Homework No. 8 (with solutions)

Problem 1. If X has the normal distribution with mean μ and variance σ^2, find $\mathbb{E}\big[X^3\big]$ (as a function of μ and σ^2), without computing any integrals.

Solution. We have

$$0 = \mathbb{E}\left[(X-\mu)^3\right] = \mathbb{E}\left[X^3\right] - 3\mu\,\mathbb{E}\left[X^2\right] + 3\mu^2\,\mathbb{E}[X] - \mu^3\,.$$

Solving we get

$$\mathbb{E}\left[X^3\right] = 3\mu\big(\sigma^2+\mu^2\big) - 3\mu^3 - \mu^3 = \mu\big(3\sigma^2 - \mu^2\big)\,.$$

Problem 2. Suppose that the weight of a person selected at random from some population is normally distributed with parameters μ and σ^2. Suppose also that $\mathbb{P}(X \le 160) = \frac{1}{2}$ and $\mathbb{P}(X \le 140) = \frac{1}{4}$. Find μ and σ. Also, find $\mathbb{P}(X \ge 200)$. (Use the *standard normal table* for this problem).

Solution. Clearly $\mu = 160$. Hence

$$\mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{-20}{\sigma}\right) = \frac{1}{4} \Longrightarrow \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{20}{\sigma}\right) = \frac{3}{4} \Longrightarrow \frac{20}{\sigma} \approx 0.675 \Longrightarrow \sigma \approx 29.5\,.$$

Next,

$$\mathbb{P}(X \ge 200) = \left(\frac{X-\mu}{\sigma} \ge \frac{200-140}{\sigma}\right) \approx \left(\frac{X-\mu}{\sigma} \ge 2\right) \approx 0.0228\,.$$

Problem 3. Let X and Y be independent exponential random variables with a common parameter λ.

(a) Find the moment generating function associated with $aX+Y$, where a is a constant.

(b) Use the result of part (a) to find the probability density function of $aX+Y$, for the case where a is positive and different than 1.

(c) Use the result of part (a) to find the probability density function of $X-Y$.

Solution. (a) Let $Z = aX+Y$. We have

$$M_Z(s) = \mathbb{E}\left[\mathrm{e}^{s(aX+Y)}\right] = \mathbb{E}\left[\mathrm{e}^{saX}\right]\,\mathbb{E}\left[\mathrm{e}^{sY}\right] = M_X(sa)M_Y(s) = \frac{\lambda}{\lambda - sa} \times \frac{\lambda}{\lambda - s}\,,$$

for $s < \lambda$. (b) We will express the moment generating function of Z in the form

$$M_Z(s) = \frac{c}{\lambda - sa} + \frac{d}{\lambda - s}\,.$$

We have

$$c = M_Z(s)(\lambda - sa)\Big|_{s = \lambda/a} = \frac{\lambda^2}{\lambda - \lambda/a} = \frac{a\lambda}{a-1}\,,$$

and

$$d = M_Z(s)(\lambda - s)\Big|_{s=\lambda} = \frac{\lambda^2}{\lambda - a\lambda} = -\frac{\lambda}{a-1}\,.$$

Thus,

$$M_Z(s) = \frac{\lambda}{a-1}\left(\frac{a}{\lambda - sa} - \frac{1}{\lambda - s}\right) = \frac{a}{a-1} \times \frac{\lambda/a}{\lambda/a - s} - \frac{1}{a-1} \times \frac{\lambda}{\lambda - s}\,.$$

We recognize this as the moment generating function associated with the probability density function

$$f_X(x) = \frac{\lambda}{a-1} e^{-\lambda x/a} - \frac{\lambda}{a-1} e^{-\lambda x}, \qquad x \ge 0\,.$$

(c) From part (a) we have that the moment generating function of $-Y$ is equal to $\lambda/(\lambda+s)$. With $Z = X - Y$, we have

$$M_Z(s) = \frac{\lambda^2}{(\lambda - s)(\lambda + s)} = \frac{1}{2}\left(\frac{\lambda}{\lambda - s} + \frac{\lambda}{\lambda + s}\right),$$

which we recognize as the moment generating function of a mixture of two random variables, one distributed as X, the other distributed as $-Y$. It follows that

$$p_Z(z) = \begin{cases} \frac{\lambda}{2}\mathrm{e}^{-\lambda z}, & \text{if } z \geq 0\,, \\ \frac{\lambda}{2}\mathrm{e}^{\lambda z}, & \text{if } z \leq 0\,, \end{cases}$$

or

$$p_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}.$$

Problem 4. Each egg laid by the hen falls onto the concrete floor of the henhouse and cracks with probability p. If the number of eggs laid today by the hen has the Poisson distribution, with parameter λ, use moment generating functions to find the probability distribution of the number of uncracked eggs.

Solution. Let N denote the number of eggs laid. Then $G_N(s) = \mathrm{e}^{\lambda(s-1)}$. Let

$$X_i \triangleq \begin{cases} 1\,, & \text{if the i}^{\text{th}} \text{ egg survives,} \\ 0\,, & \text{if the i}^{\text{th}} \text{ egg cracks.} \end{cases}$$

Thus, $G_{X_i}(s) = (1-p)s + p$. Since $Z = \sum_{i=1}^{N} X_i$ is the number of uncracked eggs, we have

$$\begin{aligned} G_Z(s) &= G_N\left(G_{X_i}(s)\right) \\ &= \exp\bigl(\lambda((1-p)s + p - 1)\bigr) = \mathrm{e}^{(1-p)\lambda(s-1)}. \end{aligned}$$

Therefore Z has the Poisson distribution with parameter $\lambda(1-p)$.

