Phys 4620, Spring 2005
Exam #3

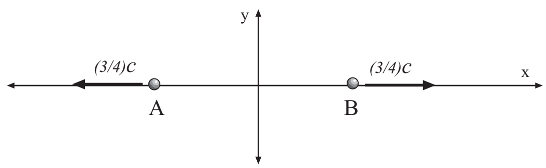
1. If we move toward a source of light we still measure the speed of the light as c but we measure a different frequency and wavelength from that measured in the source frame.

Now, we have a formula from Phys 2110 for the change in frequency when we move relative to a source[1]. Why can't we use that formula for light?

The basic Doppler shift formula assumes that we can measure the velocities of source and observer relative to the medium which carries the wave. In relativity there is *no* medmedium which carries alight wave and the only meaningful velocity is the relative velocity between source and receiver; light travels at the same speed in all inertial reference frames.

2. As seen in frame $\mathcal{S}$ particle A moves at speed $\frac{3}{4}c$ in the $-x$ direction and particle B moves at speed $\frac{3}{4}c$ in the $+x$ direction.

What is the speed of particle B as measured in the frame of particle A?



B has velocity $v = +\frac{3}{4}c$ wrt frame $\mathcal{S}$ and $\mathcal{S}$ has velocity $v = +\frac{3}{4}c$ wrt the frame of A. To find the velocity of B as measured in the frame of A, use the "Einstein velocity addition formula",

$$V_{BA} = \frac{v_{B\mathcal{S}} + v_{\mathcal{S}A}}{1 + v_{B\mathcal{S}}v_{\mathcal{S}A}/c^2} = \frac{\frac{3}{4}c + \frac{3}{4}c}{1 + \frac{9c^2/16}{c^2}} = \frac{\frac{3}{2}c}{\frac{25}{16}} = \frac{8 \cdot 3}{25}c = \frac{24}{25}c$$

3. A pion has a lifetime of 2.6×10^{-8} s (in its rest frame). If a pion has a speed of $\frac{4}{5}c$, how far (on average) do we expect it to travel after being created?

Pion has lifetime 2.6×10^{-8} s in its rest frame. If it moves at $\frac{4}{5}c$ wrt the lab then *in the lab* it lives for a time

$$\Delta t = \frac{\Delta \bar{t}}{\sqrt{1 - v^2/c^2}} = \frac{(2.6 \times 10^{-8}\ \mathrm{s})}{\sqrt{1 - \frac{(16/25)c^2}{c^2}}} = \frac{(2.6 \times 10^{-8}\ \mathrm{s})}{\sqrt{9/15}} = \frac{5}{3}(2.6 \times 10^{-8}\ \mathrm{s} = 4.33 \times 10^{-8}\ \mathrm{s}$$

Then moving at $v = \frac{4}{5}c$ it will travel a distance (in the lab) of

$$d = v\Delta t = \frac{4}{5}(3.00 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}})(4.33 \times 10^{-8}\ \mathrm{s}) = 10.4\ \mathrm{m}$$

4.a) What does it mean when we say that a certain quantity transforms as a "Lorentz vector" ?

[1] It's

$$f' = \left(\frac{v_0 \pm v_{\mathrm{obs}}}{v_0 \mp v_{\mathrm{src}}}\right) f \ .$$

When a quantity (with 4 components) tranasforms as a 4–vector we mean that the components in a *new* reference frame are reltaed tothe components in the old reference frame via

$$a^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu}$$

where Λ^{μ}_{ν} is an array of numbers dependent on the relative velocity of the two reference frames; for a new frame moving with velocity $v\hat{\mathbf{x}}$ it is

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

b) What does it mean when we say that a certain quantity is "Lorentz invariant"?

If a quantity is Lorentz invariant we mean that when its value is calculated in *any* inertial frame we get the *same* value.

Example of Lorentz invariants are $a^{\mu}a_{\mu}$, where a^{μ} is a Lorentz 4–vector, and $\mathbf{E}\cdot\mathbf{B}$.

5. A proton ($M_{\rm p}c^2 = 938$ MeV) has 800 MeV of kinetic energy.

What is its speed? (You can answer as a fraction of c.)

