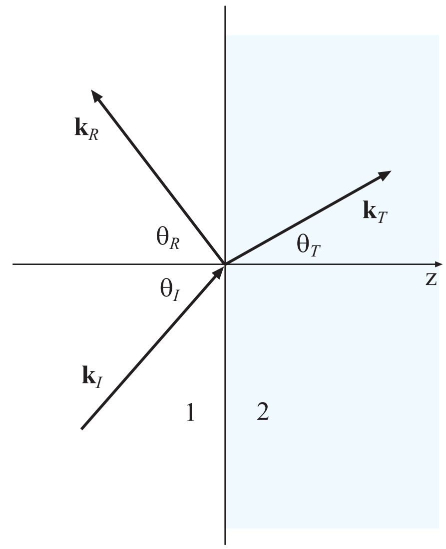
Phys 4620, Spring 2005
Exam #2, Solutions

1. When we solved for the reflected and transmitted waves for a polarized EM wave incident on the plane interface of two media we first found that the space-dependent part of the phases of terms were all equal:



$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r} \quad \text{for} \quad z = 0$$

and by considering $y = 0$ and then $x = 0$ this gave

$$(k_I)_x = (k_R)_x = (k_T)_x$$

and

$$(k_I)_y = (k_R)_y = (k_T)_y$$

Show how this result leads to the **law of reflection** and **Snell's law**.

Actually both results are the same because we intend to use the illustration as either the xz plane or the yz plane. Now, the frequencies of the waves are equal for all three, but since the speed of light differs in the two media, λ and k will differ. Using $\omega/k = v$ and $v = c/n$ we have

$$k_I = \frac{\omega}{v_1} = \frac{\omega n_1}{c} = k_R \qquad \text{and} \qquad k_T = \frac{\omega n_2}{c}$$

Then the given relations between the components of the k's gives

$$k_I \sin\theta_I = k_R \sin\theta_R \quad \longrightarrow \quad \sin\theta_I = \sin\theta_R \quad \longrightarrow \quad \theta_I = \theta_R$$

and

$$k_I \sin\theta_I = k_T \sin\theta_T \quad \longrightarrow \quad \frac{\omega n_1}{c}\sin\theta_I = \frac{\omega n_2}{c}\sin\theta_T \quad \longrightarrow \quad n_1 \sin\theta_I = n_2 \sin\theta_T$$

The first of these is the "law of reflection" and the second is Snell's law.

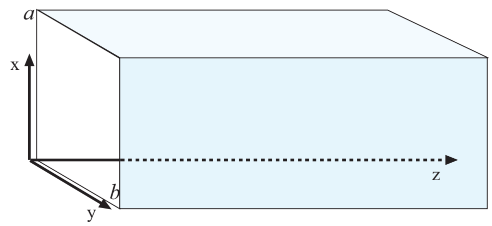
2.a) Identify two ways in which EM waves in a conductor differ qualitatively from EM waves in vacuum.

EM waves in conductors have an amplitude which decays with distance. The real part of the wave number depends on the frequency (so the waves are *dispersed* by the medium). Finally, the E and B fields do not oscillate in phase with one another in these waves.

b) What is meant by the *skin depth* of a conductor?

The skin depth d is the distance it takes for the amplitude of the EM wave to be reduced by a factor $1/e$.

3. In class and in the book we only studied TE waves in a rectangular waveguide (as shown at the right). We never studied TM waves, but now's our chance!



The waveguide runs along the z axis; the cross section is the region $0 < x < a$, $0 < y < b$ with $b < a$. The solution for $E_z(x, y)$ (from separating variables) turns out to be

$$E_z = E_0 \sin(m\pi x/a) \sin(n\pi y/b) \qquad m, n = 1, 2, 3, \ldots$$

Note, this time the indices have to start at 1, otherwise the "solution" could be zero everywhere.

a) Show that the solution for E_z does indeed satisfy the surface boundary conditions

$$\mathbf{E}^{\parallel} = 0 \qquad \text{and} \qquad B^{\perp} = 0$$

If the z–component of the amplitude $\tilde{\mathbf{E}}_0$ is given by

$$E_z = E_0 \sin(m\pi x/a) \sin(m\pi y/b) \qquad \text{with } B_z = 0$$

then the other components are given by:

$$\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2} \left(\frac{km\pi}{a} \right) E_0 \cos(m\pi x/a) \sin(m\pi y/b) \\
E_y &= \frac{i}{(\omega/c)^2 - k^2} \left(\frac{kn\pi}{b} \right) E_0 \sin(m\pi x/a) \cos(m\pi y/b) \\
B_x &= \frac{-i}{(\omega/c)^2 - k^2} \left(\frac{\omega}{c^2} \frac{n\pi}{b} \right) E_0 \sin(m\pi x/a) \cos(m\pi y/b) \\
B_y &= \frac{i}{(\omega/c)^2 - k^2} \left(\frac{\omega}{c^2} \frac{m\pi}{a} \right) E_0 \cos(m\pi x/a) \sin(m\pi y/b)
\end{aligned}$$

