1 Probability Density Functions (PDF)

For a continuous RV X with PDF $f_X(x)$,

$$P(a \le X \le b) = \int_a^b f_X(x)dx$$
$$P(X \in A) = \int_A f_X(x)dx$$

Properties:

- Nonnegativity:
$$f_X(x) \ge 0 \ \forall x$$
- Normalization:
$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

2 PDF Interpretation

Caution: $f_X(x) \neq P(X = x)$

- if X is continuous, $P(X = x) = 0 \ \ \forall x$!!
- $f_X(x)$ can be ≥ 1

Interpretation: "probability per unit length" for "small" lengths around x

$$P(x \le X \le x + \delta) \approx f_X(x)\delta$$

3 Mean and variance of a continuous RV

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
\mathrm{Var}(X) &= E\left[(X - E[X])^2\right] \\
&= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx \\
&= E[X^2] - (E[X])^2 \ (\ge 0) \\
E[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)dx \\
E[aX + b] &= aE[X] + b \\
\mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)
\end{aligned}$$

4 Cumulative Distribution Functions

Definition:
$$F_X(x) = P(X \le x)$$
monotonically increasing from 0 (at $-\infty$) to 1 (at $+\infty$).

- Continuous RV (CDF is continuous in x):
$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)dt$$
$$f_X(x) = \frac{dF_X}{dx}(x)$$
- Discrete RV (CDF is piecewise constant):
$$F_X(x) = P(X \le x) = \sum_{k \le x} p_X(k)$$
$$p_X(k) = F_X(k) - F_X(k-1)$$

5 Uniform Random Variable

If X is a uniform random variable over the interval [a,b]:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{otherwise } (x > b) \end{cases}$$

$$E[X] = \frac{b-a}{2}$$
$$\mathrm{var}(X) = \frac{(b-a)^2}{12}$$

6 Exponential Random Variable

X is an exponential random variable with parameter λ:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad \mathrm{var}(X) = \frac{1}{\lambda^2}$$

Memoryless Property: Given that $X > t$, $X - t$ is an exponential RV with parameter λ

7 Normal/Gaussian Random Variables

General normal RV: $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$
$$E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2$$

Property: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$

then $Y \sim N(a\mu + b, a^2\sigma^2)$

8 Normal CDF

Standard Normal RV: $N(0, 1)$
CDF of standard normal RV Y at y: $\Phi(y)$
- given in tables for $y \ge 0$
- for $y < 0$, use the result: $\Phi(y) = 1 - \Phi(-y)$

To evaluate CDF of a general standard normal, express it as a function of a standard normal:

$$X \sim N(\mu, \sigma^2) \Leftrightarrow \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$P(X \le x) = P\Big(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\Big) = \Phi\Big(\frac{x-\mu}{\sigma}\Big)$$

9 Joint PDF

Joint PDF of two continuous RV X and Y: $f_{X,Y}(x, y)$

$$P(A) = \int\int_A f_{X,Y}(x, y)dxdy$$

Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$
$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)dxdy$
Joint CDF: $F_{X,Y}(x, y) = P(X \le x, Y \le y)$

10 Independence

By definition,

$$X, Y \text{ independent} \Leftrightarrow f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \forall(x, y)$$

If X and Y are independent:

- $E[XY]=E[X]E[Y]$
- $g(X)$ and $h(Y)$ are independent
- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

11 Conditioning on an event

Let X be a continuous RV and A be an event with $P(A) > 0$,

$$\begin{aligned}
f_{X|A}(x) &= \begin{cases} \frac{f_X(x)}{P(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \\
P(X \in B | X \in A) &= \int_B f_{X|A}(x)dx \\
E[X|A] &= \int_{-\infty}^{\infty} x f_{X|A}(x)dx \\
E[g(X)|A] &= \int_{-\infty}^{\infty} g(x) f_{X|A}(x)dx
\end{aligned}$$

If $A_1, \ldots, A_n$ are disjoint events that form a partition of the sample space,

$$\begin{aligned}
f_X(x) &= \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x) \ (\approx \text{total probability theorem}) \\
E[X] &= \sum_{i=1}^{n} P(A_i) E[X|A_i] \ (\text{total expectation theorem}) \\
E[g(X)] &= \sum_{i=1}^{n} P(A_i) E[g(X)|A_i]
\end{aligned}$$

12 Conditioning on a RV

X, Y continuous RV

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\
f_X(x) &= \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y)dy \quad (\approx \mathit{totalprobthm})
\end{aligned}$$

Conditional Expectation:

$$\begin{aligned}
E[X|Y = y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx \\
E[g(X)|Y = y] &= \int_{-\infty}^{\infty} g(X) f_{X|Y}(x|y)dx \\
E[g(X, Y)|Y = y] &= \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x|y)dx
\end{aligned}$$

Total Expectation Theorem:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y)dy \\
E[g(X)] &= \int_{-\infty}^{\infty} E[g(X)|Y = y] f_Y(y)dy \\
E[g(X, Y)] &= \int_{-\infty}^{\infty} E[g(X, Y)|Y = y] f_Y(y)dy
\end{aligned}$$

13 Continuous Bayes' Rule

X, Y continuous RV, N discrete RV, A an event.

