Preparing for College Physics

David Murdock
TTU

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Preface!

To the Instructors:

This booklet is free.

You may download, copy and distribute it as you wish...if you find it to be of any value. It can be gotten from the URL:

`http://iweb.tntech.edu/murdock/books/PreSci.pdf`

so you will need the (equally free) Adobe Acrobat Reader to view it and print it. This piece of software is on many institutional computer systems, and if you don't have it on your machine, go get it at

`http://w1000.mv.us.adobe.com/products/acrobat/readstep2.html`

The pages are in a format which looks best when printed double-sided.

I'm giving it away to anyone in the same situation as me: You have many students in your introductory science courses who don't have adequate preparation in basic mathematics, and you want to give them something simple and friendly to read. Preferably something that gets right to the point and which costs no more than the paper it's printed on. I didn't know where I could get a document like this, so I wrote one.

You'll notice that "significant figures" have not been rigidly observed in the numerical examples. That's because this book is directed at students who need help in getting *any* correct numbers to round off.

If you find this booklet to be useful or else worth exactly what it costs and/or have any suggestions, please write to me at

`murdock@tntech.edu`

To the Students:

Your college science courses may very well require you to do some mathematics (algebra, trig) and some work with a scientific calculator. You may not have been warned about this when you wrote that check for your tuition, but it's too late now! *Bwaahh-hah-Haah!*.

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In this booklet I've tried to pick out the bits of your math courses that you will really need to get through your first courses in physics and chemistry. In addition, I give some directions on how to use an inexpensive scientific calculator.

If your college science courses seem scary because of all the math and numbers, the examples in this book may be of help. If this booklet does help you —or even if it doesn't— I'd like to hear from you.

David Murdock
Dept. of Physics, TTU
`murdock@tntech.edu`

Chapter 1

Numbers!
(Scientific Notation)

1.1 Introduction

In science we must often deal with numbers which can be very small or large compared to the entries in the bank statements of physics professors. We commonly deal with facts like:

$$\text{Mass of the Earth} \approx 5976000000000000000000000 \, \text{kg}$$

$$\text{Speed of light} = 299792500 \, \tfrac{\text{m}}{\text{s}}$$

$$\text{Mass of hydrogen atom} \approx 0.00000000000000000000000167339 \, \text{g}$$

which contain numbers much larger and smaller than those encountered in daily life or for that matter in many math courses. Science is all about making measurements and finding *quantitative* facts about the natural world, so we need a convenient way to express *any* number, large or small.

1.2 Powers of Ten

Our system of numbers is based on powers of ten. Raising 10 to the nth power means multiplying n factors of 10:

$$
\begin{aligned}
10^1 &= 10 \\
10^2 &= 10 \cdot 10 = 100 \\
10^3 &= 10 \cdot 10 \cdot 10 = 1000 \\
10^4 &= 10 \cdot 10 \cdot 10 \cdot 10 = 10,000 \\
&\cdots
\end{aligned}
$$

So 10 to the power of n (when n is positive) is a 1 followed by n zeros For consistency we also write:

$$10^0 = 1$$

When we raise 10 to a *negative* power (say, $-n$) we mean that we are dividing 1 by n factors of 10, as in

$$
\begin{aligned}
10^{-1} &= \frac{1}{10} = 0.1 \\
10^{-2} &= \frac{1}{10 \cdot 10} = 0.01 \\
10^{-3} &= \frac{1}{10 \cdot 10 \cdot 10} = 0.001 \\
10^{-4} &= \frac{1}{10 \cdot 10 \cdot 10 \cdot 10} = 0.0001 \\
&\dots
\end{aligned}
$$

The relation between positive and negative powers of 10 is:

$$\boxed{\boxed{10^{-n} = \frac{1}{10^n}}} \tag{1.1}$$

Another way to look at powers of 10 is to begin with 1.0; to get 10^n (with n positive), move the decimal point n places to the *right*; to get 10^{-n}, move the decimal point n places to the *left*.

Keep in mind that 10 raised to a negative power is still a *positive number*. For negative numbers, we see a minus sign out in front, as in

$$-100 = -10^2 \qquad \text{and} \qquad -0.01 = -10^{-2}$$

---

1. Multiply 10^5 and 10^2 and express the result as a power of 10.

The two factors are

$$10^5 = 100000 \qquad \text{and} \qquad 10^2 = 100 \ .$$

Now, multiplying any number by 100 just gives it two more zeroes on the end, so

$$10^5 \cdot 10^2 = (100000)(100) = 10000000 = 10^7$$

---

2. Divide 10^{-1} by 10^3 and express the result as a power of 10.

Since
$$10^{-1} = 0.1 \qquad \text{and} \qquad 10^3 = 1000$$
and dividing a number by 1000 moves the decimal point three places to the left, we have:

$$\frac{10^{-1}}{10^3} = \frac{0.1}{1000} = 0.0001 = 10^{-4}$$

The last two examples show how to multiply and divide powers of 10. When we multiply powers of 10 we *add* the exponents and when we divide powers of 10 we *subtract* the exponents:

$$\boxed{10^m \cdot 10^n = 10^{m+n}} \qquad \boxed{\frac{10^m}{10^n} = 10^{m-n}} \tag{1.2}$$

---

3. Divide 10^4 by 10^{-3}.

Using the rule in Eq. 1.2 we get:

$$\frac{10^4}{10^{-3}} = 10^{(4-(-3))} = 10^{4+3} = 10^7$$

1.3   Scientific Notation

Now we use the powers of ten to write large and small numbers in a compact way. The following example shows how.

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4. Write the number $206{,}000$ as 2.06 times a power of ten.

When we start with $206{,}000$ and end with 2.06 we have moved the decimal point to the left *five times* and so we make up for that with *five* factors of 10. So we have:

$$206{,}000 = 2.06 \times 10^5$$

In the last example we have taken a number written in standard form and written it as a bite–sized number (between 1 and 10 multiplied by a power of ten. This way of writing numbers is called **scientific notation**.

Here are some more examples of putting numbers into (and out of) this form:

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5. Express the number 0.00000701 **in scientific notation.**

In going from the given number to the bite–sized (yum!) number 7.01, we move the decimal point to the right 6 times, so that we need to *take away* 6 factors of 10 to make up for that. The answer is

$$0.00000701 = 7.01 \times 10^{-6}$$

---

6. Express the number 5.67×10^9 *without* **a power of** $10.$

For each power of 10 we move the decimal place to right. Filling in the zeroes (and commas, for clarity) that we need, we get:

$$5.67 \times 10^9 = 5,670,000,000$$

1.4 Scientific Arithmetic *Without* a Calculator

In the next chapter we will see how scientific calculators deal with scientific notation, but in this section we will do some simple calculations with numbers in scientific form for which you *don't need a calculator*.

To do these problems, multiply the "regular" numbers the way you always would and combine the powers of 10 using the rules in Eq. 1.2.

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7. Multiply 3.0×10^5 **and** 2.0×10^{-2}

In this problem we evaluate

$$(3.0 \times 10^5) \cdot (2.0 \times 10^{-2})$$

We don't need a calculator to multiply 3.0 and 2.0. (It gives 6.0!) Multiplying 10^5 and 10^{-2} gives 10^3. So the answer is:

$$(3.0 \times 10^5) \cdot (2.0 \times 10^{-2}) = 6.0 \times 10^3$$

---

8. Divide 8.0×10^2 **by** $2.0 \times 10^{17}.$

Again, combine the number parts and the power–of–ten parts to get:

$$\frac{8.0 \times 10^2}{2.0 \times 10^{17}} = \frac{8.0}{2.0} \cdot \frac{10^2}{10^{17}} = 4.0 \times 10^{(2-17)} = 4.0 \times 10^{-15}$$

---

9. Divide 3.0×10^2 **by** 6.0×10^{-6}**.**

Combine the "number" parts and the power–of–ten parts:

$$\frac{(3.0 \times 10^2)}{(6.0 \times 10^{-6})} = \frac{(3.0)}{(6.0)} \times 10^{(2-(-6))} = 0.50 \times 10^8$$

This answer is correct, but we can express it in scientific notation as:

$$0.50 \times 10^8 = 5.0 \times 10^7$$

Chapter 2

Calculators!

(Using a Scientific Calculator)

2.1 Introduction

A **scientific calculator** can do all the basic operations of arithmetic but in addition can "understand" and give answers which are in scientific notation and do the fancy things one learns about in math courses.

Scientific calculators don't have to be expensive. For $10 you can buy a calculator which will do everything you need in most college science courses. You can recognize them on the racks at your local discount store from the fact that they have buttons marked `sin`, `cos` and `tan`.

Now, I wish I could give you instructions that will allow you to use any scientific calculator. Alas, I cannot. Since there are many different models on the market, this is nearly impossible and the best I can do is to show you how they work in general and hope that you will look at your own scientific calculator to see how relevant my instructions are.

To make things definite I will focus on three particular calculator models which at the time of this writing (October 11, 2000) can be bought for around $10 in a discount store (Target, Wal-Mart...) or an office superstore (Staples, Office Max...). These are the TI-25X, the Casio fx–260 and the HP 6S Solar[1]. All of them are solar powered so there are never any batteries to buy and all are quite adequate for introductory science classes. The Casio and HP have more functions built into them, but the TI is smaller and simpler. The TI and Casio come with a hard plastic case to protect them inside a backpack full of heavy science textbooks.

Other calculators made by TI and Casio are similar to these.

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[1]HP also sells a 6S (no Solar) model which has the same keys but runs only on battery power. Except for its snazzy blue color I don't see any advantage to this model. Look for the 6S Solar.

2.2   Entering Numbers in Scientific Notation

First you need to locate the **exponentiation** button on the calculator. On the TI it is in the third row and is labelled `EE`. On the Casio and the HP it is labelled `EXP`. But both do the same thing, which is to tell the calculator that the next number you enter will be the *power of* 10 that goes with the digits you have already typed in. (There is probably a marking of 10^x somewhere on the calculator keys. Ignore this! We *don't* use that key, at least not yet.)

In this book I will go with the Casio and HP marking for the key and use `EXP` in my examples.

To enter a number like 6.02×10^{23} in your calculator, first type in `6.02` then hit the `EXP` key. Then type in `23`. So this sequence is:

| 6	.	0	2	EXP	2

3
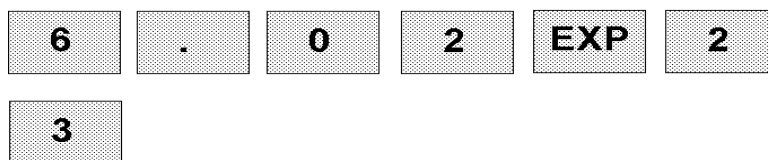
What you see on the screen after pushing the buttons should be something like:

$$6.02^{23}$$

Here, the 23 is displayed off to the side and above, which we understand to mean that it is the power of ten. If your calculator should ever display such a number as this, you should know that *on paper* you would write

$$6.02 \times 10^{23}$$

To enter a number with a negative power of ten you'll need to locate the key labelled with the symbol $^+/_-$. Let's enter the number 1.38×10^{-23}. To get the minus in the exponent, hit the $^+/_-$ key just after entering in the 23 in the `EXP` part. You should press:

1	.	3	8	EXP	2

3	+/-
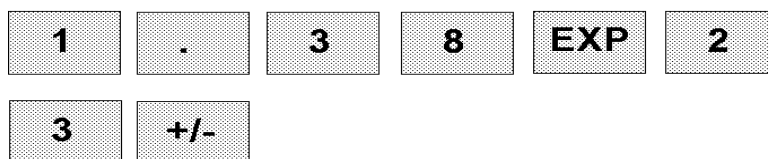
Finally, for some problems we might need to key in a *negative* number, for example the number -2.73×10^2. To do this, Type in 2.73, then press $^+/_-$; then enter `EXP` `2`. The key sequence is:

2	.	7	3	+/-	EXP

2
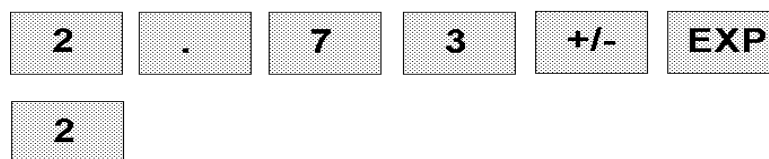
Compare this to the way we enter 9.16×10^{-54}. We press:

9	.	1	6	EXP	5

4	+ / -
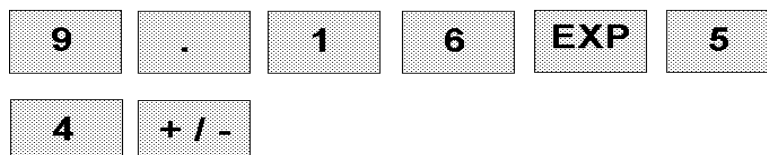
The pattern here is to hit the $^{+}/_{-}$ key *after* the number to which we want to attach the minus sign.

