Phys 2920, Spring 2012
Exam #3

1. Verify the divergence theorem for the vector field

$$\mathbf{v} = r\sin^2\phi\sin\theta\hat{\mathbf{e}}_r + r^2\hat{\mathbf{e}}_\theta + r^2\sin^2\theta\hat{\mathbf{e}}_\phi$$

for the case where the volume is the sphere of radius R centered at the origin and the surface is (of course) the boundary of that volume.

This vector field gives

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \sin^2\phi\sin\theta\frac{1}{r^2}\frac{\partial}{\partial r}(r^3) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta) + 0 \\ &= 3\sin^2\phi\sin\theta + r\cot\theta\end{aligned}$$

and the integral of the divergence over the specified volume is

$$\begin{aligned}\int_V \nabla\cdot\mathbf{v}\, d^3r &= \int_0^{2\pi}[3\sin^2\phi\sin\theta + r\cot\theta]r^2\, dr\sin\theta\, d\theta\, d\phi \\ &= 3\int_0^R r^2\, dr\int_0^{2\pi}\sin^2\phi\, d\phi\int_0^\pi \sin^2\theta\, d\theta + 0 \\ &= \frac{R^3}{3}3\left[-\tfrac{1}{4}\sin 2\phi + \frac{\phi}{2}\right]\Big|_0^{2\pi}\left[-\tfrac{1}{4}\sin 2\theta + \frac{\theta}{2}\right]\Big|_0^{\pi}\end{aligned}$$

where in the second line we recognized that the second term in $\nabla\cdot\mathbf{v}$ integrates to zero since it gives the integral of $\cos\theta$ from 0 to π. The ϕ integral gives π and the θ integral gives $\pi/2$ so that we have

$$\int_V \nabla\cdot\mathbf{v}\, d^3r = R^3\cdot\pi\cdot\pi/2 = \frac{\pi^2R^3}{2}$$

Now do the surface integral; since on this surface

$$r = R \qquad \text{and} \qquad d\mathbf{a} = r^2\sin\theta\, d\theta\, d\phi$$

so

$$\begin{aligned}\int_S \mathbf{v}\cdot d\mathbf{a} &= \int_0^{2\pi}\int_0^\pi r^3\sin\theta\, d\theta\, d\phi\Big|_{r=R} \\ &= R^3\int_0^\pi \sin^2\theta\, d\theta\sin^2\phi\, d\phi \\ &= R^3\cdot\frac{\pi}{2}\cdot\pi = \frac{\pi^2R^3}{2}\end{aligned}$$

It's the same thing.

2. Delta functions! Do the following integrals:

a)
$$\int_{-10}^{10} \frac{1}{x^2-25}\left(\delta(x-13)+\delta(x+7)\right)\,dx$$

This one splits into

$$I = \int_{-10}^{10} \frac{1}{x^2-25}\delta(x-13)\,dx + \int_{-10}^{10} \frac{1}{x^2-25}\delta(x+7)\,dx$$

but since $x = 13$ is outside the integration range, that term gives zero while the second picks out the value of $f(x)$ at $x = -7$ and this gives

$$I = \frac{1}{(-7)^2-25} = \frac{1}{24}$$

b)
$$\int_V ((\mathbf{r}+3\,\hat{\mathbf{k}})^2)\,\delta^3(\mathbf{r}-4\,\hat{\mathbf{i}}-3\,\hat{\mathbf{j}})\,d^3r\ ,$$

where the volume V is the sphere of radius 8 centered at the origin.

This one will pick out the value

$$\mathbf{r} = 4\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$$

if it is in the integration range.. and it is, because its magnitude is 5 and the sphere over which the integration takes place has radius 8. So this puts this particular value into $f(\mathbf{r})$ and we get

$$I = (4\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}) = 16+9+9 = 34$$

3. Complex arithmetic; simplify:

$$(\mathbf{a})\quad \left|\frac{1}{(2-6i)^2}\right| \qquad\qquad (\mathbf{b})\quad \frac{2+5i}{(1+3i)^2}$$

For (a) we get

$$\left|\frac{1}{(2-6i)^2}\right| = \left|\frac{1}{4-36-24i}\right| = \frac{1}{|-32-24i|} = \frac{1}{\sqrt{1600}} = \frac{1}{40}$$

For (b) we get

$$\begin{aligned} \frac{2+5i}{(1+3i)^2} &= \frac{2+5i}{-8+6i}\cdot\frac{-8-6i}{-8-6i} \\ &= \frac{-16+30-40i-12i}{64+36} = \frac{14-52i}{100} \end{aligned}$$

4. Express all values of the multi-valued function:

$$\mathrm{Ln}\,(4-3i)$$

Which one of them do you think a \$50 calculator would give?

Write the argument of the Ln in polar form:

$$4 - 3i = re^{i\phi}$$

$$r = \sqrt{3^2 + 4^2} = 5 \qquad \text{and} \qquad \phi = \tan^{-1}\frac{-3}{4} = -0.644 \text{ rad}$$

so then

$$\begin{aligned}\mathrm{Ln}\,(4-3i) &= \mathrm{Ln}(5e^{-i0.644+2\pi i}) = \ln(5) - i(0.644) + i2n\pi \\ &= 1.609 + i(-0.644 + 2\pi n)\end{aligned}$$

I would guess that a $\$150$ calculator would give the choice where $n = 0$. And mine does.

