Homework No. 4 (Due on 02/20/14)

Problem 1. Let A, B and C be three events. Show that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\ \mathbb{P}(B \mid A)\ \mathbb{P}(C \mid A \cap B)\,,$$

and

$$\mathbb{P}(A \cap C \mid B) = \mathbb{P}(A \mid B \cap C)\ \mathbb{P}(C \mid B)\,.$$

Now suppose that $\{C_1, C_2, \dots, C_n\}$ are a partition of Ω. Show that

$$\mathbb{P}(A \mid B) = \sum_{i=1}^{n} \mathbb{P}(A \mid B \cap C_i)\ \mathbb{P}(C_i \mid B)\,.$$

Problem 2. Let X be Poisson with parameter λ. Compute

(a) The mean of $\frac{1}{(X+1)(X+2)}$.

(b) $\mathbb{E}\big[X(X-1)(X-2)\big]$.

(c) $\mathbb{E}\big[\mathrm{e}^X\big]$.

Problem 3. Let N be a positive integer and let

$$p(k) = \begin{cases} c2^k\,, & k = 1, 2, \dots, N \\ 0\,, & \text{otherwise.} \end{cases}$$

Find the value of c such that p is a probability mass function.

Problem 4. There are c different types of coupon and each time you obtain a coupon it is equally likely to be any of the c types. Let Y_i be the additional number of coupons collected, after obtaining i distinct types, before a new type is collected.

(a) Show that Y_i has the geometric distribution with parameter $\frac{c-i}{c}$.

(b) Deduce from this the mean number of coupons you will need to collect before you have a complete set.

(c) Find the expected number of different types of coupon in the first n coupons received.

Problem 5. If X has the geometric distribution with parameter p, show that

$$\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)\,. \tag{1}$$

Conversely, show that if X is a non-negative integer-valued random variable, satisfying (1), then it has the geometric distribution.

Problem 6. Alice passes through four traffic lights on her way to work, and each light is equally likely to be green or red, independently of the others.

(i) What is the probability mass function, the mean, and the variance of the total number of red lights that Alice encounters?

(ii) Suppose that each red light delays Alice by two minutes. What is the variance of Alice's commuting time?

Problem 7. Suppose $Y = (3X-2)^2$, and $\mathbb{E}[X] = 2$, $\mathrm{var}(X) = 5$. Find $\mathbb{E}[Y]$.