Homework No. 6 (Due on 03/27/14)

Problem 1. Let X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} Cx^2 \mathrm{e}^{-x(1+y)} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant C and the marginal densities $f_X(x)$ and $f_Y(y)$.

(b) Find the conditional densities $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$.

(c) Compute $\mathbb{E}[X \mid Y = y]$ and $\mathbb{E}[Y \mid X = x]$.

Problem 2. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{cy}{x} & \text{if } 0 \le y < x \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{P}(X + Y \le 1)$.

Problem 3. If X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}(x+y)\mathrm{e}^{-x-y} & \text{if } x, y > 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

find the probability density function of $Z = X + Y$.

Problem 4. Random variables X, Y, and Z have joint probability density function

$$f(x,y,z) = \begin{cases} 8xyz & \text{if } 0 < x, y, z < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Are X, Y and Z independent? Find $\mathbb{P}(X > Y)$, $\mathbb{E}[Y^3 \mid Z]$, and $\mathbb{P}(X > Y > Z)$.

Problem 5. Suppose X and Y are independent random variables, each having the exponential distribution with parameter λ.

(a) If $Z = 1 - \mathrm{e}^{-\lambda X}$, find the probability density function $f_Z(z)$ and the cumulative distribution function $F_Z(z)$.

(b) Same as in part (a), if $Z = \min(X, Y)$.

(c) Same as in part (a), if $Z = X - Y$.

(d) Find $\mathbb{P}\big(\min(X,Y) \le aX\big)$ and $\mathbb{P}\big(\min(X,Y) \le a \max(X,Y)\big)$, where $a \in (0,1)$ is some constant.