Notes for Phys 3610

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These notes are just intended to give an overview of the major equations covered in class.

1 Newton's Laws

We present Newton's law as they apply to *point masses*, or *particles*. Such a particle can have translational motion but no internal degrees of freedom. Later on we will find out how the apply the law of mechanics to extended objects.

Newton's First Law: In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.

Newton's Second Law: For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration:

$$\mathbf{F} = m\mathbf{a}$$

Here, $\mathbf{a}$ is the particle's acceleration,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$$

The second law can also be expressed in terms of the momentum of the particle, defined as

$$\mathbf{p} = m\mathbf{v}$$

In classical mechanics, the mass of our idealized particle does *not* change, so that

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a}$$

so that the second law can be written as

$$\mathbf{F} = \dot{\mathbf{p}}$$

which is —in spite of what some basic physics books say— completely equivalent to $\mathbf{F} = m\mathbf{a}$.

1.1 Reference Frames

1.2 Discussion of Newton's 1st and 2nd Laws

Newton's First Law is much more profound than the way it is usually expressed in first-year physics courses. One might think that it trivially follows from the second law, but in fact it really states that there *is* a reference frame in which no acceleration takes place except when real, concrete effects known as *forces* are present. This is *not* true for many reference frames that one might choose... for example the surface of the earth! Such a reference frame is an *inertial* frame. For non-inertial frames we *do* get accelerations when there are no forces acting.

The second law applies to an inertial frame. It says

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the *total* force acting on a (point) mass m and $\mathbf{a}$ is its acceleration.

1.3 Newton's 3rd Law

Newtons's Third Law states that a force on one object is due to the presence of some other object. We will let $\mathbf{F}_{12}$ mean the force on object 1 due to object 2 and $\mathbf{F}_{21}$ mean the force on object 2 due to object 1.

Newton's Third Law: If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 always exerts a force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

2 Projectile Motion with Air Resistance, Motion in a Magnetic Field

Air exerts a force on a projectile which is opposite to the velocity and depends on the speed:

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

The simplest form of $f(v)$ has linear and quadratic parts:

$$f(v) = bv + cv^2 = f_{\rm lin} + f_{\rm quad} \tag{1}$$

The coefficients b and c for spherical projectiles of diameter D are given by

$$b = \beta D \quad \text{and} \quad c = \gamma D^2$$

where β and γ, for air at STP are

$$\beta = 1.6 \times 10^{-4}\ \mathrm{N \cdot s/m^2} \quad \text{and} \quad \gamma = 0.25\ \mathrm{N \cdot s^2/m^4}$$

It often happens that we can ignore one or the other of the terms in Eq. 1. Alas, as we'll see, for the usual conditions of projectiles moving in air, it is $f_{\rm quad}$ which is the largest, which does not have an analytic solution.

2.1 Solution for Linear Air Resistance

When we point the y axis downward, Newton's 2nd law for a projectile gives

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v}$$

where $\mathbf{g} = g\hat{\mathbf{y}}$. Separating this into components,

$$m\dot{v}_x = -bv_x \qquad\qquad m\dot{v}_y = mg - bv_y$$

$$\dot{v}_x = -kv_x \qquad \text{where} \quad k = \frac{b}{m}$$

and we also define $\tau = 1/k = m/b$.

Consider the x equation first. It is:

$$\dot{v}_x = -kv_x \qquad \text{where} \quad k = \frac{b}{m}$$

and we also define $\tau = 1/k = m/b$. Solving this differential equation gives

$$v_x(t) = v_{x0}e^{-t/\tau} \qquad \text{and} \qquad x(t) = x_\infty(1 - e^{-t/\tau})$$

where x_∞ is the limiting value of x as $t \to \infty$.

The y equation can be written as

$$\dot{v}_y = -b(v_y - v_{\text{ter}})$$

where

$$v_{\text{ter}} = \frac{mg}{b} \qquad \text{For linear drag!}$$

It has the solution

$$v_y(t) = v_{y0}e^{-t/\tau} + v_{\text{ter}}(1 - e^{-t/\tau})$$

from which we can get $y(t)$,

$$y(t) = v_{\text{ter}}t + (v_{y0} - v_{\text{ter}})\tau(1 - e^{-t/\tau})$$

It is possible to combine the x and y equations to get an equation for the *trajectory* of the projectile,

$$y = \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

but this equation must be solved numerically.

For *zero* air resistance, the range R of a projectile fired from ground level is

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{2v_{x0}v_{y0}}{g} = R_{\text{vac}}$$

For low air resistance, the range is given apporximately by

$$R \approx R_{\text{vac}}\left(1 - \frac{4}{3}\frac{v_{y0}}{v_{\text{ter}}}\right)$$

2.2 Solutions for Quadratic Air Resistance

Separating we get

$$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\,v_x \qquad m\dot{v}_y = mg - c\sqrt{v_x^2 + v_y^2}\,v_y \tag{2}$$

for which both equations contain *both* of the variables v_x and v_y.

If the motion is *purely* horizontal, the equation of motion

$$m\dot{v}_x = -cv_x^2$$

has the solution

$$v_x(t) = \frac{v_{x0}}{1 + t/\tau} \qquad \text{where} \qquad \tau = \frac{m}{cv_{x0}} \quad \text{quadratic drag}$$

and

$$x(t) = v_{x0}\tau \ln(1 + t/\tau)$$

If the motion is purely vertical, we can also solve the equation. If a baseball is dropped from a high place and again the y axis goes downward, the equation of motion is

$$m\dot{v}_y = mg - cv_y^2$$

There will be a terminal velocity, which is

$$v_{\rm ter} = \sqrt{\frac{mg}{c}}$$

which has the solutions

$$v_y(t) = v_{\rm ter}\tanh\left(\frac{gt}{v_{\rm ter}}\right) \qquad y(t) = \frac{(v_{\rm ter})^2}{g}\ln\left[\cosh\left(\frac{gt}{v_{\rm ter}}\right)\right]$$

When we need to have both horizontal and vertical motion, the equations in 2 must be solved numerically... with a computer. The popular packages Mathematica, Maple and Matlab have canned routines which will do this; you just need to learn how to set up the DE's according to their syntax.

2.3 Motion of a Charge in a Uniform Magnetic Field

If a particle of charge q and velocity $\mathbf{v}$ moves in a magnetic field $\mathbf{B}$ it experiences a force

$$\mathbf{F} = m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}$$

Consider motion in a uniform magnetic field $\mathbf{B} = b\hat{\mathbf{z}}$. From these we get the equations of motion,

$$m\dot{v}_x = qBv_y \qquad m\dot{v}_y = -qB \qquad m\dot{v}_z = 0$$

The z equation just gives $v_z = \text{const}$, so we can focus on v_x and v_y. With $\omega \equiv qB/m$, we solve

$$\dot{v}_x = \omega v_y \qquad \dot{v}_y = -\omega v_x$$

While these can be solved by conventional means (you would combine the x and y equations and use the appropriate initial conditions) Taylor goes for a creative approach with the use of complex variables. He defines

$$\eta = v_x + iv_y \qquad \text{so that} \qquad \dot{\eta} = -i\omega\eta$$

The solution is

$$\eta = Ae^{-i\omega t}$$

where A is determined from the initial conditions. Defining

$$\xi = x + iy \qquad \text{we have} \qquad \xi = \int \eta dt$$

which, with a convenient choice of coordinates gives

$$\xi = x + iy = Ce^{-i\omega t}$$

where $C = x_0 + iy_0$, with (x_0, y_0) being the initial coordinates.

From this form one sees that the particle travels in a helical path, with the radius of the orbit being $r = v/\omega = mv/(qB)$. Thus we have derived the elementary result with some rigor!

3 Linear Momentum

From the discussion of Newton's Third Law, we found

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}} \qquad \text{where} \qquad \mathbf{P} = \sum_\alpha \mathbf{p}_\alpha$$

and $\mathbf{F}^{\text{ext}}$ is the total external force on the system.

3.1 The "Rocket Problem"

Rockets accelerate by chucking mass out the back (continuously) at some speed v_{ex} relative to the rocket. We consider the change in speed of a rocket in a time interval dt. Initially the rocket has mass m and moves with velocity v. At a time dt later, its mass has changes by a (negative) amount dm. The mass dm is traveling at velocity $v - v_{\text{ex}}$ and the rocket now has velocity $v + dv$. If no external force acts on the rocket, conservation of momentum gives

$$m\, dv = -dm\, v_{\text{ex}}$$

which implies

$$\text{thrust} = m\dot{v} = -\dot{m}v_{\text{ex}}$$

where $\dot{m}$ is the rate at which the rocket expels the mass. (It's not simply a *decrease* in its mass!

