EE351K Problems—Spring 2014

Problem (Bl). The intuitive "sure-thing principle" says that if

$$\mathbb{P}(A \mid C) \geq \mathbb{P}(B \mid C) \quad \text{and} \quad \mathbb{P}(A \mid C^c) \geq \mathbb{P}(B \mid C^c)$$

then $\mathbb{P}(A) \geq \mathbb{P}(B)$. Is this always true?

Problem (Bo). N balls are thrown at random into M boxes, with multiple occupancy permitted. Show that the expected number of empty boxes is $\frac{(M-1)^N}{M^{N-1}}$.

Problem (Bp). In order to promote its products, a chocolate factory has enclosed in each package of its signature series chocolate bars, a miniature replica of one of the six different flags that have ruled over Texas. All different flags are in equal proportions, and each chocolate bar retails for \$ 2.50. How much does one need to spent on the average, to acquire the entire collection of flags?

Problem (BBa). A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives.

Problem (BBb). Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer, and the third chest has a silver coin in each drawer. A chest is drawn at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin.

Problem (De). Let X have the Poisson distribution with parameter λ. Use Chebyshev's inequality to verify the following inequalities:

$$\mathbb{P}\left(X \leq \tfrac{\lambda}{2}\right) \leq \tfrac{4}{\lambda}\,, \qquad \mathbb{P}\left(X \geq 2\lambda\right) \leq \tfrac{1}{\lambda}\,.$$

Problem (Dj). Let $X_1, X_2, \ldots, X_n$ be independent, discrete random variables, each satisfying

$$\mathbb{P}(X_i = k) = \frac{1}{N}\,, \qquad \text{for} \;\; k = 1, 2, \ldots, N\,.$$

Let U and V be random variables defined by:

$$U = \min\bigl\{X_1, X_2, \ldots, X_n\bigr\}\,, \qquad V = \max\bigl\{X_1, X_2, \ldots, X_n\bigr\}\,.$$

Calculate $\mathbb{P}(U = k)$ and $\mathbb{P}(V = k)$, for $k = 1, 2, \ldots, N$. (hint: $\mathbb{P}(U = k) = \mathbb{P}(U \geq k) - \mathbb{P}(U \geq k+1)$; also $\{U \geq k\} = \cap_j \{X_j \geq k\}$)

Problem (Dk). A binary random variable X has mean 3 and second moment equal to 25. What is the probability mass function of X?

Problem (DCb). Let N be the number of tosses of a fair coin up to and including the appearance of the first head. By conditioning on the result of the first toss, show that $E(N) = 2$.

Problem (DCc). Each time you flip a certain coin, heads appears with probability p. Suppose that you flip the coin a random number N of times, where N has the Poisson distribution with parameter $\lambda > 0$ and is independent of the outcomes of the flips. Let X and Y be the numbers of resulting heads and tails, respectively. Find the probability mass function of X (hint: it is easier to use moment generating functions). Find the joint probability mass function of X and Y. What are the distributions of X and Y? Are X and Y independent? Prove of disprove.

Problem (MAb). Suppose that X and Y have joint probability mass function

$$\mathbb{P}(X = i,\, Y = j) = \theta^{i+j+1} \qquad \text{for } i, j = 0, 1, 2\,.$$

Compute $E(XY)$ and $E(X)$ (both as a function of θ).

Problem (MAd). You are presented with two boxes. Box I contains 1 white and 2 black balls and Box II contains 2 white and 5 black balls. You pick a ball randomly from Box I and place it in Box II. Next you pick a ball randomly from Box II. What is the probability that this ball is black?

Problem (MAe). Urn I contains 4 white and 3 black balls, and Urn II contains 3 white and 7 black balls. An urn is selected at random and a ball is picked from it.

(a) What is the probability that this ball is black?

(b) If this ball is white, what is the probability that Urn I was selected?

Problem (MAg). The random variables X and Y each take values -1 and 1, and $\mathbb{P}(X = 1) = a$, $\mathbb{P}(Y = 1) = b$. Suppose $Z = XY$ and that X, Y and Z are pairwise independent? What are the values of a and b? Are $\{X, Y, Z\}$ independent?

Problem (MAk). Two players A and B take turns in tossing (repeatedly) a biased coin having $\mathbb{P}(\text{Heads}) = p$, $p \in (0, 1)$. The first to get heads wins the game. Suppose A starts first. What is the probability that A wins?

Problem (MAn). Suppose $X_1, X_2, \ldots$ are independent discrete random variables, having the same distribution, and $\mathbb{E}[X_i] > 0$, for each i. Is it true that for any two positive integers n and m,

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_m}{X_1 + \cdots + X_n}\right] = \frac{m}{n}\,?$$

Why not, or why yes?

Problem (MBa). Let X and Y be independent continuous random variables with common distribution function F and probability density function f. Find the cumulative distribution function and the probability density function of $V = \max(X, Y)$. Do the same for $U = \min(X, Y)$.

