Phys 3820, Fall 2011
Exam #2

1. In doing the variational calculation for the He atom, the claim was made that the *effective* charge of the nucleus was 1.69 rather than 2.00

a) What does this really mean? How was this figure arrived at? (Give its significance in the calculation and how we got its value.)

We used a trial wave function which was hydrogenic in *form* (and depends on the nuclear charge Z) but where we let Z be a variable parameter with no pre-set value.

b) What is the common physical "interpretation" of the smaller value?

One says that for each electron the central charge is "screened", meaning it has an *effective* value which is smaller than the value 2 of the helium nucleus. Each electron "feels" less than the true charge because the *other* electron spends some of its time *between* that electron and the nucleus so that the electric field (and its effective charge) from the nucleus is reduced.

2. In the text we solved the hydrogen molecule *ion* H_2^+. But the smallest real molecule is neutral hydrogen, H_2, a system with two electrons (and two nuclei).

How might you try to solve the neutral hydrogen molecule H_2? Assuming that the nuclei are fixed at some separation R you would need a two-electron variational wave function. Tell me something that you might try.

There is no "right" answer here; but write down something you think is a sensible choice.

One answer is to think of the work on the H_2^+ ion; there, the single electron lived in a wave function given by

$$\psi_{\rm lcao}(\mathbf{r}) = A[\psi(\mathbf{r}_A) + \psi(\mathbf{r}_B)] \qquad (1)$$

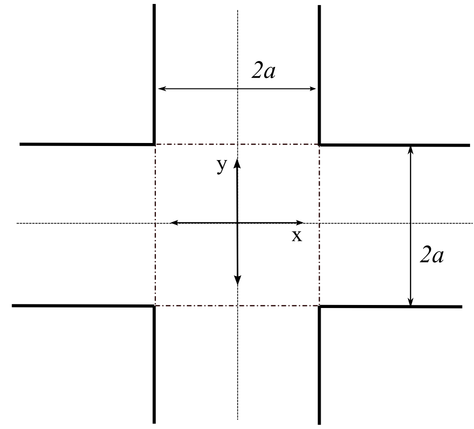
where $\mathbf{r}_A$ stands for $\mathbf{r} - \mathbf{R}_A$, with $\mathbf{R}_A$ being the location of the first nucleus, etc. and the ψ's are hydrogenic 1s wave functions. A is chosen to normalize this function.

While we don't vary the internuclear distance R unit *after* the electronic calculation, we might think of taking the nuclear charge Z to be a variable parameter in this function. (And Z will depend on the pre-set value of R.)

Now "put" two electrons *symmetrically* into the state given by Eq. 1 as long as the spin part (with which we don't concern ourselves) is antisymmetric. Thus the wave function for the H_2 atom could be

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_{\rm lcao}(\mathbf{r}_1)\psi_{\rm lcao}(\mathbf{r}_2)$$

3. In the notorious particle–trapped–at-an–intersection problem that concludes Chapter 7, ("Quantum dots", actually) two zero-potential channels of width $2a$ with "hard walls" cross and give rise to a bound state.



a) For a particle of mass m moving in this geometry the lowest energy that can propagate off to infinity is

$$E_{\text{threshold}} = \frac{\pi^2\hbar^2}{8ma^2}$$

Show this; at least discuss why the value is not zero.

The solution for a traveling wave in one of the arms will be a complex wiggle in (say) the x direction with a factor to satisfy the boundary condition along y: in the left arm this will be

$$\psi(x, y) = Ae^{ikx}\cos\left(\frac{\pi y}{2a}\right) . \qquad (2)$$

This is not really normalizable but a true state will be a combination of such states (and thus not be a true eigenstate of the energy).

The energy of this state is

$$E = -\frac{\hbar^2}{2m}\nabla^2\psi(x, y) = -\frac{\hbar^2}{2m}(-k^2 - \frac{\pi^2}{4a^2})$$

As we can make the wave number k as small as we like (corresponding to longer wavelength) the energy of a traveling (free) particle has a lower limit of

$$E_{\text{thresh}} = \frac{\pi^2\hbar^2}{8ma^2}$$

If a solution has an energy less than this it must be a bound state of the system.

b) Give a qualitative explanation for why the bound state exists.

