Phys 3820, Fall 2010
Exam #3

1. Give concise but *careful* definitions of:

a) Impact parameter (for a classical collision).

The impact parameter (usually denoted by b) is the distance from the axis at which the projectile approaches the target particle (the axis being the line which is parallel to the projectile's motion and passes through the target).

In the classical case one can in principle find a relation between b and the scattering angle θ. You can't do that in quantum mechanics.

2. Suppose you have a mono-energetic beam of particles hitting a stationary target (with very massive particles) and a particle detector of certain specific dimensions. Summarize the steps you would need (the basic measurements and basic calculations) to go through to find the *measured* differential cross section $D(\theta) = \frac{d\sigma}{d\Omega}$ at each angle.

Measure the number of particles per area per time which hit the target material; call this the Incoming Flux, which has units of $\frac{1}{\mathrm{s}\cdot\ \mathrm{m}^2}$.

Then at some angle measure the number of particle which enter your (small) detector per time; divide this by the solid angle subtended by the detector; call this the outgoing flux, which has units of $\frac{1}{\mathrm{s}\cdot\ \mathrm{sterrad}}$.

The ratio of the outgoing flux to the incoming flux has units of $\frac{\mathrm{m}^2}{\mathrm{sterrad}}$ and is denoted by $D(\theta)$ or more commonly by $\frac{d\sigma}{d\Omega}$.

3. *Show me the numbers*: Find the differential cross-section $D(\theta) = \frac{d\sigma}{d\Omega}$ for the scattering of protons of energy 5.0 MeV from a stationary point-mass target of charge $+20e$.

Make rough sketch of what the plot looks like with all axes *properly labeled*. *Show all your work*

Using 1 barn $= 10^{-28}$ m^2 and 1 fm $= 10^{-15}$ m may make things simpler in the labeling.

The formula for Coulomb scattering differential cross section is

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right]^2$$

where we have

$$e_1 = e \qquad e_2 = 20e \qquad E = 5.0\ \mathrm{MeV}$$

Putting all of this into formula and showing all of our work, we get

$$D(\theta) = \left[\frac{20(1.602\times10^{-19}\ \mathrm{C})^2}{5.0(1.602\times10^{-13}\ \mathrm{J}\cdot 16\pi(8.854\times10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2})}\right]^2 \frac{1}{\sin^4(\theta/2)}$$

Inside the square brackets the units of C^2 cancels and since $1\ \mathrm{N}\cdot\mathrm{m} = 1\ \mathrm{J}$ the units inside give m which when squared gives m^2. Punching buttons on the claculator we get

$$D(\theta) = \frac{2.07\times 10^{-30}\ \mathrm{m}^2}{\sin^4(\theta/2)} = \frac{2.07\ \mathrm{fm}^2}{\sin^4(\theta/2)}$$

As θ ranges from 0 to π this function goes from ∞ to a minimum of $2.07\ \mathrm{fm}^2 = 207\ \mathrm{barn}$.

4. What property of the Coulomb interaction makes it poorly behaved for the purposes of doing a quantum scattering calculation?

The Coulomb potential falls off as $1/r$ which is not faster than the so-called centrifugal potential and so the procedure we followed where we could get far enough from the scatterer that asymptotic solutions were the spherical Bessel functions is not valid.

In fact the basic form of our time-independent scattering wave function is not valid with the Coulomb potential and so the scattering amplitudes and phase shifts need a radical overhaul of their definitions to deal with this case.

We could have gone over a proper quantum treatment of the Coulomb potential in class but wisely voted against it.

5. Describe how a computer program calculates $f(\theta)$ for a given potential $V(r)$, that is, how it finds the "exact" solution.

I will begin your answer for you:

For each l we find the the partial–wave radial function $u_l(r)$ by choosing $u_l(0) = 0$ and $u_l(dx)$ equal to some arbitrary small value. Use the Schrödinger equation to integrate to large r...

...then at large values of x, beyond the range of the short-ranged potential, the radial solution must have the form

$$R_l(r) = \mathrm{Const}\cdot\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right]$$

so that by considering a couple values of the radial solution and its derivative at large r one can get the coefficients in this formula, and thus find the partial-wave scattering amplitude a_l and the phase shift δ_l and thereby construct the full scattering amplitude $f(\theta)$ using the standard formulae.

