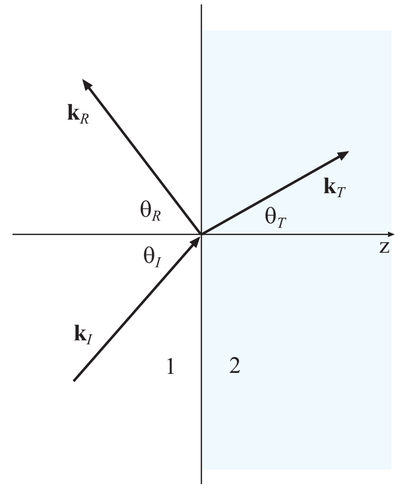
Phys 4620, Spring 2006
Exam #2

1. When we covered "oblique incidence" at a boundary between two media we first derived Snell's law (relating angles θ_I and θ_T) and then went on to derive the relations between the *intensities* of the incoming, reflected and transmitted waves, arriving at Fresnel's equations.

However for the latter we were forced to make an assumption about the nature of the incoming plane wave. What was this assumption? (It has to do with directions of the fields...)

One can make a different assumption about the incoming wave and get a different pair of Fresnel equations; what is the other case that we would consider?

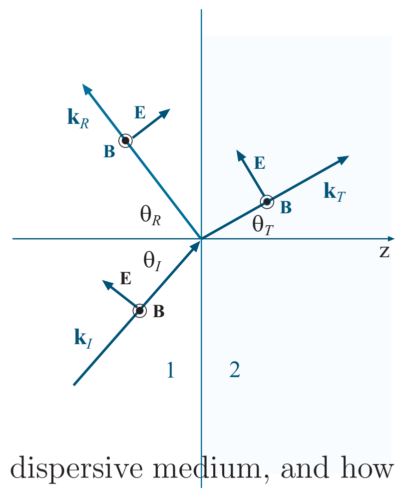
For both of these questions, a picture will help.



To relate the intensities of the three waves we need to apply the boundary conditions and to use these we have to make a choice of the *polarization* of the incoming wave. In particular we made the choice that the electric field of the incoming wave was in the plane of incidence and as a result the magnetic field was perpendicular to this plane.

See the picture at the right.

The other independent choice is to have the electric field be perpendicular to the plane of incidence in all three waves This will result in other relations of the Fesnel tyupe.



2. a) What do we mean when we say a medium (for waves) is *dispersive*?

Harmonic waves of differing frequencies may have different speeds in a (real) medium. Then $v = \omega/k$ (speed of a harmonic wave in the medium) will not be constant.

b) What is the significance of the *group velocity* for waves in a dispersive medium, and how does it differ from the *wave velocity*?

The wave velocity gives the speed of a harmonic wave i.e. an infinite sinusoidal wae. Such a wave cannot carry information as a wave *pulse* can. The group velocity, $v_g = \frac{d\omega}{dk}$ gives the speed of a wave packet constructed of waves who frequencies are close to that of some central frequency ω_0.

There are cases where the wave velocity v is greater then c, but v_g is never greater than c. Well, according to Griffiths, *almost* never.

3. For a rectangular waveguide waveguide show that the group velocity of wave in TE mode mn is

$$v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$$

where

$$\omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2}$$

For TE modes in a rectangular waveguide, we found that

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \qquad m = 0, 1, 2 \quad n = 0, 1, 2$$

(This followed from the solution

$$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$$

and the relation between k and ω.)

Now, $v_g = d\omega/dk = (dk/d\omega)^{-1}$, so

$$\frac{dk}{d\omega} = \frac{1}{2}\frac{2\omega/c^2}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}}$$

so then

$$v_g = \frac{d\omega}{dk} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} = c\sqrt{1 - \frac{c^2\pi^2}{\omega^2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

With $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$, this is

$$v_g = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}$$

4. Consider a rectangular waveguide with dimensions 3.0 cm × 2.0 cm.

a) What is the lowest frequency of TE waves that will propagate in this waveguide?

