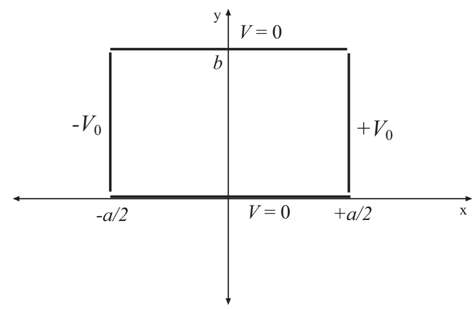
Phys 4610, Fall 2006
Exam #2



1. We would like to solve the 2-dimensionl potential problem (no z–dependence) shown at the right. The cross–section of an infinite rectangular tube has sides of length a measured along x and b measured along y. For later convenience we put the left and right sides at $x = \pm\frac{a}{2}$. The left and right sides are held at potentials $-V_0$ and $+V_0$. The sides at $y = 0$ and $y = b$ are at $V = 0$.

We want a solution for the potential everywhere inside the tube. We do this by separation of variables with a sum of solutions.

a) We first note that by the solution will necessarily be antisymmetric in x, so each term in the expansion must be antisymmetric in x. It turns out that the form of the solution must be:

$$\sum_n C_n \sinh(k_n x) \sin\left(\frac{n\pi y}{b}\right)$$

You don't have to *derive* this expansion, but explain *why* this form is consistent with the boundary conditions, and using the Laplace equation find k_n in terms of n.

$$V(x, y) = \sum_{n=1}^{\infty} C_n \sinh(k_n x) \sin\left(\frac{n\pi y}{b}\right)$$

This expression satisfies the bc's at $y = 0$ and $y = b$ because of the sin factor:

$$\sin(n\pi 0) = \sin(n\pi) = 0$$

It is antisymmetric in x since

$$\sinh(-k_n x) = -\sinh(k_n x)$$

Operating on each term with ∇^2 gives

$$\nabla^2[C_n \sinh(k_n x) \sin\left(\frac{n\pi y}{b}\right)] = \left(k_n^2 - \frac{n^2\pi^2}{b^2}\right) [C_n \sinh(k_n x) \sin\left(\frac{n\pi y}{b}\right)]$$

This must give zero (from the equation for V in a charge--free region, $\nabla^2 V = 0$) so that

$$k_n^2 = \frac{n^2\pi^2}{b^2} \qquad \Longrightarrow \qquad k_n = \frac{n\pi}{b}$$

b) Use the boundary condition at $x = a/2$ to solve for the coefficients C_n. The relation

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \begin{cases} 0 & n' \neq n \\ \frac{a}{2} & n' = n \end{cases}$$

may be useful.

Note that the boundary condition at $x = -a/2$ does not need to be treated since it is automatically satisfied by the form of the series.

At $x = a/2$ we have $V(a/2, y) = V_0$, so

$$\sum_{n=1}^{\infty} C_n \sinh(k_n a/2) \sin\left(\frac{n\pi y}{b}\right) = V_0$$

Multiply both sides by $\sin(n'\pi y/b)$ and integrate from $y = 0$ to $y = b$. The the orthogonality relation for the sin functions. Get:

$$\sum_{n=1}^{\infty} C_n \sinh(k_n a/2) \frac{b}{2} \delta_{nn'} = V_0 \int_0^b \sin\left(\frac{n'\pi y}{b}\right) dy$$

Then:

$$\begin{aligned}
C_{n'} \sinh(k_{n'} a/2)\, \frac{b}{2} &= -V_0 \frac{b}{n'\pi} \cos\left(\frac{n'\pi y}{b}\right)\Big|_0^b \\
&= -V_0 \frac{b}{n'\pi} [\cos(n'\pi) - \cos(0)] = -V_0 \frac{b}{n'pi}[(-1)^{n'} - 1] \\
&= \begin{cases} 0 & \text{if } n' \text{ even} \\ \frac{2V_0 b}{n'\pi} & \text{if } n' \text{ odd} \end{cases}
\end{aligned}$$

Then

$$C_n = \frac{2}{b} \frac{2V_0 b}{\sinh(k_n a/2) n\pi} = \frac{4V_0}{n\pi \sinh(k_n a/2)} \qquad \text{for odd } n$$

The full solution is then

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n \sinh\left(\frac{n\pi a}{2b}\right)} \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

c) What is the value of the charge density on the right wall of the tube? (Watch the signs when you find the normal derivative...)

