Homework No. 1 (Due on 01/28/14)

Problem 1. Let A, B, and C be three events. Express the following verbal statements in set notation.

(a) At least one of the events occurs.

(b) Exactly one of the events occurs.

(c) Exactly two of the events occur.

(d) Not more than two of the events occur simultaneously.

Problem 2. A six-sided die is rolled in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model $(\Omega, \mathfrak{F}, \mathbb{P})$ for a single roll of this die, and find the probability that the outcome is less than 4.

Problem 3. Suppose that A and B are events for which $\mathbb{P}(A) = x$, $\mathbb{P}(B) = y$, and $\mathbb{P}(A \cap B) = z$. Express each of the following probabilities in terms of x, y, and z.

(a) $\mathbb{P}(A^c \cup B^c)$.

(b) $\mathbb{P}(A^c \cap B)$.

(c) $\mathbb{P}(A^c \cup B)$.

(d) $\mathbb{P}(A^c \cap B^c)$.

Problem 4. Suppose A, B and C are events such that $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{4}, \mathbb{P}(A \cap B) = \mathbb{P}(C \cap B) = 0$, and $\mathbb{P}(A \cap C) = \frac{1}{8}$. Evaluate the probability that at least one of the events A, B, or C occurs.

Problem 5. There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability p, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C, i.e., $\mathbb{P}(A \leftrightarrow B \mid A \nleftrightarrow C)$.

If, in addition, there is a direct road from A to C, this road being blocked with probability p independently of the others, find the required conditional probability $\mathbb{P}(A \leftrightarrow B \mid A \nleftrightarrow C)$.

Problem 6. Two fair dice are thrown. Let A be the event the the first shows an odd number, B be the event that the second shows an even number, and C be the event the either both are odd or both are even. Show that A, B, C are pairwise independent, but not independent.

Problem 7. For any collection of events $A_1, A_2, \ldots, A_n$, the following formula holds

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1 \le i \le n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{1 \le i \le n} A_i\right).$$

Take the above formula for granted (it can be proved by induction), and use it to solve the following problem: A drunken porter tries to hang a number of N keys on their hooks but only manages to do so independently and at random (one key on each hook). Find the probability that no key was hung on its own hook. Does the value of this probability tend to 0 as $N \to \infty$? Do the calculation and draw conclusions.