Phys 3810, Spring 2010
Exam #3

1. What is meant by:

a) Stern-Gerlach experiment

An experiment in which a beam of atoms was separated according to the z component of spin of the odd electron of the atoms. A non--uniform magnetic field exerted different forces according to the spin direction and thus separated the beam.

b) Gyromagnetic ratio (for a particle).

The numerical ratio between the *angular momentum* (a mechanical quantity) and the *magnetic moment* (an electromagnetic property) of a particle. (This number must be gotten from the quantum theory, as the elementary particles are *not* simply charged particles of a finte size with a rotational motion!)

c) Pauli Exclusion Principle

Most generally, the rule that the wave function for a system of identical fermions must be antisymmetric under exchange of the labels for any two particles. For non-interacting particles, when the wave function is expressed as a product of single--particle "orbitals", it implies that no two particles can occupy the same orbital.

2. If we operate with L_- on $Y_1^{-1}(\theta, \phi)$, what do we expect to get? *Show* that we get this result. Use:

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

We expect that *lowering* (i.e. lowering the m value) of the the *lowest* Y_1 wave functions will give zero.

Since

$$Y_1^{-1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

we get

$$\begin{aligned} L_-Y_1^{-1}(\theta, \phi) &= -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta e^{i\phi} \\ &= \sqrt{\frac{3}{8\pi}}\hbar e^{-i\phi}\left(\cos\theta e^{i\phi} - i\cos\theta(i)\right)e^{i\phi} = 0 \end{aligned}$$

3. Suppose a spin-$\frac{1}{2}$ particle is in the state (written in the S_z basis):

$$\chi = \frac{1}{\sqrt{13}}\begin{pmatrix} 2i \\ -3 \end{pmatrix}$$

What are the probabilities of getting $+\hbar/2$ and $-\hbar/2$ if you measure S_y? What is the expectation value of S_y for this state?

We are looking for the expansion coefficients for the $\chi_\pm^{(y)}$ for this state. Using the spinors given at the back, we find:

$$c_+ \equiv \langle \chi_+^{(y)} | \chi \rangle = \frac{1}{\sqrt{13}} \frac{1}{\sqrt{2}} (1 \ -i) \begin{pmatrix} 2i \\ 3 \end{pmatrix} = \frac{1}{\sqrt{26}} (2i + 3i) = \frac{5i}{\sqrt{26}}$$

which clearly has $|c_+|^2 = \frac{25}{26}$ and

$$c_- \equiv \langle \chi_+^{(y)} | \chi \rangle = \frac{1}{\sqrt{13}} \frac{1}{\sqrt{2}} (1 \ i) \begin{pmatrix} 2i \\ -3 \end{pmatrix} = \frac{1}{\sqrt{26}} (2i - 3i) = \frac{-i}{\sqrt{26}}$$

which has $|c_-|{=}\frac{1}{26}$. So the probabilities for "spin-up y" and "spin-down y" are respectively $\frac{1}{26}$ and $\frac{25}{26}$ respectively from which we can quickly get the expectation value,

$$\langle S_y \rangle = \frac{25}{26}(\hbar/2) + \frac{1}{26}(-\hbar/2) = \frac{(25-1)}{52}\hbar = +\frac{6}{13}\hbar$$

We can also get this from

$$\begin{aligned} \langle S_y \rangle &= \langle \chi^\dagger S_y \chi \rangle = \frac{1}{13} \frac{\hbar}{2} (-2i \ -3) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 2i \\ -3 \end{pmatrix} \\ &= \frac{\hbar}{26} (-2i \ -3) \begin{pmatrix} 3i \\ -2 \end{pmatrix} = \frac{\hbar}{26}(6+6) = \frac{6}{13}\hbar \end{aligned}$$

and it agrees, like it should.

4. Construct S_z and S_+ matrices for a spin–2 particle. The relation

$$S_\pm |s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}|s\ (m\pm 1)$$

may be useful!