2.2.1 Examples

1. What is the key sequence needed to enter the number 1.0×10^{-4}?

To enter this number we can press:

1	EXP	4	+/-

Note: The HP 6S has the nice feature in that you don't need to type the 1 *in this sequence; if you begin typing a scientific number with* EXP, *the HP assumes you want a* 1 *in front, but the Casio and TI don't do this. (The TI leaves a zero in front; the Casio will give the number π.) Be careful.*

2. What is the key sequence needed to enter the number -7.228×10^{-20}?

To enter this number we press:

7	.	2	2	8	+/-

EXP	2	0	+/-
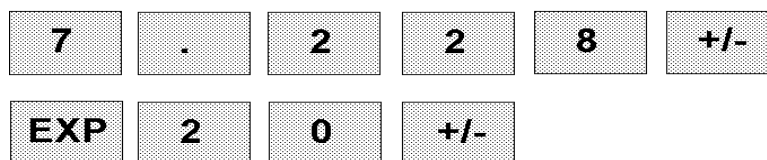
Electronic calculators are limited in the scientific numbers which they can display; for the $10 models, the power of ten factor can't be bigger than 10^{99} (which *is* pretty damn big) and can't be smaller then 10^{-99} (which is pretty damn small). For most work in beginning physics these limits don't cause a problem. But keep them in mind!

2.3   Arithmetic With Scientific Numbers

Once we know how to put a *scientific* number into a calculator, doing arithmetic is the same as on any simple calculator. To add you use a + between the numbers you are summing and an = afterwards. Simple!

2.3.1   Examples

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3. What are the keystrokes needed to multiply the two numbers 2.3×10^4 and 1.508×10^{11}? What is the result?

We do the multiplication with the keystrokes:

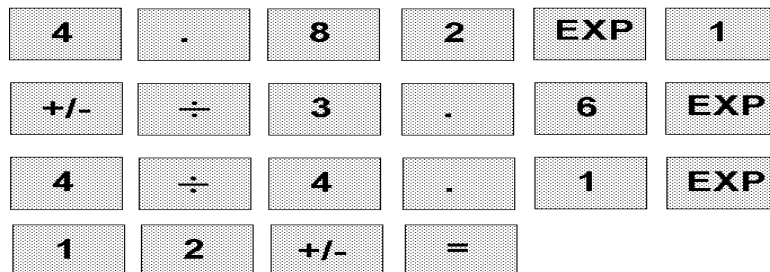|					
2	**.**	**3**	**EXP**	**4**	**X**
1	**.**	**5**	**0**	**8**	**EXP**
1	**1**	**=**			
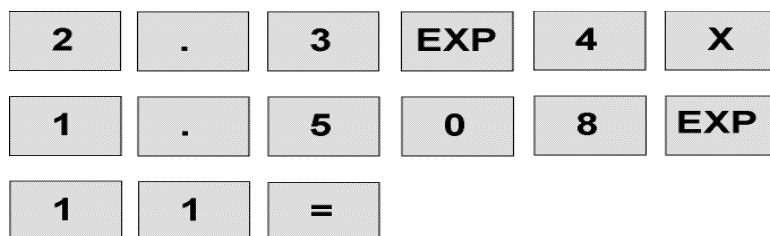
The result of the multiplication is

$$3.468 \times 10^{15} \ .$$

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4. Using your calculator, find:

$$\frac{(4.82 \times 10^{-1})}{(3.6 \times 10^4)(4.1 \times 10^{-12})}$$

The keystrokes needed are:

4	**.**	**8**	**2**	**EXP**	**1**
+/−	**÷**	**3**	**.**	**6**	**EXP**
4	**÷**	**4**	**.**	**1**	**EXP**
1	**2**	**+/−**	**=**		

[Don't make the mistake of pushing "×" between the numbers 3.6×10^4 and 4.1×10^{-12}. You press ÷ since both numbers go in the *denominator* of the whole expression.]
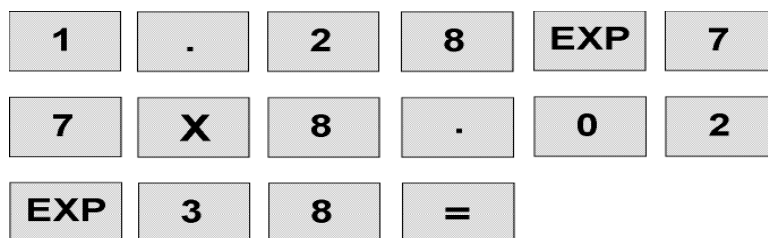
After performing these steps, the calculator gives the answer:

$$3.27 \times 10^6$$

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5. Using your \$9.95 (+ tax) calculator, multiply:

$$(1.28 \times 10^{77}) \cdot (8.02 \times 10^{38}) \ .$$

We key in:

1	.	2	8	EXP	7
7	X	8	.	0	2
EXP	3	8	=		

But the calculator gives an *error message*. (An "E" shows up somewhere on the screen.) The answer to this problem is larger than 10^{99} and that's too damn big!

Chapter 3

Algebra!

(Simple Equations, Powers and Roots)

3.1 Introduction

I'm thinking of a number...if I double this number and add 1 I get 61. What number was I thinking of?

Can you solve this puzzle? If you can, then you were probably doing **algebra** whether you knew it or not.

When you evaluate the expression

$$[2 \cdot 30] + 1 \qquad \text{(Dot means multiplication.)}$$

then you are doing *arithmetic*. But when you are solving the puzzle (equation):

$$2x + 1 = 61 \tag{3.1}$$

you are doing *algebra*. We do algebra when the world hands us *number puzzles* to solve.

To solve my number puzzle in an *organized* way, first subtract 1 from both sides of Eq. 3.1:

$$2x + 1 - 1 = 61 - 1 \qquad \Rightarrow \qquad 2x = 60$$

Now divide both sides of the last equation by 2:

$$\frac{2x}{2} = \frac{60}{2} \qquad \Rightarrow \qquad x = 30$$

Our steps have now isolated x and have given us the answer: x is 30.

I hope that was what you guessed.

3.2 Addition and Multiplication

In the example in the last section, we found it helpful to subtract the same number from both sides of the equation. (That is, it was helpful in isolating the unknown x.) We can give a general rule for this idea: If we are told that $a + b = c$ and we want to isolate a, then it will be helpful to subtract b from both sides. Then we will have

$$a + b - b = c - b \qquad \Rightarrow \qquad a = c - b$$

so then:

$$\text{If} \quad a + b = c \quad \text{then} \quad a = c - b \tag{3.2}$$

In words, you can say to yourself "We bring b to the other side as a minus".

In our example we also found it helpful to divide both sides of the equation by the same number. We can give a general rule for this as well: If we are told that $ab = c$ and we want to isolate a, it is helpful to divide both sides by b. Then we will have:

$$\frac{ab}{b} = \frac{c}{b} \qquad \Rightarrow \qquad a = \frac{c}{b}$$

so then:

$$\text{If} \quad ab = c \quad \text{then} \quad a = \frac{c}{b} \tag{3.3}$$

In words, "We put b on the other side in the denominator." Actually we need to be a little careful in performing this step because division by *zero* is not permitted in math, so one needs to be sure that b is not zero.

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1. Given:

$$12x + 7 = 121 \ ,$$

solve for x.

First, move the 7 to the other side of the equation as a -7:

$$12x = 121 - 7 \qquad \Longrightarrow \qquad 12x = 114$$

Now divide out the factor of 12:

$$x = \frac{114}{12} = 9.5$$

The solution is $x = 9.5$.

---

2. Given:

$$6.4t + 41.0 = 7.72 \ ,$$

solve for t.

Put the 41 on the other side as a negative:

$$6.4t = 7.72 - 41.0 \qquad \Longrightarrow \qquad 6.4t = -33.28$$

Then divide out the factor of 6.4:

$$t = \frac{(-33.28)}{6.4} = -5.2$$

The solution is $t = -5.2$.

---

3. Given:
$$8.2y - 10.0 = 13.6y \ ,$$

solve for y.

Here we need to combine the terms which have a y in them. Subtract $8.2y$ from both sides and get:

$$\begin{aligned} -10.0 &= 13.6y - 8.2y \\ &= (13.6 - 8.2)y \\ &= 5.4y \end{aligned}$$

so now we have:

$$5.4y = -10.0$$

Divide both sides by 5.4 and get:

$$y = \frac{(-10.0)}{(5.4)} = -1.85$$

So the solution is $y = -1.85$.
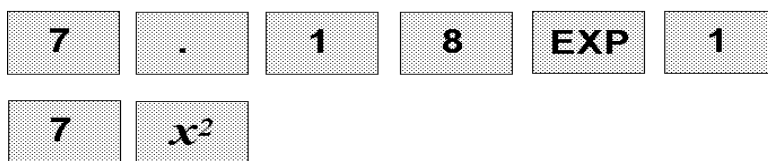
3.3 Squares!

When we **square** a number it just means that we are multiplying it by itself. We denote this operation with a small "2" written next to the number being squared:

$$3^2 = 3 \cdot 3 = 9 \qquad 19^2 = 19 \cdot 19 = 361 \qquad (91.6)^2 = (91.6) \cdot (91.6) = 8.391 \times 10^3$$

This operation is so common in science that there is a special button on your \$10 calculator that will square the number that is currently being displayed. The square of x is written x^2, so the key on your calculator is marked with the symbol x^2.

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4. What is the sequence of keys to press on your calculator which will find the square of the number 7.18×10^{17} ? What is the result?

The sequence of keys is

| 7 | . | 1 | 8 | EXP | 1 |

| 7 | x^2 |

The calculator tells me:

$$(7.18 \times 10^{17})^2 = 5.155 \times 10^{35}$$

3.4 Square Roots!

Here's another number puzzle: I thinking of a number... When I square this number, I get 9. What is it?

You might blurt out "3". That's almost right. But it's also true that the number -3 squared gives 9. So in fact this puzzle has *two* possible answers, 3 and -3.

So this (short) puzzle and its solution go as:

$$x^2 = 9 \qquad \Longrightarrow \qquad x = \pm 3$$

Some puzzles don't have such simple answers: "I'm thinking of a number which when squared, gives 7. What's the number?"

