Phys 3820, Fall 2012
Exam #3

1. Give concise but *careful* definitions of:

a) Impact parameter, b.

In classical scattering, a particle comes in along a straight line at large distance. The distance between this line and a parallel line passing through the center of force is the impact parameter.

The number loses strict meaning in quantum scattering.

b) Total cross section.

If we have the differential cross section $D(\theta)$ at all angles θ, then the integral over all solid angles gives the total cross section σ:

$$\sigma = \int D(\theta) d\Omega$$

c) Rutherford cross section.

This is the formula for the cross section for scattering of a point charge q_1 from a point charge q_2. It turns out to be the same whether calculated classically or with quantum mechanics, one of the fortunate accidents of physics.

Dependence on angle is given by a factor of $[\sin(\theta/2)]^{-4}$

d) Luminosity (of beam).

The number of particles in the stream of particles from the accelerator per unit time and area. (Particle flux.)

2. We began the study of scattering with the wave function

$$\psi(r, \theta) \approx A \left\{ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right\}$$

Though we did not arrive at this form in a rigorous way, say a few words as to why this is a *physically reasonable* form for a time-independent solution of the scattering problem.

The first term is the wave function for a free particle with momentum k in the z direction. The second term describes a wave propagating radially outward, with a strength which depends on θ and with an appropriate factor of r to get the probability flow correct.

Neither term has anything to do with the fact that in real experiments particles are produced in a beam. The wave function applies to a *single particle*.

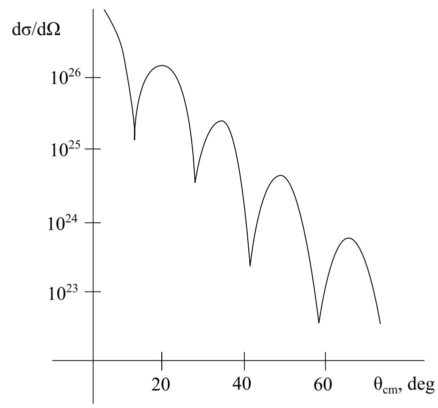
3. All through Chapter 11 we assumed that that scattering amplitude was independent of ϕ (the angle about the z axis). For what kind of scattering experiment might we have a cross section which is dependent on ϕ?

If the projectiles or target have spin, either or both can be polarized in a direction perpendicular to the scattering axis. Then the amplitude would have a ϕ dependence.

4. Suppose you have a mono-energetic beam of particles hitting a stationary target (with very massive particles) and a particle detector of certain specific dimensions. Summarize the steps (basic measurements and basic calculations) you would need to go through to find the *measured* differential cross section $D(\theta) = \frac{d\sigma}{d\Omega}$ at each angle.

Measure the luminosity of the beam (particles per area per time), measure number of scattering centers per area of the target, measure the number of particles per time entering a certain solid angle $d\Omega$ at angle (θ, ϕ). Take the ratio of the latter to the product of the first two and this will give a quantity with units of area (per solid angle), and that is $D(\theta)$.

5. What's wrong with the cross section plot shown at the right?



Units?
Units?

6. What property of the Coulomb interaction makes it poorly behaved for the purposes of doing an elementary quantum scattering calculation? (Your answer should not be too glib and should mention terms in the radial Schrödinger equation and its solutions.)

It falls off as $1/r^2$ and that is not fast enough to be valid in our formalism.

The problem arises from the radial Schrödinger equation at large r. We have the centrifugal potential $\hbar^2 l(l+1)/(2mr^2)$ which already falls off as $1/r^2$ and has the solutions of the spherical Bessel functions j_l and n_l. Keeping a term which is not small compared to this invalidates the procedure, and we need to look for other large-distance solutions which are the *Coulomb wave functions*, material for graduate school.

7. Last semester you (numerically) solved for the bound states in a certain radial potential $V(r)$; possibly you found the result for a Woods-Saxon potential, which is not possible analytically.

Describe how —in principle– you would write a computer program (or *use* a computer) to solve for the differential cross section for the scattering of particles of energy E from a Woods-Saxon potential.

For given experiment parameters, loop on partial wave l and calculate a functions proportional to u_l: Start at the origin with $u = 0$ and u' arbitrary and use the Schrödinger equation to step outward and find u at a distance large compared with the range of the potential. (Step size should be small compared to the size R of the Woods-Saxon potential.)

At this large distance, express u as a linear combination of the spherical Bessel functions using the slope and derivative of u. From the ratio of the coefficients one can deduce the partial wave amplitudes a_l or likewise the phase shifts δ_l.

Having all the a_l's (or δ_l's) necessary for convergence, use the summation formulae to get the values of $f(\theta)$ and $D(\theta)$.

8. Find the scattering amplitude in Born approximation for a central potential of the form:

$$V(r) = V_0 \frac{e^{-\alpha r^2}}{r}$$

(Don't assume low energy.)

As usual, do the math as far as you can– though you may be able to come up with a closed form for this one.

