Phys 3610, Fall 2008
Exam #2

1. Give definitions (explanations) of:

a) Stable equilibrium

Configuration (position) of a system where the force on a particle is zero but for which a small displacement from that position will send the mass moving far away, not returning.

b) Decay time of a damped oscillator

Time required for the amplitude (or position itself, if there are no oscillations to decay to a small fraction (like $1/e$) of its maximum value.

c) Resonance frequency of a driven damped system.

The frequency of the driving force (or possibly the natural frequency of the oscillator if we are looking changing *that* quantity) for the amplitude of the long-term solution is a maximum.

2. Consider the isotropic harmonic oscillator in 2 dimensions, for a particle of mass m, where we have

$$\mathbf{F} = -k\mathbf{r}$$

a) Find the general solution for the coordinates of the particle, $x(t)$ and $y(t)$.

The force law decomposes as

$$F_x = -kx \qquad \text{and} \qquad F_y = -ky$$

So they x and y coordinates separately oscillate with general solutions:

$$x(t) = A_x \cos(\omega t - \delta_x) \qquad y(t) = A_y \cos(\omega t - \delta_y)$$

b) Suppose at $t = 0$ the particle has coordinates and velocity given by

$$\mathbf{r}_0 = R\hat{\mathbf{x}} \qquad \text{and} \qquad \mathbf{v}_0 = v_0\hat{\mathbf{y}}$$

Solve for $x(t)$ and $y(t)$.

Here we have

$$x(0) = R \qquad y(0) = 0 \qquad \dot{x}(0) = 0 \qquad \dot{y}(0) = v_0$$

so, considering that

$$\dot{x}(t) = -\omega A_x \sin(\omega t - \delta_x) \qquad \dot{y}(t) = -\omega A_y \sin(\omega t - \delta_y)$$

a little cogitation gives

$$x(t) = R\cos(\omega t) \qquad y(t) = \frac{v_0}{\omega}\sin(\omega t)$$

as the correct solutions for $x(t)$ and $y(t)$.

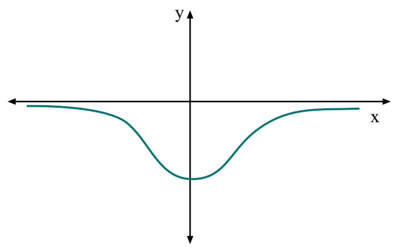
3. The potential energy for a mass m moving in one dimension is given by

$$U(x) = -\frac{A}{\cosh^2(\alpha x)}$$

where A and α are positive constants.

a) Sketch this function; what is the point of equilibrium? Is the equilibrium stable?

Since $\cosh(x)$ has its minimum at $x = 0$ then $\frac{1}{\cosh(x)}$ must have its maximum at $x = 0$ and go to zero at $x = \pm\infty$. Then the given function ought to behave as shown in the graph here.



Clearly the minimum value is at $x = 0$, as seen from the derivative of U:

$$\frac{dU}{dx} = +2A\alpha\frac{\sinh(\alpha x)}{\cosh^3(\alpha x)}$$

which is zero at $x = 0$.

b) Well, it *is* stable! For oscillations about this point find the equivalent harmonic potential and identify the force constant k.

We need to find the behavior of the function for small x. To get, find the Taylor series for the given $U(x)$ up to second order. We need the second derivtive, and we find:

$$U''(x) = \frac{2A\alpha^2}{\cosh^2(\alpha x)} - \frac{6A\alpha^2\sinh^2(\alpha x)}{\cosh^4(\alpha x)}$$

so that

$$U''(0) = 2A\alpha^2$$

Then the Taylor expansion for $U(x)$ is

$$\begin{aligned} U(x) &= U(0) + \frac{1}{1!}U'(0)x + \frac{1}{2!}U''(x)x^2 + \cdots \\ &= -A + 0 + A\alpha^2 + \cdots \end{aligned}$$

which is equivalent to a constant term plus a potential $\frac{1}{2}kx^2$ if we indentify

$$\tfrac{1}{2}k = A\alpha^2 \qquad \Longrightarrow \qquad k = 2A\alpha^2$$

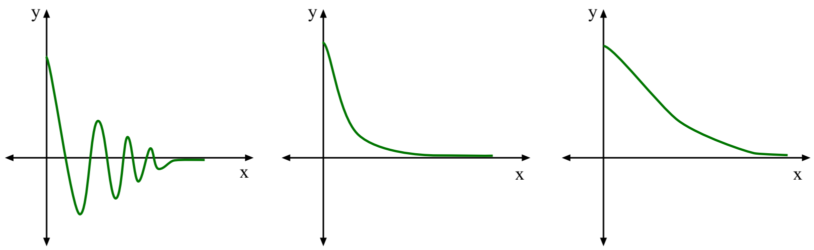
4. Sketch (reasonably accurately) a graph of the motion $x(t)$ of a damped harmonic oscillator if it has displacement $x = A$ and $v = 0$ at $t = 0$ and

a) $\beta = \frac{1}{5}\omega_0$ (Hint: What is decay time?)

b) β slightly less than ω_0.

c) $\beta = 6\omega_0$

Curves look something like:



5. The resonance curve for a driven damped oscillator is (or, can be) a plot of A^2 vs. ω_0. What is the meaning of A here? (I know it's an amplitude of some sort. Be precise.) For that matter, what is the meaning of ω_0?

A is the amplitude of the long-term solution, the one that keeps oscillating harmonically after the transient part of the total solution has died off.

