Phys 2920, Spring 2012
Exam #2

1. The operator $\mathcal{A}$ is given by

$$\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

as written in the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ basis.

a) Suppose we want to re-express our vectors and operators in the new basis of the unit (orthonormal!) unit vectors

$$\hat{\mathbf{e}}'_1 = \frac{1}{5}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \qquad \hat{\mathbf{e}}'_2 = \frac{1}{5}(-4\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$$

What is the transformation matrix S and its inverse S^{-1}? (Hint: Getting S^{-1} won't take much work.)

We read off the coefficients of the $\hat{\mathrm{e}}_i$ vectors and make these the *columns* of S. This gives

$$\mathsf{S} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \qquad \Longrightarrow \qquad \mathsf{S}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

where we have found S^{-1} from the fact that the new basis is orthonormal and the matrix S is an orthogonal matrix, so to get S^{-1}, take the transpose.

b) Express the operator A in the new basis; note, it won't (necessarily) be diagonal. Express the vector

$$\mathbf{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

in the new basis. *Check* your answer for $\mathbf{x}$.

We calculate $\mathsf{A}' = \mathsf{S}^{-1}\mathsf{A}\mathsf{S}$:

$$\mathsf{A}' = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

and with a little bit of work we get

$$\mathsf{A}' = \frac{1}{25}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 13 & -9 \\ -5 & -10 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 19 & -67 \\ -67 & 6 \end{pmatrix}$$

And we get

$$\mathsf{x}' = \mathsf{S}^{-1}\mathsf{x} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

To check, note

$$\mathsf{x}' = -\hat{\mathbf{e}}'_1 + 3\hat{\mathbf{e}}'_2 = \frac{1}{5}(-3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 12\hat{\mathbf{i}} + 9\hat{\mathbf{j}}) = -3\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

c) Explain what I would do if I wanted to find a basis in which A *is* diagonal.

We would find the eigenvectors of the matrix A and make these the new basis for the transformation. In this basis A is diagonal, with its values being the eigenvalues of A.

2. a) Consider the point given by the spherical coordinates $(4, \pi/4, \pi/2))$. What are the Cartesian (rectangular) coordinates of this point?

$$x = r\sin(\pi/4)\cos(\pi/2) = 4 \cdot \frac{1}{\sqrt{2}} \cdot 0 = 0$$

$$y = r\sin(\pi/4)\sin(\pi/2) = 4 \cdot \frac{1}{\sqrt{2}} \cdot 1 = 2\sqrt{2}$$

$$z = r\cos\theta = 4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

b) Consider the point given by the cartesian coordinates $(1, -1, 3))$. What are the spherical coordinates of this point?

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{11} \qquad \theta = \cos^{-1}(z/r) = \cos^{-1}(3/\sqrt{11}) = 25.2^\circ = 0.441 \text{ rad}$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}((-1)/(1)) = 9\pi/4 = 7.07 \text{ rad}$$

3. For the scalar field

$$\Phi = 2x^2y - 3xyz^2$$

a) Find the rate of change of Φ at the point $(1, 1, 1)$ in the direction *parallel to* the vector $(-1, -1, 1)$.

$$\nabla\Phi = (4xy - 3yz^2)\,\hat{\mathbf{i}} + (2x^2 - 3xz^2)\,\hat{\mathbf{j}} + (-6xyz)\,\hat{\mathbf{k}}$$

At $(1, 1, 1)$ this is

$$\nabla\Phi = \hat{\mathbf{i}} - \hat{\mathbf{j}} - 6\,\hat{\mathbf{k}}$$

so we dot this with the unit vector in the direction of $-\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and get

$$\nabla\Phi \cdot \frac{1}{\sqrt{3}}(-\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = \frac{1}{\sqrt{3}}(-1 + 1 - 6) = -\frac{6}{\sqrt{3}} = -2\sqrt{3}$$

b) In what direction from the point $P = (1, 1, 1)$ is the directional derivative a maximum?

