Homework No. 9 (with solutions)

Problem 1. If X has cumulative distribution function $F_X(x)$, what is the distribution function of $Y = \max\{0, X\}$. Next suppose that X be a random variable which is uniformly distributed on the interval $(-1, 1)$. Compute the cumulative distribution function of $Y = \max\{0, X\}$.

Solution.

Problem 2. Let X and Y be independent having an exponential distribution with parameters λ and μ respectively. Find the probability density function of $Z = X + Y$.

Solution. We have, assuming $\lambda \neq \mu$,

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(x) f_Y(z-x) \mathrm{d}x \\ &= \int_0^z \lambda\mu\, \mathrm{e}^{-\mu z} \mathrm{e}^{-(\lambda-\mu)x} \mathrm{d}x \\ &= \lambda\mu\, \frac{\mathrm{e}^{-\mu z} - \mathrm{e}^{-\lambda z}}{\lambda - \mu}\,, \quad z > 0\,. \end{aligned}$$

When $\lambda = \mu$, this reduces to $f_Z(z) = \mu^2 z \mathrm{e}^{-\mu z}$.

Problem 3. Suppose that the two dimensional random variable (X, Y) is uniformly distributed over the triangular region $R = \{(x, y) \mid 0 \le x \le y \le 1\}$. Find $\rho(X, Y)$.

Solution. The area of the triangle is $1/2$, so the joint probability density function of (X, Y) equals 2 on the triangle and is 0 outside the triangle. Thus for any function $g(X, Y)$,

$$\mathbb{E}[g(X,Y)] = 2 \int_0^1 \left(\int_0^y g(x,y)\, \mathrm{d}x \right) \mathrm{d}y\,.$$

Setting $g(X, Y)$ equal to X, Y, XY, X^2 and Y^2 and calculating the integral we obtain

$$\mathbb{E}[X] = \frac{1}{3}\,, \quad \mathbb{E}[Y] = \frac{2}{3}\,, \quad \mathbb{E}[XY] = \frac{1}{4}\,, \quad \mathbb{E}\left[X^2\right] = \frac{1}{6}\,, \quad \mathbb{E}\left[Y^2\right] = \frac{1}{2}\,.$$

Thus $\operatorname{cov}(X, Y) = \frac{1}{12}$, and $\rho(X, Y) = \frac{1}{\sqrt{12}}$.

Problem 4. A coin has an a priori probability X of coming up heads, where X is a random variable with probability density function

$$f_X(x) = \begin{cases} x\mathrm{e}^x, & \text{for } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Essentially this means that $\mathbb{P}(\text{Heads} \mid X = x) = x$.

(a) Find $\mathbb{P}(\text{Heads})$.

(b) If A denotes the event that the last flip came up heads, find the conditional probability density function of X given A, i.e., $f_{X|A}(x \mid A)$.

(c) Given A, find the conditional probability of heads at the next flip.

Solution. For part (a) with $H \equiv$ Heads,

$$\mathbb{P}(H) = \int_0^1 \mathbb{P}(H \mid X = x) f_X(x)\, \mathrm{d}x = \int_0^1 x^2 \mathrm{e}^x\, \mathrm{d}x = \mathrm{e} - 2\,.$$

For part (b) we use Bayes' theorem (continuous version):

$$f_{X|A}(x \mid A) = \frac{\mathbb{P}(A \mid X = x) f_X(x)}{\int_0^1 \mathbb{P}(A \mid X = x) f_X(x)\,\mathrm{d}x} = \begin{cases} \frac{x^2 \mathrm{e}^x}{\mathrm{e}-2}\,, & 0 \le x \le 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

For part (c) we combine the results in (a) and (b)

$$\begin{aligned} \mathbb{P}(H \mid A) &= \int_0^1 \mathbb{P}(H \mid X = x) f_{X|A}(x \mid A)\,\mathrm{d}x \\ &= \frac{1}{\mathrm{e}-2} \int_0^1 x^3 \mathrm{e}^x\,\mathrm{d}x \\ &= \frac{6 - 2\mathrm{e}}{\mathrm{e}-2} \\ &\approx 0.786 \end{aligned}$$

Problem 5. Let X be a geometric random variable with parameter P, where P is itself random and uniformly distributed on the interval $[0, \frac{n-1}{n}]$. Let $Z = \mathbb{E}[X \mid P]$. Find $\mathbb{E}[Z]$.

