Phys 4620, Spring 2008
Exam #2

1.a) What do we mean when we say a medium (for waves) is *dispersive*?

We mean that the speed of waves depends on the frequency. For visible light going through a transparent medium this occurs because the dielectric constant and index of refraction depend on the frequency of the wave.

b) What is the significance of the *group velocity* for waves in a dispersive medium, and how does it differ from the *wave velocity*? (Give its definition also.)

The wave velocity is the ratio of ω and k for a harmonic wave in the medium. The group velocity is defined as $v_g = \frac{d\omega}{dk}$. This quantity, evaluated at the "typical" frequency of a $wave\ packet$ gives the velocity of the wave packet. The wave velocity in a medium may be larger than c butthis does not violate causality; the group velocity (the speed at which information can travel) should be less than c (though Griffiths references a situation where it can be $> c$).

2. In the class (and text) we discussed a simple model for matter which gave a correct prediction for the behavior of n (index of refraction) with frequency.

a) Basically, what were the physical elements of this model?

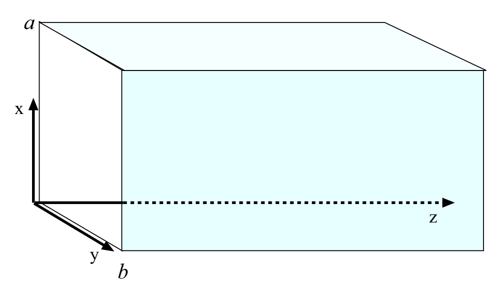
The model first considered an electron bound to a fixed nucleus by a harmonic potential with damping. (Later the possibility of different electrons bound to the same nucleus was included.) In the calculation, Griffiths produced a frequency--dependent polarization $\mathbf{P}$ and then a frequency-dependent ϵ_r and n.

b) What was the behavior that we wanted to predict?

One result of the calculation was that the predicted n increases with frequency. This agrees with the fact that purple light has a higher index of refraction than red light, i.e. it refracts more.

3. Suppose we have a (long) rectangular waveguide with dimensions 5.00 cm × 3.00 cm.

a) What is the lowest frequency of TE waves that will propagate in this waveguide?



The formula for the threshold frequencies is

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2}$$

and using $a = 0.050$ m and $b = 0.030$ m we find (recall m and n both start at 0 but one must be non-zero)

$$f_{10} = 3 \times 10^9 \text{ Hz}$$

b) Which modes will propagate at frequencies less than 9.00×10^9 Hz? (Note, this is a *frequency*, not an *angular frequency*.)

$$f_{01} = 5.0 \times 10^9 \ \mathrm{Hz} \qquad f_{11} = 5.83 \times 10^9 \ \mathrm{Hz} \qquad f_{20} = 6.0 \times 10^9 \ \mathrm{Hz}$$
$$f_{02} = 1.0 \times 10^{10} \ \mathrm{Hz} \qquad f_{21} = 7.8 \times 10^9 \ \mathrm{Hz}$$

so the modes $\{10\}$, $\{01\}$, $\{11\}$, $\{20\}$, $\{21\}$ can propagate.

c) Find the wave and group velocities for a wave of frequency 4.00 MHz.

At $f = 4.00$ GHz, we have $\omega = 2.5 \times 10^{10}\ \mathrm{s}^{-1}$. Also $\omega_{10} = 2\pi f_{10} = 1.88 \times 10^{10}\ \mathrm{s}^{-1}$. Using the (simpler) expression for k,

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{10}^2}$$

we find $k = 55.4\ \mathrm{m}^{-1}$. This gives

$$v_{\mathrm{wave}} = \frac{\omega}{k} = 4.5 \times 10^8\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

which is bigger than c but which from Question 1 does not bother us. The group velocity can be calculated as

$$v_{\mathrm{g}} = \left(\frac{dk}{d\omega}\right)^{-1} = c\sqrt{1 - (\omega_{10}/\omega)^2}$$

which I find to be

$$v_{\mathrm{g}} = 1.98 \times 10^8\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

4. Again, consider TE modes in a rectangular waveguide with sides a and b with $a \geq b$.

a) By the way, what does "TE" stand for, and what mathematical condition does it impose on the waves in the guide?

A Transverse Electric mode is one in which $E_z = 0$, (but B_z is *not* zero).

b) Find explicit expressions for the (complex) amplitudes B_x and B_y for the $(3,3)$ mode if the maximum value of the B field is B_0.

The solution for B_z for the TE wave of mode $(3,3)$ was

$$B_z = B_0 \cos(3\pi x/a) \cos(3\pi y/b)$$

then with $E_z = 0$ the equations which give all the other components from E_z and B_z give

$$B_x = \frac{-i}{(\omega/c)^2 - k^2}\left(B_0 k \frac{3\pi}{a}\right) \sin(3\pi x/a) \cos(3\pi y/a)$$

and

$$B_y = \frac{-i}{(\omega/c)^2 - k^2}\left(B_0 k \frac{3\pi}{b}\right)\cos(3\pi x/a)\sin(3\pi y/a)$$

All of these components are then put into

$$\tilde{\mathbf{B}}_0 = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$$

5. **a)** What is meant (generally) by a choice of *gauge* in electromagnetism? Give an explicit example of a "gauge condition".