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) f_X(t)dt} \\
P(A|Y = y) &= \frac{P(A) f_{Y|A}(y)}{f_Y(y)} = \frac{P(A) f_{Y|A}(y)}{f_{Y|A}(y)P(A) + f_{Y|A^c}(y)P(A^c)} \\
P(N = n|Y = y) &= \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)} = \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)}
\end{aligned}$$

14 Derived distributions

Def: PDF of a *function* of a RV X with known PDF: $Y = g(X)$.
Method:

- Get the CDF:
$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int_{x|g(x) \le y} f_X(x)dx$$
- Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Special case: if $Y = g(X) = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{x-b}{a}\right)$

15 Convolution

$W = X + Y$, with X, Y independent.

- Discrete case:
$$p_W(w) = \sum_x p_X(x)p_Y(w - x)$$
- Continuous case:
$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) \ dx$$

Graphical Method:

- put the PMFs (or PDFs) on top of each other
- flip the PMF (or PDF) of Y
- shift the flipped PMF (or PDF) of Y by w
- cross-multiply and add (or evaluate the integral)

In particular, if X, Y are independent and normal, then $W = X + Y$ is normal.

16 Law of iterated expectations

$E[X|Y = y] = f(y)$ is a number.
$E[X|Y] = f(Y)$ is a random variable
(the expectation is taken with respect to X).
To compute $E[X|Y]$, first express $E[X|Y = y]$ as a function of y.

Law of iterated expectations:

$$E[X] = E[E[X|Y]]$$

(equality between two real numbers)

17 Law of Total Variance

$\mathrm{Var}(X|Y)$ is a random variable that is a function of Y
(the variance is taken with respect to X).
To compute $\mathrm{Var}(X|Y)$, first express

$$\mathrm{Var}(X|Y = y) = E[(X - E[X|Y = y])^2|Y = y]$$

as a function of y.

Law of conditional variances:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

(equality between two real numbers)

18 Sum of a random number of iid RVs

N discrete RV, X_i i.i.d and independent of N.
$Y = X_1 + \ldots + X_N$. Then:

$$\begin{aligned}
E[Y] &= E[X]E[N] \\
\mathrm{Var}(Y) &= E[N]\mathrm{Var}(X) + (E[X])^2\mathrm{Var}(N)
\end{aligned}$$

19 Covariance and Correlation

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

- By definition, X, Y are uncorrelated $\Leftrightarrow \mathrm{Cov}(X, Y) = 0$.
- If X, Y independent $\Rightarrow$ X and Y are uncorrelated. (the converse is not true)
- In general, Var(X+Y)= Var(X)+ Var(Y)+ 2 Cov(X,Y)
- If X and Y are uncorrelated, Cov(X,Y)=0 and Var(X+Y)= Var(X)+Var(Y)

Correlation Coefficient: (dimensionless)

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X\sigma_Y} \quad \in [-1, 1]$$

$\rho = 0 \Leftrightarrow$ X and Y are uncorrelated.
$|\rho| = 1 \Leftrightarrow \quad X - E[X] = c[Y - E[Y]]$ (linearly related)

	X	$p_X(k)$	E[X]	var(X)
Bernoulli	$\begin{cases} 1 & \text{success} \\ 0 & \text{failure} \end{cases}$	$\begin{cases} p & k = 1 \\ 1 - p & k = 0 \end{cases}$	p	$p(1-p)$
Binomial	Number of successes in n Bernoulli trials	$\binom{n}{k} p^k(1-p)^{n-k}$, $k = 0, 1, \ldots, n$	np	np(1-p)
Geometric	Number of trials until first success	$(1-p)^{k-1}p$, $k = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Uniform	An integer in the interval [a,b]	$\begin{cases} \frac{1}{b-a+1} & k = a, \ldots, b \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$

Homework No. 6 (with solutions)

Problem 1. Let X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} Cx^2e^{-x(1+y)} & \text{if } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant C and the marginal densities $f_X(x)$ and $f_Y(y)$.
(b) Find the conditional densities $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$.
(c) Compute $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[Y \mid X = x]$.

Solution. Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\infty} x^2e^{-x}\left(\int_0^{\infty} e^{-xy}\mathrm{d}y\right)\mathrm{d}x = \int_0^{\infty} xe^{-x}\mathrm{d}x = 1,$$

it follows that $C = 1$. Also, by the computation above

$$f_X(x) = xe^{-x}, \quad x > 0.$$

and

$$f_Y(y) = \int_0^{\infty} x^2e^{-x(1+y)}\mathrm{d}x = \frac{2}{(1+y)^3}, \quad y > 0.$$

Thus, for part (b)

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \tfrac{1}{2}(1+y)^3x^2e^{-x(1+y)}, \quad x, y > 0,$$

and

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = xe^{-xy}, \quad x, y > 0.$$

Part (c) is a routine integration and yields

$$\mathbb{E}[X \mid Y = y] = \int_0^{\infty} \tfrac{1}{2}(1+y)^3x^3e^{-x(1+y)}\mathrm{d}x = \frac{(1+y)^3}{2}\frac{6}{(1+y)^4} = \frac{3}{1+y}.$$

and

$$\mathbb{E}[Y \mid X = x] = \int_0^{\infty} yxe^{-xy}\mathrm{d}y = \frac{1}{x}.$$

Problem 2. The random variables X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{cy}{x} & \text{if } 0 \le y < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.
(b) Find the marginal densities $f_X(x)$ and $f_Y(y)$.
(c) Find $\mathbb{P}(X + Y \le 1)$.

