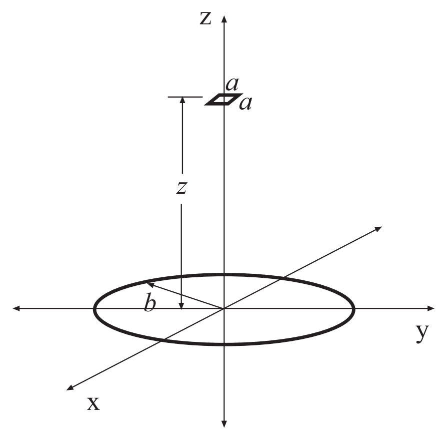
Phys 4620, Spring 2007
Exam #1



1. A small *square* loop of wire of side a and resistance R lies a distance z above the center of a much larger circular loob of radius b. The planes of the two loops are parallel and perpendicular to a common axis.

a) Find an approximate expression for the mutual inductance of the two loops. Do this the easiest way you can!

With current I in the big loop, the field at distance z from the center is

$$B_z = \frac{\mu_0 I}{2} \frac{b^2}{(b^2 + z^2)^{3/2}}$$

Since the square loop is very small, the field has roughly this same value all over the flat surface bounded by the loop so the magnetic flux thru the small loop is

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \approx B_z a^2 = \frac{\mu_0 I a^2 b^2}{2(b^2 + z^2)^{3/2}} \tag{1}$$

The definition of mutual inductance is $\Phi_2 = M_{21} I_1$, so from 1 the mutual inductance is

$$M_{21} = M_{12} = M = \frac{\mu_0 a^2 b^2}{2(b^2 + z^2)^{3/2}}$$

b) For this system, suppose the current in the big loop depends on time via:

$$I(t) = I_0[1 - e^{-\alpha t}]$$

Find an expression for the current induced in the small loop.

The magnitude of the current in the small loop is

$$I' = \frac{\mathcal{E}}{R} = \frac{1}{R}\frac{d\Phi}{dt} = \frac{\mu_0 a^2 b^2}{2(b^2 + z^2)^{3/2}} \frac{dI}{dt}$$

so with $I = I_0(1 - e^{-\alpha t})$, $\frac{dI}{dt} = I_0 \alpha e^{-\alpha t}$, so

$$I' = \frac{\mu_0 a^2 b^2}{2(b^2 + z^2)^{3/2}} I_0 \alpha e^{-\alpha t} \tag{2}$$

c) What is the total charge which circulates in the small loop from $t = 0$ to $t = \infty$?

The total charge which circulates is the integral over time of the current: $Q = q(\infty) = \int_0^\infty I(t)\,dt$. Using 2,

$$Q = q(\infty) = \frac{\mu_0 a^2 b^2}{2(b^2+z^2)^{3/2}} I_0 \alpha \int_0^\infty e^{-\alpha t}\,dt$$

Since

$$\int_0^\infty e^{-\alpha t}\,dt = -\frac{1}{\alpha} e^{-\alpha t}\Big|_0^\infty = \frac{1}{\alpha}\,,$$

we have

$$Q = \frac{\mu_0 a^2 b^2 I_0}{2(b^2+z^2)^{3/2}}$$

2. Suppose the electron is a uniformly charged spherical shell of radius a and charge $-e$.
a) (Easy; old stuff) What is the electric field everywhere in space?

From Gauss' law (and symmetry) the E field is zero inside the shell. Outside, the field is the same as that of a point charge at the origin, so for $r > a$,

$$\mathbf{E} = -\frac{e}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

b) Find the energy density of the electric field; integrate this over all space to find the total energy stored in the electric field. The answer depends on a; what happens as $a \to 0$?

The energy density is (with no magnetic field),

$$u_{\mathrm{EM}} = \frac{\epsilon_0}{2}E^2 = \frac{\epsilon_0}{2}\frac{e^2}{(4\pi\epsilon_0)^2)r^4} = \frac{e^2}{32\pi^2\epsilon_0 r^4}\,,$$

for $r > a$ *only*. Integrate this over all space and get:

$$\begin{aligned}\text{Energy} &= \frac{e^2}{32\pi^2\epsilon_0}\int_{r>a}\frac{1}{r^4}\,d\tau = \frac{e^2}{32\pi^2\epsilon_0}(4\pi)\int_a^\infty \frac{1}{r^4}r^2\,dr \\ &= \frac{e^2}{8\pi\epsilon_0}\int_a^\infty \frac{1}{r^2}dr = \frac{e^2}{8\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_a^\infty = \frac{e^2}{8\pi\epsilon_0 a}\end{aligned}$$

c) If we equate the energy stored the E field to the mass-energy of the electron ($m_e c^2$), what value does this give for the radius a? Can this theory about the electron's mass possibly be right?

