Phys 2920, Spring 2013
Exam #1

Do all matrix calculations by hand unless otherwise indicated. So you need to show your work.

1. a) Write the complex number

$$-3.2 + 5.1\,i$$

in polar ($\rho e^{i\phi}$) form.

$$\rho = \sqrt{a^2 + b^2} = 6.02 \qquad \tan\phi = \frac{b}{a} = -1.594$$

Naively hitting the $\tan^{-1}$ button on the latter number gives (in radians)

$$\tan^{-1}(-1.594) = -1.01 \text{ rad}$$

which can't be right because this complex number lies in Quadrant II. Fix by adding π, thus

$$\phi = -1.01 \text{ rad} + \pi = 2.13 \qquad \Longrightarrow \qquad \boxed{z = (6.02)e^{i(2.13)}}$$

b) Write the complex number

$$8.0\,e^{i\,3.8}$$

in $a + bi$ form.

With $\rho = 8.0$, $\phi = 3.8$, we get

$$a = \rho\cos\phi = -6.33 \quad \text{and} \quad b = \rho\sin\phi = -4.89$$

so that

$$z = \boxed{-6.33 - 4.89\,i}$$

2. Give a suitable definition of **linear independence** (of a set of vectors).

A set of vectors is **linearly independent** if it is impossible to express any one of them as a linear combination of all the other vectors. Equivalently, the set in linearly independent if it is impossible to form the zero vector by taking a linear combination of all of them.

For problems 3 – 5 use the vectors

$$\mathbf{a} = -\hat{\mathbf{\imath}} + 2\hat{\mathbf{\jmath}} + 3\hat{\mathbf{k}} \qquad \mathbf{b} = -5\hat{\mathbf{\jmath}} + 2\hat{\mathbf{k}} \qquad \mathbf{c} = -2\hat{\mathbf{\imath}} + 4\hat{\mathbf{\jmath}} - 3\hat{\mathbf{k}} \qquad \mathbf{d} = \hat{\mathbf{\imath}} + 2\hat{\mathbf{\jmath}} + \hat{\mathbf{k}}$$

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3. Find the angle between the vectors **a** and **d**.

Calculate:

$$\mathbf{a}\cdot\mathbf{d} = -1+4+3 = 6 \qquad a = \sqrt{15} \qquad d = \sqrt{6}$$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{d}}{ad} = \frac{6}{\sqrt{14}\sqrt{6}} = 0.655 \qquad \Longrightarrow \qquad \theta = \boxed{49.1^\circ}$$

4. Find a unit vector perpendicular to both **b** and **c**.

The cross product of $\mathbf{b}$ and $\mathbf{c}$ gives a vector perpendicular to both:

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -5 & 2 \\ -2 & 4 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 10\hat{\mathbf{k}}$$

As this vector has magnitude $\sqrt{165}$, a *unit* vector pointing in the same direction is

$$\hat{\mathbf{u}} = \boxed{\frac{1}{\sqrt{165}}(7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 10\hat{\mathbf{k}})}$$

5. Are the vectors **b**, **c** and **d** linearly independent? *If so, show how you know this.*

As discussed in class, if (for example) $\mathbf{b}$ is perpendicular to the cross product of $\mathbf{c}$ and $\mathbf{d}$ then it must lie in the same plane as $\mathbf{c}$ and $\mathbf{d}$ and thus is not independent of them. Thus, evaluate $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})$; as also seen in class, this the determinant of the coefficients, thus

$$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} 0 & -5 & 2 \\ -2 & 4 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 15 - 10 - 8 + 8 = 5 \neq 0$$

so the three vectors must be independent.

6. Write a compact expression for

$$\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}+\mathbf{d}))$$

in terms of the components of the four vectors. Use the ϵ's and the summation convention.

First, use the epsilon notation to write $\mathbf{b}\times(\mathbf{c}+\mathbf{d})$:

$$\mathbf{b}\times(\mathbf{c}+\mathbf{d}) = b_i(c_j+d_j)\epsilon_{ijk}\hat{\mathbf{e}}_k$$

Then the dot product of $\mathbf{a}$ with this vector is

$$\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}+\mathbf{d})) = a_k b_i(c_j+d_j)\epsilon_{ijk}$$

7. Explain the concept of a **basis** for a vector space.

A basis for an (N-dimensional) vector space is a set of N independent vectors for which we intend to form all the vectors by taking the appropriate linear combinations.

8. Recall that on one of the problem sets, I had you make *orthonormal* functions $\tilde{P}_n(x)$ out of the $P_n(x)$'s which were orthogonal, but did not have unit magnitude.

Given that

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$$

what is the general expression for $\tilde{P}_n(x)$?

