Phys 3610, Fall 2009
Exam #2

1. Define:

a) Unstable equilibrium

For a one-dimensional potential $U(x)$ the value of x is such that $\frac{dU}{dx} = 0$ (zero force) but U has a maximal value, $\frac{d^2U}{dx^2} < 0$ so that for small displacements away from the point, the forces act to push the particle *away from* the point.

b) Resonance frequency of driven damped oscillator.

The frequency at which the amplitude of the *long-term* motion is a maximum.

c) Overdamped oscillator.

The case when the coefficient of the linear resistive force is large enough that when released from some position the mass simply moves toward the origin but not go past it to make any true oscillations.

d) Homogeneous solution of a differential equation.

A differential equation generally has the form where there are operators acting on an unknown function on the left side and some given function on the right side. The homogeneous solution is one where the same operators from the left side act on the solution to give *zero*

2. If

$$U(\mathbf{r}) = A\cosh\alpha x + B\sinh\beta y$$

find the force $\mathbf{F}(\mathbf{r})$.

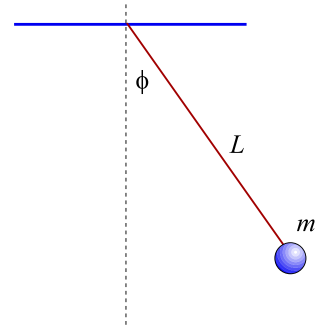
Use $\mathbf{F} = -\nabla U$ to get

$$\mathbf{F} = -\nabla U = A\alpha\sinh\alpha x\,\hat{\mathbf{x}} + B\beta\cosh\beta y\,\hat{\mathbf{y}}$$

3. Given the potential energy function of a multiparticle system, $U(\mathbf{r}_1, \mathbf{r}_2, \ldots)$, write down a mathematical expression for the force on particle 2.

$$\mathbf{F}_2 = -\nabla_2 U(\mathbf{r}_1, \mathbf{r}_2, \ldots)$$

4. How *would* one find an exact solution for the simple pendulum, whee we *don't* assume the angle ϕ is always small? It is a one-dimensional system which conserves energy, so one *can* find a "first integral" for the motion.



a) Get an expression for the total energy of the system in terms of ϕ and $\dot{\phi}$.

With the y coordinate of the bob being $y = -L\cos\phi$, the potential energy is

$$U = -mgL\cos\phi$$

and with $v = L\dot{\phi}$, the kinetic energy is

$$T = \tfrac{1}{2}mL^2\dot{\phi}^2$$

so that the (conserved) total energy is

$$E = T + U = \tfrac{1}{2}L^2\dot{\phi}^2 - mgL\cos\phi$$

b) Take the energy to be some constant value E, and write $\dot{\phi}$ in terms E, ϕ and all the other parameters..

Rearrange some stuff in the last equation and get

$$\dot{\phi}^2 = \frac{2g}{L}\cos\phi + \frac{2E}{mL^2}$$

so that

$$\dot{\phi} = \frac{d\phi}{dt} = \pm\sqrt{\frac{2g}{L}}\sqrt{\frac{E}{mgL} + \cos\phi}$$

This gives

$$dt = \pm\sqrt{\frac{L}{2g}}\frac{1}{\sqrt{\frac{E}{mgL} + \cos\phi}}$$

Integrate from $t' = 0$ to $t' = t$ on the left and on the right from $\phi' = \phi_0$ to $\phi' = \phi$:

$$t = \pm\sqrt{\frac{L}{2g}}\int_{\phi_0}^{\phi}\frac{1}{\sqrt{\frac{E}{mgL} + \cos\phi'}}\,d\phi'$$

which in principle can give ϕ in terms of t.

c) Show how one could do an integral —numerically, perhaps— to get t as a function of ϕ. In principle, this could be inverted to get ϕ in terms of t.

5. Consider the one-dimensional potential

$$U(x) = -\frac{V_0}{x^2 + a^2}\ .$$

with $V_0 > 0$.

a) What are the units of V_0 and a?

Since x and a have units of length and U must have units of energy, the coefficient V_0 must have units of

$$(\text{Energy}) \cdot (\text{Length})^2 = \ \mathrm{J} \cdot \mathrm{m}^2$$

in the MKS system.

b) It has an equilibrium point at $x = 0$; how do you know that it is stable equilibrium?

$U(x)$ clearly has a *minimum* at $x = 0$. For small displacements away from the origin the force on the particle points back to the origin, so the equilibrium is stable.

c) For small oscillations around the equilibrium point, the PE has the form $U = \frac{1}{2}kx^2$. What is k?

