Notes for Phys 2112

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These notes are just intended to give an overview of the major equations covered in class.

1 Beginning Ideas

We usually begin the study of physics with motion. We start simple; we consider the motion of **particles**, that is, objects whose size is small compared with the distances over which they travel. An important problem is physics is to find the trajectory of a particle which moves under certain physical conditions. For motion along a straight line, we try to find the position function $x(t)$.

For one-dimensional motion, the velocity a particle is given by $v = \frac{dx}{dt}$. The speed of a particle is the absolute value of the velocity: $s = |v|$.

The maximum speed which can be measured for any particle is (so far as anyone knows),

$$c = 2.998 \times 10^8 \, \tfrac{\mathrm{m}}{\mathrm{s}}$$

2 Some Formulae

The momentum of a particle (moving in one dimension) is given by

$$p = mv \qquad \text{Units are} \quad \tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

where m is the mass of the particle and v is its velocity.

The kinetic energy of a particle is given by

$$K = \tfrac{1}{2}mv^2 \qquad \text{Units are} \quad \tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}} = 1 \text{ joule} = 1 \text{ J}$$

Other units of energy are

$$1 \text{ erg} = 10^{-7} \text{ J} \qquad 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \qquad 1 \text{ cal} = 4.816 \text{ J}$$

When we discuss physics of the scale of the atom (but not really until then) we have reason to define the wavelength of a moving particle as

$$\lambda = \frac{h}{p}$$

where h is Planck's constant,

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}$$

3 The Limits of Newtonian physics

The kind of physics which makes up the substance of Phys 2110 is **Newtonian physics** (also sometimes called **classical physics**) and it is the simplest way to treat familiar objects. For the precision we usually demand, it is sufficient for this purpose and it is certainly simpler than using the more exact theories that we know of today.

However for particles which have speeds on the order of c one must use the relations from relativity, as deduced by Einstein in the early years of the 20th century. One might find that the speed of a particle is large from naively using $K = \frac{1}{2}mv^2$ if one is given the value of the kinetic energy. In that case it turns out that this is *not* the correct expression for K, so the calculated speed is somewhat wrong. But the conclusion that relativity is *needed* is correct.

Quantum mechanics is needed when the wavelength for particle motion is comparable to the spatial length over which the particle motion is known to take place.

4 One-Dimensional Motion

$$v(t) = x'(t) = \frac{dx}{dt} \qquad a(t) = v'(t) = \frac{dv}{dt}$$

If we are given the acceleration $a(t)$ we can get the velocity $v(t)$ from:

$$\int_0^t \frac{dv}{dt'}dt' = \int_0^t a(t')dt' \qquad \Longrightarrow \qquad v(t) - v(0) = \int_0^t a(t')dt'$$

$$v(t) = v_0 + \int_0^t a(t')dt'$$

Likewise if we have the velocity $v(t)$ we can get the coordinate $x(t)$ from

$$\int_0^t \frac{dx}{dt'}dt' = \int_0^t v(t')dt' \qquad \Longrightarrow \qquad x(t) - x(0) = \int_0^t v(t')dt'$$

$$x(t) = x_0 + \int_0^t v(t')dt'$$

5 Two-Dimensional Motion

When motion takes place in more dimensions

$$v_x(t) = x'(t) = \frac{dx}{dt} \qquad a_x(t) = v_x'(t) = \frac{dv_x}{dt} \qquad v_y(t) = y'(t) = \frac{dy}{dt} \qquad a_y(t) = v_y'(t) = \frac{dv_y}{dt}$$

$$v_x(t) = v_{0x} + \int_0^t a_x(t')dt' \qquad v_y(t) = v_{0y} + \int_0^t a_y(t')dt'$$

$$x(t) = x_0 + \int_0^t v_x(t')dt' \qquad y(t) = y_0 + \int_0^t v_y(t')dt'$$

The **speed** is the magnitude of the velocity vector:

$$v = \sqrt{v_x^2 + v_y^2}$$

For objects moving under the influence of gravity near the earth's surface, we have

$$a_x = 0 \qquad a_y = -9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g$$

When an object is launched at the origin with speed v_0 at an angle θ upward from the horizontal, the solutions for x and y are

$$x = v_0 \cos\theta\, t \qquad y = v_0 \sin\theta t - \tfrac{1}{2} g t^2$$

We can eliminate t from these equations to get a relation between x and y,

$$y = \tan\theta\, x - \frac{g x^2}{2 v_0^2 \cos^2\theta}$$

The is the equation of a parabola. For a freely-falling object the **trajectory** of the flight has the shape of a parabola.

6 Circular Motion

Consider an object which moves on the path of a circle in the xy plpane which is centered on the origin and has radius R. If the particle takes a time T to go around the circle, its speed is $v = \frac{2\pi R}{T}$. The frequency of its motion (the number of "cycles per second") is $f = \frac{1}{T}$.