Setting this equal to $m_{\mathrm{e}}c^2$ we can solve for a:

$$\frac{e^2}{8\pi\epsilon_0 a} = m_{\mathrm{e}}c^2 \qquad \Longrightarrow \qquad a = \frac{e^2}{8\pi\epsilon_0 m_{\mathrm{e}}c^2}$$

Plug in the numbers:

$$a = \frac{(1.602 \times 10^{-19}\ \mathrm{C})^2}{8\pi(8.854 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N \cdot m^2}})(9.11 \times 10^{-31}\ \mathrm{kg})(3.00 \times 10^8\ \frac{\mathrm{m}}{\mathrm{s}})^2} = 1.41 \times 10^{-15}\ \mathrm{m}$$

This is comparable to (in fact greater than) the nucleon radius so it is now known that this theory of mass can't be right.

3. Soon, we will be studying the E and B fields from an oscillating magnetic dipole (strength m_0, frequency ω, pointing along the z axis). We will find the time–dependent fields (in spherical coords) for an observer at (r, θ):

$$\mathbf{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\theta}}$$

Find the Poynting vector at (r, θ). Find time-averaged value of the Poynting vector. (Remember how to do this? Any squared time-dependent trig function is replaced by $\frac{1}{2}$.)

What is the physical meaning of the Poynting vector?

With $\mathbf{E}$ and $\mathbf{B}$ as given, calculate $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$. Use the relation between the unit vectors, $\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$. (This is true at each point, although the unit vectors themselves depend on location.)

Then:

$$\mathbf{S} = \frac{\mu_0 m_0^2 \omega^4}{16\pi^2 c^3}\frac{\sin^2\theta}{r^2}\cos^2[\omega(t - r/c)]\hat{\mathbf{r}} \tag{3}$$

To get the time-average of $\mathbf{S}$, since t occurs only in the $\cos^2$ factor, replace $cos^2(-)$ by $\frac{1}{2}$, then

$$\mathbf{S} = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

The Poynting vector give the rate (and direction) of the energy flow per unit area. Here the flow of energy is radially outward everywhere.

4. a) In the text, we derived the big messy equation

$$\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} = -\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$$

Explain in words what each term means, i.e. its physical significance.

The term on the lhs is the rate of change of the total momentum of the (massive) particles within the volume $\mathcal{V}$.

The first term on the rhs gives the rate of change of the total EM field momentum within the volume $\mathcal{V}$.

The last term on the rhs gives the rate of flow of momentum through the bounding surface $\mathcal{S}$. The equation gives an accounting of the momenutm associated with the massive particles and the EM field. If the particles lose it, the EM field (inside and outside the volume) gain it!

b) In electrodynamics we find that when charges q_1 and q_2 are in motion, the force of q_1 on q_2 is not "equal and opposite" to the force of q_2 on q_1. This means that the usual statement of Newtons's 3rd law is wrong.

Does it also mean that we must give up the principle of momentum conservation? Why or why not?

The total force on the particles is given by $d\mathbf{p}_{\rm mech}/dt$, and it isnot zero in general (even if they are the only particles in the universe). But by the equation in part (a), the momentum lost by the particles is gained by the EM field so that momentum is conserved when we consider both the particles *and* the EM field.

5. Write the wave function

$$f(z,t) = 8\sin(3z - 4t + \frac{\pi}{3})$$

in the form of a complex wave $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$ where

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

If the "3" really means 3.0 m^{-1} and the "4" means 4.0 s^{-1}, what is the speed of this wave?