Rectangular waveguide with $a = 3.0$ cm, $b = 2.0$ cm. The cutoff frequencies (frequencies where mode mn can propagate) are

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{and} \qquad f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

For the given dimensions, the first few frequencies are

$$f_{10} = 5.00 \times 10^9 \text{ Hz} = 5.00 \text{ GHz} \qquad f_{01} = 7.50 \text{ GHz}$$

$$f_{11} = 9.01 \text{ GHz} \qquad f_{20} = 10.0 \text{ GHz} \qquad f_{02} = 15.0 \text{ GHz} \qquad f_{21} = 12.5 \text{ GHz} \qquad f_{12} = 15.8 \text{ GHz}$$

So the lowest frequency at which waves which can propagate is 5.00 GHz.

b) What TE modes will propagate in this waveguide if the frequency is 1.30×10^{10} Hz?

For a frequency $1.30 \times 10^{10}\ \mathrm{Hz} = 13.0\ \mathrm{GHz}$, the modes $\{10\}$, $\{01\}$,$\{11\}$, $\{20\}$ and $\{21\}$ can propagate.

c) Write out the *full* wavefunctions for the fields for the $\{11\}$ mode.

For the $\{11\}$ mode we have

$$B_z = B_0 \cos(\pi x/a)\cos(\pi y/b) \qquad \text{and} \qquad E_z = 0$$

Putting this into the equations which give the transverse components in terms of the (possible) longitudinal components, we get

$$\begin{aligned} E_x &= \frac{i}{(\omega/c)^2 - k^2}\omega(-1)\frac{\pi}{b}B_0\cos(\pi x/a)\sin(\pi y/b) \\ &= \frac{-i\pi\omega}{b[(\omega/c)^2 - k^2]}B_0\cos(\pi x/a)\sin(\pi y/b) \end{aligned}$$

Likewise we find

$$E_y = \frac{+i\pi\omega}{a[(\omega/c)^2 - k^2]}B_0\sin(\pi/a)\cos(\pi y/b)$$

$$B_x = \frac{-i\pi k}{a[(\omega/c)^2 - k^2]}B_0\sin(\pi x/a)\cos(\pi y/b)$$

$$B_y = \frac{-i\pi k}{b[(\omega/c)^2 - k^2]}B_0\cos(\pi x/a)\sin(\pi y/b)$$

If we put these functions into

$$\tilde{\mathbf{E}}_0 = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} \qquad \tilde{\mathbf{B}}_0 = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}$$

and

$$\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(kz-\omega t)}$$

we get the complete expression of the field inside the waveguide.

d) Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use?

For only *one* frequency to propagate we must restrict f to

$$5.00\ \mathrm{GHz} < f < 7.50\ \mathrm{GHz}$$

5. What is meant (generally) by a choice of *gauge* in electromagnetism? Give an example of a "gauge condition".

The potentials V and $\mathbf{A}$ give the $\mathbf{E}$ and $\mathbf{B}$ fields via

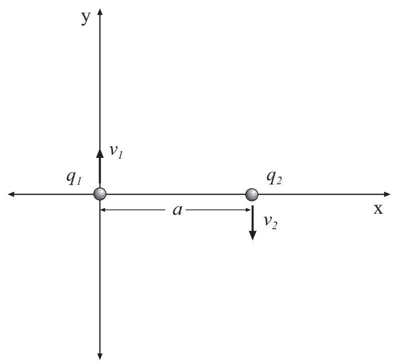
$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

But the functions V and $\mathbf{A}$ which satisfy these are *not unique*. If we have one set of V and $\mathbf{A}$ which give the correct fields we can arrive at another set by means of a "gauge transformation". The choice of V and $\mathbf{A}$ thus has some arbitrariness unless we have a reason to impose a mathematical condition on V and $\mathbf{A}$, and then we will make a particular choice of these functions to satisfy the *gauge condition*. Examples are:

$$\nabla \cdot \mathbf{A} = 0 \qquad \text{Coulomb gauge condition}$$

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \qquad \text{Lorentz gauge condition}$$

6. Two charges q_1 and q_2 move in the xy plane, as shown. (q_1 is at the origin and moves in the $+y$ with constant speed v_1. q_2 is at $x = a$ and moves in the $-y$ direction with speed v_2.)



a) Find the force of q_1 on q_2.