This differential equation can be solved to give

$$v - v_0 = v_{\text{ex}} \ln(m_0/m)$$

3.2 Center of Mass

For a system of particles, the center of mass is defined by

$$\mathbf{R} = \frac{1}{M} \sum_\alpha m_\alpha \mathbf{r}_\alpha$$

From this definition it follows that

$$\mathbf{P} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{P}}$$

so that

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

4 Angular Momentum

The angular momentum for a single particle (relative to some origin O) is

$$\ell = \mathbf{r} \times \mathbf{p}$$

The rate of change of the angular momentum is

$$\dot{\ell} = \mathbf{r} \times \mathbf{F} = \mathbf{\Gamma}$$

where $\mathbf{\Gamma}$ is the torque on the particle. (Not to be confused (yet) with the *external* torque acting on a rigid body.)

4.1 Kepler's Second Law

Kepler's second law states that the rate at which a planet sweeps out area as it orbits the sun is constant: dA/dt is constant. In fact, it follows from the conservation of angular momentum. One can show that

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r} \times \mathbf{p}| = \frac{\ell}{2m}$$

and since the force on a planet is parallel to $\hat{\mathbf{r}}$ (giving zero torque) then ℓ is constant and thus so is dA/dt.

4.2 Angular Momentum of a System of Particles

The total momentum of a system of particles is

$$\mathbf{L} = \sum_{\alpha=1}^{N} \ell_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

One can show that if the particles obey Newton's third law, *and* if the forces between them act along the line joining each pair (that is, they are *central* force) then

$$\dot{\mathbf{L}} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}}$$

i.e. the rate of change of the total momentum is just the total *external* torque acting on the system.

While we hold off on the serious rotation problems until a leter chapter, we recall some elementary facts bout rotation dynamics so that we can do a few examples.

Recall that if a rigid object turns about an axis along the z axis, its angular momentum is $L_z = I\omega$, where the moment of inertia for a set of mass points is defined as

$$I = \sum m_\alpha \rho_\alpha^2 \ ,$$

ρ being the distance of each mass point from the axis of rotation.

An under-appreciated theorem states that we can use the equation $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ even in an accelerating frame as long as the origin stays with the center of mass, that is:

$$\frac{d}{dt}\mathbf{L}(\text{about CM}) = \mathbf{\Gamma}^{\text{ext}}(\text{about CM})$$

It is only because of this theorem that we can analyze the motion of an accelerating rolling object!

5 Energy

5.1 Kinetic Energy and Work

The kinetic energy of a particle is $T = \frac{1}{2}mv^2$, which gives

$$\frac{dT}{dt} = m\dot{\mathbf{v}} \cdot \mathbf{v} = F \cdot \mathbf{v} \qquad \Longrightarrow \qquad dT = \mathbf{F} \cdot d\mathbf{r}$$

For the change in T over a finite displacement (over a particular path!) we have

$$\Delta T = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv W(1 \rightarrow 2)$$

where we've introduced W, the work done on the particle as it moves from 1 to 2 along the path of its motion. If several individual forces act on the particle, we add up the work done by each:

$$\Delta T = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r} = \sum_i W_i(1 \rightarrow 2)$$

5.2 Potential Energy, Conservative Forces

We say that a force $\mathbf{F}$ is *conservative* if the work done by $\mathbf{F}$ does not depend on the path chosen between points 1 and 2. If that is the case we can choose a reference point $\mathbf{r}_0$ and define the potential energy function $U(\mathbf{r})$ as

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \rightarrow \mathbf{r}) \equiv \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

With this definition of $U(\mathbf{r})$, we than have

$$W(\mathbf{r}_1 \rightarrow \mathbf{r}_2) = -[U(\mathbf{r}_2) - U(\mathbf{r}_1)] = -\Delta U$$

and after defining the total mechanical energy as $E = T+U$, we find that if only a single conservative force acts on the particle, then

$$\Delta E = \Delta(T + U) = 0$$

When several different conservative forces act, we can add the potential energies from the different forces to get the total mechanical energy:

$$E = T + U_1(\mathbf{r}) + U_2(\mathbf{r}) + \cdots$$

If there are nonconservative forces in the problem, we must leave them in the work–energy theorem as the work which they do:

$$\Delta T = W = W_{\mathrm{cons}} + W_{\mathrm{nc}} \qquad \Longrightarrow \qquad \Delta E = \Delta(T + U) = W_{\mathrm{nc}}$$

5.3 More About Conservative Forces

The definition given above for a conservative force is equivalent to the requirement that $\mathbf{F}$ is derivable from a potential energy, that is, there is some function $U(\mathbf{r})$ such that

$$\mathbf{F} = -\nabla U$$

It is also equivalent to the condition that $\mathbf{F}$ has zero curl:

$$\nabla \times \mathbf{F} = 0$$

5.4 If F Depends On t.

We can consider the case that the force $\mathbf{F}$ depends *explicitly* on time: $\mathbf{F}(\mathbf{r}, t)$. For such a force, a particle could *stay in one spot* and the force would change with time.

If such a force satisfies $\nabla \times \mathbf{F} = 0$ we can define a time–dependent potential by $\mathbf{F} = -\nabla U$ but that is still not good enough to make it useful because with such a potential, mechanical energy would not be conserved:

$$d(T + U) = \frac{\partial U}{\partial t} dt$$

5.5 One–Dimensional Systems

If a particle can move in just one linear dimension we can get some extra understanding of the motion of the particle. In this case the work done by $\mathbf{F}$ is

$$W(x_1 \to x_2) = \int_{x_1}^{x_2} F_x(x)\, dx$$

All we need for a conservative force in this case is that is only a function of x.

With the potential energy function given by

$$U(x) = -\int_{x_0}^{x} F_x(x')\, dx'$$

a plot of $U(x)$ will show "uphill" and "downhill" parts. The maxima and minima (where $dU/dx = 0$) are places where the force $F_x = -dU/dx$ is zero. The minima are points of *stable* equilibrium and the maxima are points of *unstable* equilibrium.

When we draw a horizontal line on graph at the value of the total energy E then motion can only take place where the values of $U(x)$ are below this line. The points where $U(x) = E$ are points where the kinetic energy and the particle must reverse its motion for these values of x; they are called the *turning points* of the motion.

5.6 Complete Solution for the Motion

If a particle moves in one dimension, acted on by a conservative force it turns out that (in principle) we can always solve for $x(t)$. We only have to be given the (constant) value of the total energy E, then:

$$E = T + U(x) = \tfrac{1}{2}m\dot{x}^2 + U(x) \qquad \Longrightarrow \qquad \dot{x}(t) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

The ambiguity in signs comes from the fact that the energy value E won't tell you which way the particle is initially moving! Choosing $+$, we have

$$dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}$$

and integrating the lhs from 0 to t and the rhs from x_0 to x, we get

$$t = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$$

which —in principle— can be worked out andthen inverted to get x as a afunction of t.

5.7 Central Forces

Though we will have more to say about this special type of force later, for now we note that some forces are of the form

$$\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$$

If the potentially is spherically symmetric then f depends only on r. In that case U depends only on r and we have

$$\mathbf{F}(\mathbf{r}) = -\hat{\mathbf{r}}\frac{\partial U}{\partial r}$$

We will find later that symmetric central forces (like that of gravity) the motion can be expressed as an equivalent sort on one-dimensional motion which will simplify the solution considerably.

5.8 Energy of Multiparticle Systems

We need to pay special attention to multiparticle systems because here we have coordinates for *all* the particles to deal with $(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N)$. The particles will exert forces on one another and also may move under the influence of external forces. The previous work of this chapter only dealt with *one* particle and we need to see precisely what changes are needed in the theory.

First off, while the force between two particles is *some* function of the form $\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2)$, in fact the interparticle forces have to be *translationally invariant* which means that it must be a function of the *difference* of the coordinates:

$$\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2)$$

Then if such force is conservative, we need to have

$$\nabla_1 \times \mathbf{F}_{12} = 0$$

where ∇_1 involves derivatives with respect to the components of $\mathbf{r}_1$.

