Phys 3820, Fall 2012
Exam #2

1. Give concise but *careful* definitions of:

a) Yukawa or "screened Coulomb" potential.

A potential of the form

$$V(r) = A\frac{e^{-\mu r}}{r}$$

which at small distances behaves the same way as a $1/r$ potential but which falls off more rapidly at large r. Reduces to the Coulomb form for $\mu = 0$.

b) LCAO method (in molecular QM)

Stands for **Linear Combination of Atomic Orbitals** and a is sensible way to construct an orbital for a simple molecular system: Take linear combinations of atomic orbitals centered on each atom in the molecule. This guarantees that when the electron is close to any one of atoms in the linear combination the behavior of its wave function is like that of the individual atom.

c) Airy functions (Ai(x) and Bi(x).)

Peculiar special functions from mathematical physics; they are the solutions to the Schrödinger equation for a linear potential. A linear potential arises in the usage of the WKB approximation but seldom elsewhere in quantum mechanics!

d) Laser.

A system where electrons are somehow excited to a state of higher energy in a *non–thermal* distribution. When this occurs, stimulated emission of radiation can give a very intense beam of light which is also coherent and polarized.

2. In class we did a variational calculation for the He atom, but obviously much more accurate calculations than this have been done.

a) Write down the variational wave function used for our calculation and note what parameter is varied.

The trial wave function used a product of H-like wave functions where nuclear charge Z *in the wave function* was treated as a variational parameter:

$$\psi_{\rm var}(\mathbf{r}_1, \mathbf{r}_2) = \frac{Z^3}{\pi a^3} e^{-Zr_1/a} e^{-Zr_2/a}$$

When we evaluate $\langle H \rangle$ the nuclear charge *the Hamiltonian* still has its God-given value.

b) I hope you said that this parameter was called Z! Anyways, it was shown in the book that for this trial wave function the expectation value of H was

$$\langle H \rangle = [-2Z^2 + (27/4)Z]E_1 \qquad \text{where} \qquad E_1 = -13.6 \text{ eV}$$

From this deduce the variational estimate of the ground state energy of the He atom. (Get a numerical answer.)

We just minimize this expression with respect to Z. Evaluate:

$$\frac{d\langle H\rangle}{dZ} = [-4Z + (27/4)]E_1 = 0 \qquad \Longrightarrow \qquad Z = \frac{27}{16} = 1.69$$

and this gives

$$E_{\rm gs} \le [-2(1.69)^2 + (27/4)(1.69)](-13.6 \text{ eV}) = -77.5 \text{ eV}$$

as an upper limit to the ground state He energy.

c) Assuming you had a computer to do *any* needed integrals, think of how you might improve the wave function in (a). That is, give some more terms along with other parameters which could be varied. (There are lots of possibilities.)

One could still use a product of single-particle wave functions, but instead of a single H-like wave function one could add one with another effective charge or even include a different r dependence:

$$\psi_1(\mathbf{r}) = A[e^{Z_1 r/a} + a_1 e^{-Z_2 r/a} + a_2 e^{-\alpha r^2}]$$

and so one, then take

$$\psi_{\rm tr} = \psi_1(\mathbf{r}_1)\psi_1(\mathbf{r}_2)$$

with parameters a_1, a_2, Z_1, Z_2, and α (A is set by normalization). These may not be the smartest choices, but they would give more freedom in the trial wave function.

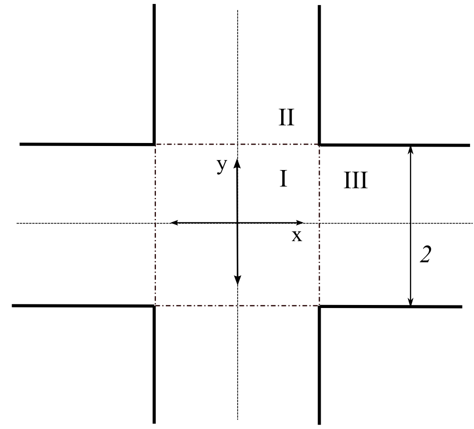
3. a) If the electron in the H_2^+ molecule-ion is replaced by a muon, how does the resulting molecule differ from the usual H_2^+ molecule-ion? Try to include an approximate number in your answer.

