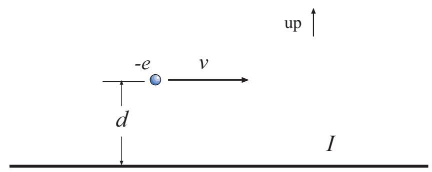
Phys 4610, Fall 2006
Exam #3

1. A particle with mass m and charge $-e$ moves to the right with speed v just above a horizontal wire (i.e. both velocity and wire are in the plane ofthe page). The particle is a distance d from the wire. The wire carries a current I; I is just sufficient to cancel out the force of gravity on the particle.



a) Which way does the current in the wire go? (How do you know this?)

Suppose the current goes to the right then at the location of the charge the field would come out of the page and the force on the charge (from $\mathbf{F} = (-e)\mathbf{v} \times \mathbf{B}$) would go up; and that is just what we want to oppose theh force of gravity (which is down, of course) so the current in the wire goes $\boxed{\text{to the right}}$.

b) Find I in terms of all other relevant parameters.

The $\mathbf{B}$ field at the location of the charge has magnitude $B = \frac{\mu_0 I}{2\pi d}$ so the magnetic force has magnitude

$$F_{\rm mag} = ev\frac{\mu_0 I}{2\pi d}$$

To cancel with gravity we need $F_{\rm mag} = mg$ which gives

$$\frac{ev\mu_0 I}{2\pi d} = mg \qquad \Longrightarrow \qquad I = \frac{2\pi dmg}{ev\mu_0}$$

To get a rough idea of the size here, suppose $d = 2.0$ cm and $v = c/10$. Then the current is

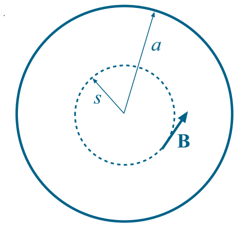
$$I = \frac{2\pi(2.0 \times 10^{-2}\ \mathrm{m})(9.11 \times 10^{-31}\ \mathrm{kg})(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(1.6 \times 10^{-19}\ \mathrm{C})(3.0 \times 10^{7}\,\frac{\mathrm{m}}{\mathrm{s}})(4\pi \times 10^{-7}\ \mathrm{N/A^2})} = 1.9 \times 10^{-13}\ \mathrm{A} \quad \text{(!!)}$$

The force of gravity is weak in comparison to EM forces!

2. A wire of circular cross-section with radius a carries a current I uniformly distributed over the cross-section.

Find the magnetic field inside the wire. This can be a semi-rigorous derivation. (Yeah, this is 2120 stuff but show me that you know how to do it.)

Assume that $\mathbf{B}$ is tangential and draw an Amperian loop of radius s. Then the current enclosed by this loop is

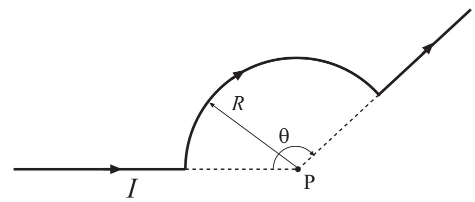


$$I_{\text{enc}} = \frac{\pi s^2}{\pi a^2} I = \frac{s^2 I}{a^2}$$

and on this path, $\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi (2\pi s)$. Then from Ampere's law,

$$B_\phi 2\pi s = \mu_0 \frac{s^2 I}{a^2} \qquad \Longrightarrow \qquad B_\phi \frac{\mu_0 s I}{2\pi a^2} \qquad \Longrightarrow \qquad \mathbf{B} = \frac{\mu_0 s I}{2\pi a^2} \hat{\boldsymbol{\phi}}$$

3. A long piece of wire is bent into an arc of radius R and subtended angle θ, as shown here. Point P is at the center of the curcular segment. The wire carries a current I. What is the magnitude and direction of the magnetic field at P?



The straight parts of the wire do not contribute to the field at P, since in the Biot-Savart law, $d\mathbf{l} \times \hat{\boldsymbol{\mathscr{r}}} / \mathscr{r}^2$ is zero ($d\mathbf{l}$ is parallel to $\hat{\boldsymbol{\mathscr{r}}}$.) We know the field at the center of a loop of radius R, namely $B = \frac{\mu_0 I}{2R}$. Of course, all bits of the loop contribute equally and here we only have a fraction $\frac{\theta}{2\pi}$ of a full loop. So the magnitude of the $\mathbf{B}$ field is

$$\frac{\theta}{2\pi} \frac{\mu_0 I}{2R} = \frac{\mu_0 I \theta}{4\pi R}$$

And here the field goes *into* the page.