5. Evaluate these the \$10 calculator way:

a) $\sinh(3-2i)$

Use "angle-addition" formulae etc. to get

$$\begin{aligned}\sinh(3-2i) &= \sinh(3)\cosh(-2i) + \cosh(3)\sinh(-2i) \\ &= \sinh(3)\cos(2) - i\cosh(3)\sin(2) = -4.169 - i\,9.154\end{aligned}$$

b) $\sin^{-1}(5)$

$$\begin{aligned}\sin^{-1}(5) &= \frac{1}{i}\ln(i5 + \sqrt{1-25}) = -i\ln(i5 + 1\sqrt{24}) \\ &= -i\ln(9.899i) = -i\ln(9.899e^{i\pi/2}) \\ &= -i(i\pi/2 + 2.29) = \frac{\pi}{2} - 2.29i = 1.57 - 2.29i\end{aligned}$$

My calculator also gives this answer.

6. Explain why for a "branch cut" is needed for the function which we would write as

$$f(z) = z^{1/3}$$

The meaning of $z^{1/3}$ (we do need to know what it means...) is to write z in polar form $re^{i\phi}$ and then take

$$z^{1/3} = r^{1/3}e^{i\phi/3}$$

but if we start with $\phi = 0$ and proceed through $\phi = 2\pi$ and $\phi = 4\pi$ then when we come back to the $\mathit{same\ point}\ z$ the value of $f(z)$ is different. We can prevent the ambiguity in value of the function by constructing a boundary running from the origin off to infinity and not permitting z to change continuously across that boundary. That is the "branch cut"; if desired we can dispense with that and allow multiple values for a function.

7. Explain what is meant when we say that the derivative of a complex function $f(z)$ exists (at a particular z).

We mean that when we evaluate the limit

$$f'(z) \equiv \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

the value *does not depend* on the path by which the complex number Δz goes to zero (just as long as $|z| \to 0$.

8. For the function

$$w(z) = \frac{1}{2z+1} \equiv u(x,y) + iv(x,y) \qquad \text{where} \quad z = x + iy$$

find the (real) functions u and v and show the Cauchy–Riemann equations are satisfied everywhere except for $z = -1/2$.

With $z = x + iy$ we have

$$\begin{aligned} f(z) &= \frac{1}{(2(x+iy)+1} = \frac{1}{(2x+1)+i2y} \cdot \frac{(2x+1)-i2y}{(2x+1)+i2y} \\ &= \frac{(2x+1)-i2y}{(2x+1)^2+4y^2} \end{aligned}$$

from which we get

$$u(x,y) = \frac{(2x+1)}{(2x+1)^2+4y^2} \qquad \text{and} \qquad v(x,y) = \frac{-2y}{(2x+1)^2+4y^2}$$

Now test the C-R conditions on these. For the first one we find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{2[(2x+1)^2+4y^2] - (2x+1)4(2x+1)}{[(2x+1)^2+4y^2]^2} \\ &= \frac{-2(2x+1)^2+8y^2}{[(2x+1)^2+4y^2]^2} \end{aligned}$$

We also get

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{-2[(2x+1)^2+4y^2] - (-2y)(8y)}{[(2x+1)^2+4y^2]^2} \\ &= \frac{-2(2x+1)^2+8y^2}{[(2x+1)^2+4y^2]^2} \end{aligned}$$

so these agree.

And then we get

$$\frac{\partial u}{\partial y} = -\frac{(2x+1)(8y)}{[(2x+1)^2+4y^2]^2} = \frac{-8y(2x+1)}{[(2x+1)^2+4y^2]^2}$$

and

$$\frac{\partial v}{\partial x} = -\frac{(-2y)(4)(2x+1)}{[(2x+1)^2+4y^2]^2} = \frac{8y(2x+1)}{[(2x+1)^2+4y^2]^2}$$

which are opposite in sign and this shows the second of the C-R relations.

9. The Taylor series for the (real) function

$$f(x) = \frac{1}{x^2+9}$$

(expanded about x=0, that is, a power series in x) only converges for $|x| < 3$.

a) Explain how we know this from the complex version of Taylor's theorem.

The denominator of the function has zeros at $z = \pm 3i$ so these are the (simple) poles of the function. Thus from the *complex* version of Taylor's Theorem, a Taylor series for the function can only converge within a circle of radius 3 centered at the origin. This implies that on the *real* axis it does not converge for $|x| > 3$.

b) Find the Laurent expansion for this function good for values of z with $|z| > 3$. Hint: Write

$$\frac{1}{z^2+9} = \frac{1}{z^2}\frac{1}{1+\frac{9}{z^2}}$$

and use the form for the geometric series or the binomial theorem on the second factor.

