Phys 3820, Fall 2011
Exam #3

1. An important part of chapter 9 was to go from the transition probability for a monochromatic polarized wave to a *transition rate* for incoherent perturbations.

I don't need you to reproduce this tricky and dreary calculation here, but *summarize* the steps that were performed to get the result for the stimulated emission rate $R_{b \to a}$. (Where did that factor of 3 come from?)

We have to account for the non-monochromatic nature of light by introducing an energy density *per unit range of frequency* and we have to account for the lack of a definite direction by averaging over all possible directions of incidence and polarizations of the light. The latter involved carefully setting up an integral over all incidence and polarization directions so that everything was counted only once! That integral gave the factor of 3.

2. We had to derive the Einstein A coefficient by means of a cheat. The A coefficient is of course the one that appears in

$$\frac{dN_b}{dt} = -N_b A - N_b B_{ba} \rho(\omega_0) + N_b B_{ab} \rho(\omega_0)$$

a) First off, give the physical meanings of A, B_{ba} and B_{ab} in this equation.

A is the rate of spontaneous emission; B_{ba} is the rate of induced emission and B_{ab} is the rate of induced absorbtion.

b) We imagined the two–state system to be in thermal equilibrium with a radiation field. This gave

$$\rho(\omega_0) = \frac{A}{(N_a/N_b)B_{ab} - B_{ba}}$$

and we substituted for some terms. What did we use to substitute for N_a/N_b ?

We used the Boltzmann ratio, giving the ratio of numbers of particles in each state for equilibrium at a given temperature.

c) What famous result from stat-mech gave an alternative expression for $\rho(\omega_0)$?

The Planck radiation law which gives the energy density per unit frequency for an EM radiation field in equilibrium with a system of atoms at temperature T.

3. In the last part of Chapter 9 much time was spent deriving the selection rules for transitions in the H atom. These were

$$\Delta m = \pm 1 \text{ or } 0 \qquad \Delta l = \pm 1$$

Summarize (very briefly) how these were obtained.

The task was to find when the dipole matrix element

$$\mathbf{p} \equiv q\langle\psi_a|\mathbf{r}|\psi_b\rangle$$

was zero. In the book this was accomplished using the commutation rules for angular momentum. For the rule on Δm, the commutators of L_z with x, Y, and Z were used. For the one on Δl the commutator of L^2 with $\mathbf{r}$ was considered. (This derivation was quite so trivial!)

4. We began the study of scattering with the wave function

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}$$

a) Say a few words as to why this is highly non-rigorous.

In the scattering process, a wave packet (for *each particle*) is incident on the target and later becomes an outgoing wave packet of a spherical type with values for all θ. So the wave function must be time dependent and in any case must be normalizable; the given wave function is neither. It can be rigorously justified, but not here.

b) Say a few words as to why this is a *physically reasonable* form for a time-independent solution of the scattering problem.

The form is reasonable as we think of the incoming particles as described by a plane wave and once they interact with the target there is a probability of going outward in all directions so that a spherical outgoing wave (modulated in θ, with an r factor downstairs to keep the probability current correct) is sensible.

5. Suppose you have a mono-energetic beam of particles hitting a stationary target (with very massive particles) and a particle detector of certain specific dimensions. Summarize the steps you would need (the basic measurements and basic calculations) to go through to find the *measured* differential cross section $D(\theta) = \frac{d\sigma}{d\Omega}$ at each angle.

6. What surprising result did we get for *quantum* "hard sphere" scattering in the low-energy limit? (Hint: Something to do with the *classical* solution.)

We found that the total cross section for quantum scattering from a hard sphere of radius a was four times the classical value (namely the obvious value πa^2). While hard spheres don't really exist in nature, this is an example of where quantum and classical theories give different predictions for the result of a macroscopic measurement (a scattering experiment).

7. Describe how a computer program calculates $f(\theta)$ for a given potential $V(r)$, that is, how it finds the "exact" solution and then gives the differential cross section $D(\theta)$.

Hint$_1$: I will begin your answer for you:

For each l we find the the partial–wave radial function $u_l(r)$ by choosing $u_l(0) = 0$ and $u_l(dx)$ equal to some arbitrary small value. Use the Schrödinger equation to integrate to large r...

