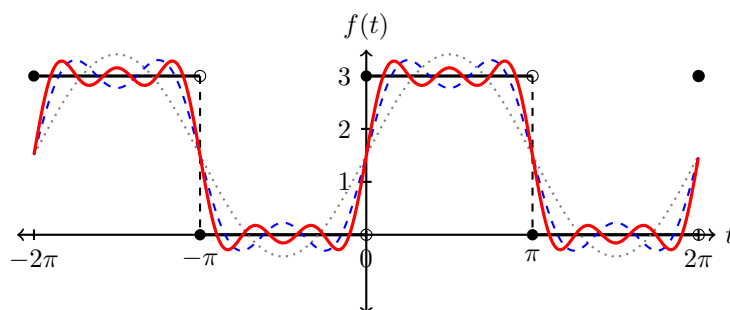


# Fourier Series & Transformation

Kirkwood Donavin – Data Scientist



## Contents

<b>1</b>	<b>The Fourier Series</b>	<b>2</b>
<b>2</b>	<b>Theory of the Fourier Series</b>	<b>2</b>
2.1	Trigonometric Identities for Fourier Series Approximation . . . . .	2
2.1.1	Integration of Sine Function for an Arbitrary Number of Periods ( $m$ ) . . . . .	3
2.1.2	Integration of Cosine Function for an Arbitrary Number of Periods ( $m$ ) . . . . .	4
2.1.3	Integration of the Products of Sine and Cosine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ ) . . . . .	4
2.1.4	Integration of Sine Times Sine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ ) . . . . .	5
2.1.5	Integration of Cosine Times Cosine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ ) . . . . .	5
<b>3</b>	<b>Derivation of Fourier Coefficients</b>	<b>6</b>
3.1	The First Term: $a_0$ . . . . .	6
3.2	The $n$ th Coefficients: $a_n$ & $b_n$ . . . . .	7
3.3	Solving for Fourier Coefficients: Approximation of a Step Function	9
3.3.1	Computing $a_0$ . . . . .	9
3.3.2	Computing $a_n$ . . . . .	9
3.3.3	Computing $b_n$ . . . . .	10
3.3.4	Summary of Fourier Series . . . . .	11

3.4 First Few Terms of the Fourier Series Expansion . . . . .	11
---	----

<b>4 References</b>	<b>12</b>
---------------------	-----------

This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

## 1 The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not easily differentiated, but instead were closely approximated by a series of sine and cosine functions. A fourier series looks like this:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \dots \quad (1)$$

$$+ b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \dots$$

Where  $t$  is time. Note that the frequency for each added sine/cosine term is increasing.

## 2 Theory of the Fourier Series

### 2.1 Trigonometric Identities for Fourier Series Approximation

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt) dt = 0 \quad (2)$$

for *any* integer  $m$

$$\int_0^{2\pi} \cos(mt) dt = 0 \quad (3)$$

for non-zero integer  $m$

$$\int_0^{2\pi} \sin(mt) \cdot \cos(nt) dt = 0 \quad (4)$$

for *any* integers  $m, n$

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) dt = 0 \quad (5)$$

for integers  $m, n$  when  $m \neq n$  or  $m \neq -n$

$$\int_0^{2\pi} \sin^2(mt) dt = \pi \quad (6)$$

for integer  $m = n \neq 0$ , note this is the edge case of  $m = n$  above

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) dt = 0 \quad (7)$$

for integers  $m, n$  when  $m \neq n$  or  $m \neq -n$

$$\int_0^{2\pi} \cos^2(mt) dt = \pi \quad (8)$$

for integer  $m = n \neq 0$

These are well known integral values, but I could use the integration review, so let us prove it. First, I will note the derivative value of sine & cosine:

$$\frac{d}{dt}[\cos(mt)] = m \cdot (-\sin(mt))$$

$$= -m \sin(mt)$$

And,

$$\frac{d}{dt}[\sin(mt)] = m \cdot (\cos(mt))$$

### 2.1.1 Integration of Sine Function for an Arbitrary Number of Periods ( $m$ )

The following is the integration of the sine function for an arbitrary number  $m$  full (i.e., integer) periods.