We can also talk about the **angular speed** of the particle. That is angle which the particle sweeps out per time, and is denoted by ω. Considering one trip around the circle, it must be given by

$$\omega = \frac{2\pi}{T}$$

From this, we can see that

$$v = \frac{2\pi}{T} R = \omega R$$

The equations of motion

$$x(t) = R\cos(\omega t) \qquad y(t) = R\sin(\omega t)$$

describe a particle moving in a circle of radius R centered at the origin. The angular speed of the motion is ω. It is easy to see that the magnitude of $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ is always R, as it should be. Taking the derivatives, the velocity is

$$v_x(t) = \frac{dx}{dt} = -\omega R \sin(\omega t) \qquad v_y(t) = \frac{dy}{dt} = \omega R \cos(\omega t)$$

from which it is easy to see that the magnitude of the vector $\mathbf{v}$ is always $v = \omega R$, which agrees with the equation above. Taking another derivative, we find

$$a_x(t) = \frac{dv_x}{dt} = -\omega^2 R \cos(\omega t) \qquad a_y(t) = \frac{dv_y}{dt} = -\omega^2 R \sin(\omega t)$$

This vector has magnitude $\omega^2 R$ and we can see that both components are proportional (with a minus sign) to x and y so that the acceleration vector is

7 Plane Polar Coordinates

Motion in a plane is often more conveniently described with plane polar coordinates, which we'll denote by the pair (ρ, ϕ). These give the distance of a point from the origin and the angle which a segment joining the point and the origin makes with the $+x$ axis. The relations between the Cartesian coordinates (x, y) and the polar coordinates (r, ϕ) are:

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\phi = \frac{y}{x}$$

where we note that when we use the last relation we need to be careful to get the right quadrant for ϕ.

As a particle moves, its x and y coordinates change with time; we will use the notation where a dot over a variable means to take $\frac{d}{dt}$, the time derivative. Thus:

$$v_x = \frac{dx}{dt} = \dot{x} \qquad a_x = \frac{dv_x}{dt} = \dot{v}_x = \ddot{x} \qquad \text{etc.}$$

Of course the coordinates r and ϕ will also change with time and from the relation between (x, y) and (r, ϕ) the rules of calculus give

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \qquad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

From these we can get expressions for $\ddot{x}$ and $\ddot{y}$.

We can define unit vectors for the polar coordinate system: Given some point $P = (r, \phi)$, $\hat{\mathbf{r}}$ is the unit vector which points outward from the origin from P; $\hat{\boldsymbol{\phi}}$ is the unit vector which points perpendicular to $\hat{\mathbf{r}}$ in the counterclockwise direction. We note that the directions of the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ depend on the point in question, whereas the Cartesian unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ always have the same direction. Because of this, we need to be more careful when doing math with the polar unit vectors.

We can show:

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$

From these one can show

$$\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

We can combine the above expressions to express the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ *completely* in terms of polar coordinates, including the polar unit vectors. One can show:

$$\mathbf{v} \;=\; \dot{x}\,\hat{\mathbf{i}} + \dot{y}\,\hat{\mathbf{j}} \;=\; \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

and

$$\mathbf{a} \;=\; \ddot{x}\,\hat{\mathbf{i}} + \ddot{y}\,\hat{\mathbf{j}} \;=\; (\ddot{r} - r\dot{\phi}^2)\,\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\boldsymbol{\phi}}$$

These expressions are useful when studying orbital motion where the force which gives the acceleration is always directed toward a "central" point in the plane. Expressing this force in Cartesian coordinates is much harder than expressing it in polar coordinates.

8 Reference Frames

A reference frame is a set of coordinates by which we measure locations and a clock by which we measure the time. A reference frame can be in motion with respect to another reference frame. It can also be *rotating* with respect to another frame, but we won't consider that (interesting and complicated) case. Thus the axes of the respective reference frames that we'll consider are all parallel to one another.

We will use the notation that $\mathcal{S}$ is our customary "lab" reference frame and $\mathcal{S}'$ is a reference frame that is moving with respect to $\mathcal{S}$ with velocity $\mathbf{v}$.

If the position of the origin of $\mathcal{S}'$ as measured in $\mathcal{S}$ is $\mathbf{R}(t)$ then we have $\mathbf{V} = \frac{d\mathbf{R}}{dt}$ and the coordinate of a point P which is measured as $\mathbf{r}$ in $\mathcal{S}$ is related to the coordinate which is measured as $\mathbf{r}'$ in $\mathcal{S}'$ by

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}$$

Now we will make the assumption that the clocks in the different reference frames all measure the same thing so that $t = t'$. If that's the case, then velocities measured in the two frames are related by

$$\mathbf{v} \;=\; \frac{d}{dt}\mathbf{r}(t) \;=\; \frac{d}{dt'}(\mathbf{r}' + \mathbf{R}) \;=\; \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{R}}{dt} \;=\; \mathbf{v}' + \mathbf{V}$$

so that

$$\mathbf{v} = \mathbf{v}' + \mathbf{V} \qquad \text{and} \qquad \mathbf{v}' = \mathbf{v} - \mathbf{V}$$

Taking the time derivative again we see that *accelerations* in the two frames are related by

$$\mathbf{a} = \frac{d\mathbf{v}'}{dt} + \frac{d\mathbf{V}}{dt} = \mathbf{a}' + \mathbf{A} \qquad \text{so} \qquad \mathbf{a}' = \mathbf{a} - \mathbf{A}$$

where $\mathbf{A} = \frac{d\mathbf{V}}{dt}$ is the acceleration of the origin of frame $\mathcal{S}'$ as observed in frame $\mathcal{S}$. This is a key relation because the laws of Newtonian physics deal with *accelerations*. From these equations we see that if one frame is not *accelerating* with respect to another one then both will agree on the acceleration of a particle even if (from the constant velocity of one frame) they don't agree on they the *velocity* of the particle.

What do Newton's laws have to do with all this? If we take Newton's laws as (roughly) $\mathbf{F} = m\mathbf{a}$ plus the Third Law then if in one reference frame a particle has no (net) force that we can tell and —according to us— is not accelerating, we need to address what is happening in a frame which is accelerating relative to us where the scientists *do* measure the particle to be accelerating but can discern no physical reason as to why. We need a deeper answer than "Just don't use that reference frame", if for no other reason than the fact that any location on the surface of the earth is surely *not* in a perfect, non-accelerating frame; so perhaps in earth-bound laboratories it is possible to have accelerations which are uncaused by any *real* physical effects coming from other objects.