Now test the boundary conditions $\mathbf{E}^{\parallel} = 0$ and $B^{\perp} = 0$ (at the boundary):

$$\begin{array}{lllllll}
E_z = 0 & \text{at} & x = 0, & x = a, & y = 0, & y = b & \checkmark \\
E_x = 0 & \text{at} & y = 0 & \text{and} & y = b & & \checkmark \\
E_y = 0 & \text{at} & x = 0 & \text{and} & x = a & & \checkmark \\
B_x = 0 & \text{at} & x = 0 & \text{and} & x = a & & \checkmark \\
B_y = 0 & \text{at} & y = 0 & \text{and} & y = b & & \checkmark
\end{array}$$

(Here we have used $\sin(n\pi) = 0$ repeatedly.)

b) If $a = 2.28$ cm and $b = 1.01$ cm find the lowest frequency at which a TM wave can propagate down the waveguide.

The DE for E_z is

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c} \right)^2 - k^2 \right] E_z = 0$$

where we choose $k = 0$ to find the threshold frequency. This gives

$$\left[-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\omega_{mn}}{c}\right)^2 \right] = 0$$

for the threshold frequency ω_{mn}, and so

$$\omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

c) Find the lowest frequency at which two different modes can propagate in this waveguide.

The *lowest* cutoff frequency is the one for $(mn) = (11)$. With $a = 2.29$ cm and $b = 1.01$ cm, we get

$$\omega_{11} = (3.00 \times 10^{10}\,\mathrm{cm/s}))\pi\sqrt{\left(\frac{1}{2.28\ \mathrm{cm}}\right)^2 + \left(\frac{1}{1.01\ \mathrm{cm}}\right)^2} = 1.02 \times 10^{11}\ \mathrm{s}^{-1}$$

or $f = 1.62 \times 10^{10}$ Hz. The next lowest cutoff has $m = 2$ and $n = 1$, giving

$$\omega_{21} = 1.25 \times 1011\ \mathrm{s}^{-1} \qquad \text{or} \qquad f_{21} = 1.98 \times 10^{10}\ \mathrm{Hz}\ .$$

This is the lwest frequency at which *two* TM wave modes can propagate in the waveguide.

4.a) What is meant (generally) by a choice of *gauge* in electromagnetism?

A choice of gauge is a particular choice of the functions $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ *within the set* of such functions which give

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

b) When the Coulomb gauge was discussed (in passing) in the text, the solution for the scalar potential V was given as:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}}\, d\tau'$$

This expression is very peculiar, especially if we are *only* thinking about V. What is peculiar about it?

The solution looks odd because the time argument of ρ inside the integral is *not the retarded time*, thus apparently (?) violating the principle that a "signal" neds time to travel.

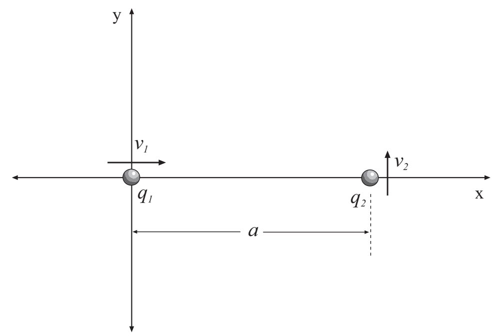
c) How do you resolve the peculiarity or paradox which you identified in part (b)?

The paradox in (b) is resolved by recalling that only the E field

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

is measurable and somehow the $\mathbf{A}$ field *will* fix things up so that $\mathbf{E}$ can only be affected by a charge element after the required propagation time.

5. Two charges are in motion in the xy plane, as shown. Charge q_1 is instantaneously located at the origin and has velocity $v_1\hat{\mathbf{x}}$. Charge q_2 is instantaneously located at $a\hat{\mathbf{x}}$ and has velocity $v_2\hat{\mathbf{y}}$.



a) Find the force of q_1 on q_2.

