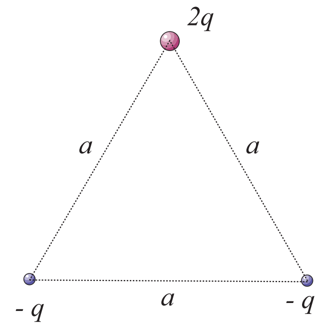
Phys 4610, Fall 2005
Exam #2, Answers

1. Three charges are arranged at the vertices of an equilateral triangle of side a, as shown.

What is the work required of an external force to bring these charges together from infinite separation?



The work required to bring a pair of charges together is $\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{a}$. For a set of charges we sum over all pairs, thus

$$\begin{aligned} W_{\rm ext} &= \frac{1}{4\pi\epsilon_0}\left(\frac{(2q)(-q)}{a} + \frac{(2q)(-q)}{a} + \frac{(-q)(-q)}{a}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{a}(-4q^2 + q^2) = -\frac{3q^2}{4\pi\epsilon_0 a} \end{aligned}$$

2. Suppose the electric potential is given by

$$V(r) = V_0 e^{-ar^2}$$

Find the electric field and charge density at all points.

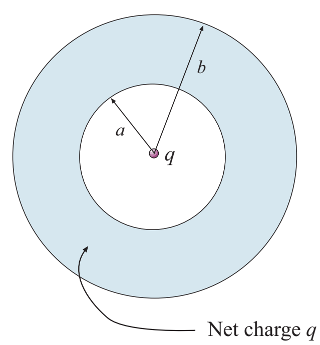
The electric field is the negative gradient of the potential V, so

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} = (-1)V_0(-2ar)e^{-ar^2}\hat{\mathbf{r}} = 2V_0 a r e^{-ar^2}\hat{\mathbf{r}}$$

The charge density is

$$\begin{aligned} \rho &= \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = 2V_0\epsilon_0 a \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 e^{-ar^2}\right] \\ &= 2V_0\epsilon_o \frac{1}{r^2}[3r^2 e^{-ar^2} - 2ar^4 e^{-ar^2}] = 2V_0\epsilon_0 a(3 - 2ar^2)e^{-ar^2} \end{aligned}$$

3. A conducting spherical shell with inner and outer radii a and b has been given a (net) charge q. At the center of the shell has been placed a point charge q.



a) Find the charge induced on the inner and outer surfaces of the conducting shell.

The E field is zero within the bulk of the conductor; a Gaussian sphere with $a < r < b$ must enclose no net charge so a charge of $-q$ must accumulate on the inner surface of the shell.

The shell has total charge q so a charge of $+2q$ must accumulate on its outer surface.

b) Find the electric fields in the regions $0 < r < a$, $a < r < b$ and $b \le r$.

For $r < a$: Gaussian sphere with $r < a$ encloses charge q so symmetry gives

$$E_r(4\pi r^2) = q/\epsilon_0 \qquad \Longrightarrow \qquad E_r = \frac{q}{4\pi\epsilon_0 r^2} \ ,$$

same as for an isolated point charge.

For $a < r < b$, $E_r = 0$ inside a conductor.

For $b < r$, a Gaussian sphere with $r > b$ encloses a total charge $2q$ so symmetry gives

$$E_r(4\pi r^2) = 2q/\epsilon_0 \qquad \Longrightarrow \qquad E_r = \frac{2q}{4\pi\epsilon_0 r^2}$$

c) Find the potential in these three regions, assuming that $V = 0$ at infinity.

Outside the shell the field is the same as that of a charge $2q$ at the origin; since we want $V = 0$ at ∞ we can use

$$V(r) = \frac{2q}{4\pi\epsilon_0 r}$$

in this region.

For $a < r < b$ the potential is constant and (from continuity) has the same value it has at $r = b$, namely

$$V(r) = \frac{2q}{4\pi\epsilon_0 b}$$

For $r < a$ the potential must have the form $\frac{q}{4\pi\epsilon_0 r} + C$ (since this gives the E_r we found there). This solution must match the $a < r < b$ solution at $r = a$, which implies

$$\frac{q}{4\pi\epsilon_0 a} + C = \frac{2q}{4\pi\epsilon_0 b} \qquad \Longrightarrow \qquad C = \frac{q}{4\pi\epsilon_0}\left(\frac{2}{b} - \frac{1}{a}\right)$$

Then for $0 < r < a$ the potential is

$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{a} + \frac{2}{b}\right)$$

4. In the homework exercise on solving the Laplace equation using a spreadsheet you used the prescription that (in the interior region) each cell should be the average of its nearest neighbors.