Hint$_2$: *You should make reference to spherical Bessel functions in your answer.*

Use the Schr odinger equation to integrate the partial wave function up to large r and then using its value and derivative match it to the appropriate linear combination of $j_l(kr)$ and $h_l^{(1)}(kr)$. This will give the value of the partial wave scattering amplitude a_l which are then collected together to produce $f(\theta)$, $D(\theta)$ and possible σ.

8. What was the advantage of using the phase shift δ_l over the scattering amplitude a_l?

The phase shift δ_l is a real number so that to describe the scattering we only need one number for each partial wave. (The fact that we can reduce the information down to one real number is a consequence of angular momentum conservation.) It also has a nice physical meaning in that it represents what it says, the amount of phase by which partial wave is shifted by the scattering process.

9. Find the scattering amplitude in Born approximation for a central potential of the form (attractive square well with barrier):

$$V(r) = \begin{cases} -V_0 & \text{for} \quad 0 < r < a \\ V_1 & \text{for} \quad a < r < b \\ 0 & \text{for} \quad b < r < \infty \end{cases}$$

(Don't assume low energy.)

As usual, do the math as far as you can– though you may be able to come up with a closed form for this one.

Use it in the formula for a spherically-symmetric potential,

$$f(\theta) = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty rV(r) \sin \kappa r \, dr$$

where $\kappa = 2k \sin(\theta/2)$, so grind away:

$$\begin{aligned} f(\theta) &= -\frac{2m}{\hbar^2 \kappa} \left[\int_0^a r(-V_0) \sin \kappa r \, dr + \int_a^b r(V_1) \sin \kappa r \, dr \right] \\ &= -\frac{2m}{\hbar^2 \kappa} \left[-V_0 \left(\frac{1}{\kappa^2} \sin \kappa r - \frac{r}{\kappa} \cos \kappa r \right) \Big|_0^a + V_1 \left(\frac{1}{\kappa^2} \sin \kappa r - \frac{r}{\kappa} \cos \kappa r \right) \Big|_a^b \right] \end{aligned}$$

More algebra gives

$$\begin{aligned} f(\theta) &= -\frac{2m}{\hbar^2 \kappa} \left[-V_0 \left(\frac{1}{\kappa^2} \sin \kappa a - \frac{a}{\kappa} \cos \kappa a \right) + V_1 \left(\frac{1}{\kappa^2} \sin \kappa b - \frac{b}{\kappa} \cos \kappa b \right) - V_1 \left(\frac{1}{\kappa^2} \sin \kappa a - \frac{a}{\kappa} \cos \kappa a \right) \right] \\ &= \frac{2m}{\hbar^2 \kappa^2} \left[(V_0 + V_1) \left[\frac{\sin \kappa a}{\kappa} - a \cos \kappa a \right] - V_1 \left(\frac{\sin \kappa b}{\kappa} - b \cos \kappa b \right) \right] \end{aligned}$$

Well, this isn't especially illuminating. But here the integral was simple and workable.

10. When we wanted to solve the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = \rho(\mathbf{r})$$

in three dimensions we found that the proper Green function to use was

$$G(\mathbf{r}_2 - \mathbf{r}_1) = \frac{e^{ik|\mathbf{r}_2 - \mathbf{r}_1|}}{4\pi|\mathbf{r}_2 - \mathbf{r}_1|}$$

This would not be true for a one-dimensional problem though. If we have the *one*–dimensional Helmholtz equation

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = f(x)$$

with a one–dimensional "source" $\rho(x)$ then the appropriate Green function is

$$-\frac{i}{2k}e^{ik|x|}$$

a) What differential equation is satisfied by this function?

It is the same as the original DE with a delta function on the right had side:

$$\left(\frac{d^2}{dx^2} + k^2\right)G(x) = \delta(x)$$

b) If I assure you that this Green function has the right boundary conditions, write down the solution for $\psi(x)$ for a given source function $f(x)$.