If a proton has kinetic energy $T = 800$ MeV, then its total energy is

$$E = mc^2 + 800 \text{ MeV} = 938 \text{ MeV} + 800 \text{ MeV} = 1738 \text{ MeV}$$

Since

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{(938 \text{ MeV})}{\sqrt{1-u^2/c^2}} = 1738 \text{ MeV} ,$$

then

$$\sqrt{1-u^2/c^2} = \frac{938}{1738} = 0.540 \qquad \Rightarrow \qquad \frac{u^2}{c^2} = 0.709 \qquad \Rightarrow \qquad \frac{u}{c} = 0.842$$

$$\Rightarrow u = (0.842)c$$

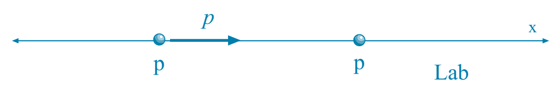
6. A proton collides with a proton at rest to produce 3 protons and an antiproton (which has the same mass, but opposite charge from the proton):

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$$

Find the smallest possible (threshold) kinetic energy for the incident proton for this reaction to take place. Use $M_{\bar{\rm p}}c^2 = M_{\rm p}c^2 = 938$ MeV and express the answer in MeV.

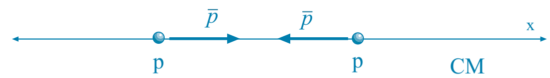
(Hint: Use *invariance* and *conservation* and use the fact that at threshold in the CM frame the final particles are at rest.)

In the lab, prior to the the collision the total momentum is p and the total energy is $E_p + m_p c^2$, where $E_p = \sqrt{p^2c^2 + m^2c^4}$. The total momentum 4–vector is

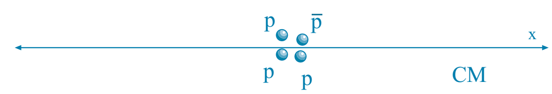


$$(E_p + m_p c^2, pc) \tag{1}$$

In the CM frame (where the total 3–momentum is zero) the momenta of both protons have magnitude $\bar{p}$ so the total momentum 4–vector is



$$(2E_{\bar{p}}, 0) \tag{2}$$



After the collision there are 4 particles of mass m_p at rest (for threshold) so the total momentum 4–vector is then

$$(4m_p c^2, 0) \tag{3}$$

Now, *conservation* of 4–momentum in the CM frame gives

$$2E_{\bar{p}} = 4m_p c^2$$

and from the *invariance* of the 4–vector magnitude, 1 and 2 give

$$-(E_p + m_p c^2)^2 + p^2c^2 = -(2E_{\bar{p}})^2 = -(4m_pc^2)^2 = -16m_p^2c^4$$

Use $E_p^2 = p^2c^2 + m^2c^4$ on the lhs and get

$$-p^2c^2 - m_p^2c^4 + 2E_p m_p c^2 - m_p^2c^4 + p^2c^2 = -16m_p^2c^4$$

$$2E_p m_p c^2 = 14 m_p c^4 \qquad \Rightarrow \qquad E_p = 7m_pc^2 = T + m_pc^2$$

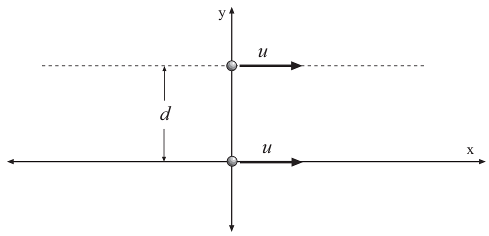
Then the kinetic energy of the incoming proton is

$$T = 6m_pc^2 = 6(938 \text{ MeV}) = 5628 \text{ MeV} = 5.63 \text{ GeV}$$

7. In the lab frame we have E and B fields which are both nonzero and point in the same direction ($\hat{\mathbf{z}}$, say). Is there any reference frame where the electric field is zero? (Give this reference frame or explain why there can't be one.)

If $\mathbf{E}$ and $\mathbf{B}$ are parallel and both nonzero then $\mathbf{E}\cdot\mathbf{B}$ is nonzero. Since $\mathbf{E}\cdot\mathbf{B}$ is *invariant*, neither $\mathbf{E}$ nor $\mathbf{B}$ can be zero in some other inertial frame; otherwise you'd get $\mathbf{E}\cdot\mathbf{B} = 0$ in that frame.

