Fourier Series & Transformation

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This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not easily differentiated, but instead were closely approximated by a series of sine and cosine functions. A fourier series looks like this:

$$\begin{aligned} f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + ... \\ + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + ... \end{aligned} \tag{1}$$

Where t is time. Note that the frequency for each added sine/cosine term is increasing.

Theory

Trigonometric Identies for Fourier Series Approximation

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt)\mathrm{d}t = 0 \tag{2}$$

for *any* integer m

$$\int_0^{2\pi} \cos(mt)\mathrm{d}t = 0 \tag{3}$$

for non-zero integer m

$$\int_0^{2\pi} \sin(mt) \cdot \cos(nt) \mathrm{d}t = 0 \tag{4}$$

for *any* integers m, n

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) = 0 \tag{5}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \sin^2(mt) \mathrm{d}t = \pi \tag{6}$$

for integer $m = n \neq 0$, note this is the edge case of $m = n$ above

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) = 0 \tag{7}$$

for integers m, n when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \cos^2(mt) \mathrm{d}t = \pi \tag{8}$$

for integer $m = n \neq 0$

These are well known integral values, but I could use the integration review, so let us prove it. First, I will note the derivitive value of sine & cosine:

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}[\cos(mt)] = m \cdot (-\sin(mt)) \\
= -m \sin(mt) \\
\text{And,} \\
\frac{\mathrm{d}}{\mathrm{d}t}[\sin(mt)] = m \cdot (\cos(mt))
\end{gathered}$$

Integration of Sine Function for an Arbitrary Number of Periods (m)

The following is the integration of the sine function for an arbitrary number m full (i.e., integer) periods.

$$
\begin{aligned}
\int_0^{2\pi} \sin(mt)\mathrm{d}t &= -\frac{1}{m}\int_0^{2\pi} -m\sin(mt)\mathrm{d}t \\
&= -\frac{1}{m}\left(\cos(mt)\right)\Big|_0^{2\pi} \\
&= -\frac{1}{m}\left(\cancel{\cos(m\cdot 2\pi)} - \cancel{\cos(m\cdot 0)}\right) \\
&= -\frac{1}{m}(1-1) \\
&= 0
\end{aligned}
$$

Integration of Cosine Function for an Arbitrary Number of Periods (m)

And, the integration of the cosine function for an arbitrary number m of full periods:

$$
\begin{aligned}
\int_0^{2\pi} \cos(mt)\mathrm{d}t &= \frac{1}{m}\int_0^{2\pi} m\cos(mt)\mathrm{d}t \\
&= \frac{1}{m}\left(\sin(mt)\right)\Big|_0^{2\pi} \\
&= \frac{1}{m}\left(\cancel{\sin(m\cdot 2\pi)} - \cancel{\sin(m\cdot 0)}\right) \\
&= -\frac{1}{m}(0-0) \\
&= 0
\end{aligned}
$$

Integration of the Products of Sine and Cosine Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of sine times cosine:

$$
\int_0^{2\pi} \sin(mt)\cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\sin((m+n)t) + \sin((m-n)t)]\mathrm{d}t
$$

by trigonometric identity

$$
= \frac{1}{2}\int_0^{2\pi} \sin((m+n)t)\mathrm{d}t + \frac{1}{2}\int_0^{2\pi} \sin((m-n)t)\mathrm{d}t
$$

Now, for integers $k = m+n$, and $l = m-n$:

$$
\begin{aligned}
&= \frac{1}{2}\cancel{\int_0^{2\pi} \sin(k\cdot t)\mathrm{d}t} + \frac{1}{2}\cancel{\int_0^{2\pi} \sin(l\cdot t)\mathrm{d}t} \\
&= 0
\end{aligned}
$$

By the integral identity of $\sin(mt)$ established above

Integration of Sine Times Sine 'Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of sine times sine function of a different number of periods:

$$\int_0^{2\pi} \sin(mt)\cdot\sin(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)]\mathrm{d}t$$

by trigonometric identity, for $m \neq n, -n$.

