**Phys 2920, Spring 2010**
**Exam #3**

**1.** Verify Stokes' theorem for the vector field

$$\mathbf{a} = r^2 \sin^2\theta\, \hat{\boldsymbol{\phi}}$$

for the surface which is the top half of the sphere of radius 2 centered at the origin and the curve bounding this surface is the circle of radius 2 in the $xy$ plane.

The curl of this vector field (note, there is $only$ an $a_\phi$ here) is

$$\begin{aligned} \nabla \times \mathbf{a} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\phi)\,\hat{\mathbf{e}}_r - \frac{1}{r}\frac{\partial}{\partial r}(r a_\phi)\,\hat{\mathbf{e}}_\phi \\ &= \frac{1}{r\sin\theta}(r^2 3\sin^2\theta\cos\theta)\,\hat{\mathbf{e}}_r - \frac{1}{r}(3r^2\sin^2\theta)\,\hat{\mathbf{e}}_\theta \\ &= 3r\sin\theta\cos\theta\,\hat{\mathbf{e}}_r - 3r\sin^2\theta\,\hat{\mathbf{e}}_\theta \end{aligned}$$

The area element (vector) for constant $r$ is

$$d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{e}}_r$$

and then the surface integral is

$$I = \int_S \mathbf{a}\cdot d\mathbf{S} = \int_S (3r\sin\theta\cos\theta) r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=2} = 24\int_0^{2\pi}\int_0^{\pi/2} \sin^2\cos\theta\, d\theta\, d\phi$$

Integration on $\phi$ gives $2\pi$ and we just have an easy $\theta$ integral which gives

$$24(2\pi)\frac{1}{3}\sin^3\theta\Big|_0^{\pi/2} = 16\pi$$

For the curve which bounds this circle, $r=2$ and $\theta=\pi/2$. For constant $r$ and $\theta$, the line element is

$$d\mathbf{r} = r\sin\theta\, d\phi\, \hat{\mathbf{e}}_\phi$$

and then the (closed) line integral is

$$\oint_C \mathbf{a}\cdot d\mathbf{r} = \int_0^{2\pi}(r^2\sin^2\theta)(r\sin\theta)\, d\phi\Big|_{r=2,\theta=\pi/2} = 8\int_0^{2\pi} d\phi = 16\pi$$

**2.** Evaluate

$$\int_0^5 (3\cos(x) - 2\sin(x))\ [\delta(x-\pi) - \delta(x-2\pi)]\ dx$$

There will be contributions from delta functions if the integration range includes $x=\pi$ or $x-2\pi$; since it only goes up to $5$, only $x=\pi$ contributes. Then the integral is

$$I = \int_0^5 (3\cos(x) - 2\sin(x))\ \delta(x-\pi)\, dx = 3\cos(\pi) - 2\sin(\pi) = -3$$

**3.** If $z_1 = 2+5i$ and and $z_2 = 3-4i$, find:

(a) $|z_1 + z_2|$ (b) $z_1 z_2^*$ (c) $\dfrac{z_1}{(z_2)^2}$

Express all answers in $a+bi$ form, i.e. simplify them.

(a) Since $z_1 + z_2 = 5 + i$, then

$$|z_1 + z_2| = |5+i| = \sqrt{25+1} = \sqrt{26}$$

(b) We get

$$z_1 z_2^* = (2+5i)(3+4i) = (6-20) + i(8+15) = -14+23i$$

(c) We get:

$$\frac{z_1}{(z_2)^2} = \frac{(2+5i)}{(3-4i)^2} = \frac{(2+5i)}{(9-16)+2i(-12)} = \frac{2+5i}{-7-24i}$$

Clean this up by multiplying top and bottom by $-7+24i$:

$$\Longrightarrow \quad = \frac{2+5i}{-7-24i} \cdot \left(\frac{-7+24i}{-7+24i}\right) = \frac{-134-13i}{625}$$

**4.** Using complex functions (and our definitions of them) prove that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

*Hint: Do something with $e^{i\alpha}$ and $e^{i\beta}$ !*

We note that

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

But this is also equal to

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \end{aligned}$$

Equating the real and imaginary parts, we get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

as well as

$$sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

So we got the addition formula for the sine for free!

