Phys 2920, Spring 2009
Exam #1

1. Write $-4.0 + 6.0i$ in polar form.

With $z = x + iy = \rho e^{i\phi}$, we have

$$\rho = \sqrt{x^2 + y^2} = \sqrt{52} = 7.21 \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6.0}{(-4.0)}\right) = 2.16$$

where of course we have chosen the right quadrant for ϕ. Then:

$$z = (7.21)e^{i(2.16)}$$

2. Find the angle between the vectors

$$\mathbf{a} = +2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} \quad \text{and} \quad \mathbf{b} = -6\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$$

Calculate:

$$a = \sqrt{4 + 9 + 25} = 6.16 \qquad b = \sqrt{36 + 25 + 4} = 8.06$$

and

$$\mathbf{a} \cdot \mathbf{b} = -12 - 15 + 10 = -17 \qquad \Longrightarrow \qquad \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \frac{-17}{(6.16)(8.06)} = -0.342$$

which gives

$$\theta = 110^\circ = 1.92$$

3. Simplify:

$$\mathbf{a} \cdot (\mathbf{b} \times (\mathbf{a} + \mathbf{c})) - \mathbf{c} \cdot (\mathbf{a} \times (\mathbf{b} + \mathbf{c}))$$

Distributing things, we have

$$\Longrightarrow \qquad = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) - \mathbf{c} \cdot (\mathbf{a} \times \mathbf{c}))$$

Here the first and last terms must be zero since the cross product is perpendicular to either of the two vectors. For the third term, we can use the relations for the scalar triple product to get

$$\Longrightarrow \qquad = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

which is zero. This expression simplifies to zero!
And you can't get any simpler than that.

4. Find a unit vector which is orthogonal to vectors **a** and **b** from Problem 2.

The cross product of the two vectors is perpendicular to both:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ +2 & -3 & -5 \\ -6 & 5 & -2 \end{vmatrix} = \mathbf{i}(6+25)+\mathbf{j}(30+4)+\mathbf{k}(10-18) = 31\,\mathbf{i}+34\,\mathbf{j}-8\,\mathbf{k}$$

the magnitude of which is $\sqrt{2181}$, so the unit vector in this direction is

$$\frac{31\,\mathbf{i}+34\,\mathbf{j}-8\,\mathbf{k}}{\sqrt{2181}}$$

5. Consider the vector space of functions defined on $[0,1]$,

$$f_n(x) = \sqrt{2}\,\sin(n\pi x)$$

Suppose I want to express the function $3x(1-x)$ in terms of these functions, that is, as:

$$3x(1-x) \;=\; \sum_{n=1}^{\infty} c_n f_n(x)$$

Tell me what I would have to do to get c_n. (You don't actually need to evaluate any integrals here, but be clear.)

First off, the given function is a proper one for the expansion, since $f(0)=0$ and $f(1)=0$. Also, the basis functions are for this case all orthonormal.

The component c_n that the function has "along" the the basis vector $f_n(x)$ is found by taking the inner product of $f(x)$ with $f_n(x)$:

$$c_n = \langle f_n | f\rangle = \int_0^1 \sqrt{2}\sin(n\pi x)(3x)(1-x)\,dx$$

The integral here actually is not hard to work (with the help of tables). It is (if I did this right):

$$\begin{aligned}
c_n &= 3\sqrt{2}\int_0^1 (x-x^2)\sin(n\pi x)\,dx \\
&= 3\sqrt{2}\left[\frac{1}{(n\pi)^2}\sin(n\pi x) - \frac{x}{n\pi}\cos(n\pi x) + \frac{2x}{(n\pi)^2}\sin(n\pi x) - \frac{(n\pi)^2x^2-2}{(n\pi)^3}\cos(n\pi x)\right]\Big|_0^1 \\
&= 3\sqrt{2}\left[-\frac{1}{n\pi}(-1)^n - \frac{[(n\pi)^2-2]}{(n\pi)^3}(-1)^n + \frac{(-2)}{(n\pi)^3}\,1\right] = -\frac{(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} + \frac{2}{(n\pi)^3}((-1)^n-1) \\
&= 3\sqrt{2}\left[-\frac{2(-1)^n}{n\pi} + \frac{2}{(n\pi)^3}((-1)^n-1)\right] = \frac{6\sqrt{2}}{n\pi}\left[(-1)^{n+1} + \frac{((-1)^n-1)}{(n\pi)^2}\right]
\end{aligned}$$

6. Show that $P_3(x)$ is orthogonal to $P_1(x)$. (Recall the P's are defined on $[-1,1]$; the first few P polynomials are given below.)