First, one finds that if the electron in the H-atom is replaced by a muon, the size of all the atomic wave functions scales inversely with the mass of the particle; thus the radius of the new H-atom is 200 times smaller. The re-sizing carries through to the calculation of the equilibrium bond length of the muon-H_2^+ molecule/ion to get a bond length which is 200 times smaller than in the electron case.

b) What is the (potential) practical application for forming such an exotic molecule?

The practical use is to form a temporary H_2^+ molecule out of fusing H nuclei and a muon; with the smaller distance between the nuclei, fusion is much more likely than in the electron case and thus could possibly be a strategy for producing controlled fusion energy.

4. In the notorious particle–trapped–at-an–intersection problem that concludes Chapter 7, ("Quantum dots", actually) two zero-potential channels of width $2a$ with "hard walls" cross and give rise to a bound state.



a) What is the condition on the energy eigenvalue such one has a bound state in *this* quantum system?

If the channel width is $2a$, the minimum energy for propagation of wave down one of the legs is just that of a particle in the ground state of a box of length $2a$, namely

$$E_{\rm min} = \frac{\pi^2\hbar^2}{2m(2a)^2} = \frac{\pi^2\hbar^2}{8ma^2}$$

because the wavelength in the direction along the channel can be arbitrarily large.

b) Give a qualitative explanation for why the bound state exists.

The bound state exists because the in vicinity of the intersection the wave function can broaden beyond the width of the channel in $both$ directions and lower the energy to get a value less than that given in (a).

5. Simple WKB problem: Calculate the approximate transmission probability for a particle of energy E that encounters a finite square barrier of height $V_0 > E$ and width $2a$.
(There's an integral to do, but it's really simple.)

Here the local momentum function (absolute value) is

$$|p(x)| = |\sqrt{2m(E - V_0)}| = \sqrt{V_0 - E}$$

so that the transmission probability through the barrier is

$$T \approx e^{-2\gamma} \, , \qquad \text{with} \qquad \gamma = \frac{1}{\hbar}\int_0^a |p(x)|\, dx$$

where the barrier (classically forbidden) goes from 0 to a. So γ is simply

$$\gamma = \frac{1}{\hbar}\int_0^{2a} \sqrt{V_0 - E}\, dx = \frac{2a}{\hbar}\sqrt{V_0 - E}$$

giving

$$T = \exp(-\frac{4a}{\hbar}\sqrt{V_0 - E}) \, .$$

6. In testing the Gamow model for α decay, you had to get the energy of the emitted α particle. How did *you* find it?

If we look up the $rest\ energies/masses$ of the various nuclei in a table of the nuclides, the difference in rest energies gives the (lab) kinetic energies of the products after the α decay. If we assume all this kinetic energy is that of the α particle and then use the $non\text{-}relativistic$ formula $T = \frac{1}{2}mv^2$ (which is OK as these KE's were much less than mc^2) we can solve for v.

In getting the nuclear masses one must be consistent; either find the $true$ masses of the nuclei of find the masses of the $neutral\ atoms$ and then when taking the differences, hope that the electron binding energies all cancel out!

7. In applying the WKB approximation to a general 1D potential problem one has to connect solutions in a classically–allowed region to one in classically–forbidden region.

What is the usual way that this is accomplished?

One isolates a region of space including the classical turning point (the boundary between forbidden and allowed regions) and makes a linear approximation for the potential there. With the parameters for this linear potential, one solves the Schrödinger equation obtaining a combination of Airy functions. Then on both sides of the selected region one matches value and slope with the WKB solutions found in both the forbidden and allowed regions. Elaborate connection formulae for this purpose were given in the text in a section that I had no desire to cover.