6. What is the definition (based on the resonance curve) of the quality factor Q?

The quality factor Q is the Full-Width at Half-Maximum for the plot of A^2 vs. ω (for fixed ω_0). One can show that for weak damping it is given by $Q \approx 2\beta$.

7. In the text and class we solved the "brachistochrone problem", in which we had to find the path which made the integral

$$\int_0^{y_2} \frac{\sqrt{x'(y)^2+1}}{\sqrt{y}}\, dy$$

stationary.

a) What was the physical premise of this problem?

The physical problem was: Given two points in the xy plane, the origin $1 = O$ and a point $2 = (x_2, y_2)$, find the path from 1 to 2 such that a a mass sliding on a frictionless track along this path (starting from rest) gets from 1 to 2 in the shortest amount of time.

b) Show how the physical problem led to the above form for the integral which was to be minimized. (A conservation law will be useful.)

The length of the path element is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}dy$$

and the speed of the mass when it is at a distance y below the x axis is, using conservation of energy,

$$\tfrac{1}{2}mv^2 = mgy \qquad \Longrightarrow \qquad v = \sqrt{2gy}$$

so if we point the y axis downward, then the time spent on each path element is the length divided by the speed:

$$dt = \frac{\sqrt{x'(y)^2+1}dy}{\sqrt{2gy}}$$

and the integral we want to minimize is

$$S = \frac{1}{\sqrt{2g}}\int_1^2 \frac{\sqrt{x'(y)^2+1}dy}{\sqrt{y}}$$

Of course we can just minimize the integral without the $1/\sqrt{2g}$ factor in front.

8. Find the equation of the path joining the origin O to the point $P(1,1)$ that makes the integral $\int_O^P (y'^2+y^2)dx$ stationary.

Here we have

$$f(y',y,x) = (y'^2+y^2)$$

and we calculate

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial y'} = 2y'$$

then the E-L equation gives

$$2y = \frac{d}{dt}(2y') = 2y'' \qquad \Longrightarrow \qquad y'' = y$$

which has the general solution

$$y(x) = A\cosh(x) + B\sinh(x)$$

Since $(0,0)$ is on the curve, this gives $0 = A$ and since $(1,1)$ is on the curve, we get

$$1 = B\sinh(1) \qquad \Longrightarrow \qquad B = \frac{1}{\sinh(1)}$$

so the solution is

$$y(x) = \frac{\sinh(x)}{\sinh(1)}$$

9. Find and describe the path $y = y(x)$ for which the integral $\displaystyle\int_{x_1}^{x_2} \sqrt{x}\sqrt{1-y'^2}\,dx$ is stationary.

(On this one, you can leave any constants of integration undetermined, since I didn't give values for x_1 and x_2.)

Here we have

$$f(y',y,x) = \sqrt{x}\sqrt{1-y'^2}$$

which we note does not depend on y so as we've seen, the E-L equation implies

$$\frac{\partial f}{\partial y'} = \text{const} \qquad \Longrightarrow \qquad \frac{2\sqrt{x}y'}{2\sqrt{1-y'^2}} = \frac{\sqrt{x}y'}{\sqrt{1-y'^2}} = C$$

This gives

$$xy'^2 = C^2(1-y'^2) \qquad \Longrightarrow \qquad (C^2+x)y'^2 = C^2 \qquad \Longrightarrow \qquad y' = \frac{C}{\sqrt{C^2+x}}$$

This integrates pretty easily to give

$$y = \frac{C_1^2}{2}\sqrt{C_1^2+x} + C_2 \qquad \Longrightarrow \qquad y - C_2 = \frac{C_1^2}{2}\sqrt{C_1^2+x}$$

Then squaring both sides gives

$$(y-C_2)^2 = \frac{C_1^4}{4}(C_1^2+x)$$

Anyway, this is a parabola.

Useful Equations

Math

$$f(x) = (x - x_0)f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!}f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!}f^{(3)}(x_0) + \cdots$$

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Physics:

$$\mathbf{v} = \dot{\mathbf{r}} \qquad \mathbf{a} = \dot{\mathbf{v}} \qquad \mathbf{p} = m\mathbf{v} \qquad \mathbf{F} = m\mathbf{a} = \dot{\mathbf{p}}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad \dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f_{\text{lin}} = bv \qquad f_{\text{quad}} = cv^2$$

$$b = \beta D \qquad c = \gamma D^2 \qquad \beta = 1.6 \times 10^{-4}\ \text{N} \cdot \text{s/m}^2 \qquad \gamma = 0.25\ \text{N} \cdot \text{s}^2/\text{m}^4$$

$$\mathbf{f} = -f(v)\hat{\mathbf{v}} \qquad f(v) = bv + cv^2 \qquad \mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \omega = \frac{qB}{m}$$

$$m\dot{v} = -\dot{m}v_{\text{ex}} \qquad \mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

$$\ell = \mathbf{r} \times \mathbf{p} \qquad \dot{\ell} = \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma} \qquad \mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \qquad \dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$$

$$L_z = I\omega \qquad I = \sum_\alpha m_\alpha \rho_\alpha^2 \qquad \frac{d}{dt}\mathbf{L}(\text{about CM}) = \mathbf{\Gamma}^{\text{ext}}(\text{about CM})$$

$$T = \tfrac{1}{2}mv^2 \qquad W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \qquad \Delta T = W(1 \to 2) \qquad \mathbf{F} = -\nabla U$$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \qquad x(t) = Ae^{-\beta t}\cos(\omega_1 - \delta) \qquad x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$A_{\text{max}} = \frac{f_0}{2\beta\omega_0} \qquad Q = \frac{\omega_0}{2\beta} = \frac{2\pi}{\omega_0}$$

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$