We must postulate that there *is* some reference frame where accelerations occur only when true physical **forces** are acting. (Undoubtedly this reference frame is not attached to any point on the earth.) But as we've seen above, since any reference frame moving at a constant velocity with respect to this "true" frame agrees with it as far as the accelerations are concerned, then these frames are *also* permissible for doing physics properly.

It is customary to say that the "true" reference frame is at rest and not rotating with respect to the "fixed stars" and leave it at that!

This then is the real substance of Newton's First Law; it is really a statement about *reference frames*. The permissible reference frames for doing Newtonian physics (that is, the true fixed reference frame and those moving at constant velocity with respect to it) are called **inertial frames**. We do not live in an inertial frame, but for most purposes (like Phys 2111 labs) it is not too far from being one.

9 Newton's Laws in an Accelerating Frame

Suppose we are in an accelerating reference frame; our frame accelerates with respect to an inertial frame by $\mathbf{A}$. In our frame, we measure the acceleration of a particle of mass m to be $\mathbf{a}'$. But because we are in a *bad frame*, it is *not* true that $\mathbf{F}_{\text{net}} = m\mathbf{a}'$.

Suppose the acceleration of the particle in an inertial frame is $\mathbf{a}$. It *is* true that $\mathbf{F}_{\text{net}} = m\mathbf{a}$. We can use

$$\mathbf{a} = \mathbf{a}' + \mathbf{A}$$

to get:

$$\mathbf{F}_{\text{net}} = m\mathbf{a} = m(\mathbf{a}' + \mathbf{A}) = m\mathbf{a}' + m\mathbf{A}$$

which we rearrange to get

$$\mathbf{F}_{\text{net}} - m\mathbf{A} = m\mathbf{a}'$$

This equation tells us how to salvage Newton's second law if we are in a non-inertial (and non-rotating!) frame. It says to add up the real forces on the mass m and then add a term $-m\mathbf{A}$. We can think of that term as another force, the **inertial force**,

$$\mathbf{F}_{\text{inertial}} = -m\mathbf{A}$$

but it is not a genuine force; such a term is often called a **pseudoforce**.

We note again that the axes our non-inertial system can not be rotating; if they are, another (much more complicated) set of pseudoforces will be introduced.

Possibly the case where one most often hears of the inertial force —and where it is most often confused— is the case of circular motion. If a mass is in uniform circular motion (speed v, radius R) about some point in an inertial frame, we can consider a set of axes that moves with the object. We would probably want one axis to always point toward the center, and if so then this set of coordinates is also *rotating* but we will ignore that part and focus on the fact that it is accelerating with respect to the inertial frame, that acceleration $\mathbf{A}$ being the centripetal acceleration of magnitude v^2/R

For us in the *revolving* frame, we can use Newton's second law for a mass m as long as we add the inertial force

$$\mathbf{F}_{\text{inertial}} = -m\mathbf{A} \;=\; -m\frac{v^2}{R}\ \text{(Inward)} \;=\; \frac{mv^2}{R}\ \text{(Outward)}$$

and in this context this inertial force is known as the **centrifugal** (outward) force. The danger of confusion is evident! For a revolving object, there is an outward force, but only in case where we *move along with the object* and where we realize it's a *phony* force.

10 Newton's Law of Gravity

Never covered properly in Phys 2110 but of great importance in physics is the law of gravitation. This was the Newton's other great contribution to mechaincs, apart from him more general three laws.

Newton's law states that between any two masses there is an attractive force with magnitude given by

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{where} \quad G = 6.67 \times 10^{-11}\ \tfrac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}} \tag{1}$$

and the force is directed from one mass to the other one. The magnitudes of the forces that the masses exert on one another are equal, in accordance with Newton's Third law.

The force between any two typical masses is very small so that this force is normally very hard to measure; Eq. 1 tells us that for two 1.0 kg masses separated by 10 cm = 0.10 m, the attractive force is of magnitude 6.7×10^{-9} N which would be challenging to measure.

The force becomes important when one of the objects is very large, as is the case when we consider the force between an object and the entire earth; this is the familiar force of gravity as used on the surface of the earth, which so far you have expressed as mg, directed downward. More on that in a bit.

When a planet goes around the sun in a *circular* orbit, it is because the centripetal force is provided by the gravitational force. (It is a good approximation to treat the much more massive sun as being motionless at the center of the circle, though that isn't exactly true.) If m is mass of the orbiting planet, M_s is the mass of the sun, R is the radius of the orbit and v is the (constant) speed of the planet in its orbit, then we have:

$$F_c = \frac{mv^2}{R} = G\frac{M_s m}{R^2}$$

algebra gives us:

$$v^2 = \frac{GM_s}{R}$$

If T is the period of the planet's orbit then $v = \frac{2\pi R}{T}$. Put this into the last equation and get

$$\frac{4\pi^2 R^2}{T^2} = \frac{GM_s}{R} \qquad \Longrightarrow \qquad T^2 = \frac{4\pi^2 R^3}{GM_s}$$

Note, the mass of the planet itself is not present.

The last relation says that for a planet in a circular orbit the *square* of the period is proportional to the *cube* of the distance from the sun. From this equation if we know both T and R we can find the mass of the sun or more generally any central which gives a circular orbit. Also, since

$$\frac{T^2}{R^3} = \frac{4\pi^2}{GM_s}$$

the ratio of T^2 to R^3 is the same for all planets in the solar system, assuming they have circular orbits.

