Notes for Phys 4620

Last revised 05/11/05

These notes are just intended to give an overview of the major equations covered in class.

1 Electrodynamics

At last we consider time–dependent phenomena in electromagnetism. In doing so we find that the electric and magnetic fields are related (through time–dependence). We will arrive at it though the phenomenon of electromagnetic induction and "Faraday's law" but to discuss this we need to learn more about how current flows in a wire. Yup, *real* currents.

1.1 Electromotive Force

The current density within a conductor is empirically proportional to the electric field within the conductor:

$$\mathbf{J} = \sigma \mathbf{E} \tag{1}$$

which is the general form of "Ohm's law". The proportionality factor σ is the conductivity of the material; its reciprocal $\rho = 1/\sigma$ is the resistivity of the material. Note, under conditions where charge is *flowing* the electric field can be non-zero within a (imperfect) conductor.

When we discuss the total current I flowing from one electrode to another we find that it is proportional to the potential difference V between them:

$$V = IR \tag{2}$$

For steady currents the charge density within the conductor is zero; excess charge must again lie on the surface.

A model for the motion of electrons which relates their drift velocity to the electric force which they "feel" (and hence the current density to the electric field in the material) gives

$$\mathbf{J} = \frac{nf\lambda q^2}{2mv_{\mathrm{thermal}}}\mathbf{E} \tag{3}$$

where n is the number density of molecules, f is the number of free electrons per molecule, λ is the mean free path for the electrons, q is the electron charge, m is their mass and v_{thermal} is their thermal speed.

The power delivered to a resistive material by a current flowing through a potential difference V is

$$P = VI = I^2R \tag{4}$$

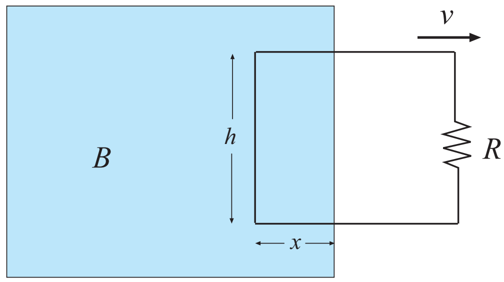

Figure 1: Simple circuit moves through a magnetic field; an emf is set up by the magnetic force on the moving charges.

Electromotive force is a name given to the line integral of the force $\mathbf{f}$ which drives the current around a circuit:

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \tag{5}$$

For *electrostatic* conditions where a battery supplies the force which drives the current around a circuit, the potential difference between the terminals of the battery is also equal to the emf of the circuit:

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} \tag{6}$$

$\mathcal{E}$ can be viewed as the work done per unit charge by the source.

1.2 Motional emf

A generator exploits the phenomenon of motional emf which can be understood using the ideas *already studied* in the course since it only involves steady fields and currents (though the circuit is in motion).

A very simple generator is shown in Fig. 1. Here there is a uniform magnetic field and a rectangular loop of wire with resistance R moves through it with speed v, as shown. The emf set up in the wire comes from the magnetic force on the charges in the wire and is given by

$$\mathcal{E} = \oint \mathbf{f}_{\rm mag} \cdot d\mathbf{l} = vBh$$

and it is related to the current in the wire by $\mathcal{E} = IR$.

The work done on the charges comes from the force which is dragging the loop.

The emf can expressed in a nice way using the magnetic flux through the loop:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \tag{7}$$

Then one can show:

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{8}$$

The flux rule 8 is more general than our basic example; it is true for wires of arbitrary shape moving in arbitrary directions in non-uniform magnetic fields. The wire can even change its shape as it moves.

We can have cases of motional emf where we can't use the flux rule. The generator shown in example 7.4 ("homopolar generator") of the text is a good example. In general a conductor which moves through a B field has eddy currents set up in it which dissipate energy.

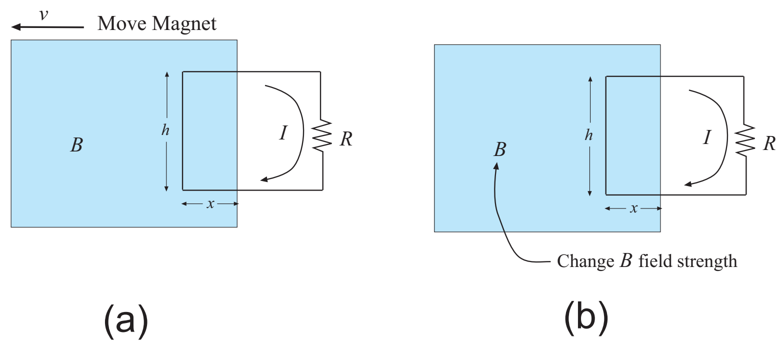

Figure 2: Two other thought experiments: (a) Loop is stationary, magnet moves. (b) Both are stationary but magnetic field changes strength.

1.3 Electromagnetic Induction

The motional emf phenomenon discussed in the last section did *not* require any new field equations other than the ones given so far in the course. But two variants of the loop experiment in the last section *do* require a new equation. These are shown in Fig. 2. In both of these a current flows in the wire but the since the loop is not moving its charges can't experience a magnetic force. If it's a force from an electric field, where does E come from? Whatever is driving the current, the *same* flux rule holds in this case: $\mathcal{E} = -\frac{d\Phi}{dt}$.

What is needed is a new law of electricity for the case of magnetic fields which change in time. It is:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9}$$

which does give the flux rule and also reduces to $\nabla \times \mathbf{E} = 0$ for the case of constant magnetic fields.

Predicting the *direction* of the induced current in a loop is made easier with Lenz's law, which states:

The induced current will flow in such a direction so as to *cancel* the original change in flux.

1.4 Inductance

Now consider two loops of wire, 1 and 2. The current in loop 1 will give a magnetic field in the vicinity of loop and hence a flux through loop 2. The magnetic field produced by 1 is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l} \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

and the flux through 2 (due to 1) is

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2$$

From these one can show that

$$\Phi_2 = M_{21} I_1 \qquad \text{where} \qquad M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\imath} \tag{10}$$

so that M_{21} is a factor which just depends on geometry. We note that $M_{21} = M_{12}$ so that we can just use the symbol M for the mutual inductance of the two loops.

The the emf induced in 2 due to changes in current in loop 1 is

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt} \tag{11}$$

It is also true that changes in current in a loop induce an emf in the *same* loop:

$$\mathcal{E} = -L\frac{dI}{dt} \tag{12}$$

Regarding this formula, we often say that a changing current in a circuit generates a back–emf in the circuit.

1.5 Energy in Magnetic Fields

When we increase the current in a circuit we are doing work against the back emf which is induced. If we start with zero current in a circuit with self-inductance L and over time build it up to a value I, the work done is

$$W = \tfrac{1}{2}LI^2 \tag{13}$$

One can show that more generally the work done in setting up a volume current distribution $\mathbf{J}$ is

$$W = \tfrac{1}{2}\int_{\mathcal{V}} (\mathbf{A}\cdot\mathbf{J})\, d\tau \tag{14}$$

This expression can be rewritten as an integral over all space involving only the B field:

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau \tag{15}$$

1.6 Maxwell's Equations

The equations governing the E and B fields *before Maxwell* were thought to be:

$$\begin{array}{ll} \text{(i)} & \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \text{(ii)} & \nabla\cdot\mathbf{B} = 0 \\ \text{(iii)} & \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \text{(iv)} & \nabla\times\mathbf{B} = \mu_0\mathbf{J} \end{array}$$

As it turns out, these equations are *inconsistent*. They may be fixed up by replacing the last of these by

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

The correction gives the important result that a changing electric field will induce a magnetic field.

The Maxwell equations are:

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{16}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{17}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{18}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{19}$$

Together with the force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

they give the entire content of classical physics, at least the part dealing with electrical forces. (Newton's law of gravity is needed for the gravitational force.)

1.7 Magnetic Charge

There is a symmetry about the Maxwell equations that is upset by the fact that there is no magnetic charge (nor a current of moving magnetic charges). If there were, the Maxwell equations would be

$$\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho_e & \text{(iii)} & \nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \dfrac{\partial \mathbf{B}}{\partial t} \\ \text{(ii)} & \nabla \cdot \mathbf{B} = \mu_0 \rho_m & \text{(iv)} & \nabla \times \mathbf{B} = +\mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{array}$$

and both kinds of charges would be conserved:

$$\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t} \qquad \nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}$$

But no one has yet found any magnetic charge, and they've looked! It turns out that the existence of magnetic charge would explain why charge is quantized.

