Homework No. 10 (Due on 05/01/14)

Problem 1. Let $X_0, X_1, X_2, \ldots$ be a Markov chain with state space $\{1, 2\}$ and transition probabilities given as follows:

$$p_{11} = 0.3, \quad p_{12} = 0.7, \quad p_{21} = 0.5, \quad p_{22} = 0.5\,.$$

Find the invariant probability π of the chain. Find $\mathbb{P}(X_2 = 1, X_3 = 2 \mid X_0 = 1)$.

Problem 2. The outcomes of successive flips of a particular coin are dependent and are found to be described fully by the conditional probabilities

$$\mathbb{P}(H_{n+1} \mid H_n) = \frac{3}{4}\,, \qquad \mathbb{P}(T_{n+1} \mid T_n) = \frac{3}{4}\,.$$

where we have used the notation: Event H_k: Heads on k^{th} toss; Event T_k: Tails on k^{th} toss. We know that the first toss came up heads. Determine the probability that the *first* tail with occur on the k^{th} toss ($k = 2, 3, 4, \ldots$). What is the probability that flip 5,000 will come up heads (approximate value)? What is the probability that flip 5,000 will come up heads and flip 5,002 will also come up heads?

Problem 3. A security guard has the only key which locks or unlocks the door to the ENS building. He visits the door once each hour on the hour. When he arrives: If the door is open, he locks it with probability 0.3. If the door is locked, he unlocks it with probability 0.8. After he has been on the job several months, is he more likely to lock the door or to unlock it on a randomly selected visit? With the process in the steady state, Joe arrived at ENS two hours ahead of Harry. What is the probability that each of them found the door in the same condition? Given the door was open at the time the security guard was hired, determine the expected value of the number of visits up to and including the one on which he unlocks the door himself for the first time.

Problem 4. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.

(a) The largest number X_n shown up to the nth roll.

(b) The number N_n of sixes in n rolls.

(c) At time r, the time C_r since the most recent six.

(d) At time r, the time B_r until the next six.

Problem 5. Let X be a random variable such that

$$M_X(s) = a + b\mathrm{e}^{2s} + c\mathrm{e}^{4s}, \quad \mathbb{E}[X] = 3, \quad \mathrm{var}(X) = 2\,,$$

where $M_X(s)$ denotes the moment generating function of X. Find a, b, and c, and the probability mass function of X.

Problem 6. Random variables X and Y are distributed according to the joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} C & \text{if } x \geq 0 \text{ and } y \geq 0 \text{ and } x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and B be the event $\{Y > X\}$.

(a) Evaluate the constant C.

(b) Calculate $\mathbb{P}(B \mid A)$.

(c) Find $\mathbb{E}[X \mid Y = 0.5]$ and the conditional probability density function $f_{X|B}(x \mid B)$.

(d) Calculate $\mathbb{E}[XY]$.