Problem 4

$$f_X(x) = \int_0^1\int_0^1 f(x,y,z)\,dy\,dz = 2x \qquad \text{for } x \in [0,1]$$
$$f_Y(y) = \int_0^1\int_0^1 f(x,y,z)\,dx\,dz = 2y \qquad \text{for } y \in [0,1]$$
$$f_Z(z) = \int_0^1\int_0^1 f(x,y,z)\,dx\,dy = 2z \qquad \text{for } z \in [0,1]$$

Since $f(x,y,z) = f_X(x)\cdot f_Y(y)\cdot f_Z(z)$

$\Rightarrow$ Independent

$$P(X>Y) = \iint_{x>y} f_{X,Y}(x,y)\,dx\,dy$$
$$= \iint_{x>y} f_X(x)\cdot f_Y(y)\,dx\,dy$$
$$= \int_0^1\left(\int_0^x 4xy\,dy\right)dx = \frac{1}{2}$$

$$P(X>Y>Z) = 8\int_0^1 x\left(\int_0^x y\left(\int_0^y z\,dz\right)dy\right)dx = \frac{1}{6}$$
$$E[Y^3|Z] = \int_0^1 y^3\cdot f_Y(y)\,dy = \frac{2}{5}$$

Problem 5

(a) $F_Z(z) = P(1-e^{-\lambda X} \le z) = P\left(X \le -\frac{1}{\lambda}\log(1-z)\right)$

$$= 1-e^{\log(1-z)} = z$$

$f_Z(z) = 1 \qquad z \in [0,1]$

(b) $P(Z>z) = P(X>z)\,P(Y>z) = e^{-2\lambda z}$

$$P(Z\le z) = 1-e^{-2\lambda z} = F_Z(z)$$
$$f_Z(z) = \frac{dF_Z(z)}{dz} = 2\lambda e^{-2\lambda z}$$

(c) $F_Z(z) = P(Z \le z_0) = \int_{-\infty}^{z_0}\left(\int_{-\infty}^{\infty} f_{X,Y}(y+z,y)\,dy\right)dz$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(y+z,y)\,dy$$
$$= \int_{-\infty}^{\infty} f_X(y+z)\,f_Y(y)\,dy$$
$$= \int_{\max(0,-z)}^{\infty} \lambda^2 e^{-\lambda y}\,e^{-\lambda(y+z)}\,dy$$
$$= \frac{\lambda}{2}e^{-\lambda|z|} \qquad -\infty < z < \infty$$

$$F_Z(z) = \begin{cases} \frac{1}{2}e^{\lambda z} & z<0 \\ 1-\frac{1}{2}e^{-\lambda z} & z\ge 0 \end{cases}$$

(d) $P(\min(X,Y) \le aX) = P(Y\le aX) = \int_0^\infty \lambda e^{-\lambda x}\left(\int_0^{ax}\lambda e^{-\lambda y}\,dy\right)dx =$

$$= \frac{a}{1+a}$$

$$P(\min(X,Y) \le a\max(X,Y)) = P(Y\le aX) + P(X\le aY) = \frac{2a}{1+a}$$

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Homework No. 6 Solutions

Problem 1 (a) Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

$$C\int_0^\infty x^2 e^{-x}\left(\int_0^\infty e^{-xy}\,dy\right)dx = C\int_0^\infty x e^{-x}\,dx = 1$$

$$\Rightarrow C = 1$$

$$f_X(x) = \int_0^\infty x^2 e^{-x(1+y)}\,dy = x^2 e^{-x}\int_0^\infty e^{-xy}\,dy = xe^{-x}, \quad x>0$$

$$f_Y(y) = \int_0^\infty x^2 e^{-x(1+y)}\,dx = \frac{2}{(1+y)^3}, \quad y>0$$

(b) $$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \tfrac{1}{2}(1+y)^3 x^2 e^{-x(1+y)}, \quad x,y>0$$

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x) = xe^{-xy}, \quad x,y>0$$

(c) $$E[X|Y=y] = \int_0^\infty x\cdot f_{X|Y}(x|y)\,dx = \frac{(1+y)^3}{2}\cdot\frac{6}{(1+y)^4} = \frac{3}{1+y}$$

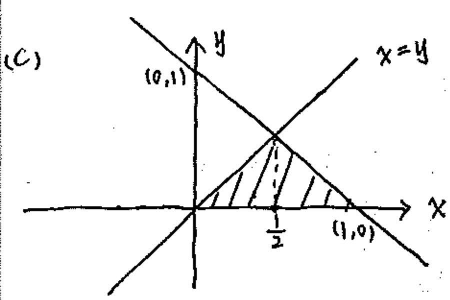
$$E[Y|X=x] = \int_0^\infty y f_{Y|X}(y|x)\,dy = \frac{1}{x}$$

Problem 2 (a) Since $\int_0^1\left(\int_0^x \frac{cy}{x}\,dy\right)dx = 1 \Rightarrow c = 4.$

(b) $f_X(x) = \int_0^x \frac{4y}{x}\,dy = 2x$ for $x\in[0,1]$

$f_Y(y) = \int_y^1 \frac{4y}{x}\,dx = -4y\log y$ for $y\in(0,1]$.

(c)



$$P(X+Y\le 1) = \int_0^{\frac{1}{2}}\left(\int_0^x \frac{4y}{x}\,dy\right)dx + \int_{1/2}^1\left(\int_0^{1-x}\frac{4y}{x}\,dy\right)dx = 2\log 2 - 1 \approx 0.39$$

Problem 3 $$F_Z(z_0) = P(Z\le z_0) = \iint_{x+y\le z_0} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{z_0}\int_0^{z_0-x}\tfrac{1}{2}(x+y)e^{-x-y}\,dy\,dx$$

$$= \tfrac{1}{2}\int_0^{z_0} e^{-x}\left(\int_0^{z_0-x} xe^{-y}\,dy + \int_0^{z_0-x} ye^{-y}\,dy\right)dx$$

$$= \tfrac{1}{2}\left(-z_0^2 e^{-z_0} - 2z_0 e^{-z_0} - 2e^{-z_0}\right)$$

$$z_0 \Rightarrow z \Rightarrow F_Z(z)$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \tfrac{1}{2}z^2 e^{-z}, \quad z\ge 0$$

Problem 6

$$X = g(Y) = 5Y/(1+Y)$$

$$\frac{dg}{dy}(y) = \frac{5}{(1+y)^2}$$

$$f_Y(y) = \left|\frac{dg}{dy}(y)\right| f_X(g(y))$$

$$= \frac{5}{4(1+y)^2}$$

As $x \in (1,5) \Rightarrow y \in (\frac{1}{4}, \infty)$.