Force of q_1 on q_2:
Use the equations for the fields from a charge moving with constant velocity (given on the exam). At the location of q_2, $\mathbf{R} = a\hat{\mathbf{x}}$, so with $\theta = 0$,

$$\mathbf{E}_1 = \frac{q-1}{4\pi\epsilon_0}\frac{(1 - v_1^2/c^2)}{(1-0)^{3/2}}\frac{\hat{\mathbf{x}}}{a^2} = \frac{q_1(1 - v_1^2/c^2)}{4\pi\epsilon_0 a^2}\hat{\mathbf{x}}$$

and the B field is

$$\mathbf{B} = \frac{1}{c}(\mathbf{v}\times\mathbf{E}) = 0$$

so

$$\mathbf{F}_{1\text{ on }2} = q_2\mathbf{E}_1 = \frac{q_1q_2(1 - v_1^2/c^2)}{4\pi\epsilon_0 a^2}\hat{\mathbf{x}}$$

b) Find the force of q_2 on q_1.

Force of q_2 on q_1: At the location of q_1, $\mathbf{R} = -a\hat{\mathbf{x}}$ and $\theta = 90°$ so

$$\mathbf{E}_2 = \frac{q_2}{4\pi\epsilon_0}\frac{(1 - v_2^2/c^2)}{(1 - 1\cdot v_2^2/c^2)^{3/2}}\frac{(-\hat{\mathbf{x}})}{a^2} = -\frac{q_2\hat{\mathbf{x}}}{4\pi\epsilon_0 a^2\sqrt{1 - v_2^2/c^2}}$$

and with $\mathbf{v}_2 = v_2\hat{\mathbf{y}}$,

$$\mathbf{B}_2 = \frac{1}{c^2}(\mathbf{v}_2\times\mathbf{E}_2) = \frac{v_2}{c^2}\frac{q_2\hat{\mathbf{z}}}{4\pi\epsilon_0 a^2\sqrt{1 - v_2^2/c^2}} = \frac{q_2v_2\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2a^2\sqrt{1 - v_2^2/c^2}}$$

Then the force on q_1 is, with $\mathbf{v}_1 = v_1\hat{\mathbf{x}}$,

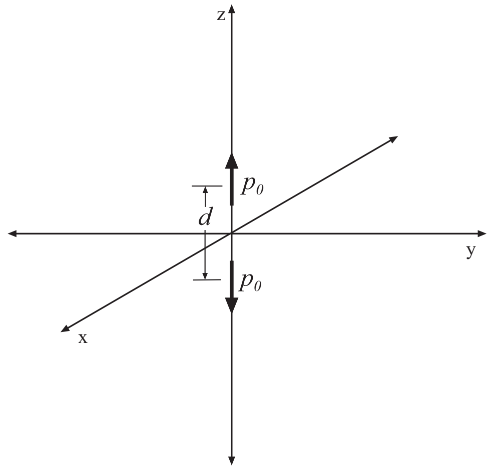
$$\begin{aligned}
\mathbf{F}_{2\text{ on }1} &= q_1\mathbf{E}_2 + q_1\mathbf{v}_1\times\mathbf{B}_2 \\
&= \frac{-q_1q_2\hat{\mathbf{x}}}{4\pi\epsilon_0 a^2\sqrt{1 - v_2^2/c^2}} - \frac{q_1q_2v_1v_2\hat{\mathbf{y}}}{4\pi\epsilon_0 c^2a^2\sqrt{1 - v_2^2/c^2}} \\
&= -\frac{q_1q_2}{4\pi\epsilon_0 a^2\sqrt{1 - v_2^2/c^2}}(\hat{\mathbf{x}} + \frac{v_1v_2}{c^2}\hat{\mathbf{y}})
\end{aligned}$$

6. Explain why the *radiation* parts of the E and B fields from a time–dependent source have to go to zero no faster than $1/r$, that is, relate the mathematical behavior of the fields to the transport of energy.

The energy flux (energy per (area·time)) can't tend to zero faster that $1/r^2$ because then the total energy loss through a large spherical surface would vanish with increasing radius and then energy would *not* be lost by the system and its fields.

Since the energy flux $\mathbf{S}$ is the *product* of $\mathbf{E}$ and $\mathbf{B}$ it follows that (assuming their behavior is the same) neither one can tend to zero faster than $1/r$.

7. In the course and in the text we studied electric dipole radiation but not electric *quadrupole* radiation. Now's our chance!



One can make a simple oscillating electric quadrupole by taking two oscillating dipoles of strength p_0 and opposite polarity separated by a (small) distance d, all directed along the z axis, as shown. This system has no net electric dipole but it does have a quadrupole moment!

