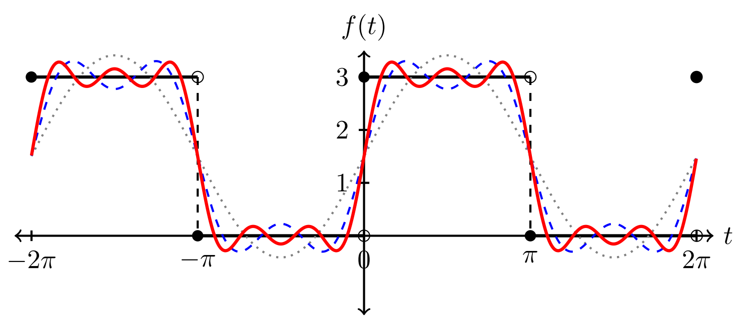
Fourier Series & Transformation

Kirkwood Donavin – Data Scientist



Contents

This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

1 The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not easily differentiated, but instead were closely approximated by a series of sine and cosine functions. A fourier series looks like this:

$$\begin{aligned} f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + ... \\ + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + ... \end{aligned} \tag{1}$$

Where t is time. Note that the frequency for each added sine/cosine term is increasing.

2 Theory of the Fourier Series

2.1 Trigonometric Identies for Fourier Series Approximation

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt)\mathrm{d}t = 0 \tag{2}$$

for *any* integer m

$$\int_0^{2\pi} \cos(mt)\mathrm{d}t = 0 \tag{3}$$

for non-zero integer m

$$\int_0^{2\pi} \sin(mt)\cdot\cos(nt)\mathrm{d}t = 0 \tag{4}$$

for *any* integers m, n

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) = 0 \tag{5}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \sin^2(mt)\mathrm{d}t = \pi \tag{6}$$

for integer $m = n \neq 0$, note this is the edge case of $m = n$ above

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) = 0 \tag{7}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \cos^2(mt)\mathrm{d}t = \pi \tag{8}$$

for integer $m = n \neq 0$

These are well known integral values, but I could use the integration review, so let us prove it. First, I will note the derivitive value of sine & cosine:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}[\cos(mt)] &= m \cdot (-\sin(mt)) \\
&= -m\sin(mt) \\
&\text{And,} \\
\frac{\mathrm{d}}{\mathrm{d}t}[\sin(mt)] &= m \cdot (\cos(mt))
\end{aligned}$$

2.1.1 Integration of Sine Function for an Arbitrary Number of Periods (m)

The following is the integration of the sine function for an arbitrary number m full (i.e., integer) periods.

$$
\begin{aligned}
\int_0^{2\pi} \sin(mt)\mathrm{d}t &= -\frac{1}{m}\int_0^{2\pi} -m\sin(mt)\mathrm{d}t \\
&= -\frac{1}{m}\left(\cos(mt)\right)\Big|_0^{2\pi} \\
&= -\frac{1}{m}\left(\cancel{\cos(m\cdot 2\pi)} - \cancel{\cos(m\cdot 0)}\right) \\
&= -\frac{1}{m}(1-1) \\
&= 0
\end{aligned}
$$

2.1.2 Integration of Cosine Function for an Arbitrary Number of Periods (m)

And, the integration of the cosine function for an arbitrary number m of full periods:

$$
\begin{aligned}
\int_0^{2\pi} \cos(mt)\mathrm{d}t &= \frac{1}{m}\int_0^{2\pi} m\cos(mt)\mathrm{d}t \\
&= \frac{1}{m}\left(\sin(mt)\right)\Big|_0^{2\pi} \\
&= \frac{1}{m}\left(\cancel{\sin(m\cdot 2\pi)} - \cancel{\sin(m\cdot 0)}\right) \\
&= -\frac{1}{m}(0-0) \\
&= 0
\end{aligned}
$$

2.1.3 Integration of the Products of Sine and Cosine Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of sine times cosine:

$$
\int_0^{2\pi} \sin(mt)\cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\sin((m+n)t) + \sin((m-n)t)]\mathrm{d}t
$$

by trigonometric identity

$$
= \frac{1}{2}\int_0^{2\pi} \sin((m+n)t)\mathrm{d}t + \frac{1}{2}\int_0^{2\pi} \sin((m-n)t)\mathrm{d}t
$$

