Phys 4610, Fall 2004
Exam #1, Answers

1. Find

a)

$$\nabla \cdot \mathbf{v}\ , \quad \text{where} \quad \mathbf{v} = 3xy\hat{\mathbf{x}} + 4yz\hat{\mathbf{y}} - zx\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(4yz) + \frac{\partial}{\partial z}(-zx) = 3y + 4z - x = -x + 3y + 4z$$

b)

$$\nabla V\ , \quad \text{where} \quad V = s^2 z \cos\phi$$

$$\nabla V = \frac{\partial V}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} = 2sz\cos\phi\,\hat{\mathbf{s}} - sz\sin\phi\,\hat{\boldsymbol{\phi}} + s^2\cos\phi\,\hat{\mathbf{z}}$$

c)

$$\nabla \times \mathbf{v}\ , \quad \text{where} \quad \mathbf{v} = r\sin\theta\,\hat{\mathbf{r}} + r\cos\theta\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned}
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\cos\theta)\,\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta) - \frac{\partial}{\partial r}(r^2\cos\theta)\right]\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial}{\partial\theta}(r\sin\theta)\,\hat{\boldsymbol{\phi}} \\
&= \frac{(\cos^2\theta - \sin^2\theta)}{\sin\theta}\,\hat{\mathbf{r}} - 2\cos\theta\,\hat{\boldsymbol{\theta}} - \cos\theta\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

d)

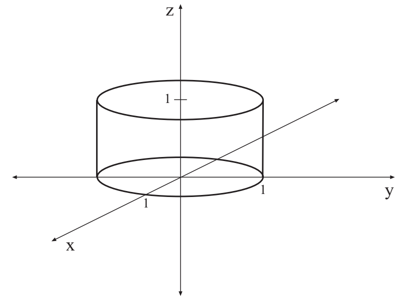
$$\nabla^2 V\ , \quad \text{where} \quad V = e^{-r/a}\cos\theta\sin\phi$$

$$\begin{aligned}
\nabla^2 V &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-r^2}{a}e^{-r/a}\cos\theta\sin\phi\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}e^{-r/a}\sin^2\theta\sin\phi - \frac{1}{r^2\sin^2\theta}e^{-r/a}\cos\theta\sin\phi \\
&= \left[-\frac{2}{ra} + \frac{1}{a^2}\right]e^{-r/a}\cos\theta\sin\phi - \frac{2}{r^2}e^{-r/a}\cos\theta\sin\phi - \frac{1}{r^2}e^{-r/a}\frac{\cos\theta}{\sin^2\theta}\sin\phi \\
&= \left[-\frac{2}{ra} + \frac{1}{a^2} - \frac{1}{r^2}\left(2 + \frac{1}{\sin^2\theta}\right)\right]e^{-r/a}\cos\theta\sin\phi
\end{aligned}$$

2. The vector field $\mathbf{v}$ is given by

$$\mathbf{v} = s\cos^2\phi\,\hat{\mathbf{s}} + s\cos\phi\,\hat{\boldsymbol{\phi}} + z^2\,\hat{\mathbf{z}}$$

a) Find the divergence of $\mathbf{v}$.



$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(s^2\cos^2\phi) + \frac{1}{s}\frac{\partial}{\partial \phi}(s\cos\phi) + \frac{\partial}{\partial z}(z^2) \\ &= 2\cos^2\phi - \sin\phi + 2z\end{aligned}$$

b) Show that the divergence theorem is satisfied using, as the volume, a cylinder of radius 1 coaxial with the z axis and extending from $z = 0$ to $z = 1$.

