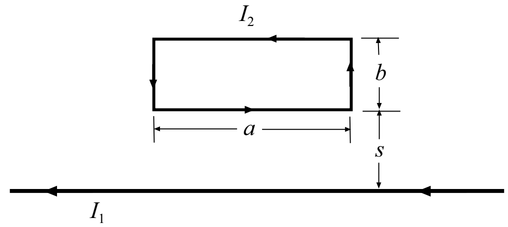
Phys 4610, Fall 2007
Exam #3

1. A rectangular wire loop of sides a and b lies parallel to a long wire. The long wire carries current I_1 and the loop carries current I_2.

Find an expression for magnitude of the force on wire loop. What is the direction of the force?



The B field at the distance of the $near$ side has magnitude $\frac{\mu_0 I_1}{2\pi s}$ and is directed $into$ the page, so the force on the near side, from $\mathbf{F} = I_2\mathbf{L}\times\mathbf{B}$ is upward and has magnitude

$$F_{\rm near} = I_2 LB = I_2 a\frac{\mu_0 I_1}{2\pi s}$$

On the sides, the forces cancel on the corresponding parts of the two wires giving a net force on the sides of zero.

On the far side, at distance $s+b$, the field has magnitude $\frac{\mu_0 I_1}{2\pi(s+b)}$ (into the page) and the force on the far segment has magnitude

$$F_{\rm far} = I_2 a\frac{\mu I_1}{2\pi(s+b)}$$

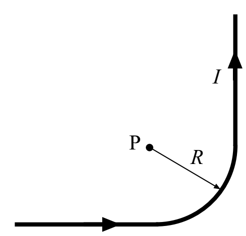
and is directed downward.

Using $\hat{\mathbf{x}}$ as the direction perp to the wire, the net force on the loop is

$$\mathbf{F}_{\rm net} = \frac{\mu_0 I_1 I_2 a}{2\pi}\left(\frac{1}{s} - \frac{1}{s+b}\right) = \frac{\mu_0 ab I_1 I_2}{2\pi s(s+b)}\hat{\mathbf{x}}$$

The total force is upward, repulsive.

2. A wire carries a current I in the geometry shown at the right. (Wire is in the plane of the page; it has two long straight parts and a quarter-circle bend.) Find the magnetic field (direction and magnitude) at the point P.



From the straight parts, the contribution to the field at P is out of the page; each is $half$ the contribution of an ∞ wire at diestance R, so together they give a contribution of a $full$ ∞ wire:

$$\mathbf{B}_{\rm straight} = \frac{\mu_0 I}{2\pi R}\hat{\mathbf{z}}$$

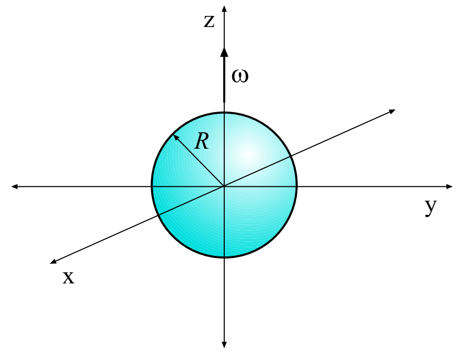
The curvy part also gives a contribution out of the page; it is $\frac{1}{4}$ the field of a circular loop (at its center), namely

$$\mathbf{B}_{\rm curvy} = \frac{1}{4}\frac{\mu_0 I}{2R}\hat{\mathbf{z}} = \frac{\mu_0 I}{8R}\hat{\mathbf{z}}$$

The total is

$$\mathbf{B}_{\rm Tot} = \frac{\mu_0 I}{2R}\left(\frac{1}{\pi} + \frac{1}{4}\right) = \frac{\mu_0 I(4+\pi)}{8\pi R}\hat{\mathbf{z}}$$

3. In Example 5.11 of the book, the example of a rotating spherical shell of charge (radius R, surface charge density σ, rotation velocity $\omega\hat{\mathbf{z}}$) was solved, and it was found that



$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 R\omega\sigma}{3} r\sin\theta\hat{\boldsymbol{\phi}} & \text{if } r \leq R \\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}} & \text{if } r > R \end{cases}$$

a) Summarize the mathematical steps used to get this result.

