Phys 3820, Fall 2009
Exam #3

1. Give concise but *careful* definitions of:
a) Spontaneous emission rate (for a given transition).

Rate at which quantum system relaxes to lower state (emitting photon) which is *independent* of the energy density of the ambient EM field. Source of the interaction which brings this about is subtle and can be *fully* understood from more sophisticated of the *quantized* EM field.

b) Induced emission rate (for a given transition).

Rate at which quantum system relaxes to lower state (emitting photon) which *depends upon* the energy density of the ambient EM field. One can study this with our elementary treatment, as the EM fields of the waves gives a perturbation which causes a change in state.

c) Selection rule for a particular transition.

A rule which gives which transitions between quantum states give non-zero values for the transitions rates *to a given order* in perturbation theory, usually first order. The rule may specify the changes in angular momentum quantum numbers which give a non-zero result.

d) Lifetime of an excited state.

Specifically, the time it takes for a big sample of atoms in the excited state to decay to $1/e$ of the original population. (It is simply realted to, but not *equal to* the half-life.)

e) Impact parameter.

Has meaning in the classical theory of scattering, it is the distance from the axis at which the incident particle moves when it is at a large distance. (where the "axis" is parallel to the particle's asymptotic motion and passes *thru* the target particle).

2. Consider a one–dimensional harmonic oscillator with angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$ and where the particle has charge q. (The oscillator potential does not come from the charge.) The system starts is in the ground ($n = 0$) state at $t = 0$. We consider *only* the states $n = 0$ and $n = 1$.

An electric field $E_0\hat{\mathbf{x}}$ is applied for a time τ so that the perturbation is

$$H'(r) = \begin{cases} -qE_0 x & \text{if} \quad 0 \le t \le \tau \\ 0 & \text{otherwise} \end{cases}$$

Using first–order perturbation theory, find the probability for the system to be in the $n = 1$ state at time $t \ge \tau$. Recall that to *first order* for such a two-level system,

$$c_a^{(1)}(t) = 1 \qquad c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\,dt'$$

As usual, go as far as you can with this. The wave functions for ψ_0 and ψ_1 are given in the equation pages.

Amplitude for system to be in state $n = 1$ after τ (since perturbation is only nonzero for $0 \le t \le \tau$) is

$$c_b = -\frac{i}{\hbar}\int_0^\tau (-qE_0)\langle\psi_1|x|\psi_0\rangle e^{i\omega_0 t'}$$

where

$$\omega_0 = \frac{E_1 - E_0}{\hbar} = \frac{\hbar\omega}{\hbar} = \omega$$

Using

$$a_+ + a_- = \frac{2m\omega}{\sqrt{2\hbar m\omega}}x = \sqrt{\frac{2m\omega}{\hbar}}x \qquad \Longrightarrow \qquad x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$$

we get

$$c_b = \frac{i}{\hbar}(-qE_0)\sqrt{\frac{\hbar}{2m\omega}}\langle\psi_1|(a_+ + a_-)|\psi_0\rangle \int_0^\tau e^{i\omega t'}\,dt'$$

Use $a_+\psi_0 = \psi_1$ and orthogonality, then

$$c_b = \frac{i}{\hbar}(-qE_0)\sqrt{\frac{\hbar}{2m\omega}}\int_0^\tau e^{i\omega t'}\,dt'$$

The time integral is

$$\int_0^\tau e^{i\omega t'}\,dt' = \frac{1}{i\omega}e^{i\omega t'}\Big|_0^\tau = \frac{1}{i\omega}\left(e^{i\omega\tau} - 1\right) = \frac{e^{i\omega\tau/2}}{\omega_0}(2i)\sin(\omega\tau/2)$$

Then

$$c_b = \frac{2qE_0}{\hbar\omega}\sin\left(\frac{\omega\tau}{2}\right)e^{i\omega\tau/2}$$

Finally, to lowest order,

$$P_{0\to 1} = |c_b|^2 = \left(\frac{2qE_0}{\hbar\omega}\right)^2\sin^2\left(\frac{\omega\tau}{2}\right)$$

3. a) In the formula for the transition rate for stimulated emission,

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\mathbf{p}|^2\rho(\omega_0)\ \ ,$$

explain in a few words why it depends on the "dipole moment" matrix elements,

$$\mathbf{p} = q\langle\psi_b|\mathbf{r}|\psi_a\rangle$$

That came from the appearance of the general matrix element

$$\langle\psi_b|H'|\psi_a\rangle$$

and for the electric field part of the EM wave (the magnetic part has little effect) if the perturbing *field* is uniform then the perturbing *potential* is proportional to $\mathbf{r}$. Thus we have a matrix element (squared) of $\mathbf{r}$ in the transition rates.

b) Do you remember how we got that factor of 3 in the denominator of $R_{a\to b}$?

