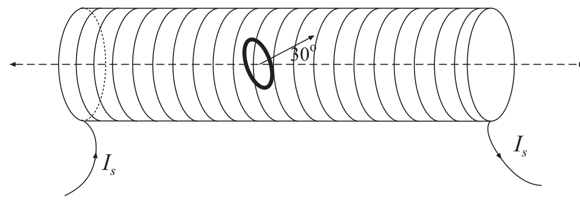
Phys 4620, Spring 2005
Exam #1, Solutions



1. a) A small circular loop of radius a and resistance R sits inside a very long solenoid of radius b (with $a < b$). The normal to the plane of the small loop makes an angle of 30° with the axis of the solenoid.

Find the current induced in the loop in terms of the rate of change of the current in the solenoid (and the other parameters of the problem).

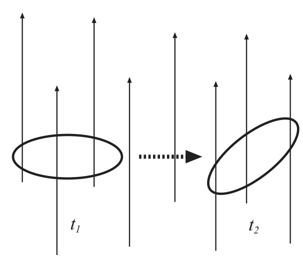
When a current I_s flows in the solenoid wire there is a uniform B field of magnitude $B = \mu_o n I_s$, where n is the number of turns per unit length. $\mathbf{B}$ is directed along the z axis; if the axis of the small loop makes an angle of 30° with $\hat{\mathbf{z}}$ then the flux through a single loop has magnitude

$$\Phi = BA\cos 30^\circ = \mu_0 n I_s \pi a^2 \frac{\sqrt{3}}{2}$$

b) What is the mutual inductance between the solenoid and the loop?

The mutual inductance is defined by $\Phi_2 = M_{21} I_1$, so applying this to the loop and the solenoid, we get

$$M = \frac{\sqrt{3}\pi\mu_0 n a^2}{2}$$



2. Suppose over a certain time interval (t_1, t_2) the magnetic flux through a planar loop changes from Φ_1 to Φ_2. (For example, this may happen because the loop changes its orientation in a uniform field, as shown here.)

Find an expression for the the total *charge* which flows though the loop during the time interval in terms of the change in flux $\Delta\Phi$. You can use the resistance R of the loop, its area A and any physical constants. And you have to show your work.

The (instantaneous) emf in the loop is given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = IR$$

where I is the instananeous current. I is the rate of charge flow, so

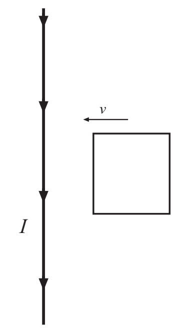
$$\frac{dQ}{dt} = I = -\frac{1}{R}\frac{d\Phi}{dt}$$

To get the total charge charge which has flowed in the loop from t_1 to t_2, integrate:

$$\begin{aligned}\int_{t_1}^{t_2} \frac{dQ}{dt}\,dt &= \Delta Q \\ &= -\frac{1}{R}\int_{t_1}^{t_2} \frac{d\Phi}{dt}\,dt = -\frac{1}{R}\left[\Phi(t_1) - \Phi(t_2)\right] = -\frac{1}{R}\Delta\Phi\end{aligned}$$

So $\Delta Q = -\frac{1}{R}\Delta\Phi$, a simple result.

3. A wire (in the plane of the page carries a current I in the direction shown. A loop of wire is approaching the straight wire from the right.



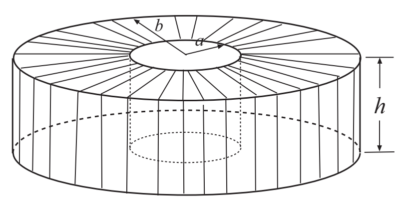
Using Lenz's law or any other valid physical reasoning, deduce which way the induced current will flow in the loop.

In the interior of the wire loop the B field from the long wire points *out of the page* and as the loop gets closer its magnitude is *increasing*.

The induced current must make a flux to *oppose* this; the magnetic field generated must go *into* the page. This will be accomplished by a *clockwise* current in the loop.

Alternately, the "positive charge" in the loop's nearest side feel an upward force; so do the ones on the far side, but that force is weaker. The current in the near side "wins out" andthe current goes clockwise.

4. Recall that the magnetic field inside a toroid is tangential around the axis of the toroid and has magnitude



$$B = \frac{\mu_0 N I}{2\pi s}$$

where s is the distance from the axis, N is the total number of turns and I is the current in the wire.

Suppose we have a toroid of rectangular cross-section with inner radius a, outer radius b and height h.

a) What is the magnetic flux through one loop of the toroid?

Find the flux through one loop: calculate $\int B\,da$:

$$\Phi = \int_a^b \frac{\mu_0 N I}{2\pi s}(h\,ds) = \frac{\mu_0 N I h}{2\pi}\ln\left(\frac{b}{a}\right)$$

b) What is the self-inductance of the toroid?