Test $\int_{\mathcal{V}} \nabla \cdot \mathbf{v} d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$ over the given volume.
The lhs is

$$\int_{z=0}^{1}\int_{\phi=0}^{2\pi}\int_{s=0}^{1}(2\cos^2\phi - \sin\phi + 2z)\, sds\, d\phi\, dz$$

The s integral can be done first: $\int_0^1 sds = \frac{1}{2}$. For the second term, the ϕ integral gives zero. This leaves:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v} d\tau = \tfrac{1}{2}\int_0^1\int_0^{2\pi}(2\cos^2\phi + 2z)\, d\phi\, dz$$

use

$$\int_0^{2\pi}\cos^2\phi\, d\phi = [\tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\phi]\Big|_0^{2\pi} = \pi \qquad \text{and} \qquad \int_0^1 2z\, dz = 1$$

Then we have:

$$\rightarrow = \pi + \tfrac{1}{2}\cdot 1 \cdot (2\pi) = 2\pi$$

Now do the rhs. There are 3 parts to the surface:
(i) Bottom: $d\mathbf{a} = -sdsd\phi\hat{\mathbf{z}}$ so

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi}\int_0^1 s\, ds\, d\phi(-z^2)\Big|_{z=0} = 0$$

(ii) Sides: $d\mathbf{a} = sd\phi dz\hat{\mathbf{s}}$, so with $s = 1$ on the sides,

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_0^1\int_0^{2\pi} 1 \cdot d\phi dz(1 \cdot \cos^2\phi) = 1 \cdot \pi = \pi$$

where we used $\int_0^{2\pi}\cos^2\phi d\phi = \pi$ again.
(iii) Top: $d\mathbf{a} = +s\, dsd\phi\hat{\mathbf{z}}$ with $z = 1$:

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi}\int_0^1 (s\, ds\, d\phi)(z^2)\Big|_{z=1} = \tfrac{1}{2}(2\pi) = \pi$$

Add them up

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \pi + \pi = 2\pi$$

The two sides of the divergence theorem agree.

3. Evaluate

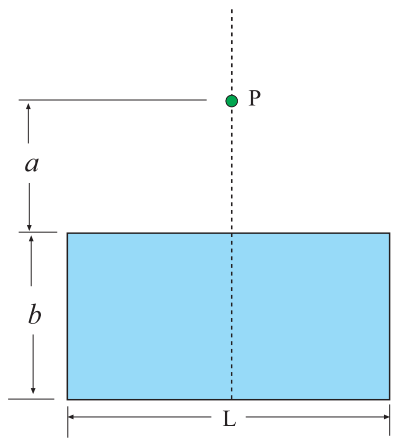
$$\int_{-3}^{3} (x^3 + 7x + 5)\delta(x-1)\,dx$$

Since the integration interval does contain $x = 1$, the δ–function "picks out" the value of the function under the integral sign at $x = 1$ to give

$$\rightarrow = (1^3 + 7\cdot 1 + 5) = 13$$

4. Consider a rectangle with a uniform surface charge density σ. The observation point P is in the plane of the rectangle on the bisector of the side of length L, a distance a from the nearest side. The other side of the rectangle has length b. See the figure.

Give the direction and magnitude of the E field at P. It will be sufficient for you to *clearly* set up any necessary integrals if they are at all difficult to work out!



Consider a strip of surface charge which lies at a distance x from P and has length L. The result from the text (given on the exam, at the end) gives the E field dE_z due to this strip. It is:

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{x\sqrt{x^2 + (L/2)^2}}$$

where λ is the linear charge density of the strip,

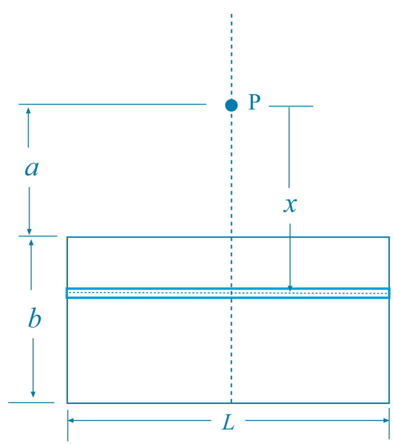
$$\lambda = \frac{\sigma(Ldx)}{L} = \sigma dx$$

Then

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\sigma L dx}{x\sqrt{x^2 (L/2)^2}}$$

