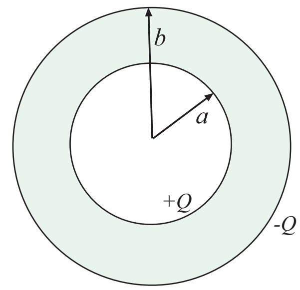
Phys 4610, Fall 2005
Exam #3



1. Two concentric conducting spherical shells of radius a and b (with $a < b$) carry charges of $+Q$ and $-Q$, respectively. The space between the shells is filled with an insulating material with permittivity ϵ.

a) Find the $\mathbf{E}$ and $\mathbf{D}$ fields in the region between the shells.

When we need to distinguish free and bound charge (like here, where there is a polarized dielectric) the useful equation is $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\mathrm{f,\,enc}}$. A Gaussian sphere of radius r with $a < r < b$ encloses a *free* charge $+Q$ so the usual manipulations give

$$(4\pi r^2)D = Q \qquad \Longrightarrow \qquad D = \frac{Q}{4\pi r^2} \qquad \text{i.e.} \quad \mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}$$

Using $\mathbf{E} = \mathbf{D}/\epsilon$, then

$$\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$$

b) Find the potential difference (ΔV) between the two conducting shells.

Find the potential difference from $\Delta V = -\int_a^b \mathbf{E} \cdot d\mathbf{r}$ which gives:

$$\begin{aligned}
\Delta V &= V(b) - V(a) = -\int_a^b \frac{Q}{4\pi\epsilon r^2}dr = -\frac{Q}{4\pi\epsilon}\left(-\frac{1}{r}\right)\Bigg|_a^b \\
&= \frac{Q}{4\pi\epsilon}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{Q(a-b)}{4\pi\epsilon ab}
\end{aligned}$$

(This is a negative number because b is at a lower potential than a.)

2. Prove that the net force on a current loop in a uniform magnetic field is zero.

The force on a (whole) current loop carrying current I is

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B})$$

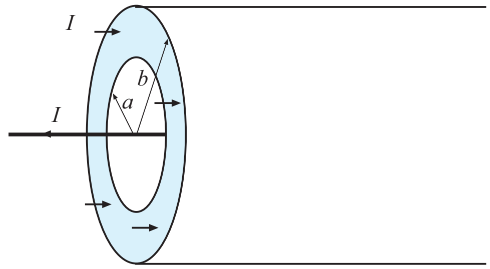
where $\mathbf{B}$ is the local value of the magnetic field. But if $\mathbf{B}$ has the same value everywhere then it can be taken outside the integral:

$$\mathbf{F} = I \left[\oint d\mathbf{l}\right] \times \mathbf{B}$$

but the line integral of the path elements $d\mathbf{l}$ over a closed loop gives zero (you come back to the same place): $\oint d\mathbf{l} = 0$, so we get $\mathbf{F} = 0$.

Hey, I didn't say it was a *hard* proof.

3. Consider the coaxial wire system shown at the right; the inner wire is very thin and carries a current I in the $\hat{\mathbf{z}}$ direction; but outer conductor has inner and outer radii a and b, and it carries a current I in the $-\hat{\mathbf{z}}$ direction which is distributed *uniformly* over its cross-section.



Find the magnetic field for $0 < s < a$, $a < s < b$ and $b < s$.

(This problem assumes there is nothing in the region between the conductors.) We will assume that the B field everywhere is tangential (in the right-handed sense) without going through all the symmetry arguments.

For $0 < s < a$ consider an Amperian loop of radius s. B_ϕ has the same value all along the path and we get

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi(2\pi r) = \mu_0 I_{\rm enc} = \mu_0 I$$

and so

$$B_\phi = \frac{\mu_0 I}{2\pi r} ,$$

the same as for a very long wire by itself.

For $a < s < b$ the Amperian loop of radius s also encloses *some* current in the outer conductor. Since the volume current density magnitude for the outer conductor is (note, it goes in the $-z$ direction)

$$J_z = -\frac{I}{\pi(b^2 - a^2)} \qquad \Longrightarrow \qquad I_{\rm enc} = I + \pi(s^2 - a^2)\left(\frac{-I}{\pi(b^2 - a^2)}\right) = I - I\frac{(s^2 - a^2)}{(b^2 - a^2)}$$

Then applying Ampere's law for the loop we get

$$B_\phi(2\pi s) = \mu_0 I_{\rm enc} = \mu_0 I\left[1 - \frac{(s^2 - a^2)}{(b^2 - a^2)}\right] = \mu_0 I\frac{(b^2 - s^2)}{(b^2 - a^2)}$$

