Phys 3610, Fall 2008
Exam #1

1. What is an "inertial reference frame"?

An inertial reference frame is a system of spatial coordinates in which which there is no acceleration of masses unless a *true* force acts on them. (In such a frame, Newton's second law holds, where the forces arise from real objects.) To be as definite as we can, we say that such a frame is at rest and not rotating with respect to the "fixed stars". Any frame moving at constant velocity and not rotating with respect to an inertial frame is also an inertial frame.

For many purposes the surface of the Earth can serve adequately as an inertial reference frame. In some cases, it cannot.

2. The examples of motion with quadratic and linear resistive forces (with gravity) each had a quantity defined as the *terminal velocity*.

Give a careful *physical* definition of terminal velocity, suitable for both cases.

A falling object can move at a constant velocity if the downward force of gravity is balanced by the opposing resistive force. Thus if the magnitude of the resistive force depends on speed via $f(v)$, at some speed we can (maybe) have $f(v_{\rm ter}) = mg$. Solving this for $v_{\rm ter}$ gives the terminal velocity in terms of the other parameters.

In fact, when we solve the problems with linear and quadratic resistive forces, we see that the terminal velocity is only approached *asymptotically*, that is, a falling mass reaches $v_{\rm ter}$ at $t = \infty$.

3. A mass m moves along the x axis; it has velocity $v_0 > 0$ at $t = 0$ and experiences a drag force given by $f(v) = -cv^{1/2}$. (This is "horizontal motion"; there is no gravity force here.)

Find v in terms of t and the given parameters. At what time (if any) does it come to rest?

If it starts at $x = 0$, can you get $x(t)$?

The equation of motion is

$$m\dot{v} = -cv^{1/2} \qquad \Longrightarrow \qquad \frac{dv}{dt} = -\frac{c}{m}v^{1/2} \qquad \Longrightarrow \qquad v^{-1/2}dv = -\frac{c}{m}dt$$

Integrate the respective sides from v_0 to v and 0 to t:

$$\int_{v_0}^{v} v'^{-1/2}dv' = -\frac{c}{m}t \qquad \Longrightarrow \qquad 2\left(v^{1/2} - v_0^{1/2}\right) = -\frac{c}{m}t$$

Solve for v:

$$v^{1/2} = -\frac{c}{2m}t + v_0^{1/2} \qquad \Longrightarrow \qquad v = \left(v_0^{1/2} - \frac{c}{2m}t\right)^2$$

The velocity is zero when

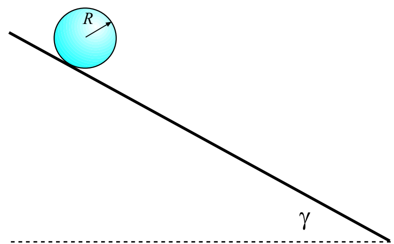
$$v_0^{1/2} = \frac{c}{2m}t \qquad \Longrightarrow \qquad t = \frac{2m}{c}v_0^{1/2}$$

In this case the mass will stay at rest for all times afterwards as the force is zero and stays zero. (This may sound funny, but the square-root form of the force is also mathematically funny, i.e. "non-analytic").

Having v(t) (at least until it stops), we can integrate again to get $x(t)$:

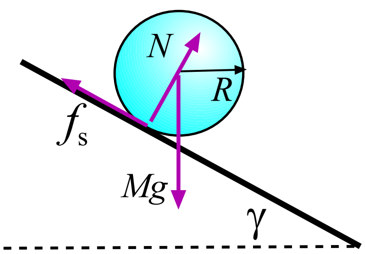
$$\begin{aligned} x &= \int_0^t v(t)dt = \int_0^t \left(v_0^{1/2} - \frac{c}{2m}t'\right)^2 dt' \\ &= -\frac{2m}{3c}\left(v_0^{1/2} - \frac{c}{2m}t'\right)^3 \Big|_0^t = \frac{2m}{3c}v_0^{3/2} - \frac{2m}{3c}\left(v_0^{1/2} - \frac{c}{2m}t\right)^3 \end{aligned}$$

One can check this to make sure that in the limit $c \to 0$ we get $x = v_0 t \ldots$



4. A solid uniform sphere of radius R rolls without slipping down an incline sloped at angle γ from the horizontal.

Find the acceleration of the sphere's center. Recall that on the homework you showed that the moment of inertia of the sphere about an axis thru its center is $\frac{2}{5}MR^2$.