The important point here is that we can define *one* potential energy function $U(\mathbf{r}_1 - \mathbf{r}_2)$ such that

$$\mathbf{F}_{12} = -\nabla_1 U(\mathbf{r}_1 - \mathbf{r}_2) \qquad \mathbf{F}_{21} = -\nabla_2 U(\mathbf{r}_1 - \mathbf{r}_2)$$

a result which we will generalize to a system of many particles. With this definition, the work–energy theorem is, with $T = T_1 + T_2$, and only our conservative force acting between the particles,

$$d(T + U) = dE = 0$$

In collisions between atomic particles, the kinetic energy is often perfectly conserved because the collision does not cause any internal excitation in the particles; such a collision is *elastic*. Collisions of macroscopic objects can often cause very little change in internal energy so they can be very nearly elastic.

The condition of elasticity gives us another constraint on the final velocities on the colliding particles.

To understand the right way to compute energy we can consider four particles (easily generalizable to "many") which interact amongst themselves via conservative forces, but may also be influenced by external forces which we'll also take to be conservative.

There will still be a principle of energy conservation but how to treat kinetic and (especially) potential energy? The total kinetic energy is

$$T = T_1 + T_2 + T_3 + T_4$$

while for each pair of particles there is a potential energy function U which depends on the difference of the coordinates of the pairs, for example

$$U_{34} = U_{34}(\mathbf{r}_3 - \mathbf{r}_4)$$

and for an external conservative force there is potential energy U for each particle, so that the total potential energy of the system is

$$U = (U_{12} + U_{13} + \cdots + U_{34}) + (U_1^{\text{ext}} + \cdots + U_1^{\text{ext}})$$

and *then* with $E = T + U$ (and no nonconservative forces acting) the total energy is conserved:

$$dE = dT + dU = 0$$

So for N particles the definitions are $T = \sum_\alpha T_\alpha$ and

$$U = U^{\text{int}} + U^{\text{ext}} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta} + \sum_\alpha U_\alpha^{\text{ext}}$$

6 Oscillations

6.1 Hooke's Law and Simple Harmonic Motion

"Hooke's law" is the same given to the linear restoring force of an ideal spring:

$$F_x = -kx$$

which gives a potential energy function of

$$U(x) = \tfrac{1}{2}kx^2$$

The "law" applies to more than springs because near the minimum of *any* one-dimensional potential, $U(x)$ can be approximated as a quadratic function with some appropriate constant k.

From this force we get the equation of motion

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x \tag{3}$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

While we know perfectly well that the solution to 3 is

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t) \tag{4}$$

it is a good idea to think about the solution in terms of complex exponential functions.

There are two ways (conceptually) to approach the solution. We can not that the functions

$$x(t) = e^{i\omega t} \qquad \text{and} \qquad e^{-i\omega t} \tag{5}$$

are solutions to 3, as is any linear combination of them. When we insist that the linear combination be *real* we get 4. That solution can also be written in the form

$$x(t) = A\cos(\omega t - \delta) \tag{6}$$

Another approach is the write down x as a linear combination of the the terms in 5 and then specify that x is the *real part* of the solution. We get the same result, and we will take this point of view when we have to solve the harder problem of a driven oscillator.

With the potential and kinetic energies given by

$$U = \tfrac{1}{2}kx^2 \qquad \text{and} \qquad T = \tfrac{1}{2}m\dot{x}^2$$

substitution of solution 6 gives

$$E = T + U = \tfrac{1}{2}kA^2\cos^2(\omega t - \delta) + \tfrac{1}{2}kA^2\sin^2(\omega t - \delta) = \tfrac{1}{2}kA^2$$

which is constant, as it should be for the conservative force we have here.

6.2 Oscillators in 2D an 3D

A (central) force of the form

$$\mathbf{F} = -k\hat{\mathbf{r}} \qquad \Longrightarrow \qquad F_x = -kx \qquad F_y = -ky$$

gives the equations of motion

$$\ddot{x} = -\omega^2 x \qquad \ddot{y} = -\omega^2 y$$

and so has solution

$$x(t) = A_x\cos(\omega t - \delta_x) \qquad y(t) = A_y\cos(\omega t - \delta_y)$$

A particle's motion in such a potential (mapped out in the xy plane) can be a straight line (if there is no difference in the phases δ) or else an ellipse.

If the potential has different force constants for the x and y motion:

$$F_x = -k_x x \qquad F_y = -k_y y$$

then there are different frequencies for the x and y coordinates:

$$\ddot{x} = -\omega_x^2 x \qquad \ddot{y} = -\omega_y^2 y$$

For this kind of force the particle's trajectory in the xy plane can be quite complicated, resulted in the famous "Lissajous" figures or a space–filling curve.

6.3 Damped Oscillations

Next we consider a one-dimensional oscillator for which there is a linear resistive force (i.e. as studied in Chapter 2) of the form $f = -b\dot{x}$. then the equation of motion is

$$m\ddot{x} = -b\dot{x} - kx \qquad \Longrightarrow \qquad m\ddot{x} + b\dot{x} + kx = 0$$

This equation has the same form as one we might write for an LRC circuit (with no driving voltage, yet!), where $q(t)$, the charge on the capacitor has the same role as $x(t)$.

Defining $2\beta \equiv \frac{b}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$, the equation to be solved is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

For this DE we need any two independent solutions which work, and then the general solution is a linear combination of them. If we try a solution of the form $x(t) = e^{rt}$, we find that it is a solution if r has either of the values

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \qquad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

To get the behavior of $x(t)$, it is most instructive to consider three separate cases:

Undamped, $\beta = 0$

In this case we just have our old solution for the simple harmonic oscillator,

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

Weak Damping, $\beta < \omega_0$.

Also called *underdamping*, things are clearer if we first write

$$\sqrt{\beta^2 - \omega_0^2} = i\omega_1$$

where ω_1 is real. Then the solution is

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

which can also be written in the form

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

Clearly, this is a wiggly trig function which falls off in amplitude due to the decaying exponential out in front. The falloff is governed by β, that is,

$$\text{Decay parameter} \quad = \beta \qquad\qquad \text{[underdamped]}$$

Strong Damping, $\beta > \omega_0$

Also called *overdamping*, the solution is now

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)}$$

which is a sum of two real exponential functions which both decrease with time. Comparing the coefficients in the exponents, we see that the first coefficient is smallest, so the first first term decays *least* slowly, so the long–term motion is governed by the first term. So here we want to define

$$\text{Decay parameter} \quad = \beta - \sqrt{\beta^2 - \omega_0^2} \qquad\qquad \text{[overdamped]}$$

Critical Damping, $\beta = \omega_0$

This singular case must be treated separtely because the two independent solutions are the same solution so the original DE must be solved again. As it so happens, two independent solutions are

$$x(t) = e^{-\beta t} \qquad \text{and} \qquad x(t) = te^{-\beta t}$$

so the general solution is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

so that both terms have the exponential falloff with

$$\text{Decay parameter} \quad = \beta = \omega_0 \qquad\qquad \text{[critical damping]}$$

6.4 Driven, Damped Oscillations

We now include the possibility of a time–dependent external force $F(t)$ that acts on the mass as it oscillates on the spring (still, with linear damping). The total force is now $-bv - ks + F(t)$ so that the equation of motion can be written

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Here there is a corresponding equation for the LRC circuit when there is a driving EMF $\mathcal{E}(t)$.

Define $2\beta = b/m$, $k/m = \omega_0^2$ and $f(t) = F(t)/m$. Then the equation of motion (and the DE we need to solve) is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \tag{7}$$

This equation has linear operators on the lhs, but the rhs is not zero (it is "inhomogeneous"); the latter fact makes things a little harder. For such equations the procedure is to find the *general* solution of the corresponding homogeneous DE (that is, 7 with the rhs replaced by zero) and then find any particular solution to the full DE. The *general* solution to the *full* DE is the sum of the two.

We would like to solve 7 for a sinusoidal driving force,

$$f(t) = f_0 \cos(\omega t)$$

Note that this function is the real part of the function $f_0 e^{i\omega t}$. We will use *that* on the rhs when we make it into a DE for a complex function.

We already have the general solution to the homogeneous equation; to get a particular solution for the sinusoidal driving force, we can try a (complex) solution of the form $z(t) = Ce^{i\omega t}$ (with the intention of taking the real part afterwards) and after a bit of algebra we find that it works, with

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

It is best to have C in the form $C = Ae^{-i\delta}$, and we find

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2} \qquad \text{and} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Of course, we just want the real part of this solution, so with A and δ as given above, the particular solution is

$$x(t) = A\cos(\omega t - \delta) \tag{8}$$

Now get the *general* solution of the driven damped harmonic oscillator by adding the general homogeneous solution to the particular solution in 8. We get:

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

which again contains a little too much to be enlightening; for the case of weak damping ($\beta < \omega_0$) it is

$$x(t) = A\cos(\omega t - \delta) + A_{\rm tr} e^{-\beta t}\cos(\omega_1 t - \delta_{\rm tr})$$

Here the "tr" is short for "transient", since it goes with an oscillation which will die off while the first term is an oscillation which goes on forever. The motion of the oscillator eventually settles down to the *same* oscillatory motion regardless of the initial conditions. This is not always the case; for nonlinear oscillators the behavior can be more complicated.

