Phys 3810, Spring 2012
Exam #3

1. What is meant by:
a) Fermion

Particle of "half-integral" spin; they must have anti-symmetric wave functions (change sign on exchange of complete coordinates of two particles). The elctron is a famous fermion.

b) Larmor precession. *Hint: This arose from considering a spin in a uniform B field...*

The rotation in the xy plane of the *expectation values* of the S_x and S_y operators when a particle with a magnetic moment is put into a uniform B field in the z direction.

c) Stern–Gerlach experiment

Experiment using a non-uniform magnetic field which will exert a *force* on particle whose sign depends on the orientation of the magnetic moment.

2. a) Find $\langle r \rangle$ for the electron in the ground state of hydrogen. (Express the answer in terms of the Bohr radius.)

Calculate

$$\langle r \rangle \int \psi_0^*(\mathbf{r})\, r\, \psi_0(\mathbf{r})\, d^3r$$

Now, the angular part of ψ, being normalized, just gives 1 in the angular integration and the remaining radial integration is

$$\begin{aligned}\langle r \rangle &= \int_0^\infty R(r)^2 r^3\, dr = \frac{2}{a^3}\int_0^\infty r^3 e^{-2r/a}\, dr \\ &= \frac{2}{a^3}(3!)\left(\frac{a}{2}\right)^4 = \frac{3a}{2}\end{aligned}$$

Note: It's not a; recall that was the most probable value of r, but it's not the average.

b) Find the probability that the electron in the H atom is at a distance less than $\langle r \rangle$ from the proton. (Is the answer about what you expected?)

Integrate the probability from 0 out to $r = \frac{3a}{2}$. This gives

$$\begin{aligned}P = \int_0^{3a/2} R(r)^2 r^2\, dr &= \frac{4}{a^3}\int_0^{3a/2} r^2 e^{-2r/a}\, dr \\ &= \frac{4}{a^3}\left[e^{-2r/a}\left(\frac{r^2}{(-2/a)} - \frac{2r}{(2/a)^2} + \frac{4}{(2/a)^3}\right)\right]\Big|_0^{3a/2} \\ &= 4\left[e^{-3}\left(-\frac{9}{8} - \frac{3}{4} - \frac{1}{4}\right) + \frac{1}{4}\right]\end{aligned}$$

Simplify this and get:

$$P = 4[-\tfrac{17}{8}e^{-3} + \tfrac{1}{4}] = 0.577$$

It's close to 0.5 as expected.

3. Show explicitly how applying the raising operator L_+ to the function $Y_1^{-1}(\theta, \phi)$ (with due regard to normalization) gives the function $Y_1^0(\theta, \phi)$.

Using the expressions for L_+ and Y_1^{-1}, we get

$$\begin{aligned} L_+ Y_1^{-1}(\theta, \phi) &= \hbar e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) (+1) \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi} \\ &= \hbar \sqrt{\frac{3}{8\pi}} e^{i\phi} \left(\cos \theta e^{-i\phi} + \cos \theta e^{-i\phi} \right) = 2\hbar \sqrt{\frac{3}{8\pi}} \cos \theta \end{aligned}$$

But the general relation for how the raising operator connects two states gives

$$L_+ |1 \ -1\rangle = \hbar \sqrt{1(2) - 0} |1 \ 0\rangle = \sqrt{2}\hbar |1 \ 0\rangle$$

Equating the two,

$$\sqrt{2}\hbar Y_1^0(\theta, \phi) = 2\hbar \sqrt{\frac{3}{8\pi}} \cos \theta$$

which gives

$$Y_1^0(\theta, \phi) = \sqrt{2} \sqrt{\frac{3}{8\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \cos \theta$$

4. Show how to construct the (3×3) matrix for the S_+ operator for spin $S = 1$.

The action of the raising operator S_+ on the states $|1 \ m\rangle$ is

$$S_+ |1 \ -1\rangle = \hbar \sqrt{2 - 0} |1 \ 0\rangle = \sqrt{2}\hbar |1 \ 0\rangle$$

$$S_+ |1 \ 0\rangle = \hbar \sqrt{2 - 0} |1 \ 1\rangle = \sqrt{2}\hbar |1 \ 1\rangle$$

$$S_+ |1 \ +1\rangle = 0$$

That is, the S_+ must take a 1 in the lowest slot and lift it to the middle position with a factor of $\sqrt{2}\hbar$, etc. This will happen with the following matrix:

