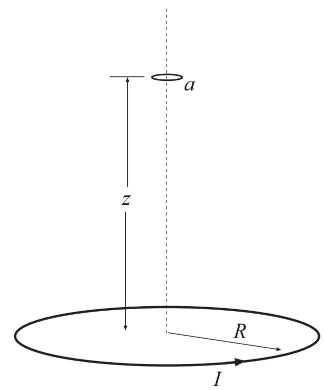
Phys 4620, Spring 2006
Exam #1



1. A ring of radius R in the xy plane carries a constant current I. A small conducting loop of radius $a \ll R$ is also perpendicular to and concentric with the z axis and is located at some value of z also much larger than its size: $a \ll z$. However, we allow z to change with time.

The small loop has resistance r.

Find magnitude of the current induced in the small loop in terms of the other variables of the problem.

If the current in the big loop is I, the magnetic field on the axis is

$$B_z(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z + 2)^{3/2}}$$

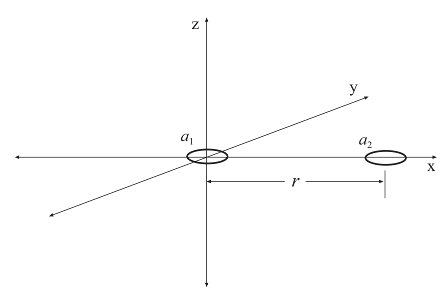
If the upper loop is small, then we can use this to approximate $\mathbf{B}$ for points in its interior, so that the magnetic flux through the small loop is

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} (\pi a^2)$$

The current which is induced in the small loop as z changes arises from the induced emf which we can get from the flux rule. Thus:

$$\begin{aligned} i &= \frac{\mathcal{E}}{r} = -\frac{1}{r}\frac{d\Phi}{dt} = -\frac{\mu_0 I R^2 \pi a^2}{2r} \cdot \frac{(-3/2)2z}{(R^2+z^2)^{5/2}} \cdot \frac{dz}{dt} \\ &= \frac{3}{2}\frac{\pi \mu_0 I R^2 a^2}{r} \frac{z}{(R^2+z^2)^{5/2}} \cdot \left(\frac{dz}{dt}\right) \end{aligned}$$

This is the current in the CCW direction (as seen from above). E.g. if dz/dt is positive, the current goes CCW, whcih is consistent with Lenz!



2. Two small circular wire loops, with radii a_1 and a_2 lie in the xy plane their centers are on the x axis, at $x = 0$ and $x = r$, where $r \gg a_1$ and $r \gg a_2$. Find their mutual inductance, M_{12}.

It will help to approximate the current loops as magnetic dipoles and to recall the formulae for the field of a dipole,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

If the current in the loop at the origin is i_1, then its magnetic dipole moment is

$$\mathbf{m}_1 = i_1(\pi a_1^2)\hat{\mathbf{z}}$$

and the magnetic field at the location of the second loop is

$$\mathbf{B} = \frac{\mu_0 m_1}{4\pi r^3}(1)\hat{\boldsymbol{\theta}} = -\frac{\mu_0 \pi i_1 a_1^2}{4\pi r^3}\hat{z}$$

because at the location of the second loop, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}$. So

$$\Phi_2 = -\frac{\mu_0 \pi i_1 a_1^2}{4\pi r^3}(\pi a_2^2) = -\frac{\mu_0 \pi a_1^2 a_2^2}{4r^3} i_1 \equiv M_{21} i_1$$

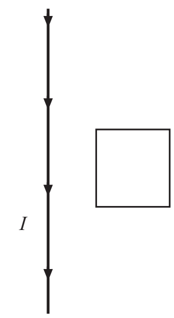
so

$$M_{21} = -\frac{\mu_0 \pi a_1^2 a_2^2}{4r^3} \ ,$$

which we know is also equal to M_{12}.

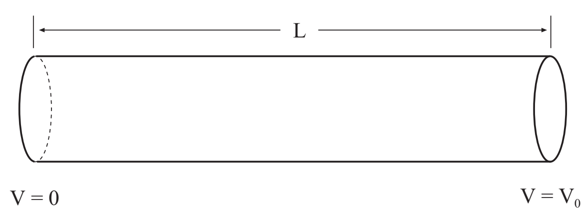
3. A wire (in the plane of the page) carries a current in the direction shown. A loop of wire also in the plane of the page, as shown. The current in the wire (which goes "down" here) is *decreasing*.

Using Lenz's law or any other valid reasoning, deduce which way the induced current will flow in the loop (clockwise or counterclockwise).



The current I is decreasing; the field in the interior of the loop is *out of the page* and *decreasing*. So the induced current must make a flux to oppose this, i.e. a flux *out* of the page. This will occur if the current in the loop goes counterclockwise.

4. A (long) conducting cylinder has length L and radius a. One end of the cylinder is held at potential $V = 0$ and the other is at $V = V_0$. We will assume that the E field inside the cylinder is uniform and directed along the cylinder axis, and since the conductivity is uniform, so is the current density J, with $I = J(\pi a^2)$.



a) What is the magnitude of the E field in the cylinder? (This ought to be easy!)

The ends of the cylinder are at potentials 0 and V_0. If we have the z axis go to the right in the figure, then from $\mathbf{E} - \nabla V$, (and the assumption that $\mathbf{E}$ is uniform) we get

$$E_z = -\frac{V_0}{L}\hat{\mathbf{z}}$$

b) What is the magnitude of the B field at the surface of the cylinder?