To get $\mathbf{A}$ from this distribution of *surface* current, we need to evaluate

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\, da'$$

over the sphere's surface. The surface current density is given by $\mathbf{K} = \sigma\mathbf{v}$ and the velocity of a piece of the sphere is given by

$$\mathbf{v} = \omega s\hat{\boldsymbol{\phi}} = \omega R\sin\theta\hat{\boldsymbol{\phi}}$$

The integration was rather tricky and Griffiths resorted to some temporary rotated coordinates.

b) Find the magnetic field both inside and outside the sphere.

Inside we have

$$A_\phi - \frac{\mu_0 R\omega\sigma}{3} r\sin\theta$$

so with $\mathbf{B} = \nabla\times\mathbf{A}$,

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\mu_0 R\omega\sigma}{3} r\sin\theta\right] = \frac{\mu_0 R\omega\sigma}{3} 2\cos\theta$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 R\omega\sigma}{3} r^2\sin\theta\right) = -\frac{2}{3}\mu_0 R\omega\sigma\sin\theta$$

Then

$$\mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0 R\omega\sigma(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})$$

Outside, we find

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin^2\theta}{r^2}\right] = \frac{2}{3}\frac{\mu_0 R^4\omega\sigma}{3r^3}\cos\theta$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r}\right] = \frac{\mu_0 R^4\omega\sigma}{3r^3}\sin\theta$$

Then

$$\mathbf{B}_{\text{out}} = \frac{\mu_0 R^4\omega\sigma}{3r^3}[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}]$$

Interestingly, since $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, the B field inside is

$$\mathbf{B}_{\rm in} = \frac{2}{3}\mu_0 R\omega\sigma\hat{\mathbf{z}}$$

i.e. a *uniform* field in the $\hat{\mathbf{z}}$ direction.

4. Suppose the magnetic field in a certain region of space is uniform and points in the $\hat{\mathbf{x}}$ direction: $\mathbf{B} = B\hat{\mathbf{x}}$ with B a constant.

Give *two* possibilities for the $\mathbf{A}$ field in this region. (Do your answers also satisfy the secondary condition $\nabla \cdot \mathbf{A} = 0$?)

Since $\mathbf{B} = \nabla \times \mathbf{A}$, then

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Two possibilities are

$$A_z = By \qquad A_y = 0 \qquad \text{so} \quad \mathbf{A} = yB\hat{\mathbf{z}}$$

$$A_z = 0 \qquad A_y = -Bz \qquad \text{so} \quad \mathbf{A} = -zB\hat{\mathbf{y}}$$

(note, these choices don't produce *other* components of $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$.)

Both choices give $\nabla \cdot \mathbf{A} = 0$.

5. What do the "boundary condition" equations for magnetism

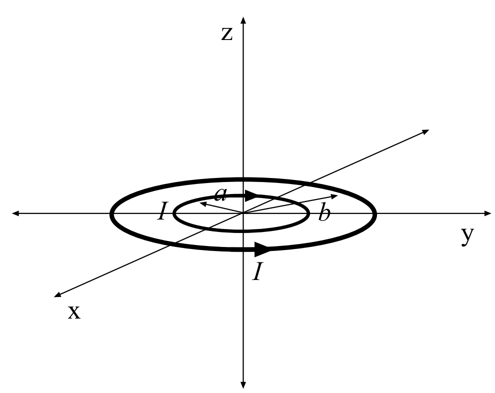
$$B^{\perp}_{\rm above} = B^{\perp}_{\rm below} \qquad \mathbf{B}_{\rm above} - \mathbf{B}_{\rm below} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$$

refer to? I.e. what do the symbols mean?

The boundary condition equations relate the components of $\mathbf{B}$ immediately above and below a two-dimensional (possibly curvy) boundary which separates two regions in space. At any point on the boundary surface we let $\hat{\mathbf{n}}$ be the normal pointing from "below" to "above". The $B^{\perp}$'s are the components along $\hat{\mathbf{n}}$, so that only the parallel components are discontinuous. $\mathbf{K}$ is the surface current in the boundary surface.

