Phys 3820, Fall 2011
Problem Set #2, Hint-o-licious Hints

1. *Griffiths QM*, **6.12** Whether we worked it or not, from Problem 4.40 (b) one has for the hydrogen atom the result

$$\langle T\rangle = -E_n \qquad \langle V\rangle = 2E_n$$

but the expectation value of $\frac{1}{r}$ is simply related to the expectation value of V:

$$\left\langle \frac{1}{r}\right\rangle = -\frac{4\pi\epsilon_0}{e^2}\langle V\rangle$$

2. *Griffiths QM*, **6.14** The relativistic perturbation to the Hamiltonian is

$$H' = -\frac{p^4}{8m^3c^2}$$

so we want to calculate

$$E^1_{n,\mathrm{rel}} = \left\langle n \left| -\frac{p^4}{8m^3c^2}\right| n\right\rangle$$

Use

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \qquad \Longrightarrow \qquad p^2 = -\frac{\hbar m\omega}{2}(a_+^2 + a_-^2 - a_+a_- - a_-a_+)$$

in

$$E^1_{n,\mathrm{rel}} = -\frac{1}{8m^3c^2}\langle p^2\psi^0_n|p^2\psi^0_n\rangle$$

and use orthogonality of the HO wave functions. When this is used in the formula for E^2_n there are only two terms in the sum and you get

$$E^1_{n,\mathrm{rel}} = -\frac{3\hbar^2\omega^2}{32mc^2}(2n^2 + 2n + 1)$$

3. *Griffiths QM*, **6.20** Use $|\mathbf{L}| \approx \hbar$ in (6.59) With this, the critical size of the B field is around 12 T.

4. *Griffiths QM*, **6.21** The fine structure is larger than the Zeeman contribution to the energy. The zero-field value of the energies is given by (6.67). Since it included the spin-orbit splitting, the energies depend on j (and n).

Now, for $n = 2$ we have the states $l = 0$ and $l = 1$. We must have states of "good" j, so we note that the $l = 0$ state is a $j = \frac{1}{2}$ states while the $l = 1$ state give $j = \frac{1}{2}$ and $j = \frac{3}{2}$.

Calculate the Landé g factor g_J for each state note, it depends on j and l, and then the weak–field Zeeman energy is

$$E^1_Z = \mu_B g_J B_{\mathrm{ext}} m_j$$

If you plot E vs. $\mu_B B_{\mathrm{ext}}$, the slope of the line is $g_J m_j$.

For the $j = \frac{1}{2}$ states we then have a pair for the state that came from $l = 0$ (with $m_j = \pm\frac{1}{2}$) and a pair for the state that came from $l = 1$. The $j = \frac{3}{2}$ state came from $l = 1$ and with $m_j = -\frac{3}{2} \ldots \frac{3}{2}$, there are four lines with their own slopes.

5. *Griffiths QM*, **6.29** The perturbation is

$$H' = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

and use this in

$$E^1_{\rm gs} = \int \psi^*(\mathbf{r}) H' \psi(\mathbf{r}) d^3\mathbf{r}$$

You can approximate the exponential as 1 to get a lowest–order answer, and that all we need to check the order of magnitude of the result. With this, show

$$\frac{E^1_{\rm gs}}{|E^0_{\rm gs}|} = \frac{4}{3}\left(\frac{b}{a}\right)^2$$

Compare with value with those from fine structure and the hyperfine splitting! (Use Table 6.1).

6. *Griffiths QM*, **7.1** (a) With $\psi(x) = Ae^{-bx^2}$ for the linear potential $V(x) = \alpha|x|$ I get

$$\langle H \rangle = \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2\pi b}}$$

which has a minimum at

$$b = \left[\frac{m^2\alpha^2}{2\pi\hbar^4}\right]^{1/3}$$

and give a bound of

$$\langle H \rangle_{\rm min} = \frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{1/3}$$

(b) For $V(x) = \alpha x^4$ I get

$$\langle H \rangle = \frac{\hbar^2 b}{2m} + \frac{3\alpha}{16b^2}$$

which has a minimum at

$$b = \left(\frac{3m\alpha}{4\hbar^2}\right)^{1/3}$$

and gives a bound

$$\langle H \rangle_{\rm min} = \frac{3}{2}\left(\frac{3\alpha\hbar^2}{4m^2}\right)^{1/3}$$

7. *Griffiths QM*, **7.4** Proof of the theorem goes as described in class; with the trial function ψ expanded as

$$\psi = \sum_{n=0}^{\infty} c_n \psi_n \qquad \text{where} \qquad H\psi_n = E_n\psi_n$$

and $n = 0$ stands for the non-degenerate ground state and $n = 1$ stands for any one of the states of the first excited energy. Find the condition on the c_n's implied by normalization.

From $\langle\psi|\psi_0\rangle = 0$ show that $c_n = 0$ then show

$$\langle H\rangle = \sum_{n=1} |c_n|^2 E_n \geq E_1$$

For the trial function of the form

$$\psi(x) = Ae^{-bx^2}$$

show that normalization gives

$$A^2 = \sqrt{\frac{32b^3}{\pi}}$$

and then after lots of careful algebra

$$\langle T\rangle = \frac{3A^2\hbar^2}{8m}\sqrt{\frac{\pi}{2b}} \qquad \langle V\rangle = \frac{A^2 m\omega^2}{b^2}\frac{3}{32}\sqrt{\frac{\pi}{2b}}$$

These lead to

$$\langle H\rangle = \frac{3}{2}\left(\frac{\hbar^2 b}{m} + \frac{m\omega^2}{4b}\right)$$

and minimizing this with respect to b gives the minimal value of $\langle H\rangle$ hence an upper bound on the first excited state. Of course, you get the exact answer as you can understand from Example 2.4.