Phys 4620, Spring 2007
Exam #3

1. Consider the picture tube in a TV set which accelerates electrons thorugh a potential difference of 10 kV over 10.0 cm.
a) Find the magnitude of the electron's acceleration (assuming it is constant) using Newtonian mechanics. How do we know we don't need relativity here? Use:

$$m_e = 9.11\times10^{-31}\ \mathrm{kg} \qquad e = 1.602\times10^{-19}\ \mathrm{C} \quad 1\ \mathrm{eV} = 1.602\times10^{-19}\ \mathrm{J}$$

Using Newtonian mechanics the final KE is $10\ \mathrm{keV}$, which is small compared to the electron mass of $511\ \mathrm{keV}$, so relativity is probably unimportant. The finall speed of the electron is:

$$\tfrac{1}{2}mv^2 = 10\ \mathrm{keV} = 1.602\times10^{-15}\ \mathrm{J} \qquad \Longrightarrow \qquad v = 5.9\times10^{7}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

(again, significantly less than c, but still pretty fast!). We can get a_x from

$$v^2 = 2a_x x \qquad \Longrightarrow \qquad a = \frac{(5.9\times10^{7}\ \frac{\mathrm{m}}{\mathrm{s}})^2}{2(0.10\ \mathrm{m})} = 1.76\times10^{16}\ \tfrac{\mathrm{m}}{\mathrm{s}^2}$$

b) Find an approximate value for the power radiated by this electron while it is accelerated.

From the Larmor formula we can get P,

$$\begin{aligned} P &= \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{(4\pi\times10^{7}\ \frac{\mathrm{N}}{\mathrm{A}^2})(1.602\times10^{-19}\ \mathrm{C})^2(1.76\times10^{16}\ \frac{\mathrm{m}}{\mathrm{s}^2})^2}{6\pi(3.00\times10^{8}\ \frac{\mathrm{m}}{\mathrm{s}})} \\ &= 1.8\times10^{-21}\ \tfrac{\mathrm{J}}{\mathrm{s}} \end{aligned}$$

2. In the lab reference frame, particle A moves in the $-x$ direction with speed $\frac{3}{4}c$ and particle B moves in the $+x$ direction with speed $\frac{3}{4}c$.
What is the speed of particle B in the reference frame of particle A?

Since the lab moves at $+\frac{3}{4}c$ wrt A and C moves at $+\frac{3}{4}c$ wrt the lab, then using the Einstein velocity addition formula, C moves at

$$v = \frac{+\frac{3}{4}c + \frac{3}{4}c}{1 + \frac{1}{c^2}(3c/4)^2} = \frac{\frac{3}{2}c}{1+\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{25}{16}}c = \frac{24}{25}c$$

3. If the proton *could* decay (as far as is known, it does *not*) we might want to consider the *possible* proton decay mode:

$$\mathrm{p} \Longrightarrow \mathrm{e}^+ + \pi^0$$

occurring in the lab frame where the proton is at rest.

Find the energies of the product positron and pion. Masses are:

$$M_{\rm p}c^2 = 938.27 \text{ MeV} \qquad m_{\rm e}c^2 = 0.5110 \text{ MeV} \qquad m_{\pi^0}c^2 = 134.98 \text{ MeV}$$

You will want to use conservation of energy/momentum, but there's only one reference frame here. If the algebra gets funky it's probably OK to neglect the mass of the electron.

Conservation of 3-momentum in this system gives

$$0 = \mathbf{p}_{\rm e} + \mathbf{p}_\pi$$

so that the final momenta have the same magnitude, say p.

Conservation of energy gives:

$$Mc^2 = E_{\rm e} + E_\pi = \sqrt{p^2c^2 + m_{\rm e}^2c^4} + \sqrt{p^2c^2 + m_\pi^2c^4}$$

where M is the proton mass. Now all we need to do is to solve this for p. Rewrite this as

$$Mc^2 - \sqrt{p^2c^2 + m_{\rm e}^2c^4} = \sqrt{p^2c^2 + m_\pi^2c^4}$$

and square both sides. This gives:

$$M^2c^4 + p^2c^2 + m_{\rm e}^2c^4 - 2Mc^2\sqrt{p^2c^2 + m_{\rm e}^2c^4} = p^2c^2 + m_\pi^2c^4$$

Cancel things and get

$$\sqrt{p^2c^2 + m_{\rm e}^2c^4} = \frac{(m_\pi^2 - m_{\rm e}^2)c^4 - M^2c^4}{(-2Mc^2)} = c^2\frac{(M^2 - m_\pi^2 + m_{\rm e}^2)}{2M}$$