Now, for integers $k = m+n$, and $l = m-n$:

$$
\begin{aligned}
&= \frac{1}{2}\cancel{\int_0^{2\pi} \sin(k\cdot t)\mathrm{d}t} + \frac{1}{2}\cancel{\int_0^{2\pi} \sin(l\cdot t)\mathrm{d}t} \\
&= 0
\end{aligned}
$$

By the integral identity of $\sin(mt)$ established above

2.1.4 Integration of Sine Times Sine ‘Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of sine times sine function of a different number of periods:

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)]\mathrm{d}t$$

by trigonometric identity, for $m \neq n, -n$.

$$= \frac{1}{2}\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t - \frac{1}{2}\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t$$

Now, for integer $k = m - n$ and $l = m + n$ we have previously established that:

$$\begin{aligned} &= \cancel{\frac{1}{2}\int_0^{2\pi} \cos(k \cdot t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi} \cos(l \cdot t)\mathrm{d}t} \\ &= 0 \end{aligned}$$

This holds for all $m \neq n, -n$. However, if $m = n$, then we have:

$$\begin{aligned} \int_0^{2\pi} \sin^2(mt)\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m - m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \\ &= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \\ &= \frac{1}{2} \cdot t\Big|_0^{2\pi} \\ &= \frac{1}{2}(2\pi - 0) \\ &= \pi \end{aligned}$$

2.1.5 Integration of Cosine Times Cosine Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of cosine times cosine of different number of periods (nearly identical math to the above):

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)]\mathrm{d}t$$

by trigonometric identity, for integers $m \neq n, -n$.

$$= \frac{1}{2}\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t - \frac{1}{2}\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t$$

Note that integer $k = m - n$ and $l = m + n$ we have previously established that:

$$\begin{aligned} &= \cancel{\frac{1}{2}\int_0^{2\pi} \cos(k \cdot t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi} \cos(l \cdot t)\mathrm{d}t} \\ &= 0 \end{aligned}$$

This holds for all integers $m \neq n, -n$. However, if $m = n$, then we have:

$$\begin{aligned} \int_0^{2\pi} \cos^2(mt)\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \\ &= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \\ &= \frac{1}{2} \cdot t\Big|_0^{2\pi} \\ &= \frac{1}{2}(2\pi - 0) \\ &= \pi \end{aligned}$$

3 Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

3.1 The First Term: a_0

First let us differentiate the infinite Fourier Series from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} f(t)\mathrm{d}t = \int_0^{2\pi} \big(&a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots + a_n\cos(nt) \\ &+ b_1\sin(t) + b_2\sin(2t) + \cdots + b_n\sin(nt)\big)\mathrm{d}t \end{aligned}$$

Using the integrated sine & cosine identities above:

$$
\begin{aligned}
&= \int_0^{2\pi} a_0 \mathrm{d}t + \cancel{\int_0^{2\pi} a_1 \cos(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} a_2 \cos(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} a_n \cos(nt)\mathrm{d}t} \\
&+ \cancel{\int_0^{2\pi} b_1 \sin(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} b_2 \sin(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} b_n \sin(nt)\mathrm{d}t} \\
&= a_0 \cdot t \Big|_0^{2\pi}
\end{aligned}
$$

$$\int_0^{2\pi} f(t) = a_0 \cdot 2\pi$$

Solving for a_0:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\mathrm{d}t$$

In other words, a_0 is equal to the *mean* of $f(t)$ for the integration period. This makes sense because sine and cosine functions oscillate between -1 and 1, and a_0 represents the center starting point for a Fourier Series representation of a periodic function.

3.2 The nth Coefficients: a_n & b_n

Now we will solve for the cosine coefficients (a_n for $n \in 1, 2, \ldots$). First, we multiply our Fourier Series by $\cos(nt)$:

$$
\begin{aligned}
f(t) \cdot \cos(nt) = a_0 \cdot \cos(nt) &+ a_1 \cos(t) \cdot \cos(nt) + a_2 \cos(2t) \cdot \cos(nt) + \ldots \\
&+ a_n \cos(nt) \cdot \cos(nt) + \ldots \\
&+ b_1 \sin(t) \cdot \cos(nt) + b_2 \sin(2t) \cdot \cos(nt) + \ldots \\
&+ b_n \sin(nt) \cdot \cos(nt) + \ldots
\end{aligned}
$$