The EM potentials are not unique; given one valid set of potentials V and $\mathbf{A}$ we can get another set from an arbitrary function $\lambda(\mathbf{r}, t)$ by

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = V - \frac{\partial\lambda}{\partial t}$$

One choice (which is always possible) is to impose the condition

$$\nabla \cdot \mathbf{A} = 0$$

which is called the Coulomb gauge condition.

b) For most of the text, the Lorentz gauge is chosen. State one advantage (aesthetic, mathematical, etc.) of making this choice.

The Lorentz gauge condition on $\mathbf{A}$ and V (again, always possible) is

$$\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$$

With this choice we simplify both (Maxwell) equations for V and $\mathbf{A}$:

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \qquad \Box^2\mathbf{A} = \mu_0\mathbf{J}$$

which have the same form and treat space and time similarly. These is appealing from our (future) study of relativity.

6. The vector potential for a non-static source can be gotten from the formula

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$

and there is also one for V.

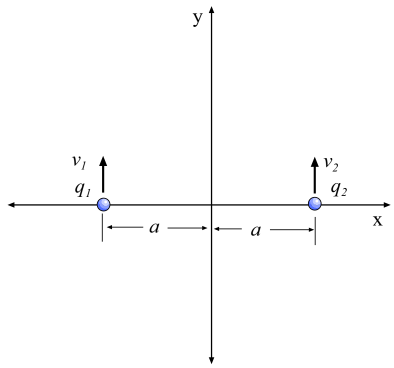
What's the deal with that t_r inside the integral? How is it defined? (How is it related to (or found from) t, $\mathbf{r}$ and $\mathbf{r}'$?)

What is the basic physical reason that we must have a t_r there instead of just t?

t_r is the "retarded time", defined by

$$t_r = t - \mathscr{r}/c \qquad \text{where} \quad \mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

7. Two charges q_1 and q_2 move in the xy plane, as shown. (q_1 is at $x = -a$ and moves in the $+y$ direction with constant speed v_1. q_2 is at $x = a$ and moves in the $+y$ direction with speed v_2.)



a) Find the force of q_1 on q_2.

Evaluate the E and B fields due to 1 at the location of 2. Use the equations for the (retarded) fields for a point charge with uniform velocity. The interesting feature of these equations was that the field points from the current position of the charge. (We do assume that the charges were moving in the same way in the recent past!) From the formula derived in the book for the E field, we use $\hat{\mathbf{R}} = 2a\hat{\mathbf{x}}$ and we get the fields at 2 (due to 1),

$$\mathbf{E}_2 = \frac{q_1}{4\pi\epsilon_0}\frac{1 - v_1^2/c^2}{(1 - v_1^2/c^2)^{3/2}}\frac{\hat{\mathbf{x}}}{(2a)^2} = \frac{q_1}{4\pi\epsilon_0(2a)^2}\frac{1}{(1 - v_1^2/c^2)^{1/2}}\hat{\mathbf{x}}$$

Then the B field at a is

$$\mathbf{B}_2 = \frac{1}{c^2}\mathbf{v}_1 \times \mathbf{E}_2 = \frac{q_1 v_1}{4\pi\epsilon_0 c^2(2a)^2}\frac{1}{(1 - v_1^2/c^2)^{1/2}}(-\hat{\mathbf{z}})$$

and now use $\mathbf{F}_2 = q_2(\mathbf{E}_2 + \mathbf{v}_2 \times \mathbf{B}_2)$. This gives (pulling out some common factors)

$$\begin{aligned}\mathbf{F}_2 &= \frac{q_1}{4\pi\epsilon_0(2a)^2(1 - v_1^2/c^2)^{1/2}}\left[q_2\hat{\mathbf{x}} + \frac{q_2 v_1 v_2}{c^2}(-\hat{\mathbf{x}})\right] \\ &= \frac{q_1 q_2\hat{\mathbf{x}}}{4\pi\epsilon_0(2a)^2(1 - v_1^2/c^2)^{1/2}}\left[1 - \frac{v_1 v_2}{c^2}\right]\end{aligned}$$

in which we recognize that for charges of the same sign, the E field gives a repulsion and the B field an attraction, but both are more complicated than the simple static results.

The forces *will* be "equal and opposite" if the speeds are the same. Otherwise, I don't think they are.

b) Find the force on q_2 on q_1.