4. If the $\mathbf{B}$ in a certain region of space is given by

$$\mathbf{B} = \alpha x \hat{\mathbf{z}}$$

where α is some constant, find a suitable expression for the vector potential $\mathbf{A}$ in that region. Does your $\mathbf{A}$ also satisfy $\nabla \cdot \mathbf{A} = 0$?

$$\mathbf{B} = \alpha x \hat{\mathbf{z}} = \nabla \times \mathbf{A} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}$$

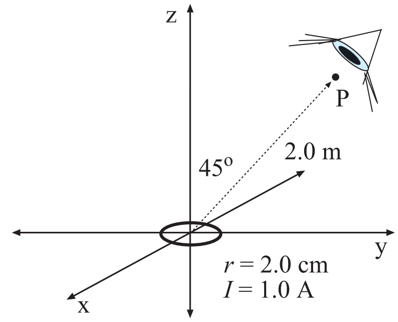
If we choose $A_y = \frac{1}{2} \alpha x^2$ we do get this result. (And we don't get an unwanted component, since $\frac{\partial A_y}{\partial z} = 0$.)

Then $\mathbf{A} = \frac{\alpha}{2} x^2 \hat{\mathbf{y}}$, which does satisfy $\nabla \cdot \mathbf{A} = 0$.

The choice $A_x = -\alpha x y$ also gives $\nabla \times \mathbf{A} = \mathbf{B}$, but it does not satisfy $\nabla \cdot \mathbf{A} = 0$.

5. A circular current is cenetered at the origin and lies in the xy plane. It carries a current of 1.0 A and has a radius of 2.0 cm.

Get a *numerical value* for the magnitude of the magnitude of the magnetic field at a distance of 2.0 m from the origin at an angle of 45° from the $+\hat{\mathbf{z}}$ axis. (Hint: What is the magnetic moment of the loop? At the given distance, shouldn't the dipole approximation for the field be OK?)



The magnitude of the loop's magnetic moment is

$$m = Ia = (1.0\ \text{A})\pi(2.0 \times 10^{-2}\ \text{m})^2 = 1.26 \times 10^{-3}\ \text{A} \cdot \text{m}^2$$

(Assume the current goes ccw so that $\mathbf{m} = m\hat{\mathbf{z}}$.)

At Jan's distance the dipole approximation should be good, so use

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

With $\theta = 45°$, the magnitude of the vector in the parenthesis is

$$\left|\frac{2}{\sqrt{2}}\hat{\mathbf{r}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\theta}}\right| = \sqrt{2 + \tfrac{1}{2}} = \sqrt{\frac{5}{2}}$$

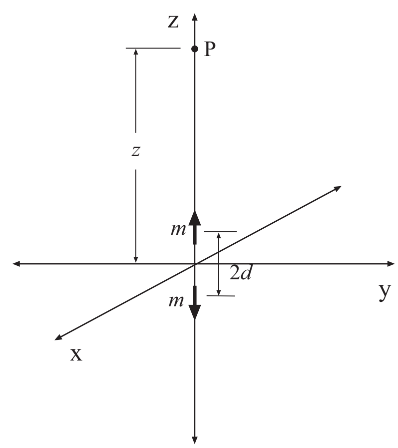
and the magnitude of $\mathbf{B}$ is then

$$B = \frac{(4\pi \times 10^{-7}\ \text{N/A}^2)(1.26 \times 10^{-3}\ \text{A} \cdot \text{m}^2)}{4\pi(2.0\ \text{m})^3}\sqrt{\frac{5}{2}} = 2.5 \times 10^{-11}\ \text{T}$$

6. A point dipole of magnitude m and pointing in the $+\hat{\mathbf{z}}$ direction is located on the z axis at $z = +d$. Another point dipole of magnitude m and pointing in the $-\hat{\mathbf{z}}$ direction is located at $z = -d$.

