Homework No. 7 (with solutions)

Problem 1. Let U and V be independent standard normal random variables, and $X = U+V$, $Y = U-2V$. Find $\mathbb{E}[X \mid Y]$, $\mathrm{cov}(X,Y)$, and the joint probability density function of X and Y.

Solution. We have

$$\mathrm{var}(Y) = \mathrm{var}(U) + 4\,\mathrm{var}(V) = 5\,,$$

and

$$\mathrm{cov}(X,Y) = \mathbb{E}\big[(U+V)(U-2V)\big] = \mathbb{E}[U^2] - 2\,\mathbb{E}[V^2] = -1\,.$$

Therefore,

$$\mathbb{E}[X \mid Y] = -\frac{1}{5}Y\,.$$

Next, we have, $\sigma_X^2 = 2$ and $\rho = -\frac{1}{\sqrt{2\times 5}} = -\frac{1}{\sqrt{10}}$. The joint probability density function is of the form $Ce^{-q(x,y)}$, where

$$C = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}\,.$$

and

$$q(x,y) = \frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right).$$

The various coefficients appearing in this formula have already been found.

Problem 2. Let the random variables X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\Big\{-\frac{x^2+y^2}{2\sigma^2}\Big\}.$$

Suppose that the random variables W and Q satisfy the relation

$$X\cos(\omega_0 t) + Y\sin(\omega_0 t) = W\cos(\omega_0 t + Q)\,,$$

for all time t, where ω_0 is a constant. Furthermore, suppose that $\mathbb{P}(W \geq 0) = 1$ and $\mathbb{P}(-\pi < Q \leq \pi) = 1$.

(a) Find the joint probability density function of W and Q.

(b) Find the marginal probability density functions of W and Q.

(c) Are W and Q independent?

Solution.

Problem 3. Consider three zero-mean random variables X, Y, and Z, with known variances and covariances. Give a formula for the linear least squares estimator of X based on Y and Z, that is, find a and b that minimize

$$\mathbb{E}\left[(X - aY - bZ)^2\right].$$

For simplicity, assume that Y and Z are uncorrelated.

Solution. For any choice of a, b, the mean squared error is equal to

$$\mathbb{E}[X^2] + a^2\,\mathbb{E}[Y^2] + b^2\,\mathbb{E}[Z^2] - 2a\,\mathbb{E}[XY] - 2b\,\mathbb{E}[XZ]\,.$$

We differentiate with respect to a and b, and set the derivatives to zero, to obtain

$$a = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}, \qquad b = \frac{\mathbb{E}[XZ]}{\mathbb{E}[Z^2]}\,.$$

Problem 4. Let X be uniformly distributed on the interval $(1,5)$. Find the probability density function of $Y = \frac{X}{5-X}$.

Solution. We have $X = g(Y) = 5Y(1+Y)^{-1}$ and $\frac{\mathrm{d}g}{\mathrm{d}y}(y) = 5(1+y)^{-2}$. Since $f_X(x) = \frac{1}{4}$, for $x \in (1,5)$, we obtain

$$\begin{aligned} f_Y(y) &= \left|\tfrac{\mathrm{d}g}{\mathrm{d}y}(y)\right| f_X\big(g(y)\big) \\ &= \frac{5}{4(1+y)^2}\,, \quad \frac{1}{4} < y < \infty\,. \end{aligned}$$

Problem 5. Let X and Y denote two points that are chosen randomly and independently from the interval $[0,1]$. Find the probability density function of $Z = |X-Y|$. Use this to calculate the mean distance between X and Y.

Solution. It is simpler to calculate $\mathbb{P}(Z > z)$. Since the locus of $\{Z > z\}$ is two rectangles of total area $(1-z)^2$, and since the pair (X,Y) is uniformly distributed on the unit square, $\mathbb{P}(Z > z) = (1-z)^2$. It follows that $F_Z(z) = 1-(1-z)^2$, $0 \le z \le 1$. Differentiating, we have $f_Z(z) = 2(1-z)$, $0 \le z \le 1$. Thus

$$\mathbb{E}[Z] = \int_0^1 2z(1-z)\,\mathrm{d}z = \frac{1}{3}\,.$$

Problem 6. Let X and Y be independent random variables, each having the exponential distribution with parameter λ. Find the joint probability density function of X and $X+Y$. Also, find the conditional probability density function of X given that $X+Y=a$.

Solution.

Problem 7. The random variables $X_1,\dots,X_n$ have common mean μ, common variance σ^2 and, furthermore, $\mathbb{E}[X_iX_j] = c$ for every pair of distinct i and j. Derive a formula for the variance of $X_1+\cdots+X_n$, in terms of μ, σ^2, c, and n.

Solution. We first note that, for distinct i and j, we have

$$\mathrm{cov}(X_i,X_j) = \mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\;\mathbb{E}[X_j] = c-\mu^2.$$

Thus, using the formula for the variance of the sum of random variables,

$$\mathrm{var}(X_1+\cdots+X_n) = n\sigma^2 + (n^2-n)(c-\mu^2).$$

Problem 8. Let Y be exponentially distributed with parameter 1, and let Z be uniformly distributed over the interval $[0,1]$. Assume Y and Z are independent. Find the probability density functions of $W = Y-Z$ and that of $X = |Y-Z|$.

Solution. Let $W = Y-Z$ and $X = |Y-Z|$. We find the probability density function of W by convolution of the exponential with parameter 1 with the uniform in the interval $[-1,0]$. We obtain

$$f_W(w) = \begin{cases} \displaystyle\int_w^{w+1} \mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{e}^{-w} - \mathrm{e}^{-(w+1)}, & \text{if } w \ge 0\,, \\ \displaystyle\int_0^{w+1} \mathrm{e}^{-x}\,\mathrm{d}x = 1 - \mathrm{e}^{-(w+1)}, & \text{if } -1 \le w \le 0\,, \\ 0, & \text{otherwise.} \end{cases}$$

We then note that

$$f_X(x) = \begin{cases} f_W(x) + f_W(-x), & \text{if } x \ge 0\,, \\ 0, & \text{otherwise.} \end{cases}$$

and obtain the probability density function of X. The validity of the above formula is established by using the relation

$$F_X(x) = \mathbb{P}\big(|W| \le x\big) = \mathbb{P}(-x \le W \le x) = F_X(x) - F_X(-x),$$

and differentiating with respect to x.