Fourier Series & Transformation

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This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not so easily differentiated as a series of sine and cosine functions. A fourier series looks like this:

$$\begin{aligned} f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + ... \\ + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + ... \end{aligned} \tag{1}$$

Where t is time. Note that the frequency for each added sine/cosine term is increasing.

Theory

Derivation of Trigonometric Identies

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt)\mathrm{d}t = 0 \tag{2}$$

for *any* integer m

$$\int_0^{2\pi} \cos(mt)\mathrm{d}t = 0 \tag{3}$$

for non-zero integer m

$$\int_0^{2\pi} \sin(mt)\cdot\cos(nt)\mathrm{d}t = 0 \tag{4}$$

for *any* integers m, n

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) = 0 \tag{5}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \sin^2(mt)\mathrm{d}t = \pi \tag{6}$$

for integer $m = n \neq 0$, note this is the edge case of $m = n$ above

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) = 0 \tag{7}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \cos^2(mt)\mathrm{d}t = \pi \tag{8}$$

for integer $m = n \neq 0$

These are well known integral values, but I could use the integration review, so let us prove it. First, I will note the derivitive value of sine & cosine:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}[\cos(mt)] &= m \cdot (-\sin(mt)) \\
&= -m\sin(mt) \\
&\text{And,} \\
\frac{\mathrm{d}}{\mathrm{d}t}[\sin(mt)] &= m \cdot (\cos(mt))
\end{aligned}$$

The following is the integration of sine function for an arbitrary number m of full periods.

$$\begin{aligned}
\int_0^{2\pi} \sin(mt)\mathrm{d}t &= -\frac{1}{m}\int_0^{2\pi} -m\sin(mt)\mathrm{d}t \\
&= -\frac{1}{m}\left(\cos(mt)\right)\Big|_0^{2\pi} \\
&= -\frac{1}{m}\left(\cancel{\cos(m \cdot 2\pi)} - \cancel{\cos(m \cdot 0)}\right) \\
&= -\frac{1}{m}(1 - 1) \\
&= 0
\end{aligned}$$

And, the integration of the cosine function for an arbitrary number m of full periods:

$$\begin{aligned}\int_0^{2\pi} \cos(mt)\mathrm{d}t &= \frac{1}{m}\int_0^{2\pi} m\cos(mt)\mathrm{d}t \\ &= \frac{1}{m}\left(\sin(mt)\right)\Big|_0^{2\pi} \\ &= \frac{1}{m}\left(\cancel{\sin(m\cdot 2\pi)} - \cancel{\sin(m\cdot 0)}\right) \\ &= -\frac{1}{m}(0-0) \\ &= 0\end{aligned}$$

And, the integration of sine times cosine:

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\cos(nt)\mathrm{d}t &= \int_0^{2\pi} \frac{1}{2}[\sin((m+n)t)+\sin((m-n)t)]\mathrm{d}t && \text{by trigo} \\ &= \frac{1}{2}\int_0^{2\pi}\sin((m+n)t)\mathrm{d}t + \frac{1}{2}\int_0^{2\pi}\sin((m-n)t)\mathrm{d}t \\ &= \cancel{\frac{1}{2}\int_0^{2\pi}\sin(k\cdot t)\mathrm{d}t} + \cancel{\frac{1}{2}\int_0^{2\pi}\sin(l\cdot t)\mathrm{d}t} && \text{where } k=m+n, \text{ and } l=m \\ &= 0 && \text{By the integral identity of } \sin(mt)\end{aligned}$$

And, the integration of sine times sine of different number of periods:

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\cdot\sin(nt)\mathrm{d}t &= \int_0^{2\pi}\frac{1}{2}[\cos((m-n)t)-\cos((m+n)t)\mathrm{d}t && \text{by trigonometric identity, for } m\neq n,-n. \\ &= \cancel{\frac{1}{2}\int_0^{2\pi}\cos((m-n)t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi}\cos((m+n)t)\mathrm{d}t}\end{aligned}$$

Note that integer $k = m-n$ and $l = m+n$. Thus, for all integers $m \neq n, -n$:

$= 0$

However, if $m = n$, then we have:

$$\begin{aligned}\int_0^{2\pi}\sin^2(mt)\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi}\cos((\cancel{m-m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi}\cos((m+m)t)\mathrm{d}t} \\ &= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \\ &= \frac{1}{2}\cdot t\Big|_0^{2\pi} \\ &= \frac{1}{2}(2\pi - 0) \\ &= \pi\end{aligned}$$

And, the integration of cosine times cosine of different number of periods (nearly identical math to above):

$$\int_0^{2\pi} \cos(mt)\cdot\cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)\mathrm{d}t \quad \text{by trigonometric identity, for integers } m$$

$$= \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t} - \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t}$$