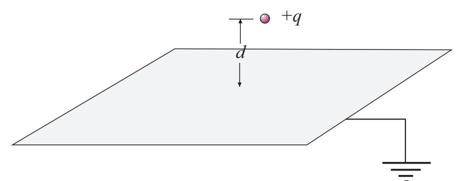
Explain *why* you used this prescription. You don't have to reproduce all the algebra in detail but just summarize the reasoning used to derive that result.

The condition that each cell is the average of its neighbors comes from discretizing the Laplace equation $\nabla^2 V = 0$. If the spatial separation of closest point is h then the numerator of $\nabla^2 V$ (on the grid) contains the combination

$$V(x+h,y) + V(x-h,y) + V(x,y+h) + V(x,y-h) - 4V(x,y)$$

which when set equal to zero gives $V(x,y)$ as the average of its four nearest neighbors.

5. In the problem of the the point charge above an infinite grounded conducting plane (the classic method–of–images problem) it was noted that the naive expression for the force on the $+q$ charge (i.e. from the point charge problem) *was* correct whereas the expression for the energy of system was *not* correct.



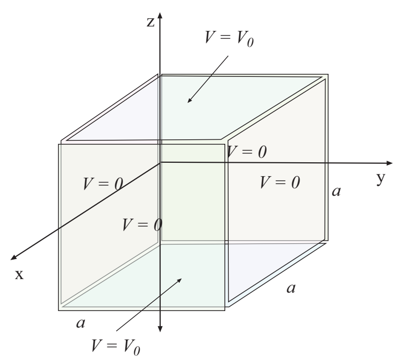
Can you give (recall) an explanation as to why the energy (work to move charge q in from infinity) was not given by the two point charges?

In the fake configuration with two charges (and no conducting plane) we have brought in *two* charges from infinity and if we say

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)}$$

we are counting the work done on the *fictional* charge. But the real problem has no second charge and there is *no* work required to charge up the conducting plane, so that the correct result for W is *half* of the naive one.

6. We shall solve the problem of the cube of side a with two opposite sides held at a constant potential. Here, the sides of the cube are planes at $x = 0$ and $x = a$ and $y = 0$ and $y = a$ but for reasons which you'll see, the top and bottom are at $z = \pm\frac{q}{2}$,



The $x = 0$ and $x = a$ sides are held at potential $V = 0$ as are the $y = 0$ and $y = a$ sides, but the $z = -a/2$ and $z = a/2$ sides are held at $V = V_0$. We want to solve for the potential inside the cube, i.e. to find $V(x, y, x)$.

To help you out with this I'll give you the general form. The solution is of the form

$$V(x,y,z) = \sum_{n,m} C_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \cosh(\alpha z)$$

a) Explain why the solution needs to be of this form and use the Laplace equation solve for α (in terms of n, m and a).

The given form *is* a solution to the Laplace equation because

$$\begin{aligned}\nabla^2 V &= \sum_{n,m} C_{n,m}\left[-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2 + \alpha^2\right] \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\cosh(\alpha z) \\ &= 0\end{aligned}$$

if

$$\alpha^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2 \qquad \text{or} \qquad \alpha_{n,m} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

The given from of V gives $V = 0$ at $x = 0, a$ and $V = 0$ at $y = 0, a$ and it is symmetric in z as the solution clearly needs to be, i.e. V satisfies

$$V(x, y, z) = V(x, y, -z)$$

b) Use orthogonality of the sin functions to solve for the coefficients $C_{n,m}$. In particular, you need to apply the boundary condition and multiply by

$$\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$$

and then integrate on x and y from 0 to a.

We still have to satisfy the boundary condition at $z = \pm\frac{a}{2}$. We have:

$$V(x, y, a/2) = V_0 = \sum_{n,m} C_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\cosh(\alpha_{n,m}\frac{a}{2})$$

Multiply both sides by

$$\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$$

and integrate on $\int_0^a\int_0^a dxdy$, using the orthogonality relations for the sin functions. The sum on n, m collapses to the single values of n', m' and the result is

$$V_0\int_0^a\int_0^a \sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)dxdy = C_{n',m'}\left(\frac{a}{2}\right)^2\cosh(\alpha_{n',m'}\frac{a}{2})$$

which gives the coefficients:

$$C_{n,m} = \frac{4V_0}{a^2}\frac{1}{\cosh(\alpha_{n,m}a/2)}\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy$$

which is the solution to the problem.

