Homework 2

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April 2018

1. **Up the Food Chain!**

   (a) Let Y be the population of tuna in the ocean in millions. Let P be the population of sharks in the ocean in hundreds of thousands. Let G be the number of fishermen in the world. And, let H be the human fishing impact at a given time. Then, the sharks, P, and tuna, Y, constitute a predator-prey relationship.

   To simplify the system I am modeling, I will model on the scale of months instead of decades so that I can make a number of assumptions about the environmental conditions and rates constants remaining relatively unchanged. I will also model an ideal environment in which food is plentiful for the prey yet scare for the predator so that the prey's growth is only dependent on its population while the predator's growth is dependent on the number of interactions between predator and prey species. Finally, I would expect fishermen to maximize their catches to maximize their profit for the number of fish they can sell (demand for food among humans is much larger than the populations of tuna and shark).

   My assumptions are then as follows:

   i. there is a limitless supply of food available for the prey
   ii. there is no competition within the prey species for food
   iii. the rate of change of population is proportional to the size of the population
   iv. the predator's only source of food is the tuna and that the predator species has limitless appetite
   v. new evolution advantages are insignificant
   vi. no major environmental changes occur which favor one species
   vii. the number of fishermen is constant
   viii. the number of fish caught by the fishermen is proportional to the populations of tuna and shark

   With these assumptions, I present a model based on the Lotka-Volterra equations. Let g_Y and g_P be growth constants, d_Y and d_P be death constants, and h_Y and h_P be fishing rate constants for the tuna and shark populations respectively.

$$Y' = g_Y Y - d_Y YP - h_Y YH$$
$$P' = -d_P P + g_P YP - h_P PH$$

   With given initial populations for each of the three species, I intend to use the Runge-Kutta 4 method for approximating the values of Y and P over the course of the simulation. Instead of the computationally simpler Euler's method, I have decided to implement the Runge-Kutta 4 algorithm which in general provides a more accurate approximation when compared on a standardized per computation basis with Euler's method (e.g. Euler's method using a smaller step size than RK4).

   (b) For the base values of the constants in the above differential equations, I've used the values in Project 3 of Chapter 4.2 of *Introduction to Computational Science* by Angela B. Shiflet and George W. Shiflet to mimic the predator-prey relationship displayed. The base values I use of the constants I introduced above are as follows:

$$g_Y = 2 \quad d_Y = 0.02 \quad g_S = 0.01 \quad d_S = 1.06$$

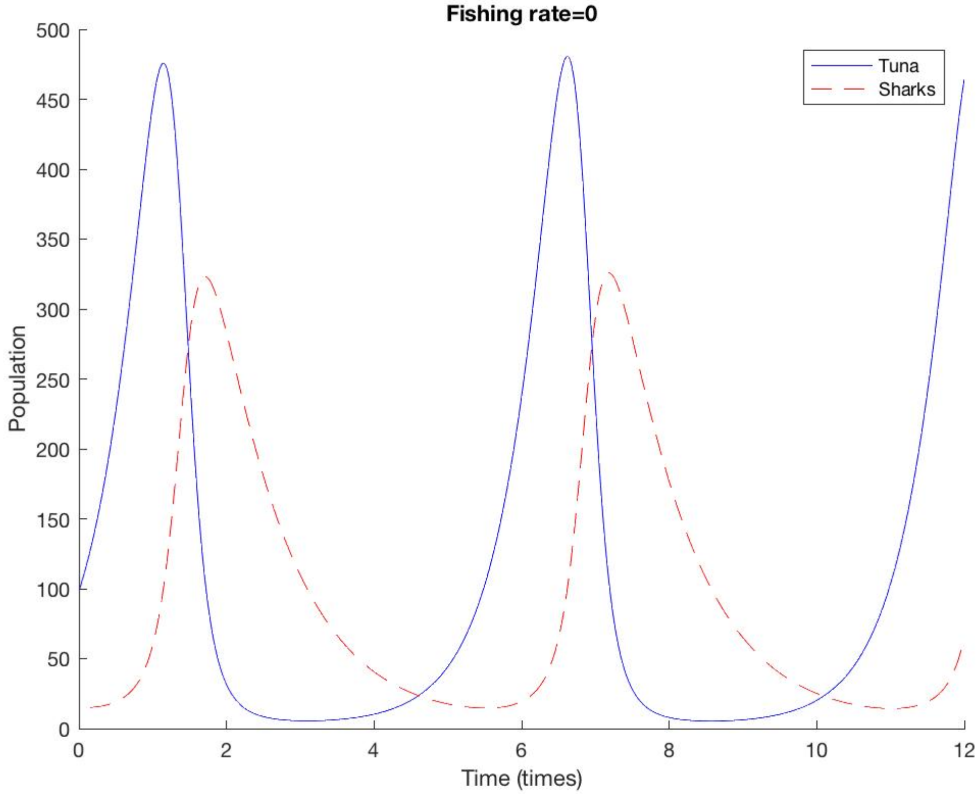

Figure 1: Plot using base constants, $g_Y = 2, d_Y = 0.02, g_S = 0.1, d_S = 1.06$, with a starting population of 100 for tuna and 15 for sharks, and a step size of 0.01 using RK4 over 12 months.

I also use a starting population of 100 for the tuna and 15 for the sharks. I use the RK4 method of approximation with a step size of 0.01 months over a total period of 12 months. In the following exercise, I will show the huge effect that modifying any parameter without in turn modifying the other parameters will has.

Over the 12 month period, the graph of tuna and shark populations without modifying any variables shows cyclic behavior. The tuna population begins higher than the shark population but peaks before the shark population peaks leading to a temporarily higher population of sharks than tuna in Figure 1.

When the tuna growth rate is reduced from 2 to 0.01 in Figure 2, both populations immediately drop significantly in size (and the sharks fall near 0). However, over time, the tuna grows, slowly, and continues cyclic behavior although with a much greater period in between peaks. The shark population follows this behavior as well, however, the shark population never becomes greater than the tuna population.

On a plot over a duration of 300 months, the pattern is much more prevalent, see Figure 3. I also noticed that the shark population does not grow significantly in size until the tuna population has reached a certain threshold and remains mostly close to 0. Further, the shark population continues to stay below the tuna population at all times.

When I increased the tuna growth rate constant to 0.505, the peaks and troughs of the tuna and shark population appeared more frequently than with lesser tuna growth rate constants (see Figure 4). The same pattern in which the shark population stays below the tuna population at all points is visible.

With a tuna growth rate constant of 1 (see Figure 5), the troughs and peaks appear even more frequently than in Figure 4 and the shark population reaches higher levels and begins to become larger than the tuna population at certain points.

I also tried changing the shark growth rate constant (when keeping the other constants at my base values). When I increased the shark growth constant to 0.055 (see Figure

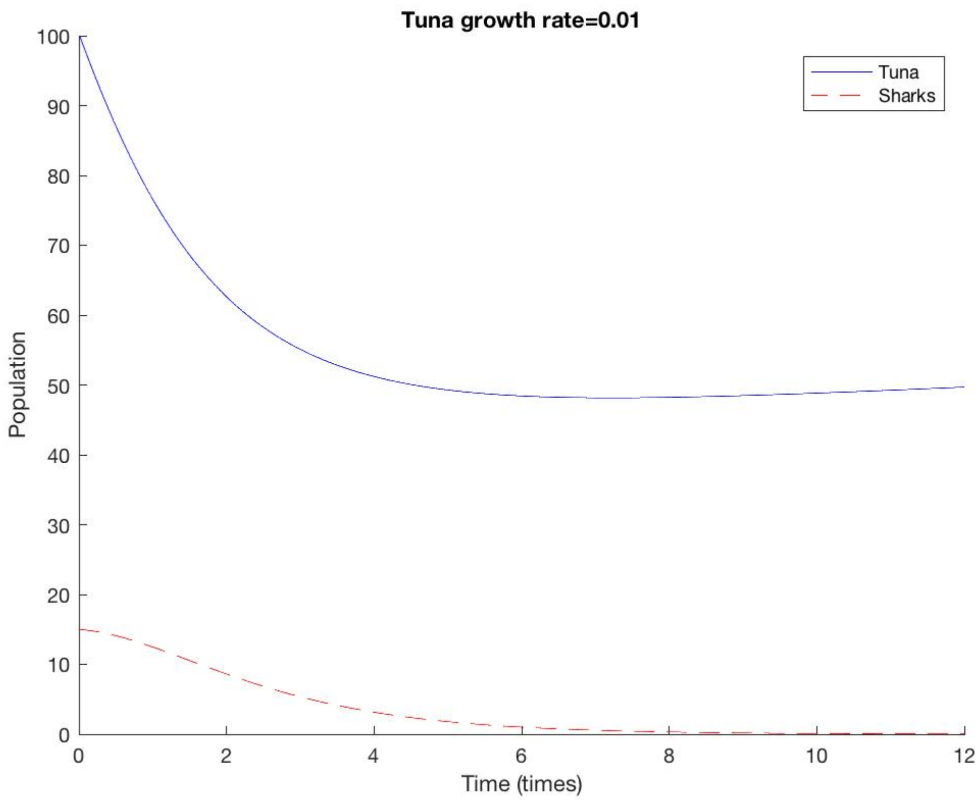

Figure 2: Plot using tuna growth constant of 0.01. Both populations immediately drop off and tuna slowly recovers.

