# Fourier Series & Transformation

Kirkwood Donavin – Data Scientist

This document is a primer on the Fourier Series and how it is used to transform complex periodic functions into a useful Fourier approximation. These notes are for personal use but may be useful to others as well.

## The Fourier Series

A Fourier series is a way to represent a *periodic* (e.g., seasonal) function as a sum of *weighted* sine and cosine waves. They were first used by Joseph Fourier to find solutions to periodic functions that are not so easily differentiated as a series of sine and cosine functions. A fourier series looks like this:

$$\begin{aligned} f(t) = a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + ... \\ + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + ... \end{aligned} \tag{1}$$

Where $t$ is time. Note that the frequency for each added sine/cosine term is increasing.

## Theory

### Derivation of Trigonometric Identies

First, let us establish some trigonometric integration identities regarding these wave functions.

$$\int_0^{2\pi} \sin(mt)\mathrm{d}t = 0 \tag{2}$$

for *any* integer $m$

$$\int_0^{2\pi} \cos(mt)\mathrm{d}t = 0 \tag{3}$$

for non-zero integer $m$

$$\int_0^{2\pi} \sin(mt)\cdot\cos(nt)\mathrm{d}t = 0 \tag{4}$$

for *any* integers $m, n$

$$\int_0^{2\pi} \sin(mt) \cdot \sin(nt) = 0 \tag{5}$$

for integers $m, n$ when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \sin^2(mt)\mathrm{d}t = \pi \tag{6}$$

for integer $m = n \neq 0$, note this is the edge case of $m = n$ above

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt) = 0 \tag{7}$$

for integers $m, n$ when $m \neq n$ or $m \neq -n$

$$\int_0^{2\pi} \cos^2(mt)\mathrm{d}t = \pi \tag{8}$$

for integer $m = n \neq 0$

These are well known integral values, but I could used the integration review, so let us prove it. First, let us acknowledge the derivitive value of sine & cosine:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\cos(mt)] &= m \cdot (-\sin(mt)) \\ &= -m\sin(mt) \\ &\text{And,} \\ \frac{\mathrm{d}}{\mathrm{d}t}[\sin(mt)] &= m \cdot (\cos(mt))\end{aligned}$$

The following is the integration of sine function for an arbitrary number $m$ of full periods.

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\mathrm{d}t &= -\frac{1}{m}\int_0^{2\pi} -m\sin(mt)\mathrm{d}t \quad (9) \\ &= -\frac{1}{m}\left(\cos(mt)\right)\Big|_0^{2\pi} \quad (10) \\ &= -\frac{1}{m}\left(\cancel{\cos(m \cdot 2\pi)} - \cancel{\cos(m \cdot 0)}\right) \quad (11) \\ &= -\frac{1}{m}(1-1) \quad (12) \\ &= 0 \quad (13)\end{aligned}$$

And, the integration of the cosine function for an arbitrary number $m$ of full periods:

$$
\begin{aligned}
\int_0^{2\pi} \cos(mt)\mathrm{d}t &= \frac{1}{m}\int_0^{2\pi} m\cos(mt)\mathrm{d}t \qquad (14)\\
&= \frac{1}{m}\left(\sin(mt)\right)\Big|_0^{2\pi} \qquad (15)\\
&= \frac{1}{m}\left(\cancel{\sin(m\cdot 2\pi)} - \cancel{\sin(m\cdot 0)}\right) \qquad (16)\\
&= -\frac{1}{m}\left(0-0\right) \qquad (17)\\
&= 0 \qquad (18)
\end{aligned}
$$

And, the integration of sine times cosine:

$$
\begin{aligned}
\int_0^{2\pi} \sin(mt)\cos(nt)\mathrm{d}t &= \int_0^{2\pi} \frac{1}{2}[\sin((m+n)t) + \sin((m-n)t)]\mathrm{d}t \qquad \text{by trig} \qquad (19)\\
&= \frac{1}{2}\int_0^{2\pi} \sin((m+n)t)\mathrm{d}t + \frac{1}{2}\int_0^{2\pi} \sin((m-n)t)\mathrm{d}t \qquad (20)\\
&= \frac{1}{2}\cancel{\int_0^{2\pi} \sin(k\cdot t)\mathrm{d}t} + \frac{1}{2}\cancel{\int_0^{2\pi} \sin(l\cdot t)\mathrm{d}t} \qquad \text{where } k = m+n\text{, and } l = \qquad (21)\\
&= 0 \qquad \text{By the integral identity of } \sin(mt \qquad (22)
\end{aligned}
$$