That is the same as the direction of the gradient at $(1, 1, 1)$ which is

$$\frac{\nabla\Phi}{|\nabla\Phi|} = \frac{1}{\sqrt{38}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} - 6\,\hat{\mathbf{k}})$$

4. Find the Laplacian (∇^2) of the scalar field

$$\Phi = 5r^4\,\sin\theta\,\sin^2\phi$$

Using the formula for ∇^2 in spherical coordinates, we get

$$\begin{aligned}\nabla^2(5r^4\sin\theta\sin^2\phi) &= 5\sin\theta\sin^2\phi\frac{1}{r^2}\frac{d}{dr}(4r^5) + 5r^2\frac{\sin^2\phi}{\sin\theta}\left[\frac{d}{d\theta}(2\sin\theta\cos\theta)\right] \\ &\quad +5r^4\sin\theta\frac{1}{r^2\sin^2\theta}\left[\frac{d}{d\phi}(2\sin\phi\cos\phi)\right]\end{aligned}$$

Use $2\sin\theta\cos\theta = \sin 2\theta$, then

$$\begin{aligned}\nabla^2\Phi &= 100\sin\theta\sin^2\phi r^2 + 5r^2\frac{\sin^2\phi}{\sin\theta}\left[\frac{d}{d\theta}(\sin 2\theta)\right] + \frac{5r^4}{r^2\sin\theta}\left[\frac{d}{d\phi}(\sin 2\phi)\right] \\ &= 100r^2\sin\theta\sin^2\phi + 5r^2\sin^2\phi\left[\frac{1}{\sin\theta}2\cos 2\theta\right] + \frac{5r^2}{\sin^2\theta}(2\cos 2\phi)\end{aligned}$$

It can be simplified a bit, but that's as far as I want to go.

5. Find the divergence of the vector field

$$2\rho^2\ z\,\hat{\mathbf{e}}_\rho + 4\cos\phi\,\hat{\mathbf{e}}_\phi - 2\rho z^2\,\hat{\mathbf{e}}_z$$

at the cylindrical point $(2, \frac{\pi}{2}, -1)$.

The divergence of this field a is

$$\nabla\cdot\mathbf{a} = 2z\frac{1}{\rho}3\rho^2 - \frac{1}{\rho}4\sin\phi - 4\rho z = 6\rho z - \frac{4\sin\phi}{\rho} - 4\rho z$$

At $(2, \frac{\pi}{2}, -1)$ this is

$$\nabla\cdot\mathbf{a} = 12(-1) - \frac{4}{2}\sin(\pi/2) - 8(-1) = -12 - 2 + 8 = -6$$

6. Find the curl of the vector field

$$\mathbf{a} = \rho\sin^2\phi\,\hat{\mathbf{e}}_\rho + 2z\,\hat{\mathbf{e}}_\phi + \rho^2\,\hat{\mathbf{e}}_z$$

We find:

$$\begin{aligned}\nabla\times\mathbf{a} &= (-2)\hat{\mathbf{e}}_\rho + (-2\rho)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[2z - 2\rho\sin\phi\cos\phi\right]\hat{\mathbf{e}}_z \\ &= -2\,\hat{\mathbf{e}}_\rho + -2\rho\,\hat{\mathbf{e}}_\phi + \left[\frac{2z}{\rho} - 2\sin\phi\cos\phi\right]\,\hat{\mathbf{e}}_z\end{aligned}$$

7. Do the line integral $\int_A^B \mathbf{a} \cdot d\mathbf{r}$ where

$$\mathbf{a} = 2y^2\,\hat{\mathbf{i}} - x^2\,\hat{\mathbf{j}}$$

where $A = (0,0)$ and $B = (1,3)$ and where the path from A to B is:

a) The line from $(0,0)$ to $(1,0)$ then from $(1,0)$ to $(1,3)$.

$$\int_C \mathbf{a} \cdot \mathbf{dr} = \int [2y^2\,dx - x^2\,dy]$$

On the path from $(0,0)$ to $(1,0)$ we have $dy = 0$ and $x : 0 \to 1$ with $y = 0$ so that the integral is

$$\int_1 \mathbf{a} \cdot \mathbf{dr} = \int 2(0)^2\,dx + 0 = 0$$

On the path from (1.0) to $(1,3)$ we have $dx = 0$ and $y : 0 \to 3$ with $x = 1$ so that the integral is

$$\int_2 \mathbf{a} \cdot \mathbf{dr} = 0 + \int_{y=0}^{y=3} (-1)\,dy = -3$$

so the entire integral is $\boxed{-3}$.

b) The straight line from $(0,0)$ to $(1,3)$.

