Notes for Phys 4610

Last revised 12/01/04

These notes are just intended to give an overview of the major equations covered in class.

1 Vector Analysis!

1.1 Products of Vectors

Dot product:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

Cross product:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\,\hat{\mathbf{x}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{y}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{z}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$|\mathbf{A} \times \mathbf{B}| = AB|\sin\theta|$$

1.2 Position Vectors

The vector

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

is reserved (usually) for the point at which we want to calculate the electromagnetic field.

The vector

$$\mathbf{r}' = x'\,\hat{\mathbf{x}} + y'\,\hat{\mathbf{y}} + z'\,\hat{\mathbf{z}}$$

is reserved for a point (often integrated over) at which we find the electric charges or electric currents.

The separation vector is the difference of the two:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}$$

1.3 Triple Products, etc. of Vectors

Scalar triple product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

1.4 The "Del" Operator

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{1}$$

- **Gradient**

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \tag{2}$$

- **Divergence**

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{3}$$

- **Curl**

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \end{aligned} \tag{4}$$

1.5 Product Rules Involving ∇

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Curls:

1.6 Second Derivatives

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

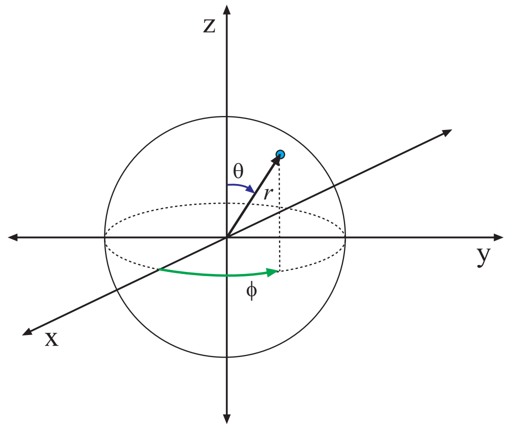

Figure 1: Spherical coordinates.

1.7 The Main Theorems of Vector Calculus

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \tag{5}$$

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \tag{6}$$

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l} \tag{7}$$

1.8 Spherical Coordinates

Spherical coordinates of a point are illustrated in Fig. 1.

Relation with Cartesian coordinates:

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{8}$$

Unit vectors:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}
\end{aligned} \tag{9}$$

Line element:

$$\begin{aligned}
d\mathbf{l} &= dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \\
&= dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}
\end{aligned} \tag{10}$$

Volume element:

$$d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi \tag{11}$$

Vector differential operators:

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{12}$$

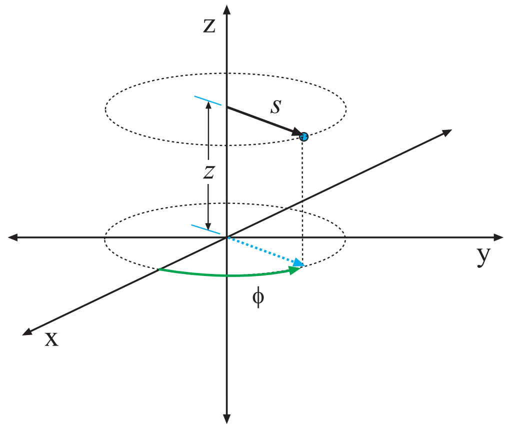

Figure 2: Cylindrical coordinates.

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{13}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{14}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{15}$$

1.9 Cylindrical Coordinates

Cylindrical coordinates of a point are illustrated in Fig. 2.

$$x = s\cos\phi \qquad y = s\sin\phi \qquad z = z \tag{16}$$

Unit vectors:

$$\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}} \end{aligned} \tag{17}$$

Line element:

$$\begin{aligned} d\mathbf{l} &= dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \\ &= ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \end{aligned} \tag{18}$$

Volume element:

$$d\tau = s\,ds\,d\phi\,dz \tag{19}$$

Vector differential operators:

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{20}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{21}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{22}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{23}$$

1.10 The Dirac Delta Function

In one dimension the delta "function" $\delta(x-c)$ has the property:

$$\int_a^b f(x)\,\delta(x-c)\,dx = \begin{cases} f(a) & \text{if } a < c < b \\ 0 & \text{otherwise} \end{cases}$$

that is, the delta function $\delta(x-c)$ "picks out" the value of $f(x)$ at c if c is contained within the interval of integration. It has the value of zero for all points except $x = c$. If there aren't any mathematicians in the room we can say:

$$\delta(x-c) = \begin{cases} 0 & x \neq c \\ \infty & x = c \end{cases}$$

so that the delta function is a big spike located at $x = c$. The spike has "area" 1:

$$\int_{-\infty}^{+\infty} \delta(x-c)\,dx = 1$$

A property of the δ function that we use from time to time is

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

The three-dimensional Dirac delta function treated as a product of three one-dimensional delta functions. The function $\delta^3(\mathbf{r}-\mathbf{a})$ with $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ is:

$$\delta^3(\mathbf{r}-\mathbf{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z) \tag{24}$$

which has the property

$$\int_{\text{all space}} f(\mathbf{r})\,\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a}) \tag{25}$$

The 3-d delta function is related to derivatives of the functions $1/r$ and $\hat{\mathbf{r}}/r^2$. With $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, we have

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}) \qquad \text{and} \qquad \nabla^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}}) \tag{26}$$

1.11 General Features of Vector Fields

In electromagnetism two vector fields will be of primary interest: the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. In the theorems about these fields we will see their divergences and curls. To this end the Helmholtz theorem is of great importance:

If the magnitude of a vector field goes to zero at infinity, then the vector field is uniquely specified by it divergence and curl.

One can also show that for a vector field $\mathbf{F}$ the following are equivalent:
(a) $\nabla \times \mathbf{F} = 0$ everywhere.
(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of the path from $\mathbf{a}$ to $\mathbf{b}$.
(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.
(d) $\mathbf{F}$ is the gradient of some scalar field:

$$\mathbf{F} = \nabla V$$

Likewise, the following are equivalent for a vector field $\mathbf{F}$:
(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.
(b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of the surface bounded by a *given* boundary line.
(c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.
(d) $\mathbf{F}$ is the curl of some vector:

$$\mathbf{F} = \nabla \times \mathbf{A}$$

2 Electrostatics

2.1 Introduction

We first deal with charges which are at rest. We first deal with the problem of finding the net force on a charge Q (the "test charge") from the presence of other point charges q_1, q_2, $q_2 \ldots$. We may think of moving our test charge Q or even changing its value but the *other* charges will remain at rest.