Solution. Since

$$1 = \int_0^1\left(\int_0^x \frac{cy}{x}\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \frac{cx}{2}\mathrm{d}x = \frac{c}{4},$$

it follows that $c = 4$. From the calculation above $f_X(x) = 2x$, for $x \in [0, 1]$. Also,

$$f_Y(y) = \int_y^1 \frac{4y}{x}\,\mathrm{d}x = -4y\log y, \quad y \in (0, 1].$$

For part (c)

$$\begin{aligned}
\mathbb{P}(X + Y \le 1) &= \int_0^{1/2}\left(\int_0^x \frac{4y}{x}\mathrm{d}y\right)\mathrm{d}x + \int_{1/2}^1\left(\int_0^{1-x} \frac{4y}{x}\mathrm{d}y\right)\mathrm{d}x \\
&= 2\log 2 - 1 \approx 0.39.
\end{aligned}$$

Problem 3. If X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}(x+y)e^{-x-y} & \text{if } x, y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

find the probability density function of $Z = X + Y$.

Solution. It is most convenient to select new coordinates $z = x + y$ and $v = \frac{1}{2}(x - y)$. Then

$$\begin{aligned}
F_Z(z_0) &= \mathbb{P}(X + Y \le z_0) \\
&= \iint_{(x+y \le z_0)} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{z_0}\left(\int_{-z/2}^{z/2} \tfrac{1}{2}ze^{-z}\mathrm{d}v\right)\mathrm{d}z \\
&= \int_0^{z_0} \tfrac{1}{2}z^2e^{-z}\mathrm{d}z,
\end{aligned}$$

from which it follows that $f_Z(z) = \frac{1}{2}z^2e^{-z}$, for $z \ge 0$.

Problem 4. Random variables X, Y, and Z have joint probability density function

$$f(x, y, z) = \begin{cases} 8xyz & \text{if } 0 < x, y, z < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Are X, Y and Z independent? Find $\mathbb{P}(X > Y)$, $\mathbb{E}[Y^3 \mid Z]$, and $\mathbb{P}(X > Y > Z)$.

Solution. Direct integration shows that $f_X(x) = 2x$, for $x \in [0, 1]$, and by symmetry Y and Z, also have the same law. Since the joint probability density function is the product of the marginal ones, X, Y, and Z are independent.

$$\begin{aligned}
\mathbb{P}(X > Y) &= \iint_{\{x>y\}} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\left(\int_0^x 4xy\,\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(X > Y > Z) &= \iiint_{(x>y>z)} f_{X,Y,Z}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= 8\int_0^1 x\left(\int_0^x y\left(\int_0^y z\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \frac{1}{6}.
\end{aligned}$$

Problem 5. Suppose X and Y are independent random variables, each having the exponential distribution with parameter λ.

(a) If $Z = 1 - e^{-\lambda X}$, find the probability density function $f_Z(z)$ and the cumulative distribution function $F_Z(z)$.
(b) Same as in part (a), if $Z = \min(X, Y)$.
(c) Same as in part (a), if $Z = X - Y$.
(d) Find $\mathbb{P}\big(\min(X, Y) \le aX\big)$ and $\mathbb{P}\big(\min(X, Y) \le a\max(X, Y)\big)$, where $a \in (0, 1)$ is some constant.

Solution. We have

$$F_Z(z) = \mathbb{P}(1 - e^{-\lambda X} \le z) = \mathbb{P}\left(X \le -\lambda^{-1}\log(1 - z)\right) = 1 - e^{\log(1-z)} = z.$$

Hence, Z is uniformly distributed on $[0, 1]$. For part (b), by independence

$$\mathbb{P}(Z > z) = \mathbb{P}(X > z)\,\mathbb{P}(Y > z) = e^{-2\lambda z}.$$

Hence Z is exponential with parameter 2λ. For part (c)

$$\mathbb{P}(Z \le z_0) = \int_0^{z_0}\left(\int_{-\infty}^{\infty} f_{X,Y}(x, z + x)\,\mathrm{d}x\right)\mathrm{d}z \tag{1}$$

Continuing with the integral inside the parenthesis in (??),

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{X,Y}(x, z + x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_X(x)f_Y(z + x)\,\mathrm{d}x \\
&= \int_{\max(0,-z)}^{\infty} \lambda^2e^{-\lambda x}e^{-\lambda(x+z)}\,\mathrm{d}x \\
&= \lambda^2e^{-\lambda z}\int_{\max(0,-z)}^{\infty} e^{-2\lambda x}\,\mathrm{d}x \\
&= \lambda^2e^{-\lambda z}\frac{1}{2\lambda}e^{-2\lambda\max(0,-z)} \\
&= \frac{\lambda}{2}e^{-\lambda|z|}, \quad -\infty < z < \infty.
\end{aligned}$$

Thus we can express F_Z as

$$F_Z(z) = \begin{cases} \frac{1}{2}e^{\lambda z}, & z < 0, \\ 1 - \frac{1}{2}e^{-\lambda z}, & z \ge 0, \end{cases}$$

or equivalenty,

$$\begin{aligned}
F_Z(z) &= \frac{1}{2}\left(1 + e^{\lambda\min(0,z)} - e^{-\lambda\max(0,z)}\right) \\
&= \frac{1}{2} + e^{-\frac{\lambda}{2}|z|}\sinh\left(\tfrac{\lambda z}{2}\right), \quad -\infty < z < \infty.
\end{aligned}$$

