Phys 4620, Spring 2008
Exam #3

1. Protons (mass 938.27 MeV/c^2) with a kinetic energy of 5.0 MeV move in a circular path of radius 1.5 m.

a) Find the speed of the protons. Are we safe in ignoring relativity? Find the time for one orbit. Find the acceleration of the protons.

Find the speed of the protons, using relativity. Suppress the factors of c throughout (let's be big boys and girls...)

$$E = M + T = 938.3 \text{ MeV} + 5.0 \text{ MeV} = 943.3 \text{ MeV} = \gamma M = \gamma(938.3 \text{ MeV}$$

Then

$$\gamma = 1.00532 = \frac{1}{\sqrt{1 - u^2/c^2}} \qquad \Longrightarrow \qquad u = 0.102c = 3.08 \times 10^7 \ \tfrac{\text{m}}{\text{s}}$$

This is about a tenth the speed of light and we are *probably* OK in ignoring relativity as long as we don't seek accuracy down to third decimal place (maybe). The time to make one orbit is

$$t = \frac{2\pi r}{v} = 3.05 \times 10^{-7} \text{ s}$$

Note that neglecting relativity and hoping for the best gives with

$$T = 5.0 \text{ MeV} = 8.0 \times 10^{-13} \text{ J}$$

from which, with $T = \frac{1}{2}mv^2$, we get $v = 3.10 \times 10^7 \ \frac{\text{m}}{\text{s}}$ which, being a tenth the speed light, justifies (sort of) the neglect of relativity.

b) Irregardless of your answer to (a) we will treat the speed of the protons as small. Calculate the total radiated power of the protons and find the fraction of the kinetic energy that it loses to radiation on each orbit.

The acceleration (centripetal) of the proton is

$$a = \frac{v^2}{r} = \frac{(3.08 \times 10^7 \ \frac{\text{m}}{\text{s}})^2}{(1.5 \text{ m})} = 6.32 \times 10^{14} \ \tfrac{\text{m}}{\text{s}^2}$$

so assuming we have a "small" speed here, the Larmor formula gives the total radiated power,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{(4\pi \times 10^{-7} \ \frac{\text{T}\cdot\text{m}}{\text{A}})(1.6 \times 10^{-19} \text{ C})^2 (6.3 \times 10^{14} \ \frac{\text{m}}{\text{s}^2})^2}{6\pi(3.00 \times 10^8 \ \frac{\text{m}}{\text{s}})} = 2.26 \times 10^{-24} \text{ W}$$

The energy lost to radiation on one orbit is

$$\Delta E = Pt = (2.26 \times 10^{-24} \text{ W})(3.05 \times 10^{-7} \text{ s}) = 6.89 \times 10^{-31} \text{ J}$$

which is a fractional loss of

$$\text{Frac} = \frac{\Delta E}{K} = 8.6 \times 10^{-19}$$

which is very small.

2. In the lab reference frame, particle C moves in the $+x$ direction with speed $\frac{3}{4}c$ and particle B moves in the $+x$ direction with speed $\frac{1}{2}c$.

What is the speed of particle C in the reference frame of particle B? (Think of A, B, C as (ground, cops, crooks) if it makes it easier!

Frame C moves at $+\frac{3}{4}c$ wrt frame A, but frame A moves at $-\frac{1}{2}c$ wrt frame B. So *these* are the velocities we add to get the velocity of C wrt B. Of course, add them the Einstein way:

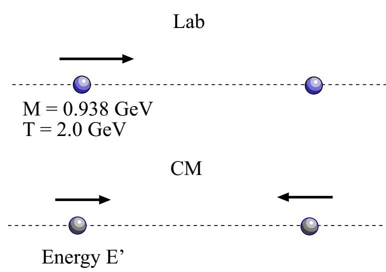
$$v_{CB} = \frac{\frac{3}{4}c - \frac{1}{2}c}{1 + \frac{1}{c^2}\left(\frac{3}{4}c\right)\left(\frac{-1}{2}c\right)} = \frac{\frac{1}{4}c}{1 - \frac{3}{8}} = \frac{2}{5}c$$

Frame C moves at $\boxed{\frac{2}{5}c}$ wrt B.