The C's will be some number times the $P_n(x)$'s. If $\tilde{P}_n(x) = AP_n(x)$ then

$$\int_{-1}^{1} [\tilde{P}_n(x)]^2 \, dx = 1 = A^2 \int_{-1}^{1} [P_n(x)]^2 \, dx = A^2 \frac{2}{2n+1}$$

which implies that $A = \sqrt{\frac{2n+1}{2}}$, hence

$$\tilde{P}_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x)$$

9. Recall the expansion of (suitable) functions on the interval $[0, 1]$ In terms of the functions $f_n(x) = \sqrt{2}\sin(n\pi x)$. (They were orthonormal.) We expanded one function which could be done "by inspection"; the function (on the interval $[0, 1]$)

$$f(x) = \begin{cases} 0 & x < \frac{1}{4} \\ 2 & \frac{1}{4} < x < \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}$$

can also be expanded in the $f_n(x)$'s but we have to do it the hard way. (First, plot this function to understand what it looks like.)

Write down an expression for how we would get n^{th} coefficient c_n in

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

and use it to find c_1, c_2 and c_3.

If $f(x)$ is a suitable function on $[0, 1]$ (here, one where $f(0) = f(1) = 0$) then it can be expanded *uniquely* as

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \, dx$$

and to get the c_n's we take the inner product of $f_n(x)$ with the given $f(x)$ (which works because the $f_n(x)$'s are orthonormal:

$$c_n = \langle f_n | f \rangle = \int_0^1 \sqrt{2}\sin(n\pi x) f(x) \, dx$$

The f here in only nonzero from $x = \frac{1}{4}$ to $x = \frac{3}{4}$, thus:

$$c_n = 2\sqrt{2}\int_{1/4}^{3/4} \sin(n\pi x)\, dx$$

which can be done easily, so

$$c_n = -\frac{2\sqrt{2}}{n\pi} \cos(n\pi x)\Big|_{1/4}^{3/4} = \frac{2\sqrt{2}}{n\pi}[\cos(n\pi/4) - \cos(3n\pi/4)]$$

The first few c_n's are

$$c_1 = \frac{2\sqrt{2}}{\pi}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = \boxed{\frac{4}{\pi}} \qquad c_2 = \frac{2\sqrt{2}}{2\pi}[0-0] = \boxed{0}$$

$$c_3 = \frac{2\sqrt{2}}{3\pi}\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] = \boxed{-\frac{4}{3\pi}}$$

10. Calculate the determinant of the matrix

$$\mathsf{A} = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ 5 & -1 & 0 & 2 \\ -1 & 0 & 2 & 4 \end{pmatrix}$$

Use the "expansion by minors" formula (watch the $(-1)^n$ factors!); this gives

$$\begin{aligned} \mathrm{Det}(\mathsf{A}) &= (-1)(2)\begin{vmatrix} 2 & -1 & 0 \\ 5 & 0 & 2 \\ -1 & 2 & 4 \end{vmatrix} + (-1)(-2)\begin{vmatrix} 2 & 0 & -1 \\ 5 & -1 & 0 \\ -1 & 0 & 2 \end{vmatrix} \\ &= -2(-8+2+20) + 2(-4+1) = -2(14) + 2(-3) \\ &= -28 - 6 = -34 \end{aligned}$$

11. If matrices A, B and C are given by

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 5 \\ 1 & 0 & 2 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -3 \\ 2 & 0 & 4 \end{pmatrix} \qquad \mathsf{C} = \begin{pmatrix} 2 & -4 \\ 2 & -3 \end{pmatrix}$$

Find:

a) AB

Do the dot--product type operations between the the rows of A and the columns of B. Get:

$$\mathsf{AB} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 5 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -3 \\ 2 & 0 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 4 & -5 \\ 11 & -3 & 22 \\ 3 & 2 & 9 \end{pmatrix}}$$

b) $\mathrm{Det}(\mathsf{A})$

For a 3×3 matrix the "diagonals" prescription gives the determinant, thus

$$\mathrm{Det}(\mathsf{A}) = \begin{vmatrix} 1 & 2 & 0 \\ -1 & -1 & 5 \\ 1 & 0 & 2 \end{vmatrix} = -2 + 10 + 4 = \boxed{12}$$

c) $\mathrm{Det}(\mathsf{B}^{-1})$

We don't need to invert the matrix B for this! Find the determinant of B:

$$\mathrm{Det}(\mathsf{B}) = \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 4 \end{vmatrix} = -4 - 12 - 2 = -18$$

