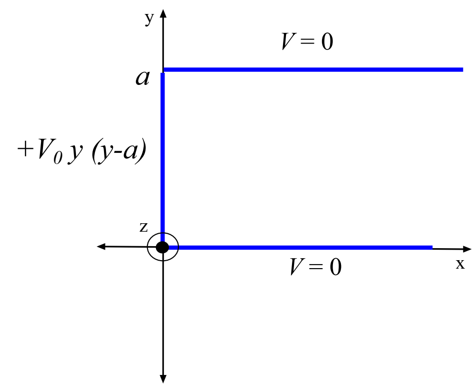
Phys 4610, Fall 2007
Exam #2



1. We would like to solve the 2-dimensional potential problem (no z–dependence) shown at the right. It is similar to the one which introduced the separation method in the text, the "slot" problem. Here, the "hot" side of the slot ($x = 0$)is (somehow) held at a y-dependent potential

$$V(0, y) = V_0 y(y - a)$$

We want a solution for the potential everywhere inside the tube. We do this by separation of variables with a sum of solutions. I will begin the solution, you finish up all the steps.

Inside the slot, the solution must be of the form

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

a) As much as you are able, justify why the solution for $V(x, y)$ must be of this form.

First, $V(x, y)$ must satisfy $\nabla^2 V = 0$ inside the slot. *Each* term of the expansion does this, since

$$\nabla^2[C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)] = \left[\left(\frac{n\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2\right] C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) = 0$$

Also, from the $\sin(n\pi y/a)$ factor, $V(x, y)$ is zero at $y = 0, a$. And from the exponential factor, $V \to 0$ as $x \to \infty$. It remains to determine the C_n's so that the boundary condition at $x = 0$ is satisfied.

b) Apply the last boundary condition and find the coefficients C_n. (Go as far as you have time; if there is an integral you can't work out, at least show clearly what you are doing.) Can you tell if any set of the C_n's are zero?

At $x = 0$ we have

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0 y(y - a)$$

Multiply both sides by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = V_0 \int_0^a y(y - a) \sin\left(\frac{m\pi y}{a}\right) dy \qquad (1)$$

Here we can use the relation given on the exam,

$$\int_0^a \sin(n\pi y/a) \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

As for the right hand side of 1, the integral is a little messy, but using tables we get:

$$V_0 \int_0^a (y^2 - ay)\sin\left(\frac{m\pi y}{a}\right) = V_0\left[-a\left(\frac{1}{\alpha^2}\sin\alpha y - \frac{y}{\alpha}\cos\alpha y\right) + \frac{2y}{\alpha^2}\sin\alpha y - \frac{(\alpha^2 y^2 - 2)}{\alpha^3}\cos\alpha y\right]\Bigg|_0^a \tag{2}$$

where $\alpha = \frac{m\pi}{a}$.

Now, $\sin\alpha a = 0$ and $\cos\alpha a = \cos m\pi = (-1)^m$. Use these in 2 and get

$$\begin{aligned}
\Rightarrow \quad &= V_0\left[\frac{a^2}{\alpha}(-1)^m - 0 - \frac{(\alpha^2 a^2 - 2)}{\alpha^3}(-1)^m + \frac{2}{\alpha^3}(1)\right] \\
&= V_0\left[\frac{a^2}{\alpha}(-1)^m - \frac{a^2}{\alpha}(-1)^m + \frac{2}{\alpha^3}[(-1)^m - 1]\right] = \frac{2V_0}{\alpha^3}[(-1)^m - 1] \\
&= \begin{cases} 0, & \text{if } m \text{ even} \\ -\frac{4V_0}{\alpha^3} & \text{if } m \text{ odd} \end{cases} = \begin{cases} 0, & \text{if } m \text{ even} \\ -\frac{4V_0 a^3}{m^3\pi^3} & \text{if } m \text{ odd} \end{cases}
\end{aligned}$$

