Group Velocity

In vacuum the speed of all EM waves is the same, c. In transparent media the speed of the waves is given by

$$v = \frac{c}{n} \qquad \text{with} \qquad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\epsilon_r}$$

since for most materials μ is close to μ_0. But for real materials ϵ_r and hence n and v will depend on the frequency ω.

For a harmonic wave, the *wave speed* is related to the frequency ω and wave number k by:

$$v = \frac{\omega}{k}$$

but a real wave have finite length; it is a wave *pulse* (or packet) of some size. A wave packet requires some Fourier analysis to construct, i.e. we need to add up a continuum of waves of differing frequency. We can get a simple picture of the motion of a wave packet by considering the addition of *two* harmonic waves with different frequencies (and different wave numbers). The sum of two such waves will give beats and we will find the speed at which the beat pattern travels. This will serve as a model for how a true wave packet would travel.

The two wave numbers will be close to some central value k; they will be chosen as

$$k + \tfrac{1}{2}\Delta k \qquad \text{and} \qquad k - \tfrac{1}{2}\Delta k$$

The frequencies of the two waves will be close to some central value ω and they will be chosen as:

$$\omega + \tfrac{1}{2}\Delta\omega \qquad \text{and} \qquad \omega - \tfrac{1}{2}\Delta\omega$$

where we mean that the Δ values are much smaller than the central values of k and ω.

The individual (complex) waves are:

$$\begin{aligned}
\tilde{f}_1(z,t) &= A\exp\left\{i(k + \tfrac{1}{2}\Delta k)z - (\omega + \tfrac{1}{2}\Delta\omega)t\right\} \\
\tilde{f}_2(z,t) &= A\exp\left\{i(k - \tfrac{1}{2}\Delta k)z - (\omega - \tfrac{1}{2}\Delta\omega)t\right\}
\end{aligned}$$

The sum of two waves can be written as

$$\begin{aligned}
\tilde{f} &= \tilde{f}_1 + \tilde{f}_2 = A\left\{\exp[i(\tfrac{1}{2}\Delta kz - \tfrac{1}{2}\Delta\omega)t] + \exp[-i(\tfrac{1}{2}\Delta kz - \tfrac{1}{2}\Delta\omega t)]\right\} e^{i(kz-\omega t)} \\
&= 2A\cos[\tfrac{1}{2}\Delta kz - \tfrac{1}{2}\Delta\omega t]e^{i(kz-\omega t)}
\end{aligned}$$

and the physical wave is the real part of this,

$$f = 2A\cos[\tfrac{1}{2}\Delta kz - \tfrac{1}{2}\Delta\omega t]\cos(kz - \omega t)$$

So what do we have here? We have a traveling "carrier" wave, $\cos(kz - \omega t)$ modulated by a travelling "signal" wave which gives the beat pulses. One can show that the separation in space of the successive beat pulses is $2\pi/\Delta k$ and that the separation in time of the beat pulses is $2\pi/\Delta\omega$. (One needs to clear here because there are factors of 2 involved in getting the separation/rate of the *pulses*.)

The ratio of these two numbers is the speed at which the pulses travel. It is:

$$v_{\text{group}} = \frac{\text{Space sep}}{\text{Time sep}} = \frac{2\pi/\Delta k}{2\pi/\Delta\omega} = \frac{\Delta\omega}{\Delta k}$$

In the limit that the Δ quantities are small,

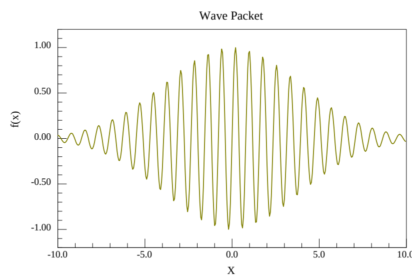
$$v_{\text{group}} \to \frac{d\omega}{dk}$$

Now for a wave in vacuum, $\omega = ck$ so that $v_{\text{group}} = c$ but if there is a more complicated dependence of ω on k we will get something else (which will depend on the central value ω).