1.8 Maxwell's Equations in Matter

The Maxwell equations 16 –19 also hold in matter but there we want to make a distinction between bound and free charges and currents and in doing so we make use of the new fields **D** and **H**.

We now have to consider time derivatives of the induced charges and currents. To deal with this we need to introduce the polarization current $\mathbf{J}_p$:

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \tag{20}$$

so that in the Maxwell equations we will substitute for ρ:

$$\rho = \rho_f + \rho_b = \rho_f = -\nabla \cdot \mathbf{P}$$

and for $\mathbf{J}$:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

With these substitutions and some algebra we get

$$\nabla \cdot \mathbf{D} = \rho_f \tag{21}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{22}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{23}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{24}$$

These equations are every bit as true as 16 –19; they only use a division of the charge and current into bound and free parts.

1.9 Boundary Conditions

Finally, we recall the boundary conditions on the fields, already derived (?!?):

$$\begin{aligned} D_1^{\perp} - D_2^{\perp} &= \sigma_f \qquad & B_1^{\perp} - B_2^{\perp} = 0 \\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} &= 0 \qquad & \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \end{aligned}$$

2 Conservation Laws: Charge, Energy and Momentum

2.1 Introduction

This section deals with the role of energy, momentum and angular momentum in EM. These quantities are conserved but we need to know how to find the energy and momentum associated with the electric and magnetic fields.

One conservation law is one we've already seen is the conservation of charge and it is worth reviewing. Within a given volume $\mathcal{V}$ the charge contained is

$$\mathcal{Q}(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau \tag{25}$$

and the total current flowing *out* through the boundary of $\mathcal{V}$ is

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} \tag{26}$$

Electric charge is conserved *locally*; this implies that $Q = -I$, or:

$$\frac{d\mathcal{Q}}{dt} = -\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} \tag{27}$$

Using the divergence theorem we get

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau \tag{28}$$

and since this is true for *any* volume we arrive (again) at the continuity equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \tag{29}$$

The continuity equation is not independent of the Maxwell equations; it is a consequence of them. (Though in the treatment of Chapter 7, continuity was *assumed* and then the "displacement current" term was discovered.

2.2 Poynting's Theorem

Consider a volume $\mathcal{V}$ and find the rate at which the electromagnetic fields do work on the charges contained in that volume. Only the E field does work, and we first find:

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau$$

but using the Maxwell equations and some vector identities one can express the rhs completely in terms of the fields, and then one gets:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \tag{30}$$

which is Poynting's theorem, the work–energy theorem as formulated in EM.

An important new quantity is the Poynting vector,

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \tag{31}$$

whose meaning is that $\mathbf{S}\cdot d\mathbf{a}$ is the energy per time crossing the area element $d\mathbf{a}$.

If we define the electromagnetic energy density as

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \tag{32}$$

and u_{emch} as the energy density of the (massive) mechanical particles, then we can write a local version of the Poynting theorem as

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla\cdot\mathbf{S} \tag{33}$$

2.3 Momentum and Electromagnetism

We can get a theorem similar (in spirit, at least) to the Poynting theorem if we consider the net force acting on the particles contained within a volume $\mathcal{V}$. This is the same as the rate of change of the total momentum of these particles. We start with

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v}\times\mathbf{B})\rho\, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\, d\tau$$

and then use the Maxwell equations and lots of vector identities to rewrite this in terms of the fields alone. The results is very hairy so some definitions are in order to make the result comprehensible.

We introduce the Maxwell stress tensor,

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{34}$$

which is a beastie with two indices and will be denoted with a double arrow: $\overleftrightarrow{\mathbf{T}}$.

When we dot a vector with $\overleftrightarrow{\mathbf{T}}$ we get a *vector* out. Thus we write:

$$(\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}})_j = \sum_{i=x,y,z} a_iT_{ij} \qquad\qquad (\nabla\cdot\overleftrightarrow{\mathbf{T}})_j = \sum_{i=x,y,z} \frac{\partial}{\partial x_i}T_{ij}$$

With this definition the total force on the charges in $\mathcal{V}$ can be expressed as

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} \tag{35}$$

Here, the first term on the rhs represents the rate of loss of the momentum contained in fields inside $\mathcal{V}$,

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0\int_{\mathcal{V}} \mathbf{S}\, d\tau$$

The second term represents the flow of momentum across the surface of that volume.

We can get a differential form of Eq. 35 with the definition of the electromagnetic momentum density:

$$\wp_{\mathrm{em}} = \mu_0 \epsilon_0 \mathbf{S} \tag{36}$$

so that if $\wp_{\mathrm{mech}}$ is the density of the momentum of the (massive) particles, we can write

$$\frac{\partial}{\partial t}(\wp_{\mathrm{mech}} + \wp_{\mathrm{em}}) = \nabla \cdot \overleftrightarrow{\mathbf{T}} \tag{37}$$

and this gives us an interpretation of $\overleftrightarrow{\mathbf{T}}$: $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction per unit area, per unit time (whew!). We also note that $\mathbf{S}$ plays a dual role: It represents a *flow* of em *energy* but is also related to the linear momentum *density*.

2.4 Electromagnetic Angular Momentum

Last but not least we note that we could carry out a similar discussion for the angular momentum of mechanical/em system. One finds that the density of angular momentum contained in the em field is

$$\ell_{\mathrm{em}} = \mathbf{r} \times \wp_{\mathrm{em}} = \epsilon_0 [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] \tag{38}$$

3 Electromagnetic Waves

3.1 Review of Waves

A wave is a travelling disturbance that maintains a fixed shape and travels at a constant velocity.

Except when it isn't. Through absorption, the size of a wave can diminish; if the medium is dispersive that parts of the wave having different frequencies will travel at different speeds and the wave will distort; and finally in the case of electromagnetic waves, the waves don't travel through a physical medium at all, they just... travel! Aside from that, it's a fine definition for a wave.

A wave function of the ideal type to be considered has the form

$$f(z,t) = g(z \pm vt) \tag{39}$$

that is, the variables *only* appear as the combination $z \pm vt$.

A stretched string is perhaps the clearest example of how the wave equation arises. If $f(z,t)$ is the displacement of the bit of the string at z away from the equilibrium position at time t, one can show that $f(z,t)$ satisfies the equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{40}$$

where

$$v = \sqrt{\frac{T}{\mu}}$$

T being the string tension and μ being the linear mass density of the string. Eq. 40 is known as the (classical) wave equation and it admits all solutions of the form

$$f(z,t) = g(z \pm vt)$$

(and also linear combinations of solutions of this type).

A harmonic (or sinusoidal) wave has the form

$$f(z,t) = A\cos[k(z - vt) + \delta] \tag{41}$$

A is the amplitude of the wave; δ is the phase constant; k is the wave number and is related to the wavelength λ by

$$\lambda = \frac{2\pi}{k}$$

We will (usually) use complex notation for waves; for the simple one-dimensional wave we would write

$$f(z,t) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}] \tag{42}$$

and do most of the mathematical operations on the complex wave function

$$\tilde{f}(z,t) = Ae^{i(kz-\omega t+\delta)} \tag{43}$$

In the end, the actual wave function will be the real part of $\tilde{f}$.

Understanding harmonic waves helps to understand *all* waves because any wave can be built up as a linear combination of sinusoidal waves by:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk \tag{44}$$

where using Fourier theory $\tilde{A}(k)$ can be obtained in terms of the initial conditions $f(z,0)$ and $\dot{f}(z,0)$.

3.2 Simple Example of Wave With Boundary Conditions

The most basic boundary value problem with waves has a harmonic waves has two different one-dimensional media with different wave speeds v_1 (for $z < 0$, region 1) and v_2 (for $z > 0$, region 2). The incident (I) wave enters from the left, going in the $+z$ direction with wave number k_1.

There will be a reflected wave (R) travelling to the right in region 1 with the same wave number k_1 and a transmitted wave (T) travelling to the right in region 2 with wave number k_2. The frequencies of oscillation of all parts of the system are the *same* (which is why the reflected wave has the same wave number) and that gives us:

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$$

The wave functions in the two regions are:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z-\omega t)} + \tilde{A}_R e^{i(-k_1 z-\omega t)}\,, & \text{for } z < 0 \\ \tilde{A}_T e^{i(k_2 z-\omega t)} & \text{for } z > 0 \end{cases} \tag{45}$$

The usual statement of the problem is: Given the amplitude and frequency of the incoming wave $\tilde{A}_I$ (and the wave speeds in the regions), what are the reflected and transmitted amplitudes? We need two equations and these conditions are the continuity of the (total) wave functions in the two regions and the continuity of the x–derivative of the wave function:

$$f(0^-,t) = f(0^+,t) \qquad \left.\frac{\partial f}{\partial z}\right|_{0^+} = \left.\frac{\partial f}{\partial z}\right|_{0^-} \tag{46}$$

For the one-dimensional two-region problem, the solution is:

$$A_R = \left(\frac{v_1 - v_2}{v_2 + v_1}\right) A_i \quad \text{and} \quad A_T = \left(\frac{2v_2}{v_2 + v_1}\right) A_I \tag{47}$$

$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{y}} \tag{48}$$

3.3 EM Waves in Vacuum

The electric and magnetic fields in vacuum also satisfy a wave equation of the form of Eq. 40.