From Ampere's law, the B field at the cylinder's surface is the same as it is a distance a from a long wire, hence:

$$B = \frac{\mu_0 I}{2\pi a} \ , \quad \text{where} \quad I = \frac{V_0}{R} \quad \text{from Ohm's law} \ .$$

Here the current goes in the $-z$ direction so we need to say

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}$$

where

$$I = J\pi a^2 = \sigma |E_z| \pi a^2 = \sigma \frac{V_0}{L}\pi a^2 = \frac{V_0}{R}$$

c) What is the magnitude and direction of the Poynting vector at the surface of the cylinder?

With

$$\mathbf{E} = -\frac{V_0}{L}\hat{\mathbf{z}} \qquad \text{and} \qquad \mathbf{B} = -\frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}$$

at the surface of the cylinder, then the Poynting vector there is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0}\left(\frac{V_0}{L}\right)\left(\frac{\mu_0 I}{2\pi a}\right)(\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}})$$

then using $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\mathbf{s}}$ we find

$$\mathbf{S} = -\frac{V_0 I}{2\pi L a}\hat{\mathbf{s}}$$

d) What is the physical meaning (in words) of the Poynting vector?

The Poynting vector represents the flux of energy of the EM field, that is, the energy per time, per area flowing in the direction of $\mathbf{S}$.

e) Calculate the rate of energy flow through the surface of the cylinder.

From our result for $\mathbf{S}$, the energy flow is radially inward; to get the *rate* of energy flow, just multiply by the surface area of the cylinder and get

$$\text{Energy flow (inward)} \quad = \frac{V_0 I}{2\pi L a}(2\pi L a) = V_0 I$$

f) Is there a way to understand this result in terms of simple 2120–type physics?

The answerto (e) is just the power dissipated in a resistor (with current I and potential difference V_0 across its ends). Energy dissipated can be *interpreted* as energy from the EM field flowing inward through the surface of the wire!

5. Given the wave

$$f(x,t) = 5\sin(kx - \omega t)$$

express this (according to our usage in 4620) as a complex wave of the form

$$\tilde{f}(x,t) = \tilde{A}e^{i(kx-\omega t)}$$

that is, find $\tilde{A}$. You might note that $\sin x = \cos(\frac{\pi}{2} - x)$.

Using $\sin(x) = \cos(x - \frac{\pi}{2})$, we note that

$$f(x,t) = 5\cos(kx - \omega t - \tfrac{\pi}{2}) = 5\mathrm{Re}[e^{i(kx-\omega t-\frac{\pi}{2})}] = \mathrm{Re}[5e^{-i\frac{\pi}{2}}e^{i(kx-\omega t-\frac{\pi}{2})}] = \mathrm{Re}[-5ie^{i(kx-\omega t)]}]$$

Thus going over to the *complex* wave notation,

$$\tilde{f}(x,t) = (-5i)e^{i(kx-\omega t)}$$

so the complex amplitude $\tilde{A}$ is $-5i$.

6. The wave equation for the electric and magnetic fields in vacuum are

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\,, \qquad \nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Start with the Maxwell equations in vacuum and show how either one of these is derived.

In vacuum, the Maxwell equations are

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Take the curl of the 3rd equation and get

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{E}}{\partial t}\right)$$

where we've used the 4th Maxwell equation in the last equality.

On the lhs use

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$$

where we've used the 1st Maxwell equation. Cancelling the minus signs, get

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

7. When we chose a general solution for the EM wave of the form

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)} , \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

(where the vectors $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are complex, constant vectors) and applied the conditions $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$ we found that the EM wave had to be transverse.

Show how this condition says the waves must be transverse.

The (complex) wave for $\mathbf{E}$ is

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kx-\omega t)} \equiv (\tilde{E}_x \hat{\mathbf{x}} + \tilde{E}_y \hat{\mathbf{y}} + \tilde{E}_z \hat{\mathbf{z}}) e^{i(kx-\omega t)}$$

where $\tilde{E}_x$, $\tilde{E}_y$ and $\tilde{E}_z$ are constants. The condition $\nabla \cdot \mathbf{E} = 0$ gives

$$\tilde{E}_z (ik) e^{i(kx-\omega t)} = 0$$

since there is $only$ a z dependence. Then $\tilde{E}_z = 0$, which means that there is no component slong the direction of propagation, i.e. the wave is transverse.

Similar steps show that the $\mathbf{B}$ wave is transverse.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\;\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times10^{-7}\,\tfrac{\text{N}}{\text{A}^2} \qquad 1\ \text{T} = 1\,\tfrac{\text{N}}{\text{A}\cdot\text{m}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\text{above}} = B^\perp_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D^\perp_1 - D^\perp_2 = \sigma_f \qquad B^\perp_1 - B^\perp_2 = 0 \qquad \mathbf{E}^\parallel_1 - \mathbf{E}^\parallel_2 = 0 \qquad \mathbf{H}^\parallel_1 - \mathbf{H}^\parallel_2 = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

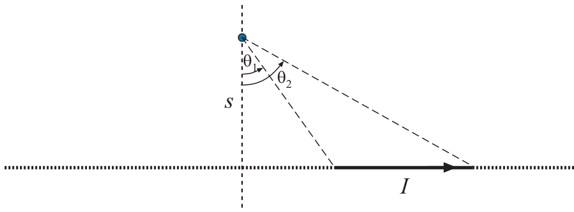
$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Specific Results:

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$



$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$