Knowing the basic law of electric force we can apply the principle of superposition for the forces, i.e. the total force is the (vector) sum of the forces from the individual charges.

2.2 Coulomb's Law

The force between two point charges in given by Coulomb's law. For a point charge q located at $\mathbf{r}'$, the force on charge Q located at $\mathbf{r}$ is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2}\, \hat{\boldsymbol{\mathscr{r}}} \tag{27}$$

where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$ and

$$\epsilon_0 = 8.854 \times 10^{-12}\, \tfrac{\mathrm{C}^2}{\mathrm{N{\cdot}m^2}} \tag{28}$$

In writing Coulomb's law in this way we are making the choice of the SI system of units where charge is measured in Coulombs (and force newtons and distance in meters). Exercise caution with the units in E&M: Most textbooks nowadays use the SI system but a few of them use the Gaussian unit system where Coulomb's law looks like:

$$F = \frac{qQ}{r^2}$$

(where the charge is measured in some other way). But the system used by most particle physicists is the Heaviside–Lorentz system, where Coulomb's law looks like

$$F = \frac{qQ}{4\pi r^2}$$

Confusion over the fact that different systems are in use can be deadly!

2.3 The Electric Field

The force $\mathbf{F}$ on a charge Q due to a set of charges q_1, $q_2 \ldots$ is proportional to the charge Q; then $\mathbf{F}/Q$ depends *only* on the positions and values of the other charges. This (vector) quantity is the electric field $\mathbf{E}$ at the position of Q. We have:

$$\mathbf{F} = Q\mathbf{E} \tag{29}$$

where

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\boldsymbol{\mathscr{r}}}_i \tag{30}$$

For a continuous charge distribution the appropriate expression is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau' \tag{31}$$

2.4 Divergence and Curl of the E Field

A field line diagram is a nice way to represent the *direction* of the E field at each point. If a diagram is drawn with the lines radiating outward from (or to) the charges in proportion to the size of the charge then the density of field lines in the diagram indicates the magnitude of the field. One should keep in mind though these diagrams just give a representative sampling of the field. You really can't "count field lines" as books so often say because the number you draw is arbitrary.

Through any surface $\mathcal{S}$ the flux of $\mathbf{E}$ through the surface is

$$\Phi_E = \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} \tag{32}$$

One can show that for any *closed* surface $\mathcal{S}$, the electric flux is related to the total enclosed charge:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\mathrm{enc}} \tag{33}$$

which is known as Gauss's law. It is equivalent to:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \tag{34}$$

Some interesting consequences of Gauss' law

- Outside of a spherically symmetric distribution of charge, the electric field is the same as if all the charge were concentrated at the center.
- Inside a spherically symmetric shell of charge the electric field is zero.
- The electric field from an *infinite* sheet of charge of charge density σ is

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} \tag{35}$$

where the unit vector $\hat{\mathbf{n}}$ is the unit normal to the sheet pointing toward the observation point.

One example of an application of Gauss's law to find the E field is that of a uniformly charged sphere of radius R. Symmetry says that the E field is radial everywhere: $\mathbf{E} = E_r\hat{\mathbf{r}}$. Using Gauss's law, we find

$$E_r(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3} & \text{If } r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & \text{If } r \geq R \end{cases}$$

2.5 The Electric Potential

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \tag{36}$$

The point $\mathcal{O}$ is the "reference point", a point chosen where we want the potential to be zero. It is arbitrary but the usual choice is to place $\mathcal{O}$ "at infinity" so that the potential gets small as we go far away from the charges.

From this it follows that

$$\mathbf{E} = -\nabla V \tag{37}$$

For many electrostatics problems it is easier to calculate V at all points in space and then take the gradient to get the electric field $\mathbf{E}$.

Since the electric field obeys the superposition principle, the electric potential does also.

The consequence of Eq. 34 and Eq. 37 is

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{38}$$

The potential of a set of n point charges is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_1} \tag{39}$$

and that of a localized charge distribution $\rho(\mathbf{r}')$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' \tag{40}$$

These expressions assume that the reference point for V is at infinity. In some idealized problems where the charge distribution extends to infinity you can't do this.

2.6 Electrostatic Boundary Conditions

One can use Gauss's law to derive some facts about the possible discontinuities in the electric field and potential as we step across a bounding surface.

First, the perpendicular component of the E field will have a discontinuity if there is a thin layer of charge:

$$E_{\perp,\text{above}} - E_{\perp,\text{below}} = \frac{\sigma}{\epsilon_0} \tag{41}$$

Secondly, the component of $\mathbf{E}$ parallel to the surface *is* continuous:

$$\mathbf{E}_{\parallel,\text{above}} = \mathbf{E}_{\parallel,\text{below}} \tag{42}$$

The potential itself is *continuous*:

$$V_{\text{above}} = V_{\text{below}} \tag{43}$$

And the condition on the normal component of the E field can be written in terms of the potential as

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0} \tag{44}$$

2.7 Work and Energy in Electrostatics

Since the electric force on a charge Q is just $\mathbf{F} = Q\mathbf{E}$, the electric field is related to work *done the charge* as we move it from one place to another.

The work required of some *outside agency* to move a charge Q from location $\mathbf{a}$ to location $\mathbf{b}$ in and electrostatic field is given by

$$W = Q[V(\mathbf{b}) - V(\mathbf{a})] \tag{45}$$

The energy associated with a collection of charges (or a charge distribution) arises from the repulsions and attractions between all the charges. In calculating this energy— by which we mean the work required of an outside agency to bring the charges together from infinity— one must be careful about counting.

For a set of point charges this energy can be expressed as

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} \tag{46}$$

and also as

$$W = \tfrac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{47}$$

For a continuous charge distribution the work to assemble it generalizes Eq. 47 to get:

$$W = \tfrac{1}{2}\int \rho V\, d\tau \tag{48}$$

which can also be written is

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau \tag{49}$$

For the moment it's worth stopping to ponder these expressions. Do they tell "where the energy is located"? Eq. 47 seems to say that the energy is all "in" the charges while Eq. 49 seems to say that the energy is "in" the space all around the charges, contained in the fields. For now it makes no difference, since these equations should be regarded as prescriptions for calculating the work necessary to assemble the charge density and the question falls in the class of a Zen koan. However, much later when we associate physical entities with the fields it *will* make sense to ask "where the energy is located".