8. For the two–level system we found the *exact* equations

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \qquad \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a \qquad \text{where} \qquad \omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

Give a brief summary of the strategy we then used to find approximate solutions for $c_a(t)$ and $c_b(t)$ individually.

We took the lowest-order solution, $c_a(t) = 1$, $c_b(t) = 0$ and put it back on the right-hand sides of both equations to get first-order equations for $\dot{c}_a$ and $\dot{c}_b$ which we then solved (formally) with an integral over a time parameter.

We repeated this process of substitution for $c_a(t)$ and $c_b(t)$, getting multiple integrals in the t variables for the higher orders. One can repeat it indefinitely.

9. In deducing our main result for $R_{b\to a}$ we had to take the result for a monochromatic wave and get the result for incoherent unpolarized light.

Summarize what was done for this. Include whatever mathematical details you remember.

The initial calculation specified a certain direction of propagation of the wave and a particular polarization of the EM wave. We needed to get an average over all propagation directions ($\mathbf{k}$ vectors) and polarizations ($\mathbf{n}$ vectors, unit vectors perpendicular to $\mathbf{k}$). A clever trick was done in the book to make this a relatively simple integral over 4π steradians; the direction of $\mathbf{k}$ was fixed and then $\mathbf{r}$ and $\hat{\mathbf{n}}$ had limited directions over which to vary, giving the simpler integral.

10. In Einstein's theory of the two–level system, he wrote down an expression for the rate of change of the population of state b,

$$\frac{dN_b}{dt} = -N_b A - N_b B_{ba}\rho(\omega_0) + N_a B_{ab}\rho(\omega_0)$$

Explain the physical significance of each term.

The number A is the rate of spontaneous emission, the emission that would take place in the absence of an external EM field. This is a quantity per atom so we need to multiply by N_b to get the contribution to $\frac{dN_b}{dt}$.

The number B_{ba} is the coefficient for induced emission. It needs a factor for the energy density of the EM field and also for the number of atoms in state b. Both of these terms give a negative contribution to $\frac{dN_b}{dt}$.

The number B_{ab} is the coefficient for induced absorption. It also needs a factor for the energy density of the EM field and the number of atoms in state a. It gives a $positive$ contribution to $\frac{dN_b}{dt}$.

b) Summarize how using this and little bit of statistical thermodynamics we related the values of A the B's.

We imagined a system in thermal equilbrium with the radiation field at some temperature T and got a relation between the A and B coefficients and $\rho(\omega, T)$. Using the Boltzmann factor for occupation of quantum states and Planck's formula for $\rho(\omega)$ we got an explicit expression for A.

11. Consider a possible radiative transition from the $n = 3$ level to the $n = 1$ level in the H atom.

a) According to the selection rules, which transitions are possible? (Give values of l and m.)

The $n = 3$ level has states with $l = 0, 1, 2$ while the $n = 1$ state has only $l = 0$. Since we must have $\Delta l = \pm 1$ then only the

$$(n = 3,\ l = 1) \ \rightarrow\ (n = 1,\ l = 0)$$

transitions are possible. We also must have $\Delta M = 0, \pm 1$ so that any of the three m_l states can go to the $(n = 1, l = 0)$ level.

b) Let's calculate the lifetime for an $n = 3$ hydrogen state for any one of these allowed $(n = 3) \rightarrow (n = 1)$ transitions... or at least get as far as we can. Choose the one you think would be the simplest and calculate

$$|\mathfrak{p}|^2$$

for that transition.

Probably easiest to compute would be the $(310) \rightarrow (100)$ transition dipole moment,

$$\mathfrak{p}_z = q \int \psi_{310}\ \mathbf{r}\ \psi_{100}\, d\mathbf{r} \quad .$$