Put all of this into 2 and get

$$\sum_{n=1}^{\infty} C_n \frac{a}{2}\delta_{nm} = \frac{a}{2}C_m = \begin{cases} 0, & \text{if } m \text{ even} \\ -\frac{4V_0 a^3}{m^3\pi^3} & \text{if } m \text{ odd} \end{cases}$$

so for even m we have

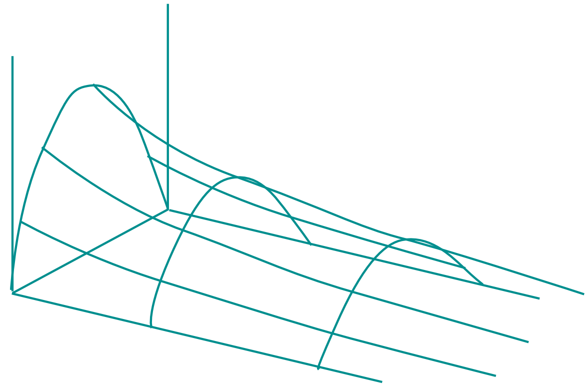
$$C_m = -\frac{8V_0 a^2}{m^3\pi^3}$$

and the full solution is

$$V(x,y) = -\frac{8V_0 a^2}{\pi^3}\sum_{m \text{ odd}} \frac{1}{m^3} e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)$$

c) Give you best sketch of what you predict the (3D) graph of $V(x,y)$ will look like.

Choosing the boundary function as positive (V_0 negative) the potential is an inverted parabola on the segment at $x = 0$, dies off at infinity and is smooth and dull everywhere. My picture is:



2. The potential on a spherical surface (radius R, centered on the origin) is given by

$$V(\theta) = V_0 \sin^2\theta$$

Find the potential $V(r,\theta)$ everywhere *outside* the sphere. (You don't need to do the inside.)

Relations involving the Legendre polynomials are found elsewhere on the exam.

Outside the sphere, the potential has the form

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

We need to solve for B_l. Note that at $r = R$, $V(R,\theta) = V_0 \sin^2\theta$. Also,

$$V_0 \sin^2\theta = V_0(1 - x^2), \qquad \text{with} \quad x = \cos\theta$$

and

$$\tfrac{2}{3}P_2(x) = \frac{2}{3}\left(\frac{3}{2}x^2 - \tfrac{1}{2}\right) = x^2 - \tfrac{1}{3}$$

so $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$, and

$$V_0(1-x^2) = V_0\left[1 - \frac{2}{3}P_2(x) - \frac{1}{3}\right] = V_0\left[\frac{2}{3}P_0(x) - \frac{2}{3}P_2(x)\right]$$

Match coefficients of the $P_l(x)$'s:

$$\frac{B_0}{R} = \frac{2}{3}V_0 \qquad \Longrightarrow \qquad B_0 = \frac{2RV_0}{3}$$

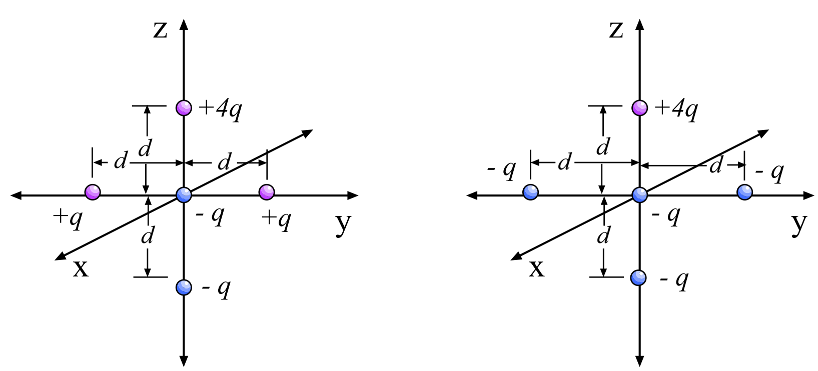
$$\frac{B_2}{R^3} = -\frac{2}{3}V_0 \qquad \Longrightarrow \qquad B_2 = -\frac{2R^3V_0}{3}$$

and all the other B_l's are zero. Then:

$$V(r,\theta) = \frac{2RV_0}{3r} - \frac{2R^3V_0}{3r^3}P_2(\cos\theta)$$

3. Shown here are two configurations of point charges. For each, give an expression for the potential at long distance ($r \gg d$), keeping only the leading order behavior (i.e. with a single power of r).

