Phys 4620, Spring 2007
Exam #2

1. In the text, there was a rather technical section where we solved for EM waves inside a conductor.

a) When we set up the Maxwell equations for EM waves inside a conductor, what new term did we have that we did *not* have for waves in vacuum and in dielectrics?

The new term (for the Maxwell in equations in matter) is the free current, $\mathbf{J}_f$. (As argued in the text, there is no free *charge* ρ_f to consider; it damps out very quickly.) A free current *can* exist in a conductor!

b) Give two ways in which the wave solutions were qualitatively different from waves in vacuum and dielectrics.

The wave solution for the conductor gave EM waves in which the amplitude *died off* with distance and in which the E and B fields were not in phase.

2. a) What do we mean when we say a medium (for waves) is *dispersive*?

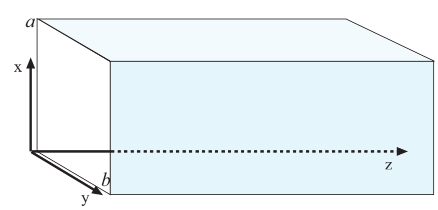
This means that the speed of waves depends on the frequency of the wave. (Or alternately the frequency of the wave depends on its wavelength in the medium.)

b) What is the significance of the *group velocity* for waves in a dispersive medium, and how does it differ from the *wave velocity*?

The group velocity gives the speed of a wave packet whose wavenumber *distribution* is *centered* at some wavenumber of interest. The wave speed for a given wavenumber (or frequency) is the speed of a harmonic wave for that frequency.

3. For TE modes in a wave guide we can generate all components of the fields from the solution

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$



for our usual coordinates for the rectancular waveguide.

Show that the solution so generated satisfies the boundary conditions appropirate for the conducting walls.

With

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \qquad \text{(TE wave)}$$

we get:

$$
\begin{aligned}
E_x &= \frac{i\omega}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial y} = \frac{-i\omega}{(\omega/c)^2 - k^2}\left(\frac{n\pi}{b}\right) B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\
E_y &= \frac{-i\omega}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial x} = \frac{i\omega}{(\omega/c)^2 - k^2}\left(\frac{m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \\
B_x &= \frac{ik}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial x} = \frac{-ik}{(\omega/c)^2 - k^2}\left(\frac{m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \\
B_y &= \frac{ik}{(\omega/c)^2 - k^2}\frac{\partial B_z}{\partial y} = \frac{-i\omega}{(\omega/c)^2 - k^2}\left(\frac{n\pi}{b}\right) B_0 \cos\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)
\end{aligned}
$$

The conditions that must be satisfied at the channel walls are $\mathbf{E}^{\parallel} = 0$ and $B^{\perp} = 0$. From these expressions, note that at $x - 0, x = a$ the "$\perp$" direction is x and $B_x = 0$ here. Also, $E^{\parallel} = E_y$ on these surfaces. (E_z is zero by assumption.)

At $y = 0, y = b$ the "$\perp$" direction is y and $B_y = 0$ here. Also, $E^{\parallel} = E_x$ and $E_x = 0$ on these surfaces.

4. Consider a rectangular waveguide of unknown dimensions...

We want to form a waveguide such that the lowest frequency for propagation of TE wave is 5.0×10^9 Hz, and for which there is only *one* mode of propagation between 5.0×10^9 Hz and 9.0×10^9 Hz. (These are *frequencies*, not *angular frequencies*.)

a) What are the dimensions of the waveguide?

The Threshold (angular) frequencies are given by

$$\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2} \qquad \text{or} \qquad f_{mn} = \omega_{mn}/(2\pi) = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2}$$

If the lowest possible mode has $f = 5.0 \times 10^9\ \mathrm{Hz}$ then (say) $m = 1, n = 0$ and

$$5.0 \times 10^9\ \mathrm{Hz} = \frac{c}{2}\frac{1}{a} \qquad \Longrightarrow \qquad a = 3.00 \times 10^{-2}\ \mathrm{m} = 3.00\ \mathrm{cm}$$

The frequency for f_{20} would have to be twice as great, namely $10.0 \times 10^9\ \mathrm{Hz}$ but we have a new threshold at $9.0 \times 10^9\ \mathrm{Hz}$ which must correspond to $m = 0, n = 1$. This gives:

$$9.0 \times 10^9\ \mathrm{Hz} = \frac{c}{2}\frac{1}{b} \qquad \Longrightarrow \qquad b = 1.67 \times 10^{-2}\ \mathrm{m} = 1.67\ \mathrm{cm}$$

The dimensions of the waveguide are $3.00\ \mathrm{cm} \times 1.67\ \mathrm{cm}$.

b) How many modes can propagate with frequencies lower than 16.0 GHz? Identify these modes with the proper labels.