6.5 Resonance

Now we focus on the long-term motion of the driven damped oscillator. In particular we look at the dependence of the amplitude A, given by

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}$$

and the phase shift δ.

The fact that the amplitude is proportional to the strength of the driving force f_0 is clear. The dependence on the driving frequency ω is a little more complicated, but it is clear that A gets big when ω is close to the natural frequency of the oscillator ω_0.

If we vary ω_0 then clearly the maximum occurs at $\omega_0 = \omega$. If we are varying ω, the maximum occurs at

$$\omega \equiv \omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$$

The Q factor is a measure of the width of the resonance peak.

The full width at half maximum of the curve of A^2 versus ω can be shown to be: FWHM$\approx \beta$. The sharpness of the peak is the ratio of the value of ω to the FWHM. The Q factor is

$$Q = \frac{\omega_0}{2\beta}$$

It can be shown that the Q factor is π times the number of cycles the oscillator makes in one decay time.

The phase of the oscillator at resonance is

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

At low ω, we find δ is small, i.e. the oscillations are in phase with the driving force.

As ω approaches ω_0, δ approaches $\pi/2$ so that at resonance the oscillations are 90° behind the driving force.

At very large ω, δ goes to π and the oscillations are (nearly) totally out of phase with the driving force.

To deal with a driving force which is *not* sinusoidal, we can still use the results from this section if the driving force is periodic. In that case we can use the mathematical machinery of *Fourier analysis* to express the driving force as a sum of sinusoidal driving forces and then put our particular solutions for the sinusoidal cse to work.

Though Taylor goes through this in the book I think I want to go on to the other important topics in classical mechanics. Of which there are lots.

7 The Calculus of Variations

Newton's laws of motion really are complete, but they can be recast in a form which has practical and aesthetic advantages. The new form is (are) known as the Lagrange equations. They are useful for working in general (and sometimes peculiar) coordinate systems such as may occur when a mass's motion is constrained in some way.

The Lagrange equations arise naturally from a study of the calculus of variations and how it is applied to mechanics. This branch of mathematics has important applications in almost every branch of physics. It's important.

7.1 Two Examples

One example is to prove that the shortest path between two points is a straight line. How do we properly state this problem? From basic calculus, if a curve $y(x)$ connects the two given points, the total length of path between points 1 and 2 is

$$L == \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)}\, dx$$

and we want the path (function) $y(x)$ such that L is a minimum.

Another problem is the find that path that light will travel between two points, which is interesting when the index of refraction n (and the speed of light) depends on the coordinates. In one version of the problem we ought to recover Snell's law for refraction.

Since the time of travel over a small distance ds is given by $dt = ds/v$, with $v = c/n$, the time of travel over a full path is

$$(\text{time of travel}) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c} \int_1^2 nds$$

which is minimized when we minimize the integral

$$\int_1^2 n(x,y)ds = \int_{x_1}^{x_2} n(x,y)\sqrt{1 + y'(x)^2}dx$$

The principle that light will follow the path that takes the least time is called Fermat's Principle, but in fact in some cases light will follow the path where the time of travel is locally a *maximum* so that the real rule is the path in one which makes the travel time *stationary*.

7.2 The Euler-Lagrange Equation

The general problem we presently want to solve in one where we have an integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx$$

where the curve $y(x)$ must join the two *given* points (x_1, y_1) and (x_2, y_2) and where we want S to be a minimum or maximum. The subject in which we study these problems is known as the calculus of variations.

One can show that the path which solves the problem satisfies

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

An interesting example solved in the test is the brachistochrone problem, the problem where we find the path on which a particle moving under gravity but without friction will get from a specified (upper) point to a specified (lower) point in the least time. The interesting result is that the path is a cycloid

We can pose the problem of extremum values of integrals where there are two or more functions which depend on a parameter (as will be the usual case in our mechanics problems, where the independent variable is the time t. We find that we get Euler-Lagrange equations for each of the dependent variables.

The material of this section is relevant for mechanics in that if we describe the motion of a particle by the coordinates $q_1(t)$, $q_2(t)$, …, we will solve for these functions of time because some function of these functions (to be called $\mathcal{L}$) will satisfy a stationary principle; that is, the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots q_n, \dot{q}_n, t)\, dt$$

is stationary. This condition will give the equations

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\,, \qquad \frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\,, \quad \cdots\,, \quad \frac{\partial \mathcal{L}}{\partial q_n} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$$

So what is this function $\mathcal{L}$?

8 Lagrange's Equations

8.1 Unconstrained Motion

First we consider the motion in 3 dimensions of a single particle. At first, we describe the motion with Cartesian coordinates.

Recall

$$T = \tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and

$$U = U(\mathbf{r}) = U(x, y, z)$$

Choosing the x coordinate, one can show that

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

and likewise for y and z. Since these equations have the same form as the E–K equations when we are minimizing the integral $S = \int \mathcal{L} dt$ we can state Hamilton's principle,

The actual path which a particle follows between points 1 and 2 in a given time interval t_1 to t_2 is such that the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

is stationary when taken along the actual path. This integral is called the action integral.

and so we have three (!) equivalent ways to get the path of a particle $\mathbf{r}(t)$: (1) Use $\mathbf{F} = m\mathbf{a}$. (2) Determine the path from the Lagrange equations (using Cartesian coordinates). (3) Determine the path from Hamilton's principle.

This demonstrates an alternate way to do mechanics when Cartesian coordinates are used, but the beauty of the way Hamilton's principle is that S is minimized for the right path, regardless of the coordinates in which it is expressed. If instead of (x, y, z) we use the set (q_1, q_2, q_3), and express the Lagrangian in terms of them, then the action integral is

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\, dt$$

and the condition that is is stationary give E–L equations in the q's,

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\,, \qquad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\,, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

When we apply this to motion described with plane polar coordinates (r, ϕ), the r equation gives

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = m\ddot{r}$$

and the ϕ equation gives

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(mr^2\dot{\phi})$$

The latter can be written as the (old) relation between torque and angular momentum:

$$\Gamma = \frac{dL}{dt}$$

The latter result is of course "Newton's second law for rotations" but we see that an equation much like "$F = ma$" has dropped out of the E–L equation for a generalized coordinate ϕ. Such an equation could have dropped out for *any* generalized coordinate q_i, and when we write out

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

we can identify the left side with a "force" associated with the coordinate q_i and on the right side we can associate a momentum for q_i with $\partial\mathcal{L}/\partial\dot{q}_i$; then we will have a force equal to a rate of change of a momentum.

The discussion generalizes to N particles. Then there are $3N$ Cartesian coordinates which can also be expressed as $3N$ generalized coordinates $q_1, \ldots, q_{3N}$, and for each of these we have a Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,, \qquad i = 1, 2, \ldots, 3N$$

8.2 Constrained Systems

As interesting as the Lagrange equations are in having the same form for generalized coordinates, their real advantage is in how they can be used to treat systems where the motion of the particles is constrained.

Examples of constrained motion are the simple pendulum (position of bob needs one coordinate; there is one degree of freedom) the double pendulum (two coordinates needed), a bead sliding on a frictionless wire, a block sliding down a smooth plane, and even a rigid body. In each, an N particle system which could possibly require the use of $3N$ coordinates needs only n coordinates to specify its configuration.

It might be supposed from considering a few simple examples that the number of degrees of freedom is always equal to the number of coordinates needed but in fact that is not the case. If it *is*, we say that the constraints are holonomic, but one can concoct examples where it is not the case; Taylor gives the example of a ball rolling (with no slipping, but also no spinning) on a horizontal table. While there are 2 degrees of freedom for the motion, the configuration of the system needs 5 (generalized) coordinates.

The study of the mathematics of such systems and their constraints is covered extensively in stuffy old English books which no one reads anymore, because we all have better things to study. In the study of such systems, one encounters such impressive biological–sounding words like *scleronomous* and *rheonomous*. It is very tedious.

For our purposes we will *only* consider holonomic systems in our problem–solving. Do the rest on your own time.

