Phys 2920, Spring 2011
Exam #3

1. Verify the divergence theorem for the vector field

$$\mathbf{v} = r\cos^2\theta\hat{\mathbf{e}}_r + r\hat{\mathbf{e}}_\theta + r\sin^2\theta\hat{\mathbf{e}}_\phi$$

for the case where the volume is the sphere of radius R centered at the origin and the surface is (of course) the boundary of that volume.

Doing the integral $\oint \mathbf{v}\cdot d\mathbf{S}$ over the surface of the sphere, and using

$$d\mathbf{S} = da_r\hat{\mathbf{e}}_r = r^2\,\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{e}}_r$$

we get

$$\oint \mathbf{v}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}(r\cos^2\theta)r^2\sin\theta\,d\theta\,d\phi\Big|_{r=R} = (2\pi)R^3\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

where the integral on ϕ just gave a factor of 2π. We can now do th ecommon trick of letting $\cos\theta \equiv$ and changing the θ integral on one on x as x goes from -1 to 1. We get:

$$\Longrightarrow \qquad = 2\pi R^3\int_{-1}^{1}x^2\,dx = 2\pi R^3\cdot\frac{2}{3} = \frac{4\pi R^3}{3}$$

The divergence of this vector field is

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta) = 3\cos^2\theta + \frac{\cos\theta}{\sin\theta}$$

Integrate over the volume and get

$$\begin{aligned}
\int_V \nabla\cdot\mathbf{v}\,dV &= \int_0^{2\pi}\int_0^{\pi}\left(3\cos^2\theta + \frac{\cos\theta}{\sin\theta}\right)r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= (2\pi)\frac{R^3}{3}\int_0^{\pi}\left(3\cos^2\theta + \frac{\cos\theta}{\sin\theta}\right)\sin\theta\,d\theta = \frac{2\pi R^3}{3}\left[\int_{-1}^{1}3x^2\,dx + \int_0^{\pi}\cos\theta\,d\theta\right] \\
&= \frac{2\pi R^3}{3}[2+0] = \frac{4\pi R^3}{3}
\end{aligned}$$

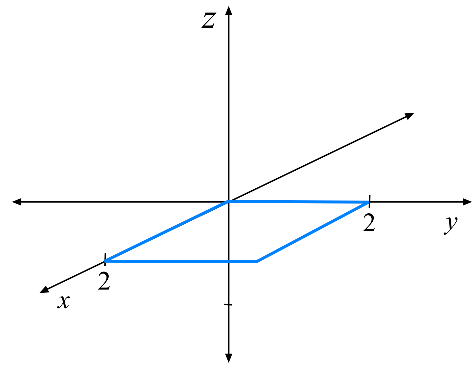
Yep. It works.

2. Demonstrate that Stokes's's's Theorem works for the vector field

$$\mathbf{v} = y^2\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + x^2\,\hat{\mathbf{k}}$$

for the case where the curve C is the square in the xy plane with $0 \le x \le 2$ and $0 \le y \le 2$ and where the surface S is the *flat* (square) area bounded by that curve.

Many of the numbers are simple but *show all of your work clearly*.



The curl of $\mathbf{v}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\,\hat{\mathbf{i}} - 2x\,\hat{\mathbf{j}} - 2y\,\hat{\mathbf{k}}$$

so that the surface integral of Stokes' theorem is (with $d\mathbf{S} = dx\,dy\,\hat{\mathbf{k}}$)

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \int_0^2 \int_0^2 (-2y)\,dx\,dy = -2(2)\int_0^2 y\,dy = -4\left(\frac{2^2}{2}\right) = -8$$

