Solutions #1

Problem 1

(a) $A \cup B \cup C$

(b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

(c) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

(d) $(A \cap B \cap C)^c$

Problem 2

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$

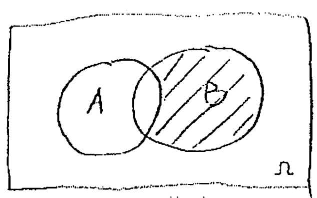
$P(\text{even}) = \frac{2}{3}$, $P(\text{odd}) = \frac{1}{3}$

$P(\text{less than } 4) = P(1) + P(2) + P(3) = \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3} = \frac{4}{9}$

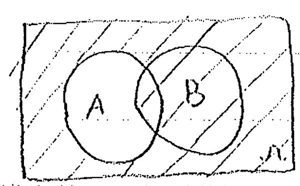
Problem 3

(a) $P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - z$

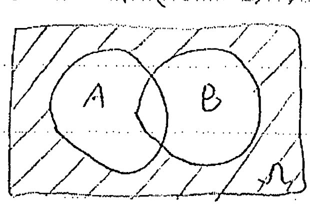
(b) $P(A^c \cap B) = y - z$



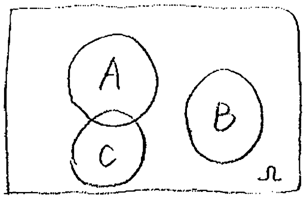
(c) $P(A^c \cup B) = 1 - (x - z) = 1 - x + z$



(d) $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - (x + y - z) = 1 - x - y + z$



Problem 4



$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-$$
$$-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$
$$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-0-0-\frac{1}{8}+0=\frac{5}{8}$$

Problem 5

① $$P(A\leftrightarrow B \mid A\not\leftrightarrow C)=\frac{P(A\leftrightarrow B\cap A\not\leftrightarrow C)}{P(A\not\leftrightarrow C)}$$
$$=\frac{P(A\leftrightarrow B\cap B\not\leftrightarrow C)}{P(A\not\leftrightarrow C)}$$
$$=\frac{P(A\leftrightarrow B)\cdot P(B\not\leftrightarrow C)}{1-P(A\leftrightarrow C)}$$
$$=\frac{P(A\leftrightarrow B)\cdot P(B\not\leftrightarrow C)}{1-P(A\leftrightarrow B)\cdot P(B\leftrightarrow C)}$$
$$=\frac{(1-p^2)\cdot p^2}{1-(1-p^2)^2}=\frac{1-p^2}{2-p^2}$$

② $$P(A\leftrightarrow B \mid A\not\leftrightarrow C)=\frac{P(A\leftrightarrow B\cap A\not\leftrightarrow C)}{P(A\not\leftrightarrow C)}$$

direct road

$$=\frac{P(A\leftrightarrow B\cap B\not\leftrightarrow C\cap A\not\leftrightarrow C)}{1-P(A-C)}$$

direct road

$$=\frac{P(A\leftrightarrow B)\cdot P(B\not\leftrightarrow C)\cdot P(A\not\leftrightarrow C)}{1-P(A\leftrightarrow C)-P(A\not\leftrightarrow C)\cdot P(A\leftrightarrow B)\cdot P(B\leftrightarrow C)}$$

direct road

$$=\frac{(1-p^2)\cdot p^2\cdot p}{1-(1-p)-p(1-p^2)^2}$$
$$=\frac{1-p^2}{2-p^2}$$

Problem 6

$P(A)=\frac{1}{2}\quad P(B)=\frac{1}{2}\quad P(C)=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

$P(A\cap B)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}=P(A)\cdot P(B)\rightarrow$ A and B independent

$P(B\cap C)=\frac{1}{4}=P(B)\cdot P(C)\rightarrow$ B and C independent

$P(A\cap C)=\frac{1}{4}=P(A)\cdot P(C)\rightarrow$ A and C independent

$P(A\cap B\cap C)=0\Rightarrow$ not independent

HW01 - #7

Let A_i be the event that key #i is in the correct hook.

Then

$$P(\text{no key in the correct hook}) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= 1 - \sum_i P(A_i) + \sum_{i<j} P(A_i \cap A_j) - \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \cdots$$

$$= 1 - \binom{n}{1}\left(\frac{1}{n}\right) + \binom{n}{2}\frac{1}{n(n-1)} - \binom{n}{3}\frac{1}{n(n-1)(n-2)} + \cdots$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

So as $n \to \infty$ this probability tends to $\frac{1}{e}$