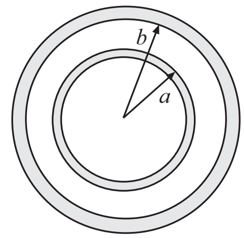
**Phys 4610, Fall 2004**
**Exam #2**

**1.** A spherical capacitor is made from two concentric spherical shells of radii $a$ and $b$ (with $a < b$).




Show that the capacitance of this system is $C = 4\pi\epsilon_0 \frac{ab}{(b-a)}$.

Suppose a charge $-Q$ is placed on the inner sphere and a charge $+Q$ on the outer sphere. From the usual Gauss law argument, the $E$ field for $a < r < b$ is the same as around a point charge $-Q$ at the origin,

$$\mathbf{E} = -\frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

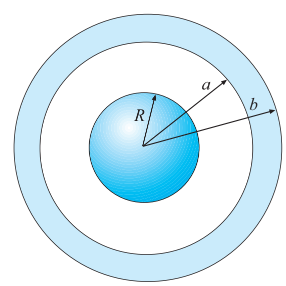
so the potential difference between the spherical surfaces is

$$\begin{aligned} V(b) - V(a) &= -\int_a^b \mathbf{E}\cdot d\mathbf{r} = +\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,dr = -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)\Big|_a^b \\ &= -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{q}{4\pi\epsilon_0}\frac{(b-a)}{ab} \end{aligned}$$

Using the definition of capacitance, $C \equiv Q/(\Delta V)$, we get:

$$C = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\frac{(b-a)}{ab}} = \frac{4\pi\epsilon_0\, ab}{(b-a)}$$

**2.** A metal sphere of radius $R$, carrying a charge $q$ is surrounded by a thick concentric metal shell (inner radius $a$, outer radius $b$, as shewn in the picture). The shell also carries a net charge of $q$.




**a)** Find the surface charge density $\sigma$ at $R$, at $a$ and at $b$.

The inner sphere has charge $q$, so $\sigma_R = \frac{q}{4\pi R^2}$. Draw a Gaussian sphere within the metal of the outer conductor; there must be zero net charge enclosed since $\oint \mathbf{E}\cdot d\mathbf{a} = 0$, hence a charge $-q$ on the inner surface of the shell, hence $\sigma_a = -\frac{q}{4\pi a^2}$.

The shell has a net charge of $q$ so there must be a charge of $2q$ on its outer surface, hence

$$\sigma_b = \frac{2q}{4\pi R^2} = \frac{q}{2\pi r R^2}$$

**b)** Find the potential at the center, using infinity as the reference point.

The net charge of both spheres is $2q$, so the $E$ field outside the chell must be $\mathbf{E}_{\rm out} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{\mathbf{r}}$, so the potential outside the shell is

$$V_{\rm out}(r) = \frac{1}{4\pi\epsilon_0}\frac{2q}{r} = \frac{q}{2\pi\epsilon_0 r}$$

so as to vanish as $r \to \infty$. Then $V(b) = \frac{q}{2\pi\epsilon_0 b}$ and this is also the value of the potential at $a$: $V(a) = \frac{q}{2\pi\epsilon_0 b}$, since the shell is an equipotential.

Between the sphere and the shell the $E$ field is $\mathbf{E}_{\rm gap} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$, so the potential must be

$$V_{\rm gap} = \frac{q}{4\pi\epsilon_0 r} + C$$

for $R < r < a$. But $V(r)$ is continuous which means that at $r = a$, $V_{\rm gap}(r)$ gives the same value as above:

$$V_{\rm gap}(a) = \frac{q}{4\pi\epsilon_0 a} + C = \frac{q}{2\pi\epsilon_0 b} = \frac{2q}{4\pi\epsilon_0 b}$$

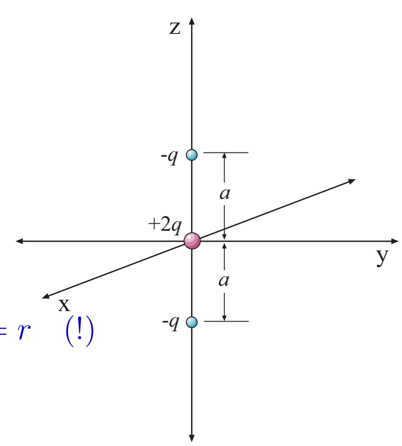
So

$$C = \frac{q}{4\pi\epsilon_0}\left(\frac{2}{b} - \frac{1}{a}\right)$$

Then

$$\begin{aligned} V(R) &= V_{\rm gap}(R) = \frac{q}{4\pi\epsilon_0 R} + \frac{q}{4\pi\epsilon_0}\left(\frac{2}{b} - \frac{1}{a}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{2}{b} - \frac{1}{a}\right) \end{aligned}$$

