Phys 3610, Fall 2009
Exam #3

1. Define:

a) Generalized momentum

Generalized momentum for coordinate q is defined by

$$p_q \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

which for the usual Cartesian coordinates (and no magnetic fields) gives the usual momentum $p_x = m\dot{x}$.

b) Generalized force

Defined by the stuff on the other side of the EL equation,

$$F_q = \frac{\partial \mathcal{L}}{\partial q}$$

which gives the usual force for Cartesian coordinates. The EL equation reproduces $F_x = ma_x$ for Cartesian coordinates for for generalized coordinates (such as angles) can give relations involving angular quantities like torque and angular momentum.

c) Reduced mass

Used for the motion of a two-mass system, the reduced mass

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

plays the role of the mass of a *single* particle whose motion is given by the relative coordinate $\mathbf{r}$.

d) Coriolis force

A pseudo-force that occurs in a rotating reference frame which is non-zero when the mass has a velocity *in the rotating frame*. It is given by

$$\mathbf{F}_{\rm cor} \equiv 2m\dot{\mathbf{r}} \times \mathbf{\Omega}$$

e) Foucault pendulum.

A big, long simple pendulum often seen in sciencey-type museums; the pivot needs to have very low friction and the pendulum may even need to have some energy input as it is meant to swing for a long time. Due to the Coriolis force in the rotating frame of the earth's surface the plane of the pendulum's motion rotates with a period depending on the latitude.

f) Principal axes.

A set of axes fixed within a rigid body (with origin at the center of mass) such that the angular velocity is parallel to the angular momentum when either one is along one of these axes.

2. A particle of mass m is confined to move freely on surface of sphere of radius R. If we include the usual gravitational force on the particle ($\mathbf{F} = -mg\hat{\mathbf{z}}$), write down the lagrangian and Lagrange equations in terms of the angles θ and ϕ.

What are the generalized momenta corresponding to these coordinates? Is either of them conserved?

With the origin at the center of the sphere, the coordinates of the particle are

$$x = R\sin\theta\cos\phi \qquad y = R\sin\theta\sin\phi \qquad z = R\cos\theta$$

giving

$$U = mgz = mgR\cos\theta$$

$$\dot{x} = R(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) \qquad \dot{y} = R(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) \qquad \dot{z} = -R\dot{\theta}\sin\theta$$

Then the kinetic energy is

$$\begin{aligned}
T &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
&= \frac{mR^2}{2}\Big(\dot{\theta}^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta) + \dot{\phi}^2(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi) \\
&\qquad + 2\dot{\theta}\dot{\phi}(-\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\sin\phi\sin\theta\cos\phi\Big) \\
&= \frac{mR^2}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)
\end{aligned}$$

So *that* could have been a lot worse! The lagrangian is

$$\mathcal{L} = T - U = \frac{mR^2}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - mgR\cos\theta$$

The EL equation for θ is found from

$$\frac{\partial\mathcal{L}}{\partial\theta} = mR^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgR\sin\theta \qquad \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad \Longrightarrow \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\ddot{\theta}$$

so then

$$mR^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgR\sin\theta = mR^2\ddot{\theta} \qquad \Longrightarrow \qquad \ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 + \frac{g}{R}\sin\theta$$

The EL equation for ϕ is found from

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mR^2\sin^2\theta\,\dot{\phi} \qquad \Longrightarrow \qquad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = C = mR^2\sin^2\theta\,\dot{\phi}$$

so then

$$\dot{\phi} = \frac{C}{mR^2 \sin^2 \theta}$$

The conjugate momenta are

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2 \dot{\theta} \qquad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2 \sin^2 \theta \dot{\phi}$$

Our second EL equation gave

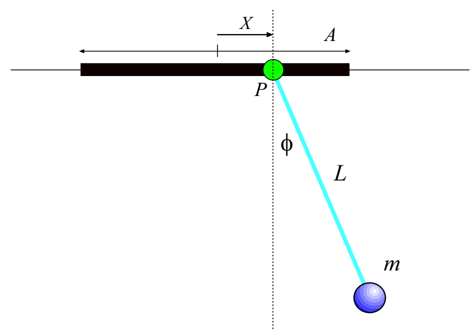
$$\frac{d}{dt}(p_\phi) = 0$$

and so p_ϕ is conserved.