Use the results for the fields from a point charge moving at constnat velocity.

The $\mathbf{E}$ field at q_2's location is (using $\mathbf{R} = a\hat{\mathbf{x}}$, $\theta = 90°$),

$$\mathbf{E}_{\text{at } 2} = \frac{q_1}{4\pi\epsilon_0} \frac{(1 - v_1^2/c^2)}{\left(1 - \frac{v_1^2}{c^2}\right)^{3/2}} \frac{\hat{\mathbf{x}}}{a^2} = \frac{q_1 \hat{\mathbf{x}}}{4\pi\epsilon_0 a^2 (1 - v_1^2/c^2)^{1/2}}$$

The $\mathbf{B}$ field at 2 is

$$\mathbf{B}_{\text{at } 2} = \frac{1}{c^2}(\mathbf{v}_1 \times \mathbf{E}_{\text{at } 2}) = \frac{-q_1 v_1 \hat{\mathbf{z}}}{4\pi\epsilon_0 c^2 a^2 (1 - v_1^2/c^2)^{1/2}}$$

Now use the force equation to get

$$\begin{aligned}
\mathbf{F}_{\text{on } 2} &= q_2 \mathbf{E}_{\text{at } 2} + q_2 \mathbf{v_2} \times \mathbf{B}_{\text{at } 2} \\
&= \frac{q_1 q_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 a^2 (1 - v_1^2/c^2)^{1/2}} + \frac{q_1 q_2 v_1 v_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 c^2 a^2 (1 - v_1^2/c^2)^{1/2}} \\
&= \frac{q_1 q_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 a^2 (1 - v_1^2/c^2)^{1/2}} \left[1 + \frac{v_1 v_2}{c^2 (1 - v_1^2/c^2)^{1/2}}\right]
\end{aligned}$$

b) Find the force on q_2 on q_1.

Likewise, with $\mathbf{R} = -a\hat{\mathbf{x}}$, for the field at 1 due to 2, we get

$$\mathbf{E}_{\text{at } 1} = \frac{-q_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 (1 - v_2^2/c^2)^{1/2}}$$

$$\mathbf{B}_{\text{at }1} = \frac{1}{c^2}(\mathbf{v}_2 \times \mathbf{E}_{\text{at }1}) = \frac{-q_2 v_2 \hat{\mathbf{z}}}{4\pi\epsilon_0 c^2 a^2 (1 - v_2^2/c^2)^{1/2}}$$

and the force on q_1 is

$$\begin{aligned}
\mathbf{F}_{\text{on }1} &= q_1 \mathbf{E}_{\text{at }1} + q_1 \mathbf{v_1} \times \mathbf{B}_{\text{at }1} \\
&= \frac{-q_1 q_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 a^2 (1 - v_2^2/c^2)^{1/2}} - \frac{q_1 q_2 v_1 v_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 c^2 a^2 (1 - v_2^2/c^2)^{1/2}} \\
&= -\frac{q_1 q_2 \hat{\mathbf{x}}}{4\pi\epsilon_0 a^2 (1 - v_2^2/c^2)^{1/2}} \left[1 + \frac{v_1 v_2}{c^2 (1 - v_2^2/c^2)^{1/2}} \right]
\end{aligned}$$

Are these forces the opposite of one another (like Newton's 3rd law would have you believe)? I don't think they are!

7. In Chap 9 it was shown that the instantaneous **E** and **B** fields for the oscillating magnetic dipole were

$$\mathbf{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)] \hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)] \hat{\boldsymbol{\theta}}$$

a) These fields are not exact for the problem of the current loop at the origin. What sorts of approximations were made in arriving at them?