On the right wall, $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$ so that

$$\begin{aligned}
\sigma(y) &= \epsilon_0 \frac{\partial V}{\partial n}\Big|_{x=a/2} = \epsilon_0 \frac{\partial V}{\partial n}\Big|_{x=a/2} \\
&= \epsilon_0 \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n \sinh\left(\frac{n\pi a}{2b}\right)} \left(\frac{n\pi}{b}\right) \cosh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \\
&= \frac{4V_0 \epsilon_0}{b} \sum_{n \text{ odd}} \frac{1}{\sinh\left(\frac{n\pi a}{2b}\right)} \cosh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)
\end{aligned}$$

2. The potential on a spherical surface (radius R, centered on the origin) is given by

$$V(\theta) = V_0 \cos^2\theta$$

Find the potential $V(r,\theta)$ everywhere *inside* the sphere.

Relations involving the Legendre polynomials are found elsewhere on the exam.

Inside the sphere, the expansion has the form

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Applying the boundary condition,

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0 \cos^2\theta \tag{1}$$

Now, with $x = \cos\theta$, and $P_0(x) = 1$, $P_3(x) = \frac{1}{2}(3x^2 - 1)$ then

$$\cos^2\theta = x^2 = \tfrac{2}{3}[\tfrac{1}{2}(3x^2-1)] + \tfrac{1}{3} = \tfrac{2}{3}P_3(x) + \tfrac{1}{3}P_0(x)$$

Equate the coefficients in 1 and get

$$l = 0: \qquad A_0 \cdot 1 = V_0 \cdot \tfrac{1}{3} \qquad \Longrightarrow \qquad A_0 = \frac{V_0}{3}$$

$$l = 3: \qquad A_3 R^3 = V_0 \cdot \tfrac{2}{3} \qquad \Longrightarrow \qquad A_3 = \frac{2V_0}{3R^3}$$

All the other A_l's are zero. The solution is then:

$$V(r,\theta) = \frac{V_0}{3} + \frac{2V_0}{3R^3} r^3 P_3(\cos\theta)$$

3. A sphere of radius R centered at the origin carries a charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-r)\cos\theta$$

where k is a constant. Find the dipole moment of the sphere. (Note, since the density is symmetric in ϕ, $\mathbf{p}$ can only have a z component.

Sphere of radius R has charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-r)\cos\theta$$

Find $\mathbf{p}$.

Use $\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})\,d\tau$. Since ρ has no x or y dependence, $\mathbf{p}$ has only a z component. Thus, evaluate

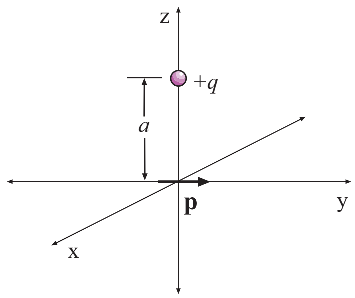
$$p_z = \int z\rho(r,\theta)\,d\tau = \int r\cos\theta k\frac{R}{r^2}(R-r)\cos\theta)\,d\tau$$

Get:

$$\begin{aligned}
p_z &= kR(2\pi)\int_{-1}^{1} x^2\,dx \int_0^R r(R-r)\,dr \\
&= kR(2\pi)\left(\frac{x^3}{3}\right)\Big|_{-1}^{1}\left(R\frac{r^2}{2}-\frac{r^3}{3}\right)\Big|_0^R = 2\pi kR(2/3)\left(\frac{R^2}{2}-\frac{R^3}{3}\right) \\
&= \frac{4}{3}\pi kR\cdot\frac{R^3}{6} = \tfrac{2}{9}\pi kR^4
\end{aligned}$$

4. A "pure" dipole p is situated at the origin, pointing in the $+y$ direction.

a) What is the force on a point charge $+q$ at $(0,0,a)$ (as shown) ?