The Galilean answer is $\frac{1}{4}c$.

3. In the lab frame a proton with kinetic energy 2.0 GeV collides with a stationary proton.

Find the kinetic energy of each proton in the center-of-momentum frame. (Proton mass 0.938 GeV$/c^2$)



Pre-collision picture of the two protons in both frames is shown here. The 4-momentum in the Lab frame, where the proton has momentum $\mathbf{p} = p\hat{\mathbf{x}}$ is

$$(E_p + M, p, 0, 0) \quad \text{where} \quad E^2 = p^2 + M^2$$

whereas in the CM frame each proton has momentum of magnitude p' and energy E'_p so that the total 4-momentum is

$$(2E'_p, 0, 0, 0)$$

Now, the total 4-momentum, being a 4-vector has an invariant magnitude; this gives

$$-(E_p + M)^2 + p^2 = -4E'^2_p$$

Expanding the left side,

$$-E_p^2 - 2E_pM - M^2 + p^2 = -4E'^2_p$$

Use $p^2 = E_p^2 - M^2$, then

$$-2E_pM - 2M^2 = -4E'^2_p$$

Now a bit of algebra gives

$$E'^2_p = \frac{M(M + E_p)}{2}$$

With $E_p = T + M = 2.0\text{ GeV} + 0.938\text{ GeV} = 2.938\text{ GeV}$, this gives

$$E'_p = 1.35\text{ GeV} \qquad T' = 0.410\text{ GeV} = 410\text{ MeV}$$

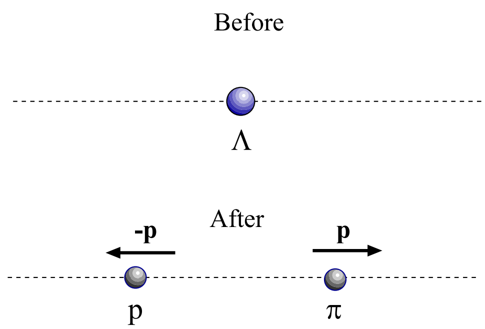
4. The Λ baryon (mass $1115.68\text{ MeV}/c^2$) primarily decays into a proton (mass $938.27\text{ MeV}/c^2$) and a pion (π^-) (mass $139.57\text{ MeV}/c^2$):

$$\Lambda \longrightarrow p + \pi^-$$

Find the momenta (or kinetic energies) of the final particles in the center-of-momentum frame.

Hintz: Draw pictures representing the system before and after the decay. Write out the total 4-momentum of the system before and after (note, the 3-momentum is always adds up to zero) and use the relation between momentum and energy, $E^2 = m^2c^4 + \mathbf{p}^2c^2$.

The decay in the CM frame is shown here; afterwards the particles have three-momenta $\mathbf{p}$ and $-\mathbf{p}$. The total 4-momentum before the decay is



$$(M_\Lambda, 0)$$

and the total four momentum after the decay is

$$(E_p + E_\pi, 0) = (\sqrt{p^2 + M_p^2} + \sqrt{p^2 + M_\pi^2}, 0)$$

There are no invariants to apply here, since there's only one frame, but the 4-momentum has to be conserved. That gives us:

$$M_\Lambda = \sqrt{p^2 + M_p^2} + \sqrt{p^2 + M_\pi^2}$$

which can be solved for p with some algebra.

