Phys 3820, Fall 2009
Exam #2

1. In the homework you found the (weak-field) Zeeman effect on $n = 2$ states of H. Doing the same for *all* the $n = 3$ states would be much more work. Pick any two states with $n = 3$ and good n, l, j and describe what happens to these (initially) degenerate sets of states as a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is turned on.

The states of good j arising from the $n = 3$ states are:

$$3\text{s}: \quad \ell = 0, j = \tfrac{1}{2}$$

$$3\text{p}: \quad \ell = 1, j = \tfrac{1}{2}, \quad \ell = 1, j = \tfrac{3}{2}$$

$$3\text{d}: \quad \ell = 2, j = \tfrac{3}{2}, \quad \ell = 2, j = \tfrac{5}{2}$$

Considering the $\ell = 2, j = \frac{3}{2}$ case we have

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} = 1 + \frac{(3/2)(5/2) - 2(3) + 3/4}{2(3/2)(5/2)} = 1 + \frac{6/4}{30/4} = \frac{6}{5}$$

and as there are 4 values of m, $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, giving energy perturbations of

$$E_Z^1 = \mu_B g_J B_{\text{ext}} m_j$$

so that the energy perturbation plotted versus $\mu_B B_{\text{ext}}$ gives 4 lines with slopes of

$$\frac{6}{5}(-3/2) = -9/5, \qquad \frac{6}{5}(-1/2) = -3/5, \qquad +3/5 \qquad +9/5$$

Other cases are similar.

2. a) What basic physical effect gives the hyperfine splitting of the H atom?

This effect arises from the difference in energy when the spins of the proton and electron are parallel or anti-parallel. It splits the 1s state into two separate states. (The fine structure corrections did not do this.)

b) *Why* is the hyperfine splitting such a small correction to the H atom energies? (I. e. in comparison with fine structure.)

The fine structure correction includes the interaction of the electron's spin magnetic moment with the magnetic field it "feels" from the proton's relative motion. This magnetic field is much larger than the magnetic field arising from the proton's magnetic moment and *that* can be ascribed to the large mass of the proton in making its gyromagnetic ratio so small.

c) The ("famous") 21- cm line of astrophysics comes from an atomic transition between two states. What states are these?

These are the states of the H atom wherein the 1s states is perturbed due to the interaction of electron and proton spin magnetic moments. The states are the singlet and triplet states, i.e. where the spins are parallel and anti-parallel.

3. Consider a Gaussian potential well in one dimension,

$$V(x) = -V_0 e^{-\alpha x^2}$$

It's not clear how to find an exact solution to this problem! But it is possible to show that it definitely has a bound state.

We'd like the get an upper bound on the ground state energy. What would you suggest using for a variational wavefunction? *Hint: It might have a similar shape to the potential itself.* How would you go about finding a minimum upper bound to the ground state energy?

Choose a wave function of the form

$$\psi(x) = Ae^{-\beta x^2}$$

Normalize:

$$1 = \int |\psi|^2 \, dx = A^2 \int_0^\infty e^{-2\beta x^2} \, dx = A^2 2\sqrt{\pi} \left(\frac{1}{2\sqrt{2\beta}} \right)$$

This gives

$$A^2 \sqrt{\frac{\pi}{2\beta}} = 1 \qquad \Longrightarrow \qquad A = \left(\frac{2\beta}{\pi} \right)^{1/4}$$

Find $\langle T \rangle$; first,

$$T\psi = -\frac{\hbar^2 A}{2m} \frac{d}{dx} (-2\beta x) e^{-\beta x^2} == -\frac{\hbar^2 A}{2m} (-2\beta + 4\beta^2 x^2) e^{-\beta x^2}$$

Then we have

$$\begin{aligned}
\langle T \rangle &= -\frac{\hbar^2 A^2}{2m} \int_\infty^\infty (-2\beta + 4\beta^2 x^2) e^{-2\beta x^2} \, dx \\
&= -\frac{\hbar^2}{2m} \sqrt{\frac{2\beta}{\pi}} \left[-2\beta \cdot 2\sqrt{\pi} \left(\frac{1}{2\sqrt{2\beta}} \right) + 4\beta^2 \cdot 2\sqrt{\pi} 2 \left(\frac{1}{2\sqrt{2\beta}} \right)^3 \right] \\
&= \left[-2\beta + 4\beta^2 \frac{1}{4\beta} \right] = -\frac{\hbar^2}{2m} [-2\beta + \beta] = \frac{\hbar^2 \beta}{2m}
\end{aligned}$$