We draw a force diagram, as show at the right. The forces acting on the sphere are gravity, Mg downward, the normal force of the surface (N applied at the point of contact) and the force of static friction, f_s, applied at the point of contact and directed up the slope.

The forces down the slope must give Ma (a being the acceleration of the center of mass) so that

$$Mg\sin\gamma - f_s = Ma$$

The (external, clockwise) torque can come only from the friction force. The torque is equal to the rate of change of $L = I\omega$, namely $I\alpha$ so that

$$\gamma = f_s R = I\alpha$$

where I is the moment of inertia about an axis thru the center of the sphere, namely $I = \frac{2}{5}MR^2$. Finally, from the kinematics of rolling, a and α are related by $a = \alpha R$. Putting this into the last equation gives

$$f_s R = \tfrac{2}{5}MR^2\frac{a}{R} \qquad \Longrightarrow \qquad f_s = \frac{2}{5}Ma$$

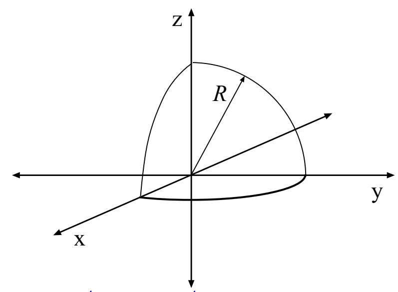
Put this into the first equation, then

$$Mg\sin\gamma - \frac{2}{5}Ma = Ma \qquad \Longrightarrow \qquad g\sin\gamma = \frac{7}{5}a$$

Finally,

$$a = \frac{5}{7}g\sin\gamma$$

5. Find the location of the center of mass of one octant of a solid uniform sphere. (Find its coordinates for the choice shown here and also its distance from the origin.)



The z coordinate of the cm is

$$\begin{aligned}\frac{1}{M}\int z dm &= \frac{\rho}{M}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (r\cos\theta)r^2\sin\theta dr\, d\theta\, d\phi \\ &= \frac{\rho}{M}\frac{\pi}{2}\left(\int_0^{\pi/2}\cos\theta\sin\theta d\theta\right)\left(\int_0^R r^3 dr\right) = \frac{\rho}{M}\cdot\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{R^4}{4} = \frac{\pi R^4\rho}{16M}\end{aligned}$$

Since the mass of the octant-sphere is

$$M = \frac{1}{8}\frac{4\pi R^3}{3}\rho = \frac{\pi R^3\rho}{6}$$

the z coord of the CM is

$$z_{CM} = \frac{\pi R^4\rho/(16M)}{\pi R^3\rho/6} = \frac{3}{8}R$$

Now from symmetry, we expect the CM values of x and y to have the same values. In that case the distance of the CM from the origin is

$$R_{CM} = \sqrt{3\left(\frac{3R}{8}\right)^2} = \frac{\sqrt{27}}{8}R$$

To check that last supposition, calculate x_{CM}. Since $x = r\sin\theta\cos\phi$, the integral, similar to the one for z is

$$\begin{aligned}\frac{1}{M}\int x dm &= \frac{\rho}{M}\int_0^{\pi/2}\int_0^{\pi/2}(r\sin\theta\cos\phi)r^2\sin\theta dr\, d\theta\, d\phi \\ &= \frac{\rho}{M}\int_0^{\pi/2}\cos\phi\, d\phi\int_0^{\pi/2}\sin^2\theta d\theta\int_0^R r^3 dr = \frac{\rho}{M}\cdot 1\cdot\frac{R^4}{4}\cdot\int_0^{\pi/2}\sin^2\theta d\theta\end{aligned}$$

The θ integral might need a tables look-up. We get:

$$-\frac{1}{4}\sin 2\theta + \frac{\theta}{2}\Big|_0^{\pi/2} = \frac{\pi}{4}$$

so using this, we get

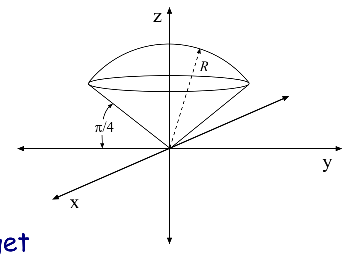
$$x_{CM} = \frac{\rho}{M}\frac{R^4}{16}$$

and with $M = \frac{\pi R^3\rho}{6}$ from before, we get

$$x_{CM} = \frac{3}{8}R$$

and I'm pretty sure y_{CM} works out the same way!