Write it as

$$M_\Lambda - \sqrt{p^2 + M_\pi^2} = \sqrt{p^2 + M_p^2}$$

and then square both sides:

$$M_\Lambda + p^2 + M_\pi^2 - 2M_\Lambda\sqrt{p^2 + M_\pi^2} = p^2 + M_p^2$$

Cancel and move things around. This gives:

$$\sqrt{p^2 + M_\pi^2} = \frac{M_\Lambda^2 - M_p^2 + M_\pi^2}{2M_\Lambda} = 172.6\text{ MeV}$$

This gives a 3-momentum of $p = 101\text{ MeV}$ and total energies of

$$E_p = 943\text{ MeV} \qquad E_\pi = 173\text{ MeV}$$

5. **a)** How is the relativistic version of Newton's second, expressed as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

different from the non-relativistic one? (Include *some* mathematical expressions and definitions.)

The definition of momentum is different from Newtonian mechanics:

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

so that as $\mathbf{u}$ changes with times both numerator and denominator change; so $d\mathbf{p}/dt$ is *not* just $m\frac{d\mathbf{u}}{dt} = m\mathbf{a}$. We can no longer write down $\mathbf{F} = m\mathbf{a}$.

b) Early on in Chapter 12 we constructed the proper velocity, given by

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

What does the symbol τ refer to here? How do we know that η^μ is a 4-vector when the ordinary velocity

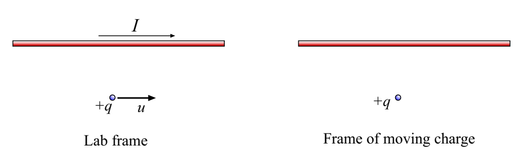
$$v^\mu = \frac{dx^\mu}{dt}$$

is not?

τ is the *proper time interval*, the time interval as measured in the frame of the moving particle. It is the same in all references because the number comes from a *particular* reference frame! With a four-vector in the numerator (dx^μ) and a scalar in the denominator the result is a 4-vector, though as the text notes it is something of a "hybrid" quantity being formed from quantities taken from two different reference frames.

The old velocity $\mathbf{v} = d\mathbf{x}/dt$ is not part of a four-vector because the denominator dt depends on the reference frame so the wuantity does not transform in a simple way.

6. In the warmup to the transformation of EM fields, we did a derivation of the origin of the magnetic force from electrostatics and relativity. We started with a charge in motion next to a current-carrying (but **electrically neutral**) wire and then considered how things were in the frame of the moving charge.



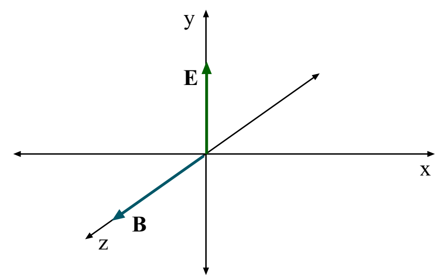
a) In the second reference frame, describe the force on the charge and how it comes about.

In the second reference frame there can only be an electric force on the charge because it is at rest. But in the second frame the charge density of the wire is *not* zero as it is in the lab. This is because the charge densities of the positive and negative components of the wire do not change in the same way due to their different velocities. So there is some net charge density of the wire in the second frame and from this there is an *electrostatic force* acting on the charge; in fact for positive q this force will pull the charge *toward* the wire.

b) When we transform back to the lab frame, the force has the same direction (do you recall why?). State why the result is consistent with what you know about magnetic force on a moving charged particle.

Back in the lab frame, there can't be an electric force on the charge because the wire is neutral. The force on the charge (toward the wire) is the magnetic force. We know that this force has to be toward the wire since at the location of the charge the B goes into the page and from that and the right-hand rule (for positive q) the force is upward, toward the wire.

7. In the lab frame there are uniform E and B field directed along the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axes, respectively. The E field has magnitude $1.0 \times 10^6 \, \frac{\mathrm{N}}{\mathrm{C}}$ and the B field has magnitude 0.050 T.



a) Is there a reference frame in which the electric field is zero? What is the value of the B field in that frame?

Consider the invariant $E^2 - c^2B^2$. Tryng the numbers, we see that $E < cB$ so that this invariant is *negative* in all reference frames. Thus we have a chance of finding a frame where E is zero. (But we can't have one where $B = 0$.)