3. Using Lagrange's equation (obviously) find the equation of motion for the following system:

A simple pendulum of mass m and length L swings freely from a point P which is constrained to oscillate horizontally, with position given by $X = A \cos \omega t$. There is only one degree of freedom here, the angle ϕ of the pendulum.

Find a differential equation for ϕ and if possible, think about its solutions!



Find the coordinates of the pendulum bob:

$$x = A \cos \omega t + L \sin \phi \qquad y = -L \cos \phi$$

Then:

$$U = mgy = -mgL \cos \phi \qquad \text{and} \qquad \dot{x} = -\omega A \sin \omega t + L\dot{\phi} \cos \phi \qquad \dot{y} = L\dot{\phi} \sin \phi$$

so that

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\omega^2 A^2 \sin^2 \omega t + L^2 \dot{\phi}^2 \cos^2 \phi - 2AL\omega\dot{\phi} \sin \omega t \, \cos \phi + L^2 \dot{\phi}^2 \sin^2 \phi) \\ &= \frac{m}{2}(\omega^2 A^2 \sin^2 \omega t + -2AL\omega\dot{\phi} \sin \omega t \, \cos \phi + L^2 \dot{\phi}^2) \end{aligned}$$

This gives

$$\mathcal{L} = \frac{m}{2}(\omega^2 A^2 \sin^2 \omega t - 2AL\omega\dot{\phi} \sin \omega t \, \cos \phi + L^2 \dot{\phi}^2) + mgL \cos \phi$$

Get the EL equation for ϕ:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgl \sin \phi \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -mAL\omega \sin \omega t + mL^2 \dot{\phi}^2$$

$$-mgl \sin \phi = \frac{d}{dt}(-mAL\omega \sin \omega t + mL^2 \dot{\phi}^2) = -mAL\omega^2 \cos \omega t + mL^2 \ddot{\phi}$$

Cancel m, rearrange:

$$\ddot{\phi} = -\frac{g}{L}\sin\phi + \frac{A\omega^2}{L}\cos\omega t$$

This is a nonlinear inhomogeneous E and so has no simple solutions, but would be a great computer project. Of course it reduces to the simple pendulum for $A = 0$. If we look at the $\sin\phi$ term as being like ϕ, then it resembles the DE for a driven oscillator but *without a friction term*. Intuitively, such a system could have very large oscillations when the driving frequency matches the natural frequency of the pendulum. (Of there is friction for a real pendulum.)

4. An asteroid is in orbit around the Sun; its distance of closest approach is $r_{\rm min} = 2.5$ AU and the eccentricity of its orbit is 0.200.

What is its farthest distance from the Sun?

From the equation for the orbit,

$$r(\phi) = \frac{c}{1+\epsilon\cos\phi}$$

and the nearest and farthest points occurring at $\phi = 0$ and $\phi = \pi$, we get

$$r_{\rm min} = \frac{c}{1+\epsilon} \qquad r_{\rm max} = \frac{c}{1-\epsilon} \qquad \Longrightarrow \qquad \frac{r_{\rm max}}{r_{\rm min}} = \frac{1+\epsilon}{1-\epsilon}$$

Plug in the given numbers,

$$\frac{r_{\rm max}}{r_{\rm min}} = \frac{1+\epsilon}{1-\epsilon} = \frac{(1+0.200)}{(1-0.200)} = \frac{1.2}{0.80} = 1.50 \qquad \Longrightarrow \qquad r_{\rm max} = 1.50(2.50\ \text{AU}) = 3.75\ \text{AU}$$

5. A particle of mass m is in "orbit" around a central (fixed) body; there is a *repulsive* central between them given by

$$F(r) = \frac{\alpha}{r^3} \qquad \text{with} \quad \alpha > 0$$

that is, a (repulsive) inverse-cubed force.