As the eigenvalues of S_z for the $S = 2$ states are

$$S_z = 2\hbar, \hbar, 0, -\hbar - 2\hbar$$

the matrix S_z (in the representation of those eigenstates!) is

$$S_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

The action of S_+ on the individual states is

$$S_+|2\ -2\rangle = \hbar\sqrt{6-(-2)(-1)}|2\ -1\rangle = 2\hbar|2\ -1\rangle$$

$$S_+|2\ -1\rangle = \hbar\sqrt{6-(-1)(-0)}|2\ 0\rangle = \sqrt{6}\hbar|2\ 0\rangle$$

$$S_+|2\ 0\rangle = \hbar\sqrt{6-(0)(1)}|2\ 1\rangle = \sqrt{6}\hbar|2\ 1\rangle$$

$$S_+|2\ 1\rangle = \hbar\sqrt{6-(1)(2)}|2\ -1\rangle = 2\hbar|2\ 2\rangle$$

$$S_+|2\ -2\rangle = 0$$

So the S_+ operator must accomplish:

$$S_+\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \hbar\begin{pmatrix} 2b \\ \sqrt{6}c \\ \sqrt{6}d \\ 2e \\ 0 \end{pmatrix}$$

which leads to

$$S_+ = \hbar\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5. Is the following a permissible wave function for a system of two non-interacting spin–$\frac{1}{2}$ particles occupying the states ψ_a and ψ_b?

$$\frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)]\ \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

If not (and of course it isn't) tell how you would fix it to make it permissible.

The wave function must be antisymmetric under an exchange of *all* the labels for the two particles (space and spin). Clearly, both factors are antisymmetric for this exchange so the overall wave function is symmetric.

To fix this (or at least get something legal) change the $-$ to a $+$ for either the space or spin part.

6. Consider a quantum state where an angular momentum $s_1 = \frac{3}{2}$ combines with an angular momentum $s_2 = 1$ to give a state with total angular momentum $s = \frac{5}{2}$ and z component $\frac{1}{2}$:

$$|s\ m\rangle\ =\ |\tfrac{5}{2}\ +\tfrac{1}{2}\rangle$$

Give the expansion of this state in terms of the states of *individual* angular momenta, that is, states of the form

$$|\tfrac{3}{2}\ m_1\rangle\,|1\ m_2\rangle$$

(A chart of C-G coefficients will be provided.)

Use the section of the table labeled $\frac{3}{2} \times 1$ and look in the column that is headed by the numbers $\frac{5}{2}\ \frac{1}{2}$. Read off the coefficients remembering that there is an implied square root. We get:

$$|\tfrac{5}{2}\ +\tfrac{1}{2}\rangle = \frac{1}{\sqrt{10}}|\tfrac{3}{2}\ +\tfrac{3}{2}\rangle\,|1\ -1\rangle + \sqrt{\frac{3}{5}}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle\,|1\ 0\rangle + \sqrt{\frac{3}{10}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle\,|1\ +1\rangle$$

7. Write out the complete Hamiltonian for the helium atom, assuming the nucleus is stationary. Give a short description of each of the terms and explain why the Schrödinger equation for this Hamiltonian is so hard to solve.

If we make the *gross* approximation of ignoring the interaction between the electrons, what are the solutions?

The correct Hamiltonian includes terms for kinetic energy, attraction to the nucleus (of charge $+2e$) and e-e repulsion:

$$H = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{2e^2}{4\pi\epsilon_0}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|}$$

It is impossible to give an "exact" solution for the wave function and its eigenvalue because the equation does not separate.

If we ignore the e-e term then the Hamiltonian does separate in the two coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ and the solutions are just products of the wave functions for the hydrogenic atoms with nuclear charge $+2e$:

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2)$$

8. In class we discussed a way to make an *estimate* of the effect of the e-e repulsion of the ground state energy of helium. How was this calculation done? (Just describe how it was set up; you don't need to work out any integrals.)