**3.** Point charges of $-q$, $+2q$ and $-q$ are located on the $z$ axis at $z = -a$, $z = 0$ and $z = +a$ respectively.




**a)** At a point given by spherical coordinates $(r, \theta)$ what is the electrical potential?

Using the law-of-cosines formaula given on the exam (use diagram at right for notation)we have

$$\mathscr{r}_1 = \sqrt{r^2 + a^2 - 2ra\cos\theta} \qquad \mathscr{r}_2 = \sqrt{r^2 + a^2 + 2ra\cos\theta} \qquad r = r \quad (!)$$

Potential from a point charge is $V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$ so at $(r, \theta)$ the potential from these 3 charges is

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{2q}{a} - \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{r^2 + a^2 + 2ra\cos\theta}}\right\}$$

**b)** Using the binomial theorem on the terms in the answer to (a), find an approximate expression for the potential at large distances $r$. Your answer just needs to have the first non-zero term proportional to some power of $r$.

In the anwer to (a), pull out a factor of $\frac{q}{r}$. Then:

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0 r}\left\{2 - [1 + \frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta]^{-1/2} - [1 + \frac{a^2}{r^2} + 2\frac{a}{r}\cos\theta]^{-1/2}\right\}$$

Now use the binomial expansion on the brackets. We will keep terms up to $\frac{a^2}{r^2}$; we know we need to go to the *next* order of approximation because the dipole moment of this charge distribution is zero:

$$\begin{aligned} V(r,\theta) \approx \frac{q}{4\pi\epsilon_0 r}\Bigg\{2 \quad &- \quad \left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta\right) + \frac{1}{2}\frac{3}{4}\left(\frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta\right)^2\right] \\ &- \quad \left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta\right) + \frac{1}{2}\frac{3}{4}\left(\frac{a^2}{r^2} + 2\frac{a}{r}\cos\theta\right)^2\right]\Bigg\} \end{aligned}$$

Keeping terms only up to $a^2/r^2$ in the curly bracket, get;

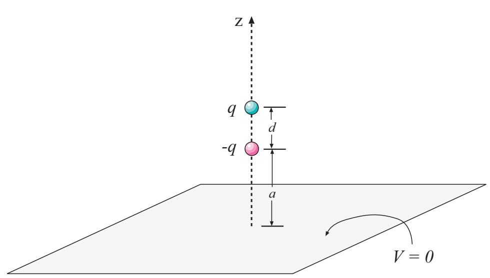
$$\frac{q}{4\pi\epsilon_0 r}\left\{2 - 1 + \frac{a^2}{2r^2} - \frac{a}{r}\cos\theta - \frac{3}{8}(4\frac{a^2}{r^2}\cos^2\theta) - 1 + \frac{a^2}{2r^2} + \frac{a}{r}\cos\theta - \frac{3}{8}(4\frac{a^2}{r^2}\cos^2\theta)\right\}$$

Simplify:

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{a^2}{r^2} - 3\frac{a^2}{r^2}\cos^2\theta\right] = \frac{qa^2}{4\pi\epsilon_0 r^3}\left[1 - 3\cos^2\theta\right]$$

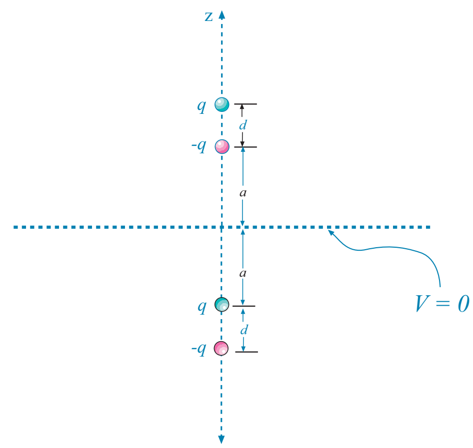
**4.** Two point charges, $\pm q$, are held above an infinite grounded conducting plane (the $xy$ plane). Both charges are on the $z$ axis; the $-q$ charge is at $z = a$ and the $= q$ charge is at $z = a + d$.

Find the potential everywhere in the region $z > 0$.




By the method of images we can solve the Laplace eqn with all the proper boundary conditions if we solve the equivalent problem of charges $\pm q$ *above* the $xy$ plane and charges $\mp q$ placed *below* the plane, oppositely charged, but equidistant with the original charges (see picture at right).

By superposition of the potentials produced by the pairs of charges (original/image) we know that the plane will have zero potential. This satisfies the boundary condition of the problem and so the solution for $z > 0$ will be that due to the four point charges.