That came from the calculation for induced emission due to incoherent radiation; we had to take an average over all the directions of the EM waves and they possible polarizations.

4. According to the selection rules, what are the possible (downward) transitions from the $(n = 3, l = 1)$ states in the H atom? (Hint: There are *two* possible transitions.)

As the selection rules for the H atom are

$$\Delta m = \pm 1; \text{or } 0 \quad \text{and} \quad \Delta l = \pm 1$$

Then the $(n = 3, l = 1)$ ("3p") state can only decay to the $(n = 2, l = 0)$ state and the $(n = 1, l = 0)$ state (note there's no restriction on n). Such a transition would satisfy the rule on m for all the possible initial values of m $(-1, 0, 1)$ so all can decay to these two states.

5. In the infamous Problem 9.11 we calculated (?) that the lifetime of the H atom in the 2s state was infinite! Is that true? Is an H atom in this state absolutely stable?

It is true only to *first order* in perturbation theory. To higher orders, the transition probability may not be zero though undoubtedly the calculation done at these higher orders will give a much *longer* lifetime for the 2s state.

6. Suppose you have a well-characterized beam of particles hitting a target (with very massive particles) and a particle detector of certain specific dimensions. Summarize the steps you would need (the basic measurements and basic calculations) to go through to find the *measured* differential cross section $\frac{d\sigma}{d\Omega}$ at each angle.

Measure the number of particles entering the detector per time, dN when the detector is at angle θ. Measure the spherical angle subtended by the detector from the scattering center, $d\Omega$. Measure the luminosity $\mathcal{L}$ of the beam, that is, the number of particles per area, per time impinging on the target. Then calculate

$$\frac{d\sigma}{d\Omega} = D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

7. What property of the Coulomb interaction makes it poorly behaved for the purposes of doing a quantum scattering calculation?

It falls off too slowly at large distances. Because of this (as more advanced treatments will show) we can't ignore the scattering potential as compared with the centrifugal potential and we must match the wave function to functions other than the spherical Bessel functions at infinity. It can be treated sensibly... in graduate school.

8. Suppose for a given potential we integrate numerically outward from the origin and find that at large r the wave function can be expressed as

$$R_l(r) = Aj_l(kr) + Bh_l^{(1)}(kr)$$

How do we get the scattering amplitude a_l from this?

Recall the asymptotic expression for $\psi(r,\theta)$,

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right]P_l(\cos\theta)$$

(Note, the P_l's are all related to the Y_l^0's by a constant factor, so we can basically take the coefficients of the P_l's as the R_l's.)

As the functions j_l and $h_l^{(1)}$ are independent the ratio of their coefficients in both expansions must be the same. Thus:

$$\frac{B}{A} = ika_l \qquad \Longrightarrow \qquad a_l = -i\frac{B}{kA}$$

and thus we can extract the partial wave scattering amplitudes a_l.

9. a) If we had to solve the differential equation

$$(\nabla^2 - k^2)\psi = f(\mathbf{r}) \qquad\qquad (k \text{ is real; } f \text{ some given function})$$

what condition would the Green function $G(\mathbf{r})$ have to satisfy? (This equation is called the "modified Helmholtz" equation.)

It would need to satisfy

$$(\nabla^2 - k^2)G(\mathbf{r}) = \delta(\mathbf{r})$$

b) A particular solution for $G(\mathbf{r}-\mathbf{r}')$ for this condition is

$$G(\mathbf{r}-\mathbf{r}') = \frac{\exp(-k|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

(you don't need to show this). If this is the appropriate Green function for the problem, write down an expression for $\psi(\mathbf{r})$ in terms of the "source function" f.

The procedure is multiply the "source function" of the DE by $G(\mathbf{r}-\mathbf{r}')$ and integrate on $\mathbf{r}'$. Thus a solution to the inhomogeneous equation is:

$$\psi(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')\,d^3\mathbf{r}' = \int \frac{\exp(-k|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}f(\mathbf{r}')\,d^3\mathbf{r}'$$

10. Find the scattering amplitude $f(\theta)$, in Born approximation for the potential

$$V(r) = \begin{cases} V_0 & \text{if} \quad r < a \\ 0 & \text{if} \quad r > a \end{cases}$$

How would you use your result to get the total cross section? (The math here may be too messy to work out.)