**5.** Using a calculator which only does *real* operations (and showing your work), evaluate:

(a) $\ln(-13+8i)$ (b) $\tan(3-4i)$ (c) $\sin^{-1}(5i)$

(a) Write the argument in polar form:

$$r = \sqrt{(13)^2 + (8)^2} = \sqrt{233} \qquad \tan\phi = \frac{8}{(-13)} \qquad \Longrightarrow \qquad \phi = 2.59$$

then the answer is

$$\ln(\sqrt{233}e^{i(2.59)}) = \ln(\sqrt{233}) + i(2.59) = 2.73 + i(2.59)$$

(b) As I didn't give an addition formula for the tangent, we can evaluate

$$\begin{aligned} \sin(3-4i) &= \sin(3)\cos(-4i) + \cos(3)\sin(-4i) = \sin(3)\cosh(-4) - i\cos(3)\sinh(4) \\ &= 3.854 + 27.02i \end{aligned}$$

Likewise,

$$\begin{aligned} \cos(3-4i) &= \cos(3)\cos(-4i) - \sin(3)\sin(-4i) = \cos(3)\cosh(4) + i\sin(3)\sinh(4) \\ &= -27.03 + i3.8512 \end{aligned}$$

With some messy arithmetic (which I'll forgive you for not doing or for using the complex mode of a calculator) this gives

$$\tan(3-4i) = \frac{3.854 + i27.02}{-27.03 + i3.8512} = (1.87 \times 10^{-4}) - i(0.9994)$$

(c) $\sin^{-1}(5i)$ is a little less messy. Use:

$$\begin{aligned} \sin^{-1}(5i) &= \frac{1}{i}\ln\left(i(5i) + \sqrt{1-(5i)^2}\right) = \frac{1}{i}\ln(-5+\sqrt{26}) \\ &= -i\ln(0.09902) = 2.3124i \end{aligned}$$

**6.** How would you make a branch cut(s) for the function

$$f(z) = (z^2 + 16)^{1/2} \quad ?$$

(That is, what imposed boundary will make this function single-valued?)

This function has *branch points* at $z = \pm 4i$ because a trip around each of these points (*individually*) will result in a different value of $f(z)$ when we get back to the starting point.

To prevent this, we can draw lines from each of the points $\pm 4i$ out to infinity or else draw a line connecting these two points.

**7.** If a function $f(z)$ has a *complex* derivative in a region it is said to be analytic. Give two further consequences of a function being analytic.

Other properties: The Cauchy-Riemann equations (partial derivatives of $u(x,y)$ and $v(x,y)$) are satisfied. The integral around a closed path which does not enclose any poles is zero. The function can be expressed as a Taylor series in a circle about a point.

**8.** Verify Cauchy-Riemann conditions for the function $f(z) = z^2 + 4iz$. (Find the functions $u(x, y)$ and $v(x, y)$ first.)

Find $u(x, y)$ and $v(x, y$:

$$\begin{aligned} f(z) &= (x+iy)^2 + 4i(x+iy) = x^2 - y^2 + 2ixy + 4ix - 4y \\ &= (x^2 - y^2 - 4y) + i(2xy + 4x) = u(x, y) + iv(x, y) \end{aligned}$$

And this gives

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x \qquad\qquad \frac{\partial u}{\partial y} = -2y - 4 \quad -\frac{\partial v}{\partial x} = -(2y+4) = -2y - 4$$

so the C-R relations are satisfied.

**9.** Find the limits

$$\text{(a)} \quad \lim_{z \to 3i} \frac{z^2 + (4-3i)z - 12i}{(z^2+9)} \qquad\qquad \text{(b)} \quad \lim_{z \to \pi i} \frac{(z - \pi i)e^z}{\sinh(z)}$$

(a) Here blindly plugging in $3i$ gives zero on top and bottom, so take a derivatives of the top and bottom. Get:

$$\lim_{z \to 3i} \left( \frac{2z + (4-3i)}{2z} \right) = \frac{2(3i) + 4 - 3i}{6i} = \frac{4+3i}{6i} = \frac{1}{2} - \frac{2}{3}i$$

(b) Here, blindly plugging in $\pi i$ again gives zero top and bottom, since $\sinh(\pi i) = i \sin(\pi) = 0$. So take derivatives. This gives:

$$\Longrightarrow \qquad = \lim_{z \to \pi i} \left( \frac{(z - i\pi)e^z + e^z}{\cosh(z)} \right) = \frac{0 + e^{i\pi}}{\cosh(i\pi)} = \frac{-1}{\cos(\pi)} = 1$$

**10.** For the function

$$f(z) = \frac{e^z}{(z+4)^2(z-3)}$$

find all the poles; give their order and find the residue of $f(z)$ at any one of them.

The denominator vanishes at $z = -4$ and $z = 3$ (on the real axis this time) so those are the poles.