To arrive at v_{group} in a more stisfying way we need to make a *real* wave packet. In particular we choose a gaussian wave packet. The game plan here is to specify the shape of the packet at $t = 0$. The packet will *not* have that shape later on because of dispersion! But with Fourier analysis we can calculate the wavefunction at later times and we can find the speed at at which the center of the packet is moving.

The wave function at $t = 0$ is chosen to be

$$\tilde{f}(x) = Ae^{-x^2/4(\Delta x_0)^2} e^{ik_0 x} \,,$$



whose real and imaginary parts have the shape illustrated here. This wave is a wiggle of wavenumber k_0 modulated by a gaussian shape of width Δx_0.

Now, if the medium had been nondispersive (with a single wave velocity v) we could substitute $x - vt$ for x and be done with it. But that's not possible with k depending on ω in some complicated way. The first thing to do is express the initial wave packet as a function of wavenumber k. This is done by evaluating the integral

$$\tilde{F}(k) = \int_{\infty}^{\infty} \tilde{f}(x) e^{-ikx}\, dx$$

$\tilde{F}(k)$ represents the contribution of the harmonic wave of wave number k (that is, e^{ikx}) to the initial packet. For our wave packet we get

$$\tilde{F}(k) = (4\pi)^{1/2} \Delta x_0 A e^{-(\Delta x_0)^2 (k-k_0)^2}$$

Now we can find the wave at all other times. We know that the harmonic wave of wavenumber k will travel at some speed v, with frequency $\omega = kv$. For *each harmonic wave* we can replace x by $x - vt$ so that the wave $\tilde{f}(x,t)$ is

$$\tilde{f}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(k) e^{ik(x-vt)}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(k) e^{i(kx-\omega t)}\, dk$$

Here, the $1/2\pi$ in front comes from the choice of normalization of the harmonic wave.

Thus we need to evaluate

$$\tilde{f}(x,t) = \frac{\Delta x_0 A}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-(\Delta x_0)^2 (k-k_0)^2} e^{i(kx-\omega t)}\, dk \tag{1}$$

which, alas, we cannot do until we know the relation between k and ω, that is, the dispersion relation!

Instead of choosing a particular dispersion relation, we will be somewhat general by using a Taylor expansion. Suppose the ω that goes with wave number k_0 is ω_0. We want the expansion for ω for values close to ω_0. It is:

$$\omega = \omega_0 + \left(\frac{d\omega}{dk}\right)_0 (k-k_0) + \frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_0 (k-k_0)^2 + \cdots \tag{2}$$

Here the subscript zero tells you the evaluate the derivative at wavenumber k_0.

Now the coefficient of the linear term is our desired expression for the group velocity. For now we will define

$$v_g = \left(\frac{d\omega}{dk}\right)_0$$

and show its significance later. We can also define

$$\alpha \equiv \frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_0$$

and with this, 2 becomes

$$\omega = \omega_0 + v_g(k-k_0) + \alpha(k-k_0)^2$$

Now we have something definite to put into the integral in 1. When we do this, we get

$$\tilde{f}(x,t) = \frac{A\Delta x_0}{\Delta x} e^{-(x-v_g t)^2/4(\Delta x)^2} e^{i(k_0 x - \omega_0 t)} \tag{3}$$

where

$$(\Delta x)^2 \equiv (\Delta x_0)^2 + i\alpha t$$

gives the (complex!) spatial spread of the wave packet.

The physical wave is the real part of 3. It's a bit of a mess but in this form we see that we have a rapid wiggle with wave number k_0 and frequency ω_0 modulated by a gaussian shape. The center of this gaussian shape is located at $x = v_g t$, hence v_g *does* represent the motion of the packet. There is also an increasing *width* to the packet given by Δx, but because Δx is complex the expression is hard to visualize completely.

The absolute magnitude of the wave, $|f(x,t)|^2$ does have a simple form:

$$|\tilde{f}(x,t)|^2 = \frac{a^2}{[1+\alpha^2 t^2/(\Delta x_0)^4]^{1/2}} \exp\left\{\frac{-(x-v_g t)^2}{2(\Delta x_0)^2[1+\alpha^2 t^2/(\Delta x_0)^4]}\right\} \tag{4}$$

and here it's easier to see the effect of the dispersion.