For part (d)

$$\mathbb{P}\big(\min(X, Y) \le aX\big) = \mathbb{P}(Y \le aX) = \int_0^{\infty} \lambda e^{-\lambda x}\left(\int_0^{ax} \lambda e^{-\lambda y}\mathrm{d}y\right)\mathrm{d}x = \frac{a}{1+a}.$$

Note that this is not continuous at $a = 1$, however

$$\mathbb{P}\big(\min(X,Y) \le a\max(X,Y)\big) = \mathbb{P}(Y \le aX) + \mathbb{P}(X \le aY) = \frac{2a}{1+a},$$

tends to 1 as $a \uparrow 1$, so it is continuous at $a = 1$.

Problem 1. Let U and V be independent standard normal random variables, and $X = U + V$, $Y = U - 2V$. Find $\mathbb{E}[X \mid Y]$, $\mathrm{cov}(X, Y)$, and the joint probability density function of X and Y.

Solution. We have

$$\mathrm{var}(Y) = \mathrm{var}(U) + 4\,\mathrm{var}(V) = 5,$$

and

$$\mathrm{cov}(X,Y) = \mathbb{E}\big[(U+V)(U-2V)\big] = \mathbb{E}[U^2] - 2\,\mathbb{E}[V^2] = -1.$$

Therefore,

$$\mathbb{E}[X \mid Y] = -\frac{1}{5}Y.$$

Next, we have, $\sigma_X^2 = 2$ and $\rho = -\frac{1}{\sqrt{2\times 5}} = -\frac{1}{\sqrt{10}}$. The joint probability density function is of the form $Ce^{-q(x,y)}$, where

$$C = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_X\sigma_Y},$$

and

$$q(x,y) = \frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right).$$

The various coefficients appearing in this formula have already been found.

Problem 2. Let the random variables X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\Big\{-\frac{x^2+y^2}{2\sigma^2}\Big\}.$$

Suppose that the random variables W and Q satisfy the relation

$$X\cos(\omega_0 t) + Y\sin(\omega_0 t) = W\cos(\omega_0 t + Q),$$

for all time t, where ω_0 is a constant. Furthermore, suppose that $\mathbb{P}(W \ge 0) = 1$ and $\mathbb{P}(-\pi < Q \le \pi) = 1$.

(a) Find the joint probability density function of W and Q.

(b) Find the marginal probability density functions of W and Q.

(c) Are W and Q independent?

Solution.

Problem 3. Consider three zero-mean random variables X, Y, and Z, with known variances and covariances. Give a formula for the linear least squares estimator of X based on Y and Z, that is, find a and b that minimize

$$\mathbb{E}\left[(X - aY - bZ)^2\right].$$

For simplicity, assume that Y and Z are uncorrelated.

Solution. For any choice of a, b, the mean squared error is equal to

$$\mathbb{E}[X^2] + a^2\,\mathbb{E}[Y^2] + b^2\,\mathbb{E}[Z^2] - 2a\,\mathbb{E}[XY] - 2b\,\mathbb{E}[XZ].$$

We differentiate with respect to a and b, and set the derivatives to zero, to obtain

$$a = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}, \qquad b = \frac{\mathbb{E}[XZ]}{\mathbb{E}[Z^2]}.$$

Problem 4. Let X be uniformly distributed on the interval $(1, 5)$. Find the probability density function of $Y = \frac{X}{5-X}$.

Solution. We have $X = g(Y) = 5Y(1+Y)^{-1}$ and $\frac{dg}{dy}(y) = 5(1+y)^{-2}$. Since $f_X(x) = \frac{1}{4}$, for $x \in (1,5)$, we obtain

$$\begin{aligned} f_Y(y) &= \left|\tfrac{dg}{dy}(y)\right| f_X(g(y)) \\ &= \frac{5}{4(1+y)^2}, \quad \frac{1}{4} < y < \infty. \end{aligned}$$

Problem 5. Let X and Y denote two points that are chosen randomly and independently from the interval $[0,1]$. Find the probability density function of $Z = |X - Y|$. Use this to calculate the mean distance between X and Y.

Solution. It is simpler to calculate $\mathbb{P}(Z > z)$. Since the locus of $\{Z > z\}$ is two rectangles of total area $(1-z)^2$, and since the pair (X,Y) is uniformly distributed on the unit square, $\mathbb{P}(Z > z) = (1-z)^2$. It follows that $F_Z(z) = 1 - (1-z)^2$, $0 \le z \le 1$. Differentiating, we have $f_Z(z) = 2(1-z)$, $0 \le z \le 1$. Thus

$$\mathbb{E}[Z] = \int_0^1 2z(1-z)\,dz = \frac{1}{3}.$$

Problem 6. Let X and Y be independent random variables, each having the exponential distribution with parameter λ. Find the joint probability density function of X and $X + Y$. Also, find the conditional probability density function of X given that $X + Y = a$.

Solution.

