Homework No. 6 (with solutions)

Problem 1. Let X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} Cx^2 \mathrm{e}^{-x(1+y)} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant C and the marginal densities $f_X(x)$ and $f_Y(y)$.

(b) Find the conditional densities $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$.

(c) Compute $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[Y \mid X = x]$.

Solution. Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\infty} x^2\mathrm{e}^{-x}\left(\int_0^{\infty}\mathrm{e}^{-xy}\mathrm{d}y\right)\mathrm{d}x = \int_0^{\infty} x\mathrm{e}^{-x}\mathrm{d}x = 1\,,$$

it follows that $C = 1$. Also, by the computation above

$$f_X(x) = x\mathrm{e}^{-x}\,, \quad x > 0\,.$$

and

$$f_Y(y) = \int_0^{\infty} x^2\mathrm{e}^{-x(1+y)}\mathrm{d}x = \frac{2}{(1+y)^3}\,, \quad y > 0\,.$$

Thus, for part (b)

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{2}(1+y)^3x^2\mathrm{e}^{-x(1+y)}\,, \quad x, y > 0\,,$$

and

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = x\mathrm{e}^{-xy}\,, \quad x, y > 0\,.$$

Part (c) is a routine integration and yields

$$\mathbb{E}[X \mid Y = y] = \int_0^{\infty}\frac{1}{2}(1+y)^3x^3\mathrm{e}^{-x(1+y)}\mathrm{d}x = \frac{(1+y)^3}{2}\frac{6}{(1+y)^4} = \frac{3}{1+y}\,.$$

and

$$\mathbb{E}[Y \mid X = x] = \int_0^{\infty} yx\mathrm{e}^{-xy}\mathrm{d}y = \frac{1}{x}\,.$$

Problem 2. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{cy}{x} & \text{if } 0 \le y < x \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{P}(X + Y \le 1)$.

Solution. Since

$$1 = \int_0^1\left(\int_0^x \frac{cy}{x}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \frac{cx}{2}\mathrm{d}x = \frac{c}{4}\,,$$

it follows that $c = 4$. From the calculation above $f_X(x) = 2x$, for $x \in [0, 1]$. Also,

$$f_Y(y) = \int_y^1 \frac{4y}{x}\,\mathrm{d}x = -4y \log y\,, \quad y \in (0, 1]\,.$$

For part (c)

$$\begin{aligned}\mathbb{P}(X + Y \leq 1) &= \int_0^{1/2} \left(\int_0^x \frac{4y}{x}\,\mathrm{d}y\right)\mathrm{d}x + \int_{1/2}^1 \left(\int_0^{1-x} \frac{4y}{x}\,\mathrm{d}y\right)\mathrm{d}x \\ &= 2\log 2 - 1 \approx 0.39\,.\end{aligned}$$

Problem 3. If X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}(x + y)\mathrm{e}^{-x-y} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

find the probability density function of $Z = X + Y$.

Solution. It is most convenient to select new coordinates $z = x + y$ and $v = \frac{1}{2}(x - y)$. Then

$$\begin{aligned} F_Z(z_0) &= \mathbb{P}(X + Y \leq z_0) \\ &= \iint_{\{x+y\leq z_0\}} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{z_0} \left(\int_{-z/2}^{z/2} \tfrac{1}{2}z\mathrm{e}^{-z}\mathrm{d}v\right)\mathrm{d}z \\ &= \int_0^{z_0} \tfrac{1}{2}z^2\mathrm{e}^{-z}\mathrm{d}z\,, \end{aligned}$$

from which it follows that $f_Z(z) = \frac{1}{2}z^2\mathrm{e}^{-z}$, for $z \geq 0$.

Problem 4. Random variables X, Y, and Z have joint probability density function

$$f(x, y, z) = \begin{cases} 8xyz & \text{if } 0 < x, y, z < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Are X, Y and Z independent? Find $\mathbb{P}(X > Y)$, $\mathbb{E}[Y^3 \mid Z]$, and $\mathbb{P}(X > Y > Z)$.

