Phys 4610, Fall 2006
Exam #1

1. a) Calculate the Laplacian of the function

$$T = \cosh 3x \sin 4y \cos 8z$$

$$\begin{aligned}
\nabla^2 T &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T \\
&= 9\cosh 3x \sin 4y \cos 8z - 16\cosh 3x \sin 4y \cos 8z - 64\cosh 3x \sin 4y \cos 8z \\
&= -71\cosh 3x \sin 4y \cos 8z = -71T
\end{aligned}$$

b) Find the curl of

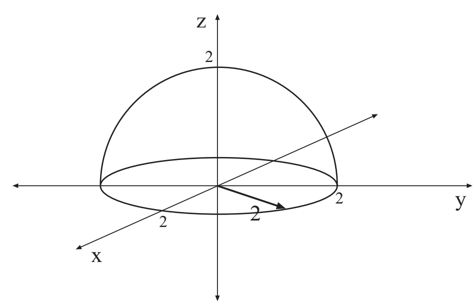
$$\mathbf{v} = r^2 \cos 2\theta \hat{\mathbf{r}} + 4r^2 \sin\theta \hat{\boldsymbol{\theta}}$$

The nonzero terms of $\nabla \times \mathbf{v}$ are

$$\begin{aligned}
\nabla \times \mathbf{v} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(4r^3 \sin\theta) - \frac{\partial}{\partial\theta}(r^2\cos 2\theta)\right]\hat{\boldsymbol{\phi}} \\
&= [12r\sin\theta + 2r\sin 2\theta]\hat{\boldsymbol{\phi}} = 2r(6\sin\theta + 2\sin\theta\cos\theta)\hat{\boldsymbol{\phi}} = 4r\sin\theta(3+\cos\theta)\hat{\boldsymbol{\phi}}
\end{aligned}$$

2. Consider the vector field

$$\mathbf{v} = 2r^2\hat{\mathbf{r}} + r^2\sin^2\theta\hat{\boldsymbol{\theta}}$$



a) Find the divergence of this function.

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(2r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin^3\theta) \\
&= 8r + \frac{1}{r\sin\theta}r^2 \cdot 3\sin^2\theta\cos\theta \\
&= 8r + 3r\sin\theta\cos\theta = r(8 + \tfrac{3}{2}\sin 2\theta)
\end{aligned}$$

b) Evaluate the integral

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

where the closed surface $\mathcal{S}$ is the one shown here, a hemisphere of radius 2 with the flat part in the xy plane and centered at the origin.

On the round part of the surface,

$$d\mathbf{a} = r^2 \sin\theta d\phi \hat{\mathbf{r}} = 4 \sin\theta d\theta d\phi \hat{\mathbf{r}}$$

so with $r = 2$ on that part,

$$\int_{\text{round}} = \int_0^{2\pi} d\phi \int_0^{\pi} (2 \cdot 2^2)(4 \sin\theta) d\theta) = 32 \cdot 2\pi \cdot 2 = 128\pi$$

On the flat part, $\theta = \frac{\pi}{2}$ and

$$d\mathbf{a} = r \sin(\frac{\pi}{2})\, dr d\phi \hat{\boldsymbol{\theta}} = r\, dr\, d\phi \hat{\boldsymbol{\theta}}$$

so the integral is

$$\begin{aligned} \int_{\text{flat}} \mathbf{v} \cdot d\mathbf{a} &= \int_0^{2\pi} \int_0^2 r^2 \sin^2(\tfrac{\pi}{2})(r\, dr) = (2\pi) \int_0^2 r^3\, dr \\ &= (2\pi)(\tfrac{1}{4} 16) = 8\pi \end{aligned}$$

The total surface integral is

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = 136\pi$$

c) Evaluate the integral

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v}\, d\tau$$

where the volume $\mathcal{V}$ is the volume of the same hemisphere.

Irregardless, I say *irregardless* of whether you can finish up parts (b) and (c), what do you expect to find when comparing these answers?

We've already got $\nabla \cdot \mathbf{v}$, so

$$\begin{aligned} \int_{\mathcal{V}} \nabla \cdot \mathbf{v}\, d\tau &= \int_0^{2\pi} d\phi \int_0^{\pi} \int_0^2 (8r + 3r \sin\theta \cos\theta) r^2 \sin\theta d\theta \\ &= (2\pi) \left[\frac{8r^4}{4} \bigg|_0^2 \cdot 2 + \int_0^2 3r^2\, dr\, \int_0^{\pi/2} \cos\theta \sin^2\theta d\theta \right] \\ &= 2\pi \left[64 + \left(\frac{3r^4}{4} \right) \bigg|_0^2 \cdot \frac{\sin^3}{3} \bigg|_0^{\pi/2} \right] = 2\pi[64 + 12 \cdot (\tfrac{1}{3})] \\ &= 2\pi(68) = 136\pi \end{aligned}$$

The is the same as the answer to part (b) and we expect this to be the case from the divergence theorem.