To use our favorite equations (which have the point dipole pointing along $\hat{\mathbf{z}}$), change the names of the axes:

$$x \longrightarrow y \qquad y \longrightarrow z \qquad z \longrightarrow x$$

and solve, then switch back. With $\mathbf{p} = p\hat{\mathbf{z}}$, the field at $\mathbf{r} = a\hat{\mathbf{x}}$ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] = \frac{1}{4\pi\epsilon_0}\frac{1}{a^3}[-p\hat{\mathbf{z}}] = -\frac{p}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}$$

Translating back to our problem, we have

$$\mathbf{E} = -\frac{p}{4\pi\epsilon_0 a^3}\hat{\mathbf{y}} \qquad \Longrightarrow \qquad \mathbf{F} = q\mathbf{E} = -\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{y}}$$

b) What is the force on a point charge $+q$ at $(0,a,0)$?

Here, using the "easy" coordinate system (where $\mathbf{r} = a\hat{\mathbf{z}}$) we would get

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] = \frac{1}{4\pi\epsilon_0}\frac{1}{a^3}[3(p)\hat{\mathbf{z}} - p\hat{\mathbf{z}}] = +\frac{p}{2\pi\epsilon_0 a^3}\hat{\mathbf{z}}$$

Translating back and getting the force,

$$\mathbf{E} = +\frac{p}{2\pi\epsilon_0 a^3}\hat{\mathbf{y}} \qquad \Longrightarrow \qquad \mathbf{F} = q\mathbf{E} = +\frac{pq}{2\pi\epsilon_0 a^3}\hat{\mathbf{y}}$$

c) How much work does it take to move q from $(0,-a,0)$ to $(0,a,0)$?

The potential along the axis of a dipole is $V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$ where θ is the angle from the direction of $\mathbf{p}$. So here $r = a$ at both points, and θ changes from π to 0. This gives:

$$\Delta V = \frac{p}{2\pi\epsilon_0 a^2}$$

and the work required is

$$W = q\Delta V = \frac{pq}{2\pi\epsilon_0 a^2}$$

5. In chapter 4 we distinguish distributions of *bound charge* and *free charge*.

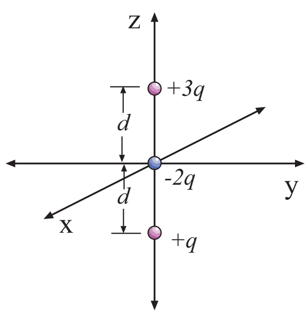
What is the difference between the two types of charge and what is the point (i.e. the utility) of making the distinction?

While both terms refer to genuine electric charge (which makes an E field and can exert force on other charges via Coulomb's law) the bound charge density arises as a charge imbalance when an insulator becomes polarized.

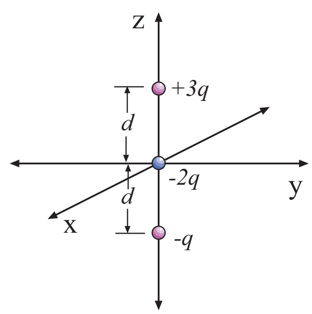
Charges that are placed on conductors (which are *free* charges) are easier to measure and control, hence it is helpful to have a version of the field equations which are written in terms of the free charges.

6. Shown here are two configurations of point charges. For each, give an expression for the potential at long distance ($r \gg d$), keeping only the leading order behavior (i.e. with a single power of r).

Be careful. There is a very *slight* trick here.



(a)



(b)

a) This charge configuration has a *net charge* of $+2q$. The predominant multipole at large distances in simply the monopole, for which we use $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$. Then:

$$V(r,\theta) \approx \frac{1}{4\pi\epsilon_0}\frac{2q}{r} = \frac{q}{2\pi\epsilon_0 r}$$

b) This charge configuration has zero net charge and a dipole moment of

$$p_z = \sum_i z_i q_i = 3qd + (-d)(-q) = 4qd$$

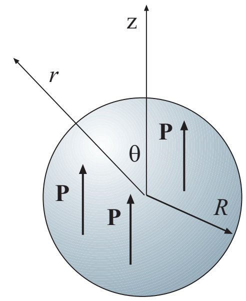
Again using the formula for the potential at long distances from a dipole (pointing along the z axis) we have

$$V(r,\theta) = \frac{4qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{qd\cos\theta}{\pi\epsilon_0 r^2}$$

7. In one of the examples of the book, Griffiths found the potential inside and outside a uniformly polarized sphere of radius R. (That is, it has a uniform "frozen–in" polarization $\mathbf{P} = P\hat{\mathbf{z}}$.)