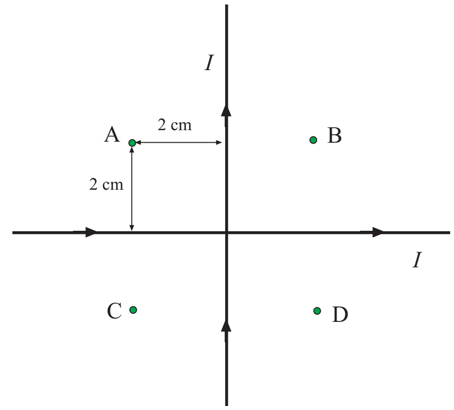
So then

$$B_\phi = \frac{\mu_0 I(b^2 - s^2)}{2\pi s(b^2 - a^2)}$$

Finally, for $b < s$ a circular Amperian loop of radius s encloses *zero* total current which implies that $B_\phi = 0$ there.

4. Phys 2120 time. Two very long wires lie in the same plane (the plane of the page here) and each carries a current $I = 3.0$ A in the directions shown. (Note, the wires don't connect at the origin!!)

The points A, B, C, D also lie in the plane of the page and each point is 2.00 cm from either wire. Find the magnitude and direction of the magnetic field at A, B, C, D. Get *numerical values*!



At a distance of 2.0 cm from a single long wire carrying a current of 3.0 A, the magnitude of the (tangential) B field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{(4\pi \times 10^{-7}\,\frac{\mathrm{T\cdot m}}{\mathrm{A}})(3.0\ \mathrm{A})}{2\pi(2.0 \times 10^{-2}\ \mathrm{m})} = 3.0 \times 10^{-5}\ \mathrm{T}$$

At point A, using the right-hand rule we have two contributions of this size coming *out of the page* (which we'll say is the $\hat{\mathbf{z}}$ direction) so that at A the B field is

$$\mathrm{A}: \qquad \mathbf{B} = +6.00 \times 10^{-5}\ \mathrm{T}\hat{\mathbf{z}}$$

At points B and C there are contributions of equal size going into and out of the page so that the B field is zero at these points.

At D the two contributions go into the page so the B field is

$$\mathrm{D}: \qquad \mathbf{B} = -6.00 \times 10^{-5}\ \mathrm{T}\hat{\mathbf{z}}$$

5. Suppose the magnetic field in some region of space is given by

$$\mathbf{B} = -\mu_0 z J \hat{\mathbf{y}}$$

where J is some constant.

Find a suitable form for the vector potential $\mathbf{A}$.

$\mathbf{A}$ *must* satisfy $\nabla \times \mathbf{A} = \mathbf{B}$. (If it also satisfies $\nabla \cdot \mathbf{A} = 0$, that's OK!). From the form of the curl in cartesian coordinates, we know that $\mathbf{A}$ must satisfy:

$$(\nabla \times \mathbf{A})_y = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = -\mu_0 z J$$

This will be satisfied if we have

$$A_x = -\frac{\mu_0 J z^2}{2}$$

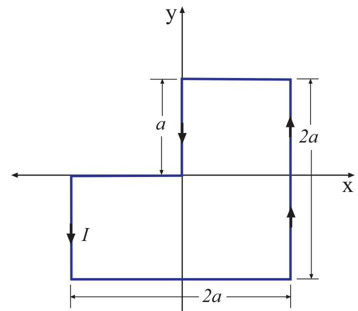
We can also try

$$A_z = \mu_0 J x z$$

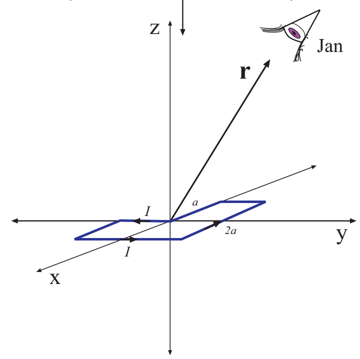
without messing up the rest of the curl because we would have a $\partial A_z/\partial y$ term elsewhere, and that would be zero. However the second choice does not give $\nabla \cdot \mathbf{A} = 0$ while the first one does. So the best solution seems to be

$$\mathbf{A} = -\frac{\mu_0 J z^2}{2}\hat{\mathbf{x}}$$

6. Consider the planar current loop shown at the right; it lies in the xy plane; the loop carries a current I and the sides of the rectangular polygon have the dimensions shown.



Now consider an observer who is at *a large distance* r from the current loop. (We mean $r \gg a$.) Find expressions for the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ at large r to *leading order*.



At large distances from a distribution of currents the leading order behavior is given by the magnetic dipole term in the multipole expansion. It involves the dipole moment $\mathbf{m}$ of the currents.