We *can* go further and evaluate the integral here (it's not hard), so use:

$$\begin{aligned}\int_0^a \sin\left(\frac{n\pi x}{a}\right)dx &= -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a = -\frac{a}{n\pi}[\cos(n\pi)-1] \\ &= \frac{a}{n\pi}[1-(-1)^n] = \begin{cases}\frac{2a}{n\pi} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}\end{aligned}$$

and then this gives

$$C_{n,m} = \frac{4V_0}{a^2}\frac{1}{\cosh(\alpha_{nm}a/2)}\frac{2a}{n\pi}\frac{2a}{m\pi} = \frac{16V_0}{\pi^2 nm\cosh(\alpha_{nm}a/2)} \qquad \text{for } n, m \text{ odd}$$

c) Write out the full solution and find the charge density σ on the bottom plate ($z = -a/2$).

Substituting for C_{nm}, the solution for the potential is

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m\text{ odd}}\frac{1}{nm\cosh(\alpha_{nm}a/2)}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\cosh(\alpha_{nm}z)$$

where $\alpha_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$.

The charge density of the lower plate is found using

$$\sigma = -\epsilon_0\frac{\partial V}{\partial z}\bigg|_{z=-a/2}.$$

Using our solution for V, this gives (using $\sinh(-x) = -\sinh(x)$)

$$\sigma = \frac{16V_0}{\pi^2}\sum_{n,m\text{ odd}}\frac{\alpha_{nm}}{nm\cosh(\alpha_{nm}a/2)}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh(\alpha_{nm}a/2)$$

which admittedly is not an intriguing result.

7. On a spherical surface of radius R (centered at the origin) the electric has the value

$$V(R,\theta) = k\sin^2\theta$$

Find the electric potential everywhere outside the surface. (Assume $V \to 0$ as $r \to \infty$).

Here you will want to use the expansion of the Laplace solution in spherical coordinates with the B_l's and note that $V(R,\theta)$ is rather easy to write down in terms of $P_2(x)$ and $P_0(x)$.

The expansion for $V)r,\theta)$ in the outer region is

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

Now, the boundary condition is

$$V(R,\theta) = k\sin^2\theta = k(1-\cos^2\theta) = k(1-x^2)$$

Use $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, then

$$\tfrac{2}{3}P_2(x) = x^2 - \tfrac{1}{3} \qquad \Longrightarrow \qquad x^2 = \tfrac{2}{3}P_2(x) + \tfrac{1}{3}$$

then

$$V(R,\theta) = k(1 - \tfrac{2}{3}P_2(x) - \tfrac{1}{3}) = k(\tfrac{2}{3} - \tfrac{2}{3}P_2(x)) = \tfrac{2}{3}k(P_0(x) - P_2(x))$$

Apply the boundary condition to the expansion:

$$\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(x) = \frac{2k}{3}P_0(x) - \frac{2k}{3}P_2(x)$$

and equate the coefficients:

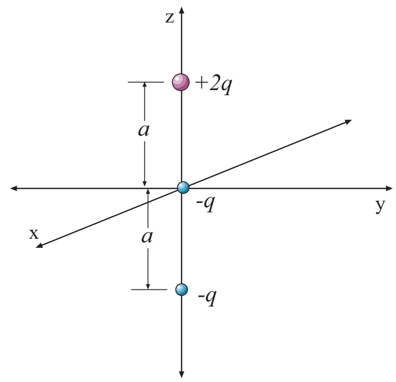
$$\frac{B_0}{R^1} = \frac{2k}{3} \qquad \Longrightarrow \qquad B_0 = \frac{2kR}{3}$$

$$\frac{B_2}{R^3} = -\frac{2k}{3} \qquad \Longrightarrow \qquad B_2 = -\frac{2kR^3}{3}$$

and all the other B_l's are zero. Substitute and get:

$$V(r,\theta) = \frac{2kR}{3}\frac{1}{r} - \frac{2kR^3}{3}\frac{1}{r^3}P_2(\cos\theta)$$

8. Charges of $+2q$, $-q$ and $-q$ are located along the z axis as shown at the right.



a) Find the monopole and dipole moments of this charge distribution.