For the line integral, the first leg has $y = 0$ and $x : 0 \to 2$ and $d\mathbf{r} = dx\,\hat{\mathbf{i}}$. This gives

$$\int_1 \mathbf{v} \cdot d\mathbf{r} = \int_0^2 y^2\,dx = 0$$

The second leg has $x = 2$ and $y : 0 \to 2$ and $d\mathbf{r} = dy\,\hat{\mathbf{j}}$. This gives

$$\int_2 \mathbf{v} \cdot d\mathbf{r} = \int_0^2 z^2\,dy = 0$$

The third leg has $y = 2$ and $x : 2 \to 0$ and $d\mathbf{r} = dx\,\hat{\mathbf{i}}$. This gives

$$\int_3 \mathbf{v} \cdot d\mathbf{r} = \int_2^0 y^2\,dx = 4\int_2^0 dx = -8$$

The fourth leg has $x = 0$ and $y : 2 \to 0$ and $d\mathbf{r} = dy\,\hat{\mathbf{j}}$. This gives

$$\int_4 \mathbf{v} \cdot d\mathbf{r} = \int_2^0 x^2\,dy = 0$$

The total line integral is -8 and the theorem checks out.

3. Delta functions! Do the following integrals:

a) $\int_0^6 \frac{\cos x}{x}\left(\delta(x-\pi) + \delta(x-2\pi)\right)\,dx$

The integration range includes π but not 2π. So only the first of the delta functions contributes to the result; its "action" inside the integral is to give the value of its multiplier at $x = \pi$,thus

$$\Longrightarrow \qquad I = \frac{\cos\pi}{\pi} = -\frac{1}{\pi}$$

b) $\int_V (3\mathbf{r}^2 - 6\mathbf{a}\cdot\mathbf{r})\delta^3(\mathbf{r} - 3\hat{\imath})\, d^3r$, the the volume V is the sphere of radius 5 centered at the origin and $\mathbf{a} = -6\hat{\imath} + 5\hat{\jmath}$.

Here the delta function has us evaluate the *vector function* multiplier at a specific point; as the point $\mathbf{r} = 3\hat{\imath}$ is enclosed by the volume V, the delta functions sets

$$\mathbf{r}^2 = 9 \qquad \mathbf{a}\cdot\mathbf{r} = (-6\hat{\imath} + 5\hat{\jmath})\cdot 3\hat{\imath} = -18$$

and the integral is

$$\Longrightarrow \qquad I = 3(9) - 6(-18) = 135$$

4. Complex arithmetic; simplify:

$$(\mathbf{a}) \quad \left|\frac{6-2i}{5+4i}\right| \qquad\qquad (\mathbf{b}) \quad \frac{6-3i}{(2-4i)^2}$$

$$(\mathbf{a}) \qquad \left|\frac{6-2i}{5+4i}\right| = \frac{|6-2i|}{|5+4i|} = \frac{\sqrt{36+4}}{\sqrt{25+16}} = \sqrt{\frac{40}{41}}$$

$$(\mathbf{b}) \qquad \frac{6-3i}{(2-4i)^2} = \frac{6-3i}{(4-16-16i)} = \frac{(-6+3i)}{(12+16i)}$$

Multiply top and bottom by $(12-16i)$:

$$\Longrightarrow \qquad = \frac{-6+3i}{(12+16i)}\cdot\frac{(12-16i)}{(12-16i)} = \frac{-72+48+i(36+96)}{(144+256)}$$

More tedious arithmetic gives:

$$\Longrightarrow \qquad = \frac{-24+132i}{400} = -\frac{3}{50} + \frac{33}{100}i$$

5. Express all values of the multi-valued function:

$$\mathrm{Ln}\,(-3+2i)$$

Which one of them do you think a \$50 calculator would give?

Write in polar form:

$$|-3+2i| = \sqrt{13} \qquad \tan\phi = \frac{2}{(-3)} \qquad \Longrightarrow \qquad \phi = 2.554$$

Then

$$\mathrm{Ln}(-3+2i) = \mathrm{Ln}(\sqrt{13}e^{i2.554}) = 1.282 + i(2.554) + 2n\pi i \quad n = 0, \pm 1, \pm 2, \ldots$$

Undoubtedly a good calculator will just return the $n = 0$ choice, $1.282 + i(2.554)$