3. The electric potential in a certain region of space is given by

$$V(s, \phi) = V_0 s^3 \cos 3\phi$$

(with cylindrical coordinates; it is independent of z).

Find an expression for the electric field in that region.

What is the charge density in this region?

The electric field is

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial s}\hat{\mathbf{s}} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

and the charge density is

$$\begin{aligned}\rho &= \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \left[\frac{1}{s}\frac{\partial}{\partial s}(sE_s) + \frac{1}{s}\frac{\partial E_\phi}{\partial \phi}\right] \\ &= \epsilon_0[-3V_0(3s)\cos 3\phi + 3V_0 s 3\cos 3\phi] = 0\end{aligned}$$

The charge density in this region is zero!

4. Evaluate

a) $\int_0^6 (x^2+2)\delta(x-5)\,dx$

The integration range contains 5 so

$$\int_0^6 (x^2+2)\delta(x-5)dx = 5^2+2 = 27$$

b) $\int_0^6 (x^7+3x^5-4x^2+6)\delta(x-2\pi)dx$

The integration range does not contain 2π so the integral is zero.

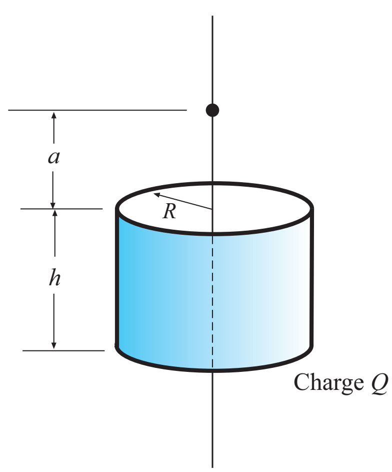
c) $\int_{-2}^2 (4x^2+8)\delta(4x)dx$

Use $\delta(4x) = \frac{1}{4}\delta(x)$, then

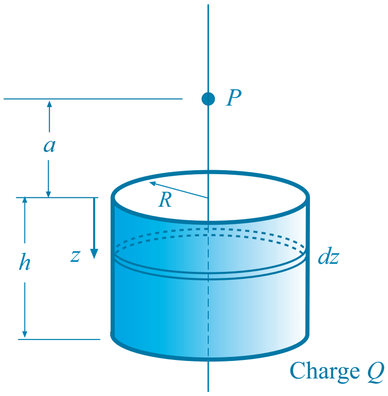
$$\text{Integral} = \tfrac{1}{4}\int_{-2}^{2}(4x^2+8)\delta(x)\,dx = \tfrac{1}{4}(0+8) = 2$$

5. A hollow cylinder of radius R and height h carries a total charge Q spread uniformly over its surface. At a point which is a distance a above the top of the cylinder, on its axis, what is the magnitude of the electric field?

We will divide up the cylinder into horizontal slices as shown.



If we measure z *downward* from the top then z goes from 0 to h. A section of thickness dz contains a charge $dq = Q\,dz/h$ and the plane of that section lies at a distance $\bar{z} = z + a$ from the point P. Using the result for the field on the axis of a ring of charge, the field at P due to this ring is



$$dE_z = \frac{dq}{4\pi\epsilon_0}\frac{\bar{z}}{(\bar{z}^2+R^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0 h}\frac{\bar{z}}{(\bar{z}^2+R^2)^{3/2}}d\bar{z}$$

Integrate from $\bar{z} = a$ to $\bar{z} = a + h$:

$$\begin{aligned} E_z &= \frac{Q}{4\pi\epsilon_0 h}\int_a^{a+h}\frac{\bar{z}}{(\bar{z}^2+R^2)^{3/2}}d\bar{z} = \frac{Q}{4\pi\epsilon_0 h}(-2)\tfrac{1}{2}(\bar{z}^2+R^2)^{-1/2}\Big|_a^{a+h} \\ &= -\frac{q}{4\pi\epsilon_0 h}\left[((a+h)^2+R^2)^{-1/2} - (a^2+R^2)^{-1/2}\right] \end{aligned}$$

That's about as far as we need to go.

6. A uniform circular ring of charge of radius R lies in the xy plane, centered at the origin. Give the charge density $\rho(\mathbf{r})$ of this system.

The answer clearly involves a Dirac delta function; you will want to use either spherical or cylindrical coordinates.

Note, this system has symmetery in both cylindrical *and* spherical coordinates. In cylindrical coords, $\rho(\mathbf{r})$ *must* be proportional to $\delta(r-R)\delta(z)$. The integral of $\rho(\mathbf{r})$ must give Q. Thus:

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} A\delta(r-R)\delta(z) r\,dr\,d\phi\,dz = AR(2\pi) = Q$$

so

$$A = \frac{Q}{2\pi R} \qquad \Longrightarrow \qquad \rho(\mathbf{r}) = \frac{Q}{2\pi R}\delta(r-R)\delta(z)$$

In spherical coords we want

$$\rho(\mathbf{r}) = A\delta(r-R)\delta(\theta - \tfrac{\pi}{2})$$

so then

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} A\delta(r-R)\delta(\theta-\tfrac{\pi}{2})r^2\sin\theta\,dr\,d\theta\,d\phi = AR^2(2\pi) = Q$$

so

$$A = \frac{1}{2\pi R^2} \qquad \Longrightarrow \qquad \rho(\mathbf{r}) = \frac{Q}{2\pi R^2}\delta(r-R)\delta(\theta-\tfrac{\pi}{2})$$

7. Gauss' law in integral form is

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Show how one can deduce the differential form of Gauss' law from this. (That is, show how it follows, using vector calculus.)