The first derivative is

$$\frac{dU}{dx} = \frac{2xV_0}{(x^2 + a^2)}$$

which of course is zero at $x = 0$. The second derivative at $x = 0$ is

$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = \left.\left\{ \frac{2V_0}{(x^2 + a^2)} + \frac{(2x)V_0(-1)(2x)}{(x^2 + a^2)^2} \right\}\right|_{x=0} = \frac{2V_0}{a^2}$$

Then using the Taylor expansion formula, for small oscillations around $x = 0$ the potential is

$$U(x) \approx U(0) + \tfrac{1}{2}U''(0)x^2 = -\frac{V_0}{a^2} + \frac{V_0}{a^2}x^2$$

so if the interesting (non-constant) part of the potential is $\frac{1}{2}kx^2$ then the effective force constant k must be

$$k = \frac{2V_0}{a^2}$$

6. A mass oscillates on an ideal spring but moves though a fluid which imposes a linear resistive force. The mass, force constant of the spring and the coefficient b for the resistive force are

$$m = 0.200\ \mathrm{kg} \qquad \mathrm{k} = 70\,\tfrac{\mathrm{N}}{\mathrm{m}} \qquad \mathrm{b} = 4.0\,\tfrac{\mathrm{kg}}{\mathrm{s}}$$

a) Do I really have the units for b right? (If not, fix them!)

It is necessary for bv to have units of force. If b is in $\frac{\mathrm{kg}}{\mathrm{s}}$ then bv is in $\ \mathrm{kg} \cdot \mathrm{m}/\ \mathrm{s}^2$ and that's a newton!

b) Find the natural frequency for the system.

Use

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{70\ \frac{\mathrm{N}}{\mathrm{m}}}{0.200\ \mathrm{kg}}} = 18.7\ \mathrm{s}^{-1}$$

c) Is the system overdamped or underdamped? In either case give a rough sketch of what the motion is like if the mass is pulled back a distance A and released.

Find β:

$$\beta = \frac{b}{2m} = \frac{4.0\ \mathrm{kg/s}}{2(0.20\ \mathrm{kg})} = 10\ \mathrm{s}^{-1}$$

We see that β is (significantly) smaller than ω_0. This is the case of weak damping; it is underdamped.

7. Suppose the damped oscillator of Problem 6 is now driven by an oscillating force of the form

$$F(t) = (5.0\ \mathrm{N})\cos((10\ \mathrm{s}^{-1})\mathrm{t})$$

Find the amplitude of the long–term (steady) motion.

Here $\omega = 10\ \mathrm{s}^{-1}$ and

$$f_0 = \frac{F_0}{m} = \frac{5.0\ \mathrm{N}}{0.200\ \mathrm{kg}} = 25.0\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

The amplitude of the long-term motion is

$$A^2 = \frac{f_0^2}{(\omega_0 - \omega)^2 + 4\beta^2\omega^2} = \frac{(25.0\ \frac{\mathrm{m}}{\mathrm{s}^2})^2}{((18.7\ \mathrm{s}^{-1})^2 - (10.0\ \mathrm{s}^{-1})^2)^2 + 4(10\ \mathrm{s}^{-1})^2(10\ \mathrm{s}^{-1})^2}$$

for which I get (I hope)

$$A = 7.8 \times 10^{-2}\ \mathrm{m}$$

About how long should it take for the transient to die off? (Give order of magnitude answer.)

The transient will "die off" at a time given by (roughly) $1/\beta$, thus after a time

$$\tau = \frac{1}{\beta} = \frac{1}{10\ \mathrm{s}^{-1}} = 0.10\ \mathrm{s}$$

8. State the brachistochrone problems and set up the integral which is to be minimized, *explaining all the steps*. Just set it up... you don't need to do any solving.

We suppose that a frictionless wire extends from the origin to the point (Y, Y). A bead slides on this wire, starting from rest at the origin. We want to find the curve $y(x)$ (or possibly $x(y)$) so that the time of travel is as small as possible. We'll suppose the y axis points downward.

By conservation of energy, the kinetic energy of the bead at vertical coordinate y is $v = \sqrt{2gy}$. If the equation of the curve is given by $y(x)$, the arclength for an element of the curve is

$$ds = \sqrt{1 + y'(x)}\, dx$$

and the time taken to traverse this element is

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + y'(x)}\, dx}{\sqrt{2gy}}$$

The total time of travel is

$$T = \int_0^X \frac{\sqrt{1 + y'(x)}\, dx}{\sqrt{2gy}}$$

and we want to minimize this T.