It's also interesting to (try to) calculate the electric energy contained in the simplest electrical system imaginable: A single point charge q. The field is $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ and Eq. 49 gives

$$W = \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}(4\pi)\int_0^\infty \left(\frac{1}{r^2}\right) r^2 dr = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}dr = \infty$$

Whoa! An infinite answer? Yes, and for that reason we have to ignore the energy required to pack a finite amount of charge into a point (?!?!). When the charge distributions are continuous, this problem does not arise.

The problem with point charges gives the answer to another odd feature of Eqs. 47 and 49. The former can clearly be positive or negative while the latter can only be positive. That is because the step in going from a point charge distribution to a continuous one ignored the poorly–behaved energy involved in assembling a point charge. In the first equation the point charges were fully formed when they were out at infinity, and we *must* use that one if we are talking about point charges.

2.8 Properties of Conductors

In a conductor some of the electrons from the atoms are completely free to roam around. The consequences of this are:

- $\mathbf{E} = 0$ inside the conductor.
- $\rho = 0$ inside the conductor so all charges reside on the surfaces.
- All points of the conductor are at the same value of V (the electric potential).
- Just outside the conductor's surface, $\mathbf{E}$ is perpendicular to the surface.

If the conductor has a cavity inside of it and the cavity is free of charge then the field in the cavity is zero regardless of the fields outside. If the cavity does contain a charge then the outer surface of the conductor just "knows" about the net charge inside but not the shape or placement of the cavity.

The electric field just outside a conductor is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \tag{50}$$

Equivalently, the charge density at some spot on the conductor's outer surface is

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \tag{51}$$

The force per unit area on the surface of a conductor (due to the other charges on the surface) is

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \tag{52}$$

or in terms of the field just outside the surface,

$$P = |\mathbf{f}| = \frac{\sigma^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2 \tag{53}$$

2.9 Capacitors

A capacitor is a system of two isolated conductors for which we intend to place a charge $+Q$ on one of them and a charge $-Q$ on the other. When we do this it turns out that the potential difference the conductors is proportional to Q and the constant of proportionality C is defined by

$$C \equiv \frac{Q}{V} \tag{54}$$

Capacitance is measured in farads, which is just a coulomb per volt. Practical capacitors are usually in the microfarad or picofarad range.

For two parallel plates of area A separated by a *small* distance d the capacitance is given approximately by

$$C = \frac{A\epsilon_0}{d} \tag{55}$$

To charge up a capacitor C to its final charge Q one must do an amount of work

$$W = \frac{Q^2}{2C} = \tfrac{1}{2}CV^2 \tag{56}$$

3 Mathematical Techniques in Electrostatics

The main problem of electrostatics reduces to solving the differential equation for V:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{57}$$

subject to the particular boundary conditions of the situation. And generally we confine our attention to the places where there is no charge so that we are solving the Laplace equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{58}$$

The mathematical techniques that we pick up in solving this equation are useful throughout physics, so there is a point to teaching impressionable young persons about solving the Laplace equation. Trust me.

3.1 Laplace Equation: Introduction

Consider the Laplace equation in one dimension,

$$\frac{d^2 V}{dx^2} = 0 \tag{59}$$

with boundary conditions of some sort. The solution is easy: A line! But we gain some insight by looking at the first numerical approximation to the derivative,

$$\frac{dV}{dx} \approx \frac{V(x+h) - V(x-h)}{2h} \qquad \Longrightarrow \qquad \frac{d^2V}{dx^2} \approx \frac{V(x+2h) + V(x-2h) - 2V(x)}{(2h)^2} \tag{60}$$

so that from this the Laplace equation implies (with $\epsilon = 2h$)

$$V(x) = \tfrac{1}{2}(V(x+\epsilon) + V(x-\epsilon)) \tag{61}$$

that is, the value of V at any point is the average of the values of V at the neighboring (!) points. As Griffiths points out, this is a prescription for making a function as "boring" as possible.

In two dimensions the Laplace equation is a partial differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{62}$$

and the boundary condition is a *function*(i.e. the value of V) given on a bounding curve. If we make a similar first approximation to the derivatives as above, we get the condition

$$V(x,y) = \tfrac{1}{4}(V(x,y+\epsilon) + V(x,y-\epsilon) + V(x+\epsilon,y) + V(x-\epsilon,y)) \tag{63}$$

that is, that value of V anyplace is the average of the 4 nearest "neighbors".

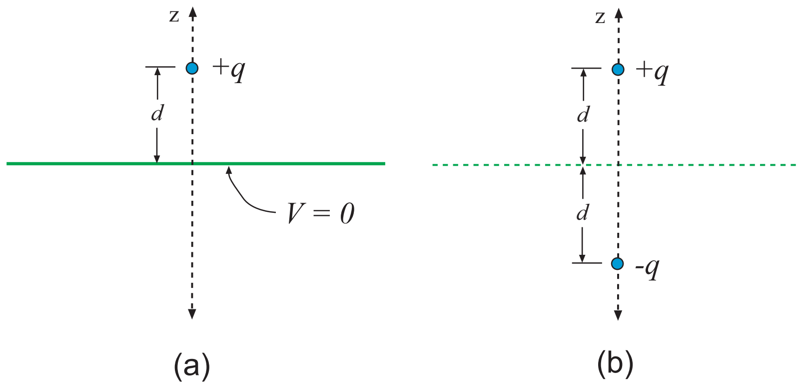

Figure 3: (a) Original electrostatics problem. (b) A different (?) electrostatics problem.

3.2 Uniqueness of Solutions

Occasionally we need to listen to the mathematicians when they tell us to sharpen up our ideas about the solutions to differential equations. It is important to know if the solution to a given differential equation (with boundary conditions) is *unique*; a study of this question will tell how much in the way of boundary conditions we need to specify for a physical problem before we know we have given all the relevant "physics".

Two theorems are of some importance in the solutions of our electrostatics problems:

First Uniqueness Theorem: The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

We note that the "outer" surface of $\mathcal{V}$ can be infinity as long as we specify the behavior of V there, and $\mathcal{V}$ could have "islands" in its interior as long as V is specified on their surfaces.

Next,

Second Uniqueness Theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given.