$$S_+ = \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix}$$

5. An electron is in the spin state (for the z basis)

$$\chi = A \begin{pmatrix} -3i \\ 2 \end{pmatrix}$$

a) Find the normalization constant A

The condition $\chi^\dagger\chi = 1$ gives

$$1 = |A|^2 \begin{pmatrix} +3i & 2 \end{pmatrix} \begin{pmatrix} -3i \\ 2 \end{pmatrix} = |A|^2(9+4) = 13|A|^2$$

which gives

$$|A|^2 = \frac{1}{13} \qquad \Longrightarrow \qquad A = \frac{1}{\sqrt{13}}$$

(With the simple choice of phase!)

b) Find the expectation values of S_z and S_x for this state.

$$\begin{aligned} \langle S_z \rangle &= \chi^\dagger S_z \chi = \frac{1}{13}\frac{\hbar}{2} \begin{pmatrix} +3i & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3i \\ 2 \end{pmatrix} \\ &= \frac{\hbar}{26} \begin{pmatrix} +3i & 2 \end{pmatrix} \begin{pmatrix} -3i \\ -2 \end{pmatrix} = \frac{\hbar}{26}(9-4) = \frac{5\hbar}{26} \end{aligned}$$

$$\begin{aligned} \langle S_x \rangle &= \chi^\dagger S_x \chi = \frac{1}{13}\frac{\hbar}{2} \begin{pmatrix} +3i & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 2 \end{pmatrix} \\ &= \frac{\hbar}{26} \begin{pmatrix} +3i & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -3i \end{pmatrix} = \frac{\hbar}{26}(6i - 6i) = 0 \end{aligned}$$

6. Consider a quantum state where an angular momentum $s_1 = 2$ combines with an angular momentum $s_2 = 1$ to give a state with total angular momentum $s = 3$ and z component $+1$.

Write out the state $|3 \; +1\rangle$ as a linear combination of states where the individual z components are determinate, that is, states of the form $|2 \; m_1\rangle \, |1 \; m_2\rangle$.

A certain table will be made available.

Read 'em off the table, if you know how to use it:

$$|3 \; 1\rangle = \frac{1}{\sqrt{15}}|2 \; 2\rangle|1 \; -1\rangle + \sqrt{\frac{8}{15}}|2 \; 1\rangle|1 \; 0\rangle + \sqrt{\frac{2}{5}}|2 \; 0\rangle|1 \; 1\rangle$$

7. The hydrogen atom is really a *two*–particle system and this fact changes the measured energy eigenvalues. Explain (detailed math not necessary.) how we start from the two particle Schrödinger equation to finally "solve" the H atom *correctly* in this one regard.

We changed variables (or discussed doing so...) from $\mathbf{r}_1$ and $\mathbf{r}_2$ to relative and center-of-mass coordinates

$$\mathbf{r} = \mathbf{r}_1 \mathbf{r}_2 \qquad R = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

and found that since the potential depends only on $\mathbf{r}$ the Schrödinger equation separates so that there is a CM motion energy which could be any positive value and an energy for relative motion which has the same *form* as when we treated the force center as stationary but with the mass m replace the reduced mass μ of the system.

8. The N_2 molecule behaves like two point masses joined by a spring of force constant $k = 3169\,\frac{\mathrm{N}}{\mathrm{m}}$. (Take the mass of one mole, 6.022×10^{23} N atoms to be 14.01 g.)

a) What is the (classical) frequency of oscillation of the system? *Hint: This is a two–body problem (with a central force).*

In the equivalent one-body problem (see previous problem!) we use the reduced mass μ for the mass of the single particle and the same potential $V(r)$. Here we have

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2} = \frac{14.01 \times 10^{-3}\ \mathrm{kg}}{2(6.022 \times 10^{23})} = 1.16 \times 10^{-26}\ \mathrm{kg}$$

which gives a classical angular frequency of

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3169\,\frac{\mathrm{N}}{\mathrm{m}}}{(1.16 \times 10^{-26}\ \mathrm{kg}}}) = 5.2957 \times 10^{14}\ \mathrm{s}^{-1}$$

and

$$f = \frac{\omega}{2\pi} = 8.31 \times 10^{13}\ \mathrm{Hz}$$

b) What is the difference in energy of the two lowest (quantum) vibrational states?