In the central region the wave function can ooze out into the wings as it were and thus have a smaller curvature than it would have to have far out in the arms. With the lower curvature the energy is less and can be less than the threshold energy found in (a).

4. a) For the classically-allowed region of the motion a particle, what assumption did the "WKB" approximation make about the form of the wave function?

It assumes that the wave function will be of the form

$$\psi(x) = A(x)e^{i\phi(x)}$$

where both $A(x)$ and $\phi(x)$ are *real* functions.

b) Under what general condition was this approximation expected to be valid?

For the classically case of the WKB we assume that in the region of interest, $E \gg V(x)$.

c) The assumption is bad near ''classical turning points" (why?) What general (mathematical) procedure is done to extend the approximate solutions to these regions?

We make a linear approximation of the potential $V(x)$ and then solve the Schr"odinger equation for a linear potential; the solutions are well known, the famous Airy functions. This solution must then me matched (in value and slope) to the wave function we were using in the region far awar from the turning point.

5. a) In Gamow's model of α decay, what form of potential did Gamow choose to keep an alpha particle trapped?

At the edge of the nucleus (called r_2 in our book), we assume that there is only the Coulomb potential between alpha particle and residual nucleus,

$$V(x) = \frac{2Ze^2}{4\pi\epsilon_0 r}$$

but within r_2 there is a deep square well of value $-V_0$. With the alpha particle having positive energy while in the nucleus, it has the possibility of tunneling and escape.

b) In applying the WKB method to the Gamow model, we actually found a tunneling *probability* from the $e^{-2\gamma}$ factor. What crude estimate did we use to convert the tunneling probability to a tunneling *rate*?

We formed the extremely hokey picture that the alpha is a particle rattling back and forth along a diameter of the nucleus, moving with speed v, to give a frequency of collisions with the wall of

$$f = \frac{1}{T_{diam}} = \frac{2r_1}{v}$$

where r_1 is the nuclear radius.

Normally such an approximation would be too silly to mention but there really are no better ideas in this case.

6. For the two–level system we found the *exact* equations

$$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \qquad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a \qquad \text{where} \qquad \omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

Give a brief summary of the strategy we then used to find approximate solutions for $c_a(t)$ and $c_b(t)$ individually.

The basic plan is to take the zeroth order solutions for both c_a and c_b and put these into the expressions on the right side, solve the resulting equations and thus get a new order of correctness for c_a and c_b.

Put *these* solutions into the right side and re-solve.

7. Suppose for our generic two–level system the perturbation takes the form

$$H'(t) = \begin{cases} 0 & t < 0 \\ Ue^{-\beta t} & t \geq 0 \end{cases}$$

with β positive and real and assume that $U_{aa} = U_{bb} = 0$ and $U_{ab} = U^*_{ba} \equiv \alpha$. (And the U's are assumed to be "small" in some sense.) If $c_a(0) = 1$ and $c_b(0) = 0$, find an expression for $c_b(t)$ (using the first–order approximation).

As usual, set up the expressions clearly and get as far as you can with the math.

Use

$$c_b^{(2)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{i\omega_0 t'}\, dt'$$

to get

$$\begin{aligned} c_b(t) &= -\frac{i}{\hbar}\int_0^t H'_{ba}(t')\, dt' \\ &= -\frac{i}{\hbar}\alpha\int_0^t e^{-\beta t'}e^{i\omega_0 t'}\, dt' = -\frac{i}{\hbar}\alpha\int_0^t e^{-\beta t' + i\omega_0 t'}\, dt' \\ &= \frac{-i\alpha}{\hbar(-\beta + i\omega_0)}\left[e^{(-\beta + i\omega_0)t} - 1\right] \end{aligned}$$

I don'te think there's anything interesting to be pulled out by working more onthe math so I'll leave it like this.