The pull of gravity from the entire earth is also what gives the familiar force of gravity on a mass m: Magnitude mg, downward. How does this come from Newton's law of gravity?

It is a problem because in the case, the entire earth is not small compared with the distance between the earth and the smaller mass m. The pull comes from *all parts* of the earth all added up!

This is a challenging mathematical problem which has an elementary answer: For a spherically symmetric distribution of mass, for the purposes of using the law of gravity you can treat all the mass as being at the *center* of the object. So for an object of mass m on the surface of the earth, the force of attraction to the *entire* earth is found by treating all the mass of the earth (M_e) as being at the center, which is a distance R_e from anything on the surface (R_e being the radius of the earth). So for objects on the surface of the earth,

$$F_{\rm grav} = G\frac{M_e m}{R_e^2}$$

But we know that this force is also equal to mg, and so

$$mg = G\frac{M_e m}{R^2} \qquad \Longrightarrow \qquad g = \frac{GM_e}{R^2}$$

This tells us where "$9.8\,\frac{\rm m}{\rm s^2}$" comes from, in terms of the mass and radius of the earth.

The same formula would hold for the gravitational acceleration on the surface on any planet:

$$g = \frac{GM}{R^2}$$

where M is the mass of the planet and R is its radius.

10.1 Gravitational Potential Energy

The gravitational force is a *conservative* force, meaning that when particles move around subjected to this force the work done on them is independent of the paths they take to get from their initial to final positions. For conservative forces it makes sense to talk about the *potential energy* due to the interaction.

For masses m_1 and m_2 separated by a distance r, the potential energy is

$$U(r) = -G\frac{m_1 m_2}{r} \tag{2}$$

We have made the sensible choice that the potential energy should be zero at $r = \infty$. With this choice the potential is always negative. Regardless, it always increases with distance. Note, there is just a single power of r in the denominator of 2.

For finite–sized spherically symmetric masses, we can just take the r to be the distance between their centers.

With this expression for U we can calculate the escape speed for a mass m on the surface of a planet with mass M and radius R. We suppose that the mass is shot straight outward from the surface of the planet with speed v. We suppose that it has just enough speed to get *very* far away from the planet; then at $r = \infty$ its speed is zero. Conservation of energy gives

$$\tfrac{1}{2}mv^2 - G\frac{Mm}{R} \;=\; \tfrac{1}{2}m(0)^2 + U(r=\infty) \;=\; 0$$

Solve for v:

$$v^2 = \frac{2GM}{R} \qquad \Longrightarrow \qquad v = \sqrt{\frac{2Gm}{R}}$$

11 Non–Circular Orbits

The orbits of the planets are *not* perfect circles as found by Kepler and expressed in his first law of planetary motion:[1] The orbit of a planet has the shape of an *ellipse* with the sun at one focus.

[1]The other two laws are, briefly: The square of a planet's period is proportional to its mean distance from the sun, and the area swept out by a line joining the sun and a planet is the same for equal periods of time.

This is the case for an object which it trapped by the sun's gravity. If an object is moving rapidly enough that it can fly away from the sun and never return, the shape of its path is a *hyperbola*[2].

We would like to *prove* that this is the case.

The force on the orbiting body is directed toward the central body and has magnitude GMm/r^2; it has only a $\hat{\mathbf{r}}$ component:

$$\mathbf{F} = F_\rho\,\hat{\mathbf{r}} + F_\phi\,\hat{\boldsymbol{\phi}} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}}$$

But from our derivation of the acceleration in polar coordinates and $\mathbf{F} = m\mathbf{a}$, we have:

$$\mathbf{F} \;=\; -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \;=\; m\mathbf{a} \;=\; m(\ddot{r} - r\dot{\phi}^2)\,\hat{\mathbf{r}} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\boldsymbol{\phi}}\;.$$

Equating the components,

$$-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\phi}^2) \qquad m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0 \tag{3}$$

Look at the second of these relations. Multiplying by r gives

$$(mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi}) = 0$$

which can be rewritten as

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0$$

which implies that $mr^2\dot{\phi}$ is a *constant*; it does not change in value while the planet orbits (though it has a different value depending on *which* orbit). We will call this constant ℓ, and it is actually the **angular momentum** of the planet. Thus:

$$mr^2\dot{\phi} \equiv \ell \qquad \Longrightarrow \qquad \dot{\phi} = \frac{\ell}{mr^2} \tag{4}$$

We substitute this last relation into the first of Eqs. 3 (cancelling the m) and get

$$-\frac{GM}{r^2} = \ddot{r} - \frac{\ell^2}{m^2r^3} \tag{5}$$

so that we have a differential equation for r as a function of time. But we need to do some work on this equation before we have it in a form that is solvable and *useful*.

The main problem is that even if we could get r as a function of time t it wouldn't tell the shape of the orbit in an obvious way; for that we need the function $r(\phi)$ which we might recognize as a familiar shape. Then we need to change 5 in to a differential equation for r as a function of ϕ. This can be done using the chain rule (relating $\dot{r} = \frac{dr}{dt}$, $\dot{\phi} = \frac{d\phi}{dt}$ and $\frac{dr}{d\phi}$) and again using our definition of ℓ to eliminate $\dot{\phi}$. It just takes some algebra. This will probably be the hardest derivation of the course, but it is well worth it, as we will *prove* that the orbits are ellipses.