If we start with the Maxwell equations in vacuum:

$$\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = 0 & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0 & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{array}$$

and apply the curl operator twice to $\mathbf{E}$ (or $\mathbf{B}$) we can substitute time derivatives of these fields and arrive at separate wave equations for the E and B fields:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}\,, \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{49}$$

We note that each of these contains *three* wave equations, i.e. one for each component of $\mathbf{E}$ and $\mathbf{B}$.

The wave equations 49 give the speed of these waves; it is

$$v = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8\ \tfrac{\mathrm{m}}{\mathrm{s}} \tag{50}$$

so that the speed of light can be deduced from the measured quantities ϵ_0 and μ_0.

As the basis for our study of EM waves we consider waves of a definite frequency which travel in the z direction and have no x or y dependence. These are monochromatic plane waves and they have the general form:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\,, \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} \tag{51}$$

where the amplitudes $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are complex-valued constant vectors.

When these waves are put into the Maxwell equations we find some restrictions on the (vector) amplitudes. First,

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

so that the directions of $\mathbf{E}$ and $\mathbf{B}$ are always perpendicular to the direction of propagation; EM waves are transverse. Secondly, the electric and magnetic fields are related via

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0)$$

More generally, we can consider a plane wave propagating in a general direction by making k into a vector; for a wave propagating in the direction of $\mathbf{k}$, polarized along direction $\hat{\mathbf{n}}$, the monochromatic plane waves are

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r},t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}} & (52) \\ \tilde{\mathbf{B}}(\mathbf{r},t) &= \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}} \times \tilde{\mathbf{E}} & (53) \end{aligned}$$

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = 0$.

The actual E and B fields for the monochromatic plane wave are

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 \cos(\mathbf{k} \cdot \mathbf{r} \omega t + \delta)\hat{\mathbf{n}} && (54) \\ \mathbf{B}(\mathbf{r}, t) &= \frac{1}{c} E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) && (55) \end{aligned}$$

3.4 Energy and momentum in EM Waves

The energy density u, Poynting vector $\mathbf{S}$ and momentum density $\wp$ for a monochromatic plane wave are all time dependent; the only sensible thing to consider is the time *average* of these quantities since the wave frequencies we consider are so large. One uses the fact that the average of a cosine-squared factor is $\frac{1}{2}$ and for a wave propagating in the z direction, one finds:

$$\langle u \rangle = \tfrac{1}{2}\epsilon_0 E_0^2 \qquad \langle \mathbf{S} \rangle = \tfrac{1}{2} c\epsilon_0 E_0^2 \hat{\mathbf{z}} \qquad \langle \wp \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \hat{\mathbf{z}} \tag{56}$$

The magnitude of the time-averaged Poynting vector is the intensity of the plane wave:

$$I \equiv \langle S \rangle = \tfrac{1}{2} c\epsilon_0 E_0^2 \tag{57}$$

I is the average power per unit area carried by the wave.

Since radiation carries momentum to a surface it exerts a pressure on the surface. For a perfect absorber the radiation pressure (force per unit area) is

$$I = \tfrac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c} \tag{58}$$

3.5 EM Waves in a Linear Medium

In a region with no free charge or no free current the Maxwell equations are

$$\begin{aligned} &\text{(i)} \quad \nabla \cdot \mathbf{D} = 0 && \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ &\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0 && \text{(iv)} \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

and if in addition it is a linear homogeneous medium then

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

with ϵ and μ constant. Then we arrive at wave equations for $\mathbf{E}$ and $\mathbf{B}$ just as before but now the wave speed is given by

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \tag{59}$$

where

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{60}$$

is the index of refraction of the material. Often, μ is close to μ_0 so that $n \approx \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$.

All the previous results for vacuum carry over with ϵ and μ replacing ϵ_0 and μ_0.

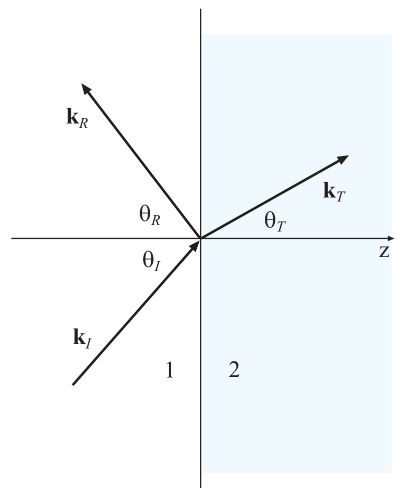

Figure 3: EM wave with wave vector $\mathbf{k}_I$ incident on interface between linear media 1 and 2.

3.6 Reflection and Transmission at an Interface (Linear Media)

The problem of oblique incidence of an EM wave at an interface of two linear media is rather technical, but it can be understood from the wave solutions and the boundary conditions and it is an *important* problem because the rules of optics come out!

The basic geometry of the plane waves is shown in Fig. 3. The incident wave has wave vector $\mathbf{k}_I$. There is a reflected wave with wave vector $\mathbf{k}_R$ and a transmitted wave with wave vector $\mathbf{k}_T$. The latter two have directions measured from the normal given by θ_R and θ_T.

In the two media the speed of light will be v_1 and v_2. In each medium the *frequency* of the wave oscillation will be the same but the wavelengths and wave numbers will differ:

$$\omega = k_I v_1 = k_R v_1 = k_T v_2$$

The incident wave is given by

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_0 e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_I}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$$

with similar expressions for the reflected and transmitted waves. We don't yet specify the polarization of the incoming wave.

The generic structure of the boundary conditions guarantees that the phases of the three waves are the same everywhere and that implies that the three wave vectors lie in the same plane; also,

$$\theta_I = \theta_R \ , \tag{61}$$

known as the law of reflection, and:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \ , \tag{62}$$

which is known as Snell's law.

To get more out of the boundary conditions we need to specify the polarization of the incoming wave. Griffiths explicitly considers only the case where the polarization (direction of the $\mathbf{E}$ field) is in the plane of the page, as shown in Fig. 4. Then using the boundary condition equations on the amplitudes (the phase information has already been used and cancels out) one can solve for

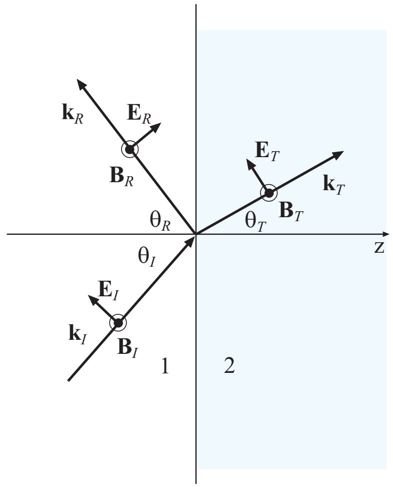

Figure 4: EM wave incident on interface is polarized in the plane of the page.

the reflected and transmitted amplitudes in terms of the amplitude of the incoming wave. After a lot of work one arrives at Fresnel's equations,

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0I} \ , \qquad \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{0I} \ , \tag{63}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \qquad \text{and} \qquad \beta \equiv \frac{\mu_1 n_2}{\mu_2 n_1}$$

[One can also get a pair of Fresnel equation for the case where the incoming wave is polarized perpendicular to the plane of the page; Griffiths leaves that as a (hard) exercise.]

For the in–the–plane polarization case, there is some intermediate angle θ_B where there is no reflected wave, i.e. it has zero amplitude. This occurs where $\alpha = \beta$, which gives:

$$\sin^2\theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2} \qquad \text{or} \qquad \tan\theta_B \approx \frac{n_2}{n_1} \tag{64}$$

where the second approximate condition is for the typical case where $\mu_1 \approx \mu_2$

3.7 EM Waves in Conductors

In a conductor there is a free current $\mathbf{J}_f$ that will enter into the Maxwell equations; if the conductor follows Ohm's law then we can use $\mathbf{J}_f = \sigma\mathbf{E}$. There is also a free charge ρ_f but one can show that in a good conductor the free charge density quickly dissipates and we can take it to be zero.