For this system, at very large r the E field turns out to be

$$\mathbf{E} = \frac{\mu_0 \omega^3 p_0 d}{4\pi c r} \sin\theta \cos\theta \sin(\omega(t - r/c))\hat{\boldsymbol{\theta}}$$

(with the real part implied) and B is given by

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{r}} \times \mathbf{E}$$

a) Find the time-averaged Poynting vector and make a crude sketch of the "intensity profile". Use $Q \equiv 2p_0 d$ to express things in terms of the *quadrupole* moment.

For the radiating quadrupole given in the problem,

$$\mathbf{E} = \frac{\mu_0 \omega^3 p_0 d}{4\pi c r} \sin\theta \cos\theta \sin(\omega(t - r/c))\,\hat{\boldsymbol{\theta}}$$

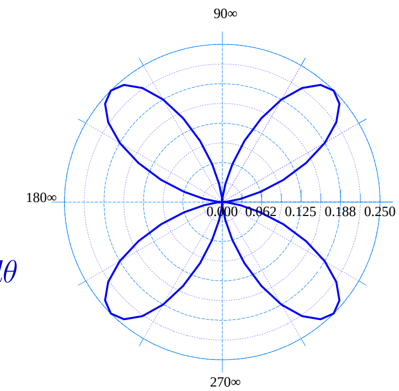
Since $\mathbf{B} = \frac{1}{c}(\hat{\mathbf{r}} \times \mathbf{E})$ and $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$, then

$$\mathbf{B} = \frac{\mu_0 \omega^3 p_0 d}{4\pi c^2 r} \sin\theta \cos\theta \sin(\omega(t - r/c))\,\hat{\boldsymbol{\phi}}$$

The time–averaged Poynting vector is, using $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$, and including a factor of $\frac{1}{2}$ for the time average,

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{\mu_0}\langle \mathbf{E} \times \mathbf{B} \rangle = \frac{\mu_0 \omega^3 p_0^2 d^2}{32\pi^2 c^3 r^2} \sin^2\theta \cos^2\theta\,\hat{\mathbf{r}} \\ &= \frac{\mu_0 \omega^6 Q^2}{128\pi^2 c^3 r^2} \sin^2\theta \cos^2\theta\,\hat{\mathbf{r}} \end{aligned}$$

A polar plot of the angular function is given at the right.



b) Find the total power radiated by the quadrupole.

Integrate over a spherical surface of radius R:

$$P = \oint_{\mathrm{Sph}\ R} \langle \mathbf{S} \rangle \cdot d\mathbf{a} = (2\pi) \frac{\mu_0 \omega^6 Q^2}{128\pi^2 c^3} \int_0^\pi \frac{1}{R^2} \sin^2\theta \cos^2\theta R^2 \sin\theta \, d\theta$$

Use:

$$\begin{aligned} \int_0^\pi \sin^2\theta \cos^2\theta (\sin\theta) \, d\theta &= \int_{-1}^{1} (1-x^2)x^2 \, dx = \int_{-1}^{1} (x^2 - x^4) \, dx \\ &= \left. \frac{x^3}{3} - \frac{x^5}{5} \right|_{-1}^{1} = 2(1/3 - 1/5) = 4/15 \end{aligned}$$

Then we have

$$P = \frac{\mu_0 \omega^6 Q^2}{64\pi c^3} \cdot \frac{4}{15} = \frac{\mu_0 \omega^6 Q^2}{240\pi c^3}$$

8. What is the importance of the *group velocity* for waves in a dispersive medium?

Waves of differing frequency do not travel at the same speed in a dispersive medium. A wave packet will change shape (broaden) as it travels, but the *center* will travel at the *group velocity* for the medium (valuated at an average frequency for the packet).

Group velocity is defined by $v_{\mathrm{g}} = d\omega/dk$.

This is to be contrasted with the *phase velocity*, which is defined for a harmonic wave and is given by $v = \omega/k = \lambda f$.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2\, \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial\lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla\cdot\mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

Waveguides:

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}} \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

$$\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \\
E_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right) \\
B_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \\
B_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)
\end{aligned}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]E_z = 0 \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]B_z = 0$$

Specific Results:

$$\langle \mathbf{S}\rangle_{\text{eldip}} = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}} \qquad \langle P\rangle_{\text{eldip}} = \int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

$$\langle \mathbf{S}\rangle_{\text{magdip}} = \left(\frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}} \qquad \langle P\rangle_{\text{magdip}} = \int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 m_0^2\omega^2}{12\pi c^3}$$