Assuming that there is such a frame where the magnitude of B is $\bar{B}$, then

$$-c^2\bar{B} = E^2 - c^2B^2 = -2.24 \times 10^{14}$$

which gives

$$\bar{B} = 4.99 \times 10^{-2} \ \mathrm{T}$$

that is, not much different.

b) The answer to the first part of (a) should be "yes". Find the velocity of that reference frame with respect to the lab.

In the frame in question, we have $\bar{E}_y = 0$ so use the transformation equation:

$$\bar{E}_y = \gamma(E_y - vB_z)$$

and if this is zero then

$$E_y = vB_z \qquad \Longrightarrow \qquad v = \frac{E_y}{B_z} = \frac{1.6 \times 10^6}{0.05} \, \frac{\mathrm{m}}{\mathrm{s}} = 3.2 \times 10^7 \, \frac{\mathrm{m}}{\mathrm{s}} = 0.107c$$

8. What does it mean to say that a mathematical object is a 4-tensor, say of the form $t^{\mu\nu}$? Specifically, what property does the object $t^{\mu\nu}$ have to have?

A tensor is a physical quantity with two indices: $t^{\mu\nu}$ so that there are 16 entries in all. Given its value(s) in one reference frame, we can find its values in another reference frame via

$$\bar{t}^{\mu\nu} = \sum_{\rho,\, \sigma} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}t^{\rho\sigma}$$

where the Λ's are the Lorentz transformation matrices.

9. Show how the relativistic form of (two of) the Maxwell equations

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}$$

gives the correct result when μ is either 1, 2, or 3. (That is, $\mu = i$ in our usual notation.) Show the steps!

Choose $\mu = 1$. Given that

$$\frac{\partial F^{1\nu}}{\partial x^{\nu}} = \mu_0 J^{1}$$

then reading off the entries for $F^{\mu\nu}$ and using $x^0 = ct$ and $J^1 = +J_x$, then

$$\frac{\partial(-E_x/c)}{\partial(ct)} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$

But we recognize two terms on the left as parts of one component of the curl, and we this gives

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\nabla \times \mathbf{B})_x = \mu_0 J_x \ .$$

With $\mu = 2$ and $\mu = 3$ we will get similar equations for y and z. With $1/c^2 = \epsilon_0\mu_0$, we get

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

which is the "Ampère-Maxwell" equation from the four Maxwell equations.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$c = 2.998\times10^8\ \frac{\text{m}}{\text{s}}$ $\qquad \epsilon_0 = 8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$ $\qquad \mu_0 = 4\pi\times10^{-7}\ \text{N/A}^2$ $\qquad e = 1.602\times10^{-19}\ \text{C}$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial\lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla\cdot\mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t \qquad \Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x$$

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \qquad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \bar{x} = \gamma(x - vt) \qquad \bar{y} = y \qquad \bar{z} = z$$

$$\bar{x}^\mu = \sum_{\nu=0}^{3}(\Lambda^\mu_\nu)x^\nu \qquad \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) \qquad \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \qquad p^\mu = (E/c, p_x, p_y, p_z) \qquad E = \gamma mc^2$$

$$p^\mu p_\mu = -m^2c^2 \qquad E^2 = p^2c^2 + m^2c^4$$

$$K^\mu = \frac{dp^\mu}{d\tau} \qquad J^\mu = (c\rho, J_x, J_y, J_z) \qquad A^\mu = (V/c, A^x, A^y, A^z) \qquad F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

$$\begin{aligned} \bar{E}_x &= E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y) \\ \bar{B}_x &= B_x \qquad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{aligned}$$

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\} \qquad G^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{array} \right\}$$

Invariants: $\mathbf{E} \cdot \mathbf{B}$, $(E^2 - c^2B^2)$

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \qquad \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_0 J^\mu \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \qquad K^\mu = q\eta_\nu F^{\mu\nu} \qquad A^{\mu\prime} = A^\mu + \frac{\partial\lambda}{\partial x_\mu}$$