We noted that with the e-e term in the Hamiltonian being

$$H_{\text{ee}} = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|}$$

if we found the *average* of this term over the "brain-dead" solution for the helium atom

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2)$$

we ought to get a good (but not exact) value for the correction to the ground state energy eigenvalue. In fact, the result was very good.

9. Suppose we wanted to treat the Fermi gas *correctly*, incorporating relativity. The correct relation between k and E is

$$E = \sqrt{(\hbar k c)^2 + m^2 c^4}$$

Write down an integral expression relating $E_{\rm tot}$ and k_F. (This integral is hard; that's why we never did it!) *Hint: Energy still increases with k and the permissible k values still correspond to points on a lattice. Some equations given below can help.*

The number of electron states in a an octant shell of "radius" k and thickness dk is (in *any* kind of dynamics!)

$$\frac{V}{\pi^2}k^2\,dk$$

and with each of these states carrying an energy $E = \sqrt{(\hbar kc)^2 + m^2c^4}$, the energy contained in the shell is

$$dE = \sqrt{(\hbar kc)^2 + m^2c^4}\,\frac{V}{\pi^2}k^2\,dk = \frac{V}{\pi^2}k^2\sqrt{(\hbar kc)^2 + m^2c^4}\,dk$$

and with the integral (still) going from $k = 0$ to the state with maximum k, that is, k_F, we have

$$E_{\rm tot} = \frac{V}{\pi^2}\int_0^{k_F} k^2\sqrt{(\hbar kc)^2 + m^2c^4}\,dk$$

Actually, this integral does have a closed form, but it's nastier than the extreme cases considered in class.

10. In solving the "Dirac comb" potential, we arrived at the equation

$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

for the "comb" with spikes $V = \alpha\delta(x - ja)$.

a) Explain what K is.

K is an index which gives the *change in phase* of the wave function ψ under a (space) translation by one lattice spacing. For our one-dimensional model it could take on the values

$$K = \frac{2\pi n}{Na} \qquad (n = 0,\ \pm1,\ \pm2,\ \dots)$$

It is used to *index* the possible states in the "crystal".

b) Why does this relation result in "energy bands" and "energy gaps" for the "Dirac comb" problem?

The solution of the TDSE for the Dirac comb results in the equation relating K and k,

$$\cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

where k essentially gives the energy of the state indexed by K. For some values of k the right-hand side has absolute value > 1 and so there is *no* state K giving that energy. This leads to "gaps" in the possible energies of states, and "bands", i.e. regions for which there are possible states.

11. The O_2 molecule behaves like a rigid rotor with point masses (the nuclei) separated by a bond length of 1.2×10^{-10} m.

a) Find the moment of inertia of the O_2 molecule (about its center of mass). Use:

$$m(\mathrm{O16}) = 15.995\,\mathrm{amu} \qquad 1\,\mathrm{amu} = 1.6605 \times 10^{-27}\,\mathrm{kg}$$

The moment of inertia is (a is the separation of the atoms and $m_{\mathrm{O}} = 2.656 \times 10^{-26}\,\mathrm{kg}$),

$$I = 2\left[m(a/2)^2\right] = \frac{ma^2}{2} = \frac{(2.656 \times 10^{-26}\,\mathrm{kg})(1.2 \times 10^{-10}\,\mathrm{m})^2}{2} = 1.91 \times 10^{-46}\,\mathrm{kg} \cdot \mathrm{m}^2$$

b) Find the difference in energy between the $n = 1$ and $n = 3$ rotational levels. (There's a reason to ask the question this way; for subtle reasons, the even-n levels (states) don't exist!)

