Phys 4610, Fall 2004
Exam #1

1. Find

a)

$$\nabla \cdot \mathbf{v}\,, \quad \text{where} \quad \mathbf{v} = 3xy\hat{\mathbf{x}} + 4yz\hat{\mathbf{y}} - zx\hat{\mathbf{z}}$$

b)

$$\nabla \mathbf{V}\,, \quad \text{where} \quad V = s^2 z \cos\phi$$

c)

$$\nabla \times \mathbf{v}\,, \quad \text{where} \quad \mathbf{v} = r\sin\theta\,\hat{\mathbf{r}} + r\cos\theta\,\hat{\boldsymbol{\phi}}$$

d)

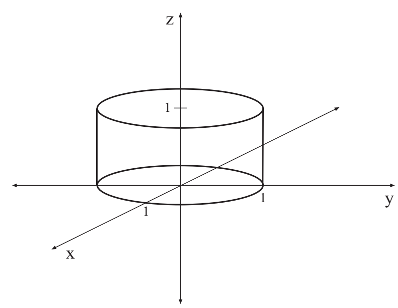
$$\nabla^2 V\,, \quad \text{where} \quad V = e^{-r/a}\cos\theta\sin\phi$$

2. The vector field $\mathbf{v}$ is given by

$$\mathbf{v} = s\cos^2\phi\,\hat{\mathbf{s}} + s\cos\phi\,\hat{\boldsymbol{\phi}} + z^2\,\hat{\mathbf{z}}$$

a) Find the divergence of $\mathbf{v}$.

b) Show that the divergence theorem is satisfied using, as the volume, a cylinder of radius 1 coaxial with the z axis and extending from $z = 0$ to $z = 1$.

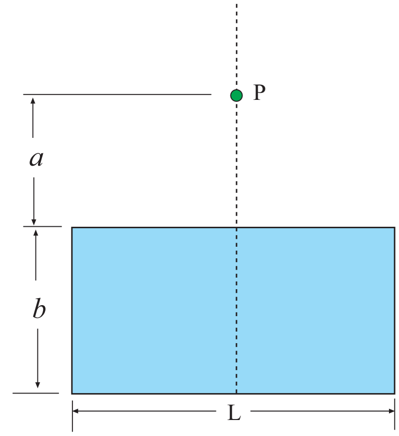


3. Evaluate

$$\int_{-3}^{3} (x^3 + 7x + 5)\delta(x-1)\,dx$$

4. Consider a rectangle with a uniform surface charge density σ. The observation point P is in the plane of the rectangle on the bisector of the side of length L, a distance a from the nearest side. The other side of the rectangle has length b. See the figure.

Give the direction and magnitude of the E field at P. It will be sufficient for you set *clearly* set up any necessary integrals if they are at all difficult to work out!



5. If the electric potential in a certain region of space is given by

$$V(\mathbf{r}) = V_0 e^{-r^2/a^2}$$

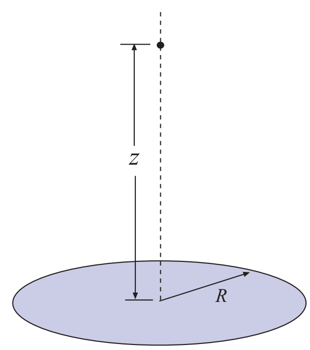
a) What is the electric field in that region?

b) What is the charge density $\rho(\mathbf{r})$?

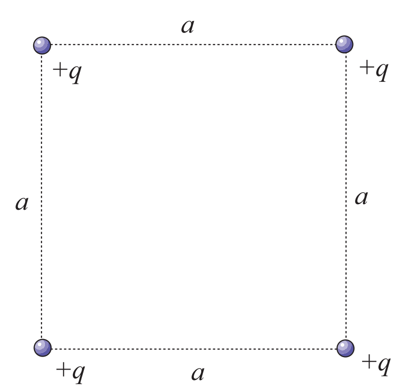
c) How much charge is contained within a sphere of radius R centered at the origin?

6. What is the electric potential at a point on the axis of a uniformly charged disk of radius R and surface charge density σ, a distance z from the center of the disk?

Assume $V = 0$ at infinity.



7. Find the work required to assemble four point charges $+q$ in a square with side a.



Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla\cdot\mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\,\hat{\mathbf{r}} + dl_\theta\,\hat{\boldsymbol{\theta}} + dl_\phi\,\hat{\boldsymbol{\phi}} \qquad d\tau = r^2\,\sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\,\hat{\mathbf{s}} + dl_\phi\,\hat{\boldsymbol{\phi}} + dl_z\,\hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

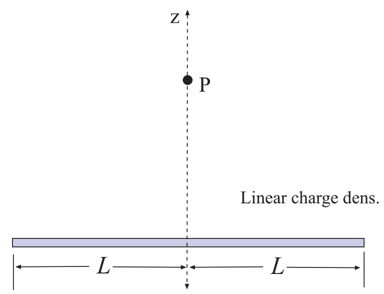
Physics:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} \qquad V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} E \cdot d\mathbf{l}$$

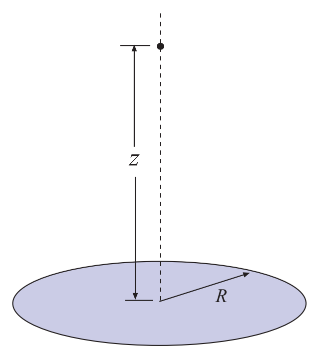
$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$E_z = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$



Charge density σ