3.3 The Method of Images

Before diving into the more general methods of function expansions for partial differential equations, we cover a method of solving problems involving conducting surfaces and simple geometries. Now that we have the uniqueness theorems we can be assured that this simple procedure gives the correct answers.

The method is best introduced with the problem of a point charge q help at a distance d above an infinite conducting plane, as shown in Fig. 3(a). The points of the plane (taken to be the xy plane) are all at the same potential which we can choose to be 0. The problem is to find the potential in the region above the plane ($z > 0$). On further thought we realize that this could be a complicated problem because the plane will acquire an induced (negative) charge which contributes to the field, and this charge is not uniform.

Now we consider the solution to a completely different (?) problem. With *no* conducting plane, consider a charge $+q$ at $z = +d$ and a charge $-q$ at $z = -d$, as shown in Fig. 3(b). The solution

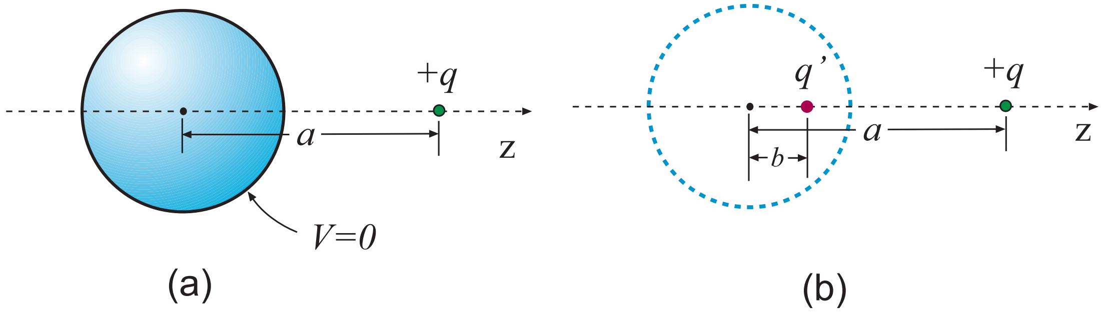

Figure 4: (a) Original electrostatics problem. (b) A different (?) electrostatics problem.

for this problem is easy,

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\mathscr{r}} - \frac{q}{\mathscr{r}'}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \end{aligned} \tag{64}$$

where we notice that we have $V = 0$ in the xy plane.

Now compare some features of the original problem and the different (?) one:

• For both, the conditions on the boundary of the $z > 0$ region are the same, i.e. $V = 0$ for $z = 0$ and $V \to 0$ as $r \to \infty$.

• In the interior of the region, $\nabla^2 V = 0$ except at the location of the charge; both regions contain the same charge distribution.

By the first uniqueness theorem, the solution to *both* problems for $z > 0$ is the same. So Eq. 64 also solves the *original* problem (with the point charge and the conducting plane).

We can find the force on the charge (it must be toward the plate since an *opposite* charge is induced on the plate); it is the same as the force of attraction between q and the image charge $-q$:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}$$

We can find the charge density σ on the plate and the total induced charge on the plate $(-q)$.

However it takes some care to compute the *work done* in bringing charge q from infinity and setting it at $z = d$. It is *not* the same as the work in forming the fictitious system since the image charge was never really brought in from infinity.

Next we consider the problem of a point charge q placed at a distance a from the center of a grounded conducting sphere, as shown in Fig. 4(a). (We need to say that the sphere is grounded, i.e. it is attached by an inconspicuous wire to a large conductor far away so that it was pull in any induced charges that it needs to be at zero potential.) The field outside the sphere will be due to q and to the induced (negative) charge on the sphere so that the solution for $V(\mathbf{r})$ for $r > R$ is not so trivial.

Now consider a different (?) problem, which is little contrived. With *no* conductors present consider a charge q on the z axis at $z = a$ and another charge $q' = -Rq/a$ located on the z axis at $z = b = R^2/a$, as shown in Fig. 4(b). One can show that with these two point charges in place, $V(r = R) = 0$.

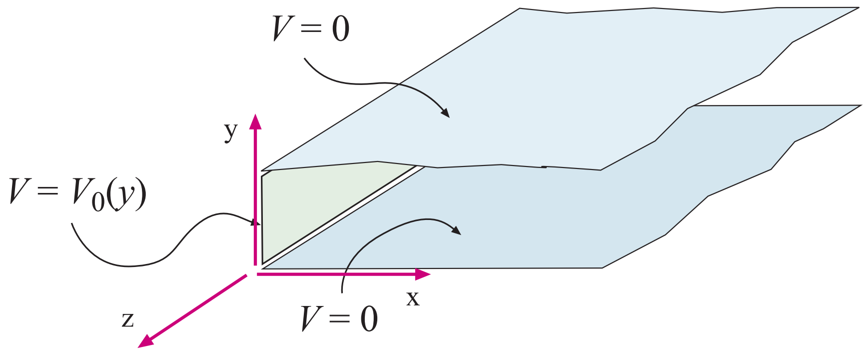

Figure 5: Find $V(x, y, z)$ within the slot.

Once again the real problem and the fake problem have the same boundary conditions for the region of interest, which here is $r > R$ and so the solution to the fake problem (which is fairly easy to write down) *is* the solution to the real problem.

3.4 Separation of Variables

We introduce the subject with a problem which contains enough of the elements of the technique that we can make generalizations later on. Consider:

• Two infinite grounded plates parallel to the xz plane, one at $y = 0$ and the other at $y = a$. We consider the space between the plates for $x > 0$. The end of the region at $x = 0$ is insulated from the grounded plates and maintained at a specific potential given by $V_0(y)$; See Fig. 5. We want to find the potential everywhere within the slot, i.e. for $x > 0$, $0 < y < a$ and all z. And then we realize that this a hard problem!