The monopole moment of this system of charges is zero since the total charge is zero. The dipole moment is

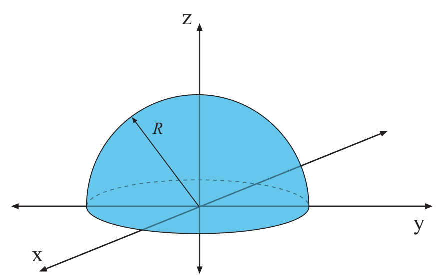
$$\mathbf{p} = \sum_i q_i \mathbf{r}_i = (2q(a) + (-q)(0) + (-q)(-a))\hat{\mathbf{z}} = 3qa\,\hat{\mathbf{z}}$$

b) Find a "leading order" expression for the the potential V at large distances from the origin.

Since the dipole moment is non-zero, the leading-order value of $V(r,\theta)$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(3qa\hat{\mathbf{z}})\cdot\hat{\mathbf{r}}}{r^2} = \frac{3qa\cos\theta}{4\pi\epsilon_0 r^2}$$

9. A solid hemisphere of a polarized material has uniform polarization $\mathbf{P} = P\hat{\mathbf{z}}$ when it is oriented as shown at the right.



Find the surface bound charge density σ_b. (Specify what it is on the boundary of the hemisphere.)

$\mathbf{P}$ is uniform (within the hemisphere) so

$$\rho_b = -\nabla\cdot\mathbf{P} = 0$$

On the curved surface, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = P\cos\theta$$

On the flat surface, $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ (i.e. the *outward* unit normal vector) so

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\hat{\mathbf{z}}\cdot(-\hat{\mathbf{z}}) = -P$$

So the bound charge has a positive θ-dependent density on the curved surface and a constant negative density on the flat surface.

Useful Equations

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \qquad \int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} \qquad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Spherical:

$$d\mathbf{l} = dl_r\, \hat{\mathbf{r}} + dl_\theta\, \hat{\boldsymbol{\theta}} + dl_\phi\, \hat{\boldsymbol{\phi}} \qquad d\tau = r^2\, \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{2}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \tag{3}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{4}$$

Cylindrical:

$$d\mathbf{l} = dl_s\, \hat{\mathbf{s}} + dl_\phi\, \hat{\boldsymbol{\phi}} + dl_z\, \hat{\mathbf{z}} \qquad d\tau = s\, ds\, d\phi\, dz$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{5}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{6}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{7}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{8}$$

More Math

Gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Product Rules:

(1) $\nabla\cdot(\nabla T)$ (Divergence of curl)

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

(2) $\nabla\times(\nabla T)$ (Curl of gradient)

$$\nabla\times(\nabla T) = 0$$

(3) $\nabla(\nabla\cdot\mathbf{v})$ (Gradient of divergence)

Nothing interesting about this; does not occur often.

(4) $\nabla\cdot(\nabla\times\mathbf{v})$

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

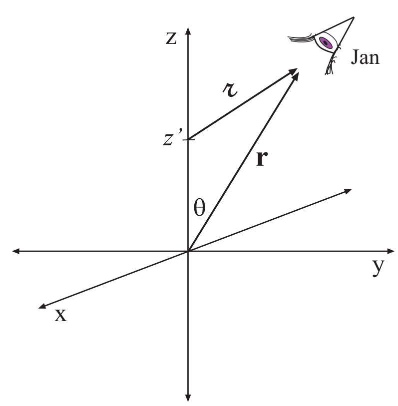
(5) $\nabla\times(\nabla\times\mathbf{v})$ (Curl of a curl)

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Even More Math

In the figure at the right,

$$\mathscr{r} = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$$



If $x < 1$ then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \qquad V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3-3x)/2$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

Physics:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \qquad \mathbf{F} = Q\mathbf{E} \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\,\hat{\boldsymbol{\mathscr{r}}}_i \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

$$\Phi_E = \int_{\mathcal{S}}\mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V \qquad -\nabla^2 V = \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \qquad W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j\\ i\neq j}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\int\rho V\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau \qquad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\epsilon_0}{2}E^2 \qquad C \equiv \frac{Q}{V}$$

$$\mathbf{p} \equiv \int\mathbf{r}'\rho(\mathbf{r}')\,d\tau' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

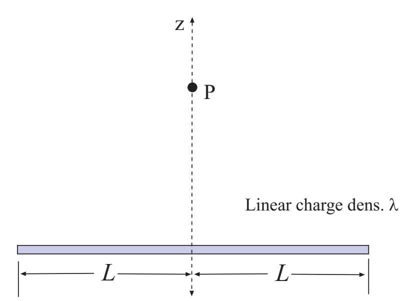
$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \qquad \nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot d\mathbf{a} = Q_{f,\text{enc}}$$

Specific Results:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &= \frac{Q}{2\pi R^2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \end{aligned}$$