Phys 3820, Fall 2010
Exam #2

1. Under what condition will a trial wave function ψ give an upper bound for the energy of the first excited state?

If the trial wave function is known to be orthogonal to the *exact* ground state (which we can probably assume is non-degenerate) then it will necessarily give an upper bound for the energy of the first excited state(s).

2. a) For the classically-allowed region of the motion a particle, what assumption did the "WKB" approximation make about the form of the wave function?

It assumed that it could be written as a plane--wave with a variable amplitude and phase:

$$\psi(x) \approx A(x)e^{i\phi(x)}$$

where A and ϕ are both real functions.

This *is* an assumption because while a general function can be expressed this way, A ϕ would not need to be real.

b) Under what general condition was this approximation expected to be valid?

It is expected to be valid for regions where the total energy is not close to $V(x)$ so that V changes slowly compared to the local wavelength of the wave function.

c) The assumption is bad near ''classical turning points" (why?) What general (mathematical) procedure is done to extend the approximate solutions to these regions?

A "classical turning point" is a place where $V(x) = E$ so this certainly violates the assumption given in (b). To deal with this, one usually makes the next best approximation, and that is to treat the potential as a linear function near this x --- certainly a good assumption is the range of x is small enough!

This necessitates solving the Schrödinger equation for a linear potential, a rather obscure problem (unlike Phys 2110!) for which the solutions are Airy functions. These have well-known mathematical properties which can be exploited in matching the solutions near the turning points to those of the regular WKB approximation in the other regions.

3. a) In Gamow's model of α decay, what form of potential did Gamow choose to keep an alpha particle trapped?

He assumed that there was a square well of some typical nuclear depth and some radius r_1. Past r_1 the potential is the Coulomb potential felt by a point charge $+2e$ outside a spherical charge $+Ze$.

b) In applying the WKB method to the Gamow model, we actually found a tunneling *probability*. What crude estimate did we use to convert the tunneling probability to a tunneling *rate*?

We assumed that the alpha would rattle back forth classically along the diagonal of a sphere of radius r_1 With the alpha moving at some speed v the collisions with the wall would take place with period $2r_1/v$. When multiplied by the probability to escape, this gave a decay rate.

4. For the two–level system we found the *exact* equations

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a \qquad \text{where} \qquad \omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

Give a brief summary of the strategy used to find approximate solutions for $c_a(t)$ and $c_b(t)$ individually.

The strategy was to begin with the $t = 0$ values, say $c_a(0) = 1$ and $c_b(0) = 0$ (which we can call zeroth-order values) and put these on the right side of the equations written above. Then the two eqns can be solved to get first--order eqns for $c_a(t)$ and $c_b(t)$. Then put these on the right side and solve again to get second-order solutions. Going on this way, we can approach the exact solution (one hopes) by recycling the previous solution.

5. Suppose for our generic two–level system the perturbation takes the form

$$H'(t) = \begin{cases} 0 & t < 0 \\ U & 0 < t < T \\ 0 & t > T \end{cases}$$

and assume that $U_{aa} = U_{bb} = 0$ and $U_{ab} = U^*_{ba} \equiv \alpha$. (And the U's are assumed to be "small" in some sense.) If $c_a(0) = 1$ and $c_b(0) = 0$, find expressions for $c_a(t)$ and $c_b(t)$ (using the second–order approximations).

As usual, set the expressions clearly and get as far as you can with the math.

(To clarify, the U here is meant as an operator,not a constant!)

Use the expression for $c_b^{(2)}(t) = c_b^{(1)}(t)$; since he perturbation is only nonzero for $0 < t < T$, then

$$c_b^{(2)}(t) = -\frac{i}{\hbar}\int_0 H'_{ba} e^{-i\omega_0 t'}\, dt' = -\frac{i}{\hbar}\frac{1}{i\omega_0} U_{ba}[e^{i\omega_0 T} - 1]$$

which can be cleaned up as

$$-\frac{U_{ba}}{\hbar\omega_0} e^{i\omega_0 T/2}[e^{i\omega_0 T/2} - e^{i\omega_0 T/2}] = -\frac{2iU_{ba}}{\hbar\omega_0} e^{i\omega_0 T/2} \sin(\omega_0 T/2)$$

and then the transition probability is

$$|c_b(t)|^2 = \frac{4|U_{ba}|^2}{(\hbar\omega_0)^2} \sin^2(\omega_0 T/2)$$

The expression for $c_a^{(2)}(t)$ is messier (and not as important, probably:

$$c_a^{(2)}(t) = 1 - \frac{i}{\hbar^2}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \left[\int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''}\, dt''\right] dt'$$