Problem 7. The random variables $X_1, \ldots, X_n$ have common mean μ, common variance σ^2 and, furthermore, $\mathbb{E}[X_i X_j] = c$ for every pair of distinct i and j. Derive a formula for the variance of $X_1 + \cdots + X_n$, in terms of μ, σ^2, c, and n.

Solution. We first note that, for distinct i and j, we have

$$\mathrm{cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\,\mathbb{E}[X_j] = c - \mu^2.$$

Thus, using the formula for the variance of the sum of random variables,

$$\mathrm{var}(X_1 + \cdots + X_n) = n\sigma^2 + (n^2 - n)(c - \mu^2).$$

Problem 8. Let Y be exponentially distributed with parameter 1, and let Z be uniformly distributed over the interval $[0,1]$. Assume Y and Z are independent. Find the probability density functions of $W = Y - Z$ and that of $X = |Y - Z|$.

Solution. Let $W = Y - Z$ and $X = |Y - Z|$. We find the probability density function of W by convolution of the exponential with parameter 1 with the uniform in the interval $[-1, 0]$. We obtain

$$f_W(w) = \begin{cases} \displaystyle\int_w^{w+1} e^{-x}\,dx = e^{-w} - e^{-(w+1)}, & \text{if } w \ge 0, \\ \displaystyle\int_0^{w+1} e^{-x}\,dx = 1 - e^{-(w+1)}, & \text{if } -1 \le w \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

We then note that

$$f_X(x) = \begin{cases} f_W(x) + f_W(-x), & \text{if } x \ge 0, \\ 0, & \text{otherwise}, \end{cases}$$

and obtain the probability density function of X. The validity of the above formula is established by using the relation

$$F_X(x) = \mathbb{P}(|W| \le x) = \mathbb{P}(-x \le W \le x) = F_X(x) - F_X(-x),$$

and differentiating with respect to x.

Problem 1. If X has the normal distribution with mean μ and variance σ^2, find $\mathbb{E}\left[X^3\right]$ (as a function of μ and σ^2), without computing any integrals.

Solution. We have

$$0 = \mathbb{E}\left[(X-\mu)^3\right] = \mathbb{E}\left[X^3\right] - 3\mu\,\mathbb{E}\left[X^2\right] + 3\mu^2\,\mathbb{E}[X] - \mu^3.$$

Solving we get

$$\mathbb{E}\left[X^3\right] = 3\mu(\sigma^2 + \mu^2) - 3\mu^3 - \mu^3 = \mu(3\sigma^2 - \mu^2).$$

Problem 2. Suppose that the weight of a person selected at random from some population is normally distributed with parameters μ and σ^2. Suppose also that $\mathbb{P}(X \le 160) = \frac{1}{2}$ and $\mathbb{P}(X \le 140) = \frac{1}{4}$. Find μ and σ. Also, find $\mathbb{P}(X \ge 200)$. (Use the *standard normal table* for this problem).

Solution. Clearly $\mu = 160$. Hence

$$\mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{-20}{\sigma}\right) = \frac{1}{4} \Longrightarrow \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{20}{\sigma}\right) = \frac{3}{4} \Longrightarrow \frac{20}{\sigma} \approx 0.675 \Longrightarrow \sigma \approx 29.5.$$

Next,

$$\mathbb{P}(X \ge 200) = \left(\frac{X-\mu}{\sigma} \ge \frac{200-140}{\sigma}\right) \approx \left(\frac{X-\mu}{\sigma} \ge 2\right) \approx 0.0228.$$

Problem 3. Let X and Y be independent exponential random variables with a common parameter λ.

(a) Find the moment generating function associated with $aX + Y$, where a is a constant.

(b) Use the result of part (a) to find the probability density function of $aX + Y$, for the case where a is positive and different than 1.

(c) Use the result of part (a) to find the probability density function of $X - Y$.

Solution. (a) Let $Z = aX + Y$. We have

$$M_Z(s) = \mathbb{E}\left[e^{s(aX+Y)}\right] = \mathbb{E}\left[e^{saX}\right]\mathbb{E}\left[e^{sY}\right] = M_X(sa)M_Y(s) = \frac{\lambda}{\lambda - sa} \times \frac{\lambda}{\lambda - s},$$

for $s < \lambda$. (b) We will express the moment generating function of Z in the form

$$M_Z(s) = \frac{c}{\lambda - sa} + \frac{d}{\lambda - s}.$$

We have

$$c = M_Z(s)(\lambda - sa)\Big|_{s=\lambda/a} = \frac{\lambda^2}{\lambda - \lambda/a} = \frac{a\lambda}{a-1},$$

and

$$d = M_Z(s)(\lambda - s)\Big|_{s=\lambda} = \frac{\lambda^2}{\lambda - a\lambda} = -\frac{\lambda}{a-1}.$$

Thus,

$$M_Z(s) = \frac{\lambda}{a-1}\left(\frac{a}{\lambda - sa} - \frac{1}{\lambda - s}\right) = \frac{a}{a-1} \times \frac{\lambda/a}{\lambda/a - s} - \frac{1}{a-1} \times \frac{\lambda}{\lambda - s}.$$

We recognize this as the moment generating function associated with the probability density function

$$f_Z(z) = \frac{\lambda}{a-1}e^{-\lambda z/a} - \frac{\lambda}{a-1}e^{-\lambda z}, \qquad z \ge 0.$$

