Potential for a Moving Point Charge Once More

In an earlier handout I arrived at the potential for a moving point charge with a trick that made working with a three-dimensional delta function a little easier. We can still work out the integral even if we don't use this trick and we *do* get the same answer! It is a good exercise to go through the derivation straight–on because one gains experience in dealing with a nasty delta-function.

Again, begin with the retarded potential of a charge distribution:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'$$

where

$$t_r = t - \frac{\mathscr{r}}{c} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

At any time, our point charge has a location in space given by $\mathbf{w}(t)$. From that, the charge density of the "distribution" at location $\mathbf{r}'$ and time t' is

$$\rho(\mathbf{r}',t') = q\delta^3(\mathbf{r}' - \mathbf{w}(t'))$$

Putting all of these equations together we have (as before)

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\,\delta^3\left(\mathbf{r}' - \mathbf{w}\left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{\mathscr{r}}\,d\tau' \tag{1}$$

which is an integral which can *not* be evaluated immediately because the delta function has such a nasty argument. So what do we do?

The only delta function we really know how to evaluate is one with a single argument, e.g. $\delta^3(\mathbf{r})$ when the integral is over all $\mathbf{r}$. What we need to do is to change variables from the vector $\mathbf{r}'$ to the variable

$$\mathbf{u} = \mathbf{r}' - \mathbf{w}\left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]$$

To change 3 variables (r'_x, r'_y, r'_z) to another set of variables $(u_x,\ u_y,\ u_z)$ where the relations where the relations between the sets are rather complicated we have to relate the volume elements of the spaces by means of the Jacobian of the transformation. The Jacobian of the transformation from (r'_x, r'_y, r'_z) to (u_x, u_y, u_z) is given by

$$\frac{\partial(u_x,u_y,u_z)}{\partial(r'_x,r'_y,r'_z)} \equiv \begin{vmatrix} \partial u_x/\partial r'_x & \partial u_x/\partial r'_y & \partial u_x/\partial r'_z \\ \partial u_y/\partial r'_x & \partial u_y/\partial r'_y & \partial u_y/\partial r'_z \\ \partial u_z/\partial r'_x & \partial u_z/\partial r'_y & \partial u_z/\partial r'_z \end{vmatrix} \tag{2}$$

where the right hand is a determinant of the array.

Once we have the determinant of the Jacobian, we use it to transform the integral into

$$V(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\int\frac{\delta^3(\mathbf{u})}{\mathscr{r}}\frac{d^3u}{\left|\frac{\partial(u_x,u_y,u_z)}{\partial(r'_x,r'_y,r'_z)}\right|}\tag{3}$$

Here we mean that $\mathscr{r}=|\mathbf{r}-\mathbf{r}'|$ will expressed as a function of $\mathbf{u}$. And note we use the *absolute value* of the Jacobian to scale the volume element.

Now, the variables $\mathbf{u}$ and $\mathbf{r}'$ are related by

$$u_i=r'_i+w_i\left[t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]$$

where

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{(r_x-r'_x)^2+(r_y-r'_y)^2+(r_z-r'_z)^2}$$

We note that u_i will depend on r'_j with $j\neq i$ through the second term only; the derivative of r'_i with respect to r'_j is δ_{ij}.

Using the fact that

$$\frac{\partial|\mathbf{r}-\mathbf{r}'|}{\partial r'_j}=\frac{1}{2}\frac{(-2)(r_j-r'_j)}{\sqrt{(r_x-r'_x)^2+(r_y-r'_y)^2+(r_z-r'_z)^2}}\;=\;-\hat{\boldsymbol{\mathscr{r}}}_j$$

and using the chain rule, we find the general element of the Jacobian matrix,

$$\begin{aligned}\frac{\partial u_i}{\partial r'_j}&=\delta_{ij}-w'_i\left[t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\left(\frac{-1}{c}\right)\frac{\partial|\mathbf{r}-\mathbf{r}'|}{\partial r'_j}\\&=\delta_{ij}-\frac{v_i(t_r)\hat{\boldsymbol{\mathscr{r}}}_j}{c}\end{aligned}\tag{4}$$

where we've written the function $w'(t)$ as $v(t)$ and used the retarded time t_r for brevity!

With the general element given by 4, the whole Jacobian is

$$\frac{\partial(u_x,u_y,u_z)}{\partial(r'_x,r'_y,r'_z)}=\begin{vmatrix}\partial u_x/\partial r'_x & \partial u_x/\partial r'_y & \partial u_x/\partial r'_z\\ \partial u_y/\partial r'_x & \partial u_y/\partial r'_y & \partial u_y/\partial r'_z\\ \partial u_z/\partial r'_x & \partial u_z/\partial r'_y & \partial u_z/\partial r'_z\end{vmatrix}=\begin{vmatrix}1-\beta_x\hat{\boldsymbol{\mathscr{r}}}_x & -\beta_x\hat{\boldsymbol{\mathscr{r}}}_y & -\beta_x\hat{\boldsymbol{\mathscr{r}}}_y\\ -\beta_y\hat{\boldsymbol{\mathscr{r}}}_x & 1-\beta_y\hat{\boldsymbol{\mathscr{r}}}_y & -\beta_y\hat{\boldsymbol{\mathscr{r}}}_z\\ -\beta_z\hat{\boldsymbol{\mathscr{r}}}_x & -\beta_z\hat{\boldsymbol{\mathscr{r}}}_y & 1-\beta_z\hat{\boldsymbol{\mathscr{r}}}_z\end{vmatrix}\tag{5}$$

where we've used $\beta_i=v_i/c$. The determinant is now in a manageable form but it still seems like it will be a mess since we have expand it out!

Not to worry. Just write out the determinant and you will find that nearly all the terms cancel. This is because the factors β_i and $\hat{\boldsymbol{\mathscr{r}}}_i$ repeat even though they appear in different combinations. Well, just do it; you'll see what I mean. We are left with terms which came from the diagonal elements, and:

$$\frac{\partial(u_x,u_y,u_z)}{\partial(r'_x,r'_y,r'_z)}=1-\beta_x\hat{\boldsymbol{\mathscr{r}}}_x-\beta_y\hat{\boldsymbol{\mathscr{r}}}_y-\beta_z\hat{\boldsymbol{\mathscr{r}}}_z=1-\frac{\mathbf{v}\cdot\hat{\boldsymbol{\mathscr{r}}}}{c}\tag{6}$$

The result is simple! Put it into 3 and we have:

$$V(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\int\frac{\delta^3(\mathbf{u})}{\mathscr{r}}\frac{d^3u}{\left|1-\frac{\mathbf{v}\cdot\hat{\boldsymbol{\mathscr{r}}}}{c}\right|}\tag{7}$$

In fact, since $v < c$, $\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c$ is always less than 1 and we can drop the absolute value in the deonominator.

And *now* the delta function collapses everything, telling us that we can take $\boldsymbol{\mathscr{r}}$ and $\mathbf{v}$ outside the integral with the stipulation that they are evaluted at the value of $\mathbf{r}'$ such that $\mathbf{u}$ is zero; but that is the retarded position of the charge, i.e. its location at the retarded time. *With this stipulation*, we then write

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 \mathscr{r}} \frac{1}{\left(1 - \frac{\mathbf{v} \cdot \hat{\boldsymbol{\mathscr{r}}}}{c}\right)} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} \tag{8}$$