Problem (MBb). Let X have the uniform distribution on [0,1]. For what function g does $Y = g(X)$ have the exponential distribution with parameter 1?

Problem (MBc). Let $Y = \mathrm{e}^X$ where X has the $N(0, 1)$ distribution. Find the probability density function of Y.

Problem (MBd). Let X and Y be independent continuous random variables each uniformly distributed on $[0, 1]$. Let $V = \max(X, Y)$ and $U = \min(X, Y)$. Find $\mathbb{E}[U]$, $\mathbb{E}[V]$, and $\mathrm{cov}(U, V)$.

Problem (MBe). Let X and Y have the joint probability density function $f_{X,Y}(x, y) = x\mathrm{e}^{-x(y+1)}$, for $x, y \geq 0$. Find $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x \mid y)$, $f_{Y|X}(y \mid x)$ and $\mathbb{E}[Y \mid X]$.

Problem (MBf). Let X and Y have the joint probability density function $f_{X,Y}(x, y) = Cx(y - x)\mathrm{e}^{-y}$ for $0 \leq x \leq y < \infty$. Find $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$. Also find $\mathbb{E}[X \mid Y]$ and $\mathbb{E}[Y \mid X]$.

Problem (MBg). Let X and Y be independent continuous random variables each uniformly distributed on $[0, 1]$. Find the probability density function of $Z = XY$.

Problem (MBh). Let X and Y independent exponential random variables with parameters λ and μ. Find $\mathbb{P}\big(X = \min(X, Y)\big)$.

Problem (MBi). Let X and Y and Z be independent continuous random variables each uniformly distributed on $[0, 1]$. Find the probability density function $f_V(v)$ of $V = X + Y + Z$.

Problem (MBj). Let X have probability density function $\frac{1}{2}\mathrm{e}^{-|x|}$, for $-\infty < x < \infty$. Find its moment generating function $M_X(s)$.

Problem (Cf). Let X and Y be independent and uniformly distributed on $[0, 1]$. Let $Z = \max(X, Y)$. Find the probability density function of Z and $\mathbb{E}[Z]$.

Problem (Cg). Let X and Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{y}\mathrm{e}^{-y-\frac{x}{y}} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathrm{cov}(X, Y)$.

Problem (Ch). The random variable X has a continuous distribution function $F_X(x)$. Suppose that $Z = F_X(X)$. What is the distribution of Z?

Problem (Cj). The *bilateral exponential* distribution has the probability density function

$$f(x) = \tfrac{c}{2}\mathrm{e}^{-c|x|}\,, \quad x \in \mathbb{R}\,,$$

where $c > 0$ is a constant. Compute its mean and its variance.

Problem (Cn). Random variables X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \mathrm{e}^{-x-y} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{P}(X + Y \le 1)$ and $\mathbb{P}(X > Y)$.

Problem (Ct). Let X, Y, Z independent and exponential random variables with parameters λ, μ, ν, respectively. Find $\mathbb{P}(X < Y)$, the probability density function of $\min\{Y, Z\}$ and $\mathbb{P}(X < Y < Z)$.

Problem (Cv). Let X and Y be independent exponential random variables with parameter 1. Find the joint probability density function of $U = X + Y$ and $V = \frac{X}{X+Y}$.

Problem (Cx). The random variables X and Y are independent and have Cauchy distributions with densities

$$f_X(x) = \frac{1}{\pi(x^2 + 1)}\,, \qquad f_Y(y) = \frac{1}{\pi(y^2 + 1)}\,.$$

Find the joint probability density function $f_{Z,W}(z, w)$ of the random variables

$$Z = 2XY \qquad W = X^2 + Y^2\,.$$

Then find the marginal probability density function s of Z and W.

Problem (Cz). The random variables X and Y have the joint probability density function :

$$f_{X,Y}(x,y) = \begin{cases} C(y-x) & \text{if } 0 \le x \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of C.

(b) Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[X]$.

(d) Find $\operatorname{cov}(X, Y)$.

Problem (CCb). Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2\,, & 0 < x \le 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

(a) A number Y is chosen at random between 0 and X, or in other words Y is a random variable uniformly distributed on $[0, X]$ (equivalently, $f_{Y|X}(y \mid x) = \frac{1}{x}$ for $0 \le y \le x$). Find the expected value of Y (i.e., $\mathbb{E}[Y]$).

(b) Suppose that Z is a random variable with:

$$\mathbb{P}(Z \le z_0 \mid Y = y,\ X = x) = \begin{cases} \dfrac{1}{x\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z_0} \exp\left(-\frac{(z-y)^2}{2x^2}\right) dz\,, & 0 < y \le x \le 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[Z \mid X = x]$ and $\mathbb{E}[Z^2 \mid X = x]$.

Problem (CCf). Find the conditional probability density function and the expectation of Y given X when they have a joint probability density function $f_{X,Y}(x,y) = x\mathrm{e}^{-x(y+1)}$, for $x,\ y \ge 0$.

Problem (CCg). Let X and Y have joint probability density function $f_{X,Y}(x,y) = Cx(y-x)\mathrm{e}^{-y}$, $0 \le x \le y < \infty$. Find the constant C, the conditional probability density function of Y given X, and $\mathbb{E}[Y \mid X]$.

Problem (CCh). Nadine is taking a quiz with 12 questions. The amount of time she spends answering a particular question, T, is exponentially distributed with $\mathbb{E}[T] = \frac{1}{3}$ of an hour. The amount of time she spends on any particular question is independent of the amount of time she spends on any other question. Once she finishes answering a question, she immediately begins answering the next question. Let N be the total number of questions she answers correctly. Let X be the total amount of time she spends on questions that she answers correctly. We know she has probability $\frac{2}{3}$ of getting any quiz question correct. Find the expectation and variance of X.

Problem (CCk). Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2\mathrm{e}^{-\lambda y}, & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_{Y|X}(y \mid x)$ and $\mathbb{E}[Y \mid X = x]$.

Problem (CCl). If X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < y < \infty\,, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[Y \mid X = x]$.

Problem (Nd). The three random variables X_1, X_2, and X_3 are jointly Gaussian, with mean values zero and covariance matrix

$$C = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 6 & 4 \\ 2 & 4 & 10 \end{pmatrix}.$$

Find the joint probability density function $f_{Z_1 Z_2}(z_1, z_2)$ of the random variables

$$Z_1 = X_1 + X_2 - X_3\,, \qquad Z_2 = 2X_1 - X_2 + 3X_3\,.$$

Problem (Ne). The random variables X and Y are independent and jointly Gaussian with $\mathrm{var}(X) = \mathrm{var}(Y) = \sigma^2$. Find the conditional probability density function $f_{XY}(x, y \mid X^2 + Y^2 > c^2)$, where c is a positive constant.

Problem (Ng). The length of a rod is normally distributed with mean 4 inches and variance 0.01 square inch. Two such rods are placed end to end and fitted into a slot. The length of the slot is 8 inches with a tolerance of -0.1 to 0.1 inch. What is the probability that the two rods will fit?

Problem (Nh). Let U and V be jointly normal. It is given that U has mean 4 and variance 5, and V has mean 2 and variance 3, and $\mathrm{cov}(X, Y) = 1$. Let $X = V + 2U$. Find $\mathbb{E}[X]$, $\mathrm{var}(X)$, and $\mathrm{cov}(X, V)$.

Problem (Gf). If X is a random variable with probability generating function $G_X(s)$ and k is a positive integer, find the probability generating functions of $Y = kX$ and $Z = X + k$.

Problem (Gj). If X has the normal distribution with mean 0 and variance 1, find the mean value of $Y = e^{2X}$.

Problem (Gk). Let X and Y be independent random variables, and $Z = X + Y$. It is given that Y has the exponential distribution with parameter 1, and Z has probability density function $f_Z(z) = ze^{-z}$, for $z \geq 0$. What is the probability density function of X?

Problem (Mf). Let X_n be a Markov chain. Which of the following are Markov chains?

(a) X_{m+r} for $r \geq 0$.

(b) X_{2m} for $m \geq 0$.

(c) The sequence of pairs (X_n, X_{n+1}) for $n \geq 0$.

Problem (Aa). We are given that $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 2$, $\mathbb{E}[X^2] = 5$, $\mathbb{E}[Y^2] = 8$, and $\mathbb{E}[XY] = 1$. Find the linear least squares estimator of Y given X.

Problem (XXb). The number of particles emitted from a radioactive source during a specified period is a random variable with a Poisson distribution. If the probability of no emissions equals $\frac{1}{3}$, what is the probability that 2 or more emissions occur?

Problem (XXg). Let X_t be equal to the number of particles emitted from a radioactive source during a time interval of length t. Assume that X_t has a Poisson distribution with parameter λt. A counting device is set up to keep count of the number of particles emitted. Suppose that there is a constant probability p that any emitted particle is not counted. If R_t equals the number of particles counted during the specified interval, what is the probability distribution of R_t ?

Problem (XXh). A bank has two tellers and they take an exponentially distributed time to deal with any customer. Let the parameters of these two service times be λ and μ respectively (assume independence). You arrive to find exactly two customers present, each occupying a teller:

1. You toss a fair coin and depending on the outcome you queue for one of the two tellers. What is the probability p that your the last of the three customers to leave the bank? (Hint: use the memoryless property)

2. Suppose that you know λ and μ and you choose to be served by the quicker teller. What is p in this case?

3. Suppose that you go to the teller that becomes free first. Find p.

Problem (Yb). If X is a continuous random variable and m is its median, show that $(\mu - m)^2 \leq \sigma^2$.

Problem (Yd). Let X and Y be independent and exponentially distributed with parameters λ and μ. Let $S = X + Y$, $R = \frac{X}{X+Y}$. Find the probability density function of R.