Be careful. There is a very *slight* trick here.



(a) (b)

(a) The charges in (a) add up to $+4q$ (not zero) so at large distances we "see" the monopole field,

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{(+4q)}{r} = \frac{q}{\pi\epsilon_0 r}$$

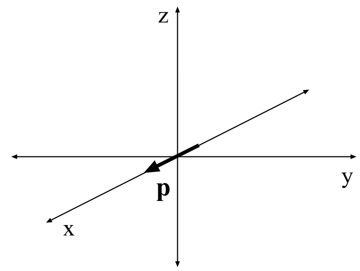
(b) The charges in (b) *do* add up to zero and the distribution has a dipole moment, since (watch the signs!)

$$\sum q_i \mathbf{r}_i = \hat{\mathbf{z}}(4qp + qd) = 5qd\hat{\mathbf{z}}$$

Then at large distances,

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{5qd\cos\theta}{r^2}$$

4. A point dipole $\mathbf{p}$ is located at the origin and is directed along the $+x$ axis.



a) Find the force on a charge q at the point $(-a,0,0)$.

We have a formula for $\mathbf{E}$ if the dipole points along the $\hat{\mathbf{z}}$ axis. At $\mathbf{r} = -a\hat{\mathbf{z}}$, *that* formula would give

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}[2(-1)\hat{\mathbf{r}}] = \frac{-p\hat{\mathbf{r}}}{2\pi\epsilon_0 a^3}$$

so *here* the field at the given point is

$$\mathbf{E} = \frac{+p\hat{\mathbf{x}}}{2\pi\epsilon_0 a^3}$$

and the force on the charge is

$$\mathbf{F} = \frac{+pq\hat{\mathbf{x}}}{2\pi\epsilon_0 a^3}$$

b) Find the force on a charge q at the point $(0,0,a)$.

Likewise at $\mathbf{r} = a\hat{\mathbf{z}}$ (here) we can use the formula for the dipole along $\hat{\mathbf{z}}$ with $\theta = \pi/2$ which would give

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{\theta}}$$

so the field *here* is

$$\mathbf{E} = -\frac{p}{4\pi\epsilon_0 a^3}\hat{\mathbf{x}}$$

and the force on the charge is

$$\mathbf{F} = -\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{x}}$$

c) Find the work required to move the charge a from $(-a,0,0)$ to $(0,0,a)$.

At $(-a, 0, 0)$ use $V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$ and get

$$V = -\frac{p}{4\pi\epsilon_0 a^2}$$

At $(0, 0, a)$ the same formula gives $V = 0$. The change in potential energy of the charge is

$$\Delta U = q\Delta V = \frac{qp}{4\pi\epsilon_0 a^2}$$

so the *work required* to move the charge is

$$W = \frac{qp}{4\pi\epsilon_0 a^2}$$

5. In chapter 4 we distinguish distributions of *bound charge* and *free charge*.

What is the difference between the two types of charge and what is the point (i.e. the utility) of making the distinction?

Bound charge, while certainly a true electric charge refers to the charge density that arises from the polarization of a dielectric material. All other charge is counted as free charge. The separation is useful because we can ofter control the amount of charge we place on a conductor (for example) whereas we can't directly control the charge densities that arise in dielecrics in response to external fields.

6. A cube of side a centered at the origin; the normals to its faces point along the x, y and z axes. It carries a uniform frozen–in polarization $\mathbf{P} = P\hat{\mathbf{z}}$.

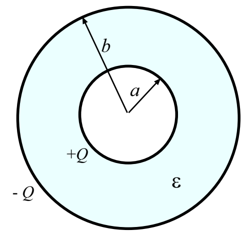
a) What are the bound charges densities σ_b and ρ_b for this system?

