Phys 2920, Spring 2010
Exam #1

1. Write $-5.0 - 8.0i$ in polar form.

With

$$\rho = \sqrt{a^2 + b^2} = 9.43 \qquad \text{and} \qquad \tan\phi = \frac{-8.0}{-5.0} = 1.6 \qquad \Longrightarrow \qquad \phi = 237^\circ = 4.15$$

we represent it as

$$-5.0 - 8.0i = (9.43)e^{i4.15}$$

2. Find the angle between the vectors

$$\mathbf{a} = +\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{b} = -6\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

With

$$a = \sqrt{1 + 9 + 16} = 5.10 \qquad \text{and} \qquad b = \sqrt{36 + 25 + 4} = 8.06$$

and

$$\mathbf{a} \cdot \mathbf{b} = -6 - 15 + 8 = -13 = ab\cos\theta$$

we get

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \frac{-13}{(5.10)(8.06)} = -0.316$$

so that

$$\theta = 108^\circ = 1.89$$

3. Simplify:

$$(2\mathbf{a} + 2\mathbf{b} - \mathbf{c}) \cdot (\mathbf{c} \times \mathbf{b}) + (\mathbf{b} - 3\mathbf{a}) \cdot (\mathbf{a} \times \mathbf{c})$$

Use

$$\mathbf{b} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \qquad \mathbf{c} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \qquad \text{etc., then}$$

$$\Longrightarrow \qquad = 2\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

Use the scalar triple product identities, then

$$\Longrightarrow \qquad = 2\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = 3\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$

4. Find a unit vector which is orthogonal to vectors **a** and **b** from Problem 2.

Since $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, evaluate

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ -6 & 5 & 2 \end{vmatrix} = \mathbf{i}(-6-20)+\mathbf{j}(-24+2)+\mathbf{k}(+5-18) = -26\mathbf{i}-22\mathbf{j}-13\mathbf{k}$$

As the magnitude of this vector is

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{1329} = 36.4$$

The unit vector is

$$\hat{\mathbf{u}} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = -0.714\mathbf{i} - 0.604\mathbf{j} - 0.357\mathbf{k}$$

5. Consider the vector space of functions defined on $[0,1]$,

$$f_n(x) = \sqrt{2}\sin(n\pi x)$$

Recall from the 3rd problem set, these functions are orthonormal.

Suppose I want to express the function $5\sin(\pi x)(1-\cos(\pi x))$ in terms of these functions, that is, as:

$$5\sin(\pi x)\,(1-\cos(\pi x)) \;=\; \sum_{n=1}^{\infty} c_n f_n(x)$$

Tell me what I would have to do to get c_n. (You don't actually need to evaluate any integrals here, but be clear.) If you can actually *find* the coefficients, do so.

Defining

$$g(x) \equiv 5\sin(\pi x)\,(1-\cos(\pi x)) \;\;,$$

to get the coefficients c_n we do the following "dot products" to project out the c_n's:

$$c_n = \langle f_n | g\rangle = \int_0^1 \sqrt{2}\sin(n\pi x) 5\sin(\pi x)\,(1-\cos(\pi x))\,dx$$

which can be done without a whole lot of trouble but in this case it is not necessary because the decomposition can be done by inspection. Using

$$\sin(\pi x)\cos(\pi x) = \tfrac{1}{2}\sin(2\pi x)$$

so our given function is

$$\begin{aligned} g(x) &= 5\sin(\pi x)\,(1-\cos(\pi x)) = 5\sin(\pi x) - \frac{5}{2}\sin(2\pi x)) \\ &= \frac{5}{\sqrt{2}}f_1(x) - \frac{5}{2\sqrt{2}}f_2(x) \end{aligned}$$

6. If matrices A and B are given by

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & 0 & 3 \end{pmatrix} \qquad\qquad \mathsf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 5 & -8 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Find (the easiest way you can):

a) Det(A^2)

We get the determinant of A fairly easily:

$$|\mathsf{A}| = (14 - 24) = -10$$

We can also use

$$|\mathsf{A}^2| = |\mathsf{A}|^2 = 100$$

b) Det(B^{-1})

Likewise, since

$$|\mathsf{B}| = (15 - 8) = 7$$

we can use

$$|\mathsf{B}^{-1}| = \frac{1}{|\mathsf{B}|} = \frac{1}{7}$$

7. If

$$\mathsf{D} = \mathsf{ABC}$$

find an expression for B in terms of the other matrices and their inverses. (This has a fairly short answer, but get it right.)