Combining the equations similar to the vacuum case, one gets *modified* wave equations

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t} \ , \qquad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t} \tag{65}$$

which now contain a term with a *single* time derivative. The equations still admit plane-wave type solutions:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)} \ , \qquad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z - \omega t)} \tag{66}$$

but now the wave number $\tilde{k}$ is complex; we will use $\tilde{k} = k + i\kappa$. Here, k determines the wavelength of wave but κ gives the length over which the wave is attenuated.

We can solve for k and κ and we get

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)}+1\right]^{1/2} \qquad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)}-1\right]^{1/2} \tag{67}$$

the distance that it takes the wave to attenuate in amplitude by $1/e$ is the skin depth d,

$$d = \frac{1}{k}$$

In a conductor the E and B fields are no longer in phase. The B field lags the E field by an amount ϕ,

$$\delta_B - \delta_E = \phi \qquad \text{where} \qquad \phi = \tan^{-1}(\kappa/k)$$

3.8 Frequency Dependence of Permittivity

The discussions so far has dealt with monochromatic plane waves where are idealizations; a real wave is a wave *packet* of finite length. Such a wave can be treated as a sum of harmonic waves (with different wavelengths) by Fourier analysis. But now we need to realize that the speed light in a medium can depend on the frequency of the wave. If there is such a dependence then we say we are dealing with a dispersive medium.

In such a medium a travelling wave will *not* retain its shape. In this situation one must be very careful about what we mean by wave speed.

For a harmonic wave with frequency ω and wavenumber k, the wave velocity is given by

$$v = \frac{\omega}{k}$$

a number which in some cases can be bigger than c! For a wave packet one can show that the "envelope" travels at a speed

$$v_g = \frac{d\omega}{dk}$$

which gives the speed of the information and energy flow, and generally v_g is smaller than c.

One can make a *very* simple model for the way that the interaction of an EM wave with matter can give rise to a dispersion relation. The model has an electron oscillating in one dimension on a spring with natural frequency ω_0 and a damping force proportional to the velocity with damping constant γ. The frequency of the EM wave is ω.

After arriving at an expression for the oscillating dipole for the system, we build up a macroscopic system by assuming that for each molecule in the substance there are f_j electrons for which the natural frequency is ω_j and the damping constant is γ_j; the number density of molecules is N. When we do this, we get complex–valued polarization for the substance:

$$\tilde{\mathbf{P}} = \frac{nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}} \tag{68}$$

which gives a complex dielectric constant.

The plane–wave type solutions are actually *attenuated* waves of the form

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{69}$$

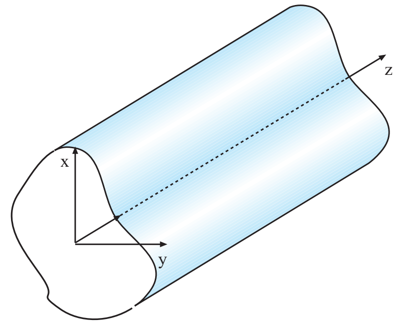

Figure 5: Waveguide extends along the z axis.

or, a plane wave where the wave number $\tilde{k}$ is complex: $\tilde{k} = k_i \kappa$. An approximate expression for $\tilde{k}$ is

$$\tilde{k} \approx \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right] \tag{70}$$

The simple model explains n generally rises with frequency except in the vicinity of a natural frequency ω_j where it can drop sharply. Away from resonances, where damping can be ignored the model gives a simple formula for n,

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2} \tag{71}$$

3.9 Guided Waves

Lastly we consider waves which are confined, in particular by conducting pipe whose cross section has an arbitrary shape, as shown in Fig. 5. (We'll assume the pipe extends along the z axis.) The relevant boundary conditions for the inner wall are:

$$\mathbf{E}^{\parallel} = 0 \qquad\qquad B^{\perp} = 0$$

Again we consider plane-wave type solutions but now they have the form

$$\begin{aligned}\tilde{\mathbf{E}}(x, y, z, t) &= \tilde{\mathbf{E}}_0(x, y)e^{i(kz - \omega t)} \\ \tilde{\mathbf{B}}(x, y, z, t) &= \tilde{\mathbf{B}}_0(x, y)e^{i(\tilde{k}z - \omega t)}\end{aligned}$$

where the coefficient of the (z, t)-dependent wiggly part has a dependence on x and y (the cross-section coordinates). We put these into the Maxwell equations with the boundary conditions and see how this restricts the functions $\tilde{\mathbf{E}}_0(x, y)$ and $\tilde{\mathbf{B}}_0(x, y)$.

With

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}} \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

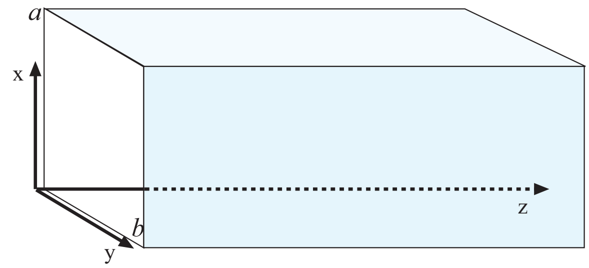

Figure 6: Rectangular waveguide which extends along the z axis. The sides are a (along x) and b (along y), with $a > b$.

we get equations for E_x, E_y, B_x and B_y:

$$
\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \\
E_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right) \\
B_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \\
B_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)
\end{aligned}
$$

so that it suffices to specify the longitudinal components E_z and B_z, since these equations will then give the other components. The longitudinal components satisfy the equations

$$
\begin{aligned}
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z &= 0 \\
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z &= 0
\end{aligned}
$$

We can solve for the case where $E_z = 0$, and such a solution is a transverse electric (TE) wave. Likewise, the case where $B_z = 0$ gives a solution which is a transverse magnetic (TM) wave. The case where both E_z and B_z are zero is called a TEM wave, but this cannot occur for a hollow waveguide.

For a rectangular waveguide whose cross-section has sides a (along x) and b (along y), with $a \geq b$, as shown in Fig. 6, we solve for the TE modes; a solution using separation of variables does the trick. If we try

$$B_z(x, y) = X(x)Y(y)$$

applying the boundary conditions gives

$$B_z(x, y) = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$$

where

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

and n and m are indices for the particular mode (solution), also called the TE_{mn} mode.

For mode mn the frequency must be greater than the cutoff frequency,

$$\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$$

Interestingly enough, the wave velocities for the various modes are greater than c:

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}$$

but the energy is transported at the group velocity, which is

$$v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$$

which is $< c$.

4 Potentials and Fields

In this section we will finally find the fields due to time-dependent charge densities and currents, *without* the fudges (quasi-static approximation) used earlier. To find out what happens in the general case, we will have to make serious use of the scalar and vector potentials (V and $\mathbf{A}$). Note, in introducing time–dependence (and obtaining the full set of Maxwell equations) we actually ignored what happened to the potentials, so that is our first order of business!

4.1 Scalar and Vector Potentials

Back in electrostatics and magnetostatics we had

$$\mathbf{E} = -\nabla V \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

but in electrodynamics the first of these is no longer true; we only wrote it down because $\mathbf{E}$ had zero curl, but that is no longer true. It *is* true that $\mathbf{B}$ has zero divergence, so the second of these still holds: Even with time dependence we still have a vector potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$.

Faraday's law tells us that it is the quantity $\mathbf{E} + \partial\mathbf{A}/\partial t$ which has zero curl (not $\mathbf{E}$) so that it is *this* quantity that we must set equal to the (negative) gradient of some scalar function V. This gives V a new meaning and the new relation between $\mathbf{E}$ an the potentials is

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{72}$$

One can write down a couple equations connecting V and $\mathbf{A}$ which follow from the Maxwell equations, the simpler of which is

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho$$

but they can be simplified with a particular *choice* for V and $\mathbf{A}$. The freedom to make such a choice is now discussed!

4.2 Gauge Transformations

The potentials V and $\mathbf{A}$ are *not uniquely determined* by the sources. There is always of choice of these functions of space and time which will give the same fields $\mathbf{E}$ and $\mathbf{B}$.

If we have potentials V and $\mathbf{A}$, it is always possible to find new potentials V' and $\mathbf{A}'$ which will also work by the formulae

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial \lambda}{\partial t} \tag{73}$$

for any function λ of space and time. This is called a gauge transformation.

There are two important conditions one can impose on V and $\mathbf{A}$ so that the equation involving the potentials are simplified.

