Homework No. 7 (Due on 04/03/14)

Problem 1. Let U and V be independent standard normal random variables, and $X = U+V$, $Y = U-2V$. Find $\mathbb{E}[X \mid Y]$, $\operatorname{cov}(X,Y)$, and the joint probability density function of X and Y.

Problem 2. Let the random variables X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} \exp\Big\{-\frac{x^2+y^2}{2\sigma^2}\Big\}.$$

Suppose that the random variables W and Q satisfy the relation

$$X\cos(\omega_0 t) + Y\sin(\omega_0 t) = W\cos(\omega_0 t + Q)\,,$$

for all time t, where ω_0 is a constant. Furthermore, suppose that $\mathbb{P}(W \geq 0) = 1$ and $\mathbb{P}(-\pi < Q \leq \pi) = 1$.

(a) Find the joint probability density function of W and Q.

(b) Find the marginal probability density functions of W and Q.

(c) Are W and Q independent?

Problem 3. Consider three zero-mean random variables X, Y, and Z, with known variances and covariances. Give a formula for the linear least squares estimator of X based on Y and Z, that is, find a and b that minimize

$$\mathbb{E}\left[(X - aY - bZ)^2\right].$$

For simplicity, assume that Y and Z are uncorrelated.

Problem 4. Let X be uniformly distributed on the interval $(1,5)$. Find the probability density function of $Y = \frac{X}{5-X}$.

Problem 5. Let X and Y denote two points that are chosen randomly and independently from the interval $[0,1]$. Find the probability density function of $Z = |X-Y|$. Use this to calculate the mean distance between X and Y.

Problem 6. Let X and Y be independent random variables, each having the exponential distribution with parameter λ. Find the joint probability density function of X and $X+Y$. Also, find the conditional probability density function of X given that $X+Y=a$.

Problem 7. The random variables $X_1, \ldots, X_n$ have common mean μ, common variance σ^2 and, furthermore, $\mathbb{E}[X_i X_j] = c$ for every pair of distinct i and j. Derive a formula for the variance of $X_1 + \cdots + X_n$, in terms of μ, σ^2, c, and n.

Problem 8. Let Y be exponentially distributed with parameter 1, and let Z be uniformly distributed over the interval $[0,1]$. Assume Y and Z are independent. Find the probability density functions of $W = Y - Z$ and that of $X = |Y - Z|$.