Phys 2920, Spring 2013
Exam #3

1. For the vector field

$$\mathbf{a} = r^2 \cos^2\theta\, \hat{\mathbf{e}}_r + r\sin\theta\, \hat{\mathbf{e}}_\theta + r^2\cos^2\theta\, \hat{\mathbf{e}}_\phi$$

verify the divergence theorem for the case where the volume is the sphere of radius 2 centered on the origin.

Using the formula for spherical coordinates, the divergence of $\mathbf{a}$ is

$$\begin{aligned}\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos^2\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) \\ &= 4r\cos^2\theta + 2\cos\theta\end{aligned}$$

The integral of $\nabla\cdot\mathbf{a}$ over the volume is

$$\begin{aligned}\int_V \nabla\cdot\mathbf{a}\, dV &= 2\pi\int_0^\pi d\theta\int_0^2 dr[4r\cos^2\theta + 2\cos\theta]\, r^2\, dr\; \sin\theta\, d\theta \\ &= 2\pi\left[\int_0^2 4r^3\, dr\int_0^\pi \cos^2\theta\sin\theta\, d\theta + 2\int_0^2 r^2\, dr\int_0^\pi \cos\theta\sin\theta\, d\theta\right] \\ &= 2\pi\left[r^4\Big|_0^2 \cdot \frac{x^3}{3}\Big|_{-1}^1 + 2\frac{r^3}{3}\Big|_0^2 \cdot \frac{x^2}{2}\Big|_0^2\right] = 2\pi[(16)\tfrac{2}{3} + 0] = \frac{64\pi}{3}\end{aligned}$$

The integral $\oint_S \mathbf{a}\cdot d\mathbf{S}$ is

$$\begin{aligned}\int_S \mathbf{a}\cdot d\mathbf{S} &= \int_0^{2\pi} d\phi\int_0^\pi d\theta\mathbf{a}\cdot(r^2\sin\theta)\Big|_{r=2} = \int_0^{2\pi} d\phi\int_0^\pi d\theta(r^2 cos^2)r^2\sin\theta\Big|_{r=2} \\ &= (2\pi)2^4\int_0^\pi \cos^2\theta\sin\theta\, d\theta = (32\pi)\int_{-1}^1 x^2\, dx = 32\pi\frac{x^3}{3}\Big|_{-1}^1 = 32\pi(\tfrac{2}{3}) \\ &= \frac{64\pi}{3}\end{aligned}$$

2. Evaluate

a)

$$\int_{-\infty}^{\infty}(x^2+3x)\,((\delta(x-2) - \delta(x+4))\; dx$$

The delta-function picks out the points $x = 2$ and $x = -4$ (which are both in the integration range!), giving two terms (the second with a minus sign):

$$I = (2^2 + 3\cdot 2) - ((-4)^2 + 3(-4)) = 10 - 4 = 6$$

b)

$$\int_V \delta^3(\mathbf{r} - 2\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}})(\mathbf{r} - 3\,\hat{\mathbf{j}})^2 \, d^3r$$

where V is a sphere centered on the origin with radius 6.

The delta function picks out the vector $\mathbf{r} = 2\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}}$ which has magnitude $\sqrt{29}$ which is less than 6 so that this vector *is* in the integration range. So evaluate the integrand function at this vector:

$$I = (2\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}} - 3\,\hat{\mathbf{j}})^2 = (-\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}})^2 = 1 + 25 = 26$$

3.

a) Evaluate

$$\left|\frac{4-2i}{(3+6i)}\right|$$

$$\left|\frac{4-2i}{(3+6i)}\right| = \frac{|4-2i|}{|3+6i|} = \frac{\sqrt{20}}{\sqrt{45}} = \frac{2\sqrt{5}}{3\sqrt{5}} = \frac{2}{3}$$

b) Put into $a + bi$ form:

$$\frac{(4+i)^2}{(2-3i)}$$

$$\begin{aligned} \Rightarrow \quad &= \frac{(16-1+8i)}{(2-3i)} = \frac{(16-1+8i)}{(2-3i)}\frac{(2+3i)}{(2+3i)} = \frac{30-24+61i}{13} \\ &= \frac{6+61i}{13} = \tfrac{6}{13} + \tfrac{61}{31}i \end{aligned}$$

4. Find

$$\mathrm{Ln}\,(-2+5i)$$

Recall that in our notation, Ln is a multi-valued function.