It is

$$\begin{aligned}\psi(x) &= \int_{-\infty}^{\infty} G(x-x')f(x')\,dx' \\ &= -\frac{i}{2k}\int_{-\infty}^{\infty} e^{ik|x-x'|f(x')}\,dx'\end{aligned}$$

11. Summarize how we get the "Born series" from the integral form of the Schrödinger equation, which Griffiths wrote as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0$$

We take the entire right-hand side of the equation and substitute it in for ψ as it appears in the last integral. This will give three terms on the right-hand side with the last term being a double integral with again ψ appearing inside that integral.

We again recycle the solution and the result is a series of term which have integrals of increasing order to perform, and when we take the final term we can use the plane wave solution ψ_0 for ψ.

12. If we are to have a meaningful theory of relativistic quantum mechanics, what features of the non-relativistic theory do we want to keep? (And what *don't* we want to keep?)

The theory must have a sensible probabilistic interpretation, so there must be a sensible definition for the probability density and probability current density. We also want to hold on to the idea of a Hilbert space of states and operators corresponding to observables. But we don't want to keep anything that was not Lorentz-invariant. The Schrödinger equation did not treat space and time in a similar way, so it has to go.

13. How did Dirac's Hamiltonian (and equation) differ from the relativistic Schrödinger equation (K-G equation)? In what way is it *consistent* with the relativistic Schrödinger equation?

The Dirac Hamiltonian is linear in both space and time (or in both energy and momentum). The relativistic Schrödinger equation had the *squares* space and time operators. To accomplish this, Dirac had to make the operators matrices and the wave function a four-component object.

The Dirac equation is consistent with the RSE in that if we apply the Hamiltonian oerator twice we arrive at the RSE for *each component* of the Dirac wave function.

14. In either of the relativistic equations, how do we change the equations for a particle of charge q moving in a region where there is a scalar potential ϕ and vector potential $\mathbf{A}$?

To the energy operator we subtract $q\phi$ and to the momentum operator we subtract $q\mathbf{A}$. (This actually has name: "Minimal substitution".)

15. Give two physical phenomena involving electrons that were predicted by the Dirac equation. (And were not "put in by hand" after the fact.)

The Dirac equation (in the right hands) *predicted* the spin angular momentum of the electron and its magnetic moment. It correctly gave the form of the spin-orbit interaction. And it predicted the existence of anti-particles.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}\text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529\times 10^{-10}\text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1,2,3,\ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2}e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-r/2a} \qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2 = 1.097{\times}10^7\text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2 f_l^m = \hbar^2 l(l+1)f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m{\pm}1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} ae^{-iE_+ t/\hbar} \\ be^{-iE_- t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m} V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n} \frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6 \text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788\times 10^{-5} \text{ eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle\mathbf{S}_p\cdot\mathbf{S}_e\rangle = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)}\end{cases}$$

$$E_{\text{gs}} \leq \langle\psi|H|\psi\rangle \equiv \langle H\rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\pm\frac{1}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v}e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar}$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$\mathbf{p} \equiv q\langle\psi_b|\mathbf{r}|\psi_a\rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\mathbf{p}|E_0}{\hbar}\right)^2 \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\mathbf{p}|^2\rho(\omega_0) \qquad A = \frac{\omega^3|\mathbf{p}|^2}{3\pi\epsilon_0\hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1; \text{or } 0$ and $\Delta l = \pm 1$

$$d\sigma = D(\theta)\,d\Omega \qquad D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad \sigma = \int D(\theta)\,d\Omega \qquad D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega}$$

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} \qquad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \qquad D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2 \qquad -\frac{\hbar^2}{2m}\frac{d^2 u_l}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u_l = E u_l$$

$$\text{Large } r\text{:} \quad \frac{d^2 u_l}{dr^2} - \frac{l(l+1)}{r^2}u_l = -k^2 u_l \qquad u_l = Arj_l(kr) + Brn_l(kr)$$

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right]P_l(\cos\theta)$$

$$a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k}e^{i\delta_l}\sin(\delta_l) \qquad f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)\,P_l(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r}) \qquad \Longrightarrow \qquad G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}$$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0$$

$$E^2 = c^2\mathbf{p}^2 + m^2c^4 \qquad (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2)\psi = E\psi$$