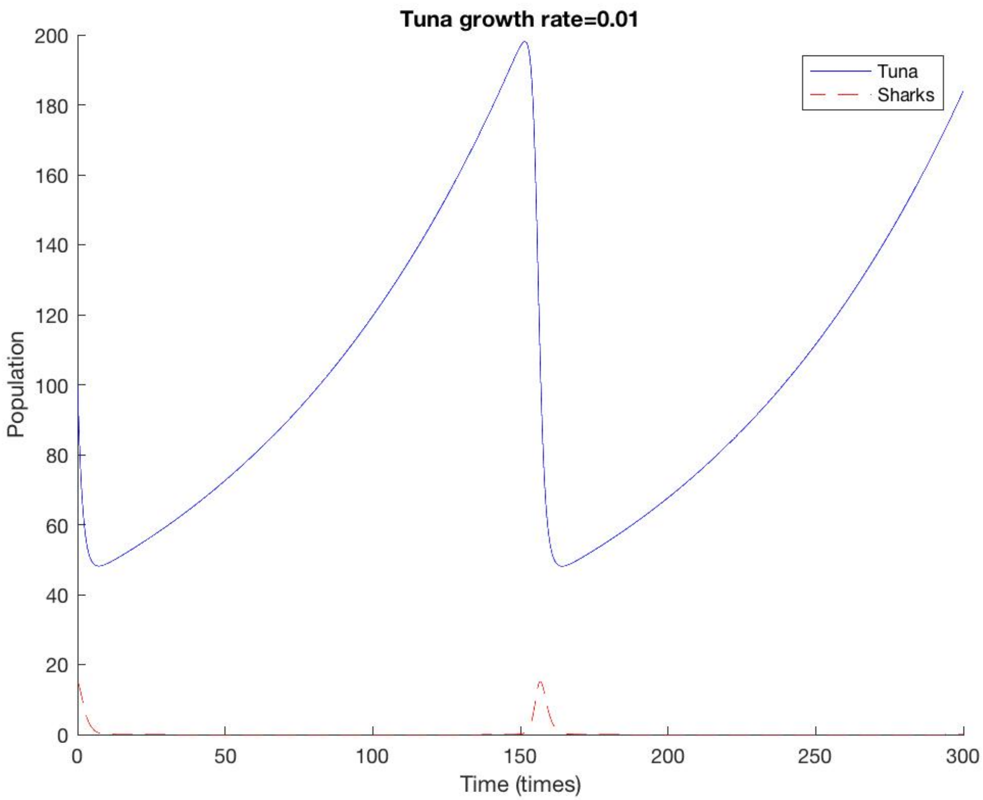

Figure 3: Plot using tuna growth rate of 0.01 over a period of 300 months. Cyclic behavior similar to Figure 1 is displayed but tuna population growth is prolonged and drop-off is much more rapid on a relative scale.