Now the distances to these charges are

$$+q \text{ (real)}: \quad \sqrt{x^2+y^2+(z-a-d)^2} \qquad -q \text{ (real)}: \quad \sqrt{x^2+y^2+(z-a)^2} \quad \text{etc.}$$

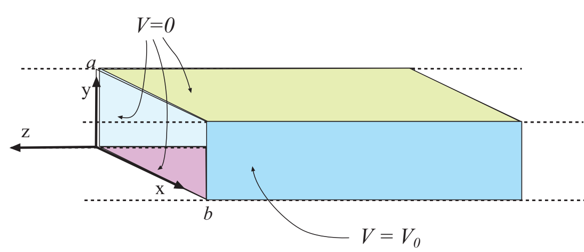
So the potential at the point $(x, y, z)$ is:

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{x^2+y^2+(z-a-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a+d)^2}}\right\}$$

Note the signs that so with the charges $\pm q$. This formula only applies to the half-space $z \geq 0$ and of course gives $V = 0$ for $z = 0$.

Maybe this can be simplified. It's not important!

**5.** We would like to solve the electrostatics problem diagrammed at the right; we have an infinite rectangular pipe which runs along the $z$ direction, where the rectangular cross-section goes from $x = 0$ to $x = b$ and $y = 0$ to $y = a$. The side at $x = b$ is held at a constant potential of $V_0$, but the other sides are at zero potential.




I will start the problem and you finish it. The potential is independent of $z$, so we are solving for $V(x, y)$, and first looking for suitable solutions of the form $X(x)Y(y)$. We find that the choices

$$X(x) = \sinh\left(\frac{n\pi x}{a}\right) \qquad Y(y) = \sin\left(\frac{n\pi y}{a}\right)$$

will do the trick, i.e. they satisfy the Laplace equation and the "zero" boundary conditions. The solution is then given by

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Apply the last boundary condition to get the $C_n$'s

The given solution

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

satisfies $V = 0$ for $y = 0$ and $y = a$ and $V = 0$ for $x = 0$. For $x = b$ we have

$$V(b,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) = V_0$$

Multiply both sides by $\sin\left(\frac{n'\pi y}{a}\right)$ and integrate from 0 to $a$. The orthogonality property given in the eam picks out the $n'$ term:

$$\sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right)\, dy = \int_0^a \sin\left(\frac{n'\pi y}{a}\right) V_0\, dy$$

$$\begin{aligned}
C_{n'} \sinh\left(\frac{n'\pi b}{a}\right)\left(\frac{a}{2}\right) &= V_0\left(\frac{a}{n'\pi}\right)\cos\left(\frac{n'\pi y}{a}\right)\Big|_0^a \\
&= -V_0\left(\frac{a}{n'\pi}\right)\left[\cos(n'\pi) - \cos(0)\right] = \begin{cases} -V_0\left(\frac{a}{n'\pi}\right)(-2) & n' \text{ odd} \\ 0 & n' \text{ even} \end{cases}
\end{aligned}$$

Then, dropping the prime on the $n'$,

$$C_n = \frac{2}{a}\frac{2}{\sinh\left(\frac{n'\pi b}{a}\right)} V_0\left(\frac{a}{n\pi}\right) = \frac{4V_0}{n\pi \sinh\left(\frac{n'\pi b}{a}\right)}$$

for *odd* $n$; otherwise $C_n$ is zero.

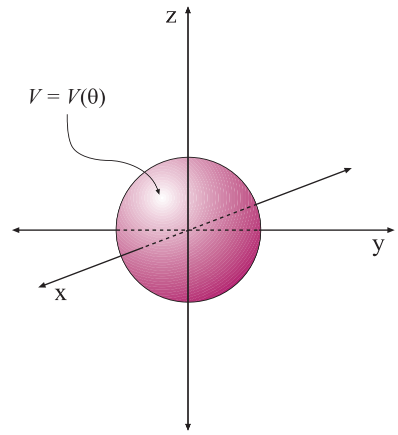
Then the full solution is

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n \sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

**6.** The potential on the surface of a sphere (radius $R$, centered on the origin) is given by

$$V_0 = k\cos^2 2\theta$$

where $k$ is a constant. Find the potential outside the sphere. (There is no charge outside the sphere.)




The problem has $\phi$ symmetry; the potential in the region $r > R$ must $\to 0$ as $r \to \infty$, so only the $B_l$ terms of our usual expnasion survive:

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The boundary condition at $r = R$ gives

$$V(R,\theta) = k\cos^2 2\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

To get the $B_l$'s, multiply both sides by $P_{l'}(\cos\theta)\sin\theta$ and integrate $\int_0^\pi$ on $\theta$. Get:

$$k\int_0^\pi \cos^2 2\theta P_{l'}(\cos\theta)\sin\theta\, d\theta = \sum_l \frac{B_l}{R^{l+1}} \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

Using the orthogonality property of the $P_l$'s, we get

$$k\int_0^\pi \cos^2 2\theta P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{B_{l'}}{R^{l+1}}\frac{2}{(2l'+1)}$$