Cancel c^2, then

$$p^2 = \left[\frac{(M^2 - m_\pi^2 + m_{\rm e}^2)^2}{4M^2} - m_{\rm e}^2\right]c^2$$

Plug in the numbers; it gives

$$pc = 459 \text{ MeV}$$

which then gives

$$E_{\rm e} \approx pc = 459 \text{ MeV} \qquad E_\pi = \sqrt{p^2c^2 + m_\pi^2c^4} = 479 \text{ MeV}$$

4. Two protons collide in the CM frame where each has a kinetic energy of 10 GeV. Consider a frame where one of those protons is initially at rest. What is the kinetic energy of the incident proton in that frame?

The *total* energy of each proton is

$$E_{\rm p} = 10 \text{ GeV} + 0.938 \text{ GeV} = 10.9 \text{ GeV}$$

and each has a 3-momentum of magnitude

$$pc = \sqrt{E_{\mathrm{p}}^2 - M^2c^4} = 10.9\ \mathrm{GeV}$$

The total energy-momentum 4-vector in the CM frame is

$$P^\mu c = (2E_{\mathrm{p}}, \mathbf{0})$$

In another frame ("lab", say) where one proton is at rest, the total energy-momentum 4-vector is

$$P^\mu_{\mathrm{lab}} c = (E_{\mathrm{lab}} + Mc^2, \mathbf{p}_{\mathrm{lab}} c)$$

The total 4--momentum is $invariant$. This gives:

$$-4E_{\mathrm{p}}^2 = -(E_{\mathrm{lab}} + Mc^2)^2 + p_{\mathrm{lab}}^2 c^2$$

But $p_{\mathrm{lab}}^2 c^2 = E_{\mathrm{lab}}^2 - m^2c^4$. Then:

$$-4E_{\mathrm{p}}^2 = -E_{\mathrm{lab}}^2 - M^2c^4 - 2E_{\mathrm{lab}}Mc^2 + E_{\mathrm{lab}}^2 - M^2c^4$$

Cancel things and get:

$$-4E_{\mathrm{p}}^2 = -2M^2c^4 - 2E_{\mathrm{lab}}Mc^2 \ ,$$

or:

$$2E_{\mathrm{p}}^2 - M^2c^4 = E_{\mathrm{lab}}Mc^2$$

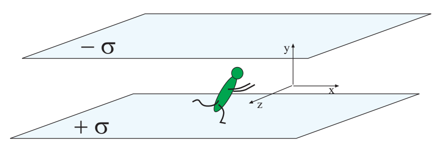
Then:

$$E_{\mathrm{lab}} = \frac{2E_{\mathrm{p}}^2}{Mc^2} - Mc^2 = 252\ \mathrm{GeV}$$

and so the kinetic energy of the moving proton in the lab is

$$T = 252\ \mathrm{GeV} - 0.983\ \mathrm{GeV} = 251\ \mathrm{GeV}$$

5. A big capacitor has (static) charge densities $\pm\sigma$ on the bottom and top plates. The direction which runs perpendicular to the plates we'll call the y direction.



a) What is the electric field between the plates?

The E field between the plates is $\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$, as can be found from Gauss' law. So $E_y = \frac{\sigma}{\epsilon_0}$ and all other components of fields are zero.

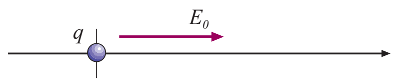
b) Inside the plates there's a green guy running along the positive x axis at speed $\frac{3}{5}c$. What are the electric and magnetic fields in his frame?

Here, the γ factor in the Lorentz transformation is $(1-(3/5)^2)^{-1/2} = \frac{5}{4}$. Then, using the field transformation equations, the field components measured by the green guy are:

$$\overline{E}_y = \gamma E_y = \frac{5}{4}\frac{\sigma}{\epsilon_0} \qquad \overline{B}_z = -\gamma\frac{v}{c^2}E_y = -\frac{5}{4}\frac{3}{5}\frac{\sigma}{c\epsilon_0} = -\frac{3}{4}\frac{\sigma}{c\epsilon_0}$$

All other components of fields are zero in the $\overline{\mathcal{S}}$ frame.

6. Whoa! Stop the presses! We never got around to re-solving the most basic EM dynamics problem of them all, the motion of a charge in a uniform E field.



