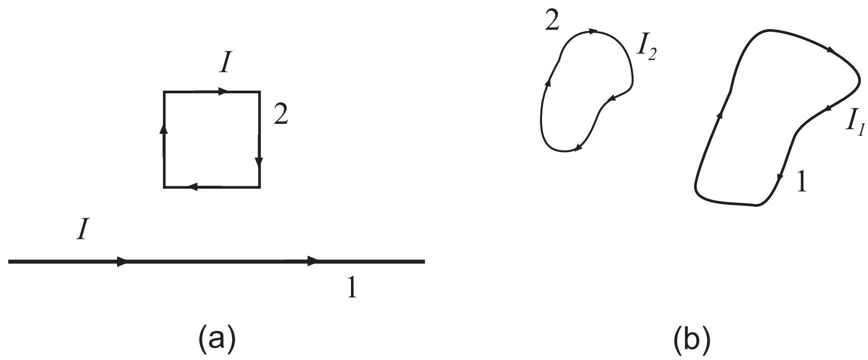

Figure 1: (a) Square loop (current 2) near a very long wire (current 1) Loop feels a magnetic force. (b) General current loop 2 near a current loop 1.

Forces Between Currents and Newton's 3rd law

In Exercise 5.10 (and other places) we find the force on a current loop in the presence of another current (in this case, in a long straight wire), shown in Fig. 1(a). To solve these problems, we consider the parts of the current loop and the forces on them which come from the magnetic field of the *other* current; we calculate it from

$$\mathbf{F}_2 = I_2 \oint_2 d\mathbf{l}_2 \times \mathbf{B}_1$$

But wait... *both* currents give rise to magnetic fields. When we find the force on the individual parts of current 2, don't we need to figure in the magnetic field generated by the *rest* of current 2?

This is an issue which concerns Newton's 3rd law since it is about the force an object exerts on itself. *Without* the long wire (wire 1) present, we would just have the field of the loop and then

$$\mathbf{F}_{\rm self} = I_2 \oint_2 d\mathbf{l}_2 \times \mathbf{B}_2 \tag{1}$$

would *have* to give zero (an object can't exert a net force on itself), though it is not so clear *why* it does. So from this, we can say that we *don't* need the field of the loop itself, as it won't contribute.

But this is not very satisfying. The integral in 1 can be calculated and it must give zero. That is, we start with the Lorentz force and Biot–Savart laws and either they *do* satisfy Newton's 3rd law or they *don't*. And in what sense do they satisfy it?

For electrostatics things were much more obvious. Point charges either attract or repel and the direction of the force is along the line which joins the two points. Big charges are built up from small point charges and for any group of charges the sum of the forces on all the individual charges from all the *other* charges within the object must cancel and so a collection of charges can exert no net force on itself. So the consistency with Newton's 3rd law for a charge distribution is fairly clear.

But we hit a problem with magnetostatics. We can *think* about "elements of current" but in reality no such objects can exist by themselves; they are always part of big currents.

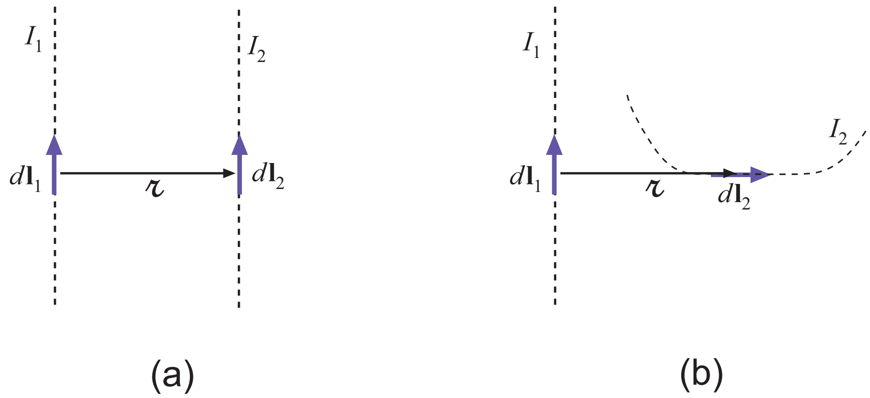

Figure 2: (a) Two current elements in two long straight wires. Newton's 3rd law seems OK. (b) Current elements in two wires. Newton's 3rd law doesn't apply to these!

Indeed, when we try to apply Biot-Savart and Lorentz to such current elements, we seem to violate Newton's third law. In Fig. 2(a) we see two current elements; we want to consider the force that element 1 exerts on elements 2 (if indeed we can). The magnetic field at 2 due to 1 comes from $Id\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}$ and points into the page. Then the force on element 2 must point to the left as it should since the elements are taken from two parallel currents, which attract. Similarly we find that the force of element 2 on element 1 is to the right, so things seem OK.

In Fig. 2(b) we again have two current elements and the field due to 1 points into the page at the location of 2. It would seem that it exerts an upward force on element 2. But the magnetic field at the location of 1 due to 2 is *zero*, since $\hat{\boldsymbol{\mathscr{r}}}$ is parallel to $d\mathbf{l}_2$ and so element 2 exerts *no* force on 1. Something's funny here as the forces are not "equal and opposite".