Parametrize the points on the line as $x = t$, $y = 3t$ so that

$$2y^2 = 18t^2 \qquad x^2 = t^2 \qquad dx = dt \qquad dy = 3dt \qquad t : 0 \to 1$$

and the integral is

$$\int_C [2y^2\,dx - x^2\,dy] = \int_0^1 [18t^2 - t^2(3)]dt = 15\int_0^1 t^2 dt = 15 \cdot \frac{1}{3} = 5$$

8. We want to evaluate the integral $\int_C \mathbf{a} \cdot d\mathbf{r}$ where $\mathbf{a}$ is the vector field

$$\mathbf{a} = (2x + 2y^2 - 3z^2)\,\hat{\mathbf{i}} + 4xy\,\hat{\mathbf{j}} - 6zx\,\hat{\mathbf{k}}$$

and the path C goes from the origin, $(0,0,0)$ to $(2,2,2)$ along the straight line joining those points.

a) Does the integral depend on which path you take? How do you know?

If we take the curl of a we find

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ (2x + 2y^2 - 3z^2) & 4xy & -6zx \end{vmatrix} = (0)\,\hat{\mathbf{i}} + (-6z + 6z)\,\hat{\mathbf{j}} + (4y - 4y)\,\hat{\mathbf{k}} = \mathbf{0}$$

so the field is $\textit{conservative}$ and it must be the gradient of some scalar field Φ. And the integral does $\boxed{\text{not}}$ depend on the path taken between the endpoints.

b) If $\mathbf{a}$ is the gradient of some scalar function, try to find that function.

Do some detective work to find what Φ must be. First,

$$2x + 2y^2 - 3z^2 = \frac{\partial \Phi}{\partial x} \qquad \Longrightarrow \qquad \Phi = x^2 + 2xy^2 - 3xz^2 + f(y,z)$$

$$4xy = \frac{\partial \Phi}{\partial y} \qquad \Longrightarrow \qquad \Phi = 2xy^2 + f(x,z)$$

Finally,

$$-6zx = \frac{\partial \Phi}{\partial z} \qquad \Longrightarrow \qquad \Phi = -3xz^2 + f(x,y)$$

All of these give

$$\Phi(x,y,z) = x^2 + 2xy^2 - 3xz^2 + C$$

c) Irregardless of your answers to the first two parts, do the integral.

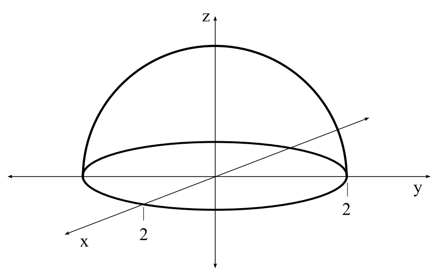
Now we know the value of the integral only comes from its $\textit{endpoints}$:

$$\int_A^B \mathbf{a} \cdot d\mathbf{r} = \int_A^B \nabla\Phi \cdot d\mathbf{r} = \Phi(B) - \Phi(A) = [4 + 16 - 24] - [0] = -4$$

9. Find $\int_V \Phi \, dV$ where Φ is the scalar field

$$\Phi(r,\theta,\phi) = 5r^2 \cos^2\theta \sin^2\phi$$

and the volume V is the upper hemisphere of radius 2.



$$\begin{aligned}
\int_V \Phi \, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 5r^2 \cos^2\theta \sin^2\phi \, r^2 \, dr \, \sin\theta \, d\theta \, d\phi \\
&= 5 \int_0^R r^4 \, dr \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} \sin^2\phi \, d\phi \\
&= 5 \left(\frac{R^5}{5}\right) \left[(-)\frac{\cos^3\theta}{3}\right]\Bigg|_0^{\pi/2} \left[-\tfrac{1}{4}\sin 2\phi + \frac{\phi}{2}\right]\Bigg|_0^{2\pi} \\
&= R^5 \frac{1}{3}\pi = \frac{\pi R^5}{3}
\end{aligned}$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{4}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{\imath}} + \sin\theta\sin\phi\,\hat{\mathbf{\jmath}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{\imath}} + \cos\theta\sin\phi\,\hat{\mathbf{\jmath}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{\imath}} + \cos\phi\,\hat{\mathbf{\jmath}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{\imath}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{\jmath}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy$$

$$\int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{x}{2} \qquad \int \cos^2 x\,dx = +\frac{1}{4}\sin 2x + \frac{x}{2}$$

Other integrals furnished upon request.