We will find that for constrained system (with their reduced number of coordinates) we will *still* get the Lagrange equations for the equations of motion but we should note that at this point in our study we have *not* proven that this is the case. This is because is our derivation using Newton's laws (and the variational principle to go over to generalized coordinates) we considered *all* the forces acting on the particles. But when we deal with constrained systems we want to ignore the forces which confine the particles to the reduced number of dimensions. So more work is needed.

8.3 Lagrange's Equations with Constraints

To accomplish the proof we need, we divide up the forces acting on our particles into those which constrain the particles and all the rest. The rest of the forces need to be conservative so that they are derivable from a potential U. The proof uses the fact that the constraint forces are normal to the path over which the particle(s) move, and finally the (obvious) fact that the generalized coordinates $q_!, q_2 \ldots, q_n$ we wish to use are *consistent* with the constraints. In the end we get an equation which *looks like* what we've written down before, but now we know that we can use it for the n coordinates of our constrained systems:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, 2, \ldots, n]$$

At this point there is no substitute for doing *lots* of problems for constrained systems, finding their equations of motion with the Lagrange equations. Examples are given in your textbook. Read them.

8.4 Generalized Momenta, Ignorable Coordinates

If for the coordinate q_i we define $\partial \mathcal{L}/\partial q_i$ as a generalized force (associated with q_i) F_i, and $\partial \mathcal{L}/\partial \dot{q}_i$ as a generalized momentum, then the Lagrange equation can be written as

$$F_i = \frac{dp_i}{dt}$$

In particular if $\mathcal{L}$ is independent of q_i then F_i is zero and the momentum p_i is constant. In such a case, we say that the coordinate q_i is ignorable or cyclic. Having ignorable coordinates in a problem simplifies the solution.

8.5 Conservation Laws and the Lagrangian

Thinking back on the law of conservation of momentum and energy, since in our previous approach we arrived at those laws from considering *forces* (which are not made explicit in the Lagrangian approach) we might ask how we get those laws from the present (variational, Lagrangian) point of view.

One can show that momentum conservation arises from the *translational invariance* of the Lagrangian, that is, if we can replace $\mathbf{r}_\alpha$ by $\mathbf{r}_\alpha + \boldsymbol{\epsilon}$ for all α. This would happen if we had a system of isolated particles; such a substitution would change the form of an *external* force but not the forces between the particles. In this way, we get a conservation law from a mathematical symmetry of the system.

Likewise, one can show that energy conservation follows if $\mathcal{L}$ has no explicit dependence on the time t. In that case, one can show that the quantity $\sum_i p_i \dot{q}_i - \mathcal{L}$ is constant. This quantity is very important in mechanics and is called the Hamiltonian of the system:

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

and for the case that the Cartesian and generalized coordinates are related by *time–independent* relations,

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \cdots, q_n)$$

then the Hamiltonian is the same as the total energy, $\mathcal{H} = T + U$, and so the total energy is conserved.

8.6 Magnetic Forces

Magnetic forces are not dissipative (they don't do any work) and they are certainly fundamental so it would be a shame if we could not incorporate them into the Lagrangian formalism. But they do depend on the velocity of the particle and that was why we did not consider them.

We *can* handle magnetic forces but we need to expand the definition of the Lagrangian; we will set up a function $\mathcal{L}$ but it won't be simply $T - U$. Nevertheless this function is intended to used in the Lagrange equation to give the equations of motion.

First, a review (?) of magnetic forces. Generally, on a charge q in an EM field, the force is

$$\mathbf{F}_{EM} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and in the general case where the fields can depend on time, the fields follow from the scalar and vector potentials:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

We claim that we will get the right force equation for a particle of mass m and charge q if the use the Lagrangian

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = \tfrac{1}{2} m \dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}) \tag{9}$$

One can note that

$$\tfrac{1}{2} m \dot{\mathbf{r}}^2 + q \dot{\mathbf{r}} \cdot \mathbf{A} \; = \frac{1}{2m}(m\dot{\mathbf{r}} + q\mathbf{A})^2 - \frac{q^2 \mathbf{A}^2}{2m}$$

and the last term on the rhs can be ignored because in the Lagrange equation either partial derivative would give zero. We see that if we define a kind of extended momentum as

$$\text{General momentum} = m\dot{\mathbf{r}} + q\mathbf{A}$$

then the new terms in Eq. 9 come from replacing

$$m\dot{\mathbf{r}} \qquad \Longrightarrow \qquad m\dot{\mathbf{r}} + q\mathbf{A}$$

When going over to quantum mechanics one finds that the momentum operator is replaced by $\frac{\hbar}{i}\nabla$. But now that we know a bit more about momentum, which does it correspond to…the old momentum or our new general momentum. It turns out it is the *new* momentum and so if we have need of an operator for the *old* momentum $m\dot{\mathbf{r}}$ in quantum mechanics we need to correct for it:

$$(m\mathbf{v})_{\mathrm{op}} = \frac{\hbar}{i}\nabla - q\mathbf{A}$$

9 Two–Body Central Force Problem

Now we turn to the case where two masses exert a central force on each other, with no external forces. This includes the case of two masses interacting through gravitation but also the problem where we might model a diatomic molecule with classical motion and forces. Of course, molecules really require quantum mechanics, but the work that we do in working out the general problem will be very useful in the QM solution.

The basic problem is that the particles need 6 coordinate to describe the motion (they have locations $\mathbf{r}_1$ and $\mathbf{r}_2$). The only forces are $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, with a potential energy $U(\mathbf{r}_1, \mathbf{r}_2)$. Examples are the gravitational force and the Coulomb force:

$$U_{\rm grav} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad \text{and} \qquad U_{\rm Coul} = \frac{kq_1q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

We will consider only central forces, in which the potential energy depends only on $|\mathbf{r}_1 - \mathbf{r}_2|$:

$$U(\mathbf{r}_1 - \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)$$

and for simplicity we will denote the relative position as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

And so $U = U(r)$. The Lagrangian is thus

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(r)$$

9.1 CM and Relative Coordinates; The Reduced Mass

We begin the task of reducing the number of coordinates that we really need to worry about!

The relative position $\mathbf{r}$ can be used as one of the variables of the problem, and the other can be taken as the CM position:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

but as we saw before, with no external force on the system, $\mathbf{P} = M\dot{\mathbf{R}}$ (with $M = m_1 + m_2$) is constant, so $\dot{\mathbf{R}}$ is constant so later we can choose an inertial reference frame where the CM is at rest. However, in the current reference frame, one can show that

$$T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2$$

where μ is an important combination of the two masses called the reduced mass:

$$\mu = \frac{m_1m_2}{m_1 + m_2} = \frac{m_1m_2}{M}$$

and so the Lagrangian has a very simple form,

$$\mathcal{L} = T - U = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \left(\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right) = \mathcal{L}_{\rm cm} + \mathcal{L}_{\rm int}$$

From the fact that the Lagrangian splits into to pieces, we can solve for the CM motion ($\mathbf{R}$) and the relative motion ($\mathbf{r}$) separately.

Using the Lagrange equation for components of the the CM coordinate $\mathbf{R}$ gives $M\ddot{\mathbf{R}} = 0$ or $\dot{\mathbf{R}}$=constant. But since this is an isolated system, we've known that for a long time! The Lagrange equation for the relative coordinate $\mathbf{r}$ gives

$$\mu\ddot{\mathbf{r}} = -\nabla U(r)$$

which is equivalent to a force problem for a *single* particle of mass μ moving in a potential $U(r)$.

It is at this point that we jump into the CM reference frame (it is also inertial, so we can do that) so that the CM is now at rest. Then the Lagrangian is simply

$$\mathcal{L} = \mathcal{L}_{\rm rel} = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

and with it we have reduced a two–body problem to (effectively) a one–body problem. But we are solving for $\mathbf{r}$, a vector which points from m_2 to m_1 and the mass of the fictional particle is μ.

If one of the masses is very large compared to the other, say $m_2 \gg m_1$, then the CM is very close to particle 2 and the vector $\mathbf{r}$ is nearly the same as $\mathbf{r}_1$. Then we very nearly have m_1 orbiting a fixed center of force.

One can show that the total angular momentum of the system is

$$\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}$$

which is *conserved* which implies that $\mathbf{r}$ and $\dot{\mathbf{r}}$ remain in a fixed plane. So we only need to consider motion in that fixed plane and thus we have reduced the problem to *two* dimension.

With the vector $\mathbf{r}$ now specified by the plane polar coordinates r and ϕ, our previous Lagrangian becomes:

$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

which is independent of ϕ. Thus ϕ is "ignorable" and its corresponding momentum is constant:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = \text{const} = \ell$$

This is of course expresses the conservation of *angular momentum*.