8. The most important time-dependent perturbation was the sinusoidal one, with

$$H'_{ab} = V_{ab}\cos(\omega t)$$

and it gave the upper occupation "amplitude"

$$c_b(t) = -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega}\right]$$

We immediately made an approximation which simplified this considerably.

a) What was this approximation?

The approximation was that we would always consider excitation frequencies ω which were very close to the transition frequency ω_0 so that

$$\omega_0 + \omega \gg |\omega_0 - \omega|$$

b) Show how this approximation gives

$$P_{a\to b}(t) \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

From the smaller size of the denominator in the second term and reasonably guessing that the numerators of both are both of order unity, only the second term survives, giving

$$c_b(t) \approx -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i)\omega_o-\omega)}-1}{\omega_0-\omega}\right]$$

Now pull out a factor of $\exp(\frac{i}{2}(\omega_0-\omega))$ from the bracket. (Note that the argument of this exponential is just half what's alread there. This gives:

$$-\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)/2}[e^{i(\omega_0-\omega)/2}-e^{-i(\omega_0-\omega)/2}]$$

But now the stuff in side the bracket is basically a sine function. (Actually $2i$ times the sine. Then

$$c_b(t) \approx -i\frac{V_{ba}}{\hbar}e^{i(\omega_0-\omega)/2}\sin[(\omega_0-\omega)/2]$$

Now the $probability$ to be found in state b is just this $amplitude$ squared. So:

$$P_{a\to b}(t) = |c_b t|^2 = \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0-\omega)/2]}{(\omega_0-\omega)^2}$$

9. Give concise but *careful* definitions of:

a) Spontaneous emission rate (for a given transition).

The rate at which the tw-state quantum system will go from the upper state b down to the lower state a in the $absence$ of any external oscillating potential.

b) Induced emission rate (for a given transition).

Rate at which emission occurs due to interaction with external potential; such a rate must be proportional to the intensity of the radiation field, whereas the spontaneous emission rate is not.

c) Laser.

A useful physical system wherein one generates a "population inversion" that is, putting members into higher quantum states than they want to be from thermal equilibrium. Radiation of the right frequency can cause stimulated emission and thereby produce light that is intense and also coherent (having a uniform phase).

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\, dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\, dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_\mathrm{e} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_\mathrm{p} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^8\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^\infty |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^\infty x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^\infty \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^\infty c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^\infty \Psi_n(x,t)$$

$$\infty \text{ Square Well:} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^\infty |c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^\infty |c_n|^2E_n$$

$$\text{Harmonic Oscillator:} \qquad V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$\text{Free particle:} \qquad \Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_\mathrm{phase} = \frac{\omega}{k} \qquad v_\mathrm{group} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta\, \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm 2i\phi} \text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2} \, \frac{r}{a}\, e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c \hbar^3} \left(\frac{c^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \phi} \right) \qquad L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s{+}1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s\ m{\pm}1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0\hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\,\psi_m^0 \qquad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1+\frac{j(j+1)-l(l+1)+3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788\times 10^{-5}\ \text{eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p\cdot\mathbf{S}_e\rangle = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \le \langle\psi|H|\psi\rangle \equiv \langle H\rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E-V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\pm\frac{1}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v}e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \qquad \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{-i\omega_0 t}c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar}$$

$$c_b^{(2)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{i\omega_0 t'}\,dt' \qquad c_a^{(2)}(t) = 1-\frac{i}{\hbar^2}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}\left[\int_0^{t'} H'_{ba}(t'')e^{i\omega_0 t''}\,dt''\right]dt'$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

$$\wp \equiv q\langle\psi_b|\mathbf{r}|\psi_a\rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\wp|E_0}{\hbar}\right)^2\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\wp|^2\rho(\omega_0) \qquad A = \frac{\omega^3|\wp|^2}{3\pi\epsilon_0\hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1;$ or 0 and $\Delta l = \pm 1$