(c) From part (a) we have that the moment generating function of $-Y$ is equal to $\lambda/(\lambda+s)$. With $Z = X - Y$, we have

$$M_Z(s) = \frac{\lambda^2}{(\lambda - s)(\lambda + s)} = \frac{1}{2}\left(\frac{\lambda}{\lambda - s} + \frac{\lambda}{\lambda + s}\right),$$

which we recognize as the moment generating function of a mixture of two random variables, one distributed as X, the other distributed as $-Y$. It follows that

$$p_Z(z) = \begin{cases} \frac{1}{2}e^{-\lambda z}, & \text{if } z \ge 0, \\ \frac{1}{2}e^{\lambda z}, & \text{if } z \le 0, \end{cases}$$

or

$$p_Z(z) = \frac{\lambda}{2}e^{-\lambda|z|}.$$

Problem 4. Each egg laid by the hen falls onto the concrete floor of the henhouse and cracks with probability p. If the number of eggs laid today by the hen has the Poisson distribution, with parameter λ, use moment generating functions to find the probability distribution of the number of uncracked eggs.

Solution. Let N denote the number of eggs laid. Then $G_N(s) = e^{\lambda(s-1)}$. Let

$$X_i \triangleq \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ egg survives,} \\ 0, & \text{if the } i^{\text{th}} \text{ egg cracks.} \end{cases}$$

Thus, $G_{X_i}(s) = (1-p)s + p$. Since $Z = \sum_{i=1}^{N} X_i$ is the number of uncracked eggs, we have

$$\begin{aligned} G_Z(s) &= G_N(G_{X_i}(s)) \\ &= \exp(\lambda((1-p)s + p - 1)) = e^{(1-p)\lambda(s-1)}. \end{aligned}$$

Therefore Z has the Poisson distribution with parameter $\lambda(1-p)$.

Problem 5. Suppose that X has moment generating function

$$M_X(s) = \frac{6 - 3s}{2(1-s)(3-s)}.$$

Find the probability density function of the associated random variable X.

Solution. $M_X(s)$ can be expanded in the form

$$M_X(s) = \frac{a}{1-s} + \frac{b}{3-s}.$$

The values of a and b can be found by plugging in particular values of s. For example, setting $s = 0$ yields

$$a + \frac{b}{3} = M_X(0) = 1,$$

and setting $s = 4$ yields

$$-\frac{a}{3} - b = M_X(4) = -1.$$

Solving for a and b, we obtain $a = b = 3/4$. It follows that

$$M_X(s) = \frac{3}{4} \times \frac{1}{1-s} + \frac{1}{4} \times \frac{3}{3-s}.$$

Thus, X is a mixture of two exponential random variables and

$$f_X(x) = \frac{3}{4}e^{-x} + \frac{1}{4}e^{-3x}, \qquad x \ge 0.$$

Problem 6. The moment generating function associated with the random variable X is

$$M(s) = ae^s + be^{4(e^s-1)}, \qquad \mathbb{E}[X] = 3.$$

Find:

(a) The scalars a and b.

(b) $\mathbb{P}(X = 1)$, $\mathbb{E}[X^2]$, and $\mathbb{E}[2^X]$.

Solution. (a) Since $M(0) = 1$, we have $a + b = 1$. Since

$$\mathbb{E}[X] = \frac{d}{ds}M(s)\Big|_{s=0} = \left(a + 4be^s e^{4(e^s-1)}\right)\Big|_{s=0} = 3,$$

we obtain $a + 4b = 3$. Solving the two equations for a and b, we have

$$a = \frac{1}{3}, \qquad b = \frac{2}{3}.$$

(b) Given the form of $M(s)$, we see that X is equal to 1, with probability $a = 1/3$, and equal to a Poisson random variable Z with parameter $\lambda = 4$, with probability $b = 2/3$. The desired quantities can be calculated starting from this observation:

$$p_X(1) = a + b\,\mathbb{P}(Z = 1) = a + 4be^{-4} = \frac{1}{3} + \frac{8}{3}e^{-4}.$$

Alternatively, we can use the series expansion

$$M(s) = ae^s + be^{4(e^s-1)} = ae^s + be^{-4}\left(1 + 4e^s + \frac{(4e^s)^2}{2} + \cdots\right),$$

to infer that X is a discrete random variable, and identify $p_X(1)$ with the coefficient of e^s, so that

$$p_X(1) = a + 4be^{-4} = \frac{1}{3} + \frac{8}{3}e^{-4}.$$

The second moment $e[X^2]$ is obtained by

$$\mathbb{E}[X^2] = \frac{d^2}{ds^2}M(s)\Big|_{s=0} = ae^s + 4b(4e^s + 1)e^{4e^s+s-4}\Big|_{s=0} = \frac{41}{3}.$$

Finally,

$$\mathbb{E}[2^X] = \mathbb{E}\left[(e^{\ln 2})^X\right] = M(\ln 2) = 2a + be^4 = \frac{2}{3} + \frac{2}{3}e^4.$$

