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Nov. 16, 2011

Phys 4900, Fall 2011
Quiz #2

1. Explain what is meant by

a) Galilean Transformation

The simple rule for translating the space and time coordinates between reference frames in relative motion which Newton and Galileo assumed. It assumes that time t is the same in all frames and that lengths and velocities add up.

b) (Relativistic) Four-vector

A four-component physical quantity whose components in a second reference frame are calculated with the *same* transformation matrix as used for the basic Lorentz transformation.

c) Relativistic "invariant" (Here, used as a noun.)

A quantity which when calculated from physical quantities in a given way yields the same value in all frames. Most famously, for a four-vector a, the quantity

$$a^2 \equiv (a^0)^2 - (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2$$

is the same in all frames.

d) Time-like vector

A four-vector for which the the time component squared is bigger than the three-vector part squared. In the choice of metric for our class this gives a *positive* value for a^2.

e) CM frame for a relativistic collision.

A reference frame wherein the total 3-momentum of the two colliding particles is zero.

f) "Choice of metric" (E.g. East or west coast, or Griffiths textbook!)

The choice of whether to use the definition

$$a^2 \equiv (a^0)^2 - (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2$$

or

$$a^2 \equiv -(a^0)^2 + (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$$

for computing the value of our usual invariant.

2. Explain what it means to say that the Maxwell equations are "invariant" (here used as an adjective) under the Lorentz transformation.

Tell me more than "They're the same in all reference frames."

This means that when we make the complete replacement of $(\mathbf{r},\ t)$ by $(\mathbf{r}',\ t')$ by the Lorentz transformation we 're left with a set of equations for the original fields and sources, for which if we then *redefine new sources and fields* as simple linear combinations of the old ones, we then have a set of equations which have the same mathematical form as the original Maxwell equations.

3. Recall that the "proper velocity" η^μ, constructed via

$$\eta^\nu \equiv \frac{dx^\mu}{d\tau} \qquad \eta = (\gamma c,\ \gamma \mathbf{v})$$

gave a relativistic four–vector.

a) Explain how we know (from the definition) that η *is* a four–vector.

As x is a four-vector so are the four components of dx; the denominator $d\tau$ is defined as a particle rest frame value so it is the *same* in all frames, hence division by $d\tau$ must give another four-vector quantity.

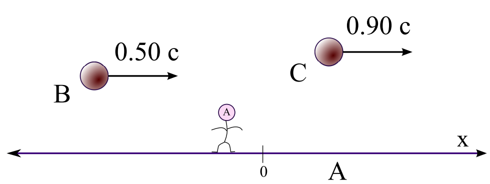
b) What is $\eta_\mu \eta^\mu$? [Derive it from $\eta = (\gamma c,\ \gamma \mathbf{v})$.]

$$\begin{aligned} \eta^2 &\equiv \eta_\mu \eta^\mu = (\eta^0)^2 - (\eta^0)^2 - (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2 \\ &= c^2\gamma^2 - \gamma^2 v^2 = c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) \\ &= c^2 \end{aligned}$$

using the definition of γ.

4. In frame A we observe two objects move in the $+x$ direction; object B moves in the $+x$ direction at a speed of $0.500c$. Object C moves in the $+x$ direction at a speed of $0.900c$.

What is the speed of object C as observed in the frame of B?



Naively, it's $0.40c$. Einstein says it ain't so. One way to get it is to start with frame B and realize that A is moving at a velocity of $-0.500c$ with respect to B, and C is moving at $0.900c$ with respect to A. Then the Einstein velocity addition formula gives

$$v_{CB} = \frac{v_{CA} + v_{AB}}{1 + v_{CA}v_{BA}/c^2} = \left(\frac{-0.500 + 0.900}{1 + (-0.500)(0.900)}\right) c = 0.727c$$

5. The quantities used in relativistic physics ought to "reduce down" to those of Newtonian mechanics in the limit of $v \to 0$.

In what way does the relativistic energy E of a massive free particle reduce to its Newtonian kinetic energy in the limit of small speed?

The energy of a free particle depends on its speed via $E = \gamma mc^2$, with $\gamma = (1 - v^2/c^2)^{-1/2}$. If we do a Taylor series expansion of this in terms of v (or v^2/c^2, actually) the first two terms are

$$E = mc^2 + \tfrac{1}{2}mv^2 + \ldots$$

where the next term will be of the order of mv^4/c^2, so it will be small for $v \ll c$. The relativistic energy thus reduces to a sum of the rest energy mc^2 and the Newtonian kinetic energy in the low-$-v$ limit. The whopping large rest-energy term actually makes no difference for Newtonian mechanics since one may always add a constant to a particle's energy as only energy *differences* matter.

6. A proton is given a kinetic energy of 500 MeV. Find its speed. (Use $m_{\rm p} = 938.272\ \mathrm{MeV}/c^2$).

The (total) energy of the proton is

$$E = Mc^2 + T = 938.272\ \mathrm{MeV} + 500.0\ \mathrm{MeV} = 1438.3\ \mathrm{MeV} = \gamma Mc^2 = \gamma(938.3\ \mathrm{MeV})$$

which gives

$$\gamma = \frac{1438.3}{938.3} = 1.533 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Solve for v:

$$1 - v^2/c^2 = 0.42558 \qquad \Longrightarrow \qquad v^2/c^2 = 0.5744 \qquad \Longrightarrow \qquad v/c = 0.758$$

and it's best left in that form!