Solution. Direct integration shows that $f_X(x) = 2x$, for $x \in [0, 1]$, and by symmetry Y and Z, also have the same law. Since the joint probability density function is the product of the marginal ones, X, Y, and Z are independent.

$$\begin{aligned} \mathbb{P}(X > Y) &= \iint_{\{x>y\}} f_{X,Y}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \left(\int_0^x 4xy\,\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2}\,, \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(X > Y > Z) &= \iiint_{\{x>y>z\}} f_{X,Y,Z}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= 8\int_0^1 x\left(\int_0^x y\left(\int_0^y z\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \frac{1}{6}\,. \end{aligned}$$

Problem 5. Suppose X and Y are independent random variables, each having the exponential distribution with parameter λ.

(a) If $Z = 1 - \mathrm{e}^{-\lambda X}$, find the probability density function $f_Z(z)$ and the cumulative distribution function $F_Z(z)$.

(b) Same as in part (a), if $Z = \min(X, Y)$.

(c) Same as in part (a), if $Z = X - Y$.

(d) Find $\mathbb{P}\big(\min(X, Y) \le aX\big)$ and $\mathbb{P}\big(\min(X, Y) \le a \max(X, Y)\big)$, where $a \in (0, 1)$ is some constant.

Solution. We have

$$F_Z(z) = \mathbb{P}(1 - \mathrm{e}^{-\lambda X} \le z) = \mathbb{P}\left(X \le -\lambda^{-1} \log(1 - z)\right) = 1 - \mathrm{e}^{\log(1-z)} = z\,.$$

Hence, Z is uniformly distributed on $[0, 1]$. For part (b), by independence

$$\mathbb{P}(Z > z) = \mathbb{P}(X > z)\ \mathbb{P}(Y > z) = \mathrm{e}^{-2\lambda z}\,.$$

Hence Z is exponential with parameter 2λ. For part (c)

$$\mathbb{P}(Z \le z_0) = \int_0^{z_0} \left(\int_{-\infty}^{\infty} f_{X,Y}(x, z + x)\, \mathrm{d}x \right) \mathrm{d}z \tag{1}$$

Continuing with the integral inside the parenthesis in (**??**),

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{X,Y}(x, z + x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_X(x) f_Y(z + x)\, \mathrm{d}x \\
&= \int_{\max(0,-z)}^{\infty} \lambda^2 \mathrm{e}^{-\lambda x} \mathrm{e}^{-\lambda(x+z)}\, \mathrm{d}x \\
&= \lambda^2 \mathrm{e}^{-\lambda z} \int_{\max(0,-z)}^{\infty} \mathrm{e}^{-2\lambda x}\, \mathrm{d}x \\
&= \lambda^2 \mathrm{e}^{-\lambda z} \frac{1}{2\lambda} \mathrm{e}^{-2\lambda \max(0,-z)} \\
&= \frac{\lambda}{2} \mathrm{e}^{-\lambda |z|}\,, \quad -\infty < z < \infty\,.
\end{aligned}$$

Thus we can express F_Z as

$$F_Z(z) = \begin{cases} \frac{1}{2}\mathrm{e}^{\lambda z}, & z < 0\,, \\ 1 - \frac{1}{2}\mathrm{e}^{-\lambda z}, & z \ge 0\,, \end{cases}$$

or equivalenty,

$$\begin{aligned}
F_Z(z) &= \frac{1}{2} \left(1 + \mathrm{e}^{\lambda \min(0,z)} - \mathrm{e}^{-\lambda \max(0,z)} \right) \\
&= \frac{1}{2} + \mathrm{e}^{-\frac{\lambda}{2}|z|} \sinh\left(\tfrac{\lambda z}{2}\right)\,, \quad -\infty < z < \infty\,.
\end{aligned}$$

For part (d)

$$\mathbb{P}\big(\min(X, Y) \le aX\big) = \mathbb{P}(Y \le aX) = \int_0^{\infty} \lambda \mathrm{e}^{-\lambda x} \left(\int_0^{ax} \lambda \mathrm{e}^{-\lambda y} \mathrm{d}y \right) \mathrm{d}x = \frac{a}{1 + a}\,.$$

Note that this is not continuous at $a = 1$, however

$$\mathbb{P}\big(\min(X,Y) \leq a\max(X,Y)\big) = \mathbb{P}(Y \leq aX) + \mathbb{P}(X \leq aY) = \frac{2a}{1+a}\,,$$

tends to 1 as $a \uparrow 1$, so it is continuous at $a = 1$.