The Lagrange equation for r gives

$$\mu r\dot{\phi}^2 - \frac{dU}{dr} = \mu\ddot{r}$$

which has the familiar *form* of $\mathbf{F} = m\mathbf{a}$

9.2 Equivalent One–Dimensional Problem

We exploit the idea that we can replace $\dot{\phi}$ with $\frac{\ell}{\mu r^2}$. Doing that, we can write the r equation as

$$\mu\ddot{r} = -\frac{dU}{dr} + \mu r\dot{\phi}^2 = -\frac{dU}{dr} + F_{\rm cf}$$

where $F_{\rm cf}$, which will be treated as a fictitious outward force, is

$$F_{\rm cf} = \mu r\dot{\phi}^2 = \frac{\ell^2}{\mu r^3}$$

Yes, you are at last old enough now learn about the word *centrifugal* and about an "outward force" for circular motion. It is only something that *acts like* a force to make simple sense of the radial equation. (And it does point outward.)

The centrifugal force can be expressed as a centrifugal *potential*:

$$F_{\rm cf} = -\frac{dU_{\rm cf}}{dr} \qquad \text{where} \quad U_{\rm cf}(r) = \frac{\ell^2}{2\mu r^2}$$

and so the *genuine* and centrifugal potentials combine to give an *effective* potential:

$$\mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{\rm cf}(r)] = -\frac{d}{dr}U_{\rm eff} \qquad \text{where} \quad U_{\rm eff} = U(r) + \frac{\ell^2}{2\mu r^2}$$

In the context of $U_{\rm eff}$ (and the equivalent one-dimensional problem), the condition of energy conservation is

$$\tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) = \text{const}$$

as one might expect.

When we plot $U_{\rm eff}(r)$ versus r for the gravitational potential $U(r) = -Gm_1m_2/r$ we get something very illuminating. The curve goes to ∞ at small r. As r increases, the potential goes negative, reaches a minimum and then approaches zero asymptotically (from the negative side). If we draw a horizontal line with a y–intercept at the value of the *total* energy of the system the line will cross the $U_{\rm eff}$ curve at the *turning points* of the motion.

A circular orbit stays at the bottom of the well at the lowest point and a single value of r. Comets usually have very elongated orbits so that the turning points are widely separated.

If $E < 0$ the motion has two turning points ($r_{\rm min}$ and $r_{\rm max}$) and the motion is bounded. For values of the total energy $E > 0$ there is only one turning point (namely $r_{\rm min}$) and the motion is unbounded.

9.3 The Equation (and Shape) of the Orbit

Go back to the radial equation and express it in terms of the actual radial force $F(r)$:

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

This contains *time* derivatives of r but our purpose here is to find r as a function of the angle, $r(\phi)$. To get to this we use a couple clever tricks, one of which is to work with the variable

$$u = \frac{1}{r} \qquad \Longrightarrow \qquad r = \frac{1}{u}$$

then to write the operator d/dt in terms of $d/d\phi$. One can show:

$$\frac{d}{dt} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

After some algebraic work, we find a DE for $u(\phi)$:

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F$$

which is good for any radial force F.

9.4 Gravity; The Kepler Orbits

Now specialize to the gravitational attraction,

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2 \qquad \text{where} \quad \gamma = Gm_1m_2$$

The solutions is now fairly easy! The DE for $u(\phi)$ is

$$u''(\phi) = -u(\phi) + \gamma\mu/\ell^2$$

which after a little more work can be shown to have the solution

$$u(\phi) = \frac{\gamma\mu}{\ell^2}(1 + \epsilon\cos\phi)$$

Since we really want $r(\phi)$, this can be written as

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \qquad \text{where} \quad c = \frac{\ell^2}{\gamma\mu}$$

For bounded orbits the previous equation gives us the perihelion and aphelion distances:

$$r_{\min} = \frac{c}{1+\epsilon} \qquad \text{and} \qquad r_{\max} = \frac{c}{1-\epsilon}$$

While to many of us it still isn't clear what shape is represented here, one can show that in the Cartesian coordinates the relation has the form

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

(that is, it is an ellipse) where

$$a = \frac{c}{1-\epsilon^2}\ , \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}\ , \qquad d = a\epsilon$$

which also give us

$$\frac{b}{a} = \sqrt{1-\epsilon^2}$$

which more clearly shows the importance of the parameter ϵ: It is the eccentricity of the ellipse.

One can show, using the result for "areal velocity" found earlier in the book, that the period and semi-major axis a are related by

$$\tau^2 = 4\pi^2\frac{a^3\mu}{\gamma} \approx \frac{4\pi^2}{GM}a^3$$

where the last approximation holds if one of the masses (M) is much bigger than the other one; such is the case with the Sun and the planets, and in that case the relation is known as "Kepler's Third Law".

The case $\epsilon = 1$ ($E = 0$) can be shown to give a parabolic trajectory. The case $\epsilon > 1$ (or $E > 0$) can be shown to give a hyperbolic path.

10 Mechanics in Noninertial Frames

Having accepted that we must be in an inertial frame to properly apply $\mathbf{F} = m\mathbf{a}$, we now deal with non-inertial frames if for no other reason than the fact that we live in one due to the rotation of the earth. We want to find how the non-inertialness (???) affects the use of Newton's 2nd law.

10.1 Accelerated Frame (No Rotation)

We first look at a relatively simple case, that where one frame (to be called $\mathcal{S}$) is accelerating uniformly with respect to an inertial frame, which we'll call $\mathcal{S}_0$ at some acceleration $\mathbf{A}$. One finds that if $\mathbf{F}$ is the total *true* force on a mass m, then in frame $\mathcal{S}$ we have

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

so that we *can* use Newton's 2nd law if we add an extra term, the "inertial force", given by

$$\mathbf{F}_{\text{inertial}} = -m\mathbf{A}$$

So that lack of suitability of the frame is accounted for by the addition of a fictitious force (called in some texts a *pseudoforce*). We will adopt the same strategy for dealing with the more complicated case of a rotating reference frame. One should note that if you are in an accelerating frame, the phony force will seem very real to you.

The use of an accelerating frame with a fictitious force can be mroe enlightening than a solution using an inertial frame as Taylor shows with the example of a suspended mass in an accelerating railroad car.

See Taylor's discussion of the tides. The tides are caused by the uneven pull of the moon on the earth, but if one thinks about it too simplistically, one (incorrectly) comes up with one bulge of the water surface on the side of the moon.

Using the idea of an inertial force in the accelerating frame of the earth one can deduce an *effective* force that pulls on the near side and pushes on the far side, giving a double bulge to the water layer. One can also deduce the shape of the equipotential surface (incorporating the inertial force!) which has a height difference of 54 cm between the high and low parts, for the tide due to the moon.

10.2 Angular Velocity

We will consider coordinate systems attached to a rigid body (again, called $\mathcal{S}$), where the origin coincides with the origin of an inertial frame (called $\mathcal{S}_0$). The coordinates of $\mathcal{S}$ rotate (possibly in a very exotic way) with respect to the coordinates of $\mathcal{S}_0$.

One can show, using a geometrical theorem known as "Euler's theorem" that the most general motion of the rigid body and the attached coordinates (relative to a fixed point O) is a rotation about some axis through O. It follows that the rate of rotation is given by an angular velocity *vector*, specified by a unit vector $\mathbf{u}$ and rotation rate ω:

$$\boldsymbol{\omega} = \omega\mathbf{u}$$

and the sign of ω is given by the right-hand rule.

One can show that if $\mathbf{r}$ is the position of some point fixed on the rotating rigid body, then its velocity is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

We note that $\mathbf{v}$ is the rate of change of the vector $\mathbf{r}$ in the inertial frame (of course $\mathbf{r}$ does not change in the frame $\mathcal{S}$). And here we arrive at the important idea that even though a vector is a single mathematical entity, it has different rates of change depending on the reference frame.

There is a more general relation for *any* vector which is fixed in the rotating body. In particular, if $\mathbf{e}$ is a unit vector fixed in the body, then

$$\frac{d\mathbf{e}}{dt} = \boldsymbol{\omega} \times \mathbf{e}$$

One can show that if we have set of rotating frames, the relative angular velocities *add*, just as their relative velocities do (in non-relativistic mechanics!). We have:

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}$$

if frame 3 rotates with respect to frame 1 with angular velocity $\boldsymbol{\omega}_{31}$, etc.