9. Find the equation of the path joining the origin O to the point $P(1,1)$ in the xy plane that makes the integral

$$\int (y'^2 + y^2 y' + 4y^2)\, dx$$

stationary.

The Euler-Lagrange equation for the integrand $f(y, y', x)$ gives

$$\frac{\partial f}{\partial y} = 2y\, y' + 8y$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}(2y' + y^2) = 2y'' + 2y\, y'$$

Equate the two and cancel the $2yy'$. Get:

$$2y'' = 8y \qquad \Longrightarrow \qquad y'' = 4y$$

The solution to the last is clearly

$$y = A \sinh 2x + B \cosh 2x$$

Apply the conditions $(0,0)$ and $(1,1)$:

$$0 = B \qquad \text{and} \qquad 1 = A \sinh 2 + B \cosh 2 = A \sinh 2 \qquad \Longrightarrow \qquad A = \frac{1}{\sinh 2}$$

and the solution for y is

$$y = \frac{1}{\sinh 2} \sinh 2x$$

10. Find and solve the differential equation for the path which makes the integral

$$S = \int_{x_1}^{x_2} x\sqrt{1 + y'^2}\, dx$$

stationary. *Hint: Note that the integrand does not involve* y*; recall that because of this, the* $\frac{d}{dx}$ *in the EL equations does not need to be taken.*

Apply the EL equations to the integrand $f = x\sqrt{1+y'^2}$: Here we have

$$\frac{\partial f}{\partial y} = 0 \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y'} = \text{constant} \equiv C$$

Then:

$$\frac{\partial f}{\partial y'} = \tfrac{1}{2}\frac{x(2y')}{\sqrt{1+y'^2}} = \frac{xy'}{\sqrt{1+y'^2}} = C$$

Do some algebra:

$$x^2{y'}^2 = C^2 \qquad \Longrightarrow \qquad (x^2 - C^2){y'}^2 = C^2$$

This gives:

$$y' = \frac{C}{\sqrt{x^2 - C^2}}$$

where we have absorbed a $\pm$ into the constant C. This equation has a general solution of the form

$$y = C\log(x + \sqrt{x^2 - C^2}) + C_1$$

and is related to the inverse $\cosh$ function, but this is as far as I'l take it.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\boldsymbol{\mathscr{r}}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$\mathbf{v} = \dot{\mathbf{r}} \qquad \mathbf{a} = \dot{\mathbf{v}} \qquad \mathbf{p} = m\mathbf{v} \qquad \mathbf{F} = m\mathbf{a} = \dot{\mathbf{p}}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad \dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f_{\text{lin}} = bv \qquad f_{\text{quad}} = cv^2$$

$$b = \beta D \qquad c = \gamma D^2 \qquad \beta = 1.6 \times 10^{-4}\ \text{N} \cdot \text{s}/\text{m}^2 \qquad \gamma = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^4$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f(v) = bv + cv^2 \qquad \mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \omega = \frac{qB}{m}$$

$$m\dot{v} = -\dot{m}v_{\text{ex}} \qquad \mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

$$\ell = \mathbf{r} \times \mathbf{p} \qquad \dot{\ell} = \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma} \qquad \mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \qquad \dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$$

$$L_z = I\omega \qquad I = \sum_\alpha m_\alpha \rho_\alpha^2 \qquad \frac{d}{dt}\mathbf{L}(\text{about CM}) = \mathbf{\Gamma}^{\text{ext}}(\text{about CM})$$

$$I_{\text{rod, ctr}} = \tfrac{1}{12}ML^2 \qquad I_{\text{rod, end}} = \tfrac{1}{3}ML^2 \qquad I_{\text{disc, ctr}} = \tfrac{1}{2}MR^2 \qquad I_{\text{sph, ctr}} = \tfrac{2}{5}MR^2$$

$$T = \tfrac{1}{2}mv^2 \qquad W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \qquad \Delta T = W(1 \to 2) \qquad \mathbf{F} = -\nabla U$$

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}} \qquad \frac{b}{m} = 2\beta$$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \qquad x(t) = Ae^{-\beta t}\cos(\omega_1 - \delta) \qquad x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$A_{\text{max}} = \frac{f_0}{2\beta\omega_0} \qquad Q = \frac{\omega_0}{2\beta} = \frac{2\pi}{\omega_0}$$

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$