$$\begin{aligned}
\int_0^{2\pi} \sin(mt) dt &= -\frac{1}{m} \int_0^{2\pi} -m \sin(mt) dt \\
&= -\frac{1}{m} (\cos(mt)) \Big|_0^{2\pi} \\
&= -\frac{1}{m} (\cos(\cancel{m \cdot 2\pi}) - \cos(\cancel{m \cdot 0})) \\
&= -\frac{1}{m} (1 - 1) \\
&= 0
\end{aligned}$$

### 2.1.2 Integration of Cosine Function for an Arbitrary Number of Periods ( $m$ )

And, the integration of the cosine function for an arbitrary number  $m$  of full periods:

$$\begin{aligned}
\int_0^{2\pi} \cos(mt) dt &= \frac{1}{m} \int_0^{2\pi} m \cos(mt) dt \\
&= \frac{1}{m} (\sin(mt)) \Big|_0^{2\pi} \\
&= \frac{1}{m} (\sin(\cancel{m \cdot 2\pi}) - \sin(\cancel{m \cdot 0})) \\
&= -\frac{1}{m} (0 - 0) \\
&= 0
\end{aligned}$$

### 2.1.3 Integration of the Products of Sine and Cosine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ )

And, the integration of sine times cosine:

$$\int_0^{2\pi} \sin(mt) \cos(nt) dt = \int_0^{2\pi} \frac{1}{2} [\sin((m+n)t) + \sin((m-n)t)] dt$$

by trigonometric identity

$$= \frac{1}{2} \int_0^{2\pi} \sin((m+n)t) dt + \frac{1}{2} \int_0^{2\pi} \sin((m-n)t) dt$$

Now, for integers  $k = m + n$ , and  $l = m - n$ :

$$\begin{aligned}
&= \frac{1}{2} \int_0^{2\pi} \cancel{\sin(k \cdot t)} dt + \frac{1}{2} \int_0^{2\pi} \cancel{\sin(l \cdot t)} dt \\
&= 0
\end{aligned}$$

By the integral identity of  $\sin(mt)$  established above

#### 2.1.4 Integration of Sine Times Sine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ )

And, the integration of sine times sine function of a different number of periods:

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) dt = \int_0^{2\pi} \frac{1}{2} [\cos((m-n)t) - \cos((m+n)t)] dt$$

by trigonometric identity, for  $m \neq n, -n$ .

$$= \frac{1}{2} \int_0^{2\pi} \cos((m-n)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((m+n)t) dt$$

Now, for integer  $k = m - n$  and  $l = m + n$  we have previously established that:

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \cancel{\cos(k \cdot t) dt} - \frac{1}{2} \int_0^{2\pi} \cancel{\cos(l \cdot t) dt} \\ &= 0 \end{aligned}$$

This holds for all  $m \neq n, -n$ . However, if  $m = n$ , then we have:

$$\begin{aligned} \int_0^{2\pi} \sin^2(mt) dt &= \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m-m)t) dt} - \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m+m)t) dt} \\ &= \frac{1}{2} \int_0^{2\pi} 1 dt \\ &= \frac{1}{2} \cdot t \Big|_0^{2\pi} \\ &= \frac{1}{2} (2\pi - 0) \\ &= \pi \end{aligned}$$

#### 2.1.5 Integration of Cosine Times Cosine Functions for Arbitrary Numbers of Periods ( $m$ & $n$ )

And, the integration of cosine times cosine of different number of periods (nearly identical math to the above):

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) dt = \int_0^{2\pi} \frac{1}{2} [\cos((m-n)t) + \cos((m+n)t)] dt$$

by trigonometric identity, for integers  $m \neq n, -n$ .

$$= \frac{1}{2} \int_0^{2\pi} \cos((m-n)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((m+n)t) dt$$

Note that integer  $k = m - n$  and  $l = m + n$  we have previously established that:

