Homework No. 5 (with solutions)

Problem 1. Let X and Y be discrete random variables with mean 0 and variance 1. Use the Cauchy-Schwartz inequality and the identity

$$\max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$$

to show that $\mathbb{E}\left[\max\{X^2, Y^2\}\right] \le 1 + \sqrt{1-\rho^2}$, where ρ is the correlation coefficient of X and Y.

Solution.

$$\begin{aligned}
\mathbb{E}\left[\max\{X^2,Y^2\}\right] &= \frac{1}{2}\,\mathbb{E}\left[X^2+Y^2\right] + \frac{1}{2}\,\mathbb{E}\left[|X^2-Y^2|\right] \\
&= 1 + \frac{1}{2}\,\mathbb{E}\left[|(X-Y)(X+Y)|\right] \\
&\le 1 + \frac{1}{2}\sqrt{\mathbb{E}\left[(X-Y)^2\right]\mathbb{E}\left[(X+Y)^2\right]} \\
&\le 1 + \frac{1}{2}\sqrt{\mathbb{E}\left[X^2+Y^2-2XY\right]\mathbb{E}\left[X^2+Y^2+2XY\right]} \\
&\le 1 + \frac{1}{2}\sqrt{(2-2\rho)(2+2\rho)} \\
&\le 1 + \sqrt{1-\rho^2}\,.
\end{aligned}$$

Problem 2. A die is rolled 4 times. Let X and Y be the number of appearances of 1 and 6 respectively. Find the joint probability mass function of (X,Y).

Solution.

$$\begin{aligned}
p_{X,Y}(x,y) &= \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{6}\right)^{x+y}\left(\frac{2}{3}\right)^{4-x-y} \\
&= \left(\frac{2}{3}\right)^4 \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{4}\right)^{x+y}, \quad 0 \le x+y \le 4\,.
\end{aligned}$$

Problem 3. Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X+Y$ and $|X-Y|$ are dependent though uncorrelated.

Solution. We have $\mathbb{P}(X+Y=1\,, |X-Y|=0) = 0$ but $\mathbb{P}(X+Y=1) = 1/2$ and $\mathbb{P}(|X-Y|=0) = 1/2$, so they are dependent.

On the other hand,

$$\mathbb{E}\left[(X+Y)|X-Y|\right] = \mathbb{E}\left[|X^2-Y^2|\right] = 1/2\,,$$

and $\mathbb{E}\left[(X+Y)\right] = 1$ while $\mathbb{E}\left[|X-Y|\right] = 1/2$, so they are uncorrelated.

Problem 4. Let X_1, X_2 and X_3, be independent dr.v.s having a geometric distribution with parameters p_1, p_2 and p_3, respectively. Show that

$$\mathbb{P}(X_1 < X_2 < X_3) = \frac{(1-p_1)(1-p_2)p_2p_3^2}{(1-p_2p_3)(1-p_1p_2p_3)}\,.$$

(hint: use the joint probability mass function to compute the probability via an appropriate summation).

Solution. The joint PMF is

$$p_{X_1,X_2,X_3}(x_1,x_2,x_3) = p_1p_2p_3q_1^{x_1-1}q_2^{x_2-1}q_3^{x_3-1}\,, \quad x_i = \{1,2,\dots\}\,,\ i=1,2,3..$$

So

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \sum_{x_1<x_2<x_3} p_{X_1,X_2,X_3}(x_1,x_2,x_3) \\
&= \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} \sum_{x_3=x_2+1}^{\infty} p_1p_2p_3q_1^{x_1-1}q_2^{x_2-1}q_3^{x_3-1} \\
&= \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} p_1p_2p_3q_1^{x_1-1}q_2^{x_2-1}\frac{q_3^{x_2}}{1-q_3} \\
&= \sum_{x_1=1}^{\infty} p_1p_2q_1^{x_1}q_3\frac{(q_2q_3)^{x_1-1}}{1-q_2q_3} \\
&= \frac{p_1p_2q_2q_3^2}{(1-q_2q_3)(1-q_1q_2q_3)} \\
&= \frac{(1-q_1)(1-q_2)q_2q_3^2}{(1-q_2q_3)(1-q_1q_2q_3)}\,.
\end{aligned}$$

Note: In the problem statement the roles of p and q were reversed.

Problem 5. Let X and Y be independent discrete random variables satisfying

$$\mathbb{P}(X=k) = \mathbb{P}(Y=k) = pq^k, \quad k = 0,1,2,\dots,$$

where $0 < p = 1-q < 1$. Calculate $\mathbb{P}(X = k \mid X+Y = n)$.

Solution. We have (using independence),

$$\begin{aligned}
\mathbb{P}(X = k \mid X+Y=n) &= \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)} \\
&= \frac{pq^k\,pq^{n-k}}{\sum_{i=1}^n pq^ipq^{n-i}} \\
&= \frac{1}{n+1}\,.
\end{aligned}$$

Problem 6. X and Y are discrete random variables, each taking only two distinct values. Suppose X and Y are uncorrelated. Are they independent?

Solution. Yes they are. Without loss of generality we can assume X and Y are Bernoulli (otherwise, shift them and scale them by constants to make them Bernoulli). Then, if they are uncorrelated, we have

$$\mathbb{P}(X=1,Y=1) = \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \mathbb{P}(X=1)\,\mathbb{P}(Y=1)\,,$$

and combinining this with the property $A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp B^c$, and the fact that X and Y take only two values they are independent.

Problem 7. You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained. Repeat part (a) for the case where you roll two dice, instead of one.

Solution. Let X_i be independent Bernoulli random variables that are equal to 1 if the i^{th} flip results in heads. Let N be the number of coin flips. We have $\mathbb{E}[X_i] = 1/2$, $\text{var}(X_i) = 1/4$, $\mathbb{E}[N] = 7/2$, and $\text{var}(N) = 35/12$. (The last equality is obtained from the formula for the variance of a discrete uniform random variable.) Therefore, the expected number of heads is

$$\mathbb{E}[X_i]\,\mathbb{E}[N] = \frac{7}{4}\,,$$

and the variance is

$$\text{var}(X_i)\,\mathbb{E}[N] + \mathbb{E}[X_i]^2\,\text{var}(N) = \frac{1}{4}\times\frac{7}{2} + \frac{1}{4}\times\frac{35}{12} = \frac{77}{48}\,.$$

The experiment with two dice can be viewed as consisting of two independent repetitions of the experiment with one die. Thus, both the mean and the variance are doubled and become $7/2$ and $77/24$, respectively.