So we see if we can solve an easier one. Can we find a solution to $\nabla^2 V = 0$ which is a product of functions of x and y individually (the solution won't depend on z)? That is, what kind of solution has the form

$$V(x, y) = X(x)Y(y)$$

One can show that with the solution we *must* have

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \qquad \text{and} \qquad \frac{1}{Y}\frac{d^2Y}{dx^2} = C_2$$

where C_1 and C_2 are constants with $C_1 + C_2 = 0$. For reasons to be seen later (!) we make the choice

$$\frac{d^2X}{dx^2} = k^2 X \qquad \text{and} \qquad \frac{d^2Y}{dx^2} = -k^2 Y$$

which have general solutions for X and Y:

$$X(x) = Ae^{kx} + Be^{-kx} \qquad Y(y) = C \sin ky + D \cos ky$$

The boundary of our particular problem would restrict these solutions to

$$X(x) = Ce^{-kx} \sin ky \qquad \text{with} \qquad k = \frac{n\pi}{a} \tag{65}$$

But in the end this solution cannot solve our example since the function $V_0(y)$ is arbitrary. But a sum of solutions of the form 65 will still satisfy the conditions at $y = 0$ and $y = a$ and *may* satisfy the one on $x = 0$:

$$V(x.y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a) \tag{66}$$

The condition at $x = 0$ requires

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) \tag{67}$$

From the theory of functions it is known that we *can* find a set of C_n's that will satisfy 67, and the method for finding them relies on the orthogonality property of the sine functions,

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \begin{cases} 0 & n' \neq n \\ a/2 & n' = n \end{cases} \tag{68}$$

Using this relation with 67 and specializing to the case V_0 =constant, one can show

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases}$$

Scraping everything together, the solution to the "slot" problem is

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a) \tag{69}$$

3.5 Separation of Variables in Spherical Coords

The Laplace equation in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{70}$$

Throughout this course we will assume azimuthal symmetry, i.e. the problem have a symmetry in ϕ so that the solution V will be independent of ϕ. Again we look for solutions which are products of r and θ:

$$V(r,\theta) = R(r)\Theta(\theta)$$

Putting all of this into the Laplace equation gives:

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = 0$$

and as before this implies that each term is equal to a constant. With hindsight we know what the form of the constant should be, and so we write

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1) \qquad \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1) \tag{71}$$

where l is a constant, and as we will see, is an integer.

The first of the equations in 71 has solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

while the second requires more work; after doing all that work —see your local "math physics" course!— we find that the solution is

$$\Theta(\theta) = P_l(\cos\theta)$$

where $P_l(x)$ is known as the Legendre polynomial of order l. $P_l(x)$ is given by the Rodriques formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{72}$$

and the first few $P_l(x)$'s are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \end{aligned}$$

These polynomials have the property

$$P_l(1) = 1 \qquad P_l(-1) = (-1)^l$$

and they are orthogonal; specifically:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases} \tag{73}$$

The general solution to a spherical problem is a linear combination of such separated solutions, so that

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{74}$$

The general solution to Laplace's equation in cylindrical coordinates is treated in Problem 3.23. (One assumes that the boundary conditions and hence the solutions are independent of z.) The result is

$$V(s,\phi) = A_0\log s + B_0 + \sum_{n=1}^{\infty}\left\{A_n r^n\cos(n\phi+\alpha_n) + B_n r^{-n}\cos(n\phi+\beta_n)\right\}$$

3.6 The Multipole Expansion

We want to consider how a localized charge distribution "looks" when we are at large distances from it. (We assume the charge distribution is centered around the origin.) We are interested in approximate, *simple* forms for the potential V are large distances. For this it will be best to work in spherical coordinates because the distance from the origin (r) is fundamental in that system.

Before taking off on the general theory, it is worth reviewing (?) the electric dipole. In the "dipole" problem, we have a charge $+q$ on the z axis at $z = +d/2$ and a charge $-q$ at $z = -d/2$. We use the binomial approximation to find an expression for the potential at large distances from the origin, where "large" means $r \gg d$. One finds:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} \tag{75}$$

The result is interesting because it show that for this system where the net charge is zero there is still a potential but it gets weaker in inverse proportion to r^2 (and not r).

Now we look for a general procedure for finding the potential at large distances from a finite charge distribution. We start with the usual

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')d\tau'$$

and use an expansion for $1/\mathscr{r}$ which is valid at large r:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)P_n(\cos\theta') \tag{76}$$

where we are assuming the vector $\mathbf{r}$ points in the z direction in our coordinate system (so its direction is fixed!). This gives:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')d\tau' \tag{77}$$

which called the multipole expansion of the charge distribution $\rho(\mathbf{r})$. One word of caution, though; it is note completely general since the direction of the field point $\mathbf{r}$ is fixed.

Explicitly,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots\right] \tag{78}$$

In 78 if the total charge is nonzero, then the dominant term is

$$\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau'\right] = \frac{Q}{4\pi\epsilon_0 r}$$

but if the total charge is zero, the second term dominates (unless it too is zero!). One can show that the second term can be written in a more general way as

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

The vector $\mathbf{p}$ is called the dipole moment of the charge distribution.

The dipole moment of a set of point charges is given by

$$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}'_i \tag{79}$$

For the dipole made of two point charges $\pm q$, we have $\mathbf{p} = q\mathbf{d}$ where $\mathbf{d}$ is the vector pointing from $-q$ to $+q$.

The electric field of a dipole is given by

$$\mathbf{E}_{\text{dip}} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \tag{80}$$

4 Electric Fields in Matter

It is an unavoidable fact of life that we have to deal normal matter, which is made of atoms with their constituent charges and which do not respond to electric fields in the simple way that conductors do. In particular, for insulators the electrons are bound to the individual atoms and do not move around in the material. Nevertheless the charge distributions of the atoms will distort under the influence of external fields and as a result will induce will produce a new electric field from this property of polarizability.

When a neutral atom is placed in an electric field $\mathbf{E}$ the charges are shifted in (average) position so as to give a dipole moment $\mathbf{p}$ proportional to the field, $\mathbf{p} = \alpha\mathbf{E}$, where α is a constant characteristic of the type of atom. (This is a simplification; the induced dipole is really given by some *matrix* α which multiplies the $\mathbf{E}$ field.)

Now is a good time to point out a couple mechanical properties of electric dipoles: If a dipole $\mathbf{p}$ is in an electric field $\mathbf{E}$, it experiences a torque

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{81}$$

and the force on the dipole is given by

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E} \tag{82}$$

Here we note that the force on a dipole arises because of the spatial variation of $\mathbf{E}$; if $\mathbf{E}$ were uniform the ∇ operator would give zero everywhere.

The potential energy of a dipole in an $\mathbf{E}$ field is

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{83}$$

When a macroscopic sample is placed in a magnetic field, the field will induce a dipole moment in all of the atoms; we can then discuss the dipole moment induced *per unit volume*, $\mathbf{P}$, which is called the polarization of the material, which, generally speaking is parallel to the direction of the original $\mathbf{E}$ field.