Find $\langle V \rangle$:

$$\begin{aligned}
\langle V \rangle &= -V_0 A^2 \int_{-\infty}^\infty e^{-\alpha x^2} e^{-2\beta x^2} \, dx = -V_0 A^2 \int_{-\infty}^\infty e^{-(\alpha + 2\beta x^2} \, dx \\
&= -V_0 A^2 \cdot 2\sqrt{\pi} \left(\frac{1}{2\sqrt{\alpha + 2\beta}} \right) \\
&= -V_0 \sqrt{\frac{2\beta}{\alpha + 2\beta}}
\end{aligned}$$

This gives an upper bound for the ground state energy

$$\langle H\rangle = \frac{\hbar^2\beta}{2m} - V_0\sqrt{\frac{2\beta}{\alpha+2\beta}}$$

which might be too messy to minimize without a computer (and values for the potential) but we would then compute:

$$\frac{d\langle H\rangle}{d\beta} = \frac{\hbar^2}{2m} - V_0\frac{1}{2}\sqrt{\frac{\alpha+2\beta}{2\beta}}\left(\frac{2}{\alpha+2\beta} - \frac{4\beta}{(\alpha+2\beta)^2}\right)$$

This gives:

$$\frac{\hbar^2}{2m} = \frac{V_0}{2}\sqrt{\frac{\alpha+2\beta}{2\beta}}\left(\frac{2\alpha}{(\alpha+2\beta)^2}\right)$$

which can obviously be simplified further but I think I'll leave it at this stage.

4. This problem is a bit open-ended in that it possibly can't be completed (reliably) but I want to see what you can do.

Consider a potential $V(r)$ in three dimensions given by

$$V(r) = \begin{cases} -V_0 & 0 \le r < a \\ 0 & a > r \end{cases}$$

where we are assured that V_0 is large enough that there is a bound state.

Work with the (hydrogenic) trial wavefunction

$$\psi(\mathbf{r}) = Ae^{-br}$$

and get as far as you can to get an upper bound for the energy of the ground state.

First, normalize the wave function (old stuff):

$$1 = \int |\psi|^2\, d^3\mathbf{r} = 4\pi A^2 \int_0^\infty r^2 e^{-2br}\, dr = 4\pi A^2 \cdot 2\left(\frac{1}{2b}\right)^3 = \frac{\pi}{b^3}$$

which gives

$$A = \sqrt{\frac{b^3}{\pi}}$$

Find $\langle V\rangle$:

$$\begin{aligned}
\langle V\rangle &= \int V(r)|\psi|^2\, d^3\mathbf{r} = 4\pi\frac{b^3}{\pi}(-V_0)\int_0^a r^2 e^{-2br} \\
&= (-4V_0b^3)\left(-\frac{1}{4b^3}\right)\left(-1+(1+ba+2a^2b^2)e^{-2ba}\right) \\
&= -V_0 - V_0(1+ba+2a^2b^2)e^{-2ba}
\end{aligned}$$

Find $\langle T \rangle$:

$$\begin{aligned}
T\psi(r) &= -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right] \\
&= -\frac{\hbar^2 A}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(-br^2e^{-br}\right)\right] \\
&= -\frac{\hbar^2 A}{2m}\frac{1}{r^2}(-2br+b^2r^2)e^{-br} = -\frac{\hbar^2 A}{2m}\left(\frac{-2b}{r}+b^2\right)e^{-br}
\end{aligned}$$

Then $\langle T \rangle$ is

$$\begin{aligned}
\langle T \rangle &= -\frac{\hbar^2}{2m}\frac{b^3}{\pi}(4\pi)\int_0^\infty(-2br+b^2r^2)e^{-2br}\,dr \\
&= -\frac{2\hbar^2b^3}{m}\left[-2b\left(\frac{1}{2b}\right)^2+b^2 2\left(\frac{1}{2b}\right)^3\right] \\
&= -\frac{2\hbar^2b^3}{m}\left[-\frac{1}{2b}+\frac{1}{4b}\right] = -\frac{2\hbar^2b^3}{m}\cdot\left(-\frac{1}{4b}\right) = +\frac{\hbar^2b^2}{2m}
\end{aligned}$$

The expression for $\langle H \rangle$ is now

$$\langle H \rangle = \frac{\hbar^2b^2}{2m} - V_0 - V_0(1+ba+2a^2b^2)e^{-2ba}$$

This indeed is an expression that would require numerics to minimize. I will leave it here.