10.3 Time Derivatives in a Rotating Frame

As Newton's 2nd law involves force vectors and time derivatives, the fact that observers in two frames can disagree on the time derivatives of vectors will be central to understanding what an observer in a rotating frame will make of the 2nd law. Here we have a rotating frame $\mathcal{S}$ which rotates with angular velocity $\boldsymbol{\Omega}$ with respect to the inertial frame $\mathcal{S}_0$.

A very special angular velocity which we'll use in applying the results of the chapter is that of earth in its rotation:

$$\Omega \approx 7.3 \times 10^{-5}\,\frac{\mathrm{rad}}{\mathrm{s}}$$

The Very Important Theorem of this section relates

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \text{(rate of change of vector } \mathbf{Q} \text{ relative to inertial frame } \mathcal{S}_0)$$

and

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} = \text{(rate of change of vector } \mathbf{Q} \text{ relative to rotating frame } \mathcal{S})$$

for any vector $\mathbf{Q}$.

One can show:

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}$$

One can use this theorem (twice, in fact) and Newton's 2nd law to calculate the second time derivative of the position $\mathbf{r}$ in the rotating frame. From now one the dot notation will stand for the time derivative *in the rotating frame* (since that's where we're interested in doing the calculations) and with that, we get:

$$m\ddot{\mathbf{r}} = \mathbf{F} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$$

which is an odd version of Newton's 2nd law, as it should be because it's not being used in an inertial frame.

The extra terms here are the Coriolis force:

$$\mathbf{F}_{\mathrm{cor}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}$$

and the centrifugal force:

$$\mathbf{F}_{\mathrm{cf}} = m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}$$

so that if we want to use physics in a rotating frame we have to add a couple terms to the *real* forces:

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}}$$

10.4 Centrifugal Force

If an object has a small velocity in the rotating frame (or if the time of observation is small) we can ignore the Coriolis force and focus on the centrifugal. We can estimate that if the speed of the object is much less than 1000 mi/h, this is safe.

We can show that on the earth's surface the centrifugal force is

$$\mathbf{F}_{\text{cf}} = m\Omega^2\rho\hat{\boldsymbol{\rho}}$$

where $\hat{\boldsymbol{\rho}}$ is the unit vector pointing outward *from the axis*.

Applying this to problems in free fall we write

$$m\ddot{\mathbf{r}} = \mathbf{F}_{\text{grav}} + \mathbf{F}_{\text{cf}} = m\mathbf{g}_0 + m\Omega^2 R\sin\theta\hat{\boldsymbol{\rho}}$$

which gives us a new effective value of the gravitational acceleration for free-fall,

$$\mathbf{g} = \mathbf{g}_0 + \Omega^2 R\sin\theta\hat{\boldsymbol{\rho}}$$

This has radial and tangential components

$$g_{\text{rad}} = g_0 - \Omega^2 R\sin^2\theta \qquad g_{\text{tang}} = \Omega^2 R\sin\theta\cos\theta$$

giving a maximum angle between **g** and the radial direction of about $0.1°$, which would seem easy to measure, but it is not, because we would have to make comparisons with a plumb line, but the plumb line points along **g**! We basically have to give up and call **g** the new "vertical" direction.

10.5 Coriolis Force

This is

$$\mathbf{F}_{\text{cor}} = 2m\mathbf{v} \times \boldsymbol{\Omega}$$

where **v** is the object's velocity in the rotating frame. For an object with speed, say $50\,\frac{\text{m}}{\text{s}}$ (a fast baseball) the maximum acceleration due to the Coriolis force is about $0.007\,\frac{\text{m}}{\text{s}^2}$, which is small compared to g, but measurable. For objects with speeds much larger than this it is more important and if it is allowed to act *for a long period of time*, as with a Foucault pendulum it can also have a large.

One can show that for objects moving in the Northern Hemisphere, the Coriolis force always deflects them to the right and in the Southern Hemisphere is deflects them to the left. A cyclone arises from this effect when air rushed into a low-pressure region and is simultaneously deflected, causing a circulating swirl.

If we let **g** be the acceleration in our rotating frame due to the actual force of gravity and the small centrifugal force, the equation of motion for a falling object is

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \boldsymbol{\Omega}$$

Choosing the axes so that the origin is on the earth's surface with z up, x East and y North one gets the equations

$$\dot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \qquad \ddot{y} = -2\Omega\dot{x}\cos\theta \qquad \ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

For an object dropped from a height h, we can get an approximate solution by first approximating

$$x = 0 \qquad y = 0 \qquad z = h - \tfrac{1}{2}gt^2$$

and then putting this result back into the equations. We find

$$x = \tfrac{1}{3}\Omega g t^3 \sin\theta$$

and that if an object is dropped by 100 m at the equator, it is deflected by 2.2 cm, a small but measurable amount.

10.6 The Foucault Pendulum

Possibly the most impressive application of the Coriolis effect, the Foucault pendulum can be seen in many fine science museums. In it, a heavy mass is suspended from a light but very long wire and set to swing freely (in a plane) for a very long time. The plane of the oscillations changes with time and the rate of change depends on the latitude of the pendulum's location.

We solve the problem by solving for the x and y motion of the bob. For small oscillations won can approximate the string tension as $T \approx mg$, and wth this we can find the components of the tension force as

$$T_x = -mgx/L \qquad T_y = -mgy/L$$

and with the Coriolis force also acting on the bob, we get the x and y equations of motion,

$$\begin{aligned} \ddot{x} &= -gx/L + 2\dot{y}\Omega\cos\theta \\ \ddot{y} &= -gy/L - 2\dot{x}\Omega\cos\theta \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 &= 0 \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 &= 0 \end{aligned}$$

where $\omega_0^2 = g/L$ and $\Omega_z = \Omega\cos\theta$.

We can solve these using a trick from back in Chapter 2: Use the complex number

$$\eta = x\,+\,iy$$

and then the coupled equations can be written as *one* equation,

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2 = 0$$

Trying a solution of the form $\eta(t) = e^{-i\alpha t}$ gives the general solution

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

Some examination of this solution shows that that plane in which the pendulum oscillates rotates at an angular rate given by Ω_z. At a latitude of 42° ($\theta = 48°$), the rate of rotation is $\frac{2}{3}\Omega$, or about 240°/day. If the pendulum can swing for 6 hours without significant damping then the effect is easily observable.

11 Rotational Motion of Rigid Bodies

A rigid body can be viewed as a collection of N particles whose motion is restricted such that the shape of the object cannot change. This means that the distance between any two masses is fixed. Whereas the N particles would normally require $3N$ coordinates to describe its motion, a rigid body needs only 6 coordinates: 3 to give the location of its center of mass and 3 mroe to give its orientation.

11.1 The Center of Mass

Recall that the position of the center of mass of a set of N particles is

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \qquad \text{where} \qquad M = \sum_{\alpha=1}^{N} m_\alpha$$

and we use the integral form for continuous masses. Recall our previous theorems,

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = M\dot{\mathbf{R}} \qquad \text{and} \qquad \mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}$$

We also want to investigate the role of the center of mass in the total angular momentum of the system. We define the position of the mass m_α relative the center of mass, $\mathbf{r}'_\alpha$ by

$$\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

.

Now, the total angular momentum of the system of particles relative to the origin O is

$$\mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \mathbf{r}_\alpha$$

Substituting for $\mathbf{r}_\alpha$ to get things in terms of coordinates relative to the center of mass, one can show

$$\mathbf{L} \;=\; \mathbf{R} \times \mathbf{P} + \sum_\alpha \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha \;=\; \mathbf{L}(\text{motion of CM}) + \mathbf{L}(\text{motion relative to CM})\ ,$$

that is, the total angular momentum can be broken down into the angular momentum of the center of mass plus the angular momentum relative to the center of mass.

One can apply this fact to the motion of the planets around the sun. For them we would break down the total angular momentum as

$$\mathbf{L} = \mathbf{L}_{\text{orbital}} + \mathbf{L}_{\text{spin}}$$

where

$$\dot{\mathbf{L}}_{\text{orbital}} = \mathbf{R} \times \mathbf{F}^{\text{ext}}$$

from which one can show

$$\dot{\mathbf{L}}_{\text{spin}} = \boldsymbol{\Gamma}^{\text{ext}}(\text{about CM})$$

so if the external torque *about* the CM is zero the spin angular momentum is separately conserved. For the earth there is a small torque about the CM which results in the precession of the equinoxes.

One can show a similar sort of separation for the kinetic energy. The result is

$$T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha'^2 \; = T(\text{motion of CM}) + T(\text{motion relative to CM})$$

For a rigid body, the second term is the kinetic energy for *rotation*.