6. Evaluate these the \$10 calculator way:

a) $\sin(7-2i)$

Using an angle addition formula and relations between $\sin$ and $\sinh$ etc. we get

$$\begin{aligned}\sin(7-2i) &= \sin(7)\cos(2i) - \cos(7)\sin(2i) \\ &= \sin(7)\cosh(2) - i\cos(7)\sinh(2) = 2.47 - 2.73\,i\end{aligned}$$

b) $\cosh^{-1}(4i)$

We can get one possible answer from:

$$\cosh^{-1}(4i) = \mathrm{Ln}(z + \sqrt{z^2-1}) = \mathrm{Ln}(4i + \sqrt{-17}) = \mathrm{Ln}(4i + 4.123i)$$

where we made the $+$ choice for hte complex square root. Then we find:

$$\Longrightarrow \qquad = \mathrm{Ln}(8.123i) = \mathrm{Ln}(8.123e^{i\pi/2}) = 2.09 + i\frac{\pi}{2}$$

which is a valid answer, but if we made the $-$ choice to go with the square root, we would get

$$\Longrightarrow \qquad = \mathrm{Ln}(4i - \sqrt{-17}) = \mathrm{Ln}(4i - 4.123i) = \mathrm{Ln}(-0.123i) = \mathrm{Ln}(0.123e^{-i\pi/2}) = -2.09 - i\frac{\pi}{2}$$

which is also correct, as cosh is an even function.

7. Explain why for some complex functions like $(1+z^2)^{1/2}$ we might need to construct a branch cut.

For the given function there is a point (two of them, in fact, $z = \pm i$) where if begin with one value of the function at a given z and then *continuously* evaluate the function as we move *around* this point we do not get the same value we don't get the same value when we return to the starting place. Such a point is a branch point.

In cases where we must have a continuous function we set up a mathematical barrier to keep us from getting multiple values of the function at the same point. Such a line starts from the branch point; it might go out to infinity or end up at another branch point.

8. Explain what is meant when we say that the derivative of a complex function $f(z)$ exists (at a particular z).

We mean that when we consider the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

we can let the *complex number* Δz approach zero arbitrarily, that is, Δz can shrink to zero along any curve in the z plane and the result is the same so that the limit *exists* as written.

9. For the function

$$w(z) = \frac{1}{z} \equiv u(x,y) + iv(x,y) \qquad \text{where} \quad z = x + iy$$

find the (real) functions u and v and show the Cauchy–Riemann equations are satisfied everywhere except for $z = 0$.

Separate the real and imaginary parts by:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

from which we identify

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = -\frac{y}{x^2+y^2}$$

Checking the C-R equations, we find:

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = -\frac{[x^2+y^2-y(2y)]}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

so that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

so that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

10. Write out the first five terms of Laurent series for

$$f(z) = \frac{e^z}{z^3}$$

and from it identify the residue of this function at $z = 0$.

As the Taylor series for e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

so that the *Laurent* series for e^z/z is

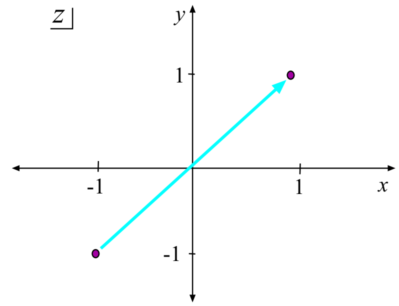
$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \cdots$$

so that the residue, the coefficient of the $1/z$ term is $\frac{1}{2}$.

11. Do the (non-closed!) contour integral

$$\int_C 4e^{2z}\,dz$$

along the path shown (a line which goes from $-1-i$ to $1+i$).



Actually, does it matter which path you take between the endpoints? Why or why not?

The integrand is analytic everywhere in the complex plane so a line integral between two points does not depend on the path (equivalently, the integral around a closed curve gives zero) and furthermore is given by the usual methods of elementary calculus:

$$\int_C 4e^{2z}\,dz = 2e^{2z}\Big|_{-1-i}^{1+i} = 2\left(e^{2+2i} - e^{-2-2i}\right) = 2(2i)\sin(2+2i)$$

Do the usual (tedious) thing to get a numerical value outof this:

$$= (4i)(\sin(2)\cos(2i) + \cos(2)\sin(2i)) = (4i)(\sin(2)\cosh(2) + i\cos(2)\sinh(2))$$

At this stage we can put it into a $10 calculator and we get:

$$\int_C 4e^{2z} = 6.037 + 13.68\,i$$

12. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 2)}$$

using a contour integral and the residue theorem.