The Born approximation for $f(\theta)$ for this potential gives:

$$f(\theta) = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty rV(r)\,\sin(\kappa r)\,dr = -\frac{2mV_0}{\hbar^2\kappa} \int_0^\infty e^{-\alpha r^2} \sin(\kappa r)\,dr$$

The expression is fairly simple, but you may need Jonathan Wheeler's calculator or a good table of integrals to get a closed form. I used the Big Russian Book of Integrals, and then this becomes:

$$f(\theta) = -\frac{2mV_0}{\hbar^2\kappa}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left(1 - e^{-\kappa^2/\alpha}\right) = -\frac{mV_0}{\hbar^2\kappa}\sqrt{\frac{\pi}{\alpha}}\left(1 - e^{-\kappa^2/\alpha}\right)$$

and I don't know if there is much more to say about the result.

9. If we want to solve the differential equation

$$\nabla^2 u(\mathbf{r}) - \lambda^2 u(\mathbf{r}) = f(\mathbf{r})$$

It turns out that the Green function to use is

$$G(|\mathbf{r}-\mathbf{r}'|) = -\frac{e^{-\lambda|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad \text{or equivalently} \qquad G(r) = -\frac{e^{-\lambda r}}{4\pi r}$$

a) What differential equation is satisfied by this Green function?

It is the same as the original DE with the rhs replaced by $\delta^3(\mathbf{r})$:

$$(\nabla^2 - \lambda^2)G(\mathbf{r}) = \delta^3(\mathbf{r})$$

b) Show that this Green function *does* satisfy this DE.

Evaluate $\nabla^2 G(r)$:

$$\nabla^2 G(r) = \nabla\cdot\nabla\left[-\frac{e^{-\lambda r}}{4\pi r}\right] = \nabla\cdot\left(+\frac{\lambda e^{-\lambda r}\hat{\mathbf{r}}}{4\pi r} + \frac{e^{-\lambda r}\hat{\mathbf{r}}}{4\pi r^2}\right)$$

Here we note that the second term does not produce a delta function; we just put an r^2 downstairs with a minus sign which does indeed blow up at the origin but is *not* zero at other points.

The divergence working on these terms requires some care. Again, the first term differentiates in the normal way but the second term is a product of a nice function $e^{-\lambda r}$ and one whose divergence gives a delta function from

$$\nabla\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

This only applies at $r = 0$ and at that point the nice function is $e^{-\lambda r} = 1$. So for the second term apply the product rule, recognizing that the divergence applied to the other factor gives a delta function. Then we get

$$\begin{aligned}\nabla^2 G(r) &= \frac{-\lambda^2 e^{-\lambda r}}{4\pi r} + \frac{\lambda e^{-\lambda r}}{4\pi r^2} - \frac{\lambda e^{-\lambda r}}{4\pi r^2} + \frac{1}{4\pi}(4\pi\,\delta^3(\mathbf{r})) \\ &= \frac{-\lambda^2 e^{-\lambda r}}{4\pi r} + \delta^3(\mathbf{r}) \;=\; \lambda^2 G(r) + \delta^3(\mathbf{r})\end{aligned}$$

from which we do indeed get

$$\nabla^2 G(r) - \lambda^2 G(r) = \delta^3(\mathbf{r})$$

c) Write down the solution for $u(\mathbf{r})$ for a given source function $f(\mathbf{r})$.

Multiply the source function by the Green function (now written as a function of $\mathbf{r} - \mathbf{r}'$ and integrate:

$$u(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}')f(\mathbf{r}'\,d^3r' = -\frac{1}{4\pi}\int \frac{e^{-\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}f(\mathbf{r}'\,d^3r'$$

d) Where did you see the function $G(r)$ before? What name did we previously give it?

The Green function for this differential equation is the same as the screened Coulomb potential from class and on the last exam. We can note that without the λ^2 term we just have the Laplace equation for which the Green function is the Coulomb potential. Apparently the λ^2 term produces a sharper falloff of the potential which, as we were given in the book, would correspond to a photon mass.

10. Summarize how we started with the integral form of the Schrödinger equation, which Griffiths wrote as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0$$

and then got the *first* Born approximation. Be as specific with the math as you can.

What was done was to substitute $\psi_0(\mathbf{r}_0)$ for $\psi(\mathbf{r}_0)$ inside the integral in the last term. While $\psi(\mathbf{r}_0)$ is the *full* solution to the Schrödinger equation, $\psi_0(\mathbf{r}_0)$ is just the incoming plane wave, namely

$$\psi_0(\mathbf{r}_0) = Ae^{ikx} = Ae^{i\mathbf{k}'\cdot\mathbf{r}} \qquad \text{where} \qquad \mathbf{k}' = k\hat{\mathbf{z}}$$

By using an approximation suitable for $r \gg r'$ this could be put into the simpler form for the Born approximation given in the text.

11. Give two physical properties of electrons that were predicted by the Dirac equation.

The Dirac equation predicted that there should exist particles with the same mass as electrons but having a charge of $+e$. The spin property of the electron is contained in the multiple components and the Dirac equation gives the proper gyromagnetic ratio (before the relatively tiny QED correction) and the spin-orbit interaction.

12. How does the hydrogen atom wave function in the Dirac theory differ from that of the (simplest) Schrödinger theory? Be as specific as you can.