Now we can integrate both sides from 0 to 2π and eliminate most terms using the trigonometric identies we built above

$$
\begin{aligned}
\int_0^{2\pi} f(t) \cdot \cos(nt)\mathrm{d}t = a_0 \cancel{\int_0^{2\pi} \cos(nt)\mathrm{d}t} &+ a_1 \cancel{\int_0^{2\pi} (\cos(t) \cdot \cos(nt))\mathrm{d}t} + a_2 \cancel{\int_0^{2\pi} (\cos(2t) \cdot \cos(nt))\mathrm{d}t} + \ldots \\
&+ a_n \int_0^{2\pi} \cos^2(nt)\mathrm{d}t + \ldots \\
&+ b_1 \cancel{\int_0^{2\pi} (\sin(t) \cdot \cos(nt))\mathrm{d}t} + b_2 \cancel{\int_0^{2\pi} (\sin(2t) \cdot \cos(nt))\mathrm{d}t} + \ldots \\
&+ b_n \cancel{\int_0^{2\pi} (\sin(nt) \cdot \cos(nt))\mathrm{d}t} + \ldots
\end{aligned}
$$

By the squared cosine integral identity established above, we have:

$$\int_0^{2\pi} f(t) \cdot \cos(nt) \mathrm{d}t = a_n \cdot \pi$$

Therfore, solving for a_n:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \cos(nt) \mathrm{d}t$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series (b_n for $n \in 1, 2, \ldots$) by multiplying each side of the series by $\sin(nt)$:

$$\begin{aligned} f(t) \cdot \sin(nt) = {} & a_0 \cdot \sin(nt) + a_1 \cos(t) \cdot \sin(nt) + a_2 \cos(2t) \cdot \sin(nt) + \ldots \\ & + a_n \cos(nt) \cdot \sin(nt) + \ldots \\ & + b_1 \sin(t) \cdot \sin(nt) + b_2 \sin(2t) \cdot \sin(nt) + \ldots \\ & + b_n \sin(nt) \cdot \sin(nt) + \ldots \end{aligned}$$

Now, as before, we can integrate both sides from 0 to 2π and eliminate most terms using the trigonometric identies we built above:

$$\begin{aligned} \int_0^{2\pi} f(t) \sin(nt) \mathrm{d}t = {} & \cancel{a_0 \int_0^{2\pi} \sin(nt) \mathrm{d}t} + \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \sin(nt)) \mathrm{d}t} + \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \sin(nt)) \mathrm{d}t} + \ldots \\ & + \cancel{a_n \int_0^{2\pi} \cos(nt) \cdot \sin(nt) \mathrm{d}t} + \ldots \\ & + \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \sin(nt)) \mathrm{d}t} + \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \sin(nt)) \mathrm{d}t} + \ldots \\ & + b_n \int_0^{2\pi} \sin^2(nt) \mathrm{d}t + \ldots \end{aligned}$$

By the squared sine integral identity established above, we have:

$$\int_0^{2\pi} f(t) \sin(nt) \mathrm{d}t = b_n \cdot \pi$$

Therfore, solving for b_n:

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \sin(nt) \mathrm{d}t$$

3.3 Solving for Fourier Coefficients: Approximation of a Step Function

Suppose we have the following periodic step function:

$$f(t) = \begin{cases} 3 & \text{if } \left\lfloor \frac{t}{\pi} \right\rfloor \text{ is even} \\ 0 & \text{if } \left\lfloor \frac{t}{\pi} \right\rfloor \text{ is odd} \end{cases}$$

This function has a period of 2π and alternates between 3 and 0 every π units. It can be visualized as follows:

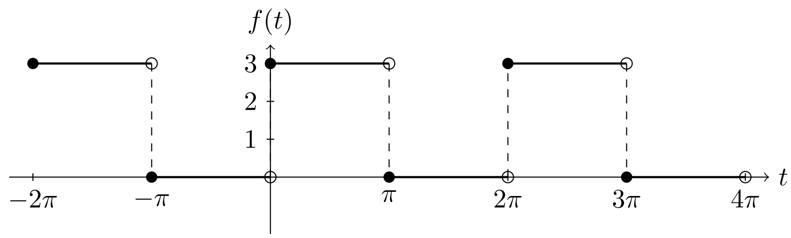


Figure 1: A periodic step function

We will compute the Fourier coefficients a_0, a_n, and b_n for this function.