Write $-2+5i$ in polar form (this should be old by now):

$$-2 + 5i = \sqrt{29}e^{i(1.95)}$$

Then we get

$$\mathrm{Ln}(-2+5i) = \ln(\sqrt{29}) + i(1.95 + 2\pi n) = 1.68 + i(1.95 + 2\pi n)$$

5. Evaluate with the "\$15 calculator method",

a) $\cos^{-1}(4)$

Use the formula for the (complex) inverse cosine; one of its values is given by

$$\cos^{-1}(4) = \frac{1}{i}\ln(4+\sqrt{16-1}) = \frac{1}{i}(2.063) = -2.063i$$

b) $\sin(1+3i)$

$$\begin{aligned}\sin(1+3i) &= \sin(1)\cos(3i) + \cos(1)\sin(3i)\\ &= \sin(1)\cosh(3) + i\cos(1)\sinh(3) = 8.472 + i\,5.413\end{aligned}$$

6. How would we induce a "branch cut(s)" so that the function

$$f(z) = (z^2+1)^{1/2}$$

is well-behaved?

This function is a fraction power (which gives multiple values) of an argument which is zero at $z = \pm i$. Thus if we walk around i or $-i$ we get a different value of the function from when we started! The way to prevent this and thus get a single valued function z is to prevent a walk around i or $-i$ *individually* and this can be done with a pair of cuts that start at $\pm i$ and go out to infinity.

In fact from the $\frac{1}{2}$ power one can walk around them *both* and get to same value so that a suitable cut can also go from $-i$ to i.

7. Explain what is meant when we say that the derivative of a complex function $f(z)$ exists.

We mean that when we consider the limit

$$\lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h}\right)$$

regardless of how *complex* h goes to zero, that is, h can go to zero from any direction in the complex plane.

8. Show that the Cauchy-Riemann conditions for the function

$$f(z) = ze^{2z}$$

are satisfied.

With $z = x+iy$ and $f(z) = u(x,y) + iv(x,y)$ (with u and v real functions), find u and v from:

$$f(z) = ze^{2z}(x+iy)e^{2(x+iy)} = (x+iy)e^{2x}e^{2iy} = (x+iy)e^{2x}(\cos 2y + i\sin 2y)$$

Then u and v are

$$f(z) = e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(y\cos 2y + x\sin 2y)$$

so then

$$u(x,y) = e^{2x}(x\cos 2y - y\sin 2y) \qquad \text{and} \qquad v(x,y) = e^{2x}(y\cos 2y + x\sin 2y)$$

These give

$$\frac{\partial u}{\partial x} = 2e^{2x}[x\cos 2y - y\sin 2y] + e^{2x}[\cos 2y]$$

and

$$\frac{\partial v}{\partial y} = e^{2x}[\cos 2y - 2y\sin 2y + 2x\cos 2y]$$

comparison of these show that the terms match so then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

Check the other C-R relation; calculate

$$\frac{\partial u}{\partial y} = e^{2x}[-2x\sin 2y - \sin 2y - 2y\cos 2y]$$

and

$$\frac{\partial v}{\partial x} 2e^{2x}[y\cos 2y + x\sin 2y] + e^{2x}[\sin 2y]$$

and comparison of these shows that the terms match with opposite signs, so that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

9. Do the following contour integral where the path C is the straight line which goes from the point $z = 2i$ to the point $z = 3$:

$$\int_C e^{2z}\, dz$$

As the integrand is analytic everywhere, the contour integral from one point to another in the complex plane does not depend on the path and can be evaluated by the formal techniques of basic calculus:

$$\int_C e^{2z}\, dz = \tfrac{1}{2}e^{2z}\Big|_{2i}^{3} = \tfrac{1}{2}[e^6 - e^{4i}]$$

Get some numbers for this expression:

$$I = \tfrac{1}{2}[e^6 - \cos 4 - i\sin 4] = \tfrac{1}{2}[404 + i\,0.757] = 202 + i\,0.378$$

10. Do the following integral using a contour integral in the complex plane; justify all the steps as much as you can.

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\, dx$$

[Actually, it can be done by conventional means, but try to do it with a single contour.]

We complete the path on the real axis with a return semi-circular path of radius R (with $R \to \infty$) in the upper half-plane and evalute

$$\oint_C \frac{z^2}{(z^2+4)(z^2+9)}\, dz$$

where C is this closed path. The integrand has poles at $z = \pm 2i$ and $z = \pm 3i$ (all of order one) and this contour encloses the ones at $z = 2i, 3i$.