And, the integration of sine times sine of different number of periods:

$$\int_0^{2\pi} \sin(mt)\cdot\sin(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)\mathrm{d}t \quad \text{by trigonometric identity, for } m \neq n, -n. \tag{23}$$

$$= \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t} - \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t} \tag{24}$$

Note that integer $k = m - n$ and $l = m + n$. Thus, for all integers $m \neq n, -n$: (25)

$$= 0 \tag{26}$$

However, if $m = n$, then we have: (27)

$$\int_0^{2\pi} \sin^2(mt)\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \frac{1}{2}\cancel{\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \tag{28}$$

$$= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \tag{29}$$

$$= \frac{1}{2}\cdot t\Big|_0^{2\pi} \tag{30}$$

$$= \frac{1}{2}(2\pi - 0) \tag{31}$$

$$= \pi \tag{32}$$

And, the integration of cosine times cosine of different number of periods (nearly identical math to above):

$$\int_0^{2\pi} \cos(mt) \cdot \cos(nt)\mathrm{d}t = \int_0^{2\pi} \frac{1}{2}[\cos((m-n)t) - \cos((m+n)t)\mathrm{d}t \quad \text{by trigonometric identity, for integers } m \tag{33}$$

$$= \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m-n)t)\mathrm{d}t} - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+n)t)\mathrm{d}t} \tag{34}$$

Note that integer $k = m - n$ and $l = m + n$. Thus, for all integers $m \neq n, -n$: (35)

$$= 0 \tag{36}$$

However, if $m = n$, then we have: (37)

$$\int_0^{2\pi} \cos^2(mt)\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \cos((\cancel{m-m})t)\mathrm{d}t - \cancel{\frac{1}{2}\int_0^{2\pi} \cos((m+m)t)\mathrm{d}t} \tag{38}$$

$$= \frac{1}{2}\int_0^{2\pi} 1\mathrm{d}t \tag{39}$$

$$= \frac{1}{2} \cdot t\Big|_0^{2\pi} \tag{40}$$

$$= \frac{1}{2}(2\pi - 0) \tag{41}$$

$$= \pi \tag{42}$$

## Derivation of Fourier Coefficients

Let us begin by solving for the first term in the Fourier Series for a periodic step function.

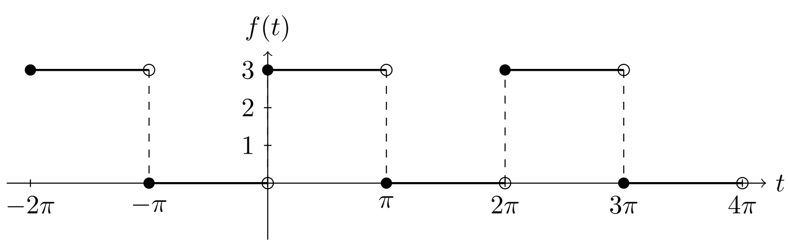


Figure 1: A periodic step function

### The First Term: $a_0$

First let us differentiate the infinite Fourier Series from 0 to $2\pi$:

$$\int_0^{2\pi} f(t)\mathrm{d}t = \int_0^{2\pi} (a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots + a_n\cos(nt) + b_1\sin(t) + b_2\sin(2t) + \cdots + b_n\sin(nt))\mathrm{d}t \tag{43}$$

Using the integrated sine & cosine identities above: (44)

$$= \int_0^{2\pi} a_0\mathrm{d}t + \cancel{\int_0^{2\pi} a_1\cos(t)\mathrm{d}t} + \cancel{\int_0^{2\pi} a_2\cos(2t)\mathrm{d}t} + \cdots + \cancel{\int_0^{2\pi} a_n\cos(nt)\mathrm{d}t} + \cancel{\int_0^{2\pi} b_1\sin(t)\mathrm{d}t} + \int_0^{2\pi} \tag{45}$$

$$= a_0 \cdot t\Big|_0^{2\pi} \tag{46}$$

$$\int_0^{2\pi} f(t) = a_0 \cdot 2\pi \tag{47}$$

Solving for $a_0$: (48)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t \tag{49}$$

In other words, $a_0$ is equal to the *mean* of $f(t)$ for the integration period. This makes sense because sine and cosine functions oscillate between $-1$ and 1, and $a_0$ represents the center starting point for a Fourier Series representation of a periodic function.