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \cancel{\cos(k \cdot t) dt} - \frac{1}{2} \int_0^{2\pi} \cancel{\cos(l \cdot t) dt} \\ &= 0 \end{aligned}$$

This holds for all integers  $m \neq n, -n$ . However, if  $m = n$ , then we have:

$$\begin{aligned} \int_0^{2\pi} \cos^2(mt) dt &= \frac{1}{2} \int_0^{2\pi} \cos(\cancel{(m-m)}t) dt - \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m+m)t) dt} \\ &= \frac{1}{2} \int_0^{2\pi} 1 dt \\ &= \frac{1}{2} \cdot t \Big|_0^{2\pi} \\ &= \frac{1}{2} (2\pi - 0) \\ &= \pi \end{aligned}$$

### 3 Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

#### 3.1 The First Term: $a_0$

First let us differentiate the infinite Fourier Series from 0 to  $2\pi$ :

$$\begin{aligned} \int_0^{2\pi} f(t) dt &= \int_0^{2\pi} (a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots + a_n \cos(nt) \\ &\quad + b_1 \sin(t) + b_2 \sin(2t) + \cdots + b_n \sin(nt)) dt \end{aligned}$$

Using the integrated sine & cosine identities above:

$$\begin{aligned}
 &= \int_0^{2\pi} a_0 dt + \int_0^{2\pi} a_1 \cos(t) dt + \int_0^{2\pi} a_2 \cos(2t) dt + \dots + \int_0^{2\pi} a_n \cos(nt) dt \\
 &+ \int_0^{2\pi} b_1 \sin(t) dt + \int_0^{2\pi} b_2 \sin(2t) dt + \dots + \int_0^{2\pi} b_n \sin(nt) dt \\
 &= a_0 \cdot t \Big|_0^{2\pi} \\
 \int_0^{2\pi} f(t) dt &= a_0 \cdot 2\pi
 \end{aligned}$$

Solving for  $a_0$ :

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

In other words,  $a_0$  is equal to the *mean* of  $f(t)$  for the integration period. This makes sense because sine and cosine functions oscillate between  $-1$  and  $1$ , and  $a_0$  represents the center starting point for a Fourier Series representation of a periodic function.

### 3.2 The $n$ th Coefficients: $a_n$ & $b_n$

Now we will solve for the cosine coefficients ( $a_n$  for  $n \in 1, 2, \dots$ ). First, we multiply our Fourier Series by  $\cos(nt)$ :

$$\begin{aligned}
 f(t) \cdot \cos(nt) &= a_0 \cdot \cos(nt) + a_1 \cos(t) \cdot \cos(nt) + a_2 \cos(2t) \cdot \cos(nt) + \dots \\
 &+ a_n \cos(nt) \cdot \cos(nt) + \dots \\
 &+ b_1 \sin(t) \cdot \cos(nt) + b_2 \sin(2t) \cdot \cos(nt) + \dots \\
 &+ b_n \sin(nt) \cdot \cos(nt) + \dots
 \end{aligned}$$

Now we can integrate both sides from 0 to  $2\pi$  and eliminate most terms using the trigonometric identities we built above

$$\begin{aligned}
 \int_0^{2\pi} f(t) \cdot \cos(nt) dt &= a_0 \int_0^{2\pi} \cos(nt) dt + a_1 \int_0^{2\pi} (\cos(t) \cdot \cos(nt)) dt + a_2 \int_0^{2\pi} (\cos(2t) \cdot \cos(nt)) dt + \dots \\
 &+ a_n \int_0^{2\pi} \cos^2(nt) dt + \dots \\
 &+ b_1 \int_0^{2\pi} (\sin(t) \cdot \cos(nt)) dt + b_2 \int_0^{2\pi} (\sin(2t) \cdot \cos(nt)) dt + \dots \\
 &+ b_n \int_0^{2\pi} (\sin(nt) \cdot \cos(nt)) dt + \dots
 \end{aligned}$$