Start from 5, where we have time derivatives of r which we wish to change to derivatives with respect to ϕ. From the chain rule, the operation $\frac{d}{dt}$ is that same as

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{mr^2}\frac{d}{d\phi}$$

[2] There is a special case where the shape of the path is a parabola.

where we have used Eq. 4. Use this to work on $\ddot{r}$:

$$\ddot{r} = \frac{d}{dt}\frac{dr}{dt} = \frac{\ell}{mr^2}\frac{d}{d\phi}\left(\frac{\ell}{mr^2}\frac{dr}{d\phi}\right)$$

We use the more compact notation $r' = \frac{dr}{d\phi}$. Then we get

$$\ddot{r} \;=\; \frac{\ell}{mr^2}\frac{d}{d\phi}\left(\frac{\ell}{mr^2}\,r'\right) \;=\; \frac{\ell^2}{m^2r^2}\left(-\frac{2}{r^3}(r')^2 + \frac{1}{r^2}r''\right) \;=\; \frac{\ell^2}{m^2}\left(-\frac{2(r')^2}{r^5} + \frac{r''}{r^4}\right)$$

Put this back into 5 and multiply both sides by r^2. This gives

$$-GM = \frac{\ell^2}{m^2}\left(-\frac{2(r')^2}{r^3} + \frac{r''}{r^3}\right) - \frac{\ell^2}{m^2 r}$$

Multiply both sides by $\frac{m^2}{\ell^2}$ to clear away the mess and get:

$$-\frac{GMm^2}{\ell^2} = -\frac{2(r')^2}{r^3} + \frac{r''}{r^2} - \frac{1}{r}$$

Next comes a series of incredibly clever steps[3]. We can write the first two terms as a single derivative:

$$-\frac{GMm^2}{\ell^2} = \frac{d}{d\phi}\left(\frac{r'}{r^2}\right) - \frac{1}{r}$$

and we *further* notice that the first term on the right contains another derivative:

$$-\frac{GMm^2}{\ell^2} = \frac{d}{d\phi}\frac{d}{d\phi}\left(-\frac{1}{r}\right) - \frac{1}{r}$$

We make two definitions to make life simpler:

$$u \equiv \frac{1}{r} \qquad\qquad \gamma \equiv \frac{GMm^2}{\ell^2}$$

This gives

$$-\gamma \;=\; -\frac{d^2}{d\phi^2}u - u \;=\; -u'' - u \qquad \Longrightarrow \qquad u'' \;=\; -u + \gamma \tag{6}$$

The fact that we have exchanged u for r is no problem since we just regard u as a function of ϕ; later we will recover $r(\phi)$.

What we have here is a **differential equation** for the function $u(\phi)$. While such an equation may be new to you, they are encountered every day in physics and engineering and you will very likely take courses in how to solve them. We are trying to find an unknown *function* by using information about its *derivatives*.

If we had been given the (simpler) differential equation

$$u'' = -u$$

then the solution would have been

$$u(\phi) = \sin\phi \qquad \text{or} \qquad u(\phi) = \cos\phi$$

[3] For which I can't take credit.

or any linear combination of these two functions. The *general* solution to that equation is

$$u(\phi) = A \sin\phi + B \cos\phi$$

Our equation is slightly more difficult because there is a constant term on the right, but one can easily see that our equation,

$$u'' + u = \gamma \tag{7}$$

has the general solution

$$u(\phi) = A\cos(\phi - \delta) + \gamma \tag{8}$$

where there are two constants we can set to get the solution corresponding to the particular orbit. Actually, the choice of δ just amounts to changing the definition of the angle ϕ so we are free to set $\delta = 0$. With this choice, we want to recover $r(\phi)$. From $r = \frac{1}{u}$,

$$\frac{1}{r(\phi)} = A\cos\phi + \gamma \qquad \Longrightarrow \qquad r(\phi) = \frac{1}{A\cos\phi + \gamma}$$

We can make this slightly simpler by dividing top and bottom of the right side by γ:

$$r(\phi) = \frac{1/\gamma}{(A/\gamma)\cos\phi + 1} = \frac{c}{1 + \epsilon\cos\theta}$$

where

$$c = \frac{1}{\gamma} = \frac{\ell^2}{GMm^2} \qquad \text{and} \qquad \epsilon = \frac{A}{\gamma}$$

so the product of all our labors —the path of the orbit— is

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \tag{9}$$

This is a simple relation and if $0 \geq \epsilon < 1$ it gives an *ellipse*. (The case $\epsilon = 0$ obviously gives a *circle* which is a special kind of ellipse.)

But Eq. 9 also predicts the shapes of orbits where the orbiting object (a comet, for example) goes away from the Sun and never comes back. The case $\epsilon = 1$ gives a parabola and the case $\epsilon > 1$ gives a hyperbola.

12 Center–of–Mass Reference Frame

For a collection of particles $1, 2, \ldots N$ with masses $m_1, m_2, \ldots m_N$ and coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N$, the center of mass is given by

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i \qquad \text{where} \qquad M = \sum_{i=1}^{N} m_i$$

In general the center of mass has its own motion, with velocity and acceleration given by

$$\mathbf{V} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{v}_i \qquad \text{and} \qquad \mathbf{A} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{a}_i$$

The total momentum of the system is

$$\mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v}_i = M\mathbf{V}$$

The center of mass gives a reference frame in which one can observe the motions of all the particles. In the center of mass frame, the particles has velocities

$$\mathbf{v}'_1 = \mathbf{v}_1 - \mathbf{V}, \quad \mathbf{v}'_2 = \mathbf{v}_2 - \mathbf{V}, \quad \ldots \quad \mathbf{v}'_N = \mathbf{v}_N - \mathbf{V}$$

The center of mass frame is unique in that in this frame the total momentum of the system is

$$\begin{aligned}
\mathbf{P}' &= \sum_{i=1}^{N} m_i \mathbf{v}'_i = \sum_{i=1}^{N} m_i(\mathbf{v}_i - \mathbf{V}) \\
&= \sum_{i=1}^{N} m_i \mathbf{v}_i - \sum_{i=1}^{N} m_i \mathbf{V} \;=\; M\mathbf{V} - \mathbf{V}\sum_{i=1}^{N} m_i \\
&= M\mathbf{V} - M\mathbf{V} \;=\; 0
\end{aligned}$$

So that the cm frame can be called the "zero momentum" frame. Analysis of collisions is usually simplest in the CM frame. For an elastic collision of two masses, when the collision is viewed in the center of mass frame, the masses simply reverse their velocities.

