Phys 3820, Fall 2009
Exam #1

1. What is the *main* physical effect which keeps a white dwarf star from collapsing further?

It is the so-called **degeneracy pressure** the requirement that electrons occupy different quantum states which effectively requires to move to higher energy states when they are forced to occupy a smaller volume.

2. How could the calculations of the radius of a white dwarf, maximum mass of a white dwarf star and radius of a neutron star (in those long messy problems you did) be improved with better physics?

For the white dwarf star (with electron energy calculated fro the free electron gas) we found that the Fermi energy was nearly relativistic so that including the relativistic relation between energy and momentum would have helped.

For the neutron star, the energies were slightly relativistic but more importantly it turns out that the neutron-neutron interactions play an important role at the densities considered so that we have to correct the model of the *free* neutron gas.

3. The Dirac comb potential gave eigenvalues of allowed energy states in a distribution which is called (a set of) energy bands.

Do the best job you can of explaining in words how these "bands" fell out of the math from solving the Dirac comb.

The energy values are of function of K. Basically, what was K?

Stationary states in a (one-dimensional) lattice potential can be characterized by the index K for which the wave function satisfies

$$\psi(x+a) = e^{iKa}\psi(x)$$

where a is the length over the potential repeats. After solving the Schrödinger equation we find that for some energies there is *no* associated value of K so that the energy spectrum has various finite gaps.

4. A particle of mass m is trapped in a one–dimensional potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases}$$

i.e. the 1–D box. To this potential we add a "small" perturbation

$$V(x) = \begin{cases} \infty & x < 0 \\ V_0 \frac{x(a-x)}{a^2} & 0 < x < a \\ \infty & x > a \end{cases}$$

a) For this to be a "small" perturbation as advertised, what should be true about V_0?

Here V_0 has units of energy and is comparable to the average value of $V(x)$. For this to be small, V_0 should be small compared with the ground state energy:

$$V_0 \ll \frac{\pi^2\hbar^2}{2ma^2}$$

b) Find the first–order correction to the energy of the ground state. (If the integrals are too tedious, you don't need to finish them, but write them out *clearly*.

With H' being the potential given above, the first-order energy correction is

$$E^1_{\rm gs} = \langle\psi_{\rm gs}|H'|\psi_{\rm gs}\rangle$$

This gives

$$E^1_{\rm gs} = \frac{2}{a}\frac{V_0}{a^2}\int_0^a \sin^2\left(\frac{\pi x}{a}\right) x(a-x)\, dx = \frac{2V_0}{a^3}\int_0^a (ax - x^2)\sin^2\left(\frac{\pi x}{a}\right)\, dx$$

That's the main answer. Of course the integral is not really difficult (just uninteresting), and using Maple gives the result

$$E^1_{\rm gs} = \frac{2V_0}{a^3}\frac{1}{12}\frac{a^3(\pi^2+3)}{\pi^2} = \frac{V_0(\pi^2+3)}{6\pi^2}$$

c) Set up some expressions to show how you would evaluate the first–order correction to the ground state wave function. (Go as far as you can with this; show me that you know how it is done.)

We need to evaluate the coefficients of the non-gs wavefunctions for the perturbed wave function:

$$c_m^{(\rm gs)} = \frac{\langle\psi^0_m|H'|\psi^0_{\rm gs}\rangle}{(E_{\rm gs} - E^0_m)}$$

Here the denominator is

$$E_{\rm gs} - E^0_m = \frac{\pi^2\hbar^2}{2ma^2} - \frac{m^2\pi^2}{2ma^a} = -\frac{\pi^2\hbar^2(m^2-1)}{2ma^2}$$

and the numerator is

$$\langle\psi^0_m|H'|\psi^0_{\rm gs}\rangle = \frac{2}{a}\frac{V_0}{a^2}\int_0^a (ax-x^2)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\, dx$$

Half of these are zero from symmetry; $\sin(\pi x/a)$ and $(ax - x^2)$ are *even* about $x = a/2$ and for even m, $\sin(m\pi x/a)$ is *odd* about $x = a/2$ so that we only have a contribution for odd m's. The first contribution is for $m = 3$; Maple gives

$$\frac{2V_0}{a^3}\int_0^a (ax-x^2)\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\, dx = \frac{2V_0}{a^3}\left(-\tfrac{3}{16}\right)\frac{a^3}{\pi^2} = -\frac{3V_0}{8\pi^2}$$

5. **a)** The poor man's relativistic correction to the Schrödinger Hamiltonian,

$$H'_{\rm rel} = -\frac{p^4}{8m^3c^2} \qquad ,$$

was used for the H atom and also for the harmonic oscillator in a homework problem.

How did we arrive at this expression for $H'_{\rm rel}$? (Give a description of the derivation.)

We took the expansion of kinetic energy T in terms of the *relativistic* momentum:

$$T = E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

and did a Taylor expansion to pull off the leading correction from the non-relativistic approximation $p^2/(2m)$. That gave a term proportional to p^4.

b) *Explain* (in words, mostly) the origin of the spin-orbit term

$$H'_{\rm so} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

The interaction can be understood in the (dubious) rest frame of the electron, wherein an orbit proton generates a magnetic field so that the electron's energy depends on whether it is spin-up or spin down. The magnetic field from the proton is proportional to its (vector) angular momentum and from the usual formula $E = -\mu\cdot\mathbf{B}$, since the magnetic moment of the electron is proportional to its vector spin $\mathbf{S}$, we get an energy proportional to $\mathbf{L}\cdot\mathbf{S}$.