$$= \frac{1}{2}\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t - \frac{1}{2}\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t$$

Now, for integer $k = m - n$ and $l = m + n$ we have previously established that:

$$\begin{aligned} &= \cancel{\frac{1}{2}\int_0^{2\pi} \cos(k\cdot t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi} \cos(l\cdot t)\mathrm{d}t} \\ &= 0 \end{aligned}$$

This holds for all $m \neq n, -n$. However, if $m = n$, then we have:

$$\begin{aligned} \int_0^{2\pi} \sin^2(mt)\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \\ &= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \\ &= \frac{1}{2}\cdot t\Big|_0^{2\pi} \\ &= \frac{1}{2}(2\pi - 0) \\ &= \pi \end{aligned}$$

Integration of Cosine Times Cosine Functions for Arbitrary Numbers of Periods (m & n)

And, the integration of cosine times cosine of different number of periods (nearly identical math to the above):

$$\int_0^{2\pi} \cos(mt)\cdot\cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)]\mathrm{d}t$$

by trigonometric identity, for integers $m \neq n, -n$.

$$= \frac{1}{2}\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t - \frac{1}{2}\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t$$

Note that integer $k = m - n$ and $l = m + n$ we have previously established that:

$$\begin{aligned}
&= \cancel{\frac{1}{2}\int_0^{2\pi} \cos(k \cdot t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi} \cos(l \cdot t)\mathrm{d}t} \\
&= 0
\end{aligned}$$

This holds for all integers $m \neq n, -n$. However, if $m = n$, then we have:

$$\begin{aligned}
\int_0^{2\pi} \cos^2(mt)\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \\
&= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \\
&= \frac{1}{2} \cdot t\Big|_0^{2\pi} \\
&= \frac{1}{2}(2\pi - 0) \\
&= \pi
\end{aligned}$$

Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

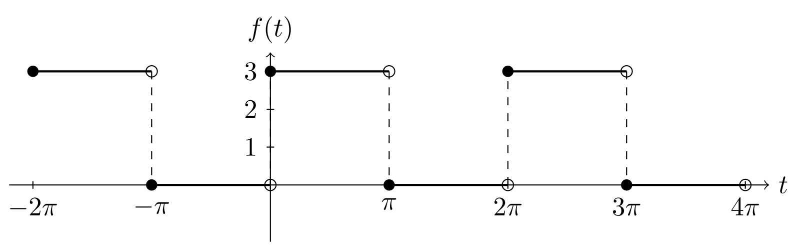

Figure 1: A periodic step function

The First Term: a_0

First let us differentiate the infinite Fourier Series from 0 to 2π:

$$
\begin{aligned}
\int_0^{2\pi} f(t)\mathrm{d}t = \int_0^{2\pi} \big(&a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots + a_n\cos(nt) \\
&+ b_1\sin(t) + b_2\sin(2t) + \cdots + b_n\sin(nt)\big)\mathrm{d}t
\end{aligned}
$$

Using the integrated sine & cosine identities above:

$$
\begin{aligned}
&= \int_0^{2\pi} a_0\mathrm{d}t + \cancel{\int_0^{2\pi} a_1\cos(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} a_2\cos(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} a_n\cos(nt)\mathrm{d}t} \\
&+ \cancel{\int_0^{2\pi} b_1\sin(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} b_2\sin(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} b_n\sin(nt)\mathrm{d}t} \\
&= a_0 \cdot t\Big|_0^{2\pi} \\
\int_0^{2\pi} f(t) &= a_0 \cdot 2\pi
\end{aligned}
$$

Solving for a_0:

$$
a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t
$$

In other words, a_0 is equal to the *mean* of $f(t)$ for the integration period. This makes sense because sine and cosine functions oscillate between -1 and 1, and a_0 represents the center starting point for a Fourier Series representation of a periodic function.