Find the leading-order behavior of the magnetic field at a point P located at coordinate z on the z axis, where $z \gg d$. The binomial expansion will be helpful here. Recall that for small x,

$$(1 + x)^n \approx 1 + nx$$



For a dipole $m\hat{\mathbf{z}}$ *at the origin*, the field on the z axis is

$$B_z = \frac{\mu_0 m}{4\pi r^3} \cdot 2 = \frac{\mu_0 m}{2\pi z^3}$$

Here we have a $+$ dipole at a distance $z - d$ from the observation point and a $-$ dipole at a distance $z + d$. Using the above formula, we find that the field for our case is

$$B_z = \frac{\mu_0 m}{2\pi(z - d)^3} - \frac{\mu_0 m}{2\pi(z + d)^3}$$

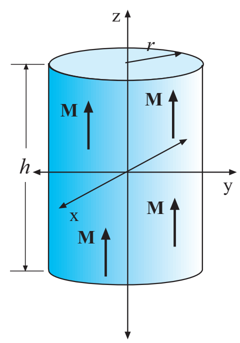
The terms nearly cancel, but not quite! With $z \gg d$, factor it as:

$$B_z = \frac{\mu_0 m}{2\pi z^3}\left[\frac{1}{(1-(d/z))^3} - \frac{1}{(1+(d/z))^3}\right] = \frac{\mu_0 m}{2\pi z^3}[(1-d/z)^{-3} - (1+d/z)^{-3}]$$

Use $(1+x)^n \approx 1 + nx$ for small x. (d/z is small.) Get:

$$B_z = \frac{\mu_0 m}{2\pi z^3}[1 + 3d/z - (1 - 3d/z)] = \frac{\mu_0 m}{2\pi z^3}\frac{6d}{z} = \frac{3\mu_0 m d}{\pi z^4}$$

7. A (finite) cylinder of radius r and height h carries a uniform built–in magnetization $\mathbf{M}$. (Shown in (a) at the right.) The cylinder is centered at the origin with its axis along z.



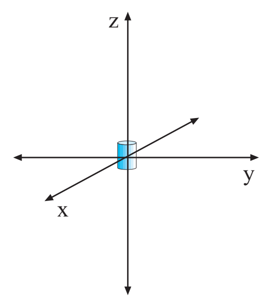
(a)

a) What are the bound volume and surface currents for the cylinder?

$\mathbf{J}_b$ is zero since $\mathbf{M}$ is uniform and

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\mathbf{z}} \times \hat{\mathbf{s}} = M\hat{\boldsymbol{\phi}}$$

on the sides, and since $\mathbf{M}$ is parallel to $\hat{\mathbf{n}}$ on the top and bottom, there is no $\mathbf{K}_b$ on those sides.



(b)

b) Give an *approximate* value for the $\mathbf{B}$ field at the center of this cylinder. (For argument, assume that $h \approx 8r$).

With only a surface current running around the side, the system can be approximated by a solenoid with current I and turns per length n such that $nI = K_b = M$. Then the $\mathbf{B}$ field near the center is roughly

$$\mathbf{B} \approx \mathbf{B}_{\mathrm{sol}} = \mu_0 n I \hat{\mathbf{z}} = \mu_0 M \hat{\mathbf{z}}$$

c) Consider the magnetic field at large distances from this cylinder (as indicated in (b)). Give an expression for the magnetic field to leading order.

At big distance, the cylinder is roughly a point dipole of magnitude (let the radius of the cylinder be s)

$$m = MV = M\pi s^2 h$$

Then

$$\begin{aligned} B_{\mathrm{dip}} &= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) = \frac{\mu_0 M\pi s^2 h}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\ &= \frac{\mu_0 M h s^2}{4r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \end{aligned}$$

8. *Micro–Essay Question:* Explain why a wire coil wrapped around a cylinder of iron will result in a *much* stronger magnetic field (near the ends) than if one just had the coil of wire (solenoid) by itself.

One good paragraph will suffice here.

The behavior of iron (and the magnetic dipoles in its atomic structure) is such that under the influence of an external $\mathbf{B}$ field the dipoles in the metal will become aligned and as a result produce an additional $\mathbf{B}$ field which is $much$ stronger than the original one. Iron differs from other materials in that there is a significant interaction between the dipoles themselves which helps in the alignment.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\epsilon_0 = 8.854\times 10^{-12}\,\tfrac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2} \qquad \Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_iq_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int\mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times 10^{-7}\,\tfrac{\mathrm{N}}{\mathrm{A}^2} \qquad 1\ \mathrm{T} = 1\,\tfrac{\mathrm{N}}{\mathrm{A}\cdot\mathrm{m}}$$

$$B_{\text{wire}} = \frac{\mu_0 I}{2\pi s} \qquad B_{\text{loop}} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \qquad B_{\text{sol}} = \mu_0 nI$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$