So consider a uniform E field in the x direction, $\mathbf{E} = E_0\hat{\mathbf{x}}$. A charge q with mass m starts at the origin; it is initially at rest.

a) (Fairly simple) What is the force on the charge? What is the solution for its motion, $u_x(t)$ and $x(t)$ using Newtonian mechanics?

$$\mathbf{F} = qE_0\hat{\mathbf{x}} = ma_x \qquad \Longrightarrow \qquad a_x = \frac{qE_0}{m}$$

This gives, with $x(0) = 0$, $\dot{x}(0) = 0$,

$$u_x(t) = a_x t = \frac{qE_0}{m}t \qquad x(t) = \tfrac{1}{2}a_x t^2 = \frac{qE_0}{2m}t^2$$

b) In relativistic mechanics, of course you must use $\mathbf{F} = \frac{d\mathbf{p}}{dt}$. Using this, find p_x as a function of t. Then find the velocity u_x as a function of time. Show that u_x never gets bigger than c.

$$F_x = \frac{dp_x}{dt} \qquad \text{so} \qquad \frac{dp_x}{dt} = qE_0$$

Then since $p_x = 0$ at $t = 0$, $p_x = qE_0 t$. Using the expression for p_x in terms of u_x,

$$p_x = \frac{mu_x}{\sqrt{1 - u_x^2/c^2}} = qE_0 t$$

Solve for u_x,

$$m^2u_x^2 = q^2E_0^2\left(1 - \frac{u_x^2}{c^2}\right)t^2 \qquad \Longrightarrow \qquad u_x^2\left(m^2 + \frac{q^2E_0^2t^2}{c^2}\right) q^2E_o^2t^2$$

then:

$$u_x = \frac{qE_0t}{\sqrt{m^2 + \frac{q^2E_0^2t^2}{c^2}}}$$

Note, as $t \to \infty$, $u_x \to \frac{qE_0t}{qE_0t}c = c$, that is is approaches c from below. So it's OK!

c) See if you can integrate this to get $x(t)$.

Since $u_x = \frac{dx}{dt}$, the answer to (b) gives

$$dx = \frac{qE_0tdt}{\left(m^2 + \frac{q^2E_0^2t^2}{c^2}\right)^{1/2}}$$

Integrate this and get:

$$x = qE_0(2)\frac{1}{2}\frac{c^2}{q^2E_0^2}\left(m^2 + \frac{q^2E_0^2t^2}{c^2}\right)^{1/2} = \frac{c^2}{qE_0}\left(m^2 + \frac{q^2E_0^2t^2}{c^2}\right)^{1/2} + C_1$$

Use $x(0) = 0$ to fix C_1, then

$$x = \frac{c^2}{qE_0}\left(m^2 + \frac{q^2E_0^2t^2}{c^2}\right)^{1/2} - \frac{c^2m}{qE_0}$$

7. What does it mean to say that a mathematical object is a 4-vector? Specifically, what property does the object a^μ have to have?

We mean that in a different ("boosted") reference frame, the components of a^μ can be found from

$$\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$$

with Λ^μ_ν given in terms of the velocity of the boosted frame by an expression found elsewhere on this exam.

So the values in one reference frame give those in another reference frme by this sort of 4-dimensional "rotation".

8. a) Show that the definition of $F^{\mu\nu}$,

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

gives the correct result for F^{03}. (Recall we write the entries down as $F^{\text{row column}}$.)

$$\begin{aligned} F^{03} &= \frac{\partial A^3}{\partial x_0} - \frac{\partial A^0}{\partial x_3} = \frac{\partial A_z}{\partial(-ct)} - \frac{\partial(V/c)}{\partial z} \\ &= \frac{1}{c}\left[-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right]_z = +E_z/c\,, \end{aligned}$$

the same as the entry in the F array.

b) If we let $\mu = 2$ in the relativistic "homogeneous" Maxwell equation, we have

$$\frac{\partial G^{2\nu}}{\partial x^\nu} = 0$$

(with an implied sum on ν). Show how this is the same as one of the Maxwell equations written in our old vector notation.

$$\frac{\partial G^{2\nu}}{\partial x^\nu} = \frac{\partial G^{20}}{\partial x^0} + \frac{\partial G^{21}}{\partial x^1} + \frac{\partial G^{23}}{\partial x^3}$$

Using the explicit form of $G^{\mu\nu}$ and $x^0 = ct$, we get

$$-\frac{\partial B_y}{\partial (ct)} + \frac{1}{c}\frac{\partial E_z}{\partial x} - \frac{1}{c}\frac{\partial E_x}{\partial z} = 0$$

