

# Fourier Series & Transformation

Kirkwood Donavin – Data Scientist

This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

## The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not so easily differentiated as a series of sine and cosine functions. A fourier series looks like this:

$$\begin{aligned} f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \dots \\ + b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \dots \end{aligned} \tag{1}$$

Where  $t$  is time. Note that the frequency for each added sine/cosine term is increasing.

## Theory

### Derivation of Trigonometric Identities

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt) dt = 0 \quad \text{for any integer } m$$

(2)

$$\int_0^{2\pi} \cos(mt) dt = 0 \quad \text{for non-zero integer } m$$

(3)

$$\int_0^{2\pi} \sin(mt) \cdot \cos(nt) dt = 0 \quad \text{for any integers } m, n$$

(4)

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) dt = 0 \quad \text{for integers } m, n \text{ when } m \neq n \text{ or } m \neq -n$$

(5)

$$\int_0^{2\pi} \sin^2(mt) dt = \pi \quad \text{for integer } m = n \neq 0, \text{ note this is the edge case of } m = n \text{ above}$$

(6)

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) dt = 0 \quad \text{for integers } m, n \text{ when } m \neq n \text{ or } m \neq -n$$

(7)

$$\int_0^{2\pi} \cos^2(mt) dt = \pi \quad \text{for integer } m = n \neq 0$$

(8)

These are well known integral values, but I could used the integration review, so let us prove it. First, let us acknowledge the derivitive value of sine & cosine:

$$\begin{aligned} \frac{d}{dt}[\cos(mt)] &= m \cdot (-\sin(mt)) \\ &= -m \sin(mt) \end{aligned}$$

And,

$$\frac{d}{dt}[\sin(mt)] = m \cdot (\cos(mt))$$

The following is the integration of sine function for an arbitrary number  $m$  of full periods.

$$\int_0^{2\pi} \sin(mt) dt = -\frac{1}{m} \int_0^{2\pi} -m \sin(mt) dt \quad (9)$$

$$= -\frac{1}{m} (\cos(mt)) \Big|_0^{2\pi} \quad (10)$$

$$= -\frac{1}{m} (\cos(\cancel{m \cdot 2\pi}) - \cos(\cancel{m \cdot 0})) \quad (11)$$

$$= -\frac{1}{m} (1 - 1) \quad (12)$$

$$= 0 \quad (13)$$

And, the integration of the cosine function for an arbitrary number  $m$  of full periods:

$$\int_0^{2\pi} \cos(mt) dt = \frac{1}{m} \int_0^{2\pi} m \cos(mt) dt \quad (14)$$

$$= \frac{1}{m} (\sin(mt)) \Big|_0^{2\pi} \quad (15)$$

$$= \frac{1}{m} (\sin(\cancel{m \cdot 2\pi}) - \sin(\cancel{m \cdot 0})) \quad (16)$$

$$= -\frac{1}{m} (0 - 0) \quad (17)$$

$$= 0 \quad (18)$$

And, the integration of sine times cosine:

$$\int_0^{2\pi} \sin(mt) \cos(nt) dt = \int_0^{2\pi} \frac{1}{2} [\sin((m+n)t) + \sin((m-n)t)] dt \quad (19)$$

by trig

$$= \frac{1}{2} \int_0^{2\pi} \sin((m+n)t) dt + \frac{1}{2} \int_0^{2\pi} \sin((m-n)t) dt \quad (20)$$

$$= \frac{1}{2} \int_0^{2\pi} \cancel{\sin(k \cdot t)} dt + \frac{1}{2} \int_0^{2\pi} \cancel{\sin(l \cdot t)} dt \quad (21)$$

where  $k = m + n$ , and  $l =$

$$= 0$$

By the integral identity of  $\sin(mt)$   
(22)

And, the integration of sine times sine of different number of periods:

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) dt = \int_0^{2\pi} \frac{1}{2} [\cos((m-n)t) - \cos((m+n)t)] dt \quad \text{by trigonometric identity, for } m \neq n, -n. \quad (23)$$

$$= \frac{1}{2} \int_0^{2\pi} \cos((m-n)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((m+n)t) dt \quad (24)$$