The first is a condition called the Coulomb gauge (also transverse or radiation gauge). Here, the condition we pick is

$$\nabla \cdot \mathbf{A} = 0 \tag{74}$$

In this gauge the scalar potential is given by

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}} \, d\tau' \tag{75}$$

which looks odd because it has V being determined by the *present* value of the charge density with no time for propagation included! However before we find the *fields* we have to include $\mathbf{A}$ and the $\mathbf{A}$ field will take care of the propagation time.

The second condition is called the Lorentz gauge. Here the condition is

$$\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \tag{76}$$

With this choice, the messy equations for V and $\mathbf{A}$ have symmetrical look:

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho \tag{77}$$

If we introduce the d'Alembertian operator

$$\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \equiv \Box^2$$

then we get simpler equations for the potentials:

$$\Box^2 V = -\frac{1}{\epsilon_0} \rho \qquad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{78}$$

From now on in the text we will assume that the Lorentz gauge condition holds, unless otherwise stated!

4.3 The Retarded Potentials

We recall that in the case of static charges and currents we had solutions for the potentials given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \tag{79}$$

but what do for the time-dependent case where we have to figure in the time for "news" to travel from the source?

If we consider a source point at $\mathbf{r}'$ the time it takes news to travel from that point to the observation point $\mathbf{r}$ is $\mathscr{r}/c$ so the news *left* the point $\mathbf{r}'$ at the retarded time,

$$t_r \equiv t - \frac{\mathscr{r}}{c} \tag{80}$$

A reasonable guess as to generalizing 79 for the time-dependent case would be to include t_r in the argument of the source function inside the integral. That would give :

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\boldsymbol{\mathscr{r}}}\, d\tau' \qquad\qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\boldsymbol{\mathscr{r}}}\, d\tau' \tag{81}$$

where we mean that $\rho(\mathbf{r}', t_r)$ is the charge density at the point $\mathbf{r}'$ at the time t_r. The potentials in 81 (which *are* the correct ones!) are called the "retarded potentials".

Some comments: In electrostatics we also had expressions for the fields $\mathbf{E}$ and $\mathbf{B}$ as integrals over the sources (Coulomb's law and the Biot–Savart law). But taking *those* integrals and adding t_r to the argument would *not* be correct because those fields would not satisfy the Maxwell equations.

Secondly, one can show that the potentials in 81 *do* satisfy the Maxwell equations and also the Lorentz condition. But the proof takes some care because of the complicated nature of the integrals. The argument t_r contains $\mathbf{r}$ and $\mathbf{r}'$ so taking the space derivatives is tricky. Griffiths shows that the $V(\mathbf{r}, t)$ given in 81 does satisfy the condition on V in 77. The proof for $\mathbf{A}$ would be similar, and he leaves it for *you* to show that they satisfy the Lorentz condition!

4.4 The *Right* Equations for E and B

Starting from the retarded potentials we can get the fields, but the result is not so simple. It is:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot\rho(\mathbf{r}', t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\mathscr{r}}\right] d\tau' &(82)\\
\mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}}\right] \times \hat{\boldsymbol{\mathscr{r}}}\, d\tau' &(83)
\end{aligned}$$

. . . two equations which are noteworthy for not having been written down explicitly in the literature until the 60's.

4.5 The Liénard–Wiechert Potentials

We close the chapter by using the formulae for the retard potentials to get the potentials from a moving point charge. One might think that a moving point charge should be very simple and indeed we *will* arrive at a definite formula for the potentials but the *derivation* of these formulae is tricky because of the singular nature of the charge distribution.

The first thing to specify is the (time–dependent) location of the charge q. It is given by the function $\mathbf{w}(t)$. At a time t we are getting the "news" of the charge's location at some earlier time t_r. (We are assured that there is only *one* such time, though one needs to think about that!) At that time the charge was at $\mathbf{w}(t_r)$ so that the time of travel of the news was $|\mathbf{r} - \mathbf{w}(t_r)|/c$. But the time of travel is the same as $t - t_r$, so that we have

$$t - t_r = |\mathbf{r} - \mathbf{w}(t_r)|/c \tag{84}$$

which generally is some equation for t_r which can be solved. (Note that this is little bit different from the previous usage of t_r where we were *given* a point $\mathbf{r}'$ and then we calculated $t_r = t - \mathscr{r}/c$ from it.)

We then put things into the formula

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\boldsymbol{\mathscr{r}}}\, d\tau'$$

which is not as easy as it looks! A careless evaluation of the integral just gives $V = q/(4\pi\epsilon_0 \mathscr{r})$ (the static Coulomb potential but evaluated at the retarded position) but that isn't correct; there are two ways to see why not.

The first way is to set up the charge density correctly with a delta function and do the integral carefully. The steps will be given in a separate handout on the web page; an additional factor arises from the complicated argument of the delta function.

The second way to see it is to consider the point charge q as the limiting case of a continuous charge distribution and then observing that we get an additional factor from the geometry of points over which the integral is done. Of course, the additional factor is the same in both cases. The upshot is that the potential we get is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \tag{85}$$

We also find (using $\mathbf{J}(\mathbf{r}', t_r) = \rho(\mathbf{r}', t_r)\mathbf{v}(t_r)$),

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t) \tag{86}$$

Equations 85 and 86 are called the Liénard–Wiechert potentials for a moving point charge. It is worth pointing out that although there is a correction to the naive delayed potentials the correction factor comes from geometry and *not* from relativity, and in any case the only element put into the general formulae for the delayed potentials was the idea of a *delay time* for the "news". (One *can* say this is really one manifestation of relativity, but that's a matter of taste)

4.6 Fields of a Moving Point Charge

We're not done yet. We would like to get the E and B fields from the moving charge considered in the last section.

First, we want to find $\mathbf{E}$ from the V and $\mathbf{A}$ in 85 and 86. This involves taking a gradient of V and a time derivative of $\mathbf{A}$, but from the complicated dependence of the variables on the retarded time these operation require a lot of care and Griffiths exercises this care over 3 full pages of the book! In the end, the result is

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \tag{87}$$

where the vector $\mathbf{u}$ is defined as $\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$ and $\mathbf{r}'$, $\mathbf{v}$ and $\mathbf{a}$ are the position, velocity and acceleration of the charge at the retarded time.

The two terms in 87

To get the B field, (carefully) take the curl of $\mathbf{A}$. The result is

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r}, t) \tag{88}$$

To finish things off, we put these into the Lorentz force law to get the total force on moving charge (charge Q, location $\mathbf{r}$, velocity $\mathbf{V}$) from another moving charge (charge q, location $\mathbf{r}'$, velocity $\mathbf{v}$, acceleration $\mathbf{a}$, where these are all evaluated at the retarded time). It is (drum roll):

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left\{ [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] + \frac{\mathbf{V}}{c} \left[\hat{\boldsymbol{\mathscr{r}}} \times [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \right] \right\} \tag{89}$$

This equation —in principle— contains all of classical EM. But it would not have been a good idea to start with this equation; Coulomb's law is easier to handle. It's so messy that it's only good for showing off my ability to put equations into LaTeX. And it looks good, doesn't it?

4.7 An Important Example

In Example 10.4 Griffiths solves the problem of the fields from a point charge q moving at constant velocity $\mathbf{v}$. (At $t = 0$ it is at the origin so that its path is given by $\mathbf{w}(t) = \mathbf{v}t$.)

Putting $\mathbf{a} = 0$ into 87 gives

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)\boldsymbol{\mathscr{r}}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \mathbf{u}$$

If one makes the definition

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

which is the vector pointing from the *present* location of the charge to the observation point, one can write $\mathbf{E}$ as

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \tag{90}$$

The B field from the moving charge is given by

$$\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}) \tag{91}$$

For the case of $v \ll c$ the fields reduce to

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{R^2} \hat{\mathbf{R}} \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q}{R^2} (\mathbf{v} \times \hat{\mathbf{R}}) \tag{92}$$

The first of these is no news since it's just the field from Coulomb's law. But the second of these gives the low-velocity approximation for the B field from a point charge. Naively one might have thought it was the *exact* answer from replacing the $\mathbf{J}d\tau$ in the Biot Savart law by $q\mathbf{v}$. But that is not the *correct* answer.

5 Radiation

We now make the connection between electromagnetic wave and the time-dependent sources that give rise to them. When charges accelerate or when currents change with time a radiation field is generated; charges moving at constant velocity and steady currents do *not* generate such a field.