Solution. We use the law of iterated expectations:

$$\mathbb{E}[Z] = \mathbb{E}\big[\mathbb{E}[X \mid P]\big] = \mathbb{E}[X].$$

We now calculate $\mathbb{E}[X \mid P = p]$, as follows:

$$\begin{aligned} \mathbb{E}[X \mid P = p] &= (1-p) \sum_{n=1}^{\infty} n p^{n-1} \\ &= (1-p) \frac{\mathrm{d}}{\mathrm{d}p} \sum_{n=1}^{\infty} p^n \\ &= (1-p) \frac{\mathrm{d}}{\mathrm{d}p} (p(1-p)^{-1}) \\ &= \frac{1}{1-p}\,. \end{aligned}$$

Next, we proceed to calculate $\mathbb{E}[Z]$:

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}\big[\mathbb{E}[X \mid P]\big] \\ &= \int_0^{(n-1)/n} \frac{n}{(n-1)(1-p)}\,\mathrm{d}p \\ &= \frac{n \log n}{n-1}\,. \end{aligned}$$

Thus, as n increases, $\mathbb{E}[Z]$ grows to infinity at the rate of $\log n$.

Problem 6. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $\mathbb{E}[X \mid Y]$ are positively correlated.

(b) Show that the correlation coefficient of Y and $\mathbb{E}[X \mid Y]$ has the same sign as the correlation coefficient of X and Y.

In this problem, do not assume that X and Y have a PDF; use only the properties of conditional expectation.

Solution. (a) We have

$$\operatorname{cov}\big(X, \mathbb{E}[X \mid Y]\big) = \mathbb{E}\big[X\, \mathbb{E}[X \mid Y]\big] = \mathbb{E}\Big[\mathbb{E}\big[X\, \mathbb{E}[X \mid Y] \mid Y\big]\Big] = \mathbb{E}\left[\big(\mathbb{E}[X \mid Y]\big)^2\right] \geq 0\,.$$

(b) We have

$$\operatorname{cov}\big(Y, \mathbb{E}[X \mid Y]\big) = \mathbb{E}\big[Y\, \mathbb{E}[X \mid Y]\big] = \mathbb{E}\big[\mathbb{E}[XY \mid Y]\big] = \mathbb{E}[XY] = \operatorname{cov}(X, Y)\,.$$

Problem 7. Random variables X and Y are distributed according to the joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} ax & \text{if } 1 \leq x \leq y \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Evaluate the constant a.

(b) Determine the marginal probability density function $f_Y(y)$.

(c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.

(d) Determine the probability density function $f_Z(z)$, if $Z = Y - X$.

Solution. Since

$$1 = \int_1^3 \left(\int_1^y ax\, \mathrm{d}x\right) \mathrm{d}y = a \int_1^3 \frac{y^2 - 1}{2}\, \mathrm{d}y = \frac{10}{3} a\,,$$

it follows that $a = 0.3$. For part (b)

$$f_Y(y) = \int_1^y ax\, \mathrm{d}x = \begin{cases} \frac{3}{20}(y^2 - 1)\,, & 1 < y \leq 3 \\ 0\,, \text{otherwise.} \end{cases}$$

For part (c), we have

$$f_{X|Y}\left(x \mid \tfrac{3}{2}\right) = \frac{f_{X,Y}\left(x, \frac{3}{2}\right)}{f_Y\left(\frac{3}{2}\right)} = \frac{8}{5} x\,, \quad 1 \leq x \leq \tfrac{3}{2}\,,$$

and hence

$$\mathbb{E}\left[X^{-1} \mid Y = \tfrac{3}{2}\right] = \int_1^{3/2} \tfrac{8}{5}\, \mathrm{d}x = \frac{4}{5}\,.$$

For part (d), we have $F_Z(z) = 0$ for $z < 0$, and for $z \in [0, 2]$ we compute

$$\begin{aligned} F_Z(z) = \mathbb{P}(Y - X \leq z) &= 1 - \mathbb{P}(Y - X > z) \\ &= 1 - \int_1^{3-z} \left(\int_{x+z}^3 \tfrac{3}{10} x\, \mathrm{d}y\right) \mathrm{d}x \\ &= \frac{9}{10} + \frac{1}{20}(3 - z)\left(3 - (3 - z)^2\right)\,. \end{aligned}$$

Thus

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = \begin{cases} \frac{3}{20} z^2 - \frac{9}{10} z + \frac{6}{5}\,, & 0 \leq z \leq 2\,, \\ 0\,, & \text{otherwise.} \end{cases}$$