4.1 Electric Field Due to a Polarized Object

We can use the potential due to a dipole $\mathbf{p}$ located at $\mathbf{r}'$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{p}}{\mathscr{r}^2}$$

and our definition of $\mathbf{P}$ as the dipole moment per unit volume of the source material: $\mathbf{p} = \mathbf{P}\, d\tau'$ to show that the potential due to the polarization of a sample is:

$$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla' \cdot \mathbf{P})\, d\tau'$$

Each term on the rhs has the same form as $V = \frac{1}{4\pi\epsilon_0}\int \frac{dq'}{\mathscr{r}}$ where there is a surface charge density

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \tag{84}$$

and a volume charge density

$$\rho_b = -\nabla \cdot \mathbf{P}\ . \tag{85}$$

so the equivalent charge distributions σ_b and ρ_b give the field due to polarization of the material. These are called the distributions of bound charge, the crude physical picture being that this charge produces an E field but stays stuck to the atoms of the material; it only became separated (detectable) because of the distortions of the electron clouds in the atoms.

Griffiths has a couple sections in the book here which are well worth reading but don't add much to the overall theory.

First he discusses what to make of the bound charge distributions which were formally found in 84 and 85. Views within the community of physics educators are not unanimous. Griffiths emphasizes that they are *real* distributions in the sense that it is charge that "cannot be removed" (!) but which sticks out in the open (uncancelled by others) when the material is polarized.

Surely the solution (as always) is to get a firm understanding of the subject and then you can view the formal quantities like σ_b in any way that you want.

Less controversial and more subtle is the discussion of the field *inside* a dielectric. The difficulty comes from the fact that while we always want to talk about average charge densities, if you are inside the material the process involves averaging charge densities which are very close to the "observation point".

4.2 The Electric Displacement

If we identify the bound charge density by Eq. 85 then the *total* charge density at any point can be written

$$\rho = \rho_b + \rho_f \tag{86}$$

where ρ_f is the density of *free* charge. Then if we define the field $\mathbf{D}$ as

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \tag{87}$$

the Gauss's law can be written as

$$\nabla \cdot \mathbf{D} = \rho_{\rm f} \tag{88}$$

Which as we now know can immediately be written in an integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\rm f,\, int} \tag{89}$$

You are cautioned against the simplistic idea of completely ditching the quantities $\mathbf{E}$ and ρ and using $\mathbf{D}$ and $\rho_{\rm f}$ instead. In essence this is because the "free charge" distribution ρ_f does not determine $\mathbf{D}$ in the way that ρ determines $\mathbf{E}$. And there is no potential like V which corresponds to the field $\mathbf{D}$.

4.3 Boundary Conditions

One can show that when dielectrics are involved, the conditions on the discontinuities in field components have to be expressed differently in order to again have a a simple form.

$$D^{\perp}_{\rm above} - D^{\perp}_{\rm below} = \sigma_f \tag{90}$$

and

$$\mathbf{E}^{\parallel}_{\rm above} - \mathbf{E}^{\parallel}_{\rm below} = 0 \tag{91}$$

4.4 Linear Dielectrics

For many substances, under a wide set of conditions the polarization field $\mathbf{P}$ is proportional to the electric field; we write this as

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{92}$$

where $\mathbf{E}$ is the *total* E field, i.e. that due to outside charges *and* the polarized material itself! Usually we work with the displacement $\mathbf{D}$, for which we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

which is expressed as

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \text{where} \quad \epsilon = \epsilon_0 (1 + \chi_e) \tag{93}$$

Finally, we extract the unitless number ϵ_r as

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{94}$$

and it is on *this* number that we focus for our studies of the properties of dielectrics. Of course, we must remember that the original linear relation was just empirical; under conditions of very strong fields it doesn't hold.

4.5 Energy and Force in Dielectric Systems

With dielectrics around we need to be even more careful with framing questions about energy in electrical systems. A calculation of the energy change in the system caused by dragging a set of free charges into place when there is polarizable material around gives

$$W = \tfrac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau \tag{95}$$

This expression would seem to contradict the one given earlier (Eq. 49) but it's a matter of knowing what energy is included. Eq. 95 includes the energy change of the dielectric material as the charges in the material get pulled apart; this energy in the system (analogous to many little springs) is present even if the polarization is uniform (and thus there is no ρ_b). But the earlier expression for the energy just counts up the energy due to the *non-zero* charge densities, that is, ρ_f, σ_f, ρ_b, σ_b.

One can think of all kinds of situations where there might be a force on a dielectric due to an external E field in which we've placed it, but one of the simplest cases is to consider the force on a dielectric slab which is partially inserted between two narrowly-separated parallel plates. This is what Griffiths does in the final section and he find that a force tends to pull the dielectric *into* the capacitor.

The derivation is interesting because it points out an error one could make by not considering *all* the energy changes in a physics problem, in this case the energy changes in the voltage source if if it attached to the capacitor to keep V constant. Also, even though the electric fields near the edge (where the field drops off from the uniform value to zero, the "fringing fields") are complicated and important in *giving rise* to the force the derivation does not need their explicit form.

5 Magnetostatics

We now move on to the study of the magnetic field: how it is produced by electric currents (charges in motion) and how it gives rise to the magnetic force on a moving charge or electric current. While the subject involves electric current (charge in motion) is still referred to as magneto*statics* because we will begin with currents which themselves don't change in time, and as a result the magnetic fields will depend on position but not on time.

5.1 Magnetic Fields and Forces

Something is going to be tricky with the mathematics of the magnetic interaction! Wires carrying parallel currents attract and those with opposite currents repel. Also a compass help near a current–carrying wire gives the direction of the magnetic field; the field lines go in circles around the wire. But these two facts taken together imply that the force on an element of current is perpendicular to both the magnetic field *and* the current. So we're in for some complicated vector mathematics. Or maybe it's *interesting* mathematics; take your pick.

5.2 Magnetic Forces

The magnetic force on a charge Q moving with speed $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_{\mathrm{mag}} = Q(\mathbf{v} \times \mathbf{B}) \tag{96}$$

known as the Lorentz force law. The magnetic force is perpendicular to *both* the velocity and the B field; it is proportional to the charge of the particle.