We will always talk about localized sources (near the origin). We know that we have electromagnetic radiation when we have an average flow of energy outward at very large distances. Since the total power passing out through a spherical surface of radius r is

$$P(r) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$

and the area of the sphere is $4\pi r^2$, the Poynting vector must decrease in magnitude no faster than $1/r^2$ to get energy transported out to infinity. If we only have static fields, that can't happen; from Coulomb's law the E field decreases at least as fast as $1/r^2$ and from the Biot-Savart law a static magnetic field (for localized sources) also goes as $1/r^2$ or faster, and that would mean that S falls off at least as fast as $1/r^4$. Thus, static fields cannot radiate. The sources and the resulting fields have to be time–dependent. Indeed, Eqs. 82 and 83 indicate that time–varying sources with non-zero $\dot{\rho}$ and $\dot{\mathbf{J}}$ give rise to E and B fields which go as $1/r$ and thus give rise to an $\mathbf{S}$ which goes as $1/r^2$.

We begin the discussion with a calculation for two simple (but important) radiating ystems, the oscillating electric dipole and the oscillating magnetic dipole. This will help to refine some ideas. Then we do a more general treatment.

5.1 Electric Dipole Radiation

The system we study has time–varying charges $\pm q(t)$ located at $z = \pm d/2$ on the z axis, with

$$q(t) = q_0 \cos(\omega t) \tag{93}$$

which gives rise to an oscillating dipole moment

$$\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}} \qquad \text{where} \qquad p_0 = q_0 d$$

As we now know, we have to find the retarded potential for observation point $\mathbf{r}$ at time t. It is:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} \tag{94}$$

We will need to make some approximations to simplify this expression or else we'll never find out if the resulting field gives radiation!

With the charge q given by 93 there are *three* length scales to consider. One is d, the size of the source and r, the distance of the observer but we also have the length c/ω, which is related to the wavelength of the resulting radiation by $c/\omega = \lambda/(2\pi)$.

We will *always* want to be far away from the source compared to its size, so that $d \ll r$. It also turns out that we will always want to consider wavelengths that are much larger that the source size, so that $d \ll \lambda$. But the relative sizes of r and λ depend on the kind of problem being solved. If we have the relation

$$d \ll r \ll \lambda$$

we say we are solving for the fields in the near (static) zone of the radiator. If we have the relation

$$d \ll r \approx \lambda$$

we say we are working in the intermediate (induction) zone. Lastly, if we want

$$d \ll \lambda \ll r$$

we are in the far (radiation) zone. For the reaminder of the chapter, we will be in the far zone, and we use these relative sizes in 94 to get a simpler expression. Some intermdiate steps using these approximations are (using $d \ll r$):

$$\mathscr{r}_\pm \approx r\left(1 \mp \frac{d}{2r}\cos\theta\right)$$

using $d \ll c/\omega$,

$$\cos\left(\frac{\omega d}{2c}\cos\theta\right) \approx 1 \qquad \sin\left(\frac{\omega d}{2c}\cos\theta\right) \approx \frac{\omega d}{2c}\cos\theta$$

and using $r \gg$anything, we drop any terms except those with the fewest powers of r downstairs.

When all is done, we get

$$V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)] \tag{95}$$

Using the same approximations for the vector potential, we can show:

$$\mathbf{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\mathbf{z}} \tag{96}$$

And finally we get the fields that result from thes V and $\mathbf{A}$. In taking the derivatives we apply the approximations again to keep only the terms with the fewest r's downstairs. One can show:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}} \tag{97}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}} \tag{98}$$

So as advertised, we do get fields which fall off as $1/r$. But even more important than the fields is the Poynting vector, which one finds to be:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2\,\hat{\mathbf{r}} \tag{99}$$

so that if we take the time average the $\cos^2$ is replaced by $\frac{1}{2}$, giving:

$$\langle \mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}} \tag{100}$$

By integrating $\langle \mathbf{S}\rangle$ over the surface of a (large) sphere, we get the total power,

$$\langle P\rangle = \int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \tag{101}$$

5.2 Magnetic Dipole Radiation

Another problem involving an elementary time-dependent source is that of the oscillating magnetic dipole. We consider a circular current loop or radius b in the xy plane. The current in the loop is given by

$$I(t) = I_0\cos(\omega t)$$

This gives rise to an oscillating magnetic dipole

$$\mathbf{m}(t) = I(t)\pi b^2\hat{\mathbf{z}} = m_0\cos(\omega t)\hat{\mathbf{z}} \qquad \text{where} \qquad m_0 = \pi b^2 I_0$$

In this problem there is no charge density anywhere so the scalar potential is zero. (And note there is never an electric dipole here so the source of the fields here is different from before.)

The retarded vector potential is calculated from

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\,d\mathbf{l}' \tag{102}$$

The algebra used to pull out the part that is relevant for radiation is long but again we use the approximations $b \ll \lambda \ll r$ and in the end we get, for the potential and the fields:

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\phi}} \tag{103}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\phi}} \tag{104}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\theta}} \tag{105}$$

One finds that the time–averaged Poynting vector is

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}} \tag{106}$$

and the total radiated power is

$$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 m_0^2 \omega^2}{12\pi c^3} \tag{107}$$

As with electric dipole radiation the radiated intensity is proportional to $\sin^2\theta$ and to ω^4.

In general for a charge distribution which has *both* oscillating electric and magnetic dipoles the electric dipole gives most of the radiation. If we identify the I_0 of the magnetic dipole problem with the maximum current $I_0 = q\omega$ of the electric dipole problem, we find:

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c} \right)^2$$

which is a small quantity because of our assumption $b \ll c/\omega$.

5.3 Radiation from an Arbitrary Source

Now we'd like to find the biggest contribution to the radiation emitted from an arbitrary (localized) distribution of charge and current located near the origin.

We want to get an approximation to

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}} \, d\tau' \tag{108}$$

for the part which will give radiation.

The derivation is long, but it uses the approximations that *any* point of the source is much closer to the origin than the observation point $\mathbf{r}$ so that $r' \ll r$ for all $\mathbf{r}'$. The one can use the approximation

$$\mathscr{r} \approx r \left(1 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} \right)$$

and one can do a Taylor expansion of ρ in its time argument, doing the expansion about the retarded time for a source at *the origin*, $t_0 = t - r/c$. One gets:

$$\rho(\mathbf{r}', t - \mathscr{r}/c) = \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0) \left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \right) + \cdots$$

Further terms of this expansion involve $\ddot{\rho}$, $\dddot{\rho}$ and so on. We can drop them as long as

$$r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \; \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \ldots$$

The result is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)}{rc} \right] \tag{109}$$

where

$$\mathbf{p}(t_0) = \int \mathbf{r}' \rho(t_0) \, d\tau'$$

It is the third term in 109 which gives radiation.

Similar approximations give for the vector potential

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r} \tag{110}$$

After taking the derivatives and tossing terms of order $1/r^2$ and smaller, the fields are given by

$$\mathbf{E}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})] \tag{111}$$

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}] \tag{112}$$

The Poynting vector is

$$\mathbf{S} = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}} \tag{113}$$

and the total radiated power is

$$P = \frac{\mu_0 \ddot{p}^2}{6\pi c} \tag{114}$$

5.4 Radiation by Point Charges

We have alreadt found the E and B fields of moving charges; we recall that the $\mathbf{E}$ field expression had a “velocity” field and an “acceleration” field. Because of the r dependence only the acceleration term can give radiation:

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\imath}}{(\boldsymbol{\imath}\cdot\mathbf{u})^3}[\boldsymbol{\imath}\times(\mathbf{u}\times\mathbf{a})] \tag{115}$$

which gives a Poynting vector of

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}E_{\text{rad}}^2\hat{\boldsymbol{\imath}} \tag{116}$$

Now we make an approximation: if the charge is instantaneously at rest the $\mathbf{u} = c\hat{\boldsymbol{\imath}}$ and the Poynting vector can be written as the relatively simple

$$\mathbf{S}_{\text{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\imath^2}\right)\hat{\boldsymbol{\imath}} \tag{117}$$

and the total radiated power is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{118}$$

This formula is called the Larmor formula.

5.5 Radiation Reaction

From here things start to get strange.

We have studied what radiation is emitted by a particle moving with a given velocity and acceleration; now we consider the effect of the radiation on the particle itself.

The fact that the particle can loos energy to radiation (and the conservation of energy) implies that there is a force from the fields on the particle itself. One can show that this force is given by

$$\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}} \tag{119}$$

which is called the Abraham–Lorentz formula.

Aside from being one of the few (only?) times in physics we need to consider the time derivative of the acceleration, it gives rise to some difficulties in classical EM theory. The problems are those of the "runaway solutions" (acceleration which increases exponentially in time) and of an *acausal* acceleration. While a simple treatment is given iin the text, the problem persists even in the most sophisticated treatments (where we *would* use relativity). Even though it is not a problem for *physics* because classical EM theory needs to be replaced by its quantum counterpart, it is a problem for a consistent *classical* EM theory.