Problem 7. The random variables X_1, X_2 and X_3 are independent and identically distributed, having the exponential distribution with parameter 1, i.e., a probability density function e^{-x}, $x > 0$. Find the probability density function of

$$Z = \tfrac{1}{2}X_1 + \tfrac{1}{3}X_2 + X_3.$$

Solution. We have

$$\begin{aligned} M_Z(s) &= M_{X_1}(s/2) \times M_{X_2}(s/3) \times M_{X_3}(s) \\ &= \frac{2}{2-s} \times \frac{3}{3-s} \times \frac{1}{1-s} \\ &= \frac{-6}{2-s} + \frac{3}{3-s} + \frac{3}{1-s}. \end{aligned}$$

Therefore,

$$\begin{aligned} f_Z(z) &= -6e^{-2z} + 3e^{-3z} + 3e^{-z} \\ &= 3e^{-z}(1 - e^{-z})^2, \quad z \ge 0. \end{aligned}$$

Problem 1. If X has cumulative distribution function $F_X(x)$, what is the distribution function of $Y = \max\{0, X\}$. Next suppose that X be a random variable which is uniformly distributed on the interval $(-1, 1)$. Compute the cumulative distribution function of $Y = \max\{0, X\}$.

Solution.

Problem 2. Let X and Y be independent having an exponential distribution with parameters λ and μ respectively. Find the probability density function of $Z = X + Y$.

Solution. We have, assuming $\lambda \ne \mu$,

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(x) f_Y(z - x)\,dx \\ &= \int_0^z \lambda\mu\, e^{-\mu z} e^{-(\lambda-\mu)x}\,dx \\ &= \lambda\mu\,\frac{e^{-\mu z} - e^{-\lambda z}}{\lambda - \mu}, \quad z > 0. \end{aligned}$$

When $\lambda = \mu$, this reduces to $f_Z(z) = \mu^2 z e^{-\mu z}$.

Problem 3. Suppose that the two dimensional random variable (X, Y) is uniformly distributed over the triangular region $R = \{(x, y) \mid 0 \le x \le y \le 1\}$. Find $\rho(X, Y)$.

Solution. The area of the triangle is $1/2$, so the joint probability density function of (X, Y) equals 2 on the triangle and is 0 outside the triangle. Thus for any function $g(X, Y)$,

$$\mathbb{E}[g(X,Y)] = 2\int_0^1 \left(\int_0^y g(x,y)\,dx\right)dy.$$

Setting $g(X, Y)$ equal to X, Y, XY, X^2 and Y^2 and calculating the integral we obtain

$$\mathbb{E}[X] = \frac{1}{3}, \quad \mathbb{E}[Y] = \frac{2}{3}, \quad \mathbb{E}[XY] = \frac{1}{4}, \quad \mathbb{E}\left[X^2\right] = \frac{1}{6}, \quad \mathbb{E}\left[Y^2\right] = \frac{1}{2}.$$

Thus $\mathrm{cov}(X, Y) = \frac{1}{12}$, and $\rho(X, Y) = \frac{1}{\sqrt{12}}$.

Problem 4. A coin has an a priori probability X of coming up heads, where X is a random variable with probability density function

$$f_X(x) = \begin{cases} xe^x, & \text{for } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Essentially this means that $\mathbb{P}(\text{Heads} \mid X = x) = x$.

(a) Find $\mathbb{P}(\text{Heads})$.

(b) If A denotes the event that the last flip came up heads, find the conditional probability density function of X given A, i.e., $f_{X|A}(x \mid A)$.

(c) Given A, find the conditional probability of heads at the next flip.

Solution. For part (a) with $H \equiv$ Heads,

$$\mathbb{P}(H) = \int_0^1 \mathbb{P}(H \mid X = x) f_X(x)\,dx = \int_0^1 x^2 e^x\,dx = e - 2.$$

For part (b) we use Bayes' theorem (continuous version):

$$f_{X|A}(x \mid A) = \frac{\mathbb{P}(A \mid X = x) f_X(x)}{\int_0^1 \mathbb{P}(A \mid X = x) f_X(x)\,dx} = \begin{cases} \frac{x^2 e^x}{e-2}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

For part (c) we combine the results in (a) and (b)

$$\begin{aligned} \mathbb{P}(H \mid A) &= \int_0^1 \mathbb{P}(H \mid X = x) f_{X|A}(x \mid A)\,dx \\ &= \frac{1}{e-2}\int_0^1 x^3 e^x\,dx \\ &= \frac{6 - 2e}{e - 2} \\ &\approx 0.786 \end{aligned}$$

Problem 5. Let X be a geometric random variable with parameter P, where P is itself random and uniformly distributed on the interval $[0, \frac{n-1}{n}]$. Let $Z = \mathbb{E}[X \mid P]$. Find $\mathbb{E}[Z]$.

Solution. We use the law of iterated expectations:

$$\mathbb{E}[Z] = \mathbb{E}\big[\mathbb{E}[X \mid P]\big] = \mathbb{E}[X].$$

We now calculate $\mathbb{E}[X \mid P = p]$, as follows:

$$\begin{aligned} \mathbb{E}[X \mid P = p] &= (1-p)\sum_{n=1}^{\infty} np^{n-1} \\ &= (1-p)\frac{d}{dp}\sum_{n=1}^{\infty} p^n \\ &= (1-p)\frac{d}{dp}(p(1-p)^{-1}) \\ &= \frac{1}{1-p}. \end{aligned}$$

Next, we proceed to calculate $\mathbb{E}[Z]$:

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}\big[\mathbb{E}[X \mid P]\big] \\ &= \int_0^{(n-1)/n} \frac{n}{(n-1)(1-p)}\,dp \\ &= \frac{n\log n}{n-1}. \end{aligned}$$

Thus, as n increases, $\mathbb{E}[Z]$ grows to infinity at the rate of $\log n$.

