Phys 2920, Spring 2011
Problem Set #2

1. If

$$\mathbf{a} = 5.62\,\hat{\mathbf{i}} - 3.58\,\hat{\mathbf{j}} + 9.41\,\hat{\mathbf{k}} \qquad \text{and} \qquad \mathbf{b} = -8.92\,\hat{\mathbf{i}} + 6.77\,\hat{\mathbf{j}} + 2.11\,\hat{\mathbf{k}}$$

find $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ using Maple or another computer–math system. (You can just include a printout of the result, or explain what you typed and what it gave.)

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2. a) Show that the set of vectors

$$\mathbf{e}'_1 = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \qquad \mathbf{e}'_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{j}} - \hat{\mathbf{k}}) \qquad \mathbf{e}'_3 = \frac{1}{\sqrt{6}}(-2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

is orthonormal and thus can serve as a basis for 3D vectors.
b) Express the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ in terms of the vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$ and $\mathbf{e}'_3$.
c) If we denote our usual unit vectors by

$$\mathbf{e}_1 = \hat{\mathbf{i}} \qquad \mathbf{e}_2 = \hat{\mathbf{j}} \qquad \mathbf{e}_3 = \hat{\mathbf{k}}$$

Express your answer in (b) as a matrix/vector relation of the form

$$\mathbf{e}_i = \sum_{j=1}^{3} \mathsf{A}_{ij}\mathbf{e}'_j$$

That is, find the matrix A here.

As an example of an abstract vector space (with an inner product), and a suitable basis, we will consider functions defined on a certain finite interval in x. In such a vector space, "adding the vectors" amounts to just adding the functions, with scalar multiplication being just simple multiplication. For each, the basis is a set of functions indexed by n, $f_n(x)$.

The inner product of two "vectors" is defined as the *integral of the product* of the two functions. (The integral is taken over the appropriate interval in x.)

3. Consider the set of functions

$$f_n(x) \equiv \sqrt{2}\sin(n\pi x) \qquad n = 1, 2, 3\ldots \qquad \text{on the interval} \quad 0 \le x \le 1$$

That is, the basis of this vector space is

$$\hat{\mathbf{e}}_n = f_n(x) = \sqrt{2}\sin(n\pi x)$$

a) Show that these basis "vectors" are orthonormal. As stated, the definition of the inner product here is

$$\langle m|n\rangle = \int_0^1 f_m(x) f_n(x)\,dx$$

b) Any smooth function which is zero at both $x = 0$ and $x = 1$ can be expressed as a vector in this space. Express the function

$$f(x) = 2\sin^3(\pi x)$$

as a linear combination of the basis functions (vectors). You just need to get the coefficients and at very least you can find them from

$$c_n = \langle n|f\rangle$$

4. A second example of a set of functions which can be treated as basis vectors can be gotten from the *Legendre polynomials*, a set of polynomials indexed by $n = 0, 1, \ldots$. They are defined on the integral $-1 \le x \le 1$. The first few are given by:

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

a) Show that the first three functions $P_n(x)$ are mutually *orthogonal*. (For the inner product, do the intergral from $x = -1$ to $x = 1$.)

b) The polynomials as given do *not* have magnitude 1 so the set is not *orthonormal*. For the first three, divide each function (vector) by its magnitude to get *unit* basis vectors. (You can call these $\tilde{P}_n(x)$, maybe.)

c) Express the polynomial $1 - x^2$ in terms of the unit basis functions $\tilde{P}_n(x)$.

5. For

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \qquad \mathsf{A} = \begin{pmatrix} 2 & -4 & 7 \\ -3 & 0 & 6 \\ 1 & 8 & -5 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ -3 & 0 & 4 \end{pmatrix}$$

find:

a) $\mathsf{A}\mathbf{x}$

b) A^2

c) AB

d) BA

e) A^{-1}. (Use a calculator or computer.)