Problem 5. Suppose that X has moment generating function

$$M_X(s) = \frac{6 - 3s}{2(1-s)(3-s)}\,.$$

Find the probability density function of the associated random variable X.

Solution. $M_X(s)$ can be expanded in the form

$$M_X(s) = \frac{a}{1-s} + \frac{b}{3-s}\,.$$

The values of a and b can be found by plugging in particular values of s. For example, setting $s = 0$ yields

$$a + \frac{b}{3} = M_X(0) = 1\,,$$

and setting $s = 4$ yields

$$-\frac{a}{3} - b = M_X(4) = -1\,.$$

Solving for a and b, we obtain $a = b = 3/4$. It follows that

$$M_X(s) = \frac{3}{4} \times \frac{1}{1-s} + \frac{1}{4} \times \frac{3}{3-s}\,.$$

Thus, X is a mixture of two exponential random variables and

$$f_X(x) = \frac{3}{4}e^{-x} + \frac{1}{4}e^{-3x}, \qquad x \geq 0\,.$$

Problem 6. The moment generating function associated with the random variable X is

$$M(s) = a\mathrm{e}^s + b\mathrm{e}^{4(\mathrm{e}^s-1)}\,, \qquad \mathbb{E}[X] = 3\,.$$

Find:

(a) The scalars a and b.

(b) $\mathbb{P}(X = 1)$, $\mathbb{E}[X^2]$, and $\mathbb{E}[2^X]$.

Solution. (a) Since $M(0) = 1$, we have $a + b = 1$. Since

$$\mathbb{E}[X] = \frac{\mathrm{d}}{\mathrm{d}s}M(s)\Big|_{s=0} = \big(a + 4b\mathrm{e}^s\mathrm{e}^{4(\mathrm{e}^s-1)}\big)\Big|_{s=0} = 3\,,$$

we obtain $a + 4b = 3$. Solving the two equations for a and b, we have

$$a = \frac{1}{3}, \qquad b = \frac{2}{3}\,.$$

(b) Given the form of $M(s)$, we see that X is equal to 1, with probability $a = {}^1\!/_3$, and equal to a Poisson random variable Z with parameter $\lambda = 4$, with probability $b = {}^2\!/_3$. The desired quantities can be calculated starting from this observation:

$$p_X(1) = a + b\,\mathbb{P}(Z = 1) = a + 4b\mathrm{e}^{-4} = \frac{1}{3} + \frac{8}{3}\mathrm{e}^{-4}.$$

Alternatively, we can use the series expansion

$$M(s) = a\mathrm{e}^s + b\mathrm{e}^{4(\mathrm{e}^s-1)} = a\mathrm{e}^s + b\mathrm{e}^{-4}\Big(1 + 4\mathrm{e}^s + \frac{(4\mathrm{e}^s)^2}{2} + \cdots\Big),$$

to infer that X is a discrete random variable, and identify $p_X(1)$ with the coefficient of e^s, so that

$$p_X(1) = a + 4b\mathrm{e}^{-4} = \frac{1}{3} + \frac{8}{3}\mathrm{e}^{-4}.$$

The second moment $\mathrm{e}[X^2]$ is obtained by

$$\mathbb{E}[X^2] = \frac{\mathrm{d}^2}{\mathrm{d}s^2}M(s)\Big|_{s=0} = a\mathrm{e}^s + 4b(4\mathrm{e}^s + 1)e^{4\mathrm{e}^s+s-4}\Big|_{s=0} = \frac{41}{3}\,.$$

Finally,

$$\mathbb{E}[2^X] = \mathbb{E}\big[(\mathrm{e}^{\ln 2})^X\big] = M(\ln 2) = 2a + b\mathrm{e}^4 = \frac{2}{3} + \frac{2}{3}\mathrm{e}^4.$$

Problem 7. The random variables X_1, X_2 and X_3 are independent and identically distributed, having the exponential distribution with parameter 1, i.e., a probability density function e^{-x}, $x > 0$. Find the probability density function of

$$Z = \tfrac{1}{2}X_1 + \tfrac{1}{3}X_2 + X_3\,.$$

Solution. We have

$$\begin{aligned}
M_Z(s) &= M_{X_1}({}^s\!/_2) \times M_{X_2}({}^s\!/_3) \times M_{X_3}(s)\\
&= \frac{2}{2-s} \times \frac{3}{3-s} \times \frac{1}{1-s}\\
&= \frac{-6}{2-s} + \frac{3}{3-s} + \frac{3}{1-s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_Z(z) &= -6\mathrm{e}^{-2z} + 3\mathrm{e}^{-3z} + 3\mathrm{e}^{-z}\\
&= 3\mathrm{e}^{-z}\big(1 - \mathrm{e}^{-z}\big)^2\,, \quad z \geq 0\,.
\end{aligned}$$