Find the general form of the solutions for $r(\phi)$. (Yes, the *can* solve the differential equation for this and in fact it's rather easy; as usual you can take ℓ and E as constants.) How would you describe the shape of these orbits (paths)?

Extra Credit: What are the paths like for an attractive force,

$$F(r) = -\frac{\beta}{r^3} \qquad \text{with} \quad \beta > 0 \quad ?$$

Recall the DE for $u = 1/r$ as a function for ϕ; for a given value of ℓ,

$$u'' = -u - \frac{\mu}{\ell^2 u^2}F \qquad \text{with} \qquad F = \frac{\alpha}{r^3} = \alpha u^3$$

This gives

$$u'' = -u - \frac{\mu}{\ell^2 u^2}\alpha u^3 = -u - \frac{\mu\alpha}{\ell^2}u = -\left(1 + \frac{\mu\alpha}{\ell^2}\right)u \equiv -\beta^2 u$$

which has the general solution

$$u = A\sin\beta\phi + B\cos\beta\phi \qquad \Longrightarrow \qquad r = \frac{1}{A\sin\beta\phi + B\cos\beta\phi}$$

It is not immediately obvious what kind of curve this is except that there will be some value of ϕ where $r \to \infty$. (Note, it is the equation of a straight line when $\beta = 1$ but the whole point here is that $\beta \neq 1$.)

If the potential is attractive then $\alpha < 0$ and then we can't necessarily write the combination $1 + \mu\alpha/\ell^2$ as the positive number β^2. For some values of α and ℓ it could be negative and then the solutions of

$$u'' = -\left(1 + \frac{\mu\alpha}{\ell^2}\right)u = +\gamma^2 u$$

will be exponentials, either growing or decaying. For such solutions, r does not blow up at any particular ϕ and the particle either spirals away from or toward the center of force.

6. Recall the equation from the text,

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \mathbf{\Omega} \times \mathbf{Q}$$

a) What does this equation tell us?

This relation tells us how the time derivative of a vector in the inertial frame is related to the time derivative of the vector in a rotating frame, where the angular velocity of rotation for the frame is $\mathbf{\Omega}$.

b) Describe how it was used to derive "Newton's 2nd law in a rotating frame".

With this relation we can relate

$$\left(\frac{d}{dt}\right)_{\text{inertial}} \quad \text{to} \quad \left(\frac{d}{dt}\right)_{\text{rotating}}$$

and then rewrite Newton's 2nd law which involves a second derivative in time of $\mathbf{r}$ for the *inertial* frame. We then pull out the second derivative in time of $\mathbf{r}$ for the rotating frame and find that in addition to the true forces there are extra terms, the centrifugal and Coriolis forces.

7. What is the direction and magnitude of the centrifugal force acting on a particle on the earth's surface? (Be sure to define all angle in the answer.)

Why is this "force" difficult to measure (so we basically just give up and redefine $\mathbf{g}$)?

The centrifugal force is given by

$$m((\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega})$$

so with $\mathbf{\Omega}$ pointing in the $\hat{\mathbf{z}}$ direction, $\mathbf{\Omega} \times \mathbf{r}$ points along $\hat{\boldsymbol{\theta}}$, and then the centrifugal force points along $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{z}}$, which is the direction known as $\hat{\boldsymbol{\rho}}$. For a point at colatitude θ the magnitude is

$$F_{\rm cen} = \Omega^2 R \sin\theta$$

The force is difficult to measure because at any location it is hard to find what the true vertical direction is except to hang a plumb line but then the hanging plumb (?) is also affected by the centrifugal force.

8. When we studied the effect of the Coriolis force on free fall, we derived the equations

$$\begin{aligned}
\ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\
\ddot{y} &= -2\Omega\dot{x}\cos\theta \\
\ddot{z} &= -g + 2\Omega\dot{x}\sin\theta
\end{aligned}$$

where z is "up", x goes East and y goes North.