Note that integer $k = m - n$ and $l = m + n$. Thus, for all integers $m \neq n, -n$:

$$= 0$$

However, if $m = n$, then we have:

$$\int_0^{2\pi} \cos^2(mt)\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t}$$

$$= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t$$

$$= \frac{1}{2}\cdot t\Big|_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0)$$

$$= \pi$$

Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

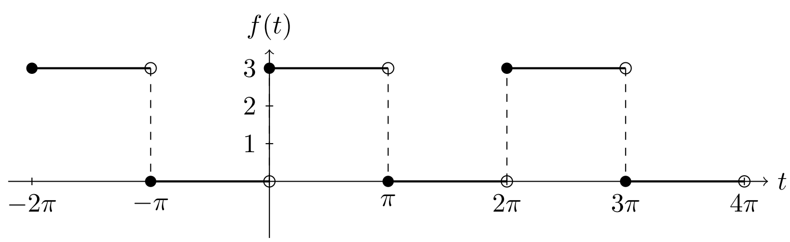

Figure 1: A periodic step function

The First Term: a_0

First let us differentiate the infinite Fourier Series from 0 to 2π:

$$\int_0^{2\pi} f(t)\mathrm{d}t = \int_0^{2\pi} (a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots + a_n\cos(nt) + b_1\sin(t) + b_2\sin(2t) + \cdots + b_n\sin(nt))\mathrm{d}t \tag{9}$$

$$\text{Using the integrated sine \& cosine identities above:} \tag{10}$$

$$= \int_0^{2\pi} a_0\mathrm{d}t + \cancel{\int_0^{2\pi} a_1\cos(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} a_2\cos(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} a_n\cos(nt)\mathrm{d}t} + \cancel{\int_0^{2\pi} b_1\sin(t)\mathrm{d}t} + \int_0^{2\pi} \tag{11}$$

$$= a_0 \cdot t\Big|_0^{2\pi} \tag{12}$$

$$\int_0^{2\pi} f(t) = a_0 \cdot 2\pi \tag{13}$$

$$\text{Solving for } a_0: \tag{14}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t \tag{15}$$

In other words, a_0 is equal to the *mean* of $f(t)$ for the integration period. This makes sense because sine and cosine functions oscillate between -1 and 1, and a_0 represents the center starting point for a Fourier Series representation of a periodic function.

The nth Coefficients: a_n & b_n

Now we will solve for the cosine coefficients (a_n for $n \in 1, 2, \ldots$). First, we multiply our Fourier Series by $\cos(nt)$:

$$
\begin{aligned}
f(t)\cos(nt) &= a_0 \cdot \cos(nt) \tag{16}\\
&+ a_1 \cos(t) \cdot \cos(nt) \tag{17}\\
&+ a_2 \cos(2t) \cdot \cos(nt) \tag{18}\\
&+ \dots \tag{19}\\
&+ a_n \cos(nt) \cdot \cos(nt) \tag{20}\\
&+ b_1 \sin(t) \cdot \cos(nt) \tag{21}\\
&+ b_2 \sin(2t) \cdot \cos(nt) \tag{22}\\
&+ \dots \tag{23}\\
&+ b_n \sin(nt) \cdot \cos(nt) \tag{24}
\end{aligned}
$$

$$\text{Now we can integrate both sides from 0 to } 2\pi \text{ and eliminate most terms using the trigonom} \tag{25}$$

$$
\begin{aligned}
\int_0^{2\pi} f(t)\cos(nt)\mathrm{d}t &= \cancel{a_0 \int_0^{2\pi} \cos(nt)\mathrm{d}t} \tag{26}\\
&+ \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \cos(nt))\mathrm{d}t} \tag{27}\\
&+ \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \cos(nt))\mathrm{d}t} \tag{28}\\
&+ \dots \tag{29}\\
&+ a_n \int_0^{2\pi} \cos^2(nt)\mathrm{d}t \tag{30}\\
&+ \dots \tag{31}\\
&+ \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \cos(nt))\mathrm{d}t} \tag{32}\\
&+ \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \cos(nt))\mathrm{d}t} \tag{33}\\
&+ \dots \tag{34}\\
&+ \cancel{b_n \int_0^{2\pi} (\sin(nt) \cdot \cos(nt))\mathrm{d}t} \tag{35}\\
&= a_n \cdot \pi \tag{36}\\
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \cos(nt)\mathrm{d}t \tag{37}
\end{aligned}
$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series (b_n for $n \in 1, 2, \dots$) by multiplying each side of the series by $\sin(nt)$:

$$f(t)\sin(nt) = a_0 \cdot \sin(nt) \tag{38}$$
$$+ a_1 \cos(t) \cdot \sin(nt) \tag{39}$$
$$+ a_2 \cos(2t) \cdot \sin(nt) \tag{40}$$
$$+ \dots \tag{41}$$
$$+ a_n \cos(nt) \cdot \sin(nt) \tag{42}$$
$$+ \dots \tag{43}$$
$$+ b_1 \sin(t) \cdot \sin(nt) \tag{44}$$
$$+ b_2 \sin(2t) \cdot \sin(nt) \tag{45}$$
$$+ \dots \tag{46}$$
$$+ b_n \sin(nt) \cdot \sin(nt) \tag{47}$$
$$+ \dots \tag{48}$$

Now we can integrate both sides from 0 to 2π and eliminate most terms using the trigonom (49)

$$\int_0^{2\pi} f(t)\sin(nt)\mathrm{d}t = \cancel{a_0 \int_0^{2\pi} \sin(nt)\mathrm{d}t} \tag{50}$$
$$+ \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \sin(nt))\mathrm{d}t} \tag{51}$$
$$+ \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \sin(nt))\mathrm{d}t} \tag{52}$$
$$+ \dots \tag{53}$$
$$+ \cancel{a_n \int_0^{2\pi} \cos(nt) \cdot \sin(nt)\mathrm{d}t} \tag{54}$$
$$+ \dots \tag{55}$$
$$+ \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \sin(nt))\mathrm{d}t} \tag{56}$$
$$+ \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \sin(nt))\mathrm{d}t} \tag{57}$$
$$+ \dots \tag{58}$$
$$+ b_n \int_0^{2\pi} \sin^2(nt)\mathrm{d}t \tag{59}$$
$$+ \dots \tag{60}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \sin(nt)\mathrm{d}t \tag{61}$$