As for the rigid rotor the energy states are given by

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} = \frac{\hbar^2}{2I} n(n+1) \quad n = 0, 1, 2, \ldots$$

We get

$$E_n = \frac{(1.054 \times 10^{-34}\,\mathrm{J} \cdot \mathrm{s})^2}{(2.656 \times 10^{-26}\,\mathrm{kg})(1.2 \times 10^{-10}\,\mathrm{m})^2} n(n+1) = (1.81 \times 10^{-4}\,\mathrm{eV}) n(n+1)$$

Between the $n = 1$ and $n = 3$ levels there is thus an energy difference of

$$(1.81 \times 10^{-4}\,\mathrm{eV})(3(4) - 1(2)) = 1.81\,\mathrm{meV}$$

12. In the computer "project" for this semester, you adjusted the energy eigenvalue by looking for a change in the behavior of the wave function solution.

What behavior *was* this and why should this procedure give the right energy eigenvalue?

The energy eigenvalue E is the constant which is put into the time-independent Sch ödinger equation such that when the wave has the right boundary condition at $r = 0$ the wave function vanishes at infinity. Since we can never input an *exact* value into a computer, a good approximate value will have the wave function blowing up at a *large* value of r. When (while changing E we see a change in the large-r behavior from blowing up *negative* to blowing up *positive* (or vice-versa) then we guess that some where in between the wave function did not blow up at all, and that is a possible value for E.

Numbers

$$\hbar = 1.05457 \times 10^{-34}\,\mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\,\mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\,\mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\,\mathrm{C} \qquad c = 2.99792 \times 10^{8}\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^* f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx-\frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}\,dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,dx$$

$$\text{Delta Fn Potl:} \qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi$$

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2}=-m^2\Phi \qquad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+[\ell(\ell+1)\sin^2\theta-m^2]\Theta=0$$

$$Y_0^0=\sqrt{\frac{1}{4\pi}} \qquad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \qquad Y_2^{\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\ \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}\text{etc.}$$

$$u(r)\equiv rR(r) \qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$$

$$a=\frac{4\pi\epsilon_0\hbar^2}{me^2}=0.529\times10^{-10}\,\mathrm{m} \qquad E_n=-\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}\equiv\frac{E_1}{n^2} \qquad \text{for} \quad n=1,2,3,\ldots$$

where $E_1=-13.6\,\mathrm{eV}$.

$$R_{10}(r)=2a^{-3/2}e^{-r/a} \qquad R_{20}(r)=\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-r/2a} \qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\ \frac{r}{a}\ e^{-r/2a}$$

$$\lambda f=c \qquad E_\gamma=hf \qquad \frac{1}{\lambda}=R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right) \qquad \text{where} \qquad R=\frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2=1.097\times10^7\,\mathrm{m}^{-1}$$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p} \qquad [L_x,L_y]=i\hbar L_z \qquad [L_y,L_z]=i\hbar L_x \qquad [L_z,L_x]=i\hbar L_y$$

$$L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm=\pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2f_l^m=\hbar^2l(l+1)f_l^m \qquad L_zf_l^m=\hbar mf_l^m$$

$$[S_x,S_y]=i\hbar S_z \qquad [S_y,S_z]=i\hbar S_x \qquad [S_z,S_x]=i\hbar S_y$$

$$S^2|s\ m\rangle=\hbar^2s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle=\hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle=\hbar\sqrt{s(s+1)-m(m\pm1)}\,|s\ m{\pm}1\rangle$$

$$\chi=\begin{pmatrix}a\\b\end{pmatrix}=a\chi_++b\chi_- \quad \text{where} \quad \chi_+=\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi_-=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathsf{S}^2=\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \mathsf{S}_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$dE = \frac{\hbar^2 k^2}{2m}\frac{V}{\pi^2}k^2\,dk \qquad E_{\text{tot}} = \frac{\hbar^2 k_F^2 V}{10\pi^2 m} \qquad P = \frac{2}{3}\frac{\hbar^2 k_F^5}{10\pi^2 m} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

$$V(x+a) = V(x) \qquad \psi(x+a) = e^{iKx}\psi(x) \qquad K = \frac{2\pi n}{Na} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} = \frac{\hbar^2}{2I}n(n+1) \quad n = 0, 1, 2, \dots$$