7. A Σ^+ particle has a lifetime *in its rest frame* of 8.02×10^{-11} s.

a) What is its *half–life* in its rest frame?

$$t_{\frac{1}{2}} = (\ln 2)\tau = (0.6931)(8.02 \times 10^{-11}\ \mathrm{s}) = 5.56 \times 10^{-11}\ \mathrm{s}$$

b) If it is moving at a speed of $0.800\,c$, what is its half-life in the lab frame, and how far do we expect half of them to travel before they decay?

From the Lorentz transformation equations, a time interval $t_{\frac{1}{2}}$ in the rest frame of the particle is measured as $\gamma t_{\frac{1}{2}}$ in the lab frame. With $v = 0.800c$ we have

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (0.800)^2}} = 1.67$$

so the half-life in the lab frame is

$$t_{\frac{1}{2},\,\mathrm{lab}} = (1.67)(5.56\times 10^{-11}\ \mathrm{s}) = 9.29\times 10^{-11}\ \mathrm{s}$$

which means that as we observe them in the lab, half of them will live to travel a distance

$$d = vt_{\frac{1}{2},\,\mathrm{lab}} = (0.800)(2.998\times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}})(9.29\times 10^{-11}\ \mathrm{s}) = 2.23\times 10^{-2}\ \mathrm{m} = 2.23\ \mathrm{cm}$$

8. Big problem:
Consider a Λ particle at rest decaying to a proton and pion via

$$\Lambda \rightarrow \mathrm{p} + \pi^{-}$$

Find the (individual) energies of the product particles.

What I'm looking for here is whether you know how to set up the equations you would need to solve the problem. The algebra and numbers may be too much for the time available, but show your *strategy* clearly.

Use:

$$m_{\lambda} = 1115.68\ \mathrm{Mev}/c^2 \qquad m_{\mathrm{p}} = 938.272\ \mathrm{MeV}/c^2 \qquad m_{\pi^-} = 139.570\ \mathrm{MeV}/c^2$$

First, we can note that the total kinetic energy of the products must be

$$T_{\mathrm{tot,\ prod}} = (1115.68 - 938.272 - 139.57)\ \mathrm{MeV} = 37.8\ \mathrm{MeV}$$

which isn't much! Anyways, we want an *exact* relativistic answer, so if the momentum four-vectors of the proton and pion are

$$(E_{\mathrm{p}}/c, \mathbf{p}_{\mathrm{p}}) \qquad \text{and} \qquad (E_{\pi}/c, \mathbf{p}_{\pi})$$

then by energy conservation we must have

$$m_{\Lambda}c^2 = E_{\mathrm{p}} + E_{\pi} \tag{1}$$

and by momentum conservation,

$$0 = \mathbf{p}_{\mathrm{p}} + \mathbf{p}_{\pi}$$

so that we can use

$$\mathbf{p}_{\mathrm{p}} = -\mathbf{p}_{\pi} \equiv \mathbf{p}$$

Then the energy conservation equation gives

$$m_{\Lambda}c^2 = \sqrt{\mathbf{p}^2c^2 + m_{\mathrm{p}}^2c^4} + \sqrt{\mathbf{p}^2c^2 + m_{\pi}^2c^4}\ , \tag{2}$$

and equation with only one unknown, but we have to a little algebra! Write it as

$$m_{\Lambda}c^2 - \sqrt{\mathbf{p}^2c^2 + m_{\pi}^2c^4} = \sqrt{\mathbf{p}^2c^2 + m_{\mathrm{p}}^2c^4}\ , \tag{3}$$

and square both sides:

$$m_\Lambda^2 c^4 - 2m_\Lambda c^2\sqrt{\mathbf{p}^2c^2 + m_\pi^2 c^4} + \mathbf{p}^2c^2 + m_\pi^2 c^4 = \mathbf{p}^2c^2 + m_\mathrm{p}^2 c^4$$

Cancel the $\mathbf{p}^2$ terms and rearrange:

$$m_\Lambda^2 c^4 - m_\mathrm{p}^2 c^4 + m_\pi^2 c^4 = 2m_\Lambda c^2\sqrt{\mathbf{p}^2c^2 + m_\pi^2 c^4}$$

Gettin' close! Divide by $2m_\Lambda c^2$ and square both sides:

$$\left(\frac{m_\Lambda^2 c^4 - m_\mathrm{p}^2 c^4 + m_\pi^2 c^4}{4m_\Lambda^2 c^4}\right)^2 = \mathbf{p}^2c^2 + m_\pi^2 c^4$$

Finally, subtract $m_\pi^2 c^4$ and get:

$$\mathbf{p}^2c^2 = \left(\frac{m_\Lambda^2 c^4 - m_\mathrm{p}^2 c^4 + m_\pi^2 c^4}{4m_\Lambda^2 c^4}\right)^2 - m_\pi^2 c^4 \tag{4}$$

One can probably do some simplifying on this, but it's good enough for a computer or calculator. It gives:

$$\mathbf{p}^2c^2 = 1.0115 \times 10^4 \ \mathrm{MeV}^2 \qquad \Longrightarrow \qquad |\mathbf{p}|c = 100.576 \ \mathrm{MeV}$$

and we get

$$E_\mathrm{p} = \sqrt{\mathbf{p}^2c^2 + m_\mathrm{p}^2 c^4} = 943.65 \ \mathrm{MeV}$$

$$E_\pi = \sqrt{\mathbf{p}^2c^2 + m_\pi^2 c^4} = 172.02 \ \mathrm{MeV}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \beta = \frac{v}{c} \qquad c = 3.00 \times 10^8 \ \tfrac{\mathrm{m}}{\mathrm{s}} \qquad x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \qquad t_{\frac{1}{2}} = (\ln 2)\tau \qquad p = (E/c, \mathbf{p}) \qquad E = \gamma mc^2 \qquad E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$$