Now, $2^2 = 4$ and $3^2 = 9$, so the answer to this puzzle must lie somewhere between 2 and 3. Your calculator can give you the answer (or rather, a good approximation for it) if you use the operation labeled by the symbol $\sqrt{}$. On most calculators this symbol is written above the key marked x^2, and the $\sqrt{}$ operation is gotten by using the `Shift` key. (On the HP this key is called `INV`. On the TI it is called `2nd`. They all serve the same function, and in all of my examples I will refer to it as "`Shift`".)

This operation is called "taking the **square root**".

Key in 7 on your calculator and then do the $\sqrt{}$ operation; that is, press the keys

| 7 | **Shift** | x^2 |

You will get something like:

$$\sqrt{7} = 2.645751311 \ .$$

Actually the digits will go on forever and they won't repeat, but that's more of an issue for your math course. For our purposes that answer is *pretty well* given by keeping a few decimal places, and with this understanding we can write

$$\sqrt{7} = 2.646$$

even though your math teacher wouldn't like it. For a science class, it's good enough!

But as before, the calculator has only given the *positive* answer to the puzzle $x^2 = 7$. The complete answer is really:

$$x = +2.646 \quad \text{or} \quad x = -2.646 \ .$$

---

5. Given the equation: $x^2 + 1 = 23$, solve for x.

To isolate x, put the "1" on the other side as a -1:

$$x^2 = 23 - 1 = 22 \qquad \Longrightarrow \qquad x^2 = 22$$

This equation has $\sqrt{22}$ as a solution; another solution is $-\sqrt{22}$:

$$x = \pm\sqrt{22}$$

To find a numerical value for $\sqrt{22}$ key 22 into your calculator and then push Shift—x^2. Then we find:

$$x = \pm 4.6904$$

---

6. Given the equation

$$(t - 3)^2 = 25 \ ,$$

solve for t.

First, we want to "undo" the square on the left side of the equation. We realize that $5^2 = (-5)^2 = 25$, so the equation is true is $(t - 3)$ is equal to 5 but it is also true if $(t - 3)$ is equal to -5:

$$t - 3 = 5 \qquad \text{or} \qquad t - 3 = -5$$

Now for each of these cases bring the -3 to the other side as $+3$ and find:

$$t = 5 + 3 = 8 \qquad \text{or} \qquad t = -5 + 3 = -2$$

The *two possible* answers to the equation are $t = 8$ and $t = -2$.

In the last couple examples we have used the rule for simplifying an equation:

$$\text{If} \quad a^2 = b^2 \quad \text{then} \quad a = \pm b \ . \tag{3.4}$$

where the "$\pm$" symbol is read "plus or minus" and conveys the idea that we have the *two* possibilities for a solution: $a = b$ or $a = -b$.

The square root of x can also be written as $x^{\frac{1}{2}}$. This makes sense because when we multiply powers of the same number (x) we *add* the exponents. So then

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\left(\frac{1}{2}+\frac{1}{2}\right)} = x^1 = x$$

so from this $x^{\frac{1}{2}}$, when multiplied by itself, gives x. So it must be the same as the square root, $\sqrt{x}$.

3.5   Cubes and Cube Roots!

3.5.1   Cubes

When we **cube** a number, we are taking the product of *three* factors of the number:

$$3^3 = 3 \cdot 3 \cdot 3 = 27 \qquad 8^2 = 8 \cdot 8 \cdot 8 = 512 \qquad (91.6)^3 = (91.6) \cdot (91.6) \cdot (91.6) = 7.686 \times 10^5$$

This operation comes up quite often too, for example when we find the volume of a cube or sphere. And some calculators *do* have a single button which will cube any number that you put in, but the \$10 models I have written about do not. That's OK... learning what to do will help us when we take higher powers.

Your calculator will have a key on which is written y^x, or possibly x^y, as on the Casio. They all work the same way (and I will refer to this key as y^x).

First key in the number you want to cube. Then hit the y^x key. Then hit **3**, then **=**. On the screen will be the cube of the number.

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7. What is the sequence of keys that will find the cube of 7.22×10^{-3} ?

Press the sequence

7	.	2	2	EXP	3

+/-	y^x	3	=
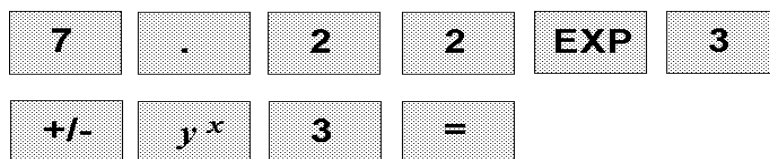
The calculator tells me:

$$(7.22 \times 10^{-3})^3 = 3.764 \times 10^{-7}$$

3.5.2   Cube Roots

Just as we did with the *square* of a number we can turn the question around for cubes: We can ask, for example, if $x^3 = 8$, then what is x? This one is fairly easy; the answer is 2. (*Only* 2 and not -2, because $(-2)^3 = -8$.)

But suppose we ask:
$$\text{If} \quad x^3 = 8.62 \quad \text{then what is} \quad x?$$

The answer to this question is the **cube root** of 8.62. It is written as

$$\sqrt[3]{(8.62)}, \qquad \text{or as} \qquad (8.62)^{1/3}$$

and it can be found easily with a scientific calculator.

Some calculators have a special key which will take a cube root but we will use a keystroke which is available on all modest calculators. On your calculator, look for a key (or shifted-key) on which is written $\sqrt[x]{y}$ (TI and HP) or $x^{1/y}$ (Casio). Regardless of what it's called, on these calculators you get the "root" function by using the Shift key along with the y^x key. To find the cube root of z first key in the number z, press the root key, then key in 3 and then hit =.

---

8. Give the keystrokes needed to find $\sqrt[3]{(8.62)}$ and give the answer.

On your calculator press (something like!) the following sequence:

| 8 | . | 6 | 2 | Shift | y^x |

| 3 | = |
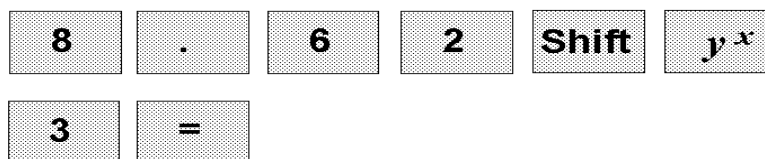
The calculator tells us::
$$\sqrt[3]{(8.62)} = 2.0504$$

There are two ways to write the cube root of x. One way is $\sqrt[3]{x}$; the other is $x^{\frac{1}{3}}$. When we multiply three factors of $x^{\frac{1}{3}}$ and use the rule of adding exponents, we get:

$$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)} = x^1 = x$$

so that $x^{\frac{1}{3}}$ must be the number which when cubed gives x; it means the same thing as the cube root of x.

---

9. If $s^3 = 4.53 \times 10^{13}$, find s.

The cube of s is 4.53×10^{13}, so s itself is the *cube root* of that number, which we can write as:
$$s = \sqrt[3]{(4.53 \times 10^{13})} \qquad \text{or} \qquad s = (4.53 \times 10^{13})^{\frac{1}{3}}$$

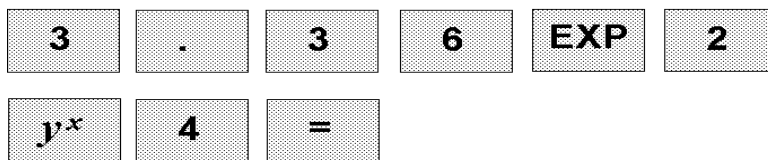Punching the right buttons on our calculators gives us:

$$s = 3.56 \times 10^4$$

3.6   More Powers and Roots

We don't often see powers larger than 3 (cubic) in beginning science courses, but when we do, we deal with them in much the same way as the cube and the cube root. Some examples:

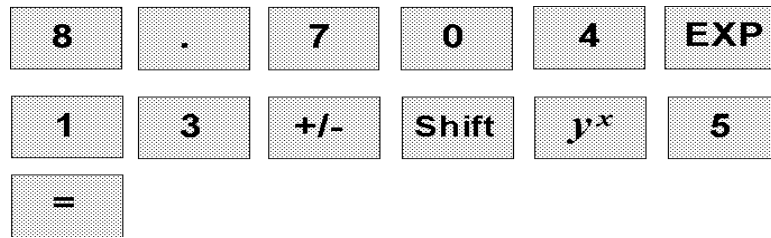**10. What keys would one press on a \$10 scientific calculator to find $(3.36 \times 10^2)^4$?**

Press:

$$\boxed{3} \quad \boxed{.} \quad \boxed{3} \quad \boxed{6} \quad \boxed{\text{EXP}} \quad \boxed{2}$$

$$\boxed{y^x} \quad \boxed{4} \quad \boxed{=}$$

We get
$$1.27 \times 10^{10} \ .$$

---

11. If $s^5 = 8.704 \times 10^{-13}$ then what is s?

If s^5 has the given value, then s must be the 5th root of this number, or equivalently, the number taken to the $\frac{1}{5}$ power.
Press the keys:

8	.	7	0	4	EXP

1	3	+/-	Shift	y^x	5

=

and find:

$$s = 3.872 \times 10^{-3} \ .$$

Chapter 4

Units!
(Metric Units, Converting, etc.)

4.1 The SI (Metric) System of Units

Science and engineering are based on *measurement*. We compare our lengths, time intervals, weights, etc. against certain **standard** lengths, times intervals and weights; we express our measurements in terms of special **units**.

In science, the **metric system** is used because it is based on powers of ten, making conversions between particular metric units very easy. More properly, the system we use is called the **SI system**.

The basic units of this system are units of **length**, **time** and **mass**. They are:

Quantity	Unit	Abbreviation	Meaning
Length	Meter	m	1 meter is roughly 39 inches
Time	Second	s	Familiar
Mass	Kilogram	kg	1 kg of mass weighs $\approx$ 2.2 lbs

The *exact* relation between "English" and metric units of measurement is

$$1\,\text{foot} = 0.3048\,\text{meter} \ .$$

4.2 Prefixes and Powers of 10

Many of the lengths, times and masses we encounter in science are either much larger or much smaller than the basic units (meter, second and kilogram). Then it becomes convenient to express them in larger or smaller units.

We form these other units from the basic ones using prefixes which correspond to different powers of 10. Some of the most common prefixes are:

Prefix	Abbreviation	Power of Ten
Tera–	T	10^{12}
Giga–	G	10^{9}
Mega–	M	10^{6}
kilo–	k	10^{3}
centi–	c	10^{-2}
milli–	m	10^{-3}
micro–	μ	10^{-6}
nano–	n	10^{-9}
pico–	p	10^{-12}
femto–	f	10^{-15}

Some examples of the use of these prefixes are:

$$1\,\text{cm} \quad = \quad 10^{-2}\,\text{m} = 1\,\text{centimeter}$$
$$1\,\mu\text{s} \quad = \quad 10^{-6}\,\text{s} = 1\,\text{microsecond}$$
$$1\,\text{mg} \quad = \quad 10^{-3}\,\text{g} = 1\,\text{milligram}$$
$$1\,\text{nm} \quad = \quad 10^{-9}\,\text{m} = 1\,\text{nanometer}$$

---

1. A certain type of light has wavelength $5.621 \times 10^{-7}\,\text{m}$. Express this length in nanometers and micrometers.