From the work done in class we note that the entire integrand has a behavior like $\frac{1}{z^2}$ for z of large magnitude, so that it falls off like $1/R^2$, and as the length of the curvy part is proportional to R, the integral on the curvy part gives a vanishing contribution as $R \to \infty$.

Find the residues at the two poles:

At $z = 2i$,

$$\begin{aligned} \text{Res} = \lim_{z \to 2i} \frac{(z-2i)z^2}{(z^2+4)(z^2+9)} &= \lim_{z \to 2i} \frac{z^2}{(z+2i)(z^2+9)} \\ &= \frac{-4}{(4i)(5)} = \frac{i}{5} \end{aligned}$$

At $z = 3i$,

$$\begin{aligned} \text{Res} = \lim_{z \to 3i} \frac{(z-3i)z^2}{(z^2+4)(z^2+9)} &= \lim_{z \to 2i} \frac{z^2}{(z^2+4)(z+3i)} \\ &= \frac{-9}{(-5)(6i)} = -\frac{3i}{10} \end{aligned}$$

This gives

$$I = 2\pi i \sum(\text{Res}) = 2\pi i \left(\frac{i}{5} - \frac{3i}{10}\right) = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Useful Equations

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \qquad \Longrightarrow \qquad c_k = \sum_{i,j=1}^{3} a_i b_j \epsilon_{ijk}$$

$$\nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \qquad \mathrm{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \\
&= \nabla\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}
\end{aligned}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{1}$$

$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{2}$$

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\mathbf{e}}_\rho + \sin\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \qquad \hat{\mathbf{k}} = \hat{\mathbf{e}}_z \tag{3}$$

$$d\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z \qquad dV = \rho\,d\rho\,d\phi\,dz \tag{4}$$

$$da_\rho = \rho\,d\phi\,dz \qquad da_\phi = d\rho\,dz \qquad da_z = \rho\,d\rho\,d\phi \tag{5}$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_z \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{6}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{i}} &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{j}} &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\hat{\mathbf{k}} &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta
\end{aligned}$$

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$da_r = r^2\sin\theta\,d\theta\,d\phi \qquad da_\theta = r\sin\theta\,dr\,d\phi \qquad da_\phi = r\,dr\,d\theta$$

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{a} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\nabla\times\mathbf{a} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,a_\phi) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial}{\partial r}(r a_\phi)\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_\theta) - \frac{\partial a_r}{\partial\theta}\right]\hat{\mathbf{e}}_\phi \\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}$$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy \qquad \int_V (\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_C \mathbf{v}\cdot d\mathbf{r}$$

$$z = x + iy = \rho e^{i\phi} \qquad |z| = \rho = \sqrt{x^2+y^2} \qquad z^* = x - iy \qquad w = \ln z = \ln r + i(\theta + 2k\pi)$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cosh z = \frac{e^z+e^{-z}}{2} \quad \sinh z = \frac{e^z-e^{-z}}{2}$$

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(z_1 \pm z_2) = \sin z_1\cos z_2 \pm \cos z_1\sin z_2 \qquad \cos(z_1 \pm z_2) = \cos z_1\cos z_2 \mp \sin z_1\sin z_2$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \mathrm{sech}^2 z \qquad \coth^2 z - 1 = \mathrm{csch}^2 z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \qquad \cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sin(iz) = i \sinh z \qquad \cos(iz) = \cosh z$$

$$\sin^{-1}(z) = \frac{1}{i} \ln(iz + \sqrt{1 - z^2}) \qquad \cos^{-1}(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}) \qquad \tan^{-1}(z) = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad w = f(z) = u(x, y) + iv(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

$$(1 + z)^p = 1 + pz + \frac{p(p - 1)}{2!} z^2 + \cdots + \frac{p(p - 1) \cdots (p - n - 1)}{n!} z^n + \cdots$$

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{(z - a)} + \frac{a_{-2}}{(z - a)^2} + \frac{a_{-3}}{(z - a)^3} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} \, dz$$

$$a_{-1} = \lim_{z \to a} (z - a) f(z) \qquad a_{-1} = \lim_{z \to a} \left(\frac{1}{(k - 1)!} \frac{d^{k-1}}{dz^{k-1}} (z - a)^k f(z) \right)$$

$$\oint_C f(z) \, dz = 2\pi i \{a_{-1} + b_{-1} + c_{-1} + \ldots\}$$