6 Relativity

6.1 Introduction

All measurements (of space and time) are tied to particular reference frames. An observer who moves at a constant velocity with respect to another observer will at least make different measurements of the coordinates in space; the difference will depend on the relative velocity of the two observers. The question is unavoidable since a lot of physics is done on the surface of the earth which is undergoing lots of complicated motions.

The big question is: Do all observers agree on the laws of physics? (And in what sense?)

We recall the fact that two of Faraday's induction experiments where the conducting loop moved near a stationary magnet and where the magnet moved in an opposite sense near a stationary loop gave the same result for the induced current. We can look at the two experiments as the same experiment taking place in two different frames (that of the magnet and that of the loop). Is it a coincidence that we get the same result?

We have not been especially explicit about how an electromagnetic wave propagates; is it like the more familiar mechanical waves which are a disturbance in some medium? If it does represent a travelling disturbance in some all–pervasive "ether", how are *we* moving with respect to this ether? If these ideas are correct, the speed of light which one measures in a laboratory on earth would depend on the direction and speed with which one is currently moving with respect to the ether.

6.2 Einstein's Postulates and the Consequences

• **Principle of Relativity:** The laws of physics (whatever they may be) apply in all inertial reference systems.

• **Universal speed of light:** The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

This seems wrong, that we should in some way add the observer's velocity onto the velocity of light that he observes to get the velocity that we observe. That is because we are used to the Galileo velocity addition rule, which goes like

$$v_{AC} = v_{AB} + v_{BC} \tag{120}$$

when in fact we need to replace it by the Einstein velocity addition rule,

$$v_{AC} = \frac{v_{ab} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \tag{121}$$

Throughout this chapter we will talk about a "lab" frame $\mathcal{S}$ and another reference frame (the "moving" frame) $\bar{\mathcal{S}}$ which moves in the $+x$ direction with speed v with respect to the lab frame.

Cleverly constructed arguments involving clocks which work using beams of light (which are no better or worse than any other physical clock) give some non-intuitive results about space and time measurements:

• Observers in different reference frames do not agree on whether events happened simultaneously.

• Time intervals between events give different results when measured. For a time interval Δt measure in the lab frame, the interval measured in the moving frame is

$$\Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t \tag{122}$$

The square root factor in 122 comes up so much that we give its reciprocal a special name,

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{123}$$

Eq. 122 says that in the moving frame the measured time interval is *shorter* so we say that moving clocks "run more slowly".

• Distances along the direction of motion of the moving frame are shortened. That is, if Δx is an $x-$distance as measured in the lab frame and $\Delta\bar{x}$ is the same $x-$distance as measured in the moving frame then

$$\Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x \tag{124}$$

• Distances perpendicular to the direction of motion are not contracted.

6.3 The Lorentz Transformations

The relation between coordinates in the frames $\mathcal{S}$ and $\bar{\mathcal{S}}$ are given by the **Lorentz transformation**,

$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \bar{x} = \gamma(x - vt) \qquad \bar{y} = y \qquad \bar{z} = z \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\ . \tag{125}$$

The reverse transformation can be gotten from 125 by switching the barred and unbarred coordinates and using $-v$ in place of v. Thus we have:

$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \qquad x = \gamma(\bar{x} + v\bar{t}) \qquad y = \bar{y} \qquad z = \bar{z}\ . \tag{126}$$

We combine the measurements of space and time into the unifying concept of **spacetime**. Instead of time t we make a zeroth coordinate x^0 defined by $x^0 \equiv ct$ to give the four spacetime coordinates.

$$x^0 = ct \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z \tag{127}$$

We will also use $\beta = \frac{v}{c}$. Then Lorentz transformation has the simpler form

$$\bar{x}^0 = \gamma(x^0 - \beta x^1) \qquad \bar{x}^1 = \gamma(x^1 - \beta x^0) \qquad \bar{x}^2 = x^2 \qquad \bar{x}^3 = x^3\ . \tag{128}$$

which can be written in matrix form as

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{129}$$

or with a summation

$$\bar{x}^{\mu} = \sum_{\nu=0}^{3} (\Lambda^{\mu}_{\nu})\ x^{\nu} \tag{130}$$

where Λ^{μ}_{ν} is the **Lorentz transformation matrix**.

The structure of 129 and 130 is similar to the transformation between coordinate systems which are rotated relative to one another; in analogy to the 3–dimensional vectors which we discuss in that context, we now discuss 4–dimensional vectors.

A **4–vector** is a set of four components which transform between reference frames in the same way as the spacetime coordinate vector x^{μ}; that is the set of numbers a^{μ} is a 4–vector if in the new reference frame it is given by

$$\bar{a}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu} \tag{131}$$

or explicitly,

$$\bar{a}^0 = \gamma(a^0 - \beta a^1) \qquad \bar{a}^1 = \gamma(a^1 - \beta a^0) \qquad \bar{a}^2 = a^2 \qquad \bar{a}^3 = a^3\ . \tag{132}$$

With 4–vectors we also have a kind of multiplication like that of the dot product for 3–vectors. If we from 4–vectors a^{μ} and b^{μ} we form the combination known as the **four–dimensional scalar product**,

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{133}$$

we find that it *does not change* under a Lorentz transformation of both vectors (even though the individual components of the vectors certainly do).

The minus sign in 133 requires special treatment and distinguishes the 4–vector version of the dot product from the 3–dimensional version; it reminds us that even though we treat all four coordinates together, the time (0^{th}) component is different from the space $(1, 2, 3)$ components.

It is found to be useful to write our conventional 4–vector with a raised index, and this is known as a **contravariant** vector:

$$a^{\mu} = (a^0, a^1, a^2, a^3) \tag{134}$$

but when we write a vector with a lowered index we have changed the sign of the zeroth component and we call this a **covariant** vector:

$$a_{\mu} = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3) \tag{135}$$

With this notation the 4–vector scalar product can be written as

$$\sum_{\mu=0}^{3} a_{\mu} b^{\mu} \tag{136}$$

It turns out that the scalar product is so common in relativistic equations that the summation in 136 will be taken as *understood* if we ever see a repeated index (raised and lowered) on one side of an equation. So we will use

$$\text{4–vector scalar product} \quad = a_{\mu} b^{\mu} \tag{137}$$

From the definitions it is also true that $a_{\mu} b^{\mu} = a^{\mu} b_{\mu}$.

It is worth mentioning here that Griffiths has made a particular choice for the 4–vector scalar product. Clearly if the scalar product in 133 is invariant then so is its negative:

$$+a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \tag{138}$$

and many authors define *this* to be the "4–vector scalar product". Clearly there is not a profound difference here, but you need to keep straight which choice has been made!

If you *do* go with the choice in 138 and you want to use a contravariant/covariant device as in 135 then you will need to change the sign of the $(1, 2, 3)$ components and we would write:

$$\text{Caution!} \qquad a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) \qquad \text{Do not use this choice!} \tag{139}$$

Aesthetically one can find reason for both of these choices so one finds them both in current textbooks. Be careful. We are using the choice of 135.

A very important quantity which is invariant between different reference frames is the **invariant interval** which is nothing more than the difference of two spacetime 4–vectors "dotted" with itself (in the 4–vector way). If we consider two events given by

$$x_A^\mu = (x_A^0, x_A^1, x_A^2, x_A^3) \qquad\qquad x_B^\mu = (x_B^0, x_B^1, x_B^2, x_B^3) \tag{140}$$

and take the difference $\Delta x^\mu \equiv x_A^\mu - x_B^\mu$ (i.e. the *distplacement* between the two events) then the scalar product of Δx^μ with itself gives the interval between the two events:

$$I \equiv (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = c^2 t^2 + d^2 \tag{141}$$

where $d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ is the square of the spatial separation of the events and t is the time separation, as measured in the reference frame $\mathcal{S}$. In the moving reference frame $\bar{\mathcal{S}}$ one will measure $\bar{d}^2$ and $\bar{t}$ for these quantities but the combination $I = -c^2\bar{t}^2 + \bar{d}^2$ will be the same.

The interval I can be positive, negative or zero depending on the relative sizes of ct and d:

• If $I < 0$, the interval is called **timelike**; there will be some reference frame where the two events occur at the same place ($d = 0$).

• If $I > 0$ the interval is called **spacelike**; there will be some reference frame where the two events occur at the same time ($t = 0$).

• If $I = 0$ the interval is called **lightlike**. For this special case the two events could be connected by a signal which travels at the speed of light.

