Phys 4900, Fall 2011
Problem Set #4

1. Some practice with transforming coordinates by hand.

The wave equation in one dimension is

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0$$

which for mechanical waves holds if we are in the rest frame of the medium; c is the speed of waves in the medium (not to be confused with the speed of light; we're already using v for the relative speed of the two frames).

The Galilean transformation between frames $\mathcal{S}$ and $\bar{\mathcal{S}}$ is

$$\bar{x} = x - vt \qquad \bar{t} = t \qquad\qquad x = \bar{x} + v\bar{t} \qquad t = \bar{t}$$

and to relate the partial derivatives in the two frames we use

$$\frac{\partial}{\partial x} = \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{t}}{\partial x}\frac{\partial}{\partial \bar{t}} \qquad\qquad \frac{\partial}{\partial t} = \frac{\partial \bar{x}}{\partial t}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{t}}{\partial t}\frac{\partial}{\partial \bar{t}}$$

Show that we get

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 V}{\partial \bar{x}^2} + \frac{2v}{c^2}\frac{\partial^2 V}{\partial \bar{t}\,\partial \bar{x}} - \frac{1}{c^2}\frac{\partial^2 V}{\partial \bar{t}^2} = 0$$

So the wave equation is not invariant under a Galilean transformation.

2. The velocity addition law of relativity looks awful:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)}$$

but it can be put into a prettier form using a somewhat useful quantity in relativity called the **rapidity**.

Define the quantity θ, to be associated with each v, by:

$$\frac{v}{c} = \tanh\theta \qquad \theta = \tanh^{-1}(v/c)$$

Then the velocity addition formula becomes

$$\tanh\theta_{AC} = \frac{\tanh\theta_{AB} + \tanh\theta_{BC}}{1 + (\tanh\theta_{AB}\tanh\theta_{BC})}$$

But using the rule for "angle addition" for hyperbolic tangents[1] we can rewrite the right-hand side of this and then get

$$\tanh\theta_{AC} = \tanh(\theta_{AB} + \theta_{BC})$$

[1] Taught wherever hyperbolic functions are taught, which doesn't seem to include TTU's math department.

and this implies

$$\theta_{AC} = \theta_{AB} + \theta_{BC} \quad .$$

So in relativity, *something* adds together, but iit's not velocities, it's *repidities*.

a) Find the rapidities corresponding to speeds of $\frac{3}{4}c$ and $\frac{2}{3}c$. Find the total rapidity.

b) Find the speed corresponding to the total rapidity found in (a). (Just leave everything as a multiple of c. We don't need $\frac{\text{m}}{\text{s}}$ here.)

c) Use the Einstein velocity formula to add the velocities the old–fashioned (!?!) way. These should agree if you've hit the right buttons on your calculator[2]. !

d) Use the identities for the hyperbolic functions (now is a good time to learn them) to write the Lorentz transformation matrix of Eq. (3.11),

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{and} \qquad \beta = \frac{v}{c}$$

in terms of the rapidity θ. The result should make you see the beauty of the mathematics of relativity. If not, you have no soul.

3. *Griffiths EP*, **3.5**

4. *Griffiths EP*, **3.6**

5. *Griffiths EP*, **3.8**

6. *Griffiths EP*, **3.9**

[2]Your calculator has buttons to calculate `tanh` and `tanh`$^{-1}$. Oh, yes it does. Look for it.