Phys 2920, Spring 2012
Problem Set #5

1. For a frame $\mathcal{S}$ which measures events as (x, y, z, t) and a frame $\mathcal{S}'$ which moves at speed v along the $+x$ axis with respect to $\mathcal{S}$ and measures events as (x', y', z', t'), the relation between the coordinates is

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

This set of relations are the **Lorentz transformations**.

Write this set of equation as a matrix/vector relation, of the form

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

where Λ is a 4×4 matrix. It is really a function of v, so it is $\Lambda(v)$. The matrix Λ will just contain γ, v and c but not x or t.

b) Show that $\Lambda(v)\Lambda(-v) = \mathbf{1}$. (The second matrix is just gotten by substituting $-v$ for v everywhere in your matrix $\Lambda(v)$; this does not change γ.)

2. *(VA 4.42)* If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2, -2, -1)$.

3. Show that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2}$$

using Cartesian coordinates, wherein we have

$$r = \sqrt{x^2 + y^2 + z^2}$$

Though this is an important gradient to take, this is not the easiest way to do it but all we've got so far is Cartesian coordinates so use *them*. Of course, $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$

4. *(VA 4.64)* In what direction from the point $(1, 3, 2)$ is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

5. *(VA 4.63)* Find the directional derivative of $P = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in a direction toward the point $(-3, 5, 6)$.

6. *(VA 4.71)* Evaluate $\nabla \cdot (2x^2 z\,\hat{\mathbf{i}} - xy^2 z\,\hat{\mathbf{j}} + 3yz^2\,\hat{\mathbf{k}})$.

7. *(VA 4.72)* If $\phi = 3x^2 z - y^2 z^3 + 4x^3 y + 2x - 3y - 5$, find $\nabla^2 \phi$.

8. *(VA 4.77)* Prove that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$

9. *(VA 4.96)* If $\mathbf{a} = yz^2\,\hat{\mathbf{i}} - 3xz^2\,\hat{\mathbf{j}} + 2xyz\,\hat{\mathbf{k}}$ and $\mathbf{b} = 3x\,\hat{\mathbf{i}} + 4z\,\hat{\mathbf{j}} - xy\,\hat{\mathbf{k}}$, and $\phi = xyz$, find:
a) $\mathbf{a} \times (\nabla \times \mathbf{b})$
c) $(\nabla \times \mathbf{a}) \times \mathbf{b}$