This gives

$$
\begin{aligned}
c_a^{(2)}(t) &= 1 - \frac{1}{\hbar^2} U_{ab} U_{ba} \int_0^T e^{-i\omega_0 t'} \int_0^{t'} e^{i\omega_0 t''} \, dt'' \, dt' \\
&= 1 - \frac{|U_{ab}|^2}{i\omega_0 \hbar^2} \int_0^T e^{-i\omega_0 t'} [e^{i\omega_0 t'} - 1] \, dt' = 1 - \frac{|U_{ab}|^2}{i\omega_0 \hbar^2} \int_0^T [1 - e^{-i\omega_0 t'}] \, dt'
\end{aligned}
$$

Note, while the t' integral goes up to T, the t'' integral goes only up to t'.

This becomes

$$
= 1 - \frac{|U_{ab}|^2}{i\omega_0 \hbar^2} \left[T + \frac{1}{i\omega_0} (e^{-i\omega_0 T} - 1) \right]
$$

at which point I would just leave it, as I don't see how to get it into some interesting form.

6. Give concise but *careful* definitions of:

a) Spontaneous emission rate (for a given transition).

Rate at which a system (say, an atom) makes a transition to a state of lower energy in the *absence* of an applied field. (Fundamental reason for this possibility must come from QED, not quantum mechanics.)

b) Induced emission rate (for a given transition).

Rate at which a system (say, an atom) makes a transition to a state of lower energy *due to* the presence of an external electromagnetic field. The rate is proportional to the energy density of the applied field.

c) Selection rule for a particular atomic transition.

A rule (as seen here, based on angular momentum of states) which state whether or not a given transition will occur *to first order in perturbation theory*. While not a rigorous rule which states that a transition *cannot* occur, a selection rule does indicate when the transition will happen very slowly (compared to the allowed ones!)

d) Lifetime of an excited state.

The time it takes for the population of an excited state to decay to $1/e$ of its initial value due to spontaneous decays.

e) Laser.

A physical system where where a population inversion is created, that is there is a highly non-statistical number of atoms in the higher-energy states. This situation serves as an amplifier for radiation of the frequency corresponding to the energy difference between the upper state and a lower state. The radiation that can be produced is very intense and is also coherent (uniform in phase).

7. Calculation of spontaneous emission rate:

a) Give the physical source of the terms of the right hand side of

$$\frac{dN_b}{dt} = -N_b A - N_b B_{ba}\rho(\omega_0) + N_a B_{ab}\rho(\omega_0)$$

First is the rate of spontaneous emission; no dependence on energy density. Second is the rate of induced emission; this is proportional to the energy density of the field. Third is the rate of induced absorption from the lower state, which is proportional to the energy density and comes with a $+$ sign as it populates state b.

b) The next step of the derivation was to set $\frac{dN_b}{dt} = 0$ and make substitutions for $\rho(\omega_0)$ and $\frac{N_a}{N_B}$; both of the formulae depend on the temperature T.

What was the source (basis) of the formula used for $\frac{N_b}{N_a}$?

This is the Boltzmann formula for the ratio of atoms in two states of higher and lower energy, at equilibrium.

c) What was the source (basis) of the formula for $\rho(\omega_0)$?

That was the celebrated formula derived by Planck for "blackbody radiation", once he had quantized the energy of the oscillators making up the solid which was in equilibrium with the radiation field.

8. Show me the numbers: When we found the lifetime of the H stom in the 2s state we found the dipole matrix element of the 210 $\rightarrow$ 100 transition:

$$\wp_z = -e\langle\psi_{210}|z|\psi_{100}\rangle = -ea\frac{256}{243\sqrt{2}}$$

with the x and y matrix elements being zero.