Evaluate $\int_0^1 P_1(x)P_3(x)\,dx$; using the expressions given below, and the definition of the inner product for "complete" sets of functions, we have

$$\begin{aligned}\int_{-1}^{1} P_1(x)P_3(x)\,dx &= \int_{-1}^{1} x\tfrac{1}{2}(5x^3-3x)\,dx = \tfrac{1}{2}\int_{-1}^{1}(5x^4-3x^2)\,dx \\ &= \tfrac{1}{2}(x^5-x^3)\big|_{-1}^{1} = \tfrac{1}{2}(0) = 0\end{aligned}$$

so they are orthogonal. (But they are not normalized.)

7. If matrices A and B are given by

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & 0 & 3 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 5 & -8 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Find $\mathrm{Det}(\mathsf{AB}) = |\mathsf{AB}|$.

The easiest way to do this is perhaps to note

$$\mathrm{Det}(\mathsf{AB}) = |\mathsf{AB}| = |\mathsf{A}|\,|\mathsf{B}|$$

The individual determinants are found to be

$$|\mathsf{A}| = (14-24) = -10 \qquad \text{and} \qquad |\mathsf{B}| = (15-8) = 7$$

Then

$$\mathrm{Det}(\mathsf{AB}) = (-10)(7) = -70$$

8. Find the eigenvalues and (unit) eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}$$

Solve for λ such that $|\mathsf{A} - \lambda\mathbf{1}| = 0$:

$$\begin{vmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda)+12 = \lambda^2+3\lambda-10+12 = \lambda^2+3\lambda+2 = (\lambda+2)(\lambda+1)$$

which has roots $\lambda = -2$ and $\lambda = -1$, which are the eigenvalues.

Find the eigenvectors: First, for $\lambda = -2$, solve:

$$\begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} a \\ b \end{pmatrix}$$

This give the equations

$$2a-3b=-2a \qquad 4a-5b=-2b$$

If $a = 1$ then both equations give $b = \frac{4}{3}$. Then the normalized eigenvector is

$$\sqrt{1 + \left(\tfrac{4}{3}\right)^2} = \sqrt{\tfrac{25}{9}} = \tfrac{5}{3} \qquad \Longrightarrow \qquad \mathbf{v}_1 = \frac{3}{5}\begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix}$$

For $\lambda = -1$, solve:

$$\begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

This give the equations

$$2a - 3b = -a \qquad 4a - 5b = -b$$

If $a = 1$ then both equations give $b = 1$. Then the normalized eigenvector is

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

9. D'oh! I tried to get the inverse of the matrix A in the last problem, but I was only able to copy down three elements before the power was shut off.

$$\mathsf{A}^{-1} = \begin{pmatrix} -\frac{5}{2} & x \\ -2 & 1 \end{pmatrix}$$

What is x?

Multiplying A and the given A^{-1} (for which we should get the unit matrix $\mathbf{1}$, we get

$$\mathsf{A}\mathsf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} -\frac{5}{2} & x \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5+6 & 2x-3 \\ -10+10 & 4x-5 \end{pmatrix} = \begin{pmatrix} 1 & 2x-3 \\ 0 & 4x-5 \end{pmatrix}$$

We get the unit matrix only if $2x - 3 = 0$, so that $x = \frac{3}{2}$. If we use this result for the last element, we do get $4x - 5 = 1$, as we have to. So $x = \frac{3}{2}$.

10. All right, do a proof for me. Prove that if A and B are two square matrices, then

$$\mathrm{Tr}(\mathsf{AB}) = \mathrm{Tr}(\mathsf{BA})$$

The trace is the sum of the diagonal elements. If $\mathsf{C} = \mathsf{AB}$, then

$$\mathsf{C}_{ij} = (\mathsf{AB})_{ij} = \sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{kj}$$

But as the trace is the sum of the diagonal elements, then

$$\mathrm{Tr}(\mathsf{AB}) = \sum_{i=1}^{N} \mathsf{C}_{ii} = \sum_{i=1}^{N}\sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{ki}$$

But since the *numbers* A_{ik} and B_{ki} commute, then

$$\Longrightarrow \qquad = \sum_{i=1}^{N}\sum_{k=1}^{N} \mathsf{B}_{ki}\mathsf{A}_{ik} = \sum_{k=1}^{N}\sum_{i=1}^{N} \mathsf{B}_{ki}\mathsf{A}_{ik} = \sum_{k=1}^{N} (\mathsf{BA})_{kk} = \mathrm{Tr}(\mathsf{BA})$$

So then $\mathrm{Tr}(\mathsf{AB}) = \mathrm{Tr}(\mathsf{BA})$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2-1) \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$(\mathsf{AB})_{ij} = \sum_{k=1}^{N} \mathsf{A}_{ik}\mathsf{B}_{kj} \qquad |\mathsf{A} - \lambda\mathbf{1}| = 0$$

$$\hat{\mathbf{e}}'_j = \sum_{i=1}^{N} \mathsf{S}_{ij}\hat{\mathbf{e}}_i \qquad \mathbf{x}' = \mathsf{S}^{-1}\mathbf{x} \qquad \mathsf{A}' = \mathsf{S}^{-1}A\mathsf{S}$$