6. When we chose states that were "good" for the perturbation $H'_{\rm so}$, we needed eigenstates of $\mathbf{L}\cdot\mathbf{S}$; these were states of "good" j, l (and $s = \frac{1}{2}$). Show that the eigenvalues of the operator $\mathbf{L}\cdot\mathbf{S}$ are

$$\frac{\hbar^2}{2}[j(j+1) - l(l+1) - \tfrac{3}{4}]$$

Use the vector relation

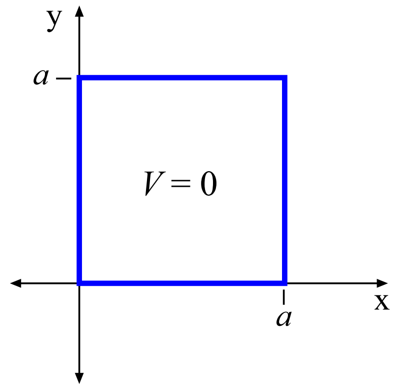
$$\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S} \qquad \Longrightarrow \qquad \mathbf{L}\cdot\mathbf{S} = \tfrac{1}{2}[J^2 - L^2 - S^2]$$

Use the eigenvalues of the squared-angular momentum operators, e.g. for L^2 it is $\hbar^2 l(l+1)$, then the *eigenvalue* of $\mathbf{L}\cdot\mathbf{S}$ is (using the value $s = \frac{1}{2}$ for the single electron,

$$\frac{\hbar^2}{2}[j(j+1) - l(l+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)] = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - \tfrac{3}{4}]$$

7. Consider a particle which moves in *two* dimensions, confined inside a potential where

$$V(x) = \begin{cases} 0 & 0 < x < a \quad \text{and} \quad 0 < y < a \\ \infty & \text{otherwise} \end{cases}$$



a) Write down (no detailed derivation necessary) the ground state wave function for this system. (Hint: Recall that the wave function for the one–dimensional box and associated eigenfunctions is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad \text{with} \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad .)$$

Clearly the Schrödinger equation for this system separates in the coordinates x and y so the eigenfunctions are products of those for x and y alone and the energies are the sums of the values from the x and y solutions. Thus the ground state wave function is

$$\psi_{\rm gs}(x, y) = \frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$

with eigenvalue

$$E_{\rm gs} = \frac{\pi^2\hbar^2}{2ma^2} + \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{ma^2}$$

b) Write down the wave functions and eigenvalue for the (degenerate!) first excited state.

For the first excited state the wave functions are

$$\psi_{\rm ex,1}(x, y) = \frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \qquad \text{and} \qquad \psi_{\rm ex,2}(x, y) = \frac{2}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$$

with eigenvalue

$$E_{\rm gs} = \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{2ma^2}$$

c) Consider the perturbation

$$H' = V_0\, a^2\, \delta(x - \tfrac{a}{4})\, \delta(y - \tfrac{a}{2})$$

where V_0 is in some sense "small". Find the first-order correction to the ground state energy.

Construct the W matrix! Using the labels 1 and 2 for the excited state wave functions as given above, we find:

$$\begin{aligned} W_{11} &= \int\int \psi_1 H' \psi_1 \, dx\, dy = \frac{4}{a^2} V_0 a^2 \int\int \sin^2\left(\frac{2\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) \delta(x - \tfrac{a}{4})\, \delta(y - \tfrac{a}{2})\, dx\, dy \\ &= 4V_0 \sin^2(\tfrac{\pi}{2}) \sin^2(\tfrac{\pi}{2}) = 4V_0 \end{aligned}$$

We also get

$$\begin{aligned} W_{22} &= \int\int \psi_2 H' \psi_2 \, dx\, dy = \frac{4}{a^2} V_0 a^2 \int\int \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{2\pi y}{a}\right) \delta(x - \tfrac{a}{4})\, \delta(y - \tfrac{a}{2})\, dx\, dy \\ &= 4V_0 \sin^2(\tfrac{\pi}{4}) \sin^2(\pi) = 0 \end{aligned}$$

d) Show how you would find the first-order corrections to the first excited states. Does the perturbation "lift" the degeneracy?

8. If the photon had a mass m_γ, the Coulomb potential would be replaced by the **Yukawa potential**,

$$V(\mathbf{r}) = -\frac{e^2}{4\pi\epsilon_0} \frac{e^{-\mu r}}{r}$$

where $\mu = m_\gamma c/\hbar$. Of course, we found eigenvalues and eigenfunctions for the massless Coulomb potential

$$V(\mathbf{r}) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$

Treat the deviation of the Yukawa potential from the Coulomb potential as a *perturbation* and find the correction to the (conventional) H atom ground state energy that would result if the photon had a mass! Just set up the main expression; the integral could be messy.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457\times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938\times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262\times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218\times 10^{-19}\ \mathrm{C} \qquad c = 2.99792\times 10^{8}\ \mathrm{\tfrac{m}{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p\to\frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty}\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty\ \text{Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta\, \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm 2i\phi} \text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c \hbar^3} \left(\frac{c^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \phi} \right) \qquad L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m{\pm}1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0\hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\,\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$