And then use

$$\mathrm{Det}(\mathsf{B}^{-1}) = [\mathrm{Det}(\mathsf{B})]^{-1} = \boxed{-\tfrac{1}{18}}$$

d) $\mathrm{Trace}(\mathsf{C})$

Sum the diagonal elements of C:

$$\mathrm{Tr}(\mathsf{C}) = 2 - 3 = \boxed{-1}$$

e) $\mathrm{Det}(\mathsf{C})$

Well, this is pretty routine by now:

$$\mathrm{Det}(\mathsf{C}) = \begin{vmatrix} 2 & -4 \\ 2 & -3 \end{vmatrix} = -6 + 8 = \boxed{2}$$

f) $\mathrm{Det}(\mathsf{C}^5)$

We don't need to raise C to the fifth power! Use

$$\mathrm{Det}(\mathsf{C}^5) = [\mathrm{Det}(\mathsf{C})]^5 = 2^5 = \boxed{32}$$

Theorems about matrix operations will be of help!

12. Find the eigenvalues and (unit) eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

Eigenvalues: Find the roots of

$$\begin{aligned}\mathrm{Det}(\mathsf{A}-\lambda\mathbf{1}) &= \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)-2 \\ &= \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5)\end{aligned}$$

so the roots (eigenvalues of A) are $\boxed{2 \text{ and } 5}$.

For the eigenvalue 2 solve for x and y in

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

which leads to an eigenvector of

$$y = -x \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For the eigenvalue 5 solve for x and y in

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

which leads to an eigenvector of

$$y = 2x \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So the normalized eigenvectors are

$$\lambda = 2 \ : \quad \boxed{\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \qquad\qquad \lambda = 5 \ : \quad \boxed{\tfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

13. a) Check if each matrix has an inverse, and if so find it (any way you can) for:

$$\mathsf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \qquad\qquad \mathsf{B} = \begin{pmatrix} -10 & -4 \\ 5 & 2 \end{pmatrix}$$

Quickly checking the determinants, we find that $\mathrm{Det}(\mathsf{A}) \neq 0$ while $\mathrm{Det}(\mathsf{B}) = 0$ so that only A has an inverse. Find it by the row--operation method discussed in class. Start with the original matrix and unit matrix placed together:

$$\left(\begin{array}{cc||cc} 1 & -2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right)$$

Now take 3 times the first row and subtract from the second; after that dived the second row by 11:

$$\Longrightarrow \quad \left(\begin{array}{cc||cc} 1 & -2 & 1 & 0 \\ 0 & 11 & -3 & 1 \end{array}\right) \quad \Longrightarrow \quad \left(\begin{array}{cc||cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{3}{11} & \frac{1}{11} \end{array}\right)$$

Add twice the second row to the first row:

$$\Longrightarrow \qquad \left(\begin{array}{cc||cc} 1 & 0 & \frac{5}{11} & \frac{2}{11} \\ 0 & 1 & -\frac{3}{11} & \frac{1}{11} \end{array}\right)$$

And that's the inverse:

$$\mathsf{A}^{-1} = \boxed{\begin{pmatrix} \frac{5}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{pmatrix}}$$

14. Explain what is meant by a *representation* of a vector or an operator.

Mathematically, vectors and the operators that act on them are really abstract objects; but the elements of the vector space can be expressed an numbers when we decide on a *basis* in which to express them. Then we express how an abstract operator acts upon these basis vectors (with results also in terms of this basis) and thus arrive at a *matrix* corresponding to the linear operator.

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \epsilon_{ijk} = \begin{cases} 1 & ijk = 123 \\ \times -1 & \text{switch indices} \\ 0 & \text{any two equal} \end{cases} \qquad \mathbf{c} = \mathbf{a}\times\mathbf{b} \quad \Rightarrow \quad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \qquad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$(\mathsf{AB})_{ij} = \sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{kj} \qquad \mathsf{AA}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathcal{I} \qquad \mathrm{Tr}\,\mathsf{A} = \sum_{i=1}^{N} A_{ii} \qquad |\mathsf{A} - \lambda\mathbf{1}| = 0$$

$$\hat{\mathbf{e}}'_j = \sum_{i=1}^{N} \mathsf{S}_{ij}\hat{\mathbf{e}}_i \qquad \mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} \qquad \mathsf{A}' = \mathsf{S}^{-1}\mathsf{AS}$$

$$|\mathsf{A}^{\mathrm{T}}| = |\mathsf{A}| \qquad |\mathsf{A}^{\dagger}| = |\mathsf{A}^{*}| = |\mathsf{A}|^{\star} \qquad |\lambda\mathsf{A}| = \lambda^n|\mathsf{A}| \qquad |\mathsf{AB}| = |\mathsf{A}|\cdot|\mathsf{B}|$$

$$(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1} \qquad \mathrm{Tr}(\mathsf{ABC}) = \mathrm{Tr}(\mathsf{BCA}) = \mathrm{Tr}(\mathsf{CAB})$$