From this calculate the life of H in the 210 state and *show me how the numbers work out*. **Be clear about the units.**

A more elegant treatment of the numbers was given in the solutions. I will use brute force here! We want to evaluate

$$\tau = \frac{1}{A} = \frac{3\pi\epsilon_0\hbar c^3}{\omega_0^3|\wp|^2}$$

The factors are:

$$\epsilon_0 = 8.55 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}$$

$$\hbar = 1.054 \times 10^{-34}\ \mathrm{J\cdot s}$$

$$\omega_0 = \frac{(E_2 - E_1)}{\hbar} = \frac{3}{4}(13.6\ \mathrm{eV})/\hbar = 1.634 \times 10^{-18}\ \mathrm{J}/\hbar = 1.55 \times 10^{16}\ \mathrm{s}^{-1}$$

$$|\wp|^2 = e^2 a^2 (0.5549) = 0.5549(1.602 \times 10^{-19}\ \mathrm{C})^2\,(0.5292 \times 10^{-10}\ \mathrm{m})^2$$

Putting all of this into the formula, the numerical part is

$$1.59 \times 10^{-9}$$

and the units are

$$\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}\cdot\frac{\mathrm{J}\cdot\mathrm{s}(\ \mathrm{m}/\ \mathrm{s})^3}{\mathrm{s}^{-3}\cdot\ \mathrm{C}^2\cdot\mathrm{m}^2}$$

Cancel the C^2, cancel $\mathrm{N}\cdot\mathrm{m}$ with J, etc. Get:

$$\frac{\mathrm{s}^4\cdot\ \mathrm{m}^3/\ \mathrm{s}^3}{\mathrm{m}\cdot\mathrm{m}^2} = \ \mathrm{s}$$

like it should be! So the lifetime is

$$\tau = 1.59\times 10^{-9}\ \mathrm{s}$$

9. According to the selection rules, what are the possible (downward) transitions from the five $(n = 3, l = 2)$ states in the H atom?

The selection rules

$$\Delta m = \pm 1,\ 0 \qquad \Delta l = \pm 1$$

restrict the five $(n = 3, l = 2)$ states to the transitions:

$$l = 2, m = -2 \qquad \Longrightarrow \qquad l = 1, m = -1$$

$$l = 2, m = -1 \qquad \Longrightarrow \qquad l = 1, m = -1 \qquad l = 1, m = 0$$

$$l = 2, m = 0 \qquad \Longrightarrow \qquad l = 1, m = -1 \qquad l = 1, m = 0 \qquad l = 1, m = 1$$

$$l = 2, m = 1 \qquad \Longrightarrow \qquad l = 1, m = 0 \qquad l = 1, m = 1$$

$$l = 2, m = 2 \qquad \Longrightarrow \qquad l = 1, m = 1$$

Note, there are no allowed transitions to the $2s$ or $1s$ states.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_\mathrm{e} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_\mathrm{p} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \mathrm{\tfrac{m}{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty\ \text{Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E + \hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E - \hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_\text{phase} = \frac{\omega}{k} \qquad v_\text{group} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\quad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta\, \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm 2i\phi} \text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \qquad L_\pm = \pm \hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \phi} \right) \qquad L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s\ m{\pm}1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0 \mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{-iE_+ t/\hbar} \\ b e^{-iE_- t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2 \psi - \frac{\hbar^2}{2\mu}\nabla_r^2 \psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2 (3\pi^2 N q)^{5/3}}{10\pi^2 m} V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle \qquad \psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{(E_n^0 - E_m^0)} \psi_m^0 \qquad E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i | H' | j \rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788 \times 10^{-5}\ \text{eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p \cdot \mathbf{S}_e \rangle = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4} \begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \le \langle \psi | H | \psi \rangle \equiv \langle H \rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{1}{\hbar}\int p(x)\, dx} \qquad \int_0^a p(x)\, dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\, dx \qquad \tau = \frac{2r_1}{v} e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar}$$

$$c_b^{(2)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ab}(t') e^{i\omega_0 t'}\, dt' \qquad c_a^{(2)}(t) = 1 - \frac{i}{\hbar^2}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \left[\int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''}\, dt'' \right] dt'$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$\wp \equiv q \langle \psi_b | \mathbf{r} | \psi_a \rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\wp| E_0}{\hbar} \right)^2 \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0 \hbar^2} |\wp|^2 \rho(\omega_0) \qquad A = \frac{\omega^3 |\wp|^2}{3\pi\epsilon_0 \hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1;$ or 0 and $\Delta l = \pm 1$