

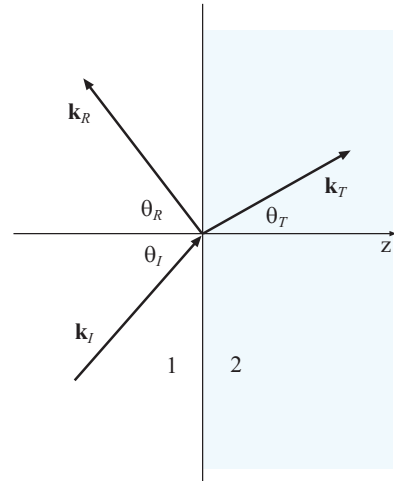
**Phys 4620, Spring 2006**  
**Exam #2**

1. When we covered “oblique incidence” at a boundary between two media we first derived Snell’s law (relating angles  $\theta_I$  and  $\theta_T$ ) and then went on to derive the relations between the *intensities* of the incoming, reflected and transmitted waves, arriving at Fresnel’s equations.

However for the latter we were forced to make an assumption about the nature of the incoming plane wave. What was this assumption? (It has to do with directions of the fields...)

One can make a different assumption about the incoming wave and get a different pair of Fresnel equations; what is the other case that we would consider?

For both of these questions, a picture will help.



2. a) What do we mean when we say a medium (for waves) is *dispersive*?

b) What is the significance of the *group velocity* for waves in a dispersive medium, and how does it differ from the *wave velocity*?

3. For a rectangular waveguide waveguide show that the group velocity of wave in TE mode  $mn$  is

$$v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$$

where

$$\omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2}$$

4. Consider a rectangular waveguide with dimensions  $3.0 \text{ cm} \times 2.0 \text{ cm}$ .

a) What is the lowest frequency of TE waves that will propagate in this waveguide?

b) What TE modes will propagate in this waveguide if the frequency is  $1.30 \times 10^{10} \text{ Hz}$ ?

c) Write out the *full* wavefunctions for the fields for the  $\{11\}$  mode.

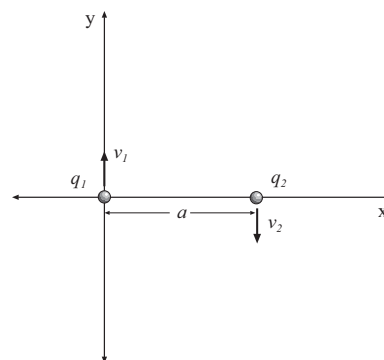
d) Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use?

5. What is meant (generally) by a choice of *gauge* in electromagnetism? Give an example of a “gauge condition”.

6. Two charges  $q_1$  and  $q_2$  move in the  $xy$  plane, as shown. ( $q_1$  is at the origin and moves in the  $+y$  with constant speed  $v_1$ .  $q_2$  is at  $x = a$  and moves in the  $-y$  direction with speed  $v_2$ .)

a) Find the force of  $q_1$  on  $q_2$ .

b) Find the force on  $q_2$  on  $q_1$ .



7. In Chap 9 it was shown that the instantaneous  $\mathbf{E}$  and  $\mathbf{B}$  fields for the oscillating magnetic dipole were

$$\mathbf{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c} \left( \frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\phi}$$

$$\mathbf{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c} \left( \frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\theta}$$

a) These fields are not exact for the problem of the current loop at the origin. What sorts of approximations were made in arriving at them?

b) Calculate the instantaneous Poynting vector  $\mathbf{S}$ . Show all the steps.

c) What is the time-averaged Poynting vector  $\langle \mathbf{S} \rangle$ ?

d) Find the total radiated power  $\langle P \rangle$ . Show the steps.

## Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \quad \int_V (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a} \quad \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

**Spherical:**

$$d\mathbf{l} = dl_r \hat{\mathbf{r}} + dl_\theta \hat{\boldsymbol{\theta}} + dl_\phi \hat{\boldsymbol{\phi}} \quad d\tau = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} \quad (1)$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (2)$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \quad (3)$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \quad (4)$$

**Cylindrical:**

$$d\mathbf{l} = dl_s \hat{\mathbf{s}} + dl_\phi \hat{\boldsymbol{\phi}} + dl_z \hat{\mathbf{z}} \quad d\tau = s \, ds \, d\phi \, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \quad (5)$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \quad (6)$$

Curl:

$$\nabla \times \mathbf{v} = \left( \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{\mathbf{s}} + \left( \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}} \quad (7)$$

Laplacian:

$$\nabla^2 T = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial T}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \quad (8)$$

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## More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

Divergences:

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

## Product Rules:

(1)  $\nabla \cdot (\nabla T)$  (Divergence of curl)

$$\nabla \cdot (\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2)  $\nabla \times (\nabla T)$  (Curl of gradient)

$$\nabla \times (\nabla T) = 0$$

(3)  $\nabla(\nabla \cdot \mathbf{v})$  (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4)  $\nabla \cdot (\nabla \times \mathbf{v})$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

(5)  $\nabla \times (\nabla \times \mathbf{v})$  (Curl of a curl)

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

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## Physics:

$$F = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r^2} \hat{\mathbf{r}} \quad \mathbf{F} = Q\mathbf{E} \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad \nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \quad -\nabla^2 V = \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$$

$$E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0} \quad \mathbf{E}_{\text{above}}^\parallel = \mathbf{E}_{\text{below}}^\parallel \quad W = \frac{1}{8\pi\epsilon_0} \sum_{\substack{i,j \\ i \neq j}}^n \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{2} \int \rho V d\tau = \frac{\epsilon_0}{2} \int E^2 d\tau \quad \mathbf{f} = \frac{1}{2\epsilon_0} \sigma^2 \hat{\mathbf{n}} \quad P = \frac{\epsilon_0}{2} E^2 \quad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}') d\tau' \quad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \quad \mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha \mathbf{E} \quad \mathbf{N} = \mathbf{p} \times \mathbf{E} \quad \mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E} \quad U = -\mathbf{p} \cdot \mathbf{E}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \quad \rho_b = -\nabla \cdot \mathbf{P} \quad \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \quad \nabla \cdot \mathbf{D} = \rho_f \quad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f, \text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}) \quad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l} \times \mathbf{B}) \quad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \quad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\tau}}}{\tau^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\tau}}}{\tau^2} \quad \mu_0 = 4\pi \times 10^{-7} \frac{\text{N}}{\text{A}^2} \quad 1 \text{ T} = 1 \frac{\text{N}}{\text{A} \cdot \text{m}}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \cdot \mathbf{A} = 0 \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\tau} d\tau'$$

$$B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \quad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}) \quad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \quad \frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \quad \text{where} \quad \mathbf{m} \equiv I \int d\mathbf{a} = I \mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2} \hat{\boldsymbol{\phi}} \quad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\tau} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{\tau} da' \quad \text{where} \quad \mathbf{J}_b = \nabla \times \mathbf{M} \quad \text{and} \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \quad \mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \nabla \times \mathbf{H} = \mathbf{J}_f \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f, \text{enc}}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \frac{1}{2} L I^2 \quad W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau$$

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \quad B_1^{\perp} - B_2^{\perp} = 0 \quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \quad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

$$\Phi_2 = M_{21} I_1 \quad \mathcal{E} = -L \frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \oint_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a} \quad \mathbf{S} \equiv \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

$$T_{ij} \equiv \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Waveguides:

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y) e^{i(kz - \omega t)} \quad \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y) e^{i(kz - \omega t)}$$

$$\tilde{\mathbf{E}}_0 = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}} \quad \tilde{\mathbf{B}}_0 = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$$

$$E_x = \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)$$

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0 \quad \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0$$

$$\text{TE solution:} \quad B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{A}' = \mathbf{A} + \nabla \lambda \quad V' = V - \frac{\partial \lambda}{\partial t}$$

$$\text{Coulomb : } \nabla \cdot \mathbf{A} = 0 \quad \text{Lorentz : } \nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{r} d\tau' \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{r} d\tau'$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{rc - \mathbf{r} \cdot \mathbf{v}} \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(rc - \mathbf{r} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(\mathbf{r} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})] \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \mathbf{r} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad \mathbf{B} = \frac{1}{c} (\hat{\mathbf{r}} \times \mathbf{E}) = \frac{1}{c^2} (\mathbf{v} \times \mathbf{E})$$

$$\mathbf{R} = \mathbf{r} - \mathbf{v}t, \theta \text{ is between } \mathbf{R} \text{ and } \mathbf{v}$$