Note that integer  $k = m - n$  and  $l = m + n$ . Thus, for all integers  $m \neq n, -n$ : (25)

$$= 0 \quad (26)$$

However, if  $m = n$ , then we have: (27)

$$\int_0^{2\pi} \sin^2(mt) dt = \frac{1}{2} \int_0^{2\pi} \cos((m-m)t) dt - \frac{1}{2} \int_0^{2\pi} \cos((m+m)t) dt \quad (28)$$

$$= \frac{1}{2} \int_0^{2\pi} 1 dt \quad (29)$$

$$= \frac{1}{2} \cdot t \Big|_0^{2\pi} \quad (30)$$

$$= \frac{1}{2} (2\pi - 0) \quad (31)$$

$$= \pi \quad (32)$$

And, the integration of cosine times cosine of different number of periods (nearly identical math to above):

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) dt = \int_0^{2\pi} \frac{1}{2} [\cos((m-n)t) - \cos((m+n)t)] dt \quad \text{by trigonometric identity, for integers } m \quad (33)$$

$$= \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m-n)t)} dt - \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m+n)t)} dt \quad (34)$$

Note that integer  $k = m - n$  and  $l = m + n$ . Thus, for all integers  $m \neq n, -n$ : (35)

$$= 0 \quad (36)$$

However, if  $m = n$ , then we have: (37)

$$\int_0^{2\pi} \cos^2(mt) dt = \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m-m)t)} dt - \frac{1}{2} \int_0^{2\pi} \cancel{\cos((m+m)t)} dt \quad (38)$$

$$= \frac{1}{2} \int_0^{2\pi} 1 dt \quad (39)$$

$$= \frac{1}{2} \cdot t \Big|_0^{2\pi} \quad (40)$$

$$= \frac{1}{2} (2\pi - 0) \quad (41)$$

$$= \pi \quad (42)$$

## Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

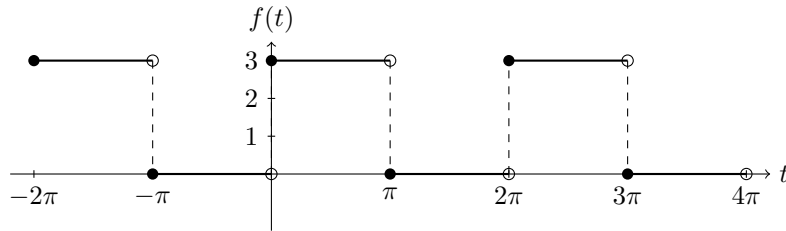


Figure 1: A periodic step function

### The First Term: $a_0$

First let us differentiate the infinite Fourier Series from 0 to  $2\pi$ :

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} (a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots + a_n \cos(nt) + b_1 \sin(t) + b_2 \sin(2t) + \cdots + b_n \sin(nt))dt \quad (43)$$

Using the integrated sine & cosine identities above: (44)

$$= \int_0^{2\pi} a_0 dt + \cancel{\int_0^{2\pi} a_1 \cos(t) dt} + \cancel{\int_0^{2\pi} a_2 \cos(2t) dt} + \cdots + \cancel{\int_0^{2\pi} a_n \cos(nt) dt} + \cancel{\int_0^{2\pi} b_1 \sin(t) dt} + \cancel{\int_0^{2\pi} b_2 \sin(2t) dt} + \cdots + \cancel{\int_0^{2\pi} b_n \sin(nt) dt} \quad (45)$$

$$= a_0 \cdot t \Big|_0^{2\pi} \quad (46)$$

$$\int_0^{2\pi} f(t) = a_0 \cdot 2\pi \quad (47)$$

Solving for  $a_0$ : (48)

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \quad (49)$$

In other words,  $a_0$  is equal to the *mean* of  $f(t)$  for the integration period. This makes sense because sine and cosine functions oscillate between  $-1$  and  $1$ , and  $a_0$  represents the center starting point for a Fourier Series representation of a periodic function.