## The $n$th Coefficients: $a_n$ & $b_n$

Now we will solve for the cosine coefficients ($a_n$ for $n \in 1, 2, \ldots$). First, we multiply our Fourier Series by $\cos(nt)$:

$$
\begin{aligned}
f(t)\cos(nt) &= a_0 \cdot \cos(nt) && (50)\\
&+ a_1 \cos(t) \cdot \cos(nt) && (51)\\
&+ a_2 \cos(2t) \cdot \cos(nt) && (52)\\
&+ \dots && (53)\\
&+ a_n \cos(nt) \cdot \cos(nt) && (54)\\
&+ b_1 \sin(t) \cdot \cos(nt) && (55)\\
&+ b_2 \sin(2t) \cdot \cos(nt) && (56)\\
&+ \dots && (57)\\
&+ b_n \sin(nt) \cdot \cos(nt) && (58)
\end{aligned}
$$

Now we can integrate both sides from 0 to $2\pi$ and eliminate most terms using the trigonom

(59)

$$
\begin{aligned}
\int_0^{2\pi} f(t)\cos(nt)\mathrm{d}t &= \cancel{a_0 \int_0^{2\pi} \cos(nt)\mathrm{d}t} && (60)\\
&+ \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \cos(nt))\mathrm{d}t} && (61)\\
&+ \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \cos(nt))\mathrm{d}t} && (62)\\
&+ \dots && (63)\\
&+ a_n \int_0^{2\pi} \cos^2(nt)\mathrm{d}t && (64)\\
&+ \dots && (65)\\
&+ \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \cos(nt))\mathrm{d}t} && (66)\\
&+ \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \cos(nt))\mathrm{d}t} && (67)\\
&+ \dots && (68)\\
&+ \cancel{b_n \int_0^{2\pi} (\sin(nt) \cdot \cos(nt))\mathrm{d}t} && (69)\\
&= a_n \cdot \pi && (70)\\
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \cos(nt)\mathrm{d}t && (71)
\end{aligned}
$$

Similarly, we can solve for the sine coefficients of the infinite Fourier series ($b_n$ for $n \in 1, 2, \dots$) by multiplying each side of the series by $\sin(nt)$:

$$
\begin{aligned}
f(t)\sin(nt) &= a_0 \cdot \sin(nt) && (72)\\
&+ a_1 \cos(t) \cdot \sin(nt) && (73)\\
&+ a_2 \cos(2t) \cdot \sin(nt) && (74)\\
&+ \dots && (75)\\
&+ a_n \cos(nt) \cdot \sin(nt) && (76)\\
&+ \dots && (77)\\
&+ b_1 \sin(t) \cdot \sin(nt) && (78)\\
&+ b_2 \sin(2t) \cdot \sin(nt) && (79)\\
&+ \dots && (80)\\
&+ b_n \sin(nt) \cdot \sin(nt) && (81)\\
&+ \dots && (82)
\end{aligned}
$$

Now we can integrate both sides from 0 to $2\pi$ and eliminate most terms using the trigonom

$$(83)$$

$$
\begin{aligned}
\int_0^{2\pi} f(t)\sin(nt)\mathrm{d}t &= \cancel{a_0 \int_0^{2\pi} \sin(nt)\mathrm{d}t} && (84)\\
&+ \cancel{a_1 \int_0^{2\pi} (\cos(t) \cdot \sin(nt))\mathrm{d}t} && (85)\\
&+ \cancel{a_2 \int_0^{2\pi} (\cos(2t) \cdot \sin(nt))\mathrm{d}t} && (86)\\
&+ \dots && (87)\\
&+ \cancel{a_n \int_0^{2\pi} \cos(nt) \cdot \sin(nt)\mathrm{d}t} && (88)\\
&+ \dots && (89)\\
&+ \cancel{b_1 \int_0^{2\pi} (\sin(t) \cdot \sin(nt))\mathrm{d}t} && (90)\\
&+ \cancel{b_2 \int_0^{2\pi} (\sin(2t) \cdot \sin(nt))\mathrm{d}t} && (91)\\
&+ \dots && (92)\\
&+ b_n \int_0^{2\pi} \sin^2(nt)\mathrm{d}t && (93)\\
&+ \dots &&\\
& && (94)
\end{aligned}
$$

$$
b_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cdot \sin(nt)\mathrm{d}t \qquad (95)
$$