For all N loops the total flux is

$$\Phi_{\text{total}} = \left[\frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)\right] I$$

so using $\Phi = LI$, the self-inductance of the toroid is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

c) What is the energy stored in the toroid (when a current I is flowing)?

We can use $W = \frac{1}{2}LI^2$ to get the energy stored in the B field, so

$$W = \tfrac{1}{2}LI^2 = \frac{\mu_0 N^2 h I^2}{4\pi} \ln\left(\frac{b}{a}\right)$$

5. In general the electromagnetic forces that moving charges exert on one another are *not* "equal and opposite", as one might think from Newton's Third law. Newton's Third law is not general enough.

What is the generalization of Newton's Third Law which *does* hold for moving charges?

The form of Newton's 3rd law which *does* hold in general is that of *momentum conservation*. Furthermore, one must take into account the *momentum of the EM field*.

When the momenta of the massive particles *and* the field are considered, the total is constant.

6. Consider an infinite parallel–plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$ and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.

a) Find all nine elements $(T_{xx}, T_{xy}, \ldots T_{zz})$ of the EM stress tensor.

The E field between the plates has the value $\mathbf{E} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}$. (This was a basic result from 1st semester, but one can recall it from

$$Q = \sigma A = CV = \left(\epsilon_0 \frac{A}{d}\right) V = \epsilon_o A E$$

since $E = V/d$, and then $E = \sigma/\epsilon_0$.)

There's no B field so we just need

$$T_{ij} = \epsilon_0 (E_i E_j - \tfrac{1}{2}\delta_{ij} E^2)$$

with $E_x = E_y = 0$, $E_z = -\sigma/\epsilon_0$ we have

$$T_{xy} = T_{yx} = 0 \quad \text{and} \quad T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0$$

The only non-zero components are

$$\begin{aligned}
T_{xx} &= -\frac{\epsilon_0}{2} E^2 = -\frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2} = -\frac{\sigma^2}{2\epsilon_0} \\
T_{yy} &= -\frac{\epsilon_0}{2} E^2 = -\frac{\sigma^2}{2\epsilon_0} \\
T_{zz} &= \epsilon_0 (E_z E_z - \tfrac{1}{2} E^2) = \epsilon_0 \left(\frac{\sigma^2}{\epsilon_0^2} - \tfrac{1}{2}\frac{\sigma^2}{\epsilon_0^2}\right) = \frac{\sigma^2}{2\epsilon_0}
\end{aligned}$$

b) What is the flux of momentum (momentum per area per time) through the xy plane?

$-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per area per time.

The momentum flux passing upward *into* the upper half plane is

$$\text{Mom Flux} = -T_{zz} = -\frac{\sigma^2}{2\epsilon_0}$$

[It is sensible that this is a *negative* value since the top plate is being pulled *downward*.

7. The wave equation for the electric and magnetic fields in vacuum are

$$\nabla^2 \mathbf{E} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}\,, \qquad \nabla^2 \mathbf{B} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Start with the Maxwell equations in vacuum and show how either one of these is derived.

The Maxwell equations in *vacuum* ($\rho = 0$, $\mathbf{J} = 0$) are

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

Start with the third of these and operate on both sides with $\nabla\times$. Get:

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B})$$

(Note, the time and space derivatives can switch in their order!)

Use the vector identity for two curls on the lhs and substitute the fourth of the Maxwell equations on the rhs. Get:

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$$

On the lhs use $\nabla\cdot\mathbf{E} = 0$. Combine the $\partial/\partial t$ operators on the rhs, and get:

$$\nabla^2 \mathbf{E} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

which is the wave equation for $\mathbf{E}$ giving the speed of the wave as

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

The derivation of the wave equation for the B field is similar.

8. When we chose a general solution for the EM wave of the form

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\,, \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

(where the vectors $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are complex, constant vectors) and applied the conditions $\nabla\cdot\mathbf{E} = 0$, $\nabla\cdot\mathbf{B} = 0$ we found that the EM wave had to be transverse.

Show how this condition says the waves must be transverse.

For EM waves of the given form the conditions $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B}$ give:

$$(\tilde{E}_0)_z (ik) e^{i(kz-\omega t)} = 0 \qquad (\tilde{B}_0)_z (ik) e^{i(kz-\omega t)} = 0$$

for all z and all t. This implies that

$$(\tilde{E}_0)_z = 0 \quad \text{and} \quad (\tilde{B}_0)_z = 0$$

So the $\mathbf{E}$ and $\mathbf{B}$ fields have *no* component along the direction of propagation of the wave, i.e. no longitudinal component, so the waves are strictly *transverse*, i.e. the components are perendicular to the direction of propagation.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \qquad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21} I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Specific Results:

$$B_{\text{sol}} = \mu_0 n I \qquad B_{\text{tor}} = B = \frac{\mu_0 N I}{2\pi s}$$