As seen,

$$f_{10} = 5.0 \times 10^9\ \mathrm{Hz} \qquad f_{20} = 10.0 \times 10^9\ \mathrm{Hz} \qquad f_{01} = 5.0 \times 9.0^9\ \mathrm{Hz}$$

We calculate, from

$$f_{mn} = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2} \qquad a = 3.0 \text{ cm} \quad b = 1.67 \text{ cm}$$

$$f_{11} = 10.2 \times 10^9 \text{ Hz} \qquad f_{30} = 15.0 \times 10^9 \text{ Hz} \qquad f_{21} = 13.5 \times 10^9 \text{ Hz} \qquad f_{02} = 18.0 \times 10^9 \text{ Hz}$$

so the last one is no good. Below 16 GHz, the only modes which can propagate are

$$f_{10}, f_{01}, f_{11}, f_{20}, f_{21}, f_{30}$$

5. What is meant (generally) by a choice of *gauge* in electromagnetism? Give an example of a "gauge condition".

Though the E and B fields have unique measurable values the potentials which give rise to them via

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

are not; by appropriate shifts of the V and $\mathbf{A}$ functions, we can produce other functions which give the *same* E and B fields by these equations. We can get a unique choice for V and $\mathbf{A}$ by imposing additional conditions; this is known as a choice of *gauge*.

6. The scalar potential for a nonstatic source is gotten from the formula

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$

and there is also one for the vector potential.

What's the deal with that t_r inside the integral? What's it mean? How is it related to (or found from) t and $\mathbf{r}'$? Why is it necessary?

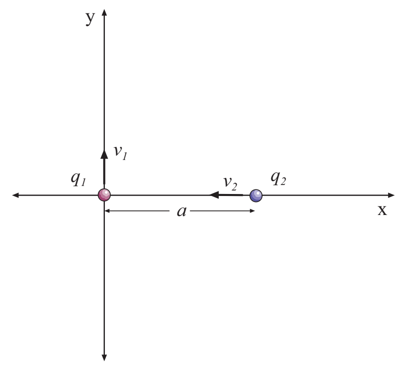
A change in a distribution of EM sources cannot cause an *instantaneous* change at an observation point some distance r away from the origin as that would violate causality. There is necessarily a delay between the time which gives the condition of (part of) the source and the time at which Jan evaluates the fields. It is sensible that that delay time is the time for a signal to travel at the speed of light from the source point to the location of the observer (Jan). At the source point $\mathbf{r}'$, at a distance $\mathscr{r}$ from Jan, the travel time is $t_{\text{travel}} = \mathscr{r}/c$ so we evaluate the condition of that source point at a time

$$t_r = t - t_{\text{travel}} = t - \frac{\mathscr{r}}{c}$$

This is the *retarded time*.

We use this time in the integration over the source points to get the potentials (*not* the fields) at the field point $\mathbf{r}$.

7. Two charges q_1 and q_2 move in the xy plane, as shown. (q_1 is at the origin and moves in the $+y$ with constant speed v_1. q_2 is at $x = a$ and moves in the $-x$ direction with speed v_2.)



a) Find the force of q_1 on q_2.

At q_2 the E field from q_1 is (using $\theta = 90^\circ$ here; angle between $\mathbf{v}_1$ and $\mathbf{R}_{21}$):

$$\mathbf{E}_1 = \frac{q_1}{4\pi\epsilon_0}\frac{(1 - v_1^2/c^2)}{(1 - v_1^2/c^2)^{3/2}}\frac{\hat{\mathbf{x}}}{a^2} = \frac{q_1\hat{\mathbf{x}}}{4\pi\epsilon_0\sqrt{1 - v_1^2/c^2}a^2}$$

and, with $\mathbf{v}_1 = v_1\hat{\mathbf{y}}$, the B field due to q_1 is

$$\mathbf{B}_1 = \frac{1}{c^2}\mathbf{v}_1 \times \mathbf{E}_1 = -\frac{q_1 v_1\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2\sqrt{1 - v_1^2/c^2}a^2}$$

The force on q_2 is, with $\mathbf{v}_2 = -v_2\hat{\mathbf{x}}$,

$$\mathbf{F}_2 = q_2\mathbf{E}_1 + q_2\mathbf{v}_2 \times \mathbf{B}_1 = \frac{q_1 q_2}{4\pi\epsilon_0\sqrt{1 - v_1^2/c^2}a^2}\left[\hat{\mathbf{x}} - \frac{v_1 v_2}{c^2}\hat{\mathbf{y}}\right]$$

b) Find the force on q_2 on q_1.

