Phys 2920, Spring 2012
Exam #1

Do all matrix calculations by hand unless otherwise indicated. So you need to show your work.

1. a) Write the complex number

$$2.2 - 5.0i$$

in polar form.

The magnitude is

$$r = \sqrt{a^2 + b^2} = 5.46$$

and the angle ϕ is

$$\tan\phi = \frac{b}{a} = -2.27 \qquad \Longrightarrow \qquad \phi = -66.3^\circ = -1.16$$

so that this number is

$$z = (5.46)e^{-(1.16)i}$$

b) Write the complex number

$$4.2e^{i5.0}$$

in $a + bi$ form.

$$z = 4.2(\cos(5.0) + i\sin(5.0)) = 1.19 + i(-4.03)$$

2. Give a suitable definition of **linear independence** (of a set of vectors).

Linear independence means we can't make one vector in the set out ofa linear combination of the *rest* of the vectors in the set.

Alternatively, it means that if we've constructed a linear combination of all the vectors such that it gives zero, then all the coefficients are zero, i.e. it is the trivial linear combination.

For problems 3 – 5 use the vectors

$$\mathbf{a} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} \qquad \mathbf{b} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}} \qquad \mathbf{c} = -2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \qquad \mathbf{d} = 6\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$$

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3. Find the angle between the vectors **a** and **b**.

We calculate:

$$a = \sqrt{17} \qquad b = \sqrt{22} \qquad \mathbf{a}\cdot\mathbf{b} = -14$$

Then from $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$ we get

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{-14}{\sqrt{17}\sqrt{22}} = -0.724 \qquad \Longrightarrow \qquad \theta = 2.38 = 136.3^\circ$$

4. Find a unit vector perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

A vector which is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{b}\times\mathbf{c}$:

$$\mathbf{b}\times\mathbf{c} = \hat{\mathbf{i}}(9+15) + \hat{\mathbf{j}}(-6+6) + \hat{\mathbf{k}}(-10-6) = 24\hat{\mathbf{i}} - 16\hat{\mathbf{k}}$$

As this vector has magnitude $\sqrt{832} = 8\sqrt{13}$, a unit vector parallel to this vector is

$$\mathbf{s} = -\frac{3}{\sqrt{13}}\hat{\mathbf{i}} + \frac{2}{\sqrt{13}}\hat{\mathbf{k}}$$

5. Are the vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ linearly independent?

One way to check: We've already found $\mathbf{b}\times\mathbf{c}$. It was $24\hat{\mathbf{i}} - 16\hat{\mathbf{k}}$. This vector points perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$. Taking the dot product of this vector with $\mathbf{d}$ gives

$$(\mathbf{b}\times\mathbf{c})\cdot\mathbf{d} = (24\hat{\mathbf{i}} - 16\hat{\mathbf{k}})\cdot(6\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 9\hat{\mathbf{k}}) = 0$$

which means that $\mathbf{d}$ must lie in the same plane which contains $\mathbf{b}$ and $\mathbf{c}$. But for that to be true it must be some linear combination of $\mathbf{b}$ and $\mathbf{c}$ and so the three vectors are *not* linearly independent.

6. Write a compact expression for

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$$

in terms of the components of the four vectors. Use the ϵ's and the summation convention.

Use

$$\mathbf{a}\times\mathbf{b} = a_i b_j \hat{\mathbf{e}}_k\,\epsilon_{ijk} \qquad \mathbf{c}\times\mathbf{d} = c_l d_m \hat{\mathbf{e}}_n\,\epsilon_{lmn}$$

The dot product of these is

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (a_i b_j \hat{\mathbf{e}}_k\,\epsilon_{ijk})(c_l d_m \hat{\mathbf{e}}_n\,\epsilon_{lmn}) = a_i b_j \epsilon_{ijk} c_l d_m \epsilon_{lmn}\delta_{kn}$$

This collapses the sum on n to give

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = a_i b_j \epsilon_{ijk} c_l d_m \epsilon_{lmk} = a_i b_j c_l d_m(\epsilon_{ijk}\epsilon_{lmk})$$

Using an identity for the ϵ's this is

$$= a_i b_j c_l d_m(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$

so that by collapsing the sums we get

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

7. Recall the expansion of (suitable) functions on the interval $[0,1]$ In terms of the functions $f_n(x) = \sqrt{2}\sin(n\pi x)$. (They were orthonormal.) We expanded one function which could be done "by inspection"; the function

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 < x < \frac{1}{2} \\ 2-2x & \frac{1}{2} < x < 1 \\ 0 & x > 1 \end{cases}$$

can also be expanded in the $f_n(x)$'s but we have to do it the hard way. (First, plot this function to understand what it looks like.)