With both electric and magnetic fields present the net force on Q is

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \tag{97}$$

Recall from Phys 2120 that when a particle has planar motion in a uniform magnetic field, the path is a circle where the radius is given by

$$QB = \frac{mv}{R}$$

(where Q, B and v are the magnitudes of the charge, magnetic field and velocity).

With E and B field both present and mutually perpendicular the motion has the shape of a cycloid.

We note that since the magnetic force is perpendicular to the velocity, a magnetic field can do *no work*, a fact which is confusing when we think of magnetic forces being used to lift things. As Griffiths points out, using some simple illustrations later on the work comes from the other forces that are present (e.g. from the electric fields); the magnetic field serves to *redirect* the directions of those other forces, and *they* are the ones which do the work.

In case you forgot, the magnetic field is measured in Tesla in the SI system but one often sees a field strength expressed in gauss; 1 Tesla equals 10^4 gauss.

5.3 Currents

We begin with the idea of a current flowing through a (thin) wire. The current I measures the amount of electric charge passing a given point of the wire per unit time. Current is measured in amperes, with 1 A = 1 C/s. In counting the electric charge which moves, a positive charge moving (say) to the right is the same as a negative charge moving to the left.

If a line charge λ were moving along a path with speed v then the current would be $I = \lambda v$. Actually, current is a *vector* where the direction is given by the orientation of the wire (and the sense in which the charge is moving) so the real relation is $\mathbf{I} = \lambda\mathbf{v}$.

The force on a current–carrying wire is then found to be

$$\mathbf{F}_{\text{mag}} = \int I(d\mathbf{l} \times \mathbf{B}) \tag{98}$$

where the integral goes over the section of the wire of interest. It's usually the case that the current I is same all through the wire and then we have $\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})$.

When charge flows over a surface it is described by the surface current density $\mathbf{K}$, defined by

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \tag{99}$$

where $dl_\perp$ is a length measured perpendicular to the flow and $d\mathbf{I}$ is the current contained in that length. If the current is made of a surface charge density σ in motion then $\mathbf{K}$ is given by $\mathbf{K} = \sigma\mathbf{v}$.

When charge flow is distributed through a three-dimensional region it is described by the volume current density $\mathbf{J}$, defined by

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \tag{100}$$

where $da_\perp$ is a small area taken perpendicular to the flow of charge and $d\mathbf{I}$ is the current contained in that area. If the current comes from a charge density ρ in motion then $\mathbf{J}$ is given by $\mathbf{J} = \rho\mathbf{v}$.

The expressions for the magnetic force on surface and volume currents are

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{K} \times \mathbf{B}) da \qquad \mathbf{F}_{\text{mag}} = \int (\mathbf{J} \times \mathbf{B}) d\tau \tag{101}$$

Conservation of charge (the idea that the loss of charge contained within a surface comes from the flow charge through that surface) gives the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{102}$$

5.4 The Biot–Savart Law

The magnetic field of a steady line current is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \tag{103}$$

where μ_0 is the permeability of free space,

$$\mu_0 = 4\pi \times 10^{-7} \tfrac{\text{N}}{\text{A}^2} \tag{104}$$

The unit of B, the Tesla, is related to the other SI units by:

$$1 \text{ T} = 1 \tfrac{\text{N}}{\text{A}\cdot\text{m}}$$

Expressions for the field arising from surface and volume currents are

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} da' \qquad \text{and} \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau' \tag{105}$$

but we note that in all cases we are evaluating the B field due to a steady *current*, not from a moving point charge (or set of point charges). A moving point charge does indeed give rise to a magnetic field but its form is too complicated to consider just now. Also the consideration of currents is much more *practical* so we don't have much to complain about.

5.5 Divergence and Curl of B

Starting with the most general expression for the **B** field arising from currents,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'$$

one can show two important facts about the magnetic field arising from steady currents:

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{106}$$

The first expresses the fact that there are no isolated "magnetic charges". The second of these is called Ampere's law and can be expressed in integral form as:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\rm enc} \tag{107}$$

where $I_{\rm enc}$ is the total current which passes through the loop; it can be found by taking $\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$ for any surface $\mathcal{S}$ bounded by the loop. Also, the direction the loop is traversed in the integral and the direction of the area vector $d\mathbf{a}$ are related by the right–hand rule.

Ampere's law can be used to calculate the magnetic field for simple geometries with the proviso that we know or can reason out the direction of the B field everywhere, because things are generally more complicated than electrostatics.

Some famous results are the B field for a very long solenoid. If the solenoid runs along the z axis and carries a current I

5.6 Magnetostatics and Electrostatics

It is worth taking time to ponder the *fundamental* equations of electricity and magnetism (both "static") we have acquired up to now. We have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0 \tag{108}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{109}$$

along with the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{110}$$

These laws show how the sources of the fields relate to the fields themselves; the E field diverges outward from point charges (Gauss's law); the B field wraps (curls) around a current (Ampere's law). But there are no point sources for the B field. The B field must be produced by moving charges and the B field can only be "felt" by a moving electric charge.

5.7 The Magnetic Vector Potential

Since $\nabla \cdot \mathbf{B} = 0$ (always) we know that there *must* be a vector field of which **B** is the curl. Here we need to think back to the electric field which, because of $\nabla \times \mathbf{E} = 0$, was *required* to be the gradient of some function which was $-V$. But there was an *ambiguity* in the choice of V which was resolved by the choice of a "reference point".

Now we introduce the *vector* potential for the magnetic field, for which we have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{111}$$

From the fact that this only gives a requirement on the *derivatives* of **A** is leaves some freedom in the choice of **A** which will be exploited later on.

From this definition and Ampere's law one can show:

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_o \mathbf{J}$$

A useful thing to do from the start is to use the freedom in the choice of **A** to demand that **A** have zero divergence:

$$\nabla \cdot \mathbf{A} = 0 \tag{112}$$

With this choice we get a simpler relation between **A** and **J**,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{113}$$

and since this is Poisson's equation, the assumption that all the currents die off at infinity allows us write the solution for **A** which is similar in form to Eq. 40,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\boldsymbol{\mathscr{r}}} \, d\tau' \tag{114}$$

which for line and surface currents is written as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\boldsymbol{\mathscr{r}}} \, dl' = \frac{\mu_o I}{4\pi} \int \frac{1}{\boldsymbol{\mathscr{r}}} \, d\mathbf{l}' \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\boldsymbol{\mathscr{r}}} \, da'$$

What's the point of introducing **A**? (Just because we *can*?) Being a vector it is trickier to compute than our electrostatic potential V, and doesn't provide as useful a tool for finding **B**. Also, at *this* stage of the course it's not clear where it can be useful, whereas the electric potential V has a direct connection to the work and energy in systems of electric charge.