In the book, we derived the equations of motion for an object dropped from a height h. We did this by writing down the *zeroth–order* solution for the motion and then putting these results back into the equations, keeping in mind that Ω is "small".

a) Suppose this time a mass m is thrown upward from ground level with speed v_0. Find the first–order solution for $x(t)$, $y(t)$ and $z(t)$.

Clearly, the $0^{\rm th}$--order solution to the problem is

$$x = 0 \qquad y = 0 \qquad z = v_0 t - \tfrac{1}{2}gt^2 \qquad \Longrightarrow \qquad \dot{z} = v_0 - gt$$

and "recycling" this solution on the right side of the exact equations gives

$$\ddot{x} = -2\Omega(v_0 - gt) = -2\Omega v_0 t + 2\Omega g t \qquad \ddot{y} = 0 \qquad \ddot{z} = -g$$

which has a (new) solution for x; since at $t = 0$, $\dot{x} = 0$ and $x = 0$,

$$\dot{x} = -2\Omega v_0 t + \Omega g t^2 \qquad \Longrightarrow \qquad x = -\Omega v_0 t^2 + \tfrac{1}{3}gt^3$$

and to the same order

$$y = 0 \qquad \text{and} \qquad z = v_0 t - \tfrac{1}{2}gt^2$$

b) From the first-order answer for the time in flight, find where the mass lands back on the ground in relation to where it started.

The ball returns to the ground when $z = 0$ for the second time, and (from elementary physics) that is at $t = \frac{2v_0}{g}$. Putting this into the x equation, the value of x at impact is

$$x = -\Omega\frac{v_0^2}{g^2} + \tfrac{1}{3}\Omega g\frac{v_0^3}{g^3} = -\frac{2}{3}\frac{\Omega v_0^3}{g^2}$$

9. What are the "Euler angles" and in what type of physical problem might they be useful?

The Euler angles (θ, ϕ, ψ) give an explicit sequence of angles through which one rotates a "stationary" set of coordinates (around particular axes) to match those stuck into a rotating rigid body. One can deduce a reasonably simple expression for the kinetic energy of the rigid body, to be used in the lagrangian. But we didn't get that far.

Useful Equations

Math

$$f(x) = (x - x_0)f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!}f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!}f^{(3)}(x_0) + \cdots$$

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

Physics:

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{F} = -\nabla U \qquad \mathbf{p} = m\mathbf{v} \qquad T = \tfrac{1}{2}mv^2 \qquad U_{\text{grav}} = mgy$$

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \qquad L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$

$$\mathcal{L} = T - U \qquad \mathcal{L} = \mathcal{L}(q_1, q_2, \ldots \dot{q}_1, \dot{q}_2, \ldots t) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\mathcal{H} = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L} \qquad (\text{gen. } \mathbf{p}) = m\mathbf{v} + q\mathbf{A}$$

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 \qquad \mathbf{L}(\text{CM}) = \mathbf{r}\times\mu\dot{\mathbf{r}}$$

$$\dot{\phi} = \frac{\ell}{\mu r^2} \qquad F_{\text{cf}} = \frac{\ell^2}{\mu r^3} \qquad \mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{\text{cf}}(r)] = -\frac{d}{dr}U_{\text{eff}}(r) \qquad U_{\text{eff}} = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$u = \frac{1}{r} \qquad u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F \qquad r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \qquad c = \frac{\ell^2}{\gamma\mu} \qquad \gamma = Gm_1m_2$$

$$E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) \qquad \tau^2 = \frac{4\pi^2}{GM_s}a^3$$

$$\mathbf{F}_{\text{inertial}} = -m\mathbf{A} \qquad \mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} \qquad \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega}\times\mathbf{Q}$$

$$m\ddot{\mathbf{r}} = \mathbf{F} = 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} + m(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega}$$

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha\mathbf{r}_\alpha \qquad \mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = M\dot{\mathbf{R}}$$

$$\mathbf{L} = \mathbf{L}(\text{CM}) + \mathbf{L}(\text{rel to CM}) \qquad T = T(\text{CM}) + T(\text{rel to CM})$$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \qquad T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \qquad (\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega}$$