Write down an expression for how we would get $n^{\rm th}$ coefficient c_n in

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

and use it to find c_1.

From the orthonormality of the basis vectors the $n^{\rm th}$ coefficient is gotten from

$$c_n = \langle n|f\rangle = \int_0^1 f_n(x) f(x)\, dx$$

From the definition of $f(x)$ this is

$$c_n = 2\sqrt{2}\int_0^{1/2} x\sin(n\pi x)\, dx + 2\sqrt{2}\int_{1/2}^1 (1-x)\sin(n\pi x)\, dx$$

In particular,

$$c_1 = 2\sqrt{2}\left[\int_0^{1/2} x\sin(\pi x)\, dx + \int_{1/2}^1 (1-x)\sin(\pi x)\, dx\right]$$

Through some grubby algebra, each of the integrals inside the bracket is equal to $\frac{1}{\pi^2}$. So

$$c_1 = 2\sqrt{2}\frac{2}{\pi^2} = \frac{4\sqrt{2}}{\pi^2}$$

8. Calculate the determinant of the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 4 & 0 & 1 & 0 \\ 6 & -1 & 2 & 5 \\ 0 & 1 & 3 & 8 \end{pmatrix}$$

We can expand by minors along th etop row, and that will have only two terms (two further determinants to evaluate). If we eliminate the first row and column the determinant to evaluate is

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 5 \\ 1 & 3 & 8 \end{vmatrix} = 5 + 8 = 13$$

Eliminating the first row and last column the determinant to evaluate is

$$\begin{vmatrix} 4 & 0 & 1 \\ 6 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -12 - 8 + 6 = -14$$

Then the determinant of the original 4×4 is

$$|\mathsf{A}| = 1(13) - 3(-14) = 55$$

9. If matrices A, B and C are given by

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & -1 & 3 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} 4 & 1 & -4 \\ 2 & -5 & 0 \\ 1 & 3 & 0 \end{pmatrix} \qquad \mathsf{C} = \begin{pmatrix} 2 & -4 \\ 2 & -3 \end{pmatrix}$$

Find:

a) AB

Using the matrix multiplication rules (what else is there to say?) we get:

$$\begin{pmatrix} 8 & -9 & -4 \\ 23 & 25 & -16 \\ 5 & 15 & -4 \end{pmatrix}$$

b) $\mathrm{Det}(\mathsf{A})$

As A is a 3×3 matrix, we can get its determinant using the diangonals. We get

$$\mathrm{Det}(\mathsf{A}) = 0 + 7 + 14 - 24 + 0 + 0 = -3$$

c) $\mathrm{Det}(\mathsf{B}^{-1})$

Since the determinant of B is

$$\mathrm{Det}(\mathsf{B}) = (-4)(2)(3) - (-4)(-5)(1) = -44$$

then as the determinant of an inverse is the same as the inverse of the determinant, we get

$$\mathrm{Det}(\mathsf{B}^{-1}) = |\mathsf{B}|^{-1} = -\frac{1}{44}$$

d) Trace(C)

$$\text{Trace}(\mathsf{C}) = 2 - 3 = -1$$

e) $\text{Det}(\mathsf{C}^5)$

Since $\text{Det}(\mathsf{C}) = -6 + 8 = 2$ then

$$\text{Det}(\mathsf{C}^5) = (\text{Det}(\mathsf{C}))^5 = 2^5 = 32$$

Theorems about matrix operations will be of help!

10. If

$$\mathsf{ABx} = \mathsf{CDy}$$

(where x and y are vectors and the others square non-singular matrices) find an expression for x in terms of the other quantities and their inverses. (This has a fairly short answer, but get it right.)

Multiply both sides by A^{-1}; get

$$\mathsf{Bx} = \mathsf{A}^{-1}\mathsf{CDy}$$

then multiply both sides by B^{-1}:

$$\mathsf{x} = \mathsf{B}^{-1}\mathsf{A}^{-1}\mathsf{CDy}$$

11. Find the eigenvalues and (unit) eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 7 & -4 \\ 5 & -5 \end{pmatrix}$$

To get the eigenvalues, solve

$$|\mathsf{A} - \Lambda\mathbf{1}| = 0 \qquad \Longrightarrow \qquad \begin{vmatrix} 7-\lambda & -4 \\ 5 & -5-\lambda \end{vmatrix} = (7-\lambda)(-5-\lambda) + 20 = 0$$