### The $n$ th Coefficients: $a_n$ & $b_n$

Now we will solve for the cosine coefficients ( $a_n$  for  $n \in 1, 2, \dots$ ). First, we multiply our Fourier Series by  $\cos(nt)$ :

$$f(t) \cos(nt) = a_0 \cdot \cos(nt) \quad (50)$$

$$+ a_1 \cos(t) \cdot \cos(nt) \quad (51)$$

$$+ a_2 \cos(2t) \cdot \cos(nt) \quad (52)$$

$$+ \dots \quad (53)$$

$$+ a_n \cos(nt) \cdot \cos(nt) \quad (54)$$

$$+ b_1 \sin(t) \cdot \cos(nt) \quad (55)$$

$$+ b_2 \sin(2t) \cdot \cos(nt) \quad (56)$$

$$+ \dots \quad (57)$$

$$+ b_n \sin(nt) \cdot \cos(nt) \quad (58)$$

Now we can integrate both sides from 0 to  $2\pi$  and eliminate most terms using the trigonometric identities (59)

$$\int_0^{2\pi} f(t) \cos(nt) dt = a_0 \int_0^{2\pi} \cos(nt) dt \quad (60)$$

$$+ a_1 \int_0^{2\pi} (\cos(t) \cdot \cos(nt)) dt \quad (61)$$

$$+ a_2 \int_0^{2\pi} (\cos(2t) \cdot \cos(nt)) dt \quad (62)$$

$$+ \dots \quad (63)$$

$$+ a_n \int_0^{2\pi} \cos^2(nt) dt \quad (64)$$

$$+ \dots \quad (65)$$

$$+ b_1 \int_0^{2\pi} (\sin(t) \cdot \cos(nt)) dt \quad (66)$$

$$+ b_2 \int_0^{2\pi} (\sin(2t) \cdot \cos(nt)) dt \quad (67)$$

$$+ \dots \quad (68)$$

$$+ b_n \int_0^{2\pi} (\sin(nt) \cdot \cos(nt)) dt \quad (69)$$

$$= a_n \cdot \pi \quad (70)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \cos(nt) dt \quad (71)$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series ( $b_n$  for  $n \in 1, 2, \dots$ ) by multiplying each side of the series by  $\sin(nt)$ :

$$f(t) \sin(nt) = a_0 \cdot \sin(nt) \quad (72)$$

$$+ a_1 \cos(t) \cdot \sin(nt) \quad (73)$$

$$+ a_2 \cos(2t) \cdot \sin(nt) \quad (74)$$

$$+ \dots \quad (75)$$

$$+ a_n \cos(nt) \cdot \sin(nt) \quad (76)$$

$$+ \dots \quad (77)$$

$$+ b_1 \sin(t) \cdot \sin(nt) \quad (78)$$

$$+ b_2 \sin(2t) \cdot \sin(nt) \quad (79)$$

$$+ \dots \quad (80)$$

$$+ b_n \sin(nt) \cdot \sin(nt) \quad (81)$$

$$+ \dots \quad (82)$$

Now we can integrate both sides from 0 to  $2\pi$  and eliminate most terms using the trigonometric identities (83)

$$\int_0^{2\pi} f(t) \sin(nt) dt = a_0 \int_0^{2\pi} \sin(nt) dt \quad (84)$$

$$+ a_1 \int_0^{2\pi} (\cos(t) \cdot \sin(nt)) dt \quad (85)$$

$$+ a_2 \int_0^{2\pi} (\cos(2t) \cdot \sin(nt)) dt \quad (86)$$

$$+ \dots \quad (87)$$

$$+ a_n \int_0^{2\pi} \cos(nt) \cdot \sin(nt) dt \quad (88)$$

$$+ \dots \quad (89)$$

$$+ b_1 \int_0^{2\pi} (\sin(t) \cdot \sin(nt)) dt \quad (90)$$

$$+ b_2 \int_0^{2\pi} (\sin(2t) \cdot \sin(nt)) dt \quad (91)$$

$$+ \dots \quad (92)$$

$$+ b_n \int_0^{2\pi} \sin^2(nt) dt \quad (93)$$

$$+ \dots \quad (94)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \sin(nt) dt \quad (95)$$