The result was

$$V(r,\theta) = \begin{cases} \frac{P}{3\epsilon_0} r\cos\theta & \text{for } r \le R \\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & \text{for } r \ge R \end{cases}$$



a) Give a brief sketch of how he (probably) did the calculation. You don't need to do any explicit math here.

First, he found the bound charge density which here ahs only a σ_b (surface) part. Then he used some previous work where, knowing the charge density $\sigma(\theta)$ on a sphere's surface he can get the potential $V(r,\theta)$ inside and out.

b) What are the $\mathbf{E}$ and $\mathbf{D}$ fields for the region outside the sphere?

Using $\mathbf{E} = -\nabla V$, inside we have

$$\mathbf{E} = \frac{P}{3\epsilon_0}\cos\theta\hat{\mathbf{r}} - \frac{P}{3\epsilon_0}\sin\theta\hat{\boldsymbol{\theta}} = \frac{P}{3\epsilon_0}\hat{\mathbf{z}}$$

Outside,

$$\mathbf{E} = -\frac{2P}{3\epsilon_0}\frac{R^3}{r^3}\cos\theta\hat{\mathbf{r}} - \frac{P}{3\epsilon_0}\frac{R^3}{r^3}\sin\theta\hat{\boldsymbol{\theta}} = -\frac{P}{3\epsilon_0}\frac{R^3}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

Since the outside is vacuum, $\mathbf{D} = \epsilon_0\mathbf{E}$ there, and

$$\text{Outside} \quad \mathbf{D} = \frac{P}{3}\hat{\mathbf{z}}\ , \qquad \text{Inside} \quad \mathbf{D} = -\frac{P}{3}\frac{R^3}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

c) Now you should have gotten a non-zero value of $\mathbf{D}$ outside the sphere. But shouldn't it be *zero*? If we draw a spherical Gaussian surface containing the sphere then the surface contains no free charge. Then wouldn't the new form of Gauss' law $\nabla \cdot \mathbf{D} = \rho_f$ imply that the $\mathbf{D}$ field is zero? (Didn't we do something like that on one of the homework problems?) What's wrong with the reasoning here?

A Gaussian surface *can* be drawn enclosing this sphere, and since there is no free charge inside, we *would* get $\oint \mathbf{D}\cdot d\mathbf{a} = 0$ on this surface.

But we can't do much with this, since the problem *does not have spherical symmetry*, i.e. if we rotate the source, we"ll know that it was rotated! So we *can't* write

$$\oint \mathbf{D}\cdot d\mathbf{a} = 4\pi R^2 D_r$$

which would have led to $D_r = 0$.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad dr\, \hat{\mathbf{r}} + rd\theta\, \hat{\boldsymbol{\theta}} + r\sin\theta\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2\, \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{2}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{3}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{4}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{5}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad ds\, \hat{\mathbf{s}} + sd\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{6}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{7}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{8}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{9}$$

Product Rules:

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Double Derivatives:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

More Math:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\mathscr{r})$$

If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3-3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

Physics:

$$\epsilon_0 = 8.854\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N{\cdot}m^2}} \qquad \mu_0 = 4\pi\times 10^{-7}\,\frac{\mathrm{T{\cdot}m}}{\mathrm{A}} \qquad c = 2.998\times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V$$

$$E^{\perp}_{\text{above}} = E^{\perp}_{\text{below}} \qquad \mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

$$V = -\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' \qquad W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \qquad W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\,d\tau$$

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \qquad Q = CV \qquad C = A\frac{\epsilon_0}{d} \qquad W = \tfrac{1}{2}CV^2$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}]$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\text{enc}}$$