The mathematics of rotations (simple ones; in a plane) involves *circles* because a rotation preserves the magnitude of position vector. In like manner if we plot spacetime events (with one spatial dimension) in the (x, ct) plane a Lorentz transformation does not preserve $(ct)^2 + x^2$, rather it preserves $-(ct)^2 + x^2$ so that the relevant shapes are *hyperbolae*.

There is much to say about the rich geometry of spacetime plots, called **Minkowski diagrams** with their hyperbolae and light cones and graphic depiction of causality but we'll have to do all of that in a course just devoted to relativity…

We need to go on to relativistic mechanics and especially the transformation of the EM field.

6.4 Relativistic Mechanics

If a particle moves at speed u (in your frame) and a time dt elapses (again, in your frame) then the **proper time** interval associated with the particle is

$$d\tau = \sqrt{1 - u^2/c^2}\, dt \tag{142}$$

If during that interval the particle moves through a displacement $d\mathbf{l}$ then the velocity of the particle is of course $\mathbf{u} = d\mathbf{l}/dt$. But it is of interest to define a **proper velocity** $\boldsymbol{\eta}$ by:

$$\boldsymbol{\eta} \equiv \frac{d\mathbf{l}}{d\tau} \tag{143}$$

which is related to the **ordinary velocity u** by

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{u} \tag{144}$$

Note the factor in front of the **u** might be called γ, but we will reserve the symbol γ for the inverse square root factor when the speed involved is that of a *reference frame*. Of course the particle *could* serve as a reference frame, but we are not emphasizing that here.

We can invent a zeroth component for $\boldsymbol{\eta}$

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}}$$

and then we have a 4–vector η defined as

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} \tag{145}$$

which *is* a 4–vector because it transforms according to Eq. 132.

The **relativistic 3–momentum** of a particle with mass m and velocity **u** is defined by

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \tag{146}$$

Here, m is the **rest mass** of the particle and we will not discuss *any other* kind of mass.

The zeroth component of a four–vector momentum must be defined as $p^0 = m\eta^0$ and this quantity is related to the **relativistic energy** by

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}} = \frac{E}{c} \tag{147}$$

or in other words,

$$E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \tag{148}$$

and so the 4–vector momentum can be written as

$$p^\mu = m\eta^\mu \tag{149}$$

Getting back to the relativistic energy, we note that when the particle is at rest (that is, $u = 0$) the energy is $E = mc^2$. But generally particles are *not* at rest and then we use Eq. 148. The motion means that the energy is larger than mc^2 and we'll call the additional part the *kinetic energy*:

$$T \equiv E_{\mathrm{kin}} = E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right) \tag{150}$$

which for relatively small values of u ($u \ll c$) can be expanded as:

$$E_{\mathrm{kin}} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots \tag{151}$$

It is a fact of physics that in a *closed* system (one where no outside agencies are exerting forces) the total (relativistic) energy and momentum are conserved, i.e. *all* components of the total 4–momentum are conserved.

We must be very careful to distinguish the concepts of *invariance* and *conservation*. Invariance means that a certain quantity is the same when calculated in different reference frames. Conservation means that a certain quantity remains the same before and after an interaction of the particles *within one particular reference frame*.

From the definitions of the components of p^μ, we find that the scalar product of p^μ with itself is[1]:

$$p^\mu p_\mu = -\frac{E^2}{c^2} + \mathbf{p}^2 = -m^2c^2 \tag{152}$$

which can be written as

$$E^2 - p^2c^2 = m^2c^4 \tag{153}$$

which lets us calculate E or p for a particle if we know either one of them.

An important application of the laws of conservation and invariance for the 4–momentum is the calculation of energies and momenta for particle collisions. In general, 4–momentum conservation tells how to find the unknown momenta or energies but it may be simplest to consider the collision as it is seen in the center-of-momentum frame, the frame where the total 3–momentum $\mathbf{P}$ is zero. Then we use invariance to relate quantities in the CM and Lab frames.

With relativity one can treat particles which have *no* rest mass, for which we have $E = pc$. (This includes the photon and to a good approximation, the neutrino.) From quantum mechanics these particles have a wavelength and frequency which depends only on their energy, via $E = h\nu = hc/\lambda$, or $p = h/\lambda$.

Newton's second law is still valid in relativity as long as we write it in the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{154}$$

where by $\mathbf{p}$ we mean the *relativistic* momentum. We *can't* use $\mathbf{F} = m\mathbf{a}$ anymore.

As in non-relativistic mechincs, the force $\mathbf{F}$ satisfies a work–energy theorem:

$$W \equiv \int \mathbf{F} \cdot d\mathbf{r} = E_{\mathrm{final}} - E_{\mathrm{initial}} \tag{155}$$

Newton's 3rd law does *not* extend to relativity. Aside from the problems with it that we have already seen (from the interchange of momentum of massive particles with the EM field) we also have the problem that observers in two different reference frames will disagree on what events are *simultaneous*, and Newton's 3rd law discusses the mutual forces on the members of a pair *at the same time*.

The force $\mathbf{F}$ is not a 4–vector, but it is possible to form a 4–vector force by taking the derivative of the momentum components with respect to proper time, to get:

$$K^\mu = \frac{dp^\mu}{d\tau} \tag{156}$$

which is called the **Minkowski force**. The spatial components are related to $\mathbf{F}$ by:

$$\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}$$

[1] Note that if we had made the other choice for the signs in the scalar product, we would get $p^\mu p_\mu = +m^2c^2$ and we avoid the minus sign in front of the mass squared.

with

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$$

However it is true that the correct force law for EM involves **F** and not **K** and that our old force law is still correct (with the proper relativisitc treatment of **F**!):

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

6.5 Relativistic Electrodynamics

A simple calcuation involving a wire with uniform charge densities moving in opposite directions shows that in different frames observers will disagree on the total charge density of the wire and thus about the force on a charge due to the electric field. It follows that if there is a current in a wire there must be a non-electric force on a moving charge, a force that we know is due to the magnetic field of the wire.

Arguments involving moving observers of the electric field due to simple sources (large parallel–plate capacitor, solenoid) show that the electric and magnetic fields transform between our frames $\mathcal{S}$ and $\bar{\mathcal{S}}$ as:

$$\bar{E}_x = E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y) \tag{157}$$

$$\bar{B}_x = B_x \qquad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \tag{158}$$

which tell us that the components of the fields *along* the relative motion of the frames don't change (!!) but perpendicular to that motion the E and B fields mix up in an interesting way. Thus, **E** and **B** don't transform as relativistic 4–vectors; something else is going on.

To understand what is going on we need to extend the idea of a 4–vector which transforms as $\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$.

A relativistic (second–rank) tensor $t^{\mu\nu}$ is an object with $4\times 4 = 16$ components which transforms between reference frames as

$$\bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma\, t^{\lambda\sigma} \tag{159}$$

A special case of these is the **antisymmetric tensor** where we have

$$t^{\mu\nu} = -t^{\nu\mu} \tag{160}$$

The electric and magnetic fields are both parts of the **field tensor** $F^{\mu\nu}$ defined as

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\} \tag{161}$$

which in another reference frame (connect to our frame $\mathcal{S}$ by the matrix Λ^μ_ν) will take on the values according to Eq. 159.

Actually, the choice of the field tensor made in Eq. 161 is not unique; one can also form the tensor

$$G^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{array} \right\} \tag{162}$$

and it too will transform between reference frames according to Eq. 159. This one is known as the **dual** field tensor.

6.6 Electromagnetism in Tensor Notation

The current density 4–vector is given by

$$J^\mu = (c\rho, J_x, J_y, J_z) \tag{163}$$

The continuity equation, $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$ can be written in 4-vector notation as

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \tag{164}$$

The Maxwell equations can be written in relativistic notation as

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \ , \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \tag{165}$$

We can also write down an equation an equation for the Minkowski force on a charge q:

$$K^\mu = q\eta_\nu F^{\mu\nu} \tag{166}$$

One can show that the spatial part of this equation gives the Lorentz force law while the 0^{th} component gives an expression for the rate of change of energy.

The scalar potential V and the vector potential $\mathbf{A}$ together make up the **4–vector potential**:

$$A^\mu = (V/c, A_x, A_y, A_z) \tag{167}$$

and the field tensor can be written in terms of the 4–vector potential as

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \tag{168}$$

Finally, we previously wrote the Lorentz gauge condition as

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$$

but we can now express it as

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 \tag{169}$$

and the differential equation for A^μ (in Lorentz gauge) becomes:

$$\Box^2 A^\mu = -\mu_0 J^\mu \tag{170}$$