We are looking for a factor of 10^{-9} in order to get nanometers. By shifting the decimal point we get:

$$5.621 \times 10^{-7}\,\text{m} = 562.1 \times 10^{-9}\,\text{m}$$

so the length is equal to $562.1\,\text{nm}$.

For micrometers we are looking for a factor of 10^{-6}, so by shifting the decimal point the other way we get:

$$5.621 \times 10^{-7}\,\text{m} = 0.5621 \times 10^{-6}\,\text{m}$$

so the length is also equal to $0.5621\,\mu\text{m}$.

4.3   Arithmetic With Units

A scientific number has two parts: A pure number and the *units* that go with it. We can *never* ignore the units or omit them in any calculation.

When we do arithmetic with scientific numbers we treat the unit symbols (like "kg", "μs" and "m") as if they were factors multiplying the numerical part. The unit symbols

will appear in the numerators and denominators of our answers but if the same unit symbol appears on top and bottom it can be cancelled out, just as we do when the same (pure) number appears on top and bottom.

---

2. Divide $3.6\,\mathrm{m}$ **by** $4.3\,\mathrm{s}$**.**

We do the division *keeping all the units*... they are a part of the answer:

$$\frac{3.6\,\mathrm{m}}{4.3\,\mathrm{s}} = 0.84\,\frac{\mathrm{m}}{\mathrm{s}}$$

The answer is $0.84\,\frac{\mathrm{m}}{\mathrm{s}}$. The units of this answer are usually read as "meters per second".

---

3. Multiply $0.626\,\mathrm{m}$ **and** $0.327\,\mathrm{m}$**.**

Do the multiplication keeping all the units:

$$(0.626\,\mathrm{m})(0.327\,\mathrm{m}) = 0.205\,\mathrm{m}^2$$

The two symbols "m" (meter) are multiplied and the result is (naturally enough) written as "m^2". These units are usually read aloud as "square meters" or "meters squared".

---

4. Divide $91.6\,\frac{\mathrm{m}}{\mathrm{s}}$ **by** $13.2\,\mathrm{s}$**.**

Again, keep all the units and combine them just like multiplicative factors:

$$\frac{\left(91.6\,\frac{\mathrm{m}}{\mathrm{s}}\right)}{13.2\,\mathrm{s}} = 6.94\frac{\left(\frac{\mathrm{m}}{\mathrm{s}}\right)}{\mathrm{s}} = 6.94\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

We have combined two factors of "s" (seconds) to give s^2. The units of this answer are usually read as "meters per second squared".

4.4   Converting Units

We can measure an area in units of ft^2 ("square feet"), and we can measure it in units of m^2 ("square meters"). The two measurements represent the *same thing*; but if we are told that a certain area is $63\,\mathrm{ft}^2$, how do we express this quantity in units of m^2? In this section we will look at an easy and foolproof method for converting units.

The method uses the two facts about scientific arithmetic:

- The unit symbols are treated as multiplicative factors as in the last section.

● When we multiply and divide by the *same number* we do not change the result...though we may change its *appearance*.

To introduce the method, take a length of 26.5 ft. We want to express this length in meters. Consider the following expression:

$$\left(\frac{0.3048 \, \text{m}}{1 \, \text{ft}}\right)$$

Since the numerator and denominator are *equal* (as given in the first section of this chapter) this expression is equal to *one*, and if we multiply any other expression with it, *nothing changes*.

Now we will multiply our 26.5 ft with this factor:

$$26.5 \, \text{ft} = (26.5 \, \text{ft}) \left(\frac{0.3048 \, \text{m}}{1 \, \text{ft}}\right) = 8.077 \, \text{m}$$

where we get the last number by multiplying and *cancelling "ft" from top and bottom*.

We can use these factors of 1 to go from one set of units to another. The only trick is to set up these factors so that the units cancel in the right way and of course to know some basic corrspondences between units.

---

5. Find the number of inches in one mile. That is, convert $1 \, \text{mi}$ **to inches.**

We use the relations:

$$1 \, \text{mi} = 5280 \, \text{ft} \qquad \text{and} \qquad 1 \, \text{ft} = 12 \, \text{in}$$

and set up two "factors of 1" which will cancel out "mi" (miles) and leave "in" (inches):

$$1 \, \text{mi} = (1 \, \text{mi}) \cdot \left(\frac{5280 \, \text{ft}}{1 \, \text{mi}}\right) \cdot \left(\frac{12 \, \text{in}}{1 \, \text{ft}}\right) = (5280) \cdot (12) \, \text{in} = 6.34 \times 10^4 \, \text{in}$$

---

6. Convert $65 \, \frac{\text{mi}}{\text{hr}}$ **to units of** $\frac{\text{ft}}{\text{s}}$**.**

To do the conversion, we use the following (well–known?) facts:

$$5380 \, \text{ft} = 1 \, \text{mi} \qquad 1 \, \text{hr} = 60 \, \text{min} \qquad 1 \, \text{min} = 60 \, \text{s} \ .$$

With these, we set up our "factors of 1" so as to cancel out the "mi" (miles) symbol on top and the "hr" (hours) symbol on the bottom leaving only "ft" and "s". Here's how we do it:

$$65 \, \frac{\text{mi}}{\text{hr}} = \left(65 \, \frac{\text{mi}}{\text{hr}}\right) \left(\frac{5280 \, \text{ft}}{1 \, \text{mi}}\right) \cdot \left(\frac{1 \, \text{hr}}{60 \, \text{min}}\right) \cdot \left(\frac{1 \, \text{min}}{60 \, \text{s}}\right) = 95.3 \, \frac{\text{ft}}{\text{s}}$$

In producing the last value we have multiplied all the numbers together and we have cancelled all the units, leaving only "ft" (feet) on top and "s" (seconds) on the bottom.

---

7. A liter is a (metric) unit of volume familiar to those of us who drink lots of bottled soda; in Europe, gasoline is sold by the liter.
 Using the relations

$$1\,\text{liter} = 0.2641\,\text{gallon} \qquad \textbf{and} \qquad 1\,\text{ft} = 0.3048\,\text{m}$$

and other well-known relations, convert $30\,\frac{\text{mi}}{\text{gal}}$ **to units of** $\frac{\text{km}}{\text{liter}}$**.**

We use the (well–known?) relations:

$$1\,\text{mi} = 5280\,\text{ft} \qquad \text{and} \qquad 1\,\text{km} = 10^3\,\text{m}$$

and set up the "factors of 1" so as to cancel the units in the right way, leaving $\frac{\text{km}}{\text{liter}}$:

$$
\begin{aligned}
30\,\tfrac{\text{mi}}{\text{gal}} &= \left(30\,\tfrac{\text{mi}}{\text{gal}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mi}}\right)\left(\frac{0.3048\,\text{m}}{1\,\text{ft}}\right)\left(\frac{1\,\text{km}}{10^3\,\text{m}}\right)\left(\frac{0.2641\,\text{gal}}{1\,\text{liter}}\right)\\
&= 12.75\,\tfrac{\text{km}}{\text{liter}}
\end{aligned}
$$

If my car gets a mileage of $30\,\frac{\text{mi}}{\text{gal}}$, then it's also getting $12.75\,\frac{\text{km}}{\text{liter}}$.

---

8. Using the fact that $1\,\text{in} = 2.54\,\text{cm}$**, convert** $43.5\,\text{in}^2$ **to** cm^2 **(square centimeters).**

To cancel out the units "in" (inches) we will need to use the factor:

$$\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)$$

but here we need to cancel out *two* factors of "in", so we need to apply this factor *twice* (that is, we need to use the *square* of this factor). So the unit conversion goes as:

$$
\begin{aligned}
43.5\,\text{in}^2 &= \left(43.5\,\text{in}^2\right)\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)^2\\
&= (43.5)(2.54)^2\,\text{cm}^2 = 280.6\,\text{cm}^2
\end{aligned}
$$

---

9. Convert $2.70\,\frac{\text{g}}{\text{cm}^3}$ **to units of** $\frac{\text{kg}}{\text{m}^3}$**.**

We will use the relations

$$1\,\text{kg} = 1000\,\text{g} \qquad \text{and} \qquad 1\,\text{m} = 100\,\text{cm}$$

But note that in the number $2.70\,\frac{\text{g}}{\text{cm}^3}$ there are *three* factors of "cm" (centimeters) that we must cancel out, so that the "factor of 1" which changes "cm" into "m" must be applied *three* times (that is, we will cube this factor).

The factors are set up as:

$$\begin{aligned}
2.70\,\tfrac{\text{g}}{\text{cm}^3} &= \left(2.70\,\frac{\text{g}}{\text{cm}^3}\right) \cdot \left(\frac{1\,\text{kg}}{1000\,\text{g}}\right) \cdot \left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^3 \\
&= (2.70) \cdot \frac{(100)^3}{(1000)} \cdot \frac{\text{kg}}{\text{m}^3} \\
&= 2.70 \times 10^3\,\frac{\text{kg}}{\text{m}^3}
\end{aligned}$$

We get $2.70 \times 10^3\,\frac{\text{kg}}{\text{m}^3}$.

Chapter 5

Shapes!

(Rectangles, Circles and Spheres)

5.1 Introduction

This chapter is all about the basic geometrical shapes we meet in doing basic physics, and the more important things that we might want to find about each one, like the area or volume associated with each.

5.2 Triangles!

We'll talk about triangles in the next chapter.

5.3 Rectangles!

A **rectangle** is a four–sided shape whose sides all meet at **right angles** (this is another way of saying "90° angles"). The two numbers to know about a rectangle are the lengths of adjoining sides, which we can call the **height** (h) and the **width** (w). The opposite sides of a rectangle have the same length:
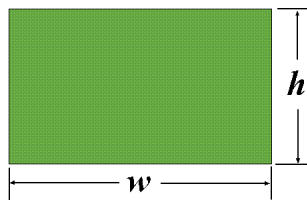
When the width w is equal to the height h, then the shape is a **square**.

The **perimeter** is the distance around on the edge of the rectangle; it is equal to the sums of the lengths of the edges:

$$\text{Perimeter} = w + h + w + h = 2w + 2h = 2(w + h)$$

The **area** of the rectangle measures the amount of space enclosed; it is equal to the product of the width and the height:

$$\boxed{\text{Area of Rectangle} = wh} \tag{5.1}$$

Since the area is a product of two lengths, the units for area in the metric system are m^2 (or any other metric unit squared, like cm^2).

---

1. A rectangle has sides of length $1.40\,\text{m}$ **and** $0.66\,\text{m}$**. What is the perimeter of the rectangle? What is its area?**

Here, the width is $1.40\,\text{m}$ and the height is $0.66\,\text{m}$ (or vice versa...it doesn't matter) so the perimeter is

$$\text{Perimeter} = 2w + 2h = 2(1.40\,\text{m}) + 2(0.66\,\text{m}) = 4.12\,\text{m}$$

and the area is

$$\text{Area} = wh = (1.40\,\text{m})(0.66\,\text{m}) = 0.924\,\text{m}^2$$

5.4 Boxes!

A **box** (or **block**) is three-dimensional shape with 6 faces, all of which are rectangles. There are now *three* numbers we need to know about a box: Its width w, height h and depth d:
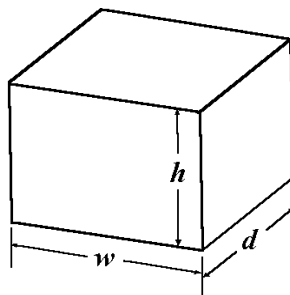
The **volume** of the box is a measure of the amount of space contained within the walls. It is the product of width, height and depth:

$$\boxed{\text{Volume of Box} = whd} \tag{5.2}$$

The volume is a product of *three* lengths, so the units of volume in the metric system are m³ (or some other metric unit cubed, like cm³).