6. Find the scattering amplitude in Born approximation for a central potential of the form

$$V(r) = \frac{V_0}{\alpha^2 + r^2}$$

As usual, do the math as far as you can– though you may be able to come up with a closed form for this one.

Griffiths gives a version of the Born approximation for a spherically symmetric potential:

$$f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0 \infty rV(r)\sin(\kappa r)\,dr$$

where $\kappa = |\mathbf{k}' - \mathbf{k}|$, the magnitude of the momentum transfer, also given by

$$\kappa = 2k \sin(\theta/2)$$

(This form is not hard to derived from the more general expression given below.) Plugging in we get:

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2 \kappa} \int_0^\infty \frac{r \sin(\kappa r)}{r^2 + \alpha^2}\, dr \;\; = -\frac{2mV_0}{\hbar^2 \kappa}\, \frac{\pi}{2}\, e^{-\kappa\alpha}$$

where we get the last expression from looking up the integral in a suitable book! (Gradsteyn and Rhyzik 3.723.3) (I'm guessing that one can show this pretty easily with a contour integral; maybe a good example for my next semester of mathematical physics!) So we get:

$$f(\theta) = -\frac{\pi m V_0}{\hbar^2 \kappa} e^{-\kappa\alpha}$$

7. When we wanted to solve the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = \rho(\mathbf{r})$$

in three dimensions we found that the proper Green function to use was

$$G(\mathbf{r}_2 - \mathbf{r}_1) = \frac{e^{ik|\mathbf{r}_2 - \mathbf{r}_1|}}{4\pi|\mathbf{r}_2 - \mathbf{r}_1|}$$

This would not be true for a two-dimensional problem though. If we have the *two*–dimensional Helmholtz equation

$$(\nabla^2 + k^2)\psi(\boldsymbol{\rho}) = f(\boldsymbol{\rho})$$

with a two–dimensional "source" $\rho(\boldsymbol{\rho})$ then the appropriate Green function is something called a Hankel function (whose details you don't need to worry about),

$$\frac{i}{4} H_0^{(1)}(\boldsymbol{\rho})$$

a) What differential equation is satisfied by this function?

It must satisfy the differential equation in two dimensions,

$$(\nabla^2 + k^2)G(\boldsymbol{\rho}) = (\nabla^2 + k^2)H_0^{1)}(\boldsymbol{\rho}) = \delta^{(2)}(\boldsymbol{\rho})$$

b) If I assure you that this Green has the right boundary conditions, write down the solution for $\psi(\boldsymbol{\rho})$ for a given source function $f(\boldsymbol{\rho})$.

Multiply the Green function (as a function of $\boldsymbol{\rho} - \boldsymbol{\rho}'$) by the source function and integrate over the entire $\boldsymbol{\rho}'$ space:

$$\psi(\boldsymbol{\rho}) = \int H_0^{1)}(\boldsymbol{\rho} - \boldsymbol{\rho}')\, f(\boldsymbol{\rho}')\, d^2\boldsymbol{\rho}'$$

8. Summarize how we get the "Born series" from the integral form of the Schrödinger equation

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, d^3\mathbf{r}_0$$

We do an iterative procedure based on the premise that the true wave function $\psi(\mathbf{r})$ has a small difference from the (incident) plane wave $\psi_0(\mathbf{r})$. If we make that substitution in the integral on the rhs we get the first Born approximation. If we take $that$ approximate solution and make the same substitution we get the second Born approximation, and so on.

It is easy to see a pattern in the series of terms we get as we make these substitutions. But the successive terms, with multiple integrals to compute are harder to evaluate.

9. If we are to have meaningful theory of relativistic quantum mechanics what features of the non-relativistic theory do we want to keep?

The theory must be able to produce a probability density and probability current density from a wave function to make contact with experiment. We want to keep the same basic form of the operators corresponding to $\mathbf{r}$, $\mathbf{p}$ and E and the mathematics of a Hilbert space of quantum states. But the wave equation itself will be different from the familiar Schrödinger equation.

In addition a relativistic must come with a procedure for transforming the operators and solution for a different inertial frame, such that it has the same mathematical form in the new frame.

10. Write down the relativistic Schrödinger equation. Explain how it generalizes the non-relativistic version.

The non-relativistic Schr. Eqn substitutes operators in the classical energy relation

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

with $\mathbf{p} \to \frac{\hbar}{i}\nabla$ and $E \to i\hbar\frac{\partial}{\partial t}$.