It's easy to calculate $\mathbf{m}$ here:

$$\mathbf{m} = I \int d\mathbf{a} = I(3a^2)\hat{\mathbf{z}} = 3Ia^2\hat{\mathbf{z}}$$

and that's what we use in the formulae. From the text we have a couple for this case, i.e. a magnetic dipole near the origin which points along $\hat{\mathbf{z}}$. We get:

$$\mathbf{A} \approx \frac{\mu_0}{4\pi}\frac{3Ia^2\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

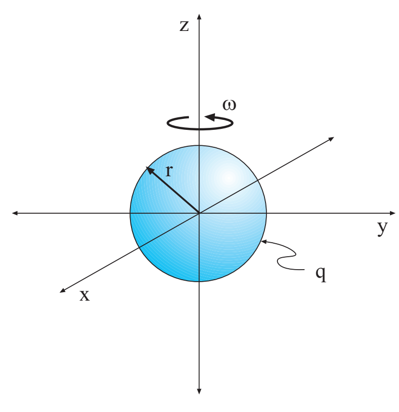
and

$$\mathbf{B} \approx \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{3\mu_0 I a^2}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

7. A shell of radius r and uniformly-distributed charge q rotates with angular velocity ω about the z axis.

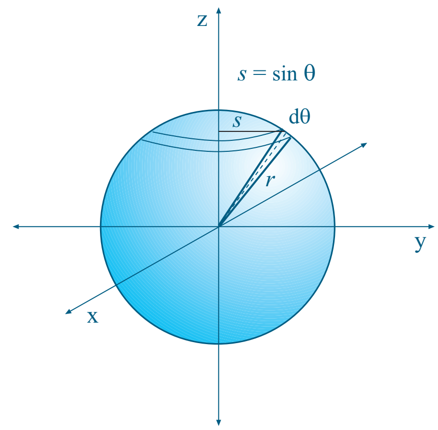
a) Show that the magnetic moment of the shell is

$$\mathbf{m} = \frac{qr^2\omega}{3}\hat{\mathbf{z}}$$



Here's one way to get through the math. Consider a sector $d\theta$ of the sphere's surface. Since the surface current density of the sphere is

$$\mathbf{K} = \sigma v \hat{\boldsymbol{\phi}} = \sigma \omega s \hat{\boldsymbol{\phi}} = \sigma \omega r \sin\theta \hat{\boldsymbol{\phi}}$$



and the transverse length through which this current travels on the sector is $dl = r\theta$ then the current in the sector is

$$dI = Kdl = \sigma \omega r^2 \sin\theta d\theta$$

The area of the sector's current loop is

$$a = \pi s^2 = \pi (r\sin\theta)^2 = \pi r^2 \sin^2\theta$$

So the magnetic moment of the sector is

$$dm = adI = \pi \sigma \omega r^4 \sin^3\theta d\theta$$

Integrate over the whole surface ($\theta : 0 \rightarrow \pi$) to get m:

$$m = \int dm = \int_0^\pi \pi \sigma \omega r^4 \sin^3\theta d\theta$$

Use a trig identity to do the integral:

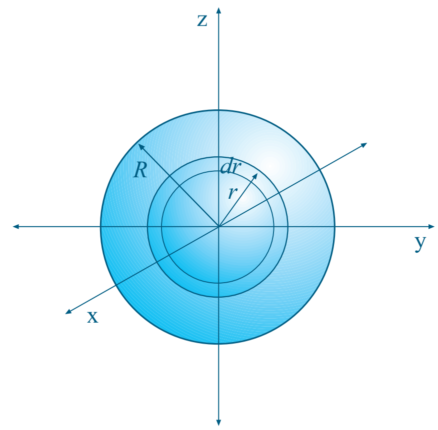
$$\begin{aligned} m &= \pi \sigma \omega r^4 \int_0^\pi \sin\theta (1 - \cos^2\theta) d\theta = \pi \sigma \omega r^4 \left[-\cos\theta + \frac{\cos^3\theta}{3} \right] \\ &= \pi \sigma \omega r^4 \left[1 + 1 - \tfrac{1}{3} - \tfrac{1}{3} \right] = \frac{q \pi \omega r^4}{4 \pi r^2} \left(\frac{4}{3} \right) = \frac{q \omega r^2}{3} \end{aligned}$$

So putting the vectors back in,

$$\mathbf{m} = \frac{q\omega r^2}{3} \hat{\mathbf{z}}$$

b) Find the magnetic moment of a uniformly-charged sphere of charge Q and radius R which rotates about the $\hat{\mathbf{z}}$ axis with angular velocity ω. The answer to part (a) can be helpful.