Its importance will come up later; one must be patient.

5.8 Magnetostatic Boundary Conditions

When there is surface current **K**, there can be a discontinuity in the B field above and below this surface current.

One can show that at such a boundary the perpendicular component of **B** will be continuous:

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \tag{115}$$

But the part of **B** which points along the surface has a discontinuity:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \tag{116}$$

where $\hat{\mathbf{n}}$ is the unit vector pointing normal to the surface, from "below" to "above".

The vector potential is continuous across a boundary:

$$\mathbf{A}_{\text{above}} + \mathbf{A}_{\text{below}} \tag{117}$$

and the derivative of **A** has a discontinuity from the surface current:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K} \tag{118}$$

5.9 Multipole Expansion of the Vector Potential

Starting with Eq. 114 and the expansion for $1/\mathscr{r}$ given way back in Eq. 76 we arrive at an expansion for $\mathbf{A}(\mathbf{r})$ in inverse powers of r:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\,d\mathbf{l}' && (119) \\ &= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta'\,d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\,d\mathbf{l}' + \cdots\right] && (120) \end{aligned}$$

where the successive terms here are called magnetic monopole, magnetic dipole, magnetic quadrupole, etc. But the first term in 120 is always zero, so the lowest-order term is the magnetic dipole. For this, the vector potential is

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathbf{l}' \tag{121}$$

which can be rewritten as

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a} \tag{122}$$

The vector $\mathbf{m}$ is the magnetic dipole moment of the current loop. $\mathbf{a}$ is the "vector area" of the loop which is the usual area if the loop is planar.

We get a pure magnetic dipole by considering a current loop whose size vanishes and current increases such that Ia is constant at a value of m. In spherical coordinates the vector potential and magnetic field of a pure magnetic dipole are

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \tag{123}$$

$$\mathbf{B}_{\rm dip}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \tag{124}$$

The B field of a magnetic dipole has the same form as the E field from an electric dipole.

6 Magnetic Fields in Matter

6.1 Introduction

Just as matter contains positive and negative charges which can be separated under the influence of an E field, matter contains tiny microscopic currents which can be influenced by a B field and thus produced magnetic dipoles in the material. In treating the magnetization of matter we will draw on our previous formalism for electric fields in matter.

Magnetic polarization has some differences with electric polarization. The magnetization of the material can be opposite to the applied field. (These are *diamagnets*; for *paramagnets* the direction of the polarization is along the direction of the applied field.) The treatment of microscopic current has the additional feature that the particles in nature have an *intrinsic* magnetic moment which does *not* come from point charges in motion. Finally, some substances retain their magnetization after the external field is removed. These are *ferromagnets* and of course are quite common.

Magnetism in matter is complicated!

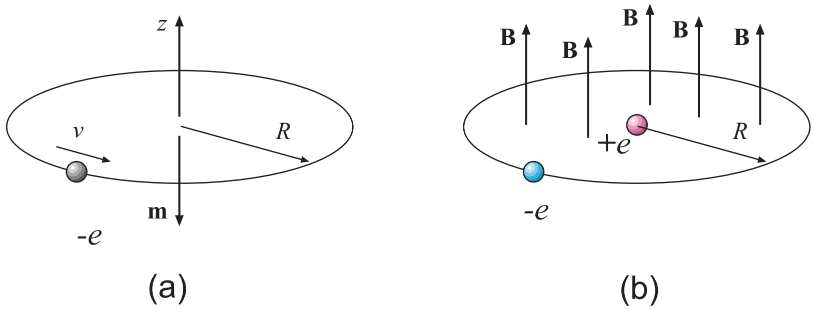

Figure 6: (a) Electron is in orbit with radius R and speed v. It gives a magnetic dipole $\mathbf{m}$. (b) Magnetic field $\mathbf{B}$ perpendicular to the orbit plane is turned on. R will be held constant. As a result, v and $\mathbf{m}$ change.

6.2 Torques and Forces on Magnetic Dipoles

Like the electric dipole, the magnetic dipole does not experience a (net) force when placed in a uniform magnetic field. But it does experience a torque, given by

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{125}$$

It does experience a force in a non-uniform field, given by

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{126}$$

With all of the similarities between magnetic dipoles and electric dipoles one might wonder if on the atomic scale the dipole might not be possibly (or better) described by two magnetic monopoles separted by a very small distance, with the proviso that for some reason the individual magnetic charges can never be separated. In fact one can get away with a model of this form if one wants, but in the end, it is no good; it is known that magnetic moments in matter are *not* generated this way; they really are made by the orbital motion of elementary charges or else by the intrinsic moments of certain particles.

6.3 A Simple ("Toy") Calculation

We consider a simplistic but revealing model of how an external magnetic field can induce a magnetic dipole moment for an orbiting electron. The model is shown in Fig. 6(a). Electron orbits a center of force at radius R and speed v. Initially the dipole moment is

$$\mathbf{m} = -\tfrac{1}{2} evR\hat{\mathbf{z}}$$

Now we introduce a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. We assume that the central force on the electron came from a charge $+e$ at the center and that the radius R stays constant. One can show that there will be an increase in speed of

$$\Delta v = \frac{eRB}{2m_e}$$

and as a result there will be a change in the magnetic moment of

$$\Delta \mathbf{m} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

which might seem odd because it in the opposite direction from the applied field $\mathbf{B}$. But that's what happens, and this example is a very simple model of what happens in diamagnetism.

6.4 Magnetization

Irregardless, yes *irregardless* of the size and direction of the magnetic dipoles $\mathbf{m}$ induced by the field, a chunk of matter influenced by a magnetic will have a magnetic moment per unit volume $\mathbf{M}$. Clearly, $\mathbf{M}$ is analogous to the electric polarization $\mathbf{P}$ in chapter 4.

As compared with the ferromagnetism, the magnetization of paramagnetic and diamagnetic materials is typically very weak.

6.5 Bound Currents

The $\mathbf{A}$ field from a pure magnetic dipole $\mathbf{m}$ located at $\mathbf{r}'$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\imath}}}{\imath^2}$$

so that the total vector potential from a magnetized object is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau' \tag{127}$$