Phys 2920, Spring 2010
Exam #2

1. Consider the new basis

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) \qquad \hat{\mathbf{e}}_2 = \mathbf{j} \qquad \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$$

which you'll note is orthonormal.

a) Find the usual "transformation matrix" S and its inverse S^{-1}. (You can get the inverse by inspection and a little thought; it's not hard. Hint, here S is an orthogonal matrix, as it was made from orthonormal vector.)

Reading off the coefficients of the old basis vectors (and using them as columns), we form the S matrix:

$$\mathsf{S} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

We now need to find the inverse of S, which satisfies $\mathsf{S}\mathsf{S}^{-1} = \mathsf{I}$. The hint that S is an orthogonal matrix should help, because the transpose of an orthogonal matrix *is* its inverse! But any way you can think of getting it, the inverse of S is

$$\mathsf{S}^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

for which it is simple to check that $\mathsf{S}\mathsf{S}^{-1} = \mathsf{I}$.

b) Find the representation of the matrix

$$\mathsf{A} = \begin{pmatrix} -1 & 2 & -1 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

in the new basis.

The matrix in the new representation is given by $\mathsf{A}' = \mathsf{S}^{-1}\mathsf{A}\mathsf{S}$. Pulling factors of $1/\sqrt{2}$ makes the arithmetic a little easier, giving

$$\mathsf{A}' = \mathsf{S}^{-1}\mathsf{A}\mathsf{S} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & -1 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

So.. okay, it's a little bit of a mess, but it gives

$$\Longrightarrow \qquad = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2\sqrt{2} & 0 \\ 0 & 0 & -4 \\ 2 & 2\sqrt{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 4\sqrt{2} & 0 \\ 0 & 0 & -4\sqrt{2} \\ 4 & 0 & 0 \end{pmatrix}$$

Finally, the answer is

$$\mathsf{A}' = \begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 0 & 0 & -2\sqrt{2} \\ 2 & 0 & 0 \end{pmatrix}$$

2. a) Find the direction in which the function

$$\phi = 3x^2y + 4z^2y - z^3$$

has its maximum rate of change when you are at the point $(-1,-1,-1)$.

Evaluate the gradient of the field ϕ:

$$\nabla\phi = (6xy)\mathbf{i} + (3x^2 + 4z^2)\mathbf{j} + (8zy - 3z^2)\mathbf{k}$$

which at the point $(-1,-1,-1)$ has the value

$$\nabla\phi = 6\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$$

The direction might be better given by the unit vector,

$$\hat{\mathbf{a}} = \frac{6\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}}{\sqrt{110}}$$

b) Find the magnitude of this greatest rate of change, $\frac{d\phi}{ds}$ along this direction.

The maximum rate of change is given by the magnitude of the gradient, so

$$\left(\frac{d\phi}{ds}\right)_{\max} = \sqrt{36 + 49 + 35} = \sqrt{110}$$

3. Do you remember the meaning of the operator (acting on the vector field $\mathbf{a}$):

$$(\mathbf{c}\cdot\nabla)\mathbf{a}$$

where $\mathbf{c}$ is a constant vector? Write it out, or explain what the expression means.

The meaning of this expression is that we evaluate the operator in the parenthesis first then apply it to the vector. The operator is

$$(\mathbf{c}\cdot\nabla) = c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}$$

and when applied to the vector field $\mathbf{a}$ we get

$$(\mathbf{c}\cdot\nabla)\mathbf{a} = \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)a_x\mathbf{i} + \left(c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right)a_y\mathbf{j} + \cdots$$

which in the end gives a *vector* and nine terms (derivatives) to evaluate.

4. Express the following scalar field in Cartesian coordinates:

$$\phi = \frac{e^{-r^2} \sin^2\theta}{r^2(1-\cos^2\theta)}$$

I goofed here... the problem is easier than intended. Since $\sin^2\theta = 1 - \cos^2\theta$ those factors cancel. (Had this not been the case, one would want to use $z = r\cos\theta$ and other relations.) Using $r = \sqrt{x^2+y^2+z^2}$ we get

$$\phi = \frac{e^{-(x^2+y^2+z^2)}}{(x^2+y^2+z^2)}$$

5. Find the divergence of the function (given in cylindrical coordinates)

$$\mathbf{a} = \rho z^2\,\hat{\mathbf{e}}_\rho + 2\rho^2 z\cos^2\phi\,\hat{\mathbf{e}}_\phi + \rho^3\sin^2\phi\,\hat{\mathbf{e}}_z$$

Use

$$\begin{aligned}
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2 z^2) - \frac{1}{\rho}4\rho^2 z\cos\phi\sin\phi + 0 \\
&= 2z^2 - 2\rho z\cos\phi\sin\phi = 2z^2 - \rho z\sin 2\phi
\end{aligned}$$

6. Find Laplacian of the function

$$\Phi(r,\theta,\phi) = \frac{\cos^2\theta}{(a^2+r^2)}$$

where a is a constant.