6. A sector of a solid sphere of radius R is spun around its symmetry axis, as shown. This chunk of the sphere is the part that goes from $\theta = 0$ to $\pi/4$.



a) Find the mass of this chunk of the sphere. Express the answer in terms of its (uniform) density ρ and R.

Integrating over the given volume (r: $0 \to R$, θ: $0 \to \pi/4$) we get

$$\begin{aligned} M &= \rho \int dV = \rho \int_0^{2\pi} \int_0^{\pi/4} \int_0^R r^2 dr \, \sin\theta d\theta d\phi \\ &= \rho(2\pi)(-\cos\theta)\Big|_0^{\pi/4} \cdot \frac{R^3}{3} = \frac{2\rho\pi R^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\pi\rho R^3(2-\sqrt{2})}{3} \end{aligned}$$

b) Find the moment of inertia of the spherical sector.

$$\begin{aligned} I &= \rho \int_0^{2\pi} \int_0^{\pi/4} \int_0^R (r\sin\theta)^2 r^2 \, dr \, \sin\theta d\theta \, d\phi = \rho(2\pi)\frac{R^5}{5}\int_0^{\pi/4} \sin^3\theta d\theta \\ &= \rho\frac{2\pi R^5}{5}\left(\frac{1}{3}\cos^3\theta - \cos\theta\right)\Big|_0^{\pi/4} = \frac{2\pi\rho R^5}{5}\left(\frac{1}{3}\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{3} + 1\right) \\ &= \frac{2\pi\rho R^5}{5}\left(\frac{1 - 6 - 2\sqrt{2} + 6\sqrt{2}}{6\sqrt{2}}\right) \\ &= \frac{2\pi\rho R^5}{5}\left(\frac{4\sqrt{2}-5}{6\sqrt{2}}\right) \end{aligned}$$

If we substitute

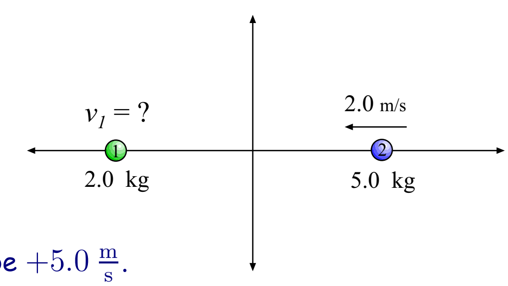
$$\rho = \frac{3M}{\pi R^3(2-\sqrt{2})}$$

then this is

$$I = \frac{MR^2}{5\sqrt{2}}\frac{(4\sqrt{2}-5)}{(2-\sqrt{2})}$$

Okay, so this one was a little messy.

7. A 2.0 kg and a 5.0 kg mass approach each other in the *center of mass* frame as shown at the right.



a) What is the initial velocity of mass 1 in the CM frame?

The total momentum is the CM frame is zero. The left particle must have momentum $10\ \frac{\text{kg}\cdot\text{m}}{\text{s}}$ so its velocity must be $+5.0\ \frac{\text{m}}{\text{s}}$.

b) If the masses have a head–on *elastic* collision, what are the final velocities of the masses in the CM frame.

In the CM frame, for an elastic collision, the velocities will simply reverse because this *will* conserve momentum and energy. Thus after the collsion mass 1 has velocity $-5.0\,\frac{\mathrm{m}}{\mathrm{s}}$ and mass 2 has velocity $+2.0\,\frac{\mathrm{m}}{\mathrm{s}}$.

c) If in the lab frame, mass 2 was initially at rest, find the initial and final velocities of the masses in the lab frame.

In the CM frame mass 2 initially has velocity $-2.0\,\frac{\mathrm{m}}{\mathrm{s}}$. If in the lab frame it is stationary, then in the Lab frame, the CM must be moving to the right with speed $2.0\,\frac{\mathrm{m}}{\mathrm{s}}$. The get velocities in the Lab frame, add $+2.0\,\frac{\mathrm{m}}{\mathrm{s}}$ to their values in the CM frame. Thus, before the collision, mass 1 had velocity $+7.0\,\frac{\mathrm{m}}{\mathrm{s}}$.