The HO states are separated in energy by $\hbar\omega$ so that energy difference is

$$\begin{aligned} \Delta E &= \hbar\omega = (1.055 \times 10^{-34})(5.227 \times 10^{14}\ \mathrm{s}^{-1}) \\ &= 5.514 \times 10^{-20}\ \mathrm{J} = 0.344\ \mathrm{eV} \end{aligned}$$

9. a) Write down the crude He atom wave function which ignores the interaction between the two electrons. Explain what the terms mean!

Give the energy eigenvalue for this wave function when we ignore the $\mathrm{e}-\mathrm{e}$ repulsion in the Hamiltonian.

The crude "bone-head" wave function for helium is the product of two hydrogenic "1s" wavefunctions for $Z = 2$:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_0(\mathbf{r}_1)\psi_0(\mathbf{r}_2)$$

where we have a separated solution only because we have ignored the e-e interaction term. The energy of this state (using the bone-head Hamiltonian) is the sum of the two individual energies, and since they are proortional to Z^2 we get

$$E_{\mathrm{bonehead}} = 2(4)(-13.6\ \mathrm{eV}) = -108.8\ \mathrm{eV}$$

We can make a crude approximation for the effect of the $e-e$ repulsion by treating each electron as moving as moving outside a point charge with a reduced value due to the presence of the other electron.

If w is the probability that either of the electrons in the naive wave function spends closer than the average distance, then we might take this point charge to be $Z' = +(2-w)e$ instead of $+2e$. You already found w in Problem 2, but if you didn't get it, just use 0.500.

The deviation of the effective nuclear charge from $+2e$ is called the "screening" effect

b) We now treat the electrons as both (still) moving independently but moving in the Coulomb field of a point nucleus of charge $Z'e$ (instead of $+2e$). What is the total energy of the helium atom now?

Again, the exact answer is -78.975 eV.

This is just another crude guess for how to find the right energy; next semester, the real thing!

We got $w = 0.288$ in Prob 2, or at least I did. This gives

$$Z' = 2 - 0.577 = 1.423$$

and our bonehead wave function is the one appropriate for a nuclear charge of $1.7116e$, but as we stil treat the electrons an independent and the energy is still proportional to the square of the central charge we noe get

$$E = 2(1.423)^2(-13.6 \text{ eV}) = -55.1 \text{ eV}$$

which makes the He atom grossly *underbound*.

10. What do we really mean when we say that the electrons in an atom move in particular "orbitals"? Why is this not exactly the case?

We mean that the many-particle wave function is constructed from a *product* of N single-particle wave functions (N being the number of electrons), where we find the optimal single-particle wave functions by taking an *average* effect of all the other electrons.

This is not the exact answer simply because there is no restriction on the exact wave function to be written as such a product, even one with the proper antisymmetrization. One actually needs an infinite set of such product wave functions to get the exact solution.

11. a) Find the numerical value for the Fermi energy for a metal with *atom* number density $6.03 \times 10^{28}/\text{ m}^3$ where each atom contributes 3 conduction electrons.

The Fermi energy E_F is given by

$$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$$

with ρ being the number density of the electrons, in this case 3 times the number density of the atoms. Plug in the numbers and be careful and get:

$$\begin{aligned} E_F &= \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \\ &= \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{2(9.11 \times 10^{-31}\ \mathrm{kg})}((3)(3)(6.03 \times 10^{28}\ \mathrm{m^{-3}}\pi^2)^{2/3} \\ &= 1.87 \times 10^{-18}\ \mathrm{J} = 11.1\ \mathrm{eV} \end{aligned}$$

b) Are we justified in not worrying about relativity in this case?

Yes, this energy is very small compared to the electron rest mass of $mc^2 = 511\ \mathrm{keV}$.

c) Calculate the "degeneracy pressure" for this Fermi gas.

$$\begin{aligned} P &= \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \\ &= 1.35 \times 10^{11}\ \frac{\mathrm{N}}{\mathrm{m^2}} \end{aligned}$$

12. What was the significance of the index K for the quantum states of the 1-D lattice problem?

The index K (as we encountered it) gave the periodicity property of the single-electron wave function for an electron in a potential lattice, via Bloch's theorem:

$$\psi(x + a) = e^{ikA}\psi(x)$$

13. From the homework problem (or associated talk) we got a basic picture of why a white dwarf star will collapse if its mass is too large.