We can use the formula for the Born approximation for a spherically symmetric potential:

$$f(\theta) \approx -\frac{2m}{\hbar^2 \kappa} \int_0^\infty rV(r) \sin(\kappa r)\, dr = -\frac{2mV_0}{\hbar^2 \kappa} \int_0^a r \sin(\kappa r)\, dr$$

Hit the tables and get:

$$-\frac{2mV_0}{\hbar^2 \kappa} \left[\frac{1}{\kappa} \sin(\kappa r) - r\cos(\kappa r) \right] \Bigg|_0^a = -\frac{2mV_0}{\hbar^2 \kappa} \left(\frac{\sin(\kappa a)}{\kappa} - a\cos(\kappa a) \right)$$

This gives

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2 \kappa}(\sin(\kappa a) - \kappa a \cos(\kappa a))$$

where

$$\kappa = 2k \sin\frac{\theta}{2} \qquad \text{and} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

To get the total cross section σ from this, evaluate the integral

$$\sigma = \int |f(\theta)|^2 \, d\Omega$$

which is quite workable but a little tedious so I'll leave it at that...

11. Summarize how we get the "Born series" from the integral form of the Schrödinger equation

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, d^3\mathbf{r}_0$$

We get a series by repeatedly substituting the entire right side of the equation for the last $\psi(\mathbf{r}_0)$ which appears inside an integral; this gives a sum of successively more complicated multiple integrals culminating in a big one with a final $\psi(\mathbf{r}_0)$ in the integrand and *that* one we approximate by $\psi_0(\mathbf{r}_0)$, i.e. the incoming plane wave.

Useful Equations

Math

$$\int_0^\infty x^n e^{-x/a} = n!\, a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\,dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \qquad \int_0^\infty x^{2n+1} e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2}$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + fg\Big|_a^b$$

Numbers

$$\hbar = 1.05457 \times 10^{-34}\ \mathrm{J\cdot s} \qquad m_\mathrm{e} = 9.10938 \times 10^{-31}\ \mathrm{kg} \qquad m_\mathrm{p} = 1.67262 \times 10^{-27}\ \mathrm{kg}$$

$$e = 1.60218 \times 10^{-19}\ \mathrm{C} \qquad c = 2.99792 \times 10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}$$

Physics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad P_{ab} = \int_a^b |\Psi(x,t)|^2\,dx \qquad p \to \frac{\hbar}{i}\frac{d}{dx}$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \qquad \langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \qquad \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \phi(t) = e^{-iEt/\hbar} \qquad \Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty}\Psi_n(x,t)$$

∞ Square Well: $$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\int \psi_m(x)^*\psi_n(x)\,dx = \delta_{mn} \qquad c_n = \int \psi_n(x)^*\,f(x)\,dx \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1 \qquad \langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2 E_n$$

Harmonic Oscillator: $$V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad \frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \qquad [A,B] = AB - BA \qquad [x,p] = i\hbar$$

$$H(a_+\psi) = (E+\hbar\omega)(a_+\psi) \qquad H(a_-\psi) = (E-\hbar\omega)(a_+\psi) \qquad a_-\psi_0 = 0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}$$

Free particle: $$\Psi_k(x) = Ae^{i(kx - \frac{\hbar k^2}{2m})t} \qquad v_\mathrm{phase} = \frac{\omega}{k} \qquad v_\mathrm{group} = \frac{d\omega}{dk}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)}\, dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx$$

Delta Fn Potl: $\qquad \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar) \qquad [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} = \hat{B}\hat{A} \qquad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad \frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + [\ell(\ell+1)\sin^2\theta - m^2]\Theta = 0$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta\, \cos\theta e^{\pm i\phi} \qquad Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm 2i\phi} \text{etc.}$$

$$u(r) \equiv rR(r) \qquad -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m} \qquad E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \equiv \frac{E_1}{n^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

where $E_1 = -13.6$ eV.

$$R_{10}(r) = 2a^{-3/2} e^{-r/a} \qquad R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) e^{-r/2a} \qquad R_{21}(r) \frac{1}{\sqrt{24}} a^{-3/2}\, \frac{r}{a}\, e^{-r/2a}$$

$$\lambda f = c \qquad E_\gamma = hf \qquad \frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \qquad \text{where} \qquad R = \frac{m}{4\pi c \hbar^3} \left(\frac{c^2}{4\pi\epsilon_0} \right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \qquad L_\pm = \pm\hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot\theta \frac{\partial}{\partial \phi} \right) \qquad L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$$