And so after the collision the velocities of the masses are:

$$\text{Mass 1:}\quad v = -5.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + 2.0\,\tfrac{\mathrm{m}}{\mathrm{s}} = -3.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

$$\text{Mass 2:}\quad v = +2.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + 2.0\,\tfrac{\mathrm{m}}{\mathrm{s}} = +4.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

8. Give a definition (any definition) of a *conservative* force.

A conservative force is one for which the work done when a particle moves from 1 to 2:

$$W(1 \to 2) = \int_1^2 = \mathbf{F} \cdot d\mathbf{r}$$

does *not depend* on the path taken from 1 to 2.

Equivalently, the work done by $\mathbf{F}$ over any closed path is zero, and also equivalently $\mathbf{F}(\mathbf{r})$ satisfies

$$\nabla \times \mathbf{F} = 0$$

Which of the following forces is conservative?

a)

$$\mathbf{F} = k(x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}})$$

b)

$$\mathbf{F} = k(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$$

If either of them is conservative, find the corresponding potential energy U.

A force $\mathbf{F}$ is conservative if $\nabla \times \mathbf{F} = 0$.

Test choice (a):

$$\nabla \times \mathbf{F} = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x & 2y & 3z \end{vmatrix} = 0$$

This one is conservative. Since $\mathbf{F} = -\nabla U(\mathbf{r})$, we need a U such that

$$F_x = kx = -\frac{\partial U}{\partial x} \qquad F_y = 2ky = -\frac{\partial U}{\partial y} \qquad F_z = 3kz = -\frac{\partial U}{\partial z}$$

Such a U is given by

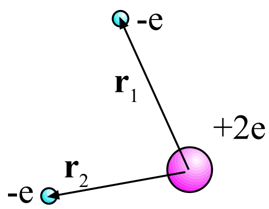
$$U = -k\left(\frac{x^2}{2} + y^2 + \frac{3}{2}z^2\right)$$

For choice (b),

$$\nabla \times \mathbf{F} = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = k\hat{\mathbf{z}}(1+1) = 2k\hat{\mathbf{z}}$$

This one is not conservative so there is no corresponding potential.

9. A helium atom –in a classical picture– can be taken to be a fixed He nucleus (of charge $+2e$) orbited by two electrons (each of charge $-e$).



Write down an expression for the total energy of the helium atom. Express the answer in terms of constants and the position vectors and velocities of the two electrons.

Recall that the potential energy for charges q_1 and q_2 is $k\frac{q_1 q_2}{r}$ where r is the distance between the charges.

With m_e as the mass of each electron, their kinetic energies are

$$T = \tfrac{1}{2}m_e\mathbf{v}_1^2 + \tfrac{1}{2}m_e\mathbf{v}_2^2$$

There is potential energy for the force between each electron and the proton and for the force between the two electrons. This gives:

$$U = k\frac{(2e)(-e)}{r_1} + k\frac{(2e)(-e)}{r_2} + k\frac{(-e)(-e)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

The total energy is

$$E = \tfrac{1}{2}m_e\mathbf{v}_1^2 + \tfrac{1}{2}m_e\mathbf{v}_2^2 + -k\frac{2e^2}{r_1} - k\frac{2e^2}{r_2} + k\frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + sd\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$\mathbf{v} = \dot{\mathbf{r}} \qquad \mathbf{a} = \dot{\mathbf{v}} \qquad \mathbf{p} = m\mathbf{v} \qquad \mathbf{F} = m\mathbf{a} = \dot{\mathbf{p}}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad \dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f_{\text{lin}} = bv \qquad f_{\text{quad}} = cv^2$$

$$b = \beta D \qquad c = \gamma D^2 \qquad \beta = 1.6 \times 10^{-4}\ \text{N} \cdot \text{s/m}^2 \qquad \gamma = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^4$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f(v) = bv + cv^2 \qquad \mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \omega = \frac{qB}{m}$$

$$m\dot{v} = -\dot{m}v_{\text{ex}} \qquad \mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

$$\ell = \mathbf{r} \times \mathbf{p} \qquad \dot{\ell} = \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma} \qquad \mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \qquad \dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$$

$$L_z = I\omega \qquad I = \sum_\alpha m_\alpha \rho_\alpha^2 \qquad \frac{d}{dt}\mathbf{L}(\text{about CM}) = \mathbf{\Gamma}^{\text{ext}}(\text{about CM})$$

$$T = \tfrac{1}{2}mv^2 \qquad W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \qquad \Delta T = W(1 \to 2) \qquad \mathbf{F} = -\nabla U$$