm{cm}} = -3.0\,\tfrac{\mathrm{m}}{\mathrm{s}} - (-0.375\,\tfrac{\mathrm{m}}{\mathrm{s}}) = -2.62\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

c) If the masses have an elastic collision, what are their final speeds as seen in the CM frame?

If momentum and kinetic energy are conserved then (in the CM frame) the final velocities will be in opposite directions with the same magnitudes as when the particles were approaching. Thus the final speeds are

$$v_1' = 4.8\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad \text{and} \qquad v_2' = 2.62\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

8. a) (Simple) *Show*, using the basic definition is that the total momentum of a system is related to the velocity of the center of mass by

$$\mathbf{P} = M\dot{\mathbf{R}}$$

$$\mathbf{P} = \sum_\alpha m_\alpha \mathbf{v}_\alpha \; = \; M\frac{d}{dt}\frac{\sum_\alpha m_\alpha \mathbf{r}_\alpha}{M} \; = \; M\frac{d}{dt}\mathbf{R} = M\dot{\mathbf{R}}$$

b) (Harder) Show that the total angular momentum of a system is related to the total angular momentum *calculated about the center of mass* and the center of mass $\mathbf{R}$ by

$$\mathbf{L} = \mathbf{L}_{\mathrm{CM}} + \mathbf{R} \times \mathbf{P}$$

You can start by considering $\mathbf{L}_{\mathrm{CM}}$, with

$$\mathbf{L}_{\mathrm{CM}} = \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha$$

where

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R} \qquad \text{and} \qquad \mathbf{p}'_\alpha = m_\alpha \mathbf{v}_\alpha - \dot{\mathbf{R}}$$

Some terms cancel to give the result.

$$\begin{aligned} \mathbf{L}_{\mathrm{cm}} &= \sum_\alpha m_\alpha (\mathbf{r}_\alpha - \mathbf{R}) \times (\mathbf{v}_\alpha - \dot{\mathbf{R}}) \\ &= \sum_\alpha m_\alpha [\mathbf{r}_\alpha \times \mathbf{v}_\alpha - \mathbf{r}_\alpha \times \dot{\mathbf{R}} - \mathbf{R} \times \mathbf{v}_\alpha + \mathbf{R} \times \dot{\mathbf{R}}] \end{aligned}$$

Work on the four terms in the bracket: The first one just gives the angular momentum about the origin of the (inertial) coordinate system. For the second we can switch the order of the cross product (with a change in sign) and pull the $\dot{\mathbf{R}}$ outside. For the third, take the $\mathbf{R}$ outside the sum, and take the fourth term outside the sum altogether. This gives:

$$\begin{aligned}
\mathbf{L}_{\rm cm} &= \mathbf{L} + \dot{\mathbf{R}} \times \sum_\alpha m_\alpha \mathbf{r}_\alpha - \mathbf{R} \times \sum_\alpha m_\alpha \mathbf{v}_\alpha + \mathbf{R} \times \dot{\mathbf{R}} \sum_\alpha m_\alpha \\
&= \mathbf{L} + \dot{\mathbf{R}} \times (M\mathbf{R}) - \mathbf{R} \times (M\dot{\mathbf{R}}) + M\mathbf{R} \times \dot{\mathbf{R}}
\end{aligned}$$

where we've used

$$\sum_\alpha m_\alpha \mathbf{r}_\alpha = M\mathbf{R} \qquad \text{and} \qquad \sum_\alpha m_\alpha \mathbf{v}_\alpha = M\dot{\mathbf{R}}$$

The third and fourth terms cancel and the anticommutivity of the cross product gives

$$\mathbf{L}_{\rm cm} = \mathbf{L} - \mathbf{R} \times (M\dot{\mathbf{R}})$$

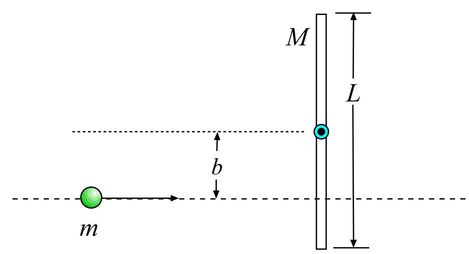
which on rearranging gives

$$\mathbf{L} = \mathbf{L}_{\rm cm} + \mathbf{R} \times (M\dot{\mathbf{R}}) = \mathbf{L}_{\rm cm} + \mathbf{R} \times \mathbf{P}$$

so that the last term can be interpreted as the angular momentum of the CM about the origin.

So angular momenta have a simple addition property as long as the point of reference is the *center of mass*.

9. A thin rod (mass M, total length L) is at rest in the xy plane, mounted on a frictionless axle at its center and initially aligned along the y axis (as shown). I throw a lump of putty (mass m) with speed v toward the rod; it moves in the $+x$ direction along a line that passes within a distance b of the origin. When the putty hits the rod, it sticks and then the two rotate together with angular velocity ω. Find ω.



Initial angular momentum of the system is just that of the incoming putty lump,

$$L_1 = |\mathbf{r} \times m\mathbf{v}| = mvb$$

The system after the collision is a rigid object rotating in a plane about an axis; the moment of inertia is

$$I = I_{\rm stick} + I_{\rm putty} = \tfrac{1}{12}ML^2 + mb^2$$

and if it rotates with angular velocity ω then $L_2 = I\omega$. There are *no external torques* about the origin so angular momentum is conserved; this gives

$$mvb = (\tfrac{1}{12}ML^2 + mb^2)\omega$$

from which we get

$$\omega = \frac{mvb}{\left(\frac{1}{12}ML^2 + mb^2\right)}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad dr\, \hat{\mathbf{r}} + r d\theta\, \hat{\boldsymbol{\theta}} + r \sin\theta\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2\, \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad ds\, \hat{\mathbf{s}} + s d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial T}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$\mathbf{v} = \dot{\mathbf{r}} \qquad \mathbf{a} = \dot{\mathbf{v}} \qquad \mathbf{p} = m\mathbf{v} \qquad \mathbf{F} = m\mathbf{a} = \dot{\mathbf{p}}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad \dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f_{\text{lin}} = bv \qquad f_{\text{quad}} = cv^2$$

$$b = \beta D \qquad c = \gamma D^2 \qquad \beta = 1.6 \times 10^{-4}\ \mathrm{N \cdot s/m^2} \qquad \gamma = 0.25\ \mathrm{N \cdot s^2/m^4}$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f(v) = bv + cv^2 \qquad \mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \omega = \frac{qB}{m}$$

$$m\dot{v} = -\dot{m}v_{\text{ex}} \qquad \mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} \qquad \dot{\boldsymbol{\ell}} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\Gamma} \qquad \mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \qquad \dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$$

$$L_z = I\omega \qquad I = \sum_\alpha m_\alpha \rho_\alpha^2 \qquad \frac{d}{dt}\mathbf{L}(\text{about CM}) = \boldsymbol{\Gamma}^{\text{ext}}(\text{about CM})$$

$$I_{\text{rod, ctr}} = \tfrac{1}{12}ML^2 \qquad I_{\text{rod, end}} = \tfrac{1}{3}ML^2 \qquad I_{\text{disc, ctr}} = \tfrac{1}{2}MR^2 \qquad I_{\text{sph, ctr}} = \tfrac{2}{5}MR^2$$

$$T = \tfrac{1}{2}mv^2 \qquad W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \qquad \Delta T = W(1 \to 2) \qquad \mathbf{F} = -\nabla U$$