The order of the matrices is important; first, multiply both sides on the left by A^{-1}, assuming that it exists:

$$\mathsf{A}^{-1}\mathsf{D} = \mathsf{BC}$$

and then multiply on the right by C^{-1}:

$$\mathsf{A}^{-1}\mathsf{D}\mathsf{C}^{-1} = \mathsf{B}$$

8. Find the eigenvalues and (unit) eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$$

Solve for $|\mathsf{A} - \lambda\mathbf{1}| = 0$, that is,

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - 6 = -4 + 3\lambda + \lambda^2 - 6 = 0$$

This gives

$$\lambda^2 + 3\lambda - 10 = 0 \qquad \Longrightarrow \qquad (\lambda + 5)(\lambda - 2) = 0$$

which has solutions $\lambda = -5,\ 2$.

For the eigenvalue -5, solve for the vector which gives

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

Then

$$\begin{aligned} x + 3y = -5x \quad &\Longrightarrow \quad 6x + 3y = 0 \quad \Longrightarrow \quad y = -2x \\ 2x - 4y = -5y \quad &\Longrightarrow \quad 2x + y = 0 \quad \Longrightarrow \quad y = -2x \end{aligned}$$

(the same result, as we expect) and a normalized eigenvector is then

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For the eigenvalue 2, solve for the vector which gives

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

Then

$$x + 3y = 2x \quad \Longrightarrow \quad x = 3y$$
$$2x - 4y = 2y \quad \Longrightarrow \quad 2x = 6y \quad \Longrightarrow \quad x = 3y$$

and a normalized eigenvector is then

$$\frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

9. a) Check if each matrix has an inverse, and if so find it (any way you can) for:

$$\mathsf{A} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

The determinants of these are

$$|\mathsf{A}| = 10 - 12 = -2 \qquad |\mathsf{B}| = 1 - 1 = 0$$

so A *will* have an inverse while B will not.

Get the inverse of A by the method shown in class: write out the matrix in question and the identity matrix side-by-side:

$$\left(\begin{array}{cc||cc} 5 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array} \right)$$

Now do some row operations that will make a unit matrix out of the left side. First, subtract $\frac{4}{5}$ times the first row from the second row. This gives

$$\left(\begin{array}{cc||cc} 5 & 3 & 1 & 0 \\ 0 & 2 - \frac{4}{5} \cdot 3 & -\frac{4}{5} & 1 \end{array} \right) = \left(\begin{array}{cc||cc} 5 & 3 & 1 & 0 \\ 0 & -\frac{2}{5} & -\frac{4}{5} & 1 \end{array} \right)$$

Now multiply the bottom row by $-\frac{5}{2}$, then subtract 3 times the second row from the first row:

$$\left(\begin{array}{cc||cc} 5 & 3 & 1 & 0 \\ 0 & 1 & 2 & -\frac{5}{2} \end{array}\right) \quad \Longrightarrow \quad \left(\begin{array}{cc||cc} 5 & 0 & -5 & +\frac{15}{2} \\ 0 & 1 & 2 & -\frac{5}{2} \end{array}\right)$$

Multiply the top row by $\frac{1}{5}$ and you've got:

$$\left(\begin{array}{cc||cc} 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 2 & -\frac{5}{2} \end{array}\right)$$

And from this we conclude:

$$\mathsf{A}^{-1} = \begin{pmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{pmatrix}$$

b) For the computer software that you used on the homework, how would you enter the matrix

$$\mathsf{A} = \begin{pmatrix} 5 & 3 & 0 \\ 4 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

and find its inverse?

For the minimal amount of typing in Maple, you can say

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a:= Matrix([[5, 3, 0],[4, 2, -3], [4, -5, 6]]);
a^(-1);
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Other answers are possible for other mathematical packages or even for... calculators.

10. Show that if λ is an eigenvalue of matrix A then λ^n is an eigenvalue of A^n.

If $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$, then

$$\mathsf{A}^n\mathbf{x} = \mathsf{A}^{n-1}\lambda\mathbf{x} = \lambda\mathsf{A}^{n-1}\mathbf{x} = \lambda^2\mathsf{A}^{n-2}\mathbf{x}$$

Keep doing this until:

$$\cdots = \lambda^{n-1}\mathsf{A}\mathbf{x} = \lambda^n\mathbf{x}$$

which shows that $\mathsf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$ so that $\mathbf{x}$ is also an eigenvector of A^n with eigenvalue λ^n.

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$(\mathsf{AB})_{ij} = \sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{kj} \qquad |\mathsf{A} - \lambda\mathbf{1}| = 0$$

$$\hat{\mathbf{e}}'_j = \sum_{i=1}^{N} \mathsf{S}_{ij}\hat{\mathbf{e}}_i \qquad \mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} \qquad \mathsf{A}' = \mathsf{S}^{-1}A\mathsf{S}$$