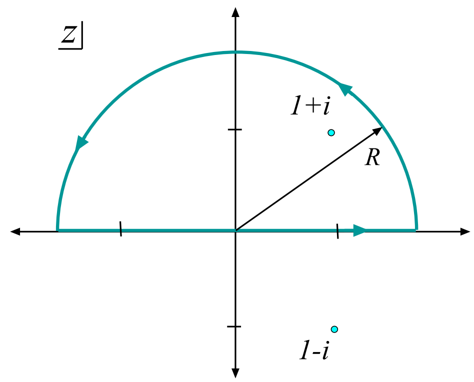
Show the contour you want to use; you can give a quick argument as to why any added bits don't matter. Find the poles of the integrand; note which poles (if any) are inside the contour. Find the residue(s) of the relevant pole(s)

Even if you can't get to the final answer, demonstrate all that you understand about working out the integral this way.

The integrand has poles where the denominator is zero; this is at

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

We will form a closed contour by adding the the part which goes along the real axis (from $-R$ to R, with $R \to \infty$) a semi-circle of radius R. We can guess that the integral on this part will go to zero in the limit because the denominator "goes as" R^2 while the length of the path of integration is proportional to R. We could also choose to close the contour in the lower plane.) So we now evaluate



$$\oint_C \frac{dz}{z^2 - 2z + 2}$$

and of course the residue theorem will be helpful!

This contour will enclose the pole at $1+i$. Find the residue at this pole. It is:

$$\mathrm{Res}(1+i) = \lim_{z \to 1+i} \frac{(z-1-i)}{z^2 - 2z + 2} = \lim_{z \to 1+i} \frac{1}{2z - 2} = \frac{1}{2}\frac{1}{(1+i) - 1} = \frac{1}{2i}$$

where we used L'Hospital rule for the limit.

The residue theorem gives

$$\oint_C \frac{dz}{z^2 - 2z + 2} = (2\pi i)\frac{1}{2i} = \pi$$

and we conclude

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 2)} = \pi$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz \tag{4}$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi \tag{5}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{6}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{\imath}} + \sin\theta\sin\phi\,\hat{\mathbf{\jmath}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{\imath}} + \cos\theta\sin\phi\,\hat{\mathbf{\jmath}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{\imath}} + \cos\phi\,\hat{\mathbf{\jmath}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{\imath}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{\jmath}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \qquad \int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$z = x + iy = \rho e^{i\phi} \qquad |z| = \rho = \sqrt{x^2+y^2} \qquad z^* = x - iy \qquad w = \ln z = \ln r + i(\theta + 2k\pi)$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cosh z = \frac{e^z+e^{-z}}{2} \quad \sinh z = \frac{e^z-e^{-z}}{2}$$

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2 \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \mathrm{sech}^2 z \qquad \coth^2 z - 1 = \mathrm{csch}^2 z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin(iz) = i \sinh z \qquad \cos(iz) = \cosh z$$

$$\sin^{-1}(z) = \frac{1}{i} \ln(iz + \sqrt{1 - z^2}) \qquad \cos^{-1}(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}) \qquad \tan^{-1}(z) = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad w = f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

$$(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots + \frac{p(p-1)\cdots(p-n-1)}{n!} z^n + \cdots$$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{(z - a)} + \frac{a_{-2}}{(z - a)^2} + \frac{a_{-3}}{(z - a)^3} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \qquad a_{-1} = \lim_{z \to a} \left(\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z - a)^k f(z) \right)$$

$$\oint_C f(z) \, dz = 2\pi i \{a_{-1} + b_{-1} + c_{-1} + \ldots\}$$