8. In system $\mathcal{S}$ charges q_A and q_B are both flying by at constant speed u on trajectories parallel to the x axis (in the xy plane), q_A on $y = 0$ and q_B on $y = d$.



a) Find the fields at q_B due to q_A and the force that q_A exerts on q_B (as, say, both cross the y axis).

In the lab frame the field due to q_A at q_B is

$$\mathbf{E}_B = \frac{q_A}{4\pi\epsilon_0} \frac{(1 - u^2/v^2)}{(1 - \frac{u^2}{c^2} \cdot 1)^{3/2}} \frac{\hat{\mathbf{y}}}{d^2} = \frac{q_a\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d^2 \sqrt{1 - u^2/c^2}}$$

and the $\mathbf{B}$ field at B is $\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E}$, or:

$$\mathbf{B}_B = \frac{q_A\,\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2 d^2 \sqrt{1 - u^2/c^2}} = \frac{\mu_0 q_A u\,\hat{\mathbf{z}}}{4\pi d^2 \sqrt{1 - u^2/c^2}}$$

and the force on q_B is

$$\begin{aligned}
\mathbf{F}_B &= q_B \mathbf{E}_B + q_B \mathbf{v} \times \mathbf{B}_B \\
&= \frac{q_A q_B\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d^2 \sqrt{1 - u^2/c^2}} - \frac{q_A q_B u^2\,\hat{\mathbf{y}}}{4\pi\epsilon_0 c^2 d^2 \sqrt{1 - u^2/c^2}} \\
&= \frac{q_A q_B}{4\pi\epsilon_0} \frac{(1 - u^2/c^2)}{\sqrt{1 - u^2/c^2}} \frac{\hat{\mathbf{y}}}{d^2} = \frac{q_A q_B}{4\pi\epsilon_0} \frac{\sqrt{1 - u^2/c^2}\,\hat{\mathbf{y}}}{d^2}
\end{aligned}$$

b) The field(s) in the frame of the moving charges is simple! Show that $(E^2 - c^2 B^2)$ is the same in both frames.

In the frame of the moving charges there is only an electrostatic field, hence

$$\mathbf{E}_B = \frac{1}{4\pi\epsilon_0} \frac{q_A \hat{\mathbf{y}}}{d^2} \qquad \mathbf{B}_B = 0$$

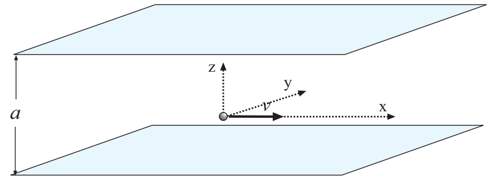
Check the invariant $E^2 - c^2 B^2$:

$$\begin{aligned}
\text{Lab:} \qquad E^2 - c^2 B^2 &= \frac{q_A^2}{(4\pi\epsilon_0)^2 d^4} \left\{ \frac{1}{(1 - u^2/c^2)} - \frac{u^2}{c^2} \frac{1}{(1 - u^2/c^2)} \right\} \\
&= \frac{q_A^2}{(4\pi\epsilon_0)^2 d^4} \frac{(1 - u^2/c^2)}{(1 - u^2/c^2)} = \frac{q_A^2}{(4\pi\epsilon_0)^2 d^4}
\end{aligned}$$

$$\text{Moving:} \qquad E^2 - c^2 B^2 = \frac{q_A^2}{(4\pi\epsilon_0)^2 d^4}$$

The same!

9. a) There are two charged plates at $z = 0$ and $z = a$ each carrying charge densities $+\sigma$ and $-\sigma$ respectively. What is the electric field between the plates?



Between big parallel plates with charge densities $\pm\sigma$ the E field is $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}$, in the lab frame. (This is old stuff; follows from Gauss law arguments.)

b) If a charge q moves in the $+\hat{\mathbf{x}}$ direction with speed v, what are the electric and magnetic fields in the frame of this charge?