Add up all the contributions as x ranges from a to $a+b$. Get:

$$E_z = \frac{\sigma L}{4\pi\epsilon_0}\int_a^{a+b}\frac{dx}{x\sqrt{x^2 + (L/2)^2}}$$



The integral is listed in tables and we get:

$$\begin{aligned} E_z &= \frac{\sigma L}{4\pi\epsilon_0}\left[-\frac{2}{L}\log\left(\frac{L/2 + \sqrt{x^2 + (L/2)^2}}{x}\right)\right]\Bigg|_a^{a+b} \\ &= \frac{\sigma}{2\pi\epsilon_0}\log\left[\frac{(a+b)}{a}\frac{(L/2 + \sqrt{a^2 + (L/2)^2})}{(L/2 + \sqrt{(a+b)^2 + (L/2)^2})}\right] \end{aligned}$$

Maybe one can simplify this...

5. If the electric potential in a certain region of space is given by

$$V(\mathbf{r}) = V_0 e^{-r^2/a^2}$$

a) What is the electric field in that region?

Use $\mathbf{E} = -\nabla V$ and the expression for the gradient in spherical coordinates:

$$\mathbf{E} = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} = +\frac{2rV_0}{a^2}e^{-r^2/a^2}\hat{\mathbf{r}} = \frac{2V_0}{a^2}re^{-r^2/a^2}\hat{\mathbf{r}}$$

b) What is the charge density $\rho(\mathbf{r})$?

Use $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$. Then

$$\begin{aligned} \rho &= \epsilon \nabla \cdot \mathbf{E} = \frac{2V_0\epsilon_0}{a^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 e^{-r^2/a^2}\right] \\ &= \frac{2V_0\epsilon_0}{a^2}\left[3 - \frac{2r^2}{a^2}\right]e^{-r^2/a^2} \end{aligned}$$

c) How much charge is contained within a sphere of radius R centered at the origin?

Use Gauss's law, $\oint \mathbf{E}\cdot d\mathbf{a} = Q_{\text{enc}}/\epsilon_0$. The E field is radial so that $\mathbf{E}$ is parallel to the area vector $d\mathbf{a}$ everywhere and the magnitude of $\mathbf{E}$ is the same everywhere on the sphere,

$$E_r(R) = \frac{2V_0}{a^2}Re^{-R^2/a^2}$$

. This gives

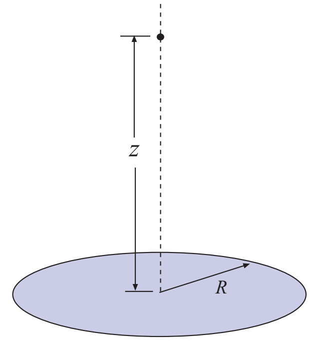
$$\oint \mathbf{E}\cdot d\mathbf{a} = (4\pi R^2)E_r(R) = \frac{8\pi R^3 V_0}{a^2}e^{-R^2/a^2} = Q_{\text{enc}}/\epsilon_0$$

Then Q_{enc} is

$$Q_{\text{enc}} = \frac{8\pi\epsilon_0 R^3 V_0}{a^2}e^{-R^2/a^2}$$

6. What is the electric potential at a point on the axis of a uniformly charged disk of radius R and surface charge density σ, a distance z from the center of the disk?

Assume $V = 0$ at infinity.