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \qquad L_z f_l^m = \hbar m f_l^m$$

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle \qquad S_z|s\ m\rangle = \hbar m|s\ m\rangle \qquad S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s\ m\pm 1\rangle$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \quad \text{where} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathsf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathsf{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \chi_+^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{B} = B_0\mathbf{k} \qquad H = -\gamma B_0 \mathsf{S}_z \qquad E_+ = -(\gamma B_0 \hbar)/2 \qquad E_- = +(\gamma B_0 \hbar)/2$$

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{-iE_+ t/\hbar} \\ b e^{-iE_- t/\hbar} \end{pmatrix}$$

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi \qquad \psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$$

$$k_F = (3\rho\pi^2)^{1/3} \qquad E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \qquad E_{\text{tot}} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m} V^{-2/3}$$

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \qquad \psi(x+a) = e^{iKa}\psi(x)$$

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \qquad \psi_n^1 = \sum_{m\neq n} \frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{(E_n^0 - E_m^0)}\,\psi_m^0 \qquad E_n^2 = \sum_{m\neq n} \frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0} \qquad W_{ij} \equiv \langle i|H'|j\rangle$$

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \qquad H'_{\text{rel}} = -\frac{p^4}{8m^3c^2} \qquad H = -\mu\cdot\mathbf{B} \qquad \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L} \qquad H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\mathbf{L}\cdot\mathbf{S}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad E_{\text{fs}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right) \qquad E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \qquad E_Z^1 = \mu_B g_J B_{\text{ext}} m_j \qquad \mu_B \equiv \frac{e\hbar}{2m} = 5.788 \times 10^{-5} \text{ eV/T}$$

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p \qquad \boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad E_{\text{hf}}^1 = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p \cdot \mathbf{S}_e \rangle = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4}\begin{cases} +1/4 & \text{(triplet)} \\ -3/4 & \text{(singlet)} \end{cases}$$

$$E_{\text{gs}} \leq \langle \psi | H | \psi \rangle \equiv \langle H \rangle \qquad \psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$p(x) \equiv \sqrt{2m[E - V(x)]]} \qquad \psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{1}{\hbar}\int p(x)\,dx} \qquad \int_0^a p(x)\,dx = n\pi\hbar$$

$$T \approx e^{-2\gamma} \qquad \gamma \equiv \frac{q}{\hbar}\int_0^a |p(x)|\,dx \qquad \tau = \frac{2r_1}{v}e^{2\gamma}$$

$$\Psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}$$

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b \qquad \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a \qquad \text{where} \quad \omega_0 \equiv \frac{E_b = E_a}{\hbar} \qquad H'_{ab} \equiv \langle \psi_a | H' | \psi_b \rangle$$

$$H'_{ab} = V_{ab}\cos(\omega t) \qquad P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$\mathbf{p} \equiv q\langle \psi_b | \psi_a | \rangle \qquad P_{a\to b}(t) = P_{b\to a}(t) = \left(\frac{|\mathbf{p}|E_0}{\hbar}\right)^2 \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$$

$$R_{b\to a} = \frac{\pi}{3\epsilon_0\hbar^2}|\mathbf{p}|^2\rho(\omega_0) \qquad A = \frac{\omega^3|\mathbf{p}|^2}{3\pi\epsilon_0\hbar c^3} \qquad \tau = \frac{1}{A}$$

No transitions occur unless $\Delta m = \pm 1;\text{or } 0$ and $\Delta l = \pm 1$

$$d\sigma = D(\theta)\,d\Omega \qquad D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad \sigma = \int D(\theta)\,d\Omega \qquad D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega}$$

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} \qquad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \qquad D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2u_l}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u_l = Eu_l$$

Large r: $\displaystyle \frac{d^2u_l}{dr^2} - \frac{l(l+1)}{r^2}u_l = -k^2u_l \qquad u_l = Arj_l(kr) + Brn_l(kr)$

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right] P_l(\cos\theta)$$

$$a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k}e^{i\delta_l}\sin(\delta_l) \qquad f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)\,P_l(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r}) \qquad \Longrightarrow \qquad G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}$$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0$$

$$f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0}V(\mathbf{r}_0)\,d^3\mathbf{r}_0 \qquad f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0^{\infty} rV(r)\sin(\kappa r)\,dr$$