What is essential difference between the behavior of the electrons in a small-mass dwarf and a larger-mass dwarf that leads to the collapse?

For larger-mass white dwarfs we can't treat the electrons as being non-relativistic. The relativistic relation between momentum and energy "takes over" and eventually it leads to the property that there is *no* radius at which the gravitational pressure is balanced by the degeneracy pressure. With further physics put into the problem, the star collapses. (We saw this by taking *all* the electrons as extreme-relativistic.)

14. In the computer "project" for this semester, explain why we could arbitrarily use the boundary condition $u'(0) = 1$ for the wave function. (Is the slope really "1" at $r = 0$?)

The wave function's normalization was never computed and in fact was never important; all we were looking for was the behavior at large r and a sign change in that behavior would occur regardless of the normalization. The choice $u'(0) = 1$ simply amounted to an arbitrary choice of normalization.

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J \cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}} \qquad N_A = 6.022 \times 10^{23}\ \mathrm{mol}^{-1}$$

Physics

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} \Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^* \psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^* f(x)\,dx \qquad \sum_{n=1}^{\infty} |c_n|^2 = 1 \qquad \langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2 x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E + \hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E - \hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$E_n = \hbar\omega(n + \tfrac{1}{2}) \qquad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)}\,dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)e^{-ikx}\,dx$$

$$\text{Delta Fn Potl:} \qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi$$

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)\qquad\frac{d^2\Phi}{d\phi^2}=-m^2\Phi\qquad\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+[\ell(\ell+1)\sin^2\theta-m^2]\Theta=0$$

$$Y_0^0=\sqrt{\frac{1}{4\pi}}\qquad Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta\qquad Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)\qquad Y_2^{\pm1}=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi}\qquad Y_2^{\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm2i\phi}\text{etc.}$$

$$u(r)\equiv rR(r)\qquad-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$$

$$a=\frac{4\pi\epsilon_0\hbar^2}{me^2}=0.529\times10^{-10}\text{ m}\qquad E_n=-\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}\equiv\frac{E_1}{n^2}\qquad\text{for}\quad n=1,2,3,\ldots$$

where $E_1=-13.6$ eV.

$$R_{10}(r)=2a^{-3/2}e^{-r/a}\qquad R_{20}(r)=\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-r/2a}\qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f=c\qquad E_\gamma=hf\qquad\frac{1}{\lambda}=R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)\qquad\text{where}\qquad R=\frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2=1.097\times10^7\text{ m}^{-1}$$

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}\qquad[L_x,L_y]=i\hbar L_z\qquad[L_y,L_z]=i\hbar L_x\qquad[L_z,L_x]=i\hbar L_y$$

$$L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi}\qquad L_\pm=\pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\cot\theta\frac{\partial}{\partial\phi}\right)\qquad L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2f_l^m=\hbar^2l(l+1)f_l^m\qquad L_zf_l^m=\hbar mf_l^m$$

$$[S_x,S_y]=i\hbar S_z\qquad[S_y,S_z]=i\hbar S_x\qquad[S_z,S_x]=i\hbar S_y$$

$$S^2|s\;m\rangle=\hbar^2s(s+1)|s\;m\rangle\qquad S_z|s\;m\rangle=\hbar m|s\;m\rangle\qquad S_\pm|s\;m\rangle=\hbar\sqrt{s(s+1)-m(m\pm1)}\,|s\;m\pm1\rangle$$

$$\chi=\begin{pmatrix}a\\b\end{pmatrix}=a\chi_++b\chi_-\quad\text{where}\quad\chi_+=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad\chi_-=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathsf{S}^2=\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\mathsf{S}_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{-iE_+ t/\hbar} \\ b e^{-iE_- t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$dE = \frac{\hbar^2 k^2}{2m}\frac{V}{\pi^2}k^2\,dk \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2 k_F^2 V}{10\pi^2 m} \qquad P = \frac{2}{3}\frac{\hbar^2 k_F^5}{10\pi^2 m} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

$$V(x+a) = V(x) \qquad \psi(x+a) = e^{iKx}\psi(x) \qquad K = \frac{2\pi n}{Na} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \cos(Ka) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)$$

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} = \frac{\hbar^2}{2I}n(n+1) \quad n = 0, 1, 2, \dots$$