Phys 3820, Fall 2012
Problem Set #2, Hint-o-licious Hints

1. *Griffiths QM*, **6.14** The relativistic perturbation to the Hamiltonian is

$$H' = -\frac{p^4}{8m^3c^2}$$

so we want to calculate

$$E^1_{n,\mathrm{rel}} = \left\langle n \left| -\frac{p^4}{8m^3c^2} \right| n \right\rangle$$

Use

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \qquad \Longrightarrow \qquad p^2 = -\frac{\hbar m\omega}{2}(a_+^2 + a_-^2 - a_+a_- - a_-a_+)$$

in

$$E^1_{n,\mathrm{rel}} = -\frac{1}{8m^3c^2}\langle p^2\psi^0_n | p^2\psi^0_n\rangle$$

and use orthogonality of the HO wave functions. When this is used in the formula for E^2_n there are only two terms in the sum and you get

$$E^1_{n,\mathrm{rel}} = -\frac{3\hbar^2\omega^2}{32mc^2}(2n^2 + 2n + 1)$$

2. *Griffiths QM*, **6.17** You need to consider two cases,

$$j = ell + \tfrac{1}{2} \qquad \text{and} \qquad j = \ell - \tfrac{1}{2}$$

and with each choice, add the two expressions for E^1_{rel} and E^1_{so} derived in the book. You'll find that after all the algebra you get the same expression in both cases! So the original degeneracy is only "lifted" for states of differing j.

3. *Griffiths QM*, **6.20** Use $|\mathbf{L}| \approx \hbar$ in (6.59) With this, the critical size of the B field is around 12 T.

4. *Griffiths QM*, **6.21** The fine structure is larger than the Zeeman contribution to the energy. The zero-field value of the energies is given by (6.67). Since it included the spin-orbit splitting, the energies depend on j (and n).

Now, for $n = 2$ we have the states $l = 0$ and $l = 1$. We must have states of "good" j, so we note that the $l = 0$ state is a $j = \frac{1}{2}$ states while the $l = 1$ state give $j = \frac{1}{2}$ and $j = \frac{3}{2}$.

Calculate the Landé g factor g_J for each state note, it depends on j and l, and then the weak–field Zeeman energy is

$$E^1_Z = \mu_B g_J B_{\mathrm{ext}} m_j$$

If you plot E vs. $\mu_B B_{\mathrm{ext}}$, the slope of the line is $g_J m_j$.

For the $j = \frac{1}{2}$ states we then have a pair for the state that came from $l = 0$ (with $m_j = \pm\frac{1}{2}$) and a pair for the state that came from $l = 1$. The $j = \frac{3}{2}$ state came from $l = 1$ and with $m_j = -\frac{3}{2} \ldots \frac{3}{2}$, there are four lines with their own slopes.

5. *Griffiths QM*, **6.29** The perturbation is

$$H' = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

and use this in

$$E^1_{\rm gs} = \int \psi^*(\mathbf{r}) H' \psi(\mathbf{r}) d^3\mathbf{r}$$

You can approximate the exponential as 1 to get a lowest–order answer, and that all we need to check the order of magnitude of the result. With this, show

$$\frac{E^1_{\rm gs}}{|E^0_{\rm gs}|} = \frac{4}{3}\left(\frac{b}{a}\right)^2$$

Compare with value with those from fine structure and the hyperfine splitting! (Use Table 6.1).

6. *Griffiths QM*, **7.1** (a) With $\psi(x) = Ae^{-bx^2}$ for the linear potential $V(x) = \alpha|x|$ I get

$$\langle H \rangle = \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2\pi b}}$$

which has a minimum at

$$b = \left[\frac{m^2\alpha^2}{2\pi\hbar^4}\right]^{1/3}$$

and give a bound of

$$\langle H \rangle_{\rm min} = \frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{1/3}$$

(b) For $V(x) = \alpha x^4$ I get

$$\langle H \rangle = \frac{\hbar^2 b}{2m} + \frac{3\alpha}{16b^2}$$

which has a minimum at

$$b = \left(\frac{3m\alpha}{4\hbar^2}\right)^{1/3}$$

and gives a bound

$$\langle H \rangle_{\rm min} = \frac{3}{2}\left(\frac{3\alpha\hbar^2}{4m^2}\right)^{1/3}$$

7. *Griffiths QM*, **7.4** Proof of the theorem goes as described in class; with the trial function ψ expanded as

$$\psi = \sum_{n=0}^{\infty} c_n\psi_n \qquad \text{where} \qquad H\psi_n = E_n\psi_n$$

and $n = 0$ stands for the non-degenerate ground state and $n = 1$ stands for any one of the states of the first excited energy. Find the condition on the c_n's implied by normalization.

From $\langle\psi|\psi_0\rangle = 0$ show that $c_n = 0$ then show

$$\langle H\rangle = \sum_{n=1} |c_n|^2 E_n \geq E_1$$

For the trial function of the form

$$\psi(x) = Ae^{-bx^2}$$

show that normalization gives

$$A^2 = \sqrt{\frac{32b^3}{\pi}}$$

and then after lots of careful algebra

$$\langle T\rangle = \frac{3A^2\hbar^2}{8m}\sqrt{\frac{\pi}{2b}} \qquad \langle V\rangle = \frac{A^2 m\omega^2}{b^2}\frac{3}{32}\sqrt{\frac{\pi}{2b}}$$

These lead to

$$\langle H\rangle = \frac{3}{2}\left(\frac{\hbar^2 b}{m} + \frac{m\omega^2}{4b}\right)$$

and minimizing this with respect to b gives the minimal value of $\langle H\rangle$ hence an upper bound on the first excited state. Of course, you get the exact answer as you can understand from Example 2.4.