13 Fluids

Here I would like to give a summary of basic fluid statics, in particular the forces which fluids exert on solid objects in contact with them. The basic *dynamics* of fluids are also covered in the introductory physics texts, but I won't discuss that here.

13.1 Density and Pressure

The density ρ of any mass is the ratio of its mass to its volume:

$$\rho = \frac{M}{V}$$

and in the SI system it has units of $\frac{\mathrm{kg}}{\mathrm{m}^3}$.

A fluid exerts a force on an object which has parts of its surface within the fluid.

Here we will mean *fluid* in a general sense which includes liquids but also the earth's atmosphere, which is... all around us!

It is true that the density of the atmosphere varies significantly with heights over which humans can travel and a fluid like water does *not* vary significantly in density but like a liquid, the air still exerts forces on objects and if we stay in one spot near the surface of the earth the density is very nearly constant.

At any point in the fluid there is a quantity called the **pressure** which gives the force of the fluid on a submerged object in the following way: For a small (flat) surface of area dA (facing any direction) the force of the fluid on that surface (and normal to it) has magnitude $dF = P\,dA$. So pressure has units of force/area or:

$$\text{Units of } P = 1\frac{\mathrm{N}}{\mathrm{m}^2} \equiv 1 \text{ pascal} = 1 \text{ Pa}$$

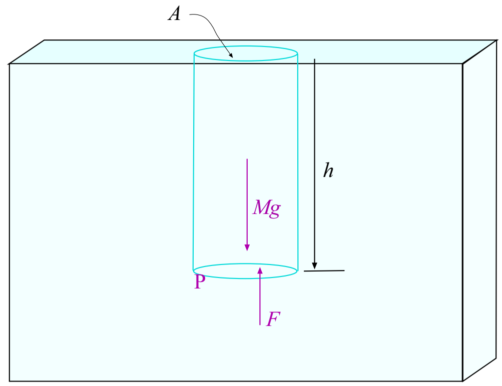

Figure 1: Forces on a fluid element.

Of course a real object has lots of bits of its surface dA that we must (vectorially) sum over to get the net force of the fluid. The pressure may be different over all the surface elements because they are at different points in the fluid; because of this the forces may *not* sum to zero and there will be a net force from the fluid on the object called the **buoyant force**; more on this later!

13.2 Pressure and Depth

Consider a fluid of uniform density ρ in which we imagine separating out a cylinder of fluid material with cross-sectional area A and height h measured downward from the surface, as shown in. Fig. 1Ṡo the volume of the cylinder is Ah and its mass is $M = \rho Ah$.

The forces on this fluid section are gravity, $Mg = \rho Ahg$ downward, and a force from the fluid F pushing *upward* on its bottom surface. For now, we assume there is nothing pushing downward on the top surface, though if there is *air* above the fluid that is not the case; we'll deal with this possibility later. There are also fluid forces pushing inward on its sides but from symmetry those will cancel.

This section of the fluid is at rest, so the forces cancel:

$$-\rho Ahg + F = 0$$

Assume that the pressure of the fluid only depends on the depth below the surface. Then the pressure is the same at all points on the lower face and upward force on the bottom surface is just $F = PA$. Then

$$\rho Ahg = F = PA \qquad \Longrightarrow \qquad P = \rho gh$$

Most of the time the surface of the liquid will be in contact with another fluid— the atmosphere— which exerts its own force on objects and in that case there will be a downward force on the top side of the cylinder in Fig. 1 . If pressure of the atmosphere (which also depends only depth) at the surface of the liquid is P_0, then the force on the top is P_0A *downward* and our condition on the forces qould be:

$$-\rho Ahg - P_0A + PA = 0 \qquad \Longrightarrow \qquad P = P_0 + \rho gh$$

that is, the atmospheric pressure P_0 adds on to the part just due to the liquid.

Atmospheric pressure can change a great deal depending on elevation and weather conditions, so P_0 is not at all a fixed number. Generally it has a value of around 1.0×10^5 Pa, which give the **atmosphere** unit of pressure. The relation between all of these is

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} = 760 \text{ torr}$$

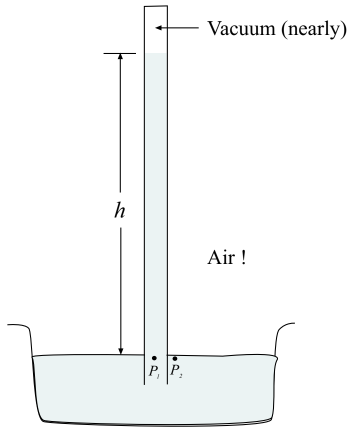

Figure 2: Barometer.

13.3 The Barometer

One can measure the (local) pressure of the atmosphere with a device which is its simplest form is a tube filled with mercury which has been inverted (without letting any air into it) and the open end of the tube placed in a bowl of mercury. When the mercury gets done resettling we have the configuration of fluids shown in Fig. 2Ṡo the volume of the cylinder is Ah and its mass is $M = \rho A h$.