Charge density σ

The E–field at point P is

$$E_z = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$

If we consier the path from ∞ to P along the z axis, the change in potential is

$$V = -\int_\infty^z \mathbf{E}(z')\cdot d\mathbf{l}$$

where $d\mathbf{l} = dz\hat{\mathbf{z}}$, so

$$V = -\int_\infty^z \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z'}{\sqrt{z'^2+R^2}}\right] = -\frac{\sigma}{2\epsilon_0}\left(z' - \sqrt{z'^2+R^2}\right)\Big|_\infty^z$$

Now while we can substitute $z' = z$ easily enough, substituting ∞ gives "$\infty - \infty$". We need to take this limit carefully:

$$\lim_{z'\to\infty}(z' - \sqrt{z'^2+R^2}) = \lim_{z'\to\infty}\left(\frac{1-\sqrt{1+R^2/z'^2}}{1/z'}\right)$$

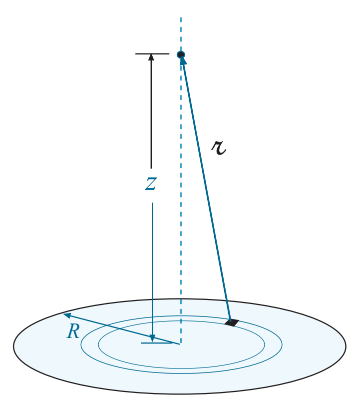
which is now of the form $0/0$, so use L'Hopital's rule: Derivatives of numerator and denominator give

$$\lim_{z'\to\infty}\left(\frac{(R^2/z'^3)\left(1+R^2/z'^2\right)^{-1/2}}{-1/z'^2}\right) = \lim_{z'\to\infty} -\frac{R^2}{z'}\left(1+R^2/z'^2\right)^{-1/2}$$

which is now clearly zero. Using this, V is:

$$V(z) = -\frac{\sigma}{2\epsilon_0}(z - \sqrt{z^2+R^2}) = \frac{\sigma}{2\epsilon_0}(\sqrt{z^2+R^2} - z)\ .$$

We could also have found the potential from scratch with

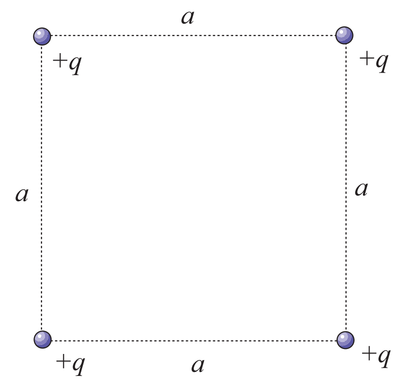


$$V(P) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma da'}{\mathscr{r}} = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{s\,ds\,d\phi}{\sqrt{s^2+z^2}}$$

This integral is easy, giving

$$V(P) = \frac{\sigma}{4\pi\epsilon_0}(2\pi)\sqrt{s^2+z^2}\Big|_0^R = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - z\right)$$

7. Find the work required to assemble four point charges $+q$ in a square with side a.



We can use

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}\frac{q_i q_j}{\mathscr{r}_{ij}}\ ,$$

(a sum over distinct pairs) then with 4 $q \cdot q$ pairs separated by a and 2 $q \cdot q$ pairs separated by $\sqrt{2}a$, the required work is

$$W = \frac{1}{4\pi\epsilon_0}\left[4\frac{q^2}{a} + 2\frac{q^2}{\sqrt{2}a}\right] = \frac{q^2}{4\pi\epsilon_0 a}[4+\sqrt{2}]$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

$$\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

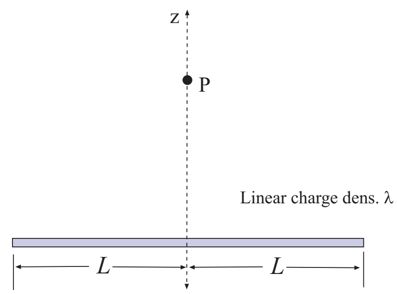
Physics:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \qquad V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} E \cdot d\mathbf{l}$$

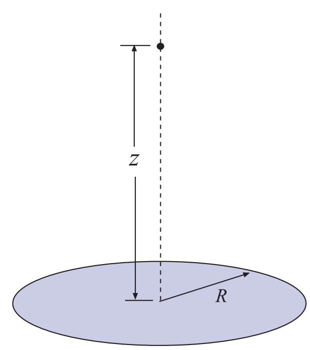
$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$E_z = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$



Charge density σ