---

2. The width, height and depth of a box are $5.32\,\text{cm}$**,** $3.22\,\text{cm}$ **and** $3.40\,\text{cm}$**. What is the volume of the box?**

The volume is the product of all three dimensions, so

$$V = (5.32\,\text{cm})(3.22\,\text{cm})(3.40\,\text{cm}) = 57.2\,\text{cm}^3 \ .$$

The volume of the box is $57.2\,\text{cm}^3$.

5.5   Circles!

A **circle** is the kind of shape that we get when we insist that all the points on a (flat) loop have the same distance from on particular point (namely the **center** of the circle). This common distance is called the **radius** (r) of the circle; the distance between any point on the circle and the center is r.
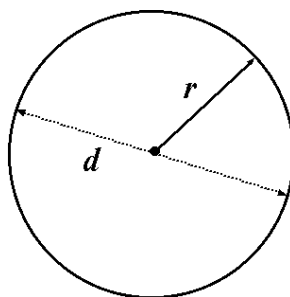
The diameter d of a circle is the distance from any point, through the center and across to the circle again. This distance is twice the radius:

$$d = 2r$$

Now consider a trip all the way around *on* the circle. How long is this path?
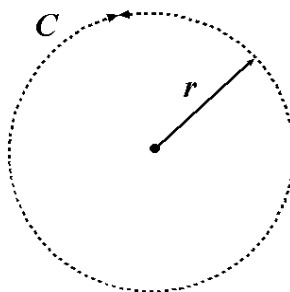
The length of this path is called the **circumference** of the circle and it is *roughly* 3 times the diameter. More precisely, it is about

$$3.14159265359$$

times the diameter. There is a special name for this ratio: π, which is the Greek letter pi (pronounced "pie"):

$$\pi \approx 3.14159265358979$$

So if C is the circumference of the circle, then

$$C = \pi d \ .$$

Since $d = 2r$, another way to express this relation is

$$\boxed{C = \pi d = 2\pi r \ .} \tag{5.3}$$

The number symbolized by π is of great importance in mathematics and is used *very* often in calculations in physics courses. It would be tiresome to have to type it in so often so this important number is *built into* every scientific calculator.

Alas, of the three calculators I am considering in this book, the number π is produced in different ways! On the TI, press `2nd` and then `EXC`. On the Casio, simply press the `EXP` key *before* pushing any number keys. On the HP, press `INV` and then `Exp`. Try this on *your* calculator. In my calculator examples I will simply say that "π" is produced!

---

3. A circle has radius 1.59 m. **What is its diameter? What is its circumference?**
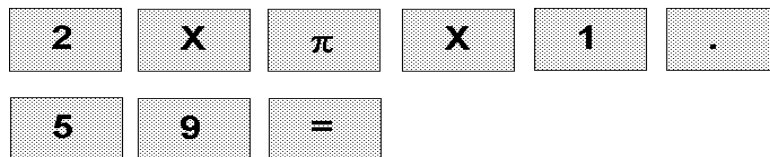
Here, $r = 1.59$ m so that

$$d = 2r = 2(1.59 \, \text{m}) = 3.18 \, \text{m}$$

and the circumference is

$$C = 2\pi r = 2\pi(1.59 \, \text{m}) \ .$$

Since the number π is built into your calculator, the key sequence you press is:

| 2 | X | π | X | 1 | . |

| 5 | 9 | = |

which gives:

$$C = 9.99 \, \text{m}$$

The amount of (flat) space contained inside the circle is its **area**. If we know the radius of a circle, we can find its area from:
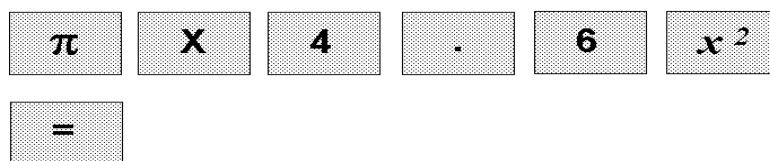
$$\boxed{\text{Area of Circle} = \pi r^2} \tag{5.4}$$

---

4. A circle has radius $4.6\,\text{cm}$**. What is its area?**

The area of a circle is given by $A = \pi r^2$. Using the given value of r,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi (4.6\,\text{cm})^2 \end{aligned}$$

The keystrokes on a calculator needed to get the answer are:

$$\boxed{\pi} \quad \boxed{\textbf{X}} \quad \boxed{\textbf{4}} \quad \boxed{\textbf{.}} \quad \boxed{\textbf{6}} \quad \boxed{x^2}$$

$$\boxed{=}$$

which gives a *numerical* answer of 66.5. Of course, we have to include the units in the answer and the units are cm^2, so the answer is

$$A = 66.5\,\text{cm}^2$$

5.6 Spheres!

A **sphere** is the shape that we get when we take *all* the points in space which lie at a common distance from a given point (the **center** of the sphere). This common distance is the **radius** of the sphere, r.
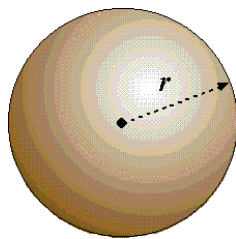
As with a circle, we can discuss the **diameter** of a sphere, which is the distance "all the way across" the sphere, and is twice the radius: $d = 2r$.

If we could peel off the outer layer of a sphere in little (flat) pieces and add up their areas we would get the **surface area** of the sphere. It is related to the radius r by:

$$\boxed{\text{Surface Area of Sphere} = 4\pi r^2} \tag{5.5}$$

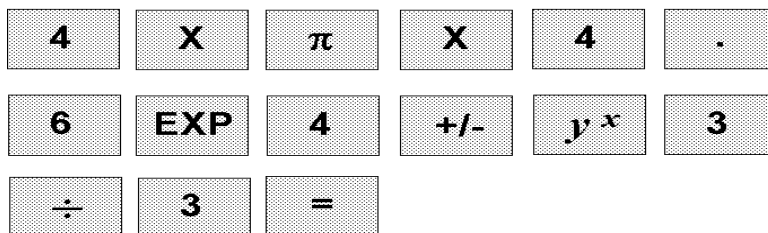The amount of space contained by the sphere is its **volume**; it is given by

$$\boxed{\text{Volume of Sphere} = \tfrac{4}{3}\pi r^3} \tag{5.6}$$

---

5. A sphere has radius 4.6×10^{-4} m. (A tiny sphere!) What is its volume?

Use the formula: $V = \frac{4}{3}\pi r^3$, with $r = 4.6 \times 10^{-4}$ m. First, write down:

$$V = \frac{4}{3}\pi(4.6 \times 10^{-4}\,\text{m})^3$$

then get the answer with the following keystrokes (or the ones which pertain to your calculator):

4	X	π	X	4	.
6	EXP	4	+/-	y^x	3
÷	3	=			

The calculator returns 4.077×10^{-10}, so the answer is:

$$V = 4.077 \times 10^{-10}\,\text{m}^3$$

---

6. A sphere has a volume of 1.57×10^{-5} m^3. Find the radius of the sphere.

Here we need to do a little algebra first.

The volume of the sphere is given as $V = 1.57 \times 10^{-5}$ m^3. We also know that $V = \frac{4}{3}\pi r^3$. The two expressions for V are the same, so we can write

$$\frac{4}{3}\pi r^3 = 1.57 \times 10^{-5}\,\text{m}^3 \quad .$$

Now isolate r. Bring the factors of 4, π and 3 over to the other side to get

$$
\begin{aligned}
r^3 &= 3\frac{(1.57 \times 10^{-5}\,\mathrm{m}^3)}{4\pi} \\
&= 3.748 \times 10^{-6}\,\mathrm{m}^3
\end{aligned}
$$

Now we have the value of r^3. To get r take the cube root of that number:

$$
r = (3.748 \times 10^{-6}\,\mathrm{m}^3)^{\frac{1}{3}} = 1.553 \times 10^{-2}\,\mathrm{m}
$$

[Here, we also take the cube root of "m^3" to get "m" (meters). A radius must have units of length!]

The radius of the sphere is $1.553 \times 10^{-2}\,\mathrm{m}$.

Chapter 6

Triangles!

(Basic Trig: Right Triangles and All That)

6.1   Angles!

An *angle* measures the steepness with which two lines join together. Angles are commonly measured in **degrees**, where there are 360 degrees (written 360°) in one complete revolution.
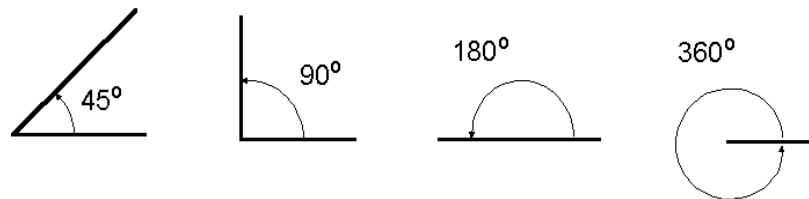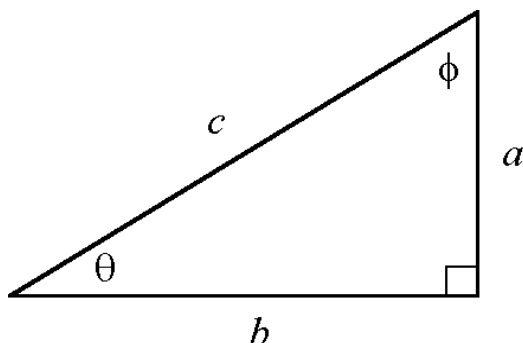

When the angle between two lines is 90° we say that the lines are **perpendicular**.

6.2   Right Triangles and That Pythago–Something.

A **right triangle** is a shape with three straight sides to it and three angles, one of which is a 90° angle. If we call the other two angles θ and ϕ (it's common to use Greek letters for these things) then the two of them must add up to 90°:

$$\theta + \phi = 90°$$

Suppose we call the length of the side opposite θ "a" and the length of the side opposite ϕ "b". We will just refer to these as the "legs" of the right triangle.

The length of the side opposite the right angle will be called "c". c is the longest side of the triangle and is called the **hypothenuse** of the triangle. This is pronounced *high–POT–uh–noos*. Say it along with me, will you? *High–POT–uh–noos*. Mmm, feels good, doesn't it? *High–POT–uh–noos*.

Using these names for the sides of a right triangle, the following relation is true:

$$\boxed{a^2 + b^2 = c^2 \ , \quad \text{or:} \quad c = \sqrt{a^2 + b^2}} \tag{6.1}$$

This result is known as the **Pythagorean Theorem**. In words, the sum of the squares of the legs is equal to the square of the hypothenuse.

6.2.1   Examples

---

1. A right triangle has sides of lengths 2.5 **and** 6.2**. What is the length of the hypothenuse?**

We are given a and b (it doesn't matter which number goes with which letter); the Pythagorean Theorem gives:

$$
\begin{aligned}
c^2 &= a^2 + b^2 \\
&= (2.5)^2 + (6.2)^2 \\
&= 44.69
\end{aligned}
$$

Take the square root to get c:

$$c = \sqrt{44.69} = 6.69$$

The hypothenuse (*high–POT–uh–noose*) has length 6.69.