Problem 6. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $\mathbb{E}[X \mid Y]$ are positively correlated.

(b) Show that the correlation coefficient of Y and $\mathbb{E}[X \mid Y]$ has the same sign as the correlation coefficient of X and Y.

In this problem, do not assume that X and Y have a PDF; use only the properties of conditional expectation.

Solution. (a) We have

$$\mathrm{cov}\big(X, \mathbb{E}[X \mid Y]\big) = \mathbb{E}\big[X\,\mathbb{E}[X \mid Y]\big] = \mathbb{E}\Big[\mathbb{E}\big[X\,\mathbb{E}[X \mid Y] \mid Y\big]\Big] = \mathbb{E}\left[\big(\mathbb{E}[X \mid Y]\big)^2\right] \ge 0.$$

(b) We have

$$\mathrm{cov}\big(Y, \mathbb{E}[X \mid Y]\big) = \mathbb{E}\big[Y\,\mathbb{E}[X \mid Y]\big] = \mathbb{E}\big[\mathbb{E}[XY \mid Y]\big] = \mathbb{E}[XY] = \mathrm{cov}(X, Y).$$

Problem 7. Random variables X and Y are distributed according to the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} ax & \text{if } 1 \le x \le y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Evaluate the constant a.

(b) Determine the marginal probability density function $f_Y(y)$.

(c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.

(d) Determine the probability density function $f_Z(z)$, if $Z = Y - X$.

Solution. Since

$$1 = \int_1^3 \left(\int_1^y ax\,dx\right)dy = a\int_1^3 \frac{y^2 - 1}{2}\,dy = \frac{10}{3}a,$$

it follows that $a = 0.3$. For part (b)

$$f_Y(y) = \int_1^y ax\,dx = \begin{cases} \frac{3}{20}(y^2 - 1), & 1 < y \le 3 \\ 0, \text{otherwise.} \end{cases}$$

For part (c), we have

$$f_{X|Y}\left(x \mid \tfrac{3}{2}\right) = \frac{f_{X,Y}\left(x, \frac{3}{2}\right)}{f_Y\left(\frac{3}{2}\right)} = \frac{8}{5}x, \quad 1 \le x \le \tfrac{3}{2},$$

and hence

$$\mathbb{E}\left[X^{-1} \mid Y = \tfrac{3}{2}\right] = \int_1^{3/2} \tfrac{8}{5}\,dx = \frac{4}{5}.$$

For part (d), we have $F_Z(z) = 0$ for $z < 0$, and for $z \in [0, 2]$ we compute

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Y - X \le z) = 1 - \mathbb{P}(Y - X > z) \\ &= 1 - \int_1^{3-z}\left(\int_{x+z}^3 \tfrac{3}{10}x\,dy\right)dx \\ &= \frac{9}{10} + \frac{1}{20}(3 - z)\left(3 - (3 - z)^2\right). \end{aligned}$$

Thus

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{3}{20}z^2 - \frac{9}{10}z + \frac{6}{5}, & 0 \le z \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Summary of Results for Special Discrete Random Variables

Discrete Uniform over [a, b]:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{a+b}{2}, \qquad var(X) = \frac{(b-a)(b-a+2)}{12}, \qquad M_X(s) = \frac{e^{sa}\left(e^{s(b-a+1)} - 1\right)}{(b-a+1)(e^s - 1)}$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1 \\ 1-p, & \text{if } k = 0 \end{cases}$$

$$E[X] = p, \qquad var(X) = p(1-p), \qquad M_X(s) = 1 - p + pe^s$$

Binomial with Parameters p and n: (Describes the num of successes in n independent Bernoulli)

$$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

$$E[X] = np, \qquad var(X) = np(1-p), \qquad M_X(s) = (1 - p + pe^s)^n$$

Geometric with Parameter p: (Describes the num of Bernoulli trials until the first success)

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

$$E[X] = \frac{1}{p}, \qquad var(X) = \frac{1-p}{p^2}, \qquad M_X(s) = \frac{pe^s}{1 - (1-p)e^s}$$

Poisson with Parameter λ: (Approx. the binomial PMF when n is large, p is small, λ=np)

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$E[X] = \lambda, \qquad var(X) = \lambda, \qquad M_X(s) = e^{\lambda(e^s - 1)}$$

Summary of Results for Special Continuous Random Variables

Continuous Uniform over [a, b]:

$$f_X(k) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{a+b}{2}, \qquad var(X) = \frac{(b-a)^2}{12}, \qquad M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$$

Exponential with parameter λ:

$$f_X(k) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}, \qquad F_X(k) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases},$$

$$E[X] = \frac{1}{\lambda}, \qquad var(X) = \frac{1}{\lambda^2}, \qquad M_X(s) = \frac{\lambda}{\lambda - s}, \qquad (s < \lambda)$$

Normal with Parameters μ and $\sigma^2 > 0$:

$$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu, \qquad var(X) = \sigma^2, \qquad M_X(s) = e^{(\sigma^2 s^2/2) + \mu s}$$