Write

$$\frac{1}{z^2+9} = \frac{1}{z^2}\frac{1}{1+\frac{9}{z^2}}$$

The general geometric series, good for complex w, with $|w| < 1$ is

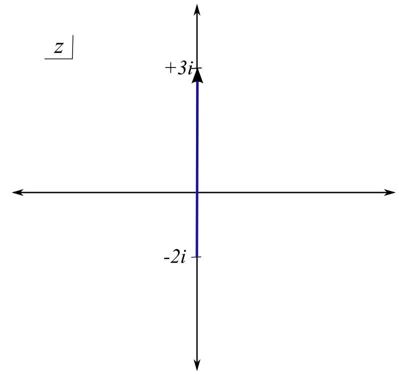
$$a + aw + aw^2 + aw^3 + \cdots = \frac{a}{1-w}$$

Using it here with $w = -\frac{9}{z^2}$ gives

$$\begin{aligned} \frac{1}{z^2+9} &= \frac{1}{z^2}\left(1 - \frac{9}{z^2} + \frac{81}{z^4} - \cdots\right) \\ &= \frac{1}{z^2} - \frac{9}{z^4} + \frac{81}{z^6} - \frac{729}{z^8} + \cdots \end{aligned}$$

10. Do the (non-closed!) contour integral

$$\int_C 5\sinh(z)\,dz$$



along the path shown (a line which goes from $-2i$ to $+3i$).

Actually, does it matter which path you take between the endpoints? Why or why not?

As the integrand $f(z) = \sinh(z)$ is an analytic function the path of integration does not matter and it can be done be find the anti-derivative of $f(z)$, i.e. the normal calculus way. This gives

$$\begin{aligned}\int_C 5\sinh(z)\,dz &= 5\left[\cosh\right]\Big|_{-2i}^{3i} \\ &= 5[\cosh(3i) - \cosh(-2i)] = 5(\cos(3) - \cos(2)] = 5(-0.5738) = -2.869\end{aligned}$$

(Note, the anti-derivative of $\sinh$ is $\cosh$!)

11. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4x + 5)}$$

using a contour integral and the residue theorem.

Hintz: Show the contour you want to use; you can give a quick argument as to why any added bits don't matter. Note, the integral already goes from $= \infty$ to $+\infty$ so that we don't need to double it. Find the poles of the integrand (when expressed as a complex function); note which poles (if any) are inside the contour. Find the residue(s) of the relevant pole(s)

Even if you can't get to the final answer, demonstrate all that you understand about working out the integral this way.

The contour we use is the one made up of the part of the real axis going from $-R$ to R closed with semi-circle of radius R in the upper plane, with $R \to \infty$. From general discussion in the book and in class we have reason to believe that from a denominator which gets "big" at large distance from the origin, the curvy part will give a vanishing contribution as $R \to \infty$.

The integrand has poles where the denominator is zero. To find these use the quadratic formula,

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = \tfrac{1}{2}(-4 \pm 2i) = -2 \pm i$$

and of these only the pole $-2 + i$ lies inside the contour.

Find the residue of the integrand (when it is made into a *complex* function at the point $z = -2+i$. We can get this from the residue formula for a simple pole,

$$a_{-1} = \lim_{z\to -2+i} \frac{[z - (-2+i)]}{[z^2 + 4z + 5]} = \lim_{z\to -2+i} \frac{1}{2z + 4} = \frac{1}{2(-2+i) + 4} = \frac{1}{2i}$$

Then by the residue theorem the integral on C is

$$\int_C \frac{dz}{(z^2+4z+5)} = 2\pi i \cdot \frac{1}{2i} = \pi$$

and so we have shown

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)} = \pi$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \text{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\text{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz \tag{4}$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi \tag{5}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{6}$$

$$\begin{aligned}\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\ \hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\ \hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{i}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{j}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\ \nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\ \nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\ \nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \qquad \int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$z = x + iy = \rho e^{i\phi} \qquad |z| = \rho = \sqrt{x^2+y^2} \qquad z^* = x - iy \qquad w = \ln z = \ln r + i(\theta + 2k\pi)$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cosh z = \frac{e^z+e^{-z}}{2} \quad \sinh z = \frac{e^z-e^{-z}}{2}$$

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2 \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \mathrm{sech}^2 z \qquad \coth^2 z - 1 = \mathrm{csch}^2 z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin(iz) = i \sinh z \qquad \cos(iz) = \cosh z$$

$$\sin^{-1}(z) = \frac{1}{i} \ln(iz + \sqrt{1 - z^2}) \qquad \cos^{-1}(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}) \qquad \tan^{-1}(z) = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad w = f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

$$(1 + z)^p = 1 + pz + \frac{p(p - 1)}{2!} z^2 + \cdots + \frac{p(p - 1) \cdots (p - n - 1)}{n!} z^n + \cdots$$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{(z - a)} + \frac{a_{-2}}{(z - a)^2} + \frac{a_{-3}}{(z - a)^3} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \qquad a_{-1} = \lim_{z \to a} \left(\frac{1}{(k - 1)!} \frac{d^{k-1}}{dz^{k-1}} (z - a)^k f(z) \right)$$

$$\oint_C f(z) \, dz = 2\pi i \{a_{-1} + b_{-1} + c_{-1} + \dots\}$$