2. The hypothenuse of a right triangle has length 0.275**. One of the legs has length** 0.166**. What is the length of the other leg?**

We are given $c = 0.275$ and $a = 0.166$, and we want to find b. Use the Pythagorean Theorem and isolate b:
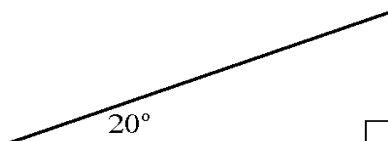
$$a^2 + b^2 = c^2 \implies b^2 = c^2 - a^2$$

so that

$$b = \sqrt{c^2 - a^2} = \sqrt{(0.275)^2 - (0.166)^2} = \sqrt{0.04807} = 0.219$$

6.3 The Basic Trig Functions

Shown here is a right triangle where the sides are not specified but where one of the smaller angles is given (namely, $20°$, so that the missing angle must be $70°$):

When we specify the angles of a right triangle (as we have done here) we determine the *shape* of the triangle, that is, we determine the *ratios* of any two sides of the triangle. In this section we will relate the angles of a right triangle to the ratios of lengths of the sides.

Here we see a right triangle with the angle θ specified:
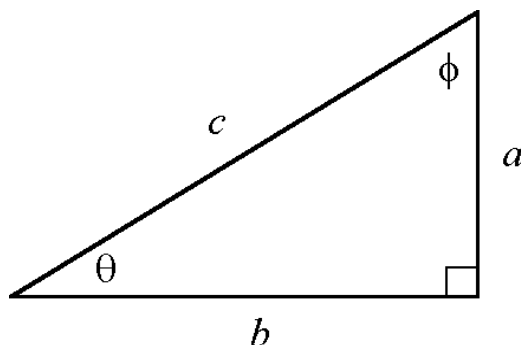
The side opposite θ is a; The other leg is b and the hypothenuse is c. If we know the angle θ, then we know the basic *shape* of the triangle. In particular, we know the ratio of a to c (the opposite side divided by the hypothenuse). This ratio is called the **sine** of the angle θ; this is written as:

$$\boxed{\sin \theta = \frac{a}{c} = \frac{\text{opp.}}{\text{hyp.}}} \tag{6.2}$$

The ratio of b to c (adjacent side divided by the hypothenuse) is called the **cosine** of the angle θ. This is written:

$$\cos\theta = \frac{b}{c} = \frac{\text{adj.}}{\text{hyp.}} \tag{6.3}$$

Finally, the ratio of the side *opposite* the angle θ to the side *adjacent* to θ is called the **tangent** of θ, and it is written as:

$$\tan\theta = \frac{a}{b} = \frac{\text{opp.}}{\text{adj.}} \tag{6.4}$$

Collectively, the sine, cosine and tangent for an angle θ are known as **trigonometric functions** (of θ).

---

3. In a right triangle, the side opposite angle θ has length 2.3. The side adjacent to θ has length 6.2. Find: (a) $\tan\theta$. (b) $\sin\theta$. (c) $\cos\theta$.

It will help to draw a picture of this triangle. It looks like:
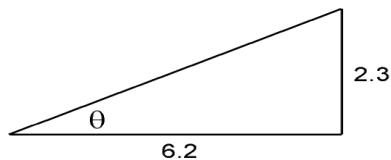
(a) From the definition of $\tan\theta$ we have:

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{2.3}{6.2} = 0.371$$

(b) To find the sine of θ we will need the length of the hypothenuse. Find it using the Pythagorean Theorem:

$$c^2 = (2.3)^2 + (6.2)^2 = 43.7 \quad \Longrightarrow \quad c = \sqrt{43.7} = 6.61$$

Now use this result to find $\sin\theta$:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{2.3}{6.61} = 0.348$$

(c) In the same way, we get $\cos\theta$:

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{6.2}{6.61} = 0.938$$

Using the Pythagorean Theorem we can demonstrate a mathematical relation between the sine and cosine of any angle. If we take any old angle θ and add the *square* of its sine to the *square* of its cosine we get:

$$(\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2}$$

But from the Pythagrean Theorem, we know that we always have $a^2 + b^2 = c^2$. If we put this fact into the last expression we get:

$$(\sin\theta)^2 + (\cos\theta)^2 = \frac{c^2}{c^2} = 1$$

It's worth mentioning now that we write the square of a trig function in a special way. We write:

$$(\sin\theta)^2 \quad \Longrightarrow \quad \sin^2\theta \ .$$

The reason for this is that if we were to write the raised "2" at the very end of "$\sin\theta$" we might think we were squaring θ and *then* finding the sine.

In our new notation, we write the last line as:

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \tag{6.5}$$

4. For some angle θ, $\sin\theta = 0.277$. What is $\cos\theta$?

We can use Eq. 6.5 and a little algebra to get $\cos^2\theta$:

$$\sin^2\theta + \cos^2\theta = 1 \quad \Longrightarrow \quad \cos^2\theta = 1 - \sin^2\theta$$

Using the given value of $\sin\theta$, we get:

$$\cos^2\theta = 1 - \sin^2\theta = 1 - (0.277)^2 = 0.9233$$

This implies

$$\cos\theta = \pm\sqrt{0.9233} = \pm 0.9609$$

But if we are considering only positive values for $\cos\theta$, as we are in this chapter, the answer must be:

$$\cos\theta = 0.9609$$

6.3.1   Calculators and Trig Functions

Before trying any of the examples in this section you need to be sure that your calculator is in "Degrees Mode". This is necessary so that when you key in the value of an angle the calculator will know that you are measuring the angle in *degrees* (there are other ways to measure angles).

For many modern electronic calculators (including the three identified in Chapter 2) you will see a small indicator saying "DEG" or "D" when the calculator thinks you want the angles in degrees. If you don't see this and instead see an indicator saying RAD or GRA then you need to make DEG light up! This is done with a key marked DRG (TI and HP) or by choosing the degrees "Mode" (Casio).

Okay, back to business.

If you are *given* some angle θ, your calculator can find the sin, cos and tan for that angle. After typing in the angle, just press SIN, COS or TAN.

---

5. Using your calculator, find:

$$\textbf{(a)}\ \ \sin 20° \qquad \textbf{(b)}\ \ \cos 20° \qquad \textbf{(c)}\ \ \tan 20°$$

To get $\sin 20°$, press the sequence:

$$\boxed{\textbf{2}} \qquad \boxed{\textbf{0}} \qquad \boxed{\textbf{SIN}}$$
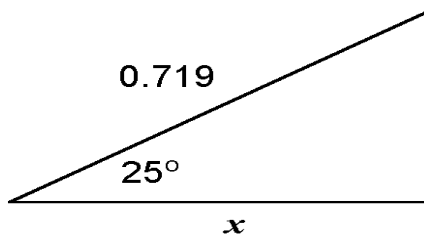
and get

$$\sin 20° = 0.34202 \ \ .$$

Likewise, we get:

$$\cos 20° = 0.93969 \qquad \text{and} \qquad \tan 20° = 0.36397$$

6.4   Using the Trig Functions

By using some algebra and the SIN, COS and TAN buttons on our calculators we can find the missing sides of triangles, as in the following examples:

---

6. For the right triangle shown here, find the length of the side marked with the x.

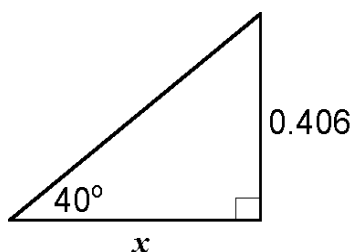Since x is the side adjacent to the angle $25°$ and we also know the hypothenuse of triangle, we can write:

$$\frac{x}{(0.719)} = \cos 25° \qquad \Longrightarrow \qquad \frac{x}{(0.719)} = 0.9063$$

where we have found the cosine of $25°$ with our trusty \$10 calculators. Using some basic algebra we can now find x:

$$x = (0.719) \cdot (0.9063) = 0.652 \quad .$$

So the "adjacent" leg of the triangle has length 0.652.

---

7. For the triangle shown here, find the length of the side marked with an x.

x is the side adjacent to the given angle of $40°$; the length of the opposite side is 0.406. We can form the *tangent* of $40°$ from these factors and write:

$$\frac{0.406}{x} = \tan 40° \qquad \Longrightarrow \qquad \frac{0.406}{x} = 0.8391$$

where we have found the tangent of $40°$ with our fine \$9.99 calculators. We do some algebra to isolate x:

$$0.406 = (0.8391)x \qquad \Longrightarrow \qquad x = \frac{(0.406)}{(0.8391)} = 0.484$$

So the length of the missing side is 0.484.

6.5   Inverse Trig Functions (That "Arc" Stuff)

In the last section, you found out how a calculator could tell you the sine, cosine and tangent for any angle.

In this section, we turn the question around; suppose I tell you:

$$\sin\theta = 0.339$$

Can you tell me what θ is?

You will need a calculator to answer this question, but the calculator *can* answer it quickly.

Let's generalize this question: If $\sin\theta = x$, then what is θ? The answer to that question is written

$$\theta = \arcsin x \qquad \text{or as} \qquad \theta = \sin^{-1} x \ ,$$

and is called the **arcsine** or the **inverse sine** of x. We need to be very careful with the notation "$\sin^{-1}$" and the term "inverse". Here we are *not* taking the sine of some angle and then taking the reciprocal (that is, $1/x$) of the result; it's an entirely different operation. Mathematical notation can be confusing and it's important to know what is meant.

The inverse trig functions on the TI, Casio and HP calculators are accessed by pushing the Shift key (labelled `2nd`, `Shift` or `Inv`) and then the corresponding trig function key. (Look above the `sin`, `cos` and `tan` keys and you will see $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.)

---

8. What is the angle θ for which $\sin\theta = 0.468$?

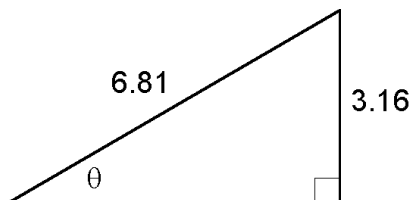The question is asking us to find the angle θ given by

$$\theta = \sin^{-1}(0.468)$$

We pick up our calculators and press the key sequence

.	4	6	8	Shift	sin
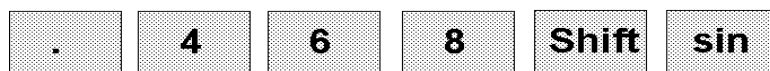
and find:

$$\theta = 27.9°$$

---

9. For the triangle shown here, find the angle θ.

Since we have the "opposite" side and the hypothenuse, we can find the sine of θ:

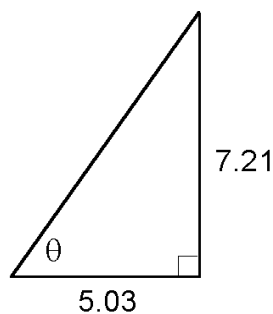$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{(3.16)}{(6.81)} = 0.464$$

Then, since θ is the angle *whose sine is* 0.464 it can be written:

$$\theta = \sin^{-1}(0.464)$$

If your calculator is still holding the result 0.464, just hit the `Shift` `SIN` keys to do the $\sin^{-1}$ operation. We get:

$$\theta = \sin^{-1}(0.4664) = 27.65°$$

---

10. For the triangle shown here, find the angle θ.

For this angle, we know the lengths of the "opposite" and "adjacent" sides, so that we can find the tangent of θ, and we can write:
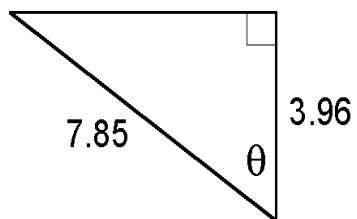
$$\tan\theta = \frac{(7.21)}{(5.03)} = 1.433$$

So θ is the angle whose tangent is 1.433. We write:

$$\theta = \tan^{-1}(1.433)$$

and we press the right buttons on our calculators to get:

$$\theta = 55.1° \ .$$

---

11. For the triangle shown here, find the angle θ.

Here we are given the lengths of the side *adjacent* to θ and also the hypothenuse. These two factors go into the cosine of θ, so we write:

$$\cos\theta = \frac{(3.96)}{(7.85)} = 0.5045$$

and so we get θ from the inverse cosine operation:

$$\theta = \cos^{-1}(0.5045) = 59.7°$$

Chapter 7

More Algebra!