Note the points S_1 and S_2 in the figure; these points are at the same height; both are just below the level of surface of the fluid in the bowl. The pressure depends only on the height so the pressures at these two points must be the same.

If we compute the pressure at S_1 just paying attention to the material above it, we note that it is at a depth h from the top of the fluid *in the tube* and there is *no* air above that surface, so our formula for pressure from depth gives

$$P_1 = \rho g h$$

while the pressure at S_2 only comes from the air which is above it, hence

$$P_2 = P_0$$

where P_0 is the current pressure of the atmosphere. Since $P_1 = P_2$, we find:

$$\rho g h = P_0$$

so that by measuring the height h of the column of mercury we can calculate the atmospheric pressure P_0.

13.4 Buoyant Force

From the fact that fluid pressure (but *not* density!) varies with depth, it follows that the fluid will exert a net upward force on an object which is totally or partially submerged. Fig. 3Ḣere a simple cylinder of cross-sectional area A and height h is completely submerged. The pressure on the top face is $P_1 = \rho g h_1 + P_0$ while that on the bottom face is $P_2 = \rho g h_2 + P_0$. The fluid exerts a *downward* force on the upper face and an *upward* force on the lower face so the net upward force (from the fluid; there is always the force of gravity and possibly other forces) is called the **buoyant force** and is given by:

$$F_{\mathrm{buoy}} = -(\rho g h_1 + P_0)A + (\rho g h_2 + P_0)A = \rho g (H_2 - h_1)A$$

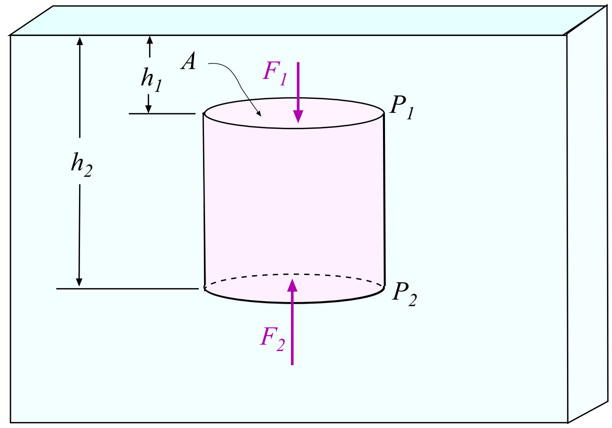

Figure 3: Difference in pressure gives buoyant force.

but we note that $(h_2 - h_1)A$ is the volume V of the object, which gives $F_{\rm buoy} = \rho g V$. But now we think of the volume V as a volume of fluid which should be occupying that space.. but isn't! If the fluid *were* there, it would have a *weight* given by $W = MV = \rho V g$, which is that same as the buoyant force.

So we arrive at a simple principle for the buoyant force: It is equal to the *weight* of the displaced fluid:

$$F_{\rm buoy} = \rho V g = W_{\rm disp} \tag{10}$$

14 Relativity

As particle speeds approach the speed light, the laws of motion as found by Newton begin to lose validity and one must use the law of motion as discovered (mostly) by Einstein.

Strictly speaking, this means that the material of Phys 2110 is *wrong* but there is little doubt that to understand the correct laws one must start with more familiar ideas; in any event the equations of relativity are more complicated than Newton's laws and it is silly to use them if they are not needed.

To review the most important things you know from Newtonian physics:

We have the same concepts of location $\mathbf{r}$, velocity $\mathbf{u}$ and acceleration $\mathbf{a}$

$$\mathbf{r}, \qquad \mathbf{u} = \frac{d\mathbf{r}}{dt}, \qquad \mathbf{a} = \frac{d\mathbf{u}}{dt}$$

where the acceleration $\mathbf{a}$ plays a central role; it is found from Newton's second law, $\mathbf{F} = m\mathbf{a}$. (Here I prefer to use $\mathbf{u}$ for the particle velocity. I have a reason for this!)

We also had the definitions of momentum $\mathbf{p}$ and kinetic energy T:

$$\mathbf{p} = m\mathbf{u} \qquad T = \tfrac{1}{2}mv^2$$

In relativity, we have the same concepts of location $\mathbf{r}$, velocity $\mathbf{u}$ and acceleration $\mathbf{a}$:

$$\mathbf{r}, \qquad \mathbf{u} = \frac{d\mathbf{r}}{dt}, \qquad \mathbf{a} = \frac{d\mathbf{u}}{dt}$$

but they are used in different ways in the physical laws. In particular, the acceleration does not play a central role as it does in Newton's mechanics, and we do *not* have $\mathbf{F} = m\mathbf{a}$ anymore.

Not all particles have mass (e.g. photons do not) but for a massive particle, the correct definition of momentum is no longer $\mathbf{p} = m\mathbf{u}$, but rather

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{11}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{12}$$

It is still true that for an isolated system, the total momentum is conserved.

The general formula for a particle's energy is

$$E^2 = p^2c^2 + m^2c^4 \tag{13}$$

One can show that for a massive particle we also get

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

These formula are not simple corrections to the Newtonian expressions because here the relativistic energy includes **rest energy** mc^2, and the Newtonian "energy" does not. For a motionless massive particle, we get $E = mc^2$ but this is not a general expression.

The kinetic energy T is given by

$$E = T + mc^2 \qquad \Longrightarrow \qquad T = E - mc^2 \tag{14}$$

The relativistic kinetic energy is the total energy with the mass energy mc^2 subtracted off. It does not have such a simple relation to the speed of a particle as in Newton's physics. When the speed of the particle is much less than c then the kinetic energy is very close to the Newtonian value $\frac{1}{2}mv^2$.