Now get the fields and force at 1 due to 2. This will largely repeat the above steps, so without further ado, we find, with $\mathbf{R} = -2a\hat{\mathbf{x}}$,

$$\mathbf{E}_1 = \frac{q_2}{4\pi\epsilon_0}\frac{1 - v_2^2/c^2}{(1 - v_2^2/c^2)^{3/2}}\frac{-\hat{\mathbf{x}}}{(2a)^2} = -\frac{q_2}{4\pi\epsilon_0(2a)^2}\frac{1}{(1 - v_2^2/c^2)^{1/2}}\hat{\mathbf{x}}$$

$$\mathbf{B}_1 = \frac{1}{c^2}\mathbf{v}_2 \times \mathbf{E}_1 = \frac{q_2 v_2}{4\pi\epsilon_0 c^2(2a)^2}\frac{1}{(1 - v_2^2/c^2)^{1/2}}(\hat{\mathbf{z}})$$

and the force on q_1 is

$$\begin{aligned}\mathbf{F}_1 &= \frac{q_2}{4\pi\epsilon_0(2a)^2(1 - v_2^2/c^2)^{1/2}}\left[-q_2\hat{\mathbf{x}} + \frac{q_1 v_1 v_2}{c^2}(\hat{\mathbf{x}})\right] \\ &= \frac{q_1 q_2\hat{\mathbf{x}}}{4\pi\epsilon_0(2a)^2(1 - v_2^2/c^2)^{1/2}}\left[-1 + \frac{v_1 v_2}{c^2}\right]\end{aligned}$$

8. What is the mathematical condition on the Poynting vector $\mathbf{S}$ such that we have *radiation* from a localized source? Specifically, what is its dependence on the distance r? Explain.

By "radiation" we mean that there is a net flow of energy away from a point source so that energy leaves and never turns. This means that if we evaluate the net energy flow through a spherical surface surrounding the source the value does *not* tend to zero as the sphere gets arbitrarily large, that is

$$P(r) = \oint \mathbf{S} \cdot d\mathbf{a}$$

remains finite as $r \to \infty$. For this to be true, the radial component of $\mathbf{S}$ must not go to zero faster than $1/r^2$, since the area of the sphere increases like r^2.

9.a) In the derivation of the radiation fields from the oscillating electric dipole, its was shown that the vector could be approximated as

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\hat{\mathbf{z}}$$

where we have

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

From this, derive the *approximate* B field in the radiation zone.

We use $\mathbf{B} = \nabla \times \mathbf{A}$ using

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

and we note that we have only $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ components here and there is no ϕ dependence. The formula for the curl in spherical coordinates then an only give a $\hat{\boldsymbol{\phi}}$ component, and we get

$$\begin{aligned} \mathbf{B} &= \hat{\boldsymbol{\phi}}\left(\frac{-\mu_0 p_0 \omega}{4\pi}\right)\left\{\frac{-\omega}{rc}\cos[\omega(t - r/c)](-\sin\theta) + \frac{1}{r}\sin[\omega(t - r/c)]\sin\theta\right\} \\ &= -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t - r/c)]\hat{\boldsymbol{\phi}} \end{aligned}$$

where we have dropped one term because it has one more power of r in the denominator.

b) The E field in the radiation zone (which is extremely similar to your answer to (a)...) is

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\theta}}$$

(Actually, from this you can derive the answer for $\mathbf{B}$).

From this and the answer to (a) derive the expression for the time-averaged Poynting vector, $\langle \mathbf{S} \rangle$.

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{\mu_0 p_0^2 \omega^4}{(4\pi)^2 c}\frac{\sin^2\theta}{r^2}\cos^2[\omega(t - r/c)]\hat{\mathbf{r}}$$

To take the time average of this, replace the $\cos^2$ term by $\frac{1}{2}$ then

$$\langle \mathbf{S} \rangle = \frac{\mu_0 p_0^2 \omega^2}{32\pi^2 c}\frac{\sin^2\theta}{r^2}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times10^{-7}\,\tfrac{\text{N}}{\text{A}^2} \qquad 1\ \text{T} = 1\,\tfrac{\text{N}}{\text{A}\cdot\text{m}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\text{above}} = B^\perp_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp - B_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \qquad \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Waveguides:

$$\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}} \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

$$\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)\\
E_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)\\
B_x &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)\\
B_y &= \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)
\end{aligned}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]E_z = 0 \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]B_z = 0$$

TE solution: $\qquad B_z = B_0\cos(m\pi x/a)\cos(n\pi y/b)$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial\lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla\cdot\mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathcal{r}}\,d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathcal{r}}\,d\tau'$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathcal{r}c - \boldsymbol{\mathcal{r}}\cdot\mathbf{v}} \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathcal{r}c - \boldsymbol{\mathcal{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathcal{r}}{(\boldsymbol{\mathcal{r}}\cdot\mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathcal{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\boldsymbol{\mathcal{r}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathcal{r}}}\times\mathbf{E}) = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$$

$\mathbf{R} = \mathbf{r} - \mathbf{v}t$, θ is between $\mathbf{R}$ and $\mathbf{v}$

$$\text{El Dipole:}\langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \qquad \langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi\ c}$$

$$\text{Mag Dipole:}\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \qquad \langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi\ c^3}$$