By the squared cosine integral identity established above, we have:

$$\int_0^{2\pi} f(t) \cdot \cos(nt) dt = a_n \cdot \pi$$

Therefore, solving for  $a_n$ :

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \cos(nt) dt$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series ( $b_n$  for  $n \in 1, 2, \dots$ ) by multiplying each side of the series by  $\sin(nt)$ :

$$\begin{aligned} f(t) \cdot \sin(nt) &= a_0 \cdot \sin(nt) + a_1 \cos(t) \cdot \sin(nt) + a_2 \cos(2t) \cdot \sin(nt) + \dots \\ &\quad + a_n \cos(nt) \cdot \sin(nt) + \dots \\ &\quad + b_1 \sin(t) \cdot \sin(nt) + b_2 \sin(2t) \cdot \sin(nt) + \dots \\ &\quad + b_n \sin(nt) \cdot \sin(nt) + \dots \end{aligned}$$

Now, as before, we can integrate both sides from 0 to  $2\pi$  and eliminate most terms using the trigonometric identities we built above:

$$\begin{aligned} \int_0^{2\pi} f(t) \sin(nt) dt &= \cancel{a_0 \int_0^{2\pi} \sin(nt) dt} + \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \sin(nt)) dt} + \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \sin(nt)) dt} + \dots \\ &\quad + \cancel{a_n \int_0^{2\pi} \cos(nt) \cdot \sin(nt) dt} + \dots \\ &\quad + \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \sin(nt)) dt} + \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \sin(nt)) dt} + \dots \\ &\quad + b_n \int_0^{2\pi} \sin^2(nt) dt + \dots \end{aligned}$$

By the squared sine integral identity established above, we have:

$$\int_0^{2\pi} f(t) \sin(nt) dt = b_n \cdot \pi$$

Therefore, solving for  $b_n$ :

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \sin(nt) dt$$



### 3.3 Solving for Fourier Coefficients: Approximation of a Step Function

Suppose we have the following periodic step function:

$$f(t) = \begin{cases} 3 & \text{if } \lfloor \frac{t}{\pi} \rfloor \text{ is even} \\ 0 & \text{if } \lfloor \frac{t}{\pi} \rfloor \text{ is odd} \end{cases}$$

This function has a period of  $2\pi$  and alternates between 3 and 0 every  $\pi$  units. It can be visualized as follows:

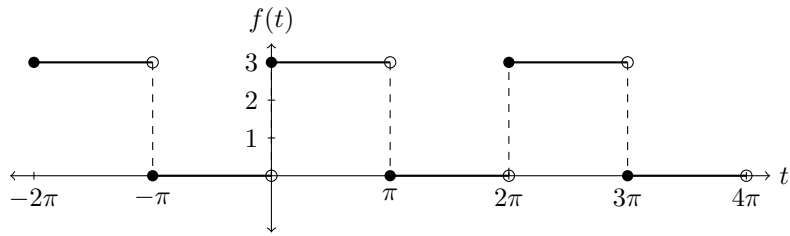


Figure 1: A periodic step function

We will compute the Fourier coefficients  $a_0$ ,  $a_n$ , and  $b_n$  for this function.

#### 3.3.1 Computing $a_0$

Recall:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$$

For our step function,  $f(t) = 3$  for  $0 \leq t < \pi$  and  $f(t) = 0$  for  $\pi \leq t < 2\pi$ :

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \left( \int_0^{\pi} 3 dt + \int_{\pi}^{2\pi} 0 dt \right) \\ &= \frac{1}{2\pi} (3 \cdot (\pi - 0) + 0) \\ &= \frac{3\pi}{2\pi} \\ &= \frac{3}{2} \end{aligned}$$

#### 3.3.2 Computing $a_n$

Recall:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt) dt$$

Again, split the integral:

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \left( \int_0^\pi 3 \cos(nt) \, dt + \int_\pi^{2\pi} 0 \cdot \cos(nt) \, dt \right) \\
 &= \frac{3}{\pi} \int_0^\pi \cos(nt) \, dt \\
 &= \frac{3}{\pi} \left[ \frac{\sin(nt)}{n} \right]_0^\pi \\
 &= \frac{3}{\pi} \left( \frac{\sin(n\pi) - \sin(0)}{n} \right) \\
 &= \frac{3}{\pi} \cdot \frac{0 - 0}{n} \\
 &= 0
 \end{aligned}$$

So, all  $a_n = 0$  for  $n \geq 1$ . That is, there are no cosine terms in the Fourier series for this step function. Observing the function graphically, this makes sense because the function is odd about its midpoint.