Split up the solid sphere into spherical shells each of radius r and thickness dr, as shown here. We will relate the parameters of this sphere to the one in part (a) so that we can get the dipole moment of each shell. The charge contained in the shell is



$$dq = (4\pi r^2 dr)\rho$$

so that part (a) gives us the dipole moment of the shell:

$$dM = \frac{(4\pi r^2 dr)\rho\omega r^2}{3}$$

so now sum up the shells with r going from 0 to R:

$$M = \int_0^R \frac{(4\pi r^2 dr)\rho\omega r^2}{3} = \frac{4\pi\rho\omega}{3}\int_0^R r^4 dr = \frac{4\pi\rho\omega}{3}\frac{R^5}{5}$$

Make the replacement $\rho = Q/[(4/3)\pi R^3]$, then

$$\mathbf{M} = \frac{4\pi Q\omega(R^5/5)}{3(4/3)\pi R^3}\hat{\mathbf{z}} = \frac{QR^2\omega}{5}\hat{\mathbf{z}}$$

8. How are the free currents $\mathbf{J}_f$ and $\mathbf{K}_f$ different from the bound currents $\mathbf{J}_b$ and $\mathbf{K}_b$?

Bound currents are a measure of the motion of charges which is due to an induced magnetization (induced dipoles) in a sample. We distinguish this from the motion of charges which can be controlled and measured by ammeters, which is the free current.

9. Suppose in Problem 3 the region between the conductors had been filled with a linear material of permeability μ. What do you *now* get for the B field between the conductors $(0 < s < a)$?

In problem 3 in the interior region we would use the Ampere law for macroscopic media, namely

$$\oint \mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

This would give the H field; assuming it is tangential and all that, we get

$$H_\phi(2\pi s) = I \qquad \Longrightarrow \qquad H_\phi = \frac{I}{2\pi s}$$

And *now* we can get the B field from $\mathbf{B} = \mu\mathbf{H}$,

$$\mathbf{B} = \mu\mathbf{H} = \frac{\mu I}{2\pi s}\hat{\boldsymbol{\phi}}$$

10. Compare and contrast *monkeyshines*, *horseplay* and *tomfoolery*. Give examples of each.

Extra Credit: Is it possible to engage in an activity which could be considered *tomfoolery* but not *horseplay*?

All three are terms describing in-class foolishness designed to get the teacher's goat. In brief:

Monkeyshines describes activity with can have a physical aspect but is carried out by *one person*. *Horseplay* always has a physical element and usually involves 2 or more people. *Tomfoolery* can involve one or more people but has only a small physical aspect; it generally involves a silly juxtaposition of ideas.

Discussing proper colors for the vectors on the chalkboard is tomfoolery but not horseplay.

Except when the professor does it. Then it's funny.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2 \, \sin\theta \, dr \, d\theta \, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

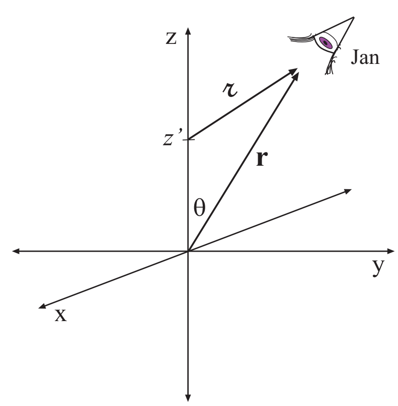
(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Even More Math

In the figure at the right,

$$\mathscr{r} = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$$



If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3-3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int\mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times10^{-7}\,\frac{\text{N}}{\text{A}^2} \qquad 1\ \text{T} = 1\,\frac{\text{N}}{\text{A}\cdot\text{m}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$$

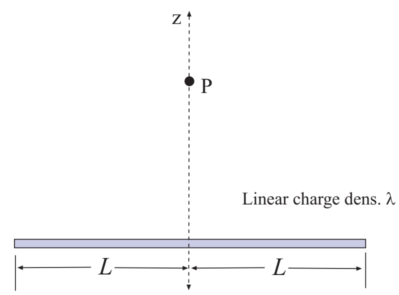
$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

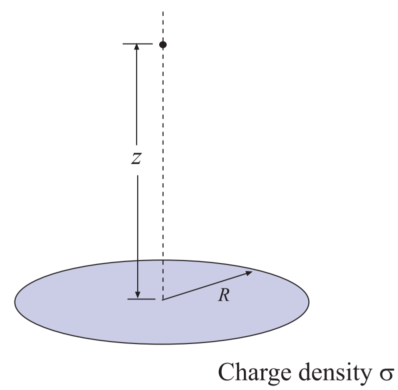
$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

Specific Results:

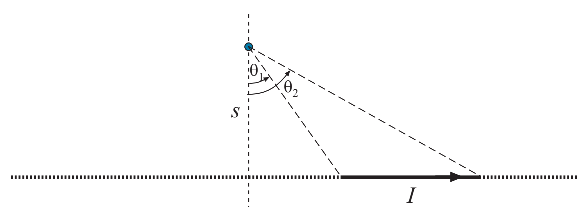
$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{Q}{2\pi R^2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \end{aligned}$$



$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$



$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$