Problem 1 (*25 points*) Random variables X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} Ce^{-y} & \text{if } 0 < x < y < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

First find the marginal density $f_X(x)$ and use it to calculate the constant C. Then find the conditional density $f_{X|Y}(x \mid y)$ and use it to calculate $\mathbb{E}[X \mid Y = y]$. (**Do not** convert $e \approx 2.718$)

$f_X(x) = C(e^{-x} - e^{-1})$	$C = \frac{e}{e-2}$	$f_{X\|Y}(x \mid y) = \frac{1}{y}$	$\mathbb{E}[X \mid Y = y] = \frac{y}{2}$

Fit your work in the box below and write the final answer in the box provided above.

$$f_X(x) = C\int_x^1 e^{-y}\,dy = C\left(e^{-x} - e^{-1}\right) \qquad 0 \le x \le 1$$

since $\int_0^1 f_X(x)\,dx = 1 \Rightarrow C\int_0^1 \left(e^{-x} - e^{-1}\right)dx = C\left(1 - 2e^{-1}\right) = 1$

Therefore: $C = \dfrac{e}{e-2}$

$$f_Y(y) = C\int_0^y e^{-y}\,dx = Cye^{-y} \qquad 0 \le y \le 1$$

$$\therefore \quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{y} \qquad 0 \le x \le y < 1$$

Therefore

$$E[X|Y=y] = \int_0^1 x\, f_{X|Y}(x|y)\,dx$$

$$= \int_0^y \frac{x}{y}\,dx = \frac{y}{2}$$

Problem 2 (*25 points*) Suppose that an electronic device has a life length X (in units of 1000 hours) which is considered as a continuous random variable with the following PDF:

$$f(x) = 0.9e^{-0.9x}, \quad x > 0.$$

Suppose that the cost of manufacturing one such item is \$2. The manufacturer sells the item for \$5, but guarantees a total refund if $X \leq 0.6$. Find the manufacturer's expected profit per item. (You may use $e^{-0.54} \approx 0.58$).

Expected profit per item = ≈ \$ 0.91

Fit your work in the box below and write the final answer in the box provided above.

$P(X \leq 0.6) = 1 - e^{-0.9 \times 0.6} \approx 0.42$

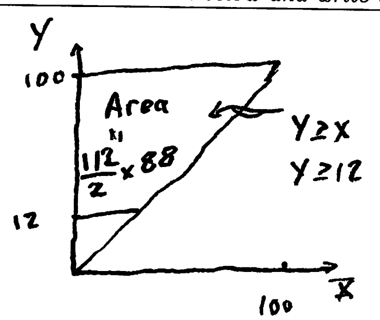
So

Expected Profit ≈ \$3 − \$5 × 0.42 = \$0.90

Problem 3 (*25 points*) Suppose that X and Y are independent random variables, each uniformly distributed on $[0, 100]$. Find the value of $\mathbb{P}(Y \geq X \mid Y \geq 12)$.

$\mathbb{P}(Y \geq X \mid Y \geq 12) = 0.56$

Fit your work in the box below and write the final answer in the box provided above.



So $P(Y \geq X \mid Y \geq 12) = \dfrac{\frac{112}{2} \times 88}{100 \times 88}$

$= 0.56$

Problem 4 (*25 points*) A bolt manufacturer knows that 5% of his production is defective. He gives a guarantee on every shipment of 10,000 parts by promising to refund the money if more than a number of N bolts are defective. How small can the manufacturer choose N and still be assured that he need not give a refund more than 1% of the time? First use Chebyshev's inequality to get an upper bound for the value of N. Then use the Central Limit Theorem (CLT) to obtain an approximate value for N. You might find the following useful:

$$\sqrt{0.0475} \approx 0.218\,, \qquad \frac{1}{2\pi}\int_0^{2.33} e^{-\frac{x^2}{2}}\,dx \approx 0.49\,.$$

Using Chebyshev: $N =$ 718	Using CLT: $N =$ 551

Fit your work in the box below and write the final answer in the box provided above.

Each bolt has Bernoulli Distribution with $p = 0.05$, Here we define

$$X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ bolt is defective} \\ 0 & \text{otherwise} \end{cases}$$

Let $S_n = X_1 + \dots + X_n$ with $n = 10{,}000$

By Chebyshev's inequality, since $\sigma_{X_i} = \sqrt{pq} \approx 0.218$

$$P(S_n \geq N) \leq P(|S_n - 500| \geq N - 500) \leq \frac{10^4 \sigma_X^2}{(N-500)^2} \approx 0.01$$

Therefore $\dfrac{10^2 \times 0.218}{N-500} = 0.1 \Rightarrow N = 718$

Using CLT

$$P(S_n \leq N) = P\left(\frac{S_n - 500}{10^2 \sigma_X} \leq \frac{N-500}{10^2 \sigma_X}\right) \geq 0.99$$

Thefore we choose

$$\frac{N-500}{100\sigma_X} = 2.33 \Rightarrow N \approx 551$$