### 3.3.3 Computing $b_n$

Recall:

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt) \, dt$$

Again, split the integral:

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \left( \int_0^\pi 3 \sin(nt) \, dt + \int_\pi^{2\pi} 0 \cdot \sin(nt) \, dt \right) \\
 &= \frac{3}{\pi} \int_0^\pi \sin(nt) \, dt \\
 &= \frac{3}{\pi} \left[ -\frac{\cos(nt)}{n} \right]_0^\pi \\
 &= \frac{3}{\pi} \left( -\frac{\cos(n\pi) - \cos(0)}{n} \right) \\
 &= \frac{3}{\pi} \left( -\frac{(-1)^n - 1}{n} \right) \\
 &= \frac{3}{\pi} \cdot \frac{1 - (-1)^n}{n}
 \end{aligned}$$

Notice that  $1 - (-1)^n$  is 0 for even  $n$  and 2 for odd  $n$ . Thus, we can express  $b_n$  as:

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{6}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

### 3.3.4 Summary of Fourier Series

The Fourier series for this step function is:

$$f(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{6}{n\pi} \sin(nt)$$

### 3.4 First Few Terms of the Fourier Series Expansion

For  $n = 1$ , the Fourier approximation is:

$$f_1(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t)$$

For  $n = 3$ , include the next odd term:

$$f_3(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t) + \frac{2}{\pi} \sin(3t)$$

For  $n = 5$ , include up to the fifth term:

$$f_5(t) = \frac{3}{2} + \frac{6}{\pi} \sin(t) + \frac{2}{\pi} \sin(3t) + \frac{6}{5\pi} \sin(5t)$$

These partial sums visualized demonstrate how the Fourier Series converges to the original step function as more terms are added:

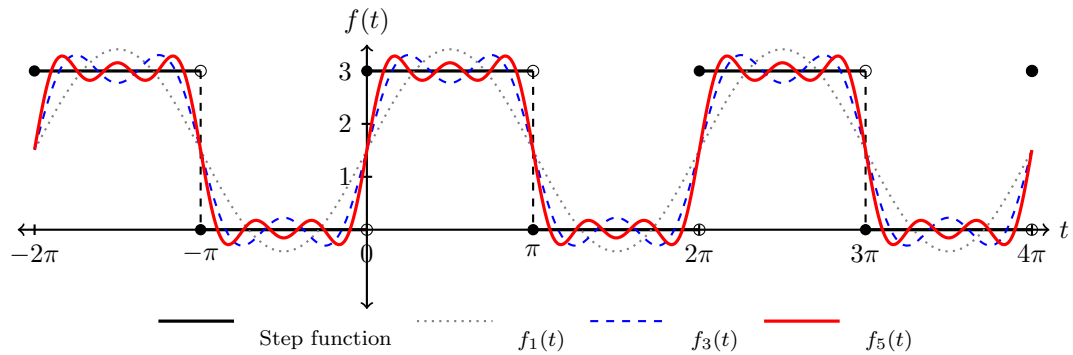


Figure 2: The periodic step function and its Fourier approximations  $f_1(t)$ ,  $f_3(t)$ , and  $f_5(t)$ . Note how the approximation improves as more terms are added.

## 4 References

- Khan Academy: *Fourier Series* course.  
<https://www.khanacademy.org/science/electrical-engineering/ee-signals/ee-fourier-series/v/ee-fourier-series-intro>