With $\mathbf{P} = P\hat{\mathbf{z}}$ (uniform), $\rho_b = -\nabla\cdot\mathbf{P} = 0$ and $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ so that:
$\sigma_b = +P$ on the upper face
$\sigma_b = -P$ on the lower face and
$\sigma_b = 0$ on all other faces.

b) How would you go about find the electric field at points along (say) the z axis? (How would you compute it, in principle?)

The problem is now reduced to finding teh E field due to a uniform square of charge of charge. For points on the axis of symmetry, this problem is easy to set up and in fact gives an answer in "closed form", as shown in one of the book's exercises.

7. A spherical capacitor is formed from concentric conducting spherical shells of radius a and b with the space in between completely filled with a dielectric of dielectric constant ϵ.



Suppose we put a charge $+Q$ on the inner conductor and a charge $-Q$ on the outer conductor. Let's calculate everything:

a) Find the $\mathbf{D}$ field at all points in space. The revised form of Gauss' law will help here? (Hint: The charge put on the conductors is the *free* charge.

Use Gauss' law for macroscopic media, $\oint \mathbf{D} \cdot d\mathbf{a} = q_{free\ encl}$. Then inside the small sphere, by our usual arguments, $\mathbf{D} = 0$. For $a < r < b$,

$$\oint \mathbf{D} \cdot d\mathbf{a} = 4\pi r^2 D_r = +Q$$

so $D_r = Q/(4\pi r^2)$. For $r > b$, $D_r = 0$. So:

$$D_r = \begin{cases} 0, & \text{if } r < a \\ \frac{Q}{4\pi r^2} & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

b) Find the $\mathbf{E}$ field at all points in space.

Since $\mathbf{E} = \mathbf{D}/\epsilon$ (with ϵ appropriate for the medium; $\epsilon = \epsilon_0$ for vacuum) then

$$E_r = \begin{cases} 0, & \text{if } r < a \\ \frac{Q}{4\pi \epsilon r^2} & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

c) Find the $\mathbf{P}$ field at all points in space and find σ_b and ρ_b.

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} = \epsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{P} = \frac{(\epsilon - \epsilon_0)}{\epsilon} \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \qquad \text{for} \quad a < r < b$$

d) Find the potential difference between the conducting shells.

$$\begin{aligned} \Delta V &= V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{r} = -\frac{Q}{4\pi\epsilon} \int_a^b \frac{1}{r^2} dr \\ &= \frac{Q}{4\pi\epsilon} \frac{1}{r}\Big|_a^b = \frac{Q}{4\pi\epsilon}\left(\frac{1}{b} - \frac{1}{a}\right) = -\frac{Q}{4\pi\epsilon} \frac{(b-a)}{ab} \end{aligned}$$

The absolute value of the voltage difference is

$$V = \frac{Q(b-a)}{4\pi\epsilon ab}$$

e) If the potential difference is V then the capacitance of this device is $C = Q/V$. What is the capacitance of this device?

With

$$V = \frac{Q}{C} = \frac{(b-a)Q}{4\pi\epsilon ab}$$

then

$$C = \frac{4\pi\epsilon ab}{b-a}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{3}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{4}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{5}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{6}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + sd\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{7}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{8}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{9}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{10}$$

Product Rules:

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Double Derivatives:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

More Math:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\mathscr{r})$$

If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3-3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

Physics:

$$\epsilon_0 = 8.854 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N{\cdot}m}^2} \qquad \mu_0 = 4\pi \times 10^{-7}\,\frac{\mathrm{T{\cdot}m}}{\mathrm{A}} \qquad c = 2.998 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V$$

$$E^{\perp}_{\text{above}} = E^{\perp}_{\text{below}} \qquad \mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

$$V = -\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' \qquad W = \frac{1}{2}\sum_{i=1}^{n}q_i V(\mathbf{r}_i) \qquad W = \frac{\epsilon_0}{2}\int_{\text{all space}}E^2\,d\tau$$

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \qquad Q = CV \qquad C = A\frac{\epsilon_0}{d} \qquad W = \tfrac{1}{2}CV^2$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\text{enc}}$$