Phys 3810, Spring 2009
Exam #2

1. Give short definitions of the terms (as used by Griffiths)

a) Hilbert space of functions.

This is the set of functions for which the modulus squared can be integrated over the entire applicable space to give a finite result, i.e. they can be normalized. (It is a vector space, following the rules for vector addition, with an inner product defined by an integral.)

b) Determinate state.

Pertaining to a specific observable, these are states where measurement of that observable always gives the same result. They are also eigenstates of the observable's operator.

c) Degenerate states.

States which are linearly independent but which have the same eigenvalue (for a particular operator/observable).

d) Compatible observables.

Compatible observables are those for which we can from *simultaneous* eigenstates, that is, states which are eigenvectors of both operators. For compatible observables, the corresponding operators must commute.

2. Give a brief but *correct* summary of the energy–time uncertainty relation $\Delta t\,\Delta E \geq \frac{\hbar}{2}$.

It isn't what most people say.

If we consider a *particular* observable, say, Q and consider the length of the time it takes for Q to change by one standard deviation (and call it Δt) then Δt and the energy uncertainty ΔE are related by

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

It's not about the amount of time for which God lets you violate conservation of energy. Sheesh.

3. (An easy one) An operator $\hat{A}$ (representing observable A) has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$ has two normalized eigenstates ϕ_1 and ϕ_2 with eigenvalues b_1 and b_2 respectively. The eigenstates are related by

$$\psi_1 = (2\phi_1 - \phi_2)/\sqrt{5} \qquad\qquad \psi_2 = (\phi_1 + 2\phi_2)/\sqrt{5}$$

a) Observable A is measured and the value a_2 obtained. What is the state of the system (immediately) after this measurement?

If A is measured and the value a_2 is the result, the system is then immediately put into state ψ_2.

b) If B is now measured, what are the possible results and what are their probabilities?

Since the system is now in state ψ_2, the possible results of a measurement of B are b_1 and b_2. Squaring the coefficients of ϕ_1 and ϕ_2 in the expression for ψ_2, these values would occur with probabilities

$$b_1 : \quad \frac{1}{5} \qquad \text{and} \qquad b_2 : \quad \frac{4}{5}$$

respcitively.

4. Evaluate the commutator of the kinetic energy and square of the coordinate operator,

$$\left[\frac{p^2}{2m}, x^2\right] .$$

Evaluate and explain all the steps.

We could use $[x,p] = i\hbar$, but here I'll derive it from scratch; consider $[p^2, x^2]$. Since $p^2 = -\hbar^2 \frac{d^2}{dx^2}$, the action of the operator $p^2 x^2$ on a function $f(x)$ gives

$$\begin{aligned} p^2(x^2 f(x)) &= -\hbar^2 \frac{d^2}{dx^2}(x^2 f(x)) \;=\; -\hbar^2 \frac{d}{dx}(2x f(x) + x^2 f'(x)) \\ &= -\hbar^2 (2f(x) + 2x f'(x) + 2x f'(x) + x^2 f''(x)) \\ &= -\hbar^2 \left(2f(x) + 4x(i/\hbar) p\, f(x) + x^2(-1/\hbar^2) p^2\right) \; f(x) \\ &= (-2\hbar^2 - 4xi\hbar p + x^2 p^2) f(x) \end{aligned}$$

Then

$$[p^2, x^2] f(x) = (p^2 x^2 - x^2 p^2) f(x) = (-2\hbar^2 - 4ix\hbar p) f(x)$$

Include the $1/2m$ factor,

$$\left[\frac{p^2}{2m}, x^2\right] = -\frac{\hbar^2}{m} - \frac{2ix\hbar}{m} p$$

5. In the space of states consisting of $|1\rangle$, $|2\rangle$, $|3\rangle$, express as a matrix the operator

$$H = a|1\rangle\langle 1| + a|2\rangle\langle 2| + c|3\rangle\langle 3| + \epsilon|1\rangle\langle 2| + \epsilon|2\rangle\langle 1| + +\epsilon|2\rangle\langle 3| + \epsilon|3\rangle\langle 2|$$

$$\text{where} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{etc.}$$

Outline how you would find the eigenvalues and eigenvectors of H. (You don't need to actually find them.)

The diagonal elements of H are (a, a, c) and "connecting" the states $|1\rangle$ and $|2\rangle$ is the coefficient ϵ, likewise for $|2\rangle$ and $|3\rangle$. We get:

$$H = \begin{pmatrix} a & \epsilon & 0 \\ \epsilon & a & \epsilon \\ 0 & \epsilon & c \end{pmatrix}$$

To get the eigenvalues we would solve the equation

$$\mathrm{Det}(H - \lambda \mathbf{1}) = 0$$

Here, this would give a cubic equation for λ. (Generally, there are three roots though in this case a couple might repeat.) For each root λ, solve for the normalized vector $\mathbf{x}$ which gives $H\mathbf{x} = \lambda\mathbf{x}$.

6. Demonstrate that the angular wave function

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$$

is indeed normalized.

