Solutions to a Few Problems in Griffiths, *Introduction to Electrodynamics, 3$^{\text{rd}}$ ed.*

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Chapter 1

Vector Analysis

1.30 Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

The sides of the tetrahedron (shown in Fig. 1.1(a)) are the planes $x = 0$, $y = 0$, $z = 0$ and the plane given by $x + y + z = 1$.

We can sum over the volume elements $dx\,dy\,dz$ by letting x run from 0 to 1; for each x let y run from 0 to $1 - x$ (see Fig. 1.1(b)); for each (x, y) pair let z run from 0 to $1 - x - y$.

Sum up z^2 over these volume elements. The integral is:

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{z^3}{3}\right]\Bigg|_0^{1-x-y} dy\,dx$$

$$= \frac{1}{3}\int_0^1\int_0^{1-x}(1-x-y)^3 dydx \;=\; -\frac{1}{3}\int_0^1 \frac{(1-x-y)^4}{4}\Bigg|_{y=0}^{y=1-x} dx$$

$$= -\frac{1}{12}\int_0^1 [0-(1-x)^4]dx \;=\; \frac{1}{12}\left[-\frac{(1-x)^5}{5}\right]\Bigg|_0^1 \;=\; -\frac{1}{60}(0-1) = \frac{1}{60}$$

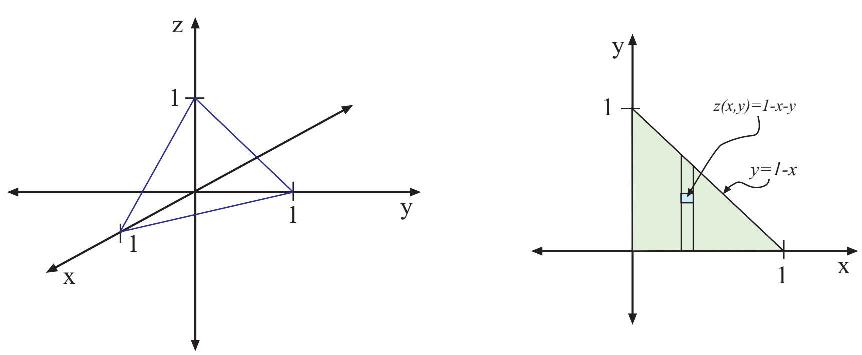

Figure 1.1: Problem 1.30

Chapter 2

The Electric Field

2.28 Calculate the potential inside a uniformly charged solid sphere of radius R and charge q using

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' \tag{2.1}$$

where $\mathscr{r} = \mathbf{r} - \mathbf{r}'$.

For simplicity choose the observation point $\mathbf{r}$ to lie on the z axis at $\mathbf{r} = r\hat{\mathbf{z}}$, as shown in Fig. 2.1. (We're free to make this choice since by symmetry $V(\mathbf{r}$ can only depend on r (and not θ or ϕ).) Then using (1.62) for the Cartesian coordinates of $\mathbf{r}'$ we have

$$\begin{aligned} \mathscr{r} &= |\mathbf{r} - \mathbf{r}'| = \sqrt{r'^2 \sin^2\theta' \cos^2\phi' + r'^2 \sin^2\theta' \sin^2\phi' + (r - r'\cos\theta')^2} \\ &= \sqrt{r'^2 \sin^2\theta' + r^2 + r'^2\cos^2\theta' - 2rr'\cos\theta'} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'} \end{aligned}$$

Then using the fact that ρ is constant, the integral in Eq. 2.1 is

$$V(r) = \frac{\rho}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} r'^2 \sin\theta' dr' d\theta' d\phi' \tag{2.2}$$

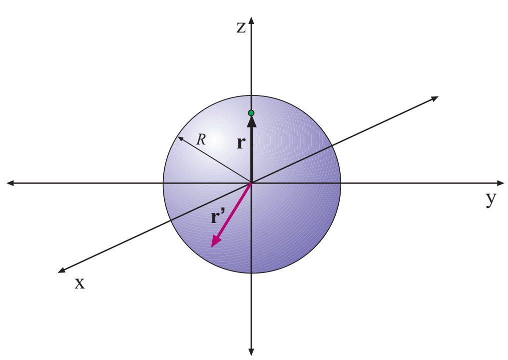

Figure 2.1: Problem 2.28

The ϕ' integral gives a factor of 2π. the integral on θ' can be changed to an integral on $x' = \cos\theta'$:

$$\int_0^\pi \sin\theta' d\theta' \quad \Rightarrow \quad \int_{-1}^1 dx'$$

This gives:

$$V(r) = \frac{\rho}{2\epsilon_0}\int_{-1}^1\int_0^R \frac{1}{\sqrt{r^2+r'^2-2rr'x}} r'^2 dr' dx' \tag{2.3}$$

The x' integral is elementary, giving

$$\begin{aligned} V(r) &= \frac{\rho}{2\epsilon_0}\int_0^R \frac{2}{(-2rr')}(r^2+r'^2-2rr'x')^{1/2}\Big|_{-1}^{1} r'^2 dr' \\ &= -\frac{\rho}{2\epsilon_0}\int_0^R \frac{r'}{r}\left\{(r^2+r'^2-2rr')^{1/2}-(r^2+r'^2+2rr')^{1/2}\right\} dr' \end{aligned} \tag{2.4}$$

The second term in the curly braces in $(r + r')$, since this is always positive. The first term is

$$(r^2+r'^2-2rr')^{1/2} = |r-r'|$$

which for $r' < r$ is $r - r'$ and for $r' > r$ is $r' - r$. So we need to break up the r' integral into parts with $0 < r' < r$ and $r < r' < R$. Then Eq. 2.4 becomes:

$$\begin{aligned} V(r) &= \frac{\rho}{2\epsilon_0}\left\{\int_0^r r'[(r+r')-(r-r')]dr' + \int_r^R r'[(r+r')-(r'-r)]dr'\right\} \\ &= \frac{\rho}{2\epsilon_0 r}\left\{\int_0^r 2r'^2 dr' + \int_r^R 2rr' dr'\right\} = \frac{\rho}{2\epsilon_0 r}\left\{\frac{2}{3}r'^3\Big|_0^r + rr'^2\Big|_r^R\right\} \\ &= \frac{\rho}{2\epsilon_0 r}\left\{\frac{2}{3}r^3 + r(r^2-r^2)\right\} = \frac{\rho}{2\epsilon_0}\left[-\frac{1}{3}r^2+R^2\right] \end{aligned} \tag{2.5}$$

Now use $\rho = \frac{q}{(4/3)\pi R^3}$, then

$$V(r) = \frac{q}{4\pi\epsilon_0 R^3}\left(\frac{3}{2}\right)\left[R^2-\frac{r^2}{3}\right] = \frac{q}{4\pi\epsilon_0 R}\left[\frac{3}{2}-\frac{r^2}{2R^2}\right] \tag{2.6}$$

This agrees with the potential for (for $r < R$) found in Prob 2.21.