6. Two concentric loops of radius a and b lie in the xy plane; each carries a current I but in opposite directions.

Produce expressions for the $\mathbf{A}$ and $\mathbf{B}$ fields at large ($r \gg b$) distances from the origin.



The long-distance behavior of the fields will come from the magnetic dipole moment of this current distribution, and *that* is the sum of the magnetic moments of the two current loops. From $\mathbf{m} = I\mathbf{a}$, we get

$$\mathbf{m}_{\rm Tot} = I\pi b^2\hat{\mathbf{z}} - I\pi a^2\hat{\mathbf{z}} = (b^2 - a^2)I\pi\hat{\mathbf{z}}$$

Using relations for the case where $\mathbf{m}$ points along $\hat{\mathbf{z}}$, we get

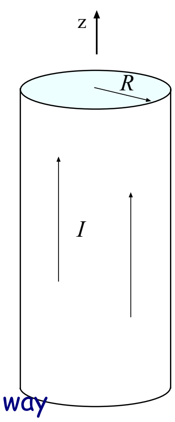
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\sin\theta}{r^2}(b^2 - a^2)I\pi\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

7. For a material with uniform magnetization $\mathbf{M}$ there is no bound volume current but there is still a surface bound current. Explain how this comes about, treating atomic currents as being small current loops. Pictures may help.

Consider a slab of uniformly magnetized material. In the interior, the currents from neighboring loops will cancel but there is nothing to cancel the parts of the loops at the edge of the slab. This results in a surface current $\mathbf{K}_b$. (It is indeed a current even though in this picture the charges never move off their individual atomic loops.)

8. A long copper rod of radius R carries a uniformly distributed *free* current I. (Copper is a linear material with susceptibility χ_m.)



Find:

a) The field $\mathbf{B}$ and $\mathbf{H}$ inside and outside the rod.

With the current I uniformly distributed over the wire, the free current density must be

$$\mathbf{J}_f = \frac{I}{\pi R^2}\hat{\mathbf{z}}$$

and the new form of Ampere's law gives $\mathbf{H}$ when we consider an Amperian loop (of radius s) and the *free* current enclosed. Assuming $\mathbf{H}$ is tangential, we get in the usual way

$$\oint \mathbf{H}\cdot d\mathbf{l} = 2\pi s H_\phi = I_{\text{f, enc}} = \pi s^2 \frac{I}{\pi R^2} = I\frac{s^2}{R^2}$$

Or,

$$H_\phi = \frac{Is}{2\pi R^2} \qquad \text{inside}$$

Outside, we have

$$2\pi s H_\phi - I \quad \text{so} \quad H = \frac{I}{2\pi s} \quad \text{outside}$$

Since in general $\mathbf{B} = \mu\mathbf{H}$, with $\mu = \mu_0(1+\chi_m)$, then

$$B_\phi = \frac{\mu I s}{2\pi R^2} \quad \text{inside, and} \quad B_\phi = \frac{\mu_0 I}{2\pi s} \quad \text{outside}$$

b) The magnetization $\mathbf{M}$.

From $\mathbf{M} = \chi_m\mathbf{H}$, we have

$$M_\phi = \frac{\chi_m I s}{2\pi r^2} \quad \text{outside, and} \quad \mathbf{M} = 0 \quad \text{outside}$$

c) The free and bound current densities $\mathbf{J}_f$ and $\mathbf{J}_b$.

From before, $\mathbf{J}_f = \frac{I}{\pi R^2}\hat{\mathbf{z}}$. We also have

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f$$

This gives

$$\mathbf{J}_b = \frac{\chi_m I}{\pi R^2}\hat{\mathbf{z}}$$

inside, and of course $\mathbf{J}_b = \mathbf{J}_f = 0$ outside.

9. Give two ways in which the magnetic properties of ferromagnetic material are different from the paramagnetic/diamagnetic materials with conventional values of χ_m.