Harder Equations; The Quadratic Formula

7.1   Introduction

In Chapters 3 and 6 we solved some fairly simple equations; as we study more math and science, we have to deal with more difficult ones.

Suppose one day you see the equation

$$x^2 - x - 6 = 0 \tag{7.1}$$

where we want to find x. None of the simple steps we have used to isolate x will do us any good; we can't use them to put x on one side of the $=$ sign and numbers on the other side!

The problem with Eq. 7.1 comes from the fact that we have a term like ax^2 *and* a term like bx *and* a pure number term c all at once. If we'd had only two of these we *could* solve it more easily.

If an equation has terms up to (and including) an x^2 term, it is called a **quadratic equation**. The equations we solved back in Chapter 3 (like $4x - 7 = 3x$) only had terms up to x; these are called **linear equations**.

7.2   The Quadratic Formula

Help comes from a thing called the **quadratic formula**. Suppose we are given an equation in which all three kinds of terms show up:

$$ax^2 + bx + c = 0 \tag{7.2}$$

where a, b and c stand for ordinary numbers which are *given* to us. (They can be positive *or* negative.) We'd like to be able to solve *any* equation which has the form of Eq. 7.2.

Now, Eq. 7.2 is a strange sort of animal. There can be two, one, or *no* real–number solutions to this equation. But we are guaranteed to find any possible solutions by using the following scary but useful formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7.3}$$

where a, b and c are the numbers that appear in Eq. 7.2; they are the **coefficients** in the quadratic equation. The $\pm$ sign in Eq. 7.3 tells us that the choices of $+$ and $-$ can be solutions.

If you can plug numbers into Eq. 7.3, you can solve any equation that looks like Eq. 7.2. Now, you might recall from a math course that equations of this form can often be solved by the "factoring" method. This is certainly true when the coefficients a, b and c are nice round numbers, but Eq. 7.3 will give the solutions *all* the time.

The only time we can run into trouble with the quadratic formula is when $b^2 - 4ac$ is a negative number. When that happens, Eq. 7.3 is telling you to take the square root of a negative number! In this case there are no *real* solutions and we'll save the discussion of this case for your math courses!

7.2.1   Examples

---

1. Solve the equation
$$x^2 - x - 6 = 0$$

Here, $a = 1$, $b = -1$ and $c = -6$. Put these into the quadratic formula and get:

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} \\
&= \frac{1 \pm \sqrt{1 + 24}}{2} \\
&= \frac{1 \pm \sqrt{25}}{2}
\end{aligned}$$

Now, $\sqrt{25} = 5$ so the two possible answers are:

$$x = \frac{1 \pm 5}{2} \qquad \text{that is,} \quad x = \frac{1 + 5}{2} \quad \text{or} \quad x = \frac{1 - 5}{2}$$

This gives:

$$x = \frac{6}{2} \quad \text{or} \quad x = \frac{-4}{2}$$

and the final answer is

$$x = 3 \quad \text{or} \quad x = -2$$

---

2. You are given:

$$3s^2 + 2s - 2 = 0$$

Solve for s.

Here, the coefficients are $a = 3$, $b = 2$ and $c = -2$, and the unknown is s. The quadratic formula gives:

$$\begin{aligned} s &= \frac{-2 \pm \sqrt{4 - 4(3)(-2)}}{2(3)} \\ &= \frac{-2 \pm \sqrt{28}}{6} \end{aligned}$$

Since this is a book about doing *science* problems, a numerical answer is preferred! Using our \$9.99 calculators, we find that the (two) solutions are:

$$s = \frac{-2 + \sqrt{28}}{6} = 0.549$$

and

$$s = \frac{-2 - \sqrt{28}}{6} = -1.215$$

---

3. Solve the equation:

$$4x^2 - 3x + 1 = 0 \ .$$

With $a = 4$, $b = -3$, and $c = 1$, the quadratic formula gives:

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(1)}}{2(4)} \\ &= \frac{9 \pm \sqrt{9 - 16}}{8} \\ &= \frac{9 \pm \sqrt{-7}}{8} \end{aligned}$$

For the purposes of this book, we can stop here since $\sqrt{-7}$ is not a real number. So the equation has *no* real solutions.

---

4. Solve for t:
$$22t - 4.9t^2 = -50$$

The equation is more recognizable when we move all the terms to one side of the $=$ sign and write it as:
$$-4.9t^2 + 22t + 50 = 0$$

which we see is a quadratic equation with $a = -4.9$, $b = 22$ and $c = 50$. The quadratic formula gives us the (possible) solutions for t:

$$
\begin{aligned}
t &= \frac{-22 \pm \sqrt{(22)^2 - 4(-4.9)(50)}}{2(-4.9)} \\
&= \frac{-22 \pm \sqrt{(22)^2 + 980}}{(-9.8)} \\
&= \frac{-22 \pm \sqrt{1464}}{(-9.8)}
\end{aligned}
$$

The two solutions for t are
$$t = \frac{-22 + \sqrt{1464}}{(-9.8)} = -1.659$$

and
$$t = \frac{-22 - \sqrt{1464}}{(-9.8)} = 6.149$$

If our original equation had come from a physics problem for which there was only *one* answer, we would have to know more about the problem to know which of these was the correct one.

---

5. Solve the equation
$$5x^2 = 30x - 45$$

Move all the terms to one side of the "$=$" and get:

$$5x^2 - 30x + 45 = 0$$

which we see is a quadratic equation with $a = 5$, $b = -10$ and $c = 5$. Actually, we could divide both sides by 5 to make things simpler, but forging ahead and applying the quadratic formula, we get:

$$
\begin{aligned}
x &= \frac{-(-30) \pm \sqrt{(-30)^2 - 4(5)(45)}}{2(5)} \\
&= \frac{30 \pm \sqrt{900 - 900}}{10} = \frac{30 \pm 0}{10}
\end{aligned}
$$

Eh? What's this? The square root part has vanished! Since $30 \pm 0 = 30$, both of the "$\pm$" choices give the same thing,

$$
x = \frac{30 \pm 0}{10} = \frac{30}{10} = 3
$$

In this case there is only *one* solution, $x = 3$.

7.3   **Why Does the Quadratic Formula Give the Solutions?

(Special Section for Enquiring Minds!)

For enquiring minds who want to know *why* Eq. 7.3 gives the answers to a quadratic equation we give the proof here.

Begin with a quadratic equation:

$$
ax^2 + bx + c = 0 \tag{7.4}
$$

Here we are assuming that a is not zero, otherwise we would have a linear equation and it would be easy! So we can divide through by a:

$$
x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{7.5}
$$

Now we start to do some manipulations which will seem mysterious but which will make sense in a little while. Put the c/a term on the other side:

$$
x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{7.6}
$$

Now we will add a special term to both sides. It is $(b/2a)^2$, which is really equal to $b^2/(4a^2)$. Why? I'll write it one way on the left side and another way on the right, but with this step we get:

$$
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{7.7}
$$

First work on the right side of Eq. 7.7. Making common denominators we combine the terms to get:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{7.8}$$

Now the reason for adding the funny term comes out: We can rewrite the left side of Eq. 7.8 as the *square* of a simple linear term. The equation becomes:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{7.9}$$

The left side is a square, so it (and the right side) are assumed to be positive numbers. Then in Eq. 7.9 we can remove the square on the left side by equating $x + \frac{b}{2a}$ equal to *plus or minus* the square root of the right side:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{7.10}$$

Now the square root on the right-hand-side of Eq. 7.10 can be "split" between the numerator and denominator. Then we will have a $\sqrt{4a^2}$ on the bottom, and this is equal to $\pm 2a$ (remember, a could be negative). But since we've got a plus-or-minus in front of the whole right side anyway, it changes nothing to write

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{7.11}$$

Now take the $b/(2a)$ from the left side and put it on the right:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{7.12}$$

and since there are common denominators on the right side, combine the terms to get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7.13}$$

And this is the solution for x. We are done.

Chapter 8

Even More Algebra!

(Powers and Logarithms)

[This chapter is under construction...October 11, 2000]

8.1   More on Powers

In this section we will review and extend some of the material in Chapter 3. In that chapter, we worked with whole-number powers of a number x:

$$x^2, \quad x^3, \quad x^4, \quad \ldots$$

and also with *fractional* powers of a number x, which were the same as the *roots* of x:

$$\sqrt{x} = x^{\frac{1}{2}}, \quad \sqrt[3]{x} = x^{\frac{1}{3}}, \quad \sqrt[4]{x} = x^{\frac{1}{4}} \ldots$$

Fractions can be represented in *decimal* form, and that is what we want to emphasize in this chapter. For example, since $\frac{1}{3} = 0.33333$ we can also write the cube root of 91.6 as

$$\sqrt[3]{91.6} = (91.6)^{\frac{1}{3}} = (91.6)^{0.3333333}$$

and if we punch this into our calculators using the y^x key (instead of the root key) we find
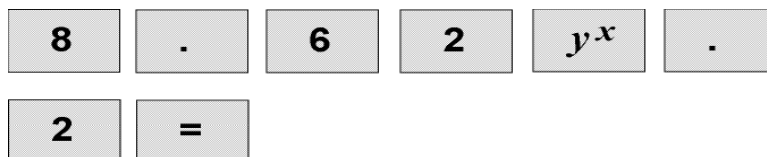
$$(91.6)^{0.3333333} = 4.508$$

---

1. Find $\sqrt[5]{(8.62)}$ using the *power* key (y^x) instead of the root key ($\sqrt[x]{y}$ or $x^{\frac{1}{y}}$).

We want to find

$$\sqrt[5]{(8.62)} = (8.62)^{\frac{1}{5}} \ .$$

Writing the power as a decimal number: $\frac{1}{5} = 0.20$ we take 8.62 to the 0.20 power. The keystrokes for this are:

| 8 | . | 6 | 2 | y^x | . |

| 2 | = |

   and we get
$$(8.62)^{0.2} = 1.5385$$

   When working with powers it is useful to combine terms using the following rules:

$$
\begin{array}{llll}
\text{(a)} & x^a\,x^b = x^{a+b} & \text{(b)} & \dfrac{x^a}{x^b} = x^{a-b} \\[2ex]
\text{(c)} & (xy)^a = x^a\,y^a & \text{(d)} & (x^a)^b = x^{ab}
\end{array}
\tag{8.1}
$$

---

2. Using the rules for powers in Eqs. 8.1 write the following as a *single* power of 3:

   $\quad$ **(a)** $\dfrac{3^{7.1}}{3^{8.5}}$ $\qquad$ **(b)** $\dfrac{1}{3^{11.5}}$ $\qquad$ **(c)** $\left(3^{-0.712}\right)^4$ $\qquad$ **(d)** $9^{6.18}$

(a) For this expression, use rule (b) of Eq. 8.1:
$$\frac{3^{7.1}}{3^{8.5}} = 3^{7.1-8.5} = 3^{-1.4}$$

(b) Since $a^{-x} = \frac{1}{a^x}$, we have
$$\frac{1}{3^{11.5}} = 3^{-11.5}$$

(c) Using rule (d) we have:
$$\left(3^{-0.712}\right)^4 = 3^{[(-0.712)(4)]} = 3^{-2.848}$$

(d) This one looks a bit strange since there's no 3... but wait, $9 = 3^2$. Using this fact and rule (d) from Eqs. 8.1 we can write:
$$9^{6.18} = \left(3^2\right)^{6.18} = 3^{2\cdot(6.18)} = 3^{12.36}$$

8.2 Powers of 10 Revisited!

Now that we know the rules for dealing with all powers of all (positive) numbers, we focus on the powers of a very special number: the number 10, the base of our number system.