If we start from the relativistic equation for a free particle,

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4$$

then the substitutions give the wave equation

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left[-\hbar^2 c^2 \nabla^2 + m^2 c^4\right]\psi$$

And we have to leave the equation as it stands since there is no clear way to take square roots of operators.

If we have an EM potential in the problem then we make the substitutions

$$E \to E - q\phi \qquad \text{and} \qquad \mathbf{p} \to \mathbf{p} - \frac{q}{c}\mathbf{A}$$

11. How did Dirac's Hamiltonian (and equation) differ from the relativistic Schrödinger equation?

The Dirac Hamiltonian is linear in both the space and time derivatives. To bring this about the Hamiltonian is a combination of the usual derivative and multiplicative operators and $matrices$. It can only have meaning if the wave function is a multi-component object.

12. In the "low–energy" representation used is class, what is the physical meaning of the components of the Dirac wave function ψ?

The four components correspond to the four combinations of the two physical properties of particle/antiparticle and spin-up/spin-down.

13. Compare the radial part of the wave function in the Dirac equation (and deducing the energy eigenvalues) with what is done in the non-relativistic theory.

For the low-energy representation used in class the wave function divides into two upper and two lower components, both eigenfunctions of the same $total$ angular momentum operator but with different eigenvalues of L^2. Thus there is one radial function for the upper two components and another for the lower two components.

The Dirac equation (after being put into a form with radial operators) gives differential equations for the two radial functions; we solve for them simultaneously in a fashion which similar to that of the non-relativistic theory. The algebra is somewhat more comlicated but in the end the energy eigenvalues are deduced from the condition that the entire wave function must die off at infinity.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2} a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\, dx = -\int_a^b \frac{df}{dx}\, g\, dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\, dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^\infty |\Psi(x,t)|^2\, dx = 1 \qquad \langle x\rangle = \int_{-\infty}^\infty x|\Psi(x,t)|^2\, dx \qquad \langle p\rangle = \int_{-\infty}^\infty \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\, dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^\infty c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^\infty \Psi_n(x,t)$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^\infty e^{-ipx/\hbar}\Psi(x,t)\, dx$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}\text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529\times 10^{-10}\ \text{m} \qquad E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2}e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-r/2a} \qquad R_{21}(r)\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3}\left(\frac{c^2}{4\pi\epsilon_0}\right)^2 = 1.097\times 10^7\ \text{m}^{-1}$$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m\pm 1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0\hbar)/2 \qquad E_- = +(\gamma B_0\hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788\times 10^{-5}\text{ eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle\mathbf{S}_p\cdot\mathbf{S}_e\rangle = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \leq \langle\psi|H|\psi\rangle \equiv \langle H\rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v}e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_at/\hbar} + c_b(t)\psi_b e^{-iE_bt/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \qquad \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{-i\omega_0 t}c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar}$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

$$\mathbf{p} \equiv q\langle\psi_b|\psi_a|\rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\mathbf{p}|E_0}{\hbar}\right)^2\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\mathbf{p}|^2\rho(\omega_0) \qquad A = \frac{\omega^3|\mathbf{p}|^2}{3\pi\epsilon_0\hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1; \text{or } 0$ and $\Delta l = \pm 1$

$$d\sigma = D(\theta)\,d\Omega \qquad D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad \sigma = \int D(\theta)\,d\Omega \qquad D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega}$$

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} \qquad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \qquad D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$D(\theta) = \left[\frac{q_1q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right]^2 \qquad -\frac{\hbar^2}{2m}\frac{d^2u_l}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u_l = Eu_l$$

$$\text{Large } r\text{:} \quad \frac{d^2u_l}{dr^2} - \frac{l(l+1)}{r^2}u_l = -k^2u_l \qquad u_l = Arj_l(kr) + Brn_l(kr)$$

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_lh_l^{(1)}(kr)\right]P_l(\cos\theta)$$

$$a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k}e^{i\delta_l}\sin(\delta_l) \qquad f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)\,P_l(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r}) \qquad \Longrightarrow \qquad G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}$$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0 \qquad f(\theta) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(\mathbf{r}')\,d^3\mathbf{r}'$$

$$E^2 = c^2\mathbf{p}^2 + m^2c^4 \qquad \boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2$$