The problem is that are not considering anything *real* here: isolated current elements don't exist. In reality we can only consider the field from a *complete* current. So Newton's 3rd law is only required if we consider the forces *from* and *on* complete current loops.

So now we consider the force on current loop 2 from current loop 1. It is:

$$\mathbf{F}_2 = I_2 \oint_2 d\mathbf{l}_2 \times \mathbf{B}_1 \qquad \text{where} \qquad \mathbf{B}_1 = \frac{\mu_0 I_2}{4\pi} \oint_1 \frac{d\mathbf{l}_1 \times \boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3} \tag{2}$$

and here we mean $\boldsymbol{\mathscr{r}}_{21} = \mathbf{r}_2 - \mathbf{r}_1$. (We will also use $\boldsymbol{\mathscr{r}}_{12} = \mathbf{r}_1 - \mathbf{r}_2 = -\boldsymbol{\mathscr{r}}_{21}$.)

Combining the two equations, we have the force on loop 2 from loop 1:

$$\mathbf{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_2 d\mathbf{l}_2 \times \oint_1 \frac{d\mathbf{l}_1 \times \boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \boldsymbol{\mathscr{r}}_{21})}{\mathscr{r}_{21}^3} \tag{3}$$

But loop 2 also exerts a force on loop 1 which we get just by switching the "1" and "2" in 3:

$$\mathbf{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \boldsymbol{\mathscr{r}}_{12})}{\mathscr{r}_{12}^3} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \boldsymbol{\mathscr{r}}_{21})}{\mathscr{r}_{21}^3} \tag{4}$$

And now we can ask if $\mathbf{F}_2 = -\mathbf{F}_1$, i.e. if $\mathbf{F}_2 + \mathbf{F}_1 = 0$. If so, then Newton's 3rd law is fine as long as we apply to the *entire* loop.

So is this true?

Because of the funny business with the cross products in Eqs. 3 and 4 it is *not obvious* that it is true. First, let's write out the triple cross products in 3 and 4 for comparison:

$$d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \boldsymbol{\mathscr{r}}_{21}) = d\mathbf{l}_1(d\mathbf{l}_2 \cdot \boldsymbol{\mathscr{r}}_{21}) - \boldsymbol{\mathscr{r}}_{21}(d\mathbf{l}_1 \cdot d\mathbf{l}_2)$$

$$-d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \boldsymbol{\mathscr{r}}_{21}) = -d\mathbf{l}_2(d\mathbf{l}_1 \cdot \boldsymbol{\mathscr{r}}_{21}) + \boldsymbol{\mathscr{r}}_{21}(d\mathbf{l}_1 \cdot d\mathbf{l}_2)$$

When we form the integral which gives $\mathbf{F}_2 + \mathbf{F}_1$, the last term will cancel and leave us with:

$$\mathbf{F}_2 + \mathbf{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \left[\left(d\mathbf{l}_2 \cdot \frac{\boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3} \right) d\mathbf{l}_1 - \left(d\mathbf{l}_1 \cdot \frac{\boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3} \right) d\mathbf{l}_2 \right] \tag{5}$$

which is *still* not obviously zero.

But now we notice one thing. In the first term we see the combination $\boldsymbol{\mathscr{r}}_{21}/\mathscr{r}_{21}^3$ and we recall the relation

$$\nabla \left(\frac{1}{\mathscr{r}} \right) = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$$

where (to be clear) the ∇ means to differentiate with respect to the unprimed coordinate. Similarly, if by ∇_2 we mean to differentiate with respect to $\mathbf{r}_2$, then we have

$$\nabla_2 \left(\frac{1}{\mathscr{r}_{21}} \right) = -\frac{\boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3}$$

and then the integral for the first term can be written

$$\oint_1 d\mathbf{l}_1 \oint_2 \frac{\boldsymbol{\mathscr{r}}_{21}}{\mathscr{r}_{21}^3} \cdot d\mathbf{l}_2 = -\oint_1 d\mathbf{l}_1 \left[\oint_2 \nabla_2 \left(\frac{1}{\mathscr{r}_{21}} \right) \cdot d\mathbf{l}_2 \right]$$

But here within the bracket is the integral over a loop in $\mathbf{r}_2$ space of the *gradient* of a function of $\mathbf{r}_2$. By the gradient theorem *this must give zero*. So now we can go back and see that the first term in 5 is zero. A similar argument shows why the second term is zero, and behold, we have now shown

$$\mathbf{F}_2 + \mathbf{F}_1 = 0 \ ,$$

the forces exerted by the loops on one another are equal and opposite. Newton's 3rd law is satisfied as long as we consider the *entire* current loops; unlike the case of electrostatics, we have to apply it in a *global* sense.

It also follows from this theorem that the force of a current loop on itself is zero: in that case the particular *paths* of the integrals $\oint_1$ and $\oint_2$ are the same and $I_1 = I_2$, and then $\mathbf{F}_1 = \mathbf{F}_2 = 0$ since the double integral is the negative of itself.