The pole at $z = 3$ is of order one (simple) and has residue

$$a_{-1} = \lim_{z \to 3} \frac{(z-3)e^z}{(z+4)^2(z-3)} = \frac{e^3}{(3+4)^2} = \frac{e^3}{49}$$

The pole at $z = -4$ is of order two and has residue

$$a_{-1} = \lim_{z \to -4} \left( \frac{d}{dz} \frac{(z+4)^2 e^z}{(z+4)^2(z-3)} \right) = \lim_{z \to -4} \left( \frac{d}{dz} \frac{e^z}{(z-3)} \right)$$

Doing the derivative,

$$\Longrightarrow \qquad = \lim_{z\to -4}\left(\frac{e^z}{(z-3)} - \frac{e^2}{(z-3)^2}\right) = e^{-4}\left(-\frac{1}{7}-\frac{1}{49}\right) = -e^{-4}\frac{8}{49}$$

**11.** If we do a Taylor expansion of the function

$$f(z) = \frac{4}{z^2+5}$$

about the origin, that is, write it in the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

in what region will the series converge?

The function $f(z)$ clearly has a (simple) poles at $z = \pm i\sqrt{5}$, both of which lie at a distance of $\sqrt{5}$ from the origin. From Taylor's theorem, the circle of convergence for the Taylor series is a circle of radius $\sqrt{5}$ centered at the origin.

**12.** Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$$

using a contour integral and the residue theorem.

Even if you can't get to the final answer, demonstrate all that you understand about working out the integral this way.

From our experience with similar integrals, we clearly want to to the integral as

$$I = \oint_C \frac{dz}{(z^2+4)^2}$$

where $C$ will go along the $x$ axis from $-R$ to $+R$ where $R \to \infty$ and then return along a semi-circle of radius $R$; we will assume that the integral on the semi-circle will vanish in the limit.

The integrand here has poles at $z = \pm 2i$, and clearly they are poles of *order two*. A contour closed in the upper half-plane will enclose the pole at $z = +2i$, so find the residue there. If we note that the denominator factors as

$$(z^2+4) = (z+2i)(z-2i)$$

then the residue formula with $k = 2$ gives

$$\begin{aligned} a_{-1} &= \lim_{z\to 2i}\left(\frac{1}{1}\frac{d}{dz}\frac{(z-2i)^2}{(z^2+4)^2}\right) = \lim_{z\to 2i}\left(\frac{d}{dz}\frac{1}{(z+2i)^2}\right) \\ &= \lim_{z\to 2i}\left(\frac{-2}{(z+2i)^2}\right) = \frac{-2}{(4i)^3} = \frac{-2}{-64i} = \frac{1}{32i} \end{aligned}$$

Then by the residue theorem,

$$\oint_C \frac{dz}{(z^2+4)^2} \;=\; (2\pi i)a_{-1} \;=\; (2\pi i)\frac{1}{32i} \;=\; \frac{\pi}{16}$$

and with the semi-circle giving no contribution we conclude

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}$$

Though you couldn't check the answer during the exam, Maple agrees! And note, the answer has a $\pi$ in it, though it did not come from $explicit$ use of an inverse trig function.

## Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \qquad \text{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\text{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \qquad \hat{\mathbf{z}} = \mathbf{k} \tag{2}$$

$$\mathbf{i} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{j} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \mathbf{k} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz \tag{4}$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi \tag{5}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{6}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\mathbf{i} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad \int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$z = x + iy = \rho e^{i\phi} \qquad |z| = \rho = \sqrt{x^2 + y^2} \qquad z^* = x - iy \qquad w = \ln z = \ln r + i(\theta + 2k\pi)$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cosh z = \frac{e^z + e^{-z}}{2} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2 \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \mathrm{sech}^2 z \qquad \coth^2 z - 1 = \mathrm{csch}^2 z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin(iz) = i \sinh z \qquad \cos(iz) = \cosh z$$

$$\sin^{-1}(z) = \frac{1}{i} \ln(iz + \sqrt{1 - z^2}) \qquad \cos^{-1}(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}) \qquad \tan^{-1}(z) = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad w = f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

$$(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots + \frac{p(p-1)\cdots(p-n-1)}{n!} z^n + \cdots$$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{(z - a)} + \frac{a_{-2}}{(z - a)^2} + \frac{a_{-3}}{(z - a)^3} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \qquad a_{-1} = \lim_{z \to a} \left( \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z - a)^k f(z) \right)$$

$$\oint_C f(z) \, dz = 2\pi i \{a_{-1} + b_{-1} + c_{-1} + \ldots\}$$