Homework 4 Solutions

1) $P(A\cap B\cap C) \overset{?}{=} P(A)\,P(B|A)\,P(C|A\cap B)$

$$= \cancel{P(A)}\,\frac{\cancel{P(B\cap A)}}{\cancel{P(A)}}\,\frac{P(A\cap B\cap C)}{\cancel{P(A\cap B)}}$$ ✓

$P(A\cap C|B) \overset{?}{=} P(A|B\cap C)P(C|B)$

$$= \frac{P(A\cap C|B)}{P(C|B)}\cdot P(C|B)$$ ✓

$P(A\cap B) \overset{?}{=} \sum_{i=1}^{n} P(A|B\cap C_i)\,P(C_i|B)$

$$= \sum_{i=1}^{n} \frac{P(A\cap C_i|B)}{\cancel{P(C_i|B)}}\,\cancel{P(C_i|B)} = \sum_{i=1}^{n} P(A\cap C_i|B)$$

Since $\bigcup_{i=1}^{n} C_i = \Omega \Rightarrow P(A\cap B) = \sum_{i=1}^{n} P(A\cap C_i|B) = P(A\cap\Omega|B)$

2) a) $E\left[\frac{1}{(X+1)}\frac{1}{(X+2)}\right] = \sum_{k=0}^{\infty}\frac{1}{k+1}\cdot\frac{1}{k+2}\frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty}\frac{\lambda^k e^{-\lambda}}{(k+2)!} = \lambda^{-2}\underbrace{\sum_{l=2}^{\infty}\frac{\lambda^l e^{-\lambda}}{l!}}_{1-e^{-\lambda}-\lambda e^{-\lambda}}$

b) $E[X(X-1)(X-2)] = \sum_{k=0}^{\infty} k(k-1)(k-2)\frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=3}^{\infty}\frac{\lambda^k e^{-\lambda}}{(k-3)!} = \lambda^3\sum_{l=0}^{\infty}\frac{\lambda^l e^{-\lambda}}{l!} = \lambda^3$

c) $E[e^X] = \sum_{k\geq 0} e^k\lambda^k\frac{e^{-\lambda}}{k!} = \left(\sum_{k=0}^{\infty}\frac{(e\lambda)^k}{k!}\right)e^{-\lambda} = e^{e\lambda-\lambda}$

3) $\sum_{k=1}^{N} c\,2^k = 1$ and $\sum_{k=0}^{N} 2^k = \frac{1-2^{N+1}}{1-2} \Rightarrow \sum_{k=1}^{N} 2^k = 2^{N+1}-2$

$$\Rightarrow c = \frac{1}{\sum_{k=1}^{N} 2^k} = \frac{1}{2^{N+1}-2}$$

4) a) $\underbrace{i \text{ distinct types}}$ + Y_i new coupons $\Rightarrow$ $i+1$ distinct types

$\sum_{j=0}^{i-1} Y_j$: numbers of coupons collected till i distinct types

$$\Rightarrow P(Y_i = k) = \underbrace{\left(\frac{c-i}{c}\right)}_{\substack{\text{probability of} \\ \text{getting } i+1^{\text{th}} \\ \text{distinct type}}} \left(1 - \frac{c-i}{c}\right)^{k-1} \Rightarrow Y_i \text{ is geometric}\left(\frac{c-i}{c}\right)$$

(given there are i distinct coupons)
& " c " " in total

b) $$E[Y] = E\left[\sum_{j=0}^{c-1} Y_j\right] = \sum_{j=0}^{c-1} E[Y_j] = \sum_{j=0}^{c-1} \frac{c}{c-j} \quad \text{let } c-j = l$$

$$= c \sum_{l=1}^{c} \frac{1}{l} = cH_c$$ where H_c is c^{th} harmonic number

Problem 4) part c)

$$E[\# \text{ different types of coupons}] = E\left[\sum_{i=1}^{c} 1_{\left\{\substack{\text{Coupon type } i \text{ is collected in the} \\ \text{first } n \text{ coupons received}}\right\}}\right]$$

$$= \sum_{i=1}^{c} E\left[1_{\left\{\substack{\text{Coupon type } i \text{ is collected in the} \\ \text{first } n \text{ coupons received}}\right\}}\right]$$

$$= \sum_{i=1}^{c} 1 - P(\text{coupon type } i \text{ is } \underline{\text{not}} \text{ collected in the first } n \text{ coupons received})$$

$$= \sum_{i=1}^{c} 1 - \left(\frac{c-1}{c}\right)^n$$

$$= c\left(1 - \left(\frac{c-1}{c}\right)^n\right)$$

5) ($\Rightarrow$) Given X has geometric distribution $= P(X=k) = q^{k-1}p$, $k \geq 1$
$q = 1-p$

$$\Rightarrow P(X > m+n \mid X > m) = \frac{P(X > m+n)}{P(X > m)}$$

$$P(X > n) = \sum_{k=n+1}^{\infty} P(X=k) = \sum_{k=n+1}^{\infty} q^{k-1} p = p \sum_{l=n}^{\infty} q^{l} = p\left(\frac{1}{1-q} - \frac{1-q^{n}}{1-q}\right) = q^{n}$$

$$\Rightarrow P(X > m+n \mid X > m) = \frac{q^{m+n}}{q^{m}} = q^{n} = P(X > n)$$

($\Leftarrow$) Now, given $P(X > m+n \mid X > m) = P(X > n)$ and X is non-negative & integer valued random variable: $P(X > m+n) = P(X > m) P(X > n)$

let $m = n = 0 \Rightarrow P(X > 0) \cancel{=} P(X > 0)^2 = 1$

$m=1, n=1 \Rightarrow P(X > 2) = P(X > 1)^2 = \alpha^2$ let $P(X > 1) = \alpha$

$m=1, n=2$ $P(X > 3) = P(X > 2) P(X > 1) = P(X > 1)^3 = \alpha^3$

$\vdots$

$P(X > n-1) = P(X > 1)^{n-1} = \alpha^{n-1}$

$P(X > n) = P(X > 1)^{n} = \alpha^{n}$

$\Rightarrow P(X=1) + P(X > 1) = 1 \Rightarrow P(X=1) = 1 - \alpha$

$P(X=2) = P(X > 1) - P(X > 2) = q - q^2$

$P(X=3) = P(X > 2) - P(X > 3) = q^2 - q^3$

$\vdots$

$P(X=n) = P(X > n-1) - P(X > n) = q^{n-1} q^{n} = q^{n-1}(1-q)$ $n \geq 1$ $\Rightarrow$ Geometric distribution.

$\blacksquare$

6) Let $X = \#$ red lights Alice encounters

a) $P(X=k) = \binom{4}{k}\left(\frac{1}{2}\right)^4 \rightarrow \text{Binomial}(4, \frac{1}{2})$

$E[X] = np = 4 \cdot \frac{1}{2} = 2$

$Var[X] = np(1-p) = 1$

b) Let Y = Commuting time
D = delay $= 2X$

fixed

$Var[Y] = Var[T + 2X]$
$= Var[2X]$
$= 4 Var[X] = 4$

7) $Y = (3X-2)^2$

$E[X] = 2$, $Var[X] = 5$ $\Rightarrow E[X^2] = 9$

$\Rightarrow E[Y] = E[9X^2 - 12X + 4]$
$= 9 \cdot 9 - 12 \cdot 2 + 4 = 61$