Integrate $Y_2^{0*}\, Y_2^0$ over the entire range of angle variables. Get:

$$\int Y_2^{0*}\, Y_2^0\, d\Omega = \frac{5}{16\pi} \int_0^{2\pi} \int_0^{\pi} (3\cos^2\theta - 1)^2 \sin\theta\, d\theta\, d\phi$$

The ϕ integration gives a factor of 2π. For the θ integration, this is a perfect time to use the substitution $x = \cos\theta$. (Change $\cos\theta$ to x in the integrated function and limits to $x : -1 \rightarrow 1$:

$$\begin{aligned} \int Y_2^{0*}\, Y_2^0\, d\Omega &= \frac{5}{16\pi}(2\pi) \int_{-1}^{1} (3x^2 - 1)^2\, dx = \frac{5}{8} \int_{-1}^{1} (9x^4 - 6x^2 - 1)\, dx \\ &= \frac{5}{8}(\tfrac{9}{5}x^5 - 2x^2 - x)\Big|_{-1}^{1} = \frac{5}{8}(\tfrac{18}{5} - 0 - 2) = \frac{5}{4}(\tfrac{9}{5} - 1) = \frac{5}{4}\frac{4}{5} = 1 \end{aligned}$$

7. Give a brief summary of the part of the derivation of the energy eigenvalues for the H atom which gave a condition on the (discrete) permitted values for the energy (and the allowed energy eigenfunctions).

When solving for the radial wave function, we first pulled off the parts which give the behavior at large and small r. We were left with a series for which we found a recurrence relation. We found that if the series is allowed to have an infinite number of terms it will overpower the decaying exponential factor and blow up at large r. Thus the series must terminate and the condition that the series only have, say, n terms gives a condition on the possible values of the energy.

8. Find $\langle r^2 \rangle$ for ground state of H atom.

Evaluate

$$\langle r^2 \rangle = \int \psi_{100}(r,\theta,\phi)^*\, r^2\, \psi_{100}(r,\theta,\phi)\, dV$$

Using $R_{10}(r) = 2a^{-3/2}e^{-r/a}$, and integrating out the (normalized) angular part, get

$$\begin{aligned} \langle r^2 \rangle &= \frac{4}{a^3} \int_0^{\infty} e^{-r/a} r^2 e^{-r/a} r^2\, dr \\ &= \frac{4}{a^3} \int_0^{\infty} r^4 e^{-2r/a}\, dr = \frac{4}{a^3}(24)\left(\frac{a}{2}\right)^5 = 3a^2 \end{aligned}$$

9. Find the most probable value of r for an electron in the $n = 2$, $\ell = 1$ state of the H atom.

Using $R_{21}(r) = \frac{a^{-3/2}}{\sqrt{24}}(r/a)\exp(-r/2a)$, the probability distribution to be found at r within dr is (as seen on a problem set) $R(r)^2 r^2$. Thus

$$P(r) = R_{21}(r)^2 r^2 = \frac{1}{24a^3}\frac{r^4}{a^2}e^{-r/a}$$

To find the maximum, take d/dr and set equal to zero:

$$\frac{dP}{dr} = \frac{1}{24a^5}(4r^3 - \frac{r^4}{a})e^{-r/a} = 0$$

which has solution at

$$4r^3 - \frac{r^4}{a} = 0 \qquad \Longrightarrow \qquad r = 4a$$

10. One can form a hydrogen–like "atom" by replacing the electron with a muon, a short-lived particle with charge $-e$ and a mass of 105.7 MeV/c^2, that is, 207 times the mass of the electron. We could consider a muon in a bound state from the Coulomb field of the proton, and call the system atomic **muonium**.

a) For the following questions, we will still consider the proton to be stationary; say a few words about why that approximation is more questionable for muonium.

Since the muon has about a ninth the mass of the proton, it is still a *reasonable* approximation to suppose that the more massive proton is motionless, but as we would guess from classical mechanics the motion must (in some sense) take place around the *center of mass* of the system, which is about one--ninth of the way from the proton to the muon. And we would guess that the mass m in all the results must be the *reduced mass* of the system.

b) Find the binding energy of the ground state of the muonium atom.

It is sufficient to find the dependence on m in all of the results and see what happens when m_{e} is replaced by $207m_{\mathrm{e}}$.

The binding energy E_n is *proportional* to m, so the binding energy of the ground state of muonium is

$$-E_1^{(\mu)} = (207)(13.6\ \mathrm{eV}) = 2.82 \times 10^3\ \mathrm{eV} = 2.82\ \mathrm{k}eV$$

c) Find the expectation value of r for the muonium ground state.

The expectation value of r for the H atom ground state one would think would be close to the Bohr radius a, but to derive this result, use the radial wavefunction to evaluate

$$\begin{aligned}\langle r \rangle &= \int_0^\infty R_{10}(r)^2 r^3 dr = \frac{2}{a^3}\int_0^\infty r^3 e^{-2r/a} \\ &= \frac{4}{a^3}6\frac{a^4}{2^4} = \tfrac{3}{2}a\end{aligned}$$

Since a is $inversely$ proportional to the mass, we have

$$\langle r \rangle_1 = \left(\frac{1}{207}\right) \tfrac{3}{2}(0.529 \times 10^{-10}\ \mathrm{m}) = 3.83 \times 10^{-13}\ \mathrm{m}$$

d) Find the wave length of the first Lyman transition in muonium.

The wave length is inversely proportional to R which in turn is proportional to m. So λ is inversely proportional to m. As the wavelength of the first Lyman transition is

$$\frac{1}{\lambda} = R(1 - \tfrac{1}{4}) \qquad \Longrightarrow \qquad \lambda = 1.2 \times 10^{-7}\ \mathrm{m}$$

the corresponding wavelength for muonium is

$$\lambda^{(\mu)} = \left(\frac{1}{207}\right)(1.21 \times 10^{-7}\ \mathrm{m}) = 5.9 \times 10^{-10}\ \mathrm{m}$$

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\bigg|_a^b \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_{\mathrm{e}} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_{\mathrm{p}} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \mathrm{\tfrac{m}{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty}\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E + \hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E - \hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_{\mathrm{phase}} = \frac{\omega}{k} \qquad v_{\mathrm{group}} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A},\hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \, \cos\theta e^{\pm i\phi} \qquad \text{etc.}$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c\hbar^3} \left(\frac{c^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$