It's noteworthy that the past formula applies for any point fixed within the body, not just the CM (with the $\mathbf{r}'$'s being relative to that point).

11.2 Rotation About a Fixed Axis

By way of starting the present (grown-up) discussion of rotating objects, we consider an arbitrary rigid body rotating about the z axis, e.g. a block of wood spinning on a fixed rod that had stuck through it!

Here the angular velocity (of the object and any mass point) is $\boldsymbol{\omega} = (0, 0, \omega)$. When we evaluate the total angular momentum $\mathbf{L}$ we find something that may surprise us if we take our elementary physics too seriously. The vector $\mathbf{L}$ doesn't just have a z component. In fact,

$$L_z = I_{zz}\omega \qquad L_x = I_{xz}\omega \qquad L_y = I_{yz}\omega$$

where

$$I_{zz} = \sum m_\alpha(x_\alpha^2 + y_\alpha^2) \qquad I_{xz} = -\sum m_\alpha x_\alpha z_\alpha \qquad I_{yz} = -\sum m_\alpha y_\alpha z_\alpha$$

Thus the z component of $\mathbf{L}$ is the old business from Phys 2110 but the x and y components are not and they involve something *like* the old moment of inertia, but where we have a product of different coordinates (like xy).

We need to solve the general situation of rotation about an arbitrary axis, that is, the case where the angular velocity of the rigid body is

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$$

We need to have a form for the general relation between $\boldsymbol{\omega}$ (which is a description of the *kinematics* of the rigid body) and the total angular momentum $\mathbf{L}$ (which is perhaps the most important *physical* quantity for the body).

One can show that

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

where

$$I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2)$$

with similar definitions for I_{yy} and I_{zz}, and

$$I_{xy} = -\sum m_\alpha x_\alpha y_\alpha$$

with similar definitions for I_{yz} and so on. We then have

$$L_i = \sum_{j=1}^{3} I_{ij}\omega_j$$

which, using the notation

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

can be written

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

Here, $\mathbf{I}$ is the moment of inertia tensor, which says a bit more than just noting that it is a matrix (which it is). $\mathbf{L}$ and $\boldsymbol{\omega}$ are of course the angular momentum and angular velocity vectors.

The matrix $\mathbf{I}$ is symmetric, meaning that it is equal to its tranpose:

$$I_{ij} = I_{ji} \qquad \text{or} \qquad \mathbf{I} = \tilde{\mathbf{I}}$$

Calculating the elements of $\mathbf{I}$ gives some practice with some funky integrals in three dimensions. Often we find that some of elements of $\mathbf{I}$ due to symmetries of the rigid body with respect to the chosen axes.

11.3 Principal Axes of Inertia

In general $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$, but one can come up with examples (often for objects with lots of symmetry) where it is. A principal axis of an object is an axis such that if $\boldsymbol{\omega}$ points along it then $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$:

$$\mathbf{L} = \lambda\boldsymbol{\omega}$$

Then λ is the (simple) moment of inertia about the axis in question. If there are three mutually perpendicular principal axes then when $\mathbf{I}$ is evaluated for these axes, it is diagonal:

$$\mathbf{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and then λ_1, λ_2, and λ_3 are the principal moments.

In fact, *any* rigid body rotating around a point has three principal axes.

It can be shown that the kinetic energy of a rotating body is

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$$

and if the axes we use are principal axes, this simplifies to

$$T = \tfrac{1}{2}(\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3)$$

For a rotation about a principal axis we have

$$\mathbf{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega} \qquad \Longrightarrow \qquad (\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$$

This equation is true if $\boldsymbol{\omega} = 0$, but we don't want that! For nonzero $\boldsymbol{\omega}$, it must be true that

$$\det(\mathbf{I} - \lambda\mathbf{1}) = 0$$

an equation often called the characteristic (or secular) equation for the matrix $\mathbf{I}$. Generally it gives a cubic equation for the *number* λ; there are generally 3 solutions, λ_1, λ_2, λ_3, which are then the principal moments. Eigenvalues for matrices can be easily found on a computer if you have to resort to that.

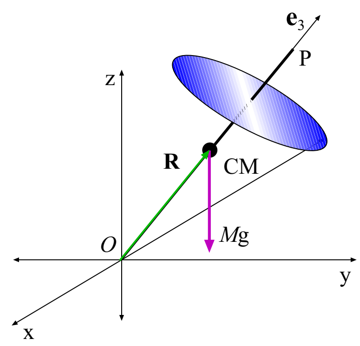

Figure 1: Spinning top is made from rod OP fastened through the center of a uniform disc; it is freely pivoted at O.

11.4 Precession of a Top due to Weak Torque

We will have to develop some dynamical equations for the motion of rigid bodies but already we have enough to do an important example.

A symmetric top has an axis of symmetry and as a result, two of the principal moments are the same. With the z axis along the axis of symmetry, we have

$$\mathbf{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

The top is set spinning so that its angular velocity is $\boldsymbol{\omega} = \omega \mathbf{e}_3$. Then its initial angular momentum is

$$\mathbf{L} = \lambda_3 \boldsymbol{\omega} = \lambda_3 \omega \mathbf{e}_3$$

and it would keep this angular momentum if there were no torque. The vector $\mathbf{e}_3$ points at an angle θ for the inertial frame's z axis.

Now introduce gravity, which acts as shown in Fig. 1. Gravity causes a torque $\boldsymbol{\Gamma} = \mathbf{R} \times M\mathbf{g}$, which has magnitude $RMg \sin\theta$ and is perpendicular to both the z axis and the axis of the top ($\mathbf{e}_3$).

The torque will give a change in $\mathbf{L}$so that the components of the angular velocity ω_1 and ω_2 will be non-zero, but if the effect of the torque is small enough, we can expect ω_1 and ω_2 to remain small so that $\mathbf{L} = \lambda_3 \omega \mathbf{e}_3$ continues to hold approximately. Then the equation $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$ gives

$$\lambda_2 \omega \dot{\mathbf{e}}_3 = \mathbf{R} \times M\mathbf{g}$$

and with $\mathbf{R} = R\mathbf{e}_3$ and $\mathbf{g} = -g\hat{\mathbf{z}}$ we can show

$$\dot{\mathbf{e}}_3 = \boldsymbol{\Omega} \times \mathbf{e}_3$$

where

$$\boldsymbol{\Omega} = \frac{MgR}{\lambda_3 \omega} \hat{\mathbf{z}}$$

and this says that the axis of the top, $\mathbf{e}_3$, rotates with angular velocity $\boldsymbol{\Omega}$ about the $\hat{\mathbf{z}}$ direction. We have all observed this effect in a spinning ans precessing top.

11.5 Euler's Equations

Now we set up the equations of motion for a rigid body; we assume that the body has a fixed point (i.e. fixed in the inertial frame), or else that we are analyzing the motion in the CM frame of the rigid body so that we take the fixed point to be the CM.

We have to understand that things are a little complicated... the principal axes of the body are fixed (so the angular momentum is simple) in the body frame, but these axes rotate with respect to the space (inertial) frame.

In the space frame, as we have seen, it is true that

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} = \mathbf{\Gamma}$$

but from the relation between rates of change of vectors in the space and body frames, we have

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} \equiv \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}$$

where as before, the dot means a time derivative in the rotating frame.

This is called Euler's equation. When resolved into components, it gives

$$\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned}$$

which give the motion of a spinning body *as seen in a frame fixed in the body*. They can be difficult to use because the components of the torque here are those in the body frame, and thus they are (unknown!) functions of time.

For the case of the symmetric top, with $\lambda_1 = \lambda_2$, and torque caused by gravity for which Γ_3 is zero, we get

$$\lambda_3\dot{\omega}_3 = 0$$

which we had assumed in order to analyze the precession of the spinning top.

11.6 Euler's Equations for Zero Torque

In the zero-torque case, Euler's equations have the simple form

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 \qquad \lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1 \qquad \lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$$

We discuss these first for the case where all three principal moments are different.

11.7 The Euler Angles

The Euler angles are three angles which specify the orientation of a rigid body with respect to an inertial (or "lab") reference frame.

For the case $\lambda_1 = \lambda_2$ one can express the angular velocity in terms of these angles and axes as

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\mathbf{e}'_1 + \dot{\theta}\mathbf{e}'_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3$$

The angular momentum is

$$\mathbf{L} = (-\lambda_1\dot{\phi}\sin\theta)\mathbf{e}'_1 + \lambda_1\dot{\theta}\mathbf{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3$$

The kinetic energy is

$$T = \tfrac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$