Use

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

where things are simplified a little from the fact that there is no ϕ dependence. Taking $\partial\Phi/\partial r$ is not too hard, nor is $\partial\Phi/\partial\theta$, so doing this in the first step we get

$$\begin{aligned}
\nabla^2\Phi &= \frac{\cos^2\theta}{r^2}\frac{\partial}{\partial r}\left(\frac{-2r^3}{(a^2+r^2)^2}\right) - \frac{1}{r^2\sin\theta}\frac{1}{(a^2+r^2)}\frac{\partial}{\partial\theta}(2\cos\theta\sin^2\theta) \\
&= \frac{\cos^2\theta}{r^2}\left(-\frac{6r^2}{(a^2+r^2)^2} + \frac{(-2)(-4r^4)}{(a^2+r^2)^3}\right) - \frac{2}{r^2\sin^2\theta(a^2+r^2)}\left(-\sin^3\theta + 2\cos^2\theta\sin\theta\right)
\end{aligned}$$

At this point, though it could obviously be simplified further, I'll consider the problem done!

7. Find the curl of the vector field

$$\mathbf{a} = r^3 \sin\phi\,\hat{\mathbf{r}} + 2r^2 \sin\theta\cos\phi\,\hat{\boldsymbol{\theta}} + r^2 \sin\theta\sin\phi\,\hat{\boldsymbol{\phi}}$$

Use the formula for the curl in spherical coordinates,

$$\nabla\times\mathbf{a} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi$$

Do the components separately; the $\hat{\mathbf{e}}_r$ component is

$$\frac{1}{r\sin\theta}\left[r^2 2\sin\theta\cos\theta\sin\phi + 2r^2\sin\theta\sin\phi\right]\hat{\mathbf{e}}_r$$

The $\hat{\mathbf{e}}_\theta$ part is

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\cos\phi - 3r^2\sin\theta\sin\phi\right]\hat{\mathbf{e}}_\theta$$

The $\hat{\mathbf{e}}_\phi$ part is

$$\frac{1}{r}[6r^2\sin\theta\cos\phi]\hat{\mathbf{e}}_\phi$$

These can each be simplified and the curl written as the sum of the three parts, but this is enough!

8. a) Find a function ϕ for which

$$\mathbf{F} = (6xy)\mathbf{i} + 3x^2\mathbf{j} + 3z^2\mathbf{k}$$

is the gradient.

The x component tells us that if the function we want is $f(x,y,z)$ then

$$f(x,y,z) = 3x^2y + g(y,z)$$

while the y component tells us that

$$f(x,y,z) = 3x^2y + h(x,z)$$

which are consistent with the last term being z^3 to give us the z component. We get:

$$f(x,y,z) = 3x^2y + z^3 + C$$

b) By the way, before you did part (a) is there a way you could have been sure that $\mathbf{F}$ *was* the gradient of some function?

We know this would be be true if the curl of $\mathbf{F}$ is zero. To check this,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 6xy & 3x^2 & 3z^2 \end{vmatrix} = \mathbf{k}(6x - 6x) = \mathbf{0}$$

c) Evaluate the line integral

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} \qquad \text{where} \qquad A=(0,0,0) \qquad \text{and} \qquad B=(1,1,1)$$

and where the path is the straight line from $(0,0,0)$ to $(1,1,1)$.

