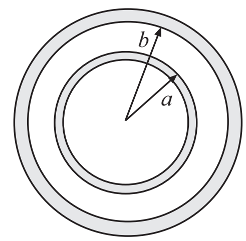
Phys 4610, Fall 2004
Exam #2

1. A spherical capacitor is made from two concentric spherical shells of radii a and b (with $a < b$).

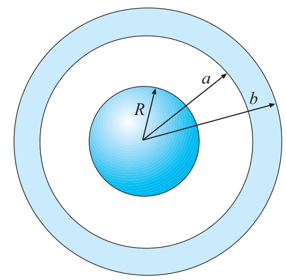
Show that the capacitance of this system is $C = 4\pi\epsilon_0 \frac{ab}{(b-a)}$.



2. A metal sphere of radius R, carrying a charge q is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as shewn in the picture). The shell also carries a net charge of q.

a) Find the surface charge density σ at R, at a and at b.

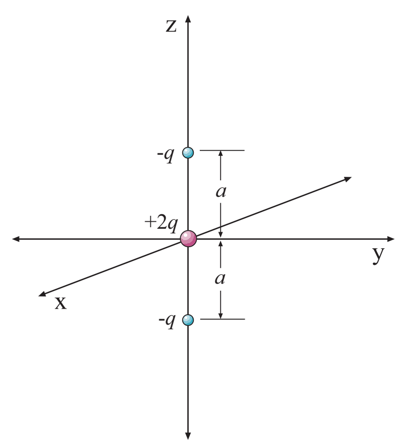
b) Find the potential at the center, using infinity as the reference point.



3. Point charges of $-q$, $+2q$ and $-q$ are located on the z axis at $z = -a$, $z = 0$ and $z = +a$ respectively.

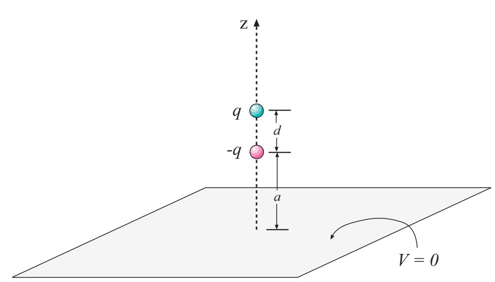
a) At a point given by spherical coordinates (r, θ) what is the electrical potential?

b) Using the binomial theorem on the terms in the answer to (a), find an approximate expression for the potential at large distances r. Your answer just needs to have the first non-zero term proportional to some power of r.

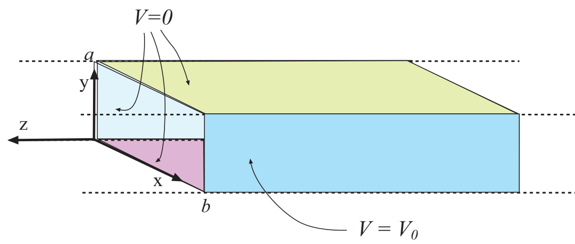


4. Two point charges, $\pm q$, are held above an infinite grounded conducting plane (the xy plane). Both charges are on the z axis; the $-q$ charge is at $z = a$ and the $= q$ charge is at $z = a + d$.

Find the potential everywhere in the region $z > 0$.



5. We would like to solve the electrostatics problem diagrammed at the right; we have an infinite rectangular pipe which runs along the z direction, where the rectangular cross-section goes from $x = 0$ to $x = b$ and $y = 0$ to $y = a$. The side at $x = b$ is held at a constant potential of V_0, but the other sides are at zero potential.



I will start the problem and you finish it. The potential is independent of z, so we are solving for $V(x, y)$, and first looking for suitable solutions of the form $X(x)Y(y)$. We find that the choices

$$X(x) = \sinh\left(\frac{n\pi x}{a}\right) \qquad Y(y) = \sin\left(\frac{n\pi y}{a}\right)$$

will do the trick, i.e. they satisfy the Laplace equation and the "zero" boundary conditions. The solution is then given by

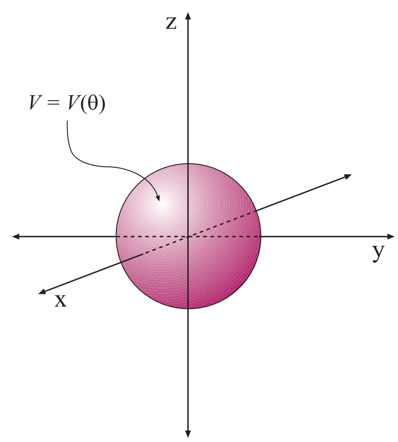
$$V(x, y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Apply the last boundary condition to get the C_n's

6. The potential on the surface of a sphere (radius R, centered on the origin) is given by

$$V_0 = k \cos^2 2\theta$$

where k is a constant. Find the potential outside the sphere. (There is no charge outside the sphere.)

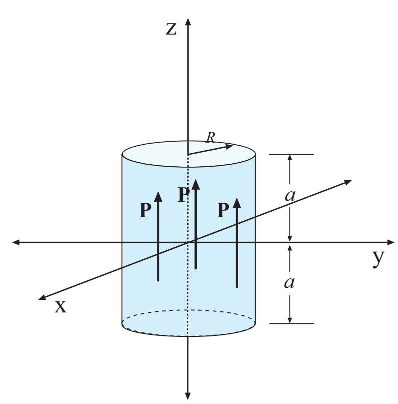


7. Explain the difference between a "bound" charge density and a "free" charge density. (Agreed, there is a little bit of arbitrariness in the distinction, but explain why we make it for practical situations involving dielectrics.)

8. A right circular cylinder of dielectric material has a frozen–in polarization (but no free charge on it). The cylinder has radius R, length $2a$ and a uniform polarization of magnitude P directed along the axis of the cylinder. The cylinder's axis lies along the z axis and it is centered on the origin.

a) What is the distribution of bound charge in this system?

b) At a location $z = b$ (with $b > a$) on the z axis, what is the value of the electric field?



Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

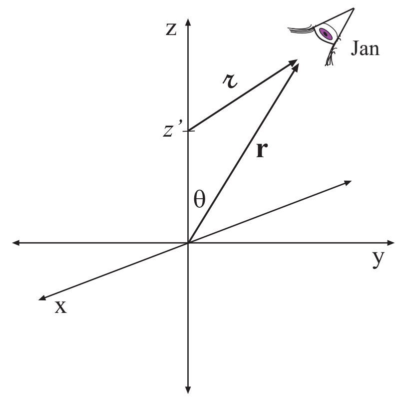
Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

In the figure at the right,

$$\mathcal{r} = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$$



If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathcal{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3 - 3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

Physics:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathcal{r}^2}\hat{\boldsymbol{\mathcal{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\boldsymbol{\mathcal{r}}}}{\mathcal{r}^2} \qquad V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} E\cdot d\mathbf{l}$$

$$\nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathcal{r}}\,d\tau'$$

$$E^\perp_{\text{above}} - E^\perp_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^\parallel_{\text{above}} = \mathbf{E}^\parallel_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j \\ i\neq j}}^{n}\frac{q_i q_j}{\mathcal{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

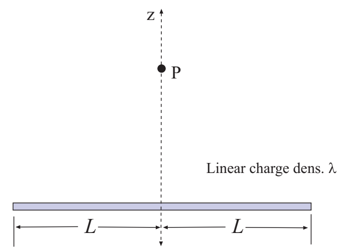
$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}$$



$$E_z = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right]$$