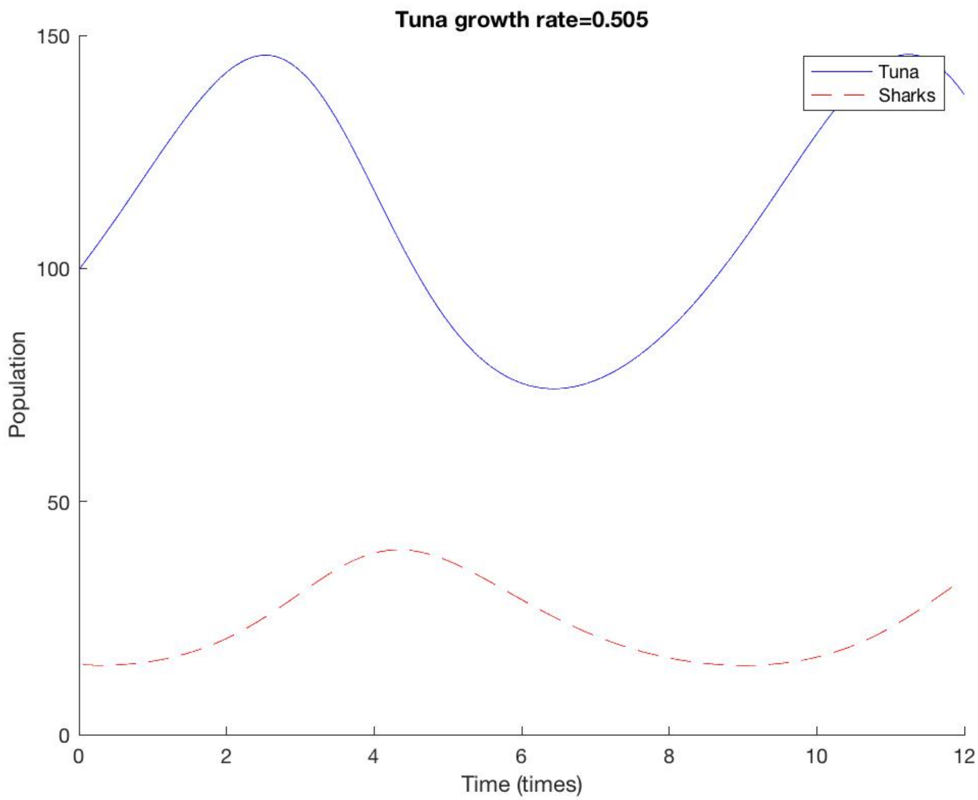

Figure 4: Plot using tuna growth rate constant of 0.505. The peaks and troughs of the tuna and shark population appear more frequently than with lesser tuna growth rate constants. The shark population stays below the tuna population at all points.

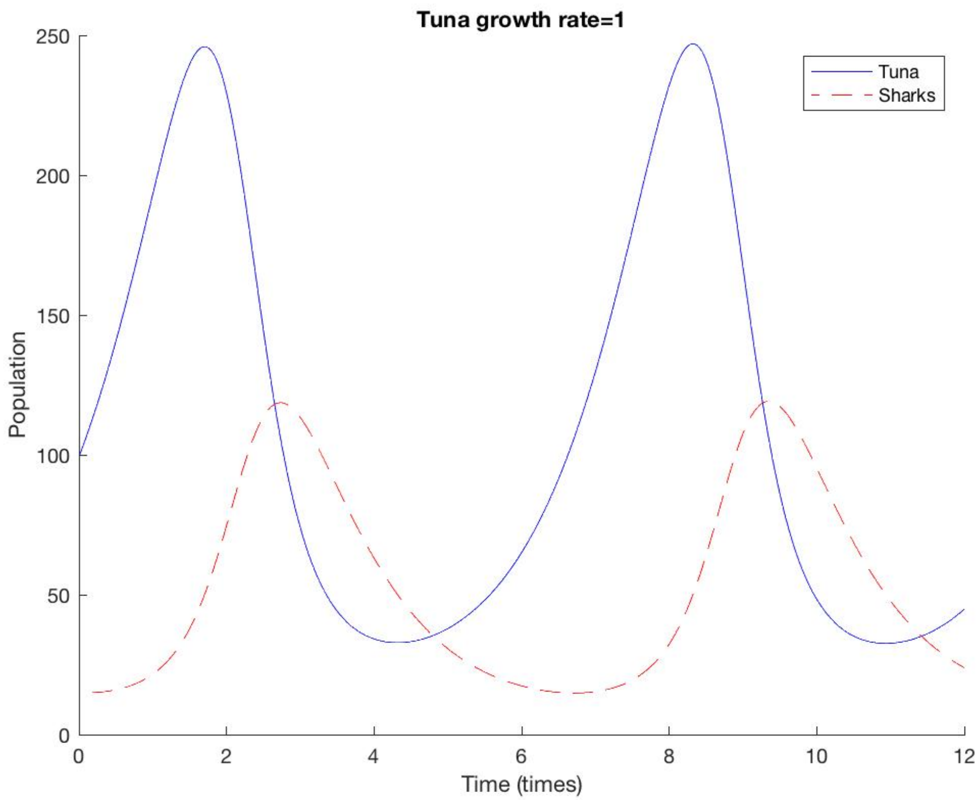

Figure 5: The troughs and peaks appear even more frequently than in Figure 4 and the shark population reaches higher levels and begins to become larger than the tuna population at certain points.