5. In the variational calculation for the He atom, a trial wave function of the form

$$\psi(\mathbf{r}_1,\mathbf{r}_2) \equiv \frac{Z^3}{\pi a^3}e^{-Z(r_2+r_2)/a}$$

was chosen with Z as variational parameter.

Roughly what optimal value was found for Z and what is the basic phyiscal interpretation of that result?

The optimal Z was found to be about 1.7. The meaning that can be taken away from this is that both electrons "see" an effective nuclear charge of $+(1.7)e$ instead of $2e$ because for a significant part of the time the second electron is between the first one and the nucleus, in effect cancelling some of its charge.

6. Under what condition will a trial wave function ψ give an upper bound for the energy of the first excited state?

As we saw on one homework problem, if we can be assured that the trial wave function is orthogonal to the $true$ ground state of the system, then for this state $\langle H \rangle$ will be an upper bound for the energy of the first excited state(s).

7. Before even *beginning* the variational calculation for the H_2^+ ion, we had to make a very reasonable approximation for the system which sometimes is called the Born–Oppenheimer approximation.

What approximation is this, and why is it reasonable?

This is the approximation that we can treat the nuclei as being stationary and solve for the states of the electrons in the resulting (Coulombic) potential. It is reasonable since the nuclei have masses which are thousands of times larger that that of the electrons so the effects of their "recoil" is small. In other words, their motion is much slower and we can later treat their motion as governed by the effective potential arising from the solutions found for the electrons.

8. The WKB method is a method for finding wave functions for one-dimensional potentials. Under what circumstances is the WKB approximation *not* valid?

It is not valid for x such that the energy of the particle is comparable to $V(x)$. For those regions (near the classical "turning points") it is necessary to use the "connection formulae" which we were far too busy to cover this semester.

9. In the Gamow model of alpha emission, the WKB method produced a *probability* for transmission through the nuclear/Coulomb barrier. What crude model for the alpha's motion before the escape was used to convert this to a transition *rate* to get a lifetime?

We used the $crude$ model of an alpha particle bouncing back and forth in the potential well (along a diameter of the nucleus) moving with a constant speed v. Multiplying the bounce frequency by the transition $probability$ gives the frequency of escape of the alpha particle.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457\times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938\times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262\times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218\times 10^{-19}\ \mathrm{C} \qquad c = 2.99792\times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}\,dk \qquad \phi(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,dx$$

Delta Fn Potl: $\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$ $\qquad E=-\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x)=\frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}]\equiv\hat{A}\hat{B}=\hat{B}\hat{A} \qquad \Phi(p,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2\geq\left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p\geq\frac{\hbar}{2} \qquad \Delta E\Delta t\geq\frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi$$

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2}=-m^2\Phi \qquad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+[\ell(\ell+1)\sin^2\theta-m^2]\Theta=0$$

$$Y_0^0=\sqrt{\frac{1}{4\pi}} \qquad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \qquad Y_2^{\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi} \qquad Y_2^{\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm2i\phi}\text{etc.}$$

$$u(r)\equiv rR(r) \qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$$

$$a=\frac{4\pi\epsilon_0\hbar^2}{me^2}=0.529\times10^{-10}\text{ m} \qquad E_n=-\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}\equiv\frac{E_1}{n^2} \qquad \text{for}\quad n=1,2,3,\ldots$$

where $E_1=-13.6$ eV.

$$R_{10}(r)=2a^{-3/2}e^{-r/a} \qquad R_{20}(r)=\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-r/2a} \qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f=c \qquad E_\gamma=hf \qquad \frac{1}{\lambda}=R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right) \qquad \text{where} \qquad R=\frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2=1.097\times10^7\text{ m}^{-1}$$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p} \qquad [L_x,L_y]=i\hbar L_z \qquad [L_y,L_z]=i\hbar L_x \qquad [L_z,L_x]=i\hbar L_y$$

$$L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm=\pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s\ m\pm1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0\hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\,\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788 \times 10^{-5}\ \text{eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p \cdot \mathbf{S}_e \rangle = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \leq \langle \psi | H | \psi \rangle \equiv \langle H \rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{1}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v} e^{2\gamma}$$