Use the identity $\sin(x) = \cos(x - \frac{\pi}{2})$, then

$$\begin{aligned} f(z,t) &= 8\sin(3z - 4t + \frac{\pi}{3}) = 8\cos(3z - 4t - \frac{\pi}{6}) \\ &= \mathrm{Re}[8e^{i(3z-4t-\pi/6)}] = \mathrm{Re}[(8e^{-i\pi/6})e^{i(3z-4t)}] \end{aligned}$$

So

$$\tilde{f}(z,t) = \mathrm{Re}[\tilde{A}e^{i(kz-wt)}] \qquad \text{where} \qquad \tilde{A} = 8e^{-i\pi/6}, \quad k = 3.0\ \mathrm{m}^{-1}, \quad \omega = 4.0\ \mathrm{s}^{-1}$$

The speed of the wave is

$$v = \frac{\omega}{k} = \frac{4.0\ \mathrm{s}^{-1}}{3.0\ \mathrm{m}^{-1}} = 1.33\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

6. Starting from the Maxwell equations, show how we get the wave equation for the $\mathbf{B}$ field (in vacuum),

$$\nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

In vacuum, $\nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$. Take the the curl of both sides, get

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = \mu_0\epsilon_0\frac{\partial}{\partial t}(-\frac{\partial\mathbf{B}}{\partial t}) = -\mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

On the lhs use

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = -\nabla^2\mathbf{B} \ ,$$

since $\nabla \cdot \mathbf{B} = 0$. Then:

$$-\nabla^2\mathbf{B} = -\mu_0\epsilon_0 \frac{\partial^2\mathbf{B}}{\partial t^2} \qquad \Longrightarrow \qquad \nabla^2\mathbf{B} = \mu_0\epsilon_0 \frac{\partial^2\mathbf{B}}{\partial t^2}$$

7. Suppose we choose the form of the plane wave for the $\mathbf{E}$ field as

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$

where the vector is $\tilde{\mathbf{E}}_0$ is complex and a constant.

Show how the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

gives

$$\tilde{\mathbf{B}} = \frac{k}{\omega}\hat{\mathbf{z}} \times \tilde{\mathbf{E}}$$

Be *very clear* about the steps.

The E field is given by the plane wave

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} \qquad \text{where} \quad \tilde{\mathbf{E}}_0 = E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}} + E_{0z}\hat{\mathbf{z}}$$

(The coefficients are *complex* constants.)

Thus each component of $\tilde{\mathbf{E}}$ depends on the coordinates only thru the e^{ikz} factor. We get:

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -\frac{\partial E_y}{\partial z}\hat{\mathbf{x}} + \frac{\partial E_x}{\partial z}\hat{\mathbf{y}} = -ikE_y\hat{\mathbf{x}} + ikE_x\hat{\mathbf{y}} \\ &= ik \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ E_x & E_y & E_z \end{vmatrix} = ik\hat{\mathbf{z}} \times \tilde{\mathbf{E}} = -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

Since $\tilde{\mathbf{B}}$ must depend on time via the factor $e^{-i\omega t}$, this gives

$$ik\hat{\mathbf{z}} \times \tilde{\mathbf{E}} = i\omega\tilde{\mathbf{B}} \qquad \Longrightarrow \qquad \tilde{\mathbf{B}} = \frac{k}{\omega}\hat{\mathbf{z}} \times \tilde{\mathbf{E}}$$

8. For an EM wave obliquely incident on a transparent dielectric with the E field polarized *in* the scattering plane, there is a special angle called Brewster's angle. What's so special about that angle?

Brewster's angle is the incidence angle for which the reflected amplitude is *zero*; the wave is completely trasmitted (for in--plane polarization).

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{4}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{5}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{6}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{7}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{8}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{9}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{10}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{11}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\hat{\mathscr{r}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\rm dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\rm enc}$$

$$\mathbf{F}_{\rm mag} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\rm mag} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times 10^{-7}\,\tfrac{\rm N}{\rm A^2} \qquad 1\ \mathrm{T} = 1\,\tfrac{\rm N}{\rm A\cdot m}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\rm enc} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\rm above} = B^\perp_{\rm below} \qquad \mathbf{B}_{\rm above}-\mathbf{B}_{\rm below} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\rm above} = \mathbf{A}_{\rm below} \qquad \frac{\partial\mathbf{A}_{\rm above}}{\partial n} - \frac{\partial\mathbf{A}_{\rm below}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\rm dip}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp - B_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \qquad \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

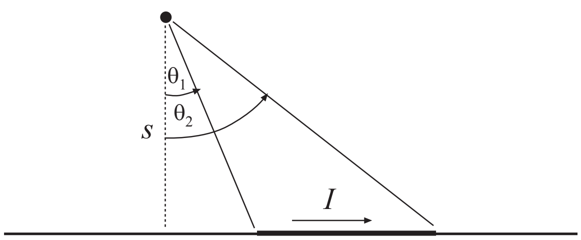
$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \qquad \tilde{\mathbf{B}} = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}$$

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0R} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \qquad \tan\theta_B = \frac{n_2}{n_1}$$

Specific Results:

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$



$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$