In getting these field we used the approximations that:

1) The size of the radiating system (a magnetic dipole here; radius of the current loop) is much less than the wavelength, i.e. c/ω.

2) The distance of the observer is much greater than the wavelength, i.e. $r \gg c/\omega$.

Terms having extra factors of these ratios were tossed to get the fields.

b) Calculate the instantaneous Poynting vector **S**. Show all the steps.

Use $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$, and $\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$,then

$$\mathbf{S} = \frac{\mu_0}{c} \left(\frac{m_0 \omega^2}{4\pi c} \right)^2 \left(\frac{\sin\theta}{r} \right)^2 \cos^2[\omega(t - r/c)] \hat{\mathbf{r}}$$

c) What is the time–averaged Poynting vector $\langle \mathbf{S} \rangle$?

The time-averaged S simply replaces any $\cos^2$ by $\frac{1}{2}$:

$$\langle \mathbf{S} \rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \left(\frac{\sin\theta}{r} \right) \hat{\mathbf{r}}$$

d) Find the total radiated power $\langle P \rangle$. Show the steps.

The total radiated power is the integral of $\langle \mathbf{S} \rangle$ over a big sphere of radius R. Evaluate:

$$P = \oint_{\mathcal{S}} \langle \mathbf{S} \rangle \cdot d\mathbf{a} = (2\pi) \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \frac{1}{R^2} R^2 \int_{-1}^{1} \sin^2\theta d(\cos\theta)$$

The integral here is

$$\int_{-1}^{1} (1 - x^2) dx = \left(x - \frac{x^3}{3} \right) \Big|_{-1}^{1} = \frac{4}{3}$$

and this gives

$$P = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\ \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau' \qquad V_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\rm dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f,\,\rm enc}$$

$$\mathbf{F}_{\rm mag} = Q(\mathbf{v}\times\mathbf{B}) \qquad \mathbf{F}_{\rm mag} = \int I(d\mathbf{l}\times\mathbf{B}) \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \qquad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \mu_0 = 4\pi\times10^{-7}\,\tfrac{\rm N}{\rm A^2} \qquad 1\ {\rm T} = 1\,\tfrac{\rm N}{\rm A\cdot m}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\rm enc} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$B^\perp_{\rm above} = B^\perp_{\rm below} \qquad \mathbf{B}_{\rm above} - \mathbf{B}_{\rm below} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \qquad \mathbf{A}_{\rm above} = \mathbf{A}_{\rm below} \qquad \frac{\partial\mathbf{A}_{\rm above}}{\partial n} - \frac{\partial\mathbf{A}_{\rm below}}{\partial n} = -\mu_0\mathbf{K}$$

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \text{where} \qquad \mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \qquad \mathbf{B}_{\rm dip}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\mathbf{N} = \mathbf{m}\times\mathbf{B} \qquad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da' \qquad \text{where} \qquad \mathbf{J}_b = \nabla\times\mathbf{M} \qquad \text{and} \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad \nabla\times\mathbf{H} = \mathbf{J}_f \qquad \oint\mathbf{H}\cdot d\mathbf{l} = I_{f,\,enc}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

$$W = \tfrac{1}{2}LI^2 \qquad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp - B_2^\perp = 0 \qquad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \qquad \mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f\times\hat{\mathbf{n}}$$

$$\Phi_2 = M_{21}I_1 \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Waveguides:

$$\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y) e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y) e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}} \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

$$\begin{aligned}
E_x &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right) \\
E_y &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right) \\
B_x &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right) \\
B_y &= \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)
\end{aligned}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0 \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0$$

$$\text{TE solution:} \qquad B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla \lambda \qquad V' = v - \frac{\partial \lambda}{\partial t}$$

$$\text{Coulomb}: \quad \nabla \cdot \mathbf{A} = 0 \qquad \text{Lorentz}: \quad \nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \boldsymbol{\mathscr{r}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta / c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \qquad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

$\mathbf{R} = \mathbf{r} - \mathbf{v}t$, θ is between $\mathbf{R}$ and $\mathbf{v}$