Cancel the c note that

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}$$

is the y component of the curl of $\mathbf{E}$. Then we get

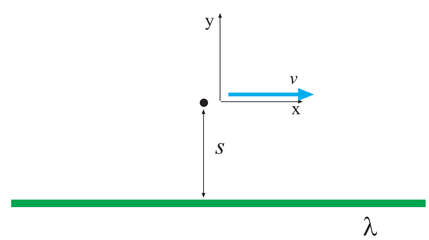
$$-\frac{\partial B_y}{\partial t} - (\nabla \times \mathbf{E})_y = 0 \qquad \text{or} \qquad (\nabla \times \mathbf{E})_y = -\frac{\partial B_y}{\partial t}$$

This is the y component of the (vector) Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

9. Consider a stationary (for our frame) infinite line charge with linear charge density λ. Consider a point a distance s from the line.

a) (Somewhat easy:) What are the E and B fields in this (our) frame?



In the lab frame there is no current, just the static line charge. We ge the usual result (if necessary, rederive with Gauss),

$$E_y = \frac{\lambda}{2\pi s \epsilon_0}\,, \qquad \text{all else} = 0.$$

b) What are the charge density and current (at the same point) in a frame that moves with speed v to the right? You may want to use the fact that charge density and current form a four-vector.)

This part was badly planned since the *linear* charge density λ was given. But if we assume the wire has a small but nonzero radius a and the charge is uniformly distributed then the volume charge density and current for the wire is

$$\rho = \frac{\lambda}{\pi a^2}, \qquad \text{and} \qquad \mathbf{J} = 0$$

So

$$J^\mu(c\rho,\, \mathbf{0}) = \left(\frac{\lambda c}{\pi a^2},\, \mathbf{0}\right)$$

In the boosted frame, this 4-vector transforms to:

$$\bar{J}^\mu = (\gamma c\rho,\, -\gamma\frac{v}{c}(cp),\, 0,\, 0) = (\gamma c\rho, -\gamma v\rho,\, 0,\, 0)$$

So the new charge density is $\bar{\rho} = \gamma\rho$ so that the new line charge density is $\bar{\lambda} = \bar{\rho}\pi a^2 = \gamma\lambda$ and there is a *current*,

$$\bar{I} = \pi a^2 \bar{J} = -\gamma v \lambda$$

c) You'll note (I hope) that the current in the new frame is not just λv. How come it isn't? Didn't we always say the current is just charge density time velocity? Give a clear explanation to resolve this "paradox"

The current is *not* just $-v\lambda$ (moving frame moves in the $+x$ direction) because the charge density of the wire is not just λ in the moving frame. With relativity, careless assumptions about the charge density can be wrong!

d) What are the E and B fields at the given point in the moving frame?

$$\bar{E}_y = \gamma E_y = \gamma \frac{\lambda}{2\pi\epsilon_0} = \frac{\bar{\lambda}}{2\pi\epsilon_0} \qquad \bar{B}_z = -\gamma \frac{v}{c^2} \frac{\lambda}{2\pi\epsilon_0} = -\frac{\mu_0 \bar{I}}{2\pi s}$$

as expected.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$c = 2.998\times10^8\ \frac{\mathrm{m}}{\mathrm{s}}$ $\qquad \epsilon_0 = 8.854\times10^{-12}\ \mathrm{C^2/N{\cdot}m^2}$ $\qquad \mu_0 = 4\pi\times10^{-7}\ \mathrm{N/A^2}$ $\qquad e = 1.602\times10^{-19}\ \mathrm{C}$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial\lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla\cdot\mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t \qquad \Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x$$

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \qquad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \bar{x} = \gamma(x - vt) \qquad \bar{y} = y \qquad \bar{z} = z$$

$$\bar{x}^\mu = \sum_{\nu=0}^{3}(\Lambda^\mu_\nu)x^\nu \qquad \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) \qquad \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \qquad p^\mu = (E/c, p_x, p_y, p_z) \qquad E = \gamma mc^2$$

$$p^\mu p_\mu = -m^2c^2 \qquad E^2 = p^2c^2 + m^2c^4$$

$$K^\mu = \frac{dp^\mu}{d\tau} \qquad J^\mu = (c\rho, J_x, J_y, J_z) \qquad A^\mu = (V/c, A^x, A^y, A^z) \qquad F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

$$\begin{aligned} \bar{E}_x &= E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y) \\ \bar{B}_x &= B_x \qquad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{aligned}$$

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

Invariants: $\mathbf{E} \cdot \mathbf{B}$, $(E^2 - c^2B^2)$

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \qquad \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_0 J^\mu \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \qquad K^\mu = q\eta_\nu F^{\mu\nu}$$