Substitute for the total charge enclosed and get:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r})\, d\tau$$

where $\mathcal{V}$ is the volume of the surface bounded by $\mathcal{S}$. Use the divergence rule on the first integral and get:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r})\, d\tau$$

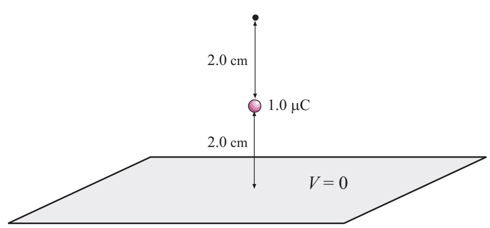
If we consider a very small volume $\mathcal{V}$ centered on the point $\mathbf{r}$ the integrands are "roughly" constant and the factor of the volume element cancels from both sides to give

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\epsilon_0$$

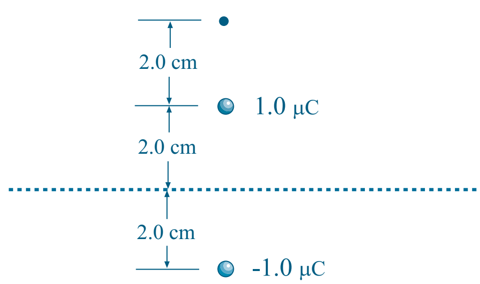
which is the differential form of Gauss' law.

8. A charge of $+1.00\,\mu\mathrm{C}$ is 2.00 cm above an infinite grounded conducting plane. (That is, its potential is 0 V.) Find the potential at a point 4.0 cm above the plane, directly above the charge.

Get a numerical answer!



The potential everywhere is found from the method of images: Remove the conductor and put a $-1.0\,\mu\mathrm{C}$ charge $2.0\ \mathrm{cm}$ below the plane. Then the potential at the given point is due to the two point charges (use $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ for each):



$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left(\frac{(1.0\times 10^{-6}\ \mathrm{C})}{(2.0\times 10^{-2}\ \mathrm{cm})} - \frac{(1.0\times 10^{-6}\ \mathrm{C})}{(6.0\times 10^{-2}\ \mathrm{cm})}\right) \\ &= \frac{1}{4\pi(8.854\times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})}\left(\tfrac{1}{3}\times 10^{-4}\ \tfrac{\mathrm{C}}{\mathrm{m}}\right) = 3.00\times 10^{5}\ \mathrm{V} \end{aligned}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + sd\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\boldsymbol{\mathscr{r}}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

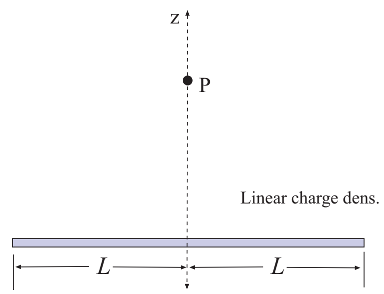
$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V$$

$$V = -\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' \qquad W = \frac{1}{2}\sum_{i=1}^{n}q_iV(\mathbf{r}_i) \qquad W = \frac{\epsilon_0}{2}\int_{\text{all space}}E^2\,d\tau$$

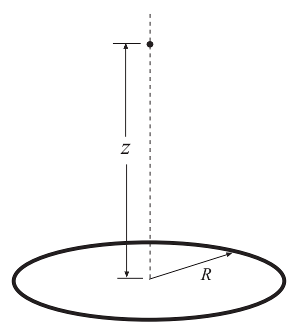
$$\epsilon_0 = 8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}} \qquad \mu_0 = 4\pi\times10^{-7}\,\frac{\mathrm{T\cdot m}}{\mathrm{A}} \qquad c = 2.998\times10^{8}\,\frac{\mathrm{m}}{\mathrm{s}}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$

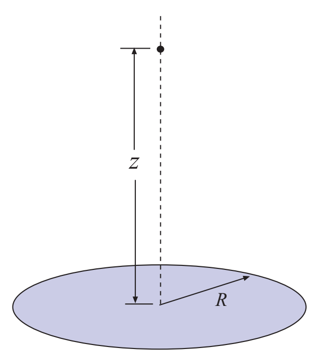


$$E_z = \frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}$$



Charge Q

$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{Q}{2\pi R^2\epsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right] \end{aligned}$$



Charge density σ