Your calculator has a button which makes it fast and easy to evaluate 10^x for any x. Of course, one *could* do it using the y^x key, but if you enter the number x and then press Shift–LOG (on our standard \$9.99 calculators), you will get 10^x. (As stated before, the Shift key is also called INV and 2nd on some calculators.) The LOG operation is *related* to taking powers of 10; we will discuss it in the next section!

---

3. Find $10^{-3.402}$.

The key sequence

$$\boxed{3} \quad \boxed{.} \quad \boxed{4} \quad \boxed{0} \quad \boxed{2} \quad \boxed{+/-}$$

$$\boxed{\textbf{Shift}} \quad \boxed{\textbf{LOG}}$$
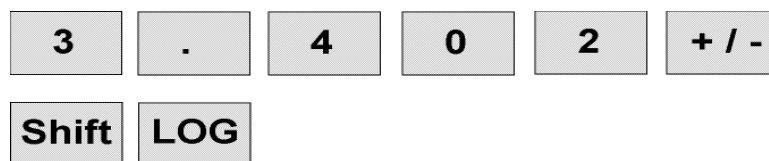
which tells us:
$$10^{-3.402} = 3.963 \times 10^{-4} \ .$$

Because our number system is based on powers of 10, it turns out that we can express *any* number as a power of 10 just by knowing the values of 10^x with x between 0.0 and 1.0. For example, suppose we are told that

$$10^{0.5416} = 3.48$$

(Check it out on your calculator!) We can use this to express 348 as a power of ten:

$$348 = 3.48 \times 10^2 = (10^{0.5416}) \cdot (10^2) = 10^{2+0.5416} = 10^{2.5416}$$

and we can also use it to express 0.348 as a power of ten:

$$0.348 = 3.48 \times 10^{-1} = (10^{0.5416}) \cdot (10^{-1}) = 10^{0.5416-1} = 10^{-0.4584}$$

---

4. Given that $10^{0.9619} = 9.16$, express each of the following numbers as a single power of 10:

$$\textbf{(a)} \quad 9160 \qquad \textbf{(b)} \quad 0.0916$$

(a) First, write 9160 as 9.16 times a power of 10:

$$9160 = 9.16 \times 10^3 = (10^{0.9619}) \cdot (10^3) = 10^{3+0.9619} = 10^{3.9619}$$

(b) Write 0.0916 as 9.16 times a power of 10:

$$0.0916 = 9.16 \times 10^{-2} = (10^{0.9619}) \cdot 10^{-2} = 10^{-2+0.9619} = 10^{-1.038}$$

---

5. Do the following example *exactly* like the last one. If there is any extra piece of information which you need, stand up on your chair and demand to have it.
   Express each of the following numbers as a single power of 10:

$$\textbf{(a)} \quad 8250 \qquad \textbf{(b)} \quad 0.0734$$

(a) Hmm... 8250. *Well I could do this part if somebody would tell me the power of 10 which gives 8.25.*

Sorry. It so happens that $10^{0.9165} = 8.25$.

Thank you. Then if $10^{0.9165} = 8.25$, we can write:

$$8250 = 8.25 \times 10^3 = (10^{0.9165}) \cdot (10^3) = 10^{3+0.9165} = 10^{3.9165}$$

(b) Hmm.. 0.0734 . Well, ex*cuse* me, I *could* do this part if someone would tell me the value of x for which $10^x = 7.34$.

Oops. Sorry. It so happens that $10^{0.8657} = 7.34$.

Oh, thank you. Then we can write 0.0743 as 7.34 times a power of 10 to get:

$$0.0734 = 7.34 \times 10^{-2} = (10^{0.8657}) \cdot (10^{-2}) = 10^{0.8657-2} = 10^{-1.134}$$

8.3 Logarithms, Base 10

Remember the indignant questions which you asked in the last example of the last section: If I have some number, say, 4.02, what is the power of 10 which gives 4.02, that is, if

$$10^x = 4.02$$

then what is x?

The answer to this question is called[1] the **logarithm (base 10)** of 4.02. More generally, we have:

$$\text{If} \quad 10^y = x \qquad \text{then} \qquad y = \log_{10} x \tag{8.2}$$

Now, if you really got the drift of the last examples in the last section, you realize that we really only need the logarithms of numbers between 1.0 and 10.0, because all other numbers differ by a factor of a power of ten, and we can easily take care of that. Indeed, in the olden days before cheap electronic calculators people had to look up the base–10 logarithms of numbers between 1.0 and 10.0 in big tables.

But not any more. Today this is no longer necessary since your $9.95 scientific calculator can easily find the logarithm (base 10) of any number. You can type in any (positive) number into a calculator and do the operation usually called `log` at the touch of a button.

Which button? That, of course, varies a bit depending on which calculator you have! Of the three calculators discussed in Chapter 2, all have a `LOG` key. Press it right after entering the number for which you want to find the base–10 log.

Before moving on, we note one fact which follows from the definition in Eq. 8.2. This is:

$$\log_{10}\left[10^x\right] = x \qquad \text{and} \qquad 10^{\log_{10} x} = x \tag{8.3}$$

We'll need these relations when we solve equations involving logs.

The logarithm function also satisfies the following rules:

$$\log_{10}(xy) = \log_{10}(x) + \log_{10}(y) \qquad \log_{10}\left(\frac{x}{y}\right) = \log_{10}(x) - \log_{10}(y) \qquad \log_{10}\left(x^n\right) = n\log_{10}(x) \tag{8.4}$$

In words: The log of a product is the *sum* of the logs, and when we take the log of a number raised to a power n, we can pull the factor n out in front.

---

[1]Another name used for the base–10 logarithm is the **common logarithm**.

8.4   The special number e.

The last section was about finding the right power of *ten* which gives a certain number. It could have been about finding the powers of any other number; the number 10 is special because of our base–10 system of numbers.

It turns out that just as the number π has special prominence in all mathematics involving angles, a rather funny number called e is important in all mathematics involving powers. The value of the number e is

$$e \approx 2.718281828459045233536 \ .$$

The digits of e do not repeat, and we have given quite enough of them here!

It is hard to understand the special nature of the number e without calculus. For now, you can use your calculator to check the fact that we get number very close to e when we evalulate

$$\left(1 + \frac{1}{m}\right)^m \tag{8.5}$$

for very large values of m. The larger the value of m in Eq. 8.5, the closer we get to e.

---

6. Let's try $m = 5000$ in the expression given in 8.5 What keystrokes can we use and what do we get?

The following keystrokes will do it: (We haven't talked about the parenthesis or $1/x$ keys yet, but they are on your calculator.)
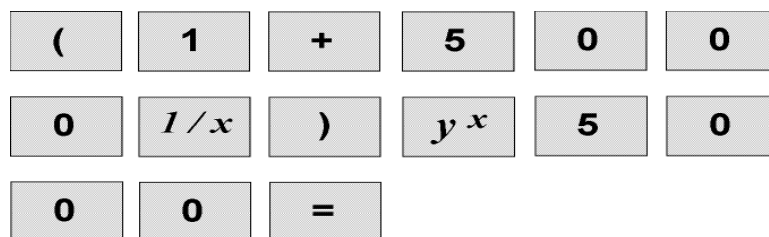
| (| 1 | + | 5 | 0 | 0 |
| 0 | $1/x$ |) | y^x | 5 | 0 |
| 0 | 0 | = |

The calculator gives

$$\left(1 + \frac{1}{5000}\right)^{5000} = 2.71801005 \ .$$

Not too bad... it gets the first four figures right!

To take the number e to any power, use the e^x operation on your calculator. On all three of the calculators of choice in this book, this operation is gotten by pushing the sequence `Shift`–`LN`. (We will discuss what "Ln" means shortly!)

---

7. Find $e^{-3.46}$.

Push the keys:

3	.	4	6	+ / -	Shift

ln
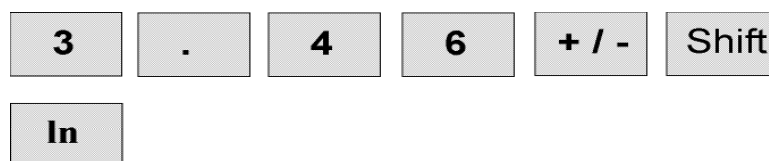
The calculator gives
$$e^{-3.46} = 0.03143 \ .$$

Raising e to the power of x is often known simply as the **exponential function**.

8.5 Logarithms, Base e

Now we want to reverse the operation of taking powers of e, just as we did in the case of powers of 10.

If I know that e^y gives a certain number, can I find y? Yup, and the operation is called taking the **natural logarithm** of the number, and is usually denoted by `ln`. For your calculator, it is a piece of cake.

Generally,

$$\text{If} \quad e^y = x \quad \text{then} \quad y = \ln x \tag{8.6}$$

To get $\ln x$, just type in x and push `LN`.

From Eq. 8.6 we get the relations:

$$\ln\left[e^x\right] = x \quad \text{and} \quad e^{\ln x} = x \ , \tag{8.7}$$

which will be useful when we solve equations which have exponentials.

The natural logarithm satisfies the same kind of rule as in Eq. 8.4:

$$\ln(xy) = \ln(x) + \ln(y) \qquad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y) \qquad \ln\left(x^n\right) = n\ln(x) \tag{8.8}$$

A word on notation for logs: Watch out! In this booklet I have always used "$\log_{10} x$" for a base–10 logarithm and "$\ln x$" for a natural (base e) logarithm. But sometimes people are sloppy; they will write "$\log x$" in cases when it's not clear *which* kind of log they mean. In some applications to experimental work it can mean $\log_{10}$ while in more mathematical works it might mean ln. If you're not sure, go up to the person who wrote "log" and say, "Hey, *you*! What kinda log are you talking about???"

8.6   Equations Involving Exponentials and Logs

---

8. Solve for x:
$$1 - e^{-x} = 0.70 \ .$$

We can put the exponential term on the right side of the equation (with a change of sign) and bring the 0.70 term to the left side of the equation (with a change of sign) to get:

$$1 - 0.70 = e^{-x} \qquad \Longrightarrow \qquad 0.30 = e^{-x}$$

Now take the natural (base e) logarithm of both sides. Recalling Eq. 8.7, we get:

$$\ln(0.30) = -x \qquad \Longrightarrow \qquad -1.204 = -x$$

where we have use our calculators to evaluate $\ln(0.30)$.

So then we have the solution,
$$x = 1.204$$

---

9. Solve for x:
$$(3.20)^x = 57.9$$

Here x appears in an exponent so it might be helpful to take the *log* of both sides. The natural log will do just fine, and using the third of the rules in Eq. 8.8, taking the "ln" of both sides gives:
$$x \ln(3.20) = \ln(57.9)$$

Now we isolate x and find:

$$x = \frac{\ln(57.9)}{\ln(3.20)} = \frac{(4.059)}{(1.163)} = 3.489$$