The Dirac wave function has *four components*; in the usual solution, the top two components are different from the lower two in that they have different radial functions.

For the angular part, the wave functions are *not* eigenstates of $\mathbf{L}^2$ or L_z or of S_z but rather are eigenstates of $\mathbf{J}^2$ and J_z. So the respective angular parts are combinations of spherical harmonics and our old Pauli spinors which combine different Y's and χ's to give eigenfunctions of the same J and J_z.

13. In *either* of the relativistic wave equations, how do we change the equations for a *free* particle of charge q to the case where it is moving in a region where there is a scalar potential ϕ and vector potential $\mathbf{A}$?

We replace the energy E by $E - q\phi$ and the momentum $\mathbf{p}$ by

$$\mathbf{p}_{\mathrm{can}} = \mathbf{p}_{\mathrm{kin}} - \frac{q}{c}\mathbf{A} \ .$$

That is, the ∇ operator now produces $\mathbf{p}_{\mathrm{can}}$, not $\mathbf{p}_{\mathrm{kin}}$. For the Schrödinger equation we then recognize that we want a (non-relativistic) kinetic energy operator but this is no longer given by ∇^2, but rather by

$$T_{\mathrm{op}} = \frac{1}{2m}(\mathbf{p}_{\mathrm{can}} + \frac{q}{c}\mathbf{A})^2 = \frac{1}{2m}(\frac{\hbar}{i}\nabla + \frac{q}{c}\mathbf{A})^2$$

which gives lots of interesting terms involving $\mathbf{A}$ and the ∇ operator.

14. What mathematical feature of quantum mechanicical "motion" in a vector potential did Aharonov and Bohm point out? Be as clear as you can and tell what hypothetical experiment could be done (or *was* done, actually) to verify their observation.

They pointed out that one could have (quantum) particle motion *only* in a region where the B field is zero but the vector potential is non-zero and the motion will still be influenced by the presence of the magnetic field. Thus the vector potential is truly the fundamental field as far as quantum mechanics is concerned, and the ambiguity in $\mathbf{A}$ as far as the gauge choice is concerned does not influence the physics.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\,a^{n+1} \qquad \int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

$$\int_0^\infty x^{2n}e^{-x^2/a^2}\,dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1}e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2}$$

$$\nabla^2 = (\nabla\cdot\nabla)_{\mathrm{op}} \qquad \nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}) \qquad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \cdots \qquad \nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \cdots$$

Numbers

$$\hbar = 1.05457\times10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938\times10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262\times10^{-27}\ \mathrm{kg}$$

$$e = 1.60218\times10^{-19}\ \mathrm{C} \qquad c = 2.99792\times10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty}\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}\text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529\times 10^{-10}\ \text{m} \qquad E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2}e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-r/2a} \qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = 1.097\times 10^7\ \text{m}^{-1}$$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m\pm 1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} ae^{-iE_+ t/\hbar} \\ be^{-iE_- t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6 \text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788\times 10^{-5} \text{ eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle\mathbf{S}_p\cdot\mathbf{S}_e\rangle = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \leq \langle\psi|H|\psi\rangle \equiv \langle H\rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\pm\frac{1}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v}e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar}$$

$$H'_{ab} = V_{ab} \cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$\mathbf{p} \equiv q\langle \psi_b | \mathbf{r} | \psi_a \rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\mathbf{p}| E_0}{\hbar} \right)^2 \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0 \hbar^2} |\mathbf{p}|^2 \rho(\omega_0) \qquad A = \frac{\omega^3 |\mathbf{p}|^2}{3\pi \epsilon_0 \hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1; \text{or } 0$ and $\Delta l = \pm 1$

$$d\sigma = D(\theta)\, d\Omega \qquad D(\theta) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| \qquad \sigma = \int D(\theta)\, d\Omega \qquad D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

$$\psi(r,\theta) \approx A \left\{ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right\} \qquad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \qquad D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)} \right]^2 \qquad -\frac{\hbar^2}{2m} \frac{d^2 u_l}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u_l = E u_l$$

$$\text{Large } r\text{:} \quad \frac{d^2 u_l}{dr^2} - \frac{l(l+1)}{r^2} u_l = -k^2 u_l \qquad u_l = A r j_l(kr) + B r n_l(kr)$$

$$\psi(r,\theta) = A \sum_{l=0}^{\infty} i^l (2l+1) \left[j_l(kr) + ik a_l h_l^{(1)}(kr) \right] P_l(\cos\theta)$$

$$a_l = \frac{1}{2ik} \left(e^{2i\delta_l} - 1 \right) = \frac{1}{k} e^{i\delta_l} \sin(\delta_l) \qquad f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin(\delta_l)\, P_l(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2(\delta_l)$$

$$(\nabla^2 + k^2) G(\mathbf{r}) = \delta^3(\mathbf{r}) \qquad \Longrightarrow \qquad G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}$$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0) \psi(\mathbf{r}_0)\, d^3\mathbf{r}_0$$

$$E^2 = c^2 \mathbf{p}^2 + m^2 c^4 \qquad (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m c^2) \psi = E\psi$$