At q_1, the E field from q_2 is (with $\theta = 0$, since $\mathbf{v}_2 \| \mathbf{R}$),

$$\mathbf{E}_2 = \frac{q_2}{4\pi\epsilon_0}\frac{(1 - v_2^2/c^2)}{(1 - 0)^{3/2}}\left(\frac{-\hat{\mathbf{x}}}{a^2}\right) = -\frac{q_2(1 - v_2^2/c^2)\hat{\mathbf{x}}}{4\pi\epsilon_0 a^2}$$

The B field from q_2 is

$$\mathbf{B}_2 = \frac{1}{c^2}(\mathbf{v}_2 \times \mathbf{E}_2) = 0\ ,$$

since $\mathbf{v}_2 \| \mathbf{E}_2$. So the force on q_1 is

$$\mathbf{F}_1 = q_1\mathbf{E}_2 = -\frac{q_1 q_2(1 - v_2^2/c^2)}{4\pi\epsilon_0 a^2}\hat{\mathbf{x}}$$

8. The instantaneous $\mathbf{E}$ and $\mathbf{B}$ fields for a particular oscillating electric *quadrupole* are

$$\mathbf{E} = \frac{\mu_0 p_0 d\omega^3}{4\pi c}\left(\frac{\sin\theta\cos\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\theta}}$$

$$\mathbf{B} = +\frac{\mu_0 p_0 d\omega^3}{4\pi c^2}\left(\frac{\sin\theta\cos\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\phi}}$$

(The quadrupole is constructed from two point dipoles of strength p_0 along the z axis separated by d.)

a) Calculate the instantaneous Poynting vector $\mathbf{S}$.

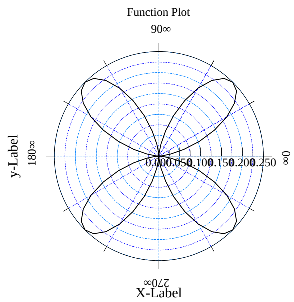
$$\mathbf{S} = \mathbf{E} \times \mathbf{B} = \frac{\mu_0 p_0^2 d^2 \omega^6}{16\pi^2 c^3} \frac{\sin^2\theta\cos^2\theta}{r^2} \sin^2[\omega(t - r/c)]\hat{\mathbf{r}}$$

b) What is the time–averaged Poynting vector $\langle \mathbf{S} \rangle$? What does the "intensity profile" look like? (Make a crude polar plot.)

Replacing the $\sin^2(-)$ factor in the answer for (a) by $\frac{1}{2}$, get:

$$\langle \mathbf{S} \rangle = \frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3} \frac{\sin^2\theta\cos^2\theta}{r^2}\hat{\mathbf{r}}$$

The polar plot of this function is shown. It has maxima at $\theta = 45°$ and $\theta = 135°$.



c) Find the total radiated power $\langle P \rangle$.

$$\begin{aligned}
\langle P \rangle &= \int_{\text{sphere},R} \mathbf{S} \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3} \frac{1}{R^2} \int \sin^2\theta\cos^2\theta R^2 d\Omega \\
&= \frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3} (2\pi) \int_{-1}^{1} (1 - x^2) x^2 dx
\end{aligned}$$

where we have made the usual substitution $x = \cos\theta$ in that integral. The integral is

$$\frac{x^3}{3} - \frac{x^5}{5}\Big|_{-1}^{1} = 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}$$

giving

$$\langle P \rangle = \frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3} \cdot \frac{4}{15} = \frac{\mu_0 p_0^2 d^2 \omega^6}{60\pi^2 c^3}$$

d) How does (this type of) quadrupole radiation differ from dipole radiation? (Give any difference that occurs to you.)

Aside from the various arbitrary source strengths and numerical factors, the total radiated power is proportional to ω^6, not ω^4 as in dipole radiation. Also, the intensity profile has two lobes to it, a more complex shape than we encountered in our examples of dipole radiation.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\,ds\,d\phi\,dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\text{enc}}$$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times 10^{-7}\,\tfrac{\text{N}}{\text{A}^2} \qquad 1\ \text{T} = 1\,\tfrac{\text{N}}{\text{A}\cdot\text{m}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\text{above}} = B^\perp_{\text{below}} \qquad \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \qquad \frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint \mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp - B_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \qquad \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Waveguides:

$$\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y) e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y) e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}} \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

$$\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right) \\
E_y &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right) \\
B_x &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right) \\
B_y &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)
\end{aligned}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0 \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0$$

$$\text{TE solution:} \qquad B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = v - \frac{\partial \lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla \cdot \mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c} \boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta / c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

$$\mathbf{R} = \mathbf{r} - \mathbf{v}t \text{ , } \theta \text{ is between } \mathbf{R} \text{ and } \mathbf{v}$$

$$\text{El Dipole:} \langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}} \qquad \langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi\ c}$$

$$\text{Mag Dipole:} \langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}} \qquad \langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi\ c^3}$$