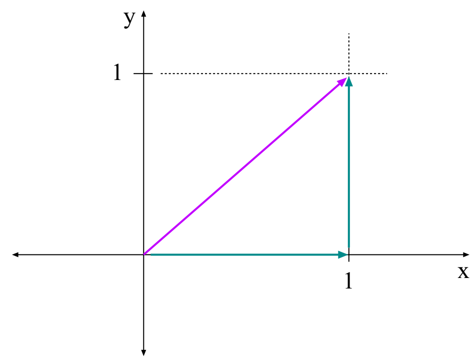
Having the scalar "anti-derivative" of the field $\mathbf{F}$, we find

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = (3x^2y+z^3)\Big|_{(0,0,0)}^{(1,1,1)} = 4$$

9. Do the line integral

$$\int_A^B \mathbf{a}\cdot d\mathbf{r} \qquad \text{for} \qquad \mathbf{a}=(y^2-x)\mathbf{i}+(3x^2y)\mathbf{j}$$

from $A=(0,0)$ to $B=(1,1)$ for the two paths:



a) The line from $(0,0)$ to $(1,0)$ then from $(1,0)$ to $(1,1)$.

On the line from $(0,0)$ to $(1,0)$, we have $d\mathbf{r}=dx\mathbf{i}$ and $y=0$. This gives

$$\int_1 \mathbf{a}\cdot d\mathbf{r} = \int_0^1 (0-x)\,dx = -\tfrac{1}{2}$$

On the line from $(1,0)$ to $(1,1)$ we have $d\mathbf{r}=dy\mathbf{j}$ and $x=1$. This gives

$$\int_2 \mathbf{a}\cdot d\mathbf{r} = \int_0^1 (3y)\,dy = \frac{3}{2}$$

The total integral is

$$\int_C \mathbf{a}\cdot d\mathbf{r} = -\frac{1}{2}+\frac{3}{2}=1$$

b) The straight line from A to B.

This path can be parameterized by $x=t$, $y=t$ as t runs from 0 to 1. Then

$$d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}=dt\mathbf{i}+dt\mathbf{j}$$

and all the substitutions give

$$\int_C \mathbf{a}\cdot\mathbf{r} = \int_0^1 (t^2-t)\,dt + \int_0^1 (3t^3)\,dt$$

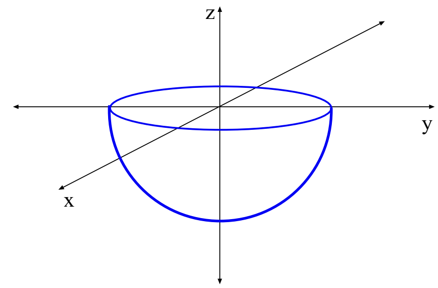
which gives

$$\Longrightarrow \qquad = \left(\frac{t^3}{3}-\frac{t^2}{2}\right)\Big|_0^1 + \frac{3t^4}{4}\Big|_0^1 = -\frac{1}{6}+\frac{3}{4}=\frac{7}{12}$$

10. Consider the vector field (expressed in spherical coordinates)

$$\mathbf{a} = 2r^2(1-\cos^3\theta)\hat{\mathbf{r}} - 3r^2\sin^2\theta\,\hat{\boldsymbol{\phi}}$$

Find the integral $\int_S \mathbf{a}\cdot d\mathbf{S}$ where S is the bottom half of the sphere of radius 2.



The area element on this surface ($r=2$) is

$$d\mathbf{S} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$$

where θ goes from $\pi/2$ to π and ϕ goes from 0 to 2π so that the surface integral is

$$\int_0^{2\pi}\int_{\pi/2}^{\pi} 2r^2(1-\cos^3\theta)r^2\sin\theta\,d\theta\,d\phi\Big|_{r=2}$$

Take the constants outside and use the fact that the integral on ϕ gives a factor of 2π. This gives

$$\implies \quad = (2\pi)(32)\int_{\pi/2}^{\pi}(1-\cos^3\theta)\sin\theta\,d\theta$$

With the substitution $x=\cos\theta$, $dx=-\sin\theta\,d\theta$ and the integral goes from $x=0$ to $x=-1$. Use the minus sign to reverse the integration limits and get

$$\implies \quad = 64\pi\int_{-1}^{0}(1-x^3)\,dx = 64\pi\left(x-\frac{x^4}{4}\right)\Big|_{-1}^{0} = 64\pi(-1)\left(-1-\tfrac{1}{4}\right) = \frac{5}{4}(64\pi) = 80\pi$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad \hat{\mathbf{z}} = \mathbf{k} \tag{2}$$

$$\mathbf{i} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{j} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{k} = \hat{\mathbf{e}}_z \tag{3}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{4}$$

$$
\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{i} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}
$$

$$
d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi
$$

$$
da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta
$$

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
$$

$$
\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy
$$

$$
\int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}
$$