3.3.1 Computing a_0

Recall:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,\mathrm{d}t$$

For our step function, $f(t) = 3$ for $0 \le t < \pi$ and $f(t) = 0$ for $\pi \le t < 2\pi$:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \left(\int_0^{\pi} 3\,\mathrm{d}t + \int_{\pi}^{2\pi} 0\,\mathrm{d}t \right) \\ &= \frac{1}{2\pi} (3 \cdot (\pi - 0) + 0) \\ &= \frac{3\pi}{2\pi} \\ &= \frac{3}{2} \end{aligned}$$

3.3.2 Computing a_n

Recall:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt)\,\mathrm{d}t$$

Again, split the integral:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left(\int_0^{\pi} 3\cos(nt)\,\mathrm{d}t + \int_{\pi}^{2\pi} 0\cdot\cos(nt)\,\mathrm{d}t\right) \\
&= \frac{3}{\pi}\int_0^{\pi}\cos(nt)\,\mathrm{d}t \\
&= \frac{3}{\pi}\left[\frac{\sin(nt)}{n}\right]_0^{\pi} \\
&= \frac{3}{\pi}\left(\frac{\sin(n\pi)-\sin(0)}{n}\right) \\
&= \frac{3}{\pi}\cdot\frac{0-0}{n} \\
&= 0
\end{aligned}
$$

So, all $a_n = 0$ for $n \geq 1$. That is, there are no cosine terms in the Fourier series for this step function. Observing the function graphically, this makes sense because the function is odd about its midpoint.

3.3.3 Computing b_n

Recall:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,\mathrm{d}t$$

Again, split the integral:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left(\int_0^{\pi} 3\sin(nt)\,\mathrm{d}t + \int_{\pi}^{2\pi} 0\cdot\sin(nt)\,\mathrm{d}t\right) \\
&= \frac{3}{\pi}\int_0^{\pi}\sin(nt)\,\mathrm{d}t \\
&= \frac{3}{\pi}\left[-\frac{\cos(nt)}{n}\right]_0^{\pi} \\
&= \frac{3}{\pi}\left(-\frac{\cos(n\pi)-\cos(0)}{n}\right) \\
&= \frac{3}{\pi}\left(-\frac{(-1)^n-1}{n}\right) \\
&= \frac{3}{\pi}\cdot\frac{1-(-1)^n}{n}
\end{aligned}
$$

Notice that $1-(-1)^n$ is 0 for even n and 2 for odd n. Thus, we can express b_n as:

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{6}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

3.3.4 Summary of Fourier Series

The Fourier series for this step function is:

$$f(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{6}{n\pi} \sin(nt)$$

3.4 First Few Terms of the Fourier Series Expansion

For $n = 1$, the Fourier approximation is:

$$f_1(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t)$$

For $n = 3$, include the next odd term:

$$f_3(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t) + \frac{2}{\pi} \sin(3t)$$

For $n = 5$, include up to the fifth term:

$$f_5(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t) + \frac{2}{\pi} \sin(3t) + \frac{6}{5\pi} \sin(5t)$$

These partial sums visualized demonstrate how the Fourier Series converges to the original step function as more terms are added:

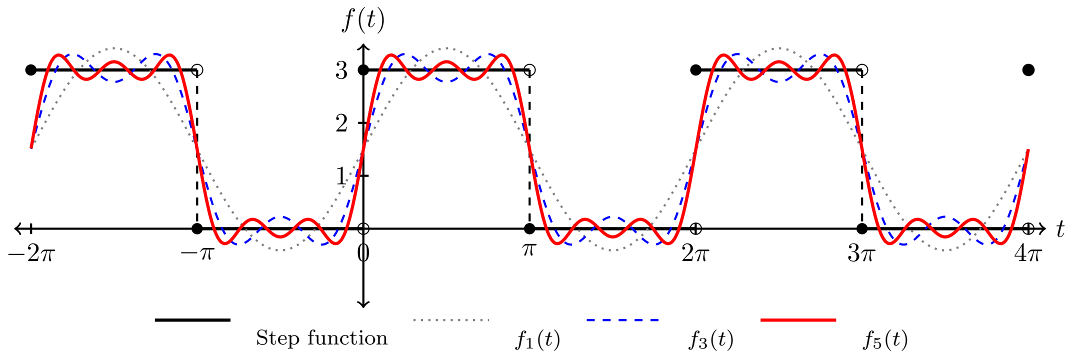


Figure 2: The periodic step function and its Fourier approximations $f_1(t)$, $f_3(t)$, and $f_5(t)$. Note how the approximation improves as more terms are added.

4 References

- Khan Academy: *Fourier Series* course.
  https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-series-intro