Then, dropping the primes,

$$B_l = \frac{kR^{l+1}(2l+1)}{2}\int_0^\pi \cos^2 2\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

This is sufficient for giving the "solution" to the problem. However it isn't too much ore work to actually get the $B_l$'s. Writing the integral in terms of $x = \cos\theta$, and using

$$\cos^2 2\theta = (2\cos^2\theta - 1)^2 = (2x^2-1)62 = 4x^4 - 4x^2 + 1$$

then

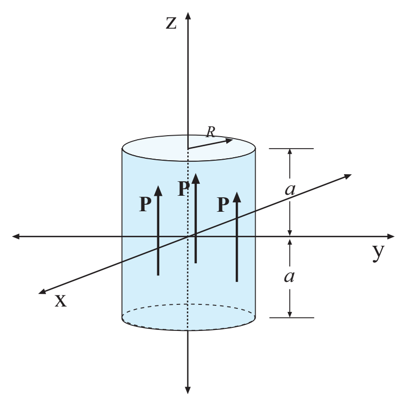
$$B_l = \frac{kR^{l+1}(2l+1)}{2}\int_{-1}^1 (4x^4 - 4x^2 + 1)P_x(x)\, dx \ .$$

One can decompose $4x^4 - 4x^2 + 1$ in terms of the $P_l$'s; it *must* be some linear combination of $P_0$, $P_2$ and $P_4$. Alas, I didn't give $P_4(x)$ on the exam! So we'll leave $B_l$ in this form.

**7.** Explain the difference between a "bound" charge density and a "free" charge density. (Agreed, there is a little bit of arbitrariness in the distinction, but explain why we make it for practical situations involving dielectrics.)

While both "bound" and "free" charge represent *real* accumulations of electric charge, the bound charge can trace its origin to the dielectric material in the problem so it has a special relation to the polarization $\mathbf{P}$ of the material. The free charge has been introduced to the system independently of the dielectric materials.

**8.** A right circular cylinder of dielectric material has a frozen–in polarization (but no free charge on it). The cylinder has radius $R$, length $2a$ and a uniform polarization of magnitude $P$ directed along the axis of the cylinder. The cylinder's axis lies along the $z$ axis and it is centered on the origin.




**a)** What is the distribution of bound charge in this system?

In the cylinder, $\rho_b = -\nabla \cdot \mathbf{P} = 0$ because $\mathbf{P}$ is *uniform*. On the side(s) of the cuylinder $\sigma_b = 0$ because $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ and $\mathbf{P}$ is perpendicular to $\hat{\mathbf{n}}$ there.

On the top surface $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ so $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P$ there. On the bottom surface, $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ so $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = -P$ there.

**b)** At a location $z = b$ (with $b > a$) on the z axis, what is the value of the electric field?

At $z = b$ (with $b > a$) the observation point is a distance $b - a$ from the center of the upper surface (which has charge density $P$) and $b + a$ from the center of the lower surface (which has charge density $-P$). Use the formula for $E$ field from disk of uniform charge (given on exam) for both surfaces to get:

$$E_z = \frac{P}{2\epsilon_0}\left[1 - \frac{(b-a)}{\sqrt{(b-a)^2 + R^2}}\right] - \frac{P}{2\epsilon_0}\left[1 - \frac{(b+a)}{\sqrt{(b+a)^2 + R^2}}\right]$$

One can probably get some interesting cancellations in this if we consider $b \gg a$ but I will leave it at this.

## Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

**Spherical:**

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{4}$$

**Cylindrical:**

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

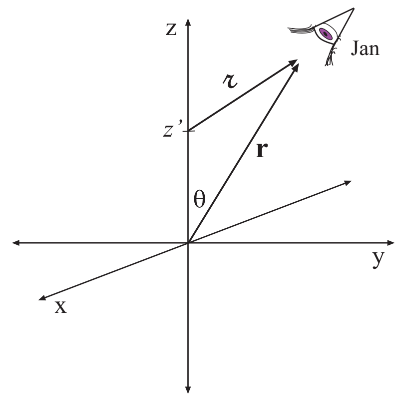
$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

**More Math**

In the figure at the right,




$$\mathscr{r} = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$$

If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3-3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

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**Physics:**

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} E\cdot d\mathbf{l}$$

$$\nabla\times\mathbf{E} = 0 \qquad \mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j \\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

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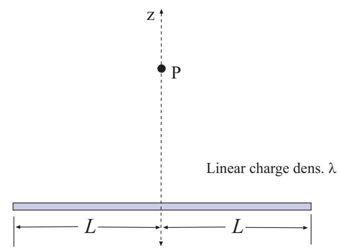
**Specific Results:**

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}$$




$$E_z = \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right]$$