Factor and get

$$-15 - 2\lambda + \lambda^2 = 0 \qquad \Longrightarrow \qquad (\lambda - 5)(\lambda + 3) = 0$$

so the eigenvalues are

$$\lambda = -3,\ 5$$

For the eigenvalue -3, we solve

$$\begin{pmatrix} 7 & -4 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$$

This gives

$$7x - 4y = -3x \qquad \Longrightarrow \qquad -4y = -10x \qquad \Longrightarrow \qquad y = \tfrac{5}{2}x$$

so that an eigenvector and the normalized version is

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \qquad \Longrightarrow \qquad \frac{1}{\sqrt{29}}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

For the eigenvalue 5, we solve

$$\begin{pmatrix} 7 & -4 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

This gives

$$7x - 4y = 5x \qquad \Longrightarrow \qquad -4y = -2x \qquad \Longrightarrow \qquad x = 2y$$

so that an eigenvector and the normalized version is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \Longrightarrow \qquad \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

12. a) Check if each matrix has an inverse, and if so find it (any way you can) for:

$$\mathsf{A} = \begin{pmatrix} 4 & -8 \\ 3 & -6 \end{pmatrix} \qquad\qquad \mathsf{B} = \begin{pmatrix} 3 & 0 \\ -4 & 1 \end{pmatrix}$$

The determinants of these matrices are

$$\mathrm{Det}(\mathsf{A}) = -24 + 24 = 0 \qquad \mathrm{Det}(\mathsf{B}) = 3 - 0 = 3$$

so the first one does not have an inverse but the second one does. Start the procedure for finding its inverse by writing

$$\left(\begin{array}{cc||cc} 3 & 0 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{array}\right)$$

Multiply the first row by $\frac{4}{3}$ and add to the second row. Get:

$$\left(\begin{array}{cc||cc} 3 & 0 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & 1 \end{array}\right)$$

Multiply the top row by $\frac{1}{3}$:

$$\left(\begin{array}{cc||cc} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{3} & 1 \end{array}\right)$$

The inverse of B is the matrix on the right side.

b) For the computer software *or* calculator that you used on the homework, *explain* how you would find the inverse of the matrix

$$\mathsf{A} = \begin{pmatrix} 5 & 3 & 0 \\ 4 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$$

In Maple the syntax would be

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b := Matrix([[5, 3, 0],[4, 2, -3],[4, -5, 6]])
b^{-1}
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13.a) Explain what is meant by a *representation* of a vector or an operator.

A representation of a vector or operator refers to the choice of *basis vectors* so that a vector expressed as a linear combination of those basis vectors can be written as a *specific* set of numbers. Likewise an operator is written as a specific *matrix* (set of numbers) corresponding to a choice of basis.

b) Explain the related idea of a *similarity transformation*

In setting up a similarity transformation we are changing the *expression* of the vectors and operators to one for an old basis to one for a new basis. The transformation determines a matrix S which is used to change the expression of vectors and matrices.

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \qquad \epsilon_{ijk} = \begin{cases} 1 & ijk = 123 \\ \times -1 & \text{switch indices} \\ 0 & \text{any two equal} \end{cases} \qquad \mathbf{c} = \mathbf{a}\times\mathbf{b} \quad \Rightarrow \quad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2-1) \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$(\mathsf{AB})_{ij} = \sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{kj} \qquad \mathsf{AA}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathcal{I} \qquad \mathrm{Tr}\,\mathsf{A} = \sum_{i=1}^{N} A_{ii} \qquad |\mathsf{A} - \lambda\mathbf{1}| = 0$$

$$\hat{\mathbf{e}}'_j = \sum_{i=1}^{N} \mathsf{S}_{ij}\hat{\mathbf{e}}_i \qquad \mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} \qquad \mathsf{A}' = \mathsf{S}^{-1}\mathsf{AS}$$

$$|\mathsf{A}^{\mathrm{T}}| = |\mathsf{A}| \qquad |\mathsf{A}^{\dagger}| = |\mathsf{A}^{*}| = |\mathsf{A}|^{\star} \qquad |\lambda\mathsf{A}| = \lambda^n|\mathsf{A}| \qquad |\mathsf{AB}| = |\mathsf{A}|\cdot|\mathsf{B}|$$

$$(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1} \qquad \mathrm{Tr}(\mathsf{ABC}) = \mathrm{Tr}(\mathsf{BCA}) = \mathrm{Tr}(\mathsf{CAB})$$