Actually, the need to rewrite Newton's laws was found from studying electromagnetism (which you will study next semester) in particular the form of the basic equations of electromagnetism called the **Maxwell equations**. When the issue of *reference frames* was considered for these equations there was something a little puzzling: Whereas Newton's equation $\mathbf{F} = m\mathbf{a}$ holds in any inertial reference frame when the transformation $\mathbf{r}' = \mathbf{r} - \mathbf{V}t$ is applied, the Maxwell equations are *not* true in any inertial frame under this transformation. This led physicists to think that there is one true reference frame as far as electromagnetism is concerned.

The idea is not so crazy; perhaps like sound (for which there definitely *is* a preferred reference frame, that of the stationary air) electromagnetism occurs because of deformations is some kind of abstract fluid which one might call the **ether**. If that is the case,then it turns out that by careful measurements of the speed of light in different circumstances one can find the velocity of the earth with respect to this ether.

Attempts to measure this velocity all failed. All kinds of reasons were tried to account for this but in the end the winning explanation was a radical one offered by Einstein: He postulated that it is *Maxwell's* equations that re correct in all frames, but they require a different transformation law for the coordinates between moving frames. That being the case, it is *Newton's* laws that need rewriting. This makes the correct laws of mechanics significantly more complicated to write down and solve, and physics teachers continue to teach Newton's laws since for most practical applications they work perfectly well.

But the transformation law for space coordinates and time between reference frames also leads to some surprising and very non-intuitive results which go by the names of **time dilation** and **length contraction**, etc. It's worth going over a few of these but (in my opinion) it is wise not to get sidetracked into thinking that they are the *main* result of this revolution in the ideas of physics brought about by Einstein, known as **special relativity**. Relativity really makes itself known in the motion of very rapid *elementary particles* and in the *forms* of the fundamental physical laws which have any chance of being correct.

15 The Lorentz Transformation

When we considered a reference frame $\mathcal{S}'$ moving in the $+x$ direction with respect to us (who live in frame $\mathcal{S}$) we arrived at the transformation equations between the coordinates

$$t' = t \qquad x' = x - vt \qquad y' = y \qquad z' = z \tag{15}$$

where the origins of the systems coincide at $t = 0$. We got these by "common sense" and a little bit of algebra; the relations in 15 are known as the **Galilean transformation**. But as we now understand the role of space and time in physics measurements, they are incorrect. One of the problems is that the frames $\mathcal{S}$ and $\mathcal{S}'$ do *not* agree on the time t or t' at which an event takes place.

As is now understood, using the definition

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{16}$$

then we have

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \tag{17}$$

and

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \qquad x = \gamma(x' + vt') \qquad y = y' \qquad z = z' \tag{18}$$

and the relations in 17 and 18 are known as the **Lorentz transformations**.

15.1 Relativistic Addition of Velocities

We will suppose that a particle's velocity is measured in both the $\mathcal{S}$ and the $\mathcal{S}'$ frames, and the respective values are $\mathbf{u}$ and $\mathbf{u}'$. If frame $\mathcal{S}'$ is moving with velocity $\mathbf{v}$ with respect to $\mathcal{S}$ then as we saw, the Galilean transformation led us to expect

$$\mathbf{u} = \mathbf{v} + \mathbf{u}' \tag{19}$$

With the velocity $\mathbf{v}$ directed along the x axes of both frames, the important relation is

$$u_x = v + u'_x$$

so the velocities transform as

$$u_x = v + u'_x \qquad u_y = u'_y \qquad u_z = u'_z \tag{20}$$

but we will get a very different result if we accept that coordinates and time transform via the Lorentz equations of Eqs. 17 and 18.

First, let's understand *why* we got Eqs. 19 and 20 in the first place. A moving object will undergo a change in position (dx, dy, dz) in a time dt in the frame $\mathcal{S}$ and a change in position (dx', dy', dz') in a time dt' in the frame $\mathcal{S}'$ In the respective frames, the x–velocities are really defined as

$$u_x = \frac{dx}{dt} \qquad\qquad u'_x = \frac{dx'}{dt'}$$

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{d(x' + vt')}{d(t')} = \frac{dx' + vdt}{dt'} \\ &= \frac{dx'}{dt'} + v = v + \frac{dx'}{dt'} = v + u'_x \end{aligned}$$

If we now start with $u_x = \frac{dx}{dt}$ and use Eqs 18 to express it in terms of the primed quantities, we find:

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{\gamma d(x' + vt')}{\gamma d(t' + \frac{v}{c^2}x')} = \frac{dx' + vdt'}{dt' + \frac{v}{c^2}dx'} \\ &= \frac{v + u'_x}{1 + \frac{v}{c^2}u'_x} = \frac{v + u'_x}{1 + vu'_x/c^2} \end{aligned}$$

In fact, the other components of the velocity will also transform, due to that fact that in relativity we also have a *time* transformation. We get:

$$\begin{aligned} u_y &= \frac{dy}{dt} = \frac{d(y')}{\gamma d(t' + \frac{v}{c^2}x')} = \frac{dy'}{\gamma(dt' + \frac{v}{c^2}dx')} \\ &= \frac{u'_y}{\gamma(1 + \frac{v}{c^2}u'_x)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \end{aligned}$$

which has a very nasty form. The z component has a similar form, so we summarize the velocity addition as:

$$u_x = \frac{v + u'_x}{1 + vu'_x/c^2} \qquad u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \qquad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)} \tag{21}$$