The nth Coefficients: a_n & b_n

Now we will solve for the cosine coefficients (a_n for $n \in 1, 2, \ldots$). First, we multiply our Fourier Series by $\cos(nt)$:

$$
\begin{aligned}
f(t)\cdot\cos(nt) = \ &a_0\cdot\cos(nt) + a_1\cos(t)\cdot\cos(nt) + a_2\cos(2t)\cdot\cos(nt) + \ldots \\
&+ a_n\cos(nt)\cdot\cos(nt) + \ldots \\
&+ b_1\sin(t)\cdot\cos(nt) + b_2\sin(2t)\cdot\cos(nt) + \ldots \\
&+ b_n\sin(nt)\cdot\cos(nt) + \ldots
\end{aligned}
$$

Now we can integrate both sides from 0 to 2π and eliminate most terms using the trigonometric identies we built above

$$\begin{aligned}
\int_0^{2\pi} f(t)\cdot\cos(nt)\mathrm{d}t = a_0 &\cancel{\int_0^{2\pi}\cos(nt)\mathrm{d}t} + a_1 \cancel{\int_0^{2\pi}(\cos(t)\cdot\cos(nt))\mathrm{d}t} + a_2 \cancel{\int_0^{2\pi}(\cos(2t)\cdot\cos(nt))\mathrm{d}t} + \dots \\
&+ a_n \int_0^{2\pi}\cos^2(nt)\mathrm{d}t + \dots \\
&+ b_1 \cancel{\int_0^{2\pi}(\sin(t)\cdot\cos(nt))\mathrm{d}t} + b_2 \cancel{\int_0^{2\pi}(\sin(2t)\cdot\cos(nt))\mathrm{d}t} + \dots \\
&+ b_n \cancel{\int_0^{2\pi}(\sin(nt)\cdot\cos(nt))\mathrm{d}t} + \dots
\end{aligned}$$

By the squared cosine integral identity established above, we have:

$$\int_0^{2\pi} f(t)\cdot\cos(nt)\mathrm{d}t = a_n\cdot\pi$$

Therfore, solving for a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cdot\cos(nt)\mathrm{d}t$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series (b_n for $n \in 1, 2, \dots$) by multiplying each side of the series by $\sin(nt)$:

$$\begin{aligned}
f(t)\cdot\sin(nt) = a_0 &\cdot \sin(nt) + a_1\cos(t)\cdot\sin(nt) + a_2\cos(2t)\cdot\sin(nt) + \dots \\
&+ a_n\cos(nt)\cdot\sin(nt) + \dots \\
&+ b_1\sin(t)\cdot\sin(nt) + b_2\sin(2t)\cdot\sin(nt) + \dots \\
&+ b_n\sin(nt)\cdot\sin(nt) + \dots
\end{aligned}$$

Now, as before, we can integrate both sides from 0 to 2π and eliminate most terms using the trigonometric identies we built above:

$$\begin{aligned}
\int_0^{2\pi} f(t)\sin(nt)\mathrm{d}t = a_0 &\cancel{\int_0^{2\pi}\sin(nt)\mathrm{d}t} + a_1 \cancel{\int_0^{2\pi}(\cos(t)\cdot\sin(nt))\mathrm{d}t} + a_2 \cancel{\int_0^{2\pi}(\cos(2t)\cdot\sin(nt))\mathrm{d}t} + \dots \\
&+ a_n \cancel{\int_0^{2\pi}\cos(nt)\cdot\sin(nt)\mathrm{d}t} + \dots \\
&+ b_1 \cancel{\int_0^{2\pi}(\sin(t)\cdot\sin(nt))\mathrm{d}t} + b_2 \cancel{\int_0^{2\pi}(\sin(2t)\cdot\sin(nt))\mathrm{d}t} + \dots \\
&+ b_n \int_0^{2\pi}\sin^2(nt)\mathrm{d}t + \dots
\end{aligned}$$

By the squared sine integral identity established above, we have:

$$\int_0^{2\pi} f(t)\sin(nt)\mathrm{d}t = b_n \cdot \pi$$

Therfore, solving for b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cdot \sin(nt)\mathrm{d}t$$