Use the equations for transformation of the fields to get E and B in the moving frame:

$$\bar{E}_z = \gamma(E_z + vb_y) = \gamma E_z = \frac{\sigma/\epsilon_0}{\sqrt{1 - v^2/c^2}}$$

$$\bar{B}_y = \gamma(B_y + \frac{v}{c^2}E + z) = \frac{v/c^2}{\sqrt{1 - v^2/c^2}}\frac{\sigma}{\epsilon_0} = \frac{\mu_0 v\sigma}{\sqrt{1 - v^2/c^2}}$$

where we've used $\epsilon_0 c^2 = \frac{1}{\mu_0}$.

10. The Maxwell equations can be written in the form

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

Show how the first of these, with the choice $\mu = 1$ (x) gives a certain component of one of the Maxwell equations.

With $\mu = 1$, we have (use $F^{11} = 0$)

$$\begin{aligned}
\frac{\partial F^{1\nu}}{\partial x^\nu} &= \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \frac{\partial(-E_x/c)}{\partial(ct)} + \frac{\partial B_z}{\partial y} + \frac{\partial(-B_y)}{\partial z} \\
&= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = -\frac{1}{c^2}\frac{\partial E^x}{\partial t} + (\nabla \times \mathbf{B})^x \\
&= \mu_0 J^x
\end{aligned}$$

This is the x–component of the Maxwell eqn we've been calling Ampere's law with maxwell's correction.

11. Evaluate $F^{\mu\nu}F_{\mu\nu}$.

It will make things easier to use the antisymmetry of $F^{\mu\nu}$ and to break up the values of $F^{\mu\nu}$ into F^{0i} and F^{ij}. Recall that when you lower a "0" index you get a minus sign.

$F^{\mu\nu}F_{\mu\nu}$ is a sum on both μ and ν, but we can just sum on unique *pairs* of the indices, with $\mu < \nu$. From antisymmetry the term for $\nu < \mu$ is the *same* from two sign switches, $(-1)(-1)$. Breaking up $F^{\mu\nu}$ term into F_{0i} and F^{ij}, the explicit sum is

$$F^{\mu\nu}F_{\mu\nu} = 2[F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + F^{12}F_{12} + F^{23}F_{23} + F^{13}F_{13}]$$

Note, $F_{01} = -F^{01}$ from lowering just one zero index, but $F_{12} = F^{12}$. Get:

$$F^{\mu\nu}F_{\mu\nu} = 2[-E_x^2/c^2 - E_y^2/c^2 - E_z^2/c^2 + B_x^2 + B_y^2 + B_z^2] = \frac{2}{c^2}[-E^2 + c^2B^2]$$

which can also be expressed as

$$F^{\mu\nu}F_{\mu\nu} = -\frac{2}{c^2}[E^2 - c^2B^2]$$

This is quantity is invariant, as it must be since $F^{\mu\nu}F_{\mu\nu}$ is a Lorentz scalar.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{4}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{5}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{6}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{7}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{8}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{9}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{10}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{11}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial\lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla\cdot\mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\boldsymbol{\mathscr{r}}\times\mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}) = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t \qquad \Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x$$

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \qquad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \bar{x} = \gamma(x - vt) \qquad \bar{y} = y \qquad \bar{z} = z$$

$$\bar{x}^\mu = \sum_{\nu=0}^{3}(\Lambda^\mu_\nu)x^\nu \qquad \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) \qquad \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \qquad p^\mu = (E/c, p_x, p_y, p_z) \qquad E = \gamma mc^2$$

$$p^\mu p_\mu = -m^2c^2 \qquad E^2 = p^2c^2 + m^2c^4$$

$$K^\mu = \frac{dp^\mu}{d\tau} \qquad J^\mu = (c\rho, J_x, J_y, J_z) \qquad A^\mu = (V/c, A^x, A^y, A^z)$$

$$\begin{aligned} \bar{E}_x &= E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y) \\ \bar{B}_x &= B_x \qquad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{aligned}$$

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\} \qquad F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Invariants: $\mathbf{E}\cdot\mathbf{B}\,,$ $(E^2 - c^2B^2)$

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \qquad \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_0 J^\mu \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \qquad K^\mu = q\eta_\nu F^{\mu\nu}$$