We see from symmetry that this has only the z component, as both of the wave functions have no x or y dependence so those integrals will give zero. Noting that

$$z = r\cos\theta = r\sqrt{\frac{4\pi}{3}}Y_1^0 \qquad \text{and} \qquad Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

we have

$$\mathfrak{p}_z = q \int R_{31} Y_1^{0*} \left(r\sqrt{\frac{4\pi}{3}}Y_1^0 \right) R_{00}\frac{1}{\sqrt{4\pi}}\ d^3r$$

which now lets us do the angular integral easily... it gives 1, as Y_1^0 is normalized ! We now have

$$\wp_z = \frac{q}{\sqrt{3}} \int_0^\infty R_{31}(r) R_{00}(r) r^3 \, dr$$

Using the radial functions from the tables and throwing the constants out in front we get

$$\wp_z = \frac{q}{\sqrt{3}} \frac{2}{a^{3/2}} \frac{8}{27\sqrt{6}} \frac{1}{a^{3/2}} \int_0^\infty \left(1 - \frac{1}{6}\frac{r}{a}\right) \frac{r}{a} e^{-r/3a} e^{-r/a} r^3 \, dr$$

and we can simplify things by using a dimensionless variable $x \equiv \frac{r}{a}$ and with this we have

$$\wp_z = \frac{q}{\sqrt{3}} \frac{16}{27\sqrt{6}} \frac{a^4}{a^3} \int_0^\infty x^4 (1 - x/6) e^{-4x/3} \, dx$$

Getting close to an answer! The integral is not too bad to do by hand, but I put it into Maple. With this we have (set $q = e$)

$$\wp_z = \frac{16}{81\sqrt{2}} (ae) \left(\frac{2187}{1024}\right) = (0.29831) ae$$

c) Calculate the spontaneous emission rate A for the transition, and then get the lifetime τ.

The frequency of the transition is

$$\omega_0 = \frac{\Delta E}{\hbar} = \frac{(8.9)(13.6 \text{ eV})}{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})} = 1.836 \times 10^{16} \text{ s}^{-1}$$

Then for the A coefficient this gives

$$A = \frac{\omega_0^3 |\boldsymbol{\wp}|^2}{3\pi\epsilon_0 \hbar c^3} = \frac{\omega_0^3 (0.29831)^2 (ae)^2}{3\pi\epsilon_0 \hbar c^3}$$

I get:

$$A = 1.668 \times 10^8 \text{ s}^{-1}$$

giving a lifetime of

$$\tau = \frac{1}{A} = 5.99 \times 10^{-9} \text{ s}$$

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2} a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\bigg|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}} \qquad \epsilon_0 = 8.854 \times 10^{-12}\ \tfrac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_{\pm} \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

$$\text{Delta Fn Potl:} \qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \, \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi} \text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$R_{30} = \frac{2}{\sqrt{27}} a^{-3/2} \left(1 - \tfrac{2}{3}(r/a) + \tfrac{2}{27}(r/a)^2 \right) \exp(-r/3a)$$

$$R_{31} = \frac{8}{27\sqrt{6}} a^{-3/2} \left(1 - \tfrac{1}{6}(r/a) \right) \left(\frac{r}{a} \right) \exp(-r/3a) \qquad R_{32} = \frac{4}{81\sqrt{30}} a^{-3/2} \left(\frac{r}{a} \right)^2 \exp(-r/3a)$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c \hbar^3} \left(\frac{c^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m{\pm}1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0\hbar)/2 \qquad E_- = +(\gamma B_0\hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E^1_{\text{fs}} = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E^1_Z = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788\times 10^{-5}\ \text{eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E^1_{\text{hf}} = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p \cdot \mathbf{S}_e\rangle = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \le \langle \psi | H | \psi \rangle \equiv \langle H \rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v} e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

$$c_b^{(2)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ab}(t') e^{i\omega_0 t'}\,dt' \qquad c_a^{(2)}(t) = 1 - \frac{i}{\hbar^2}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'}\left[\int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''}\,dt''\right] dt'$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$\wp \equiv q\langle \psi_b | \mathbf{r} | \psi_a \rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\wp| E_0}{\hbar}\right)^2 \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\wp|^2 \rho(\omega_0) \qquad A = \frac{\omega^3 |\wp|^2}{3\pi\epsilon_0 \hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1$; or 0 and $\Delta l = \pm 1$