In a ferromagnetic material, application of an external B field causes an enormous alignment of the magnetic moments of the atoms of the material resulting in an $\mathbf{M}$ which is much larger than $\mathbf{H}$, so that $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ basically comes from the magnetization.

Secondly, the magnetization depends on the "history" of the fields applied to the sample so it is not a linear material with a well-defined χ_m. In particular, with fields applied and later turned off the sample can remain magnetized.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad ds\,\hat{\mathbf{s}} + sd\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

And:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Physics:

$$F=\frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F}=Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\epsilon_0=8.854\times10^{-12}\,\tfrac{\mathrm{C}^2}{\mathrm{N\cdot m}^2} \qquad \Phi_E=\int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a}=\frac{Q_{\mathrm{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E}=0$$

$$\mathbf{E}=-\nabla V \qquad -\nabla^2V=\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \qquad V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\mathrm{above}}-E^{\perp}_{\mathrm{below}}=\frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\mathrm{above}}=\mathbf{E}^{\parallel}_{\mathrm{below}} \qquad W=\frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_iq_j}{\mathscr{r}_{ij}}$$

$$W=\frac{1}{2}\int\rho V\,d\tau=\frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f}=\frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P=\frac{\epsilon_0}{2}E^2 \qquad C\equiv\frac{Q}{V}$$

$$\mathbf{p}\equiv\int\mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\mathrm{dip}}(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\mathrm{dip}}(r,\theta)=\frac{p}{4\pi\epsilon_0r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p}=\alpha\mathbf{E} \qquad \mathbf{N}=\mathbf{p}\times\mathbf{E} \qquad \mathbf{F}=(\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U=-\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b=\mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b=-\nabla\cdot\mathbf{P} \qquad \mathbf{P}=\epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}=\epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D}=\rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a}=Q_{f,\mathrm{enc}}$$

$$\mathbf{F}_{\mathrm{mag}}=Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\mathrm{mag}}=\int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K}\equiv\frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J}\equiv\frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl'=\frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0=4\pi\times10^{-7}\,\tfrac{\mathrm{N}}{\mathrm{A}^2} \qquad 1\ \mathrm{T}=1\,\tfrac{\mathrm{N}}{\mathrm{A\cdot m}}$$

$$B_{\mathrm{wire}}=\frac{\mu_0I}{2\pi s} \qquad B_{\mathrm{loop}}=\frac{\mu_0I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \qquad B_{\mathrm{sol}}=\mu_0nI$$

$$\nabla\cdot\mathbf{B}=0 \qquad \nabla\times\mathbf{B}=\mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l}=\mu_0I_{\mathrm{enc}} \qquad \mathbf{B}=\nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A}=0 \qquad \nabla^2\mathbf{A}=-\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^{\perp}_{\mathrm{above}}=B^{\perp}_{\mathrm{below}} \qquad \mathbf{B}_{\mathrm{above}}-\mathbf{B}_{\mathrm{below}}=\mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\mathrm{above}}=\mathbf{A}_{\mathrm{below}} \qquad \frac{\partial\mathbf{A}_{\mathrm{above}}}{\partial n}-\frac{\partial\mathbf{A}_{\mathrm{below}}}{\partial n}=-\mu_0\mathbf{K}$$

$$\mathbf{A}_{\mathrm{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m}\equiv I\int d\mathbf{a}=I\mathbf{a}$$

$$\mathbf{A}_{\mathrm{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\mathrm{dip}}(\mathbf{r})=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]=\frac{\mu_0m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N}=\mathbf{m}\times\mathbf{B} \qquad \mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau'+\frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b=\nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J}=\mathbf{J}_b+\mathbf{J}_f \qquad \mathbf{H}\equiv\frac{1}{\mu_0}\mathbf{B}-\mathbf{M} \qquad \nabla\times\mathbf{H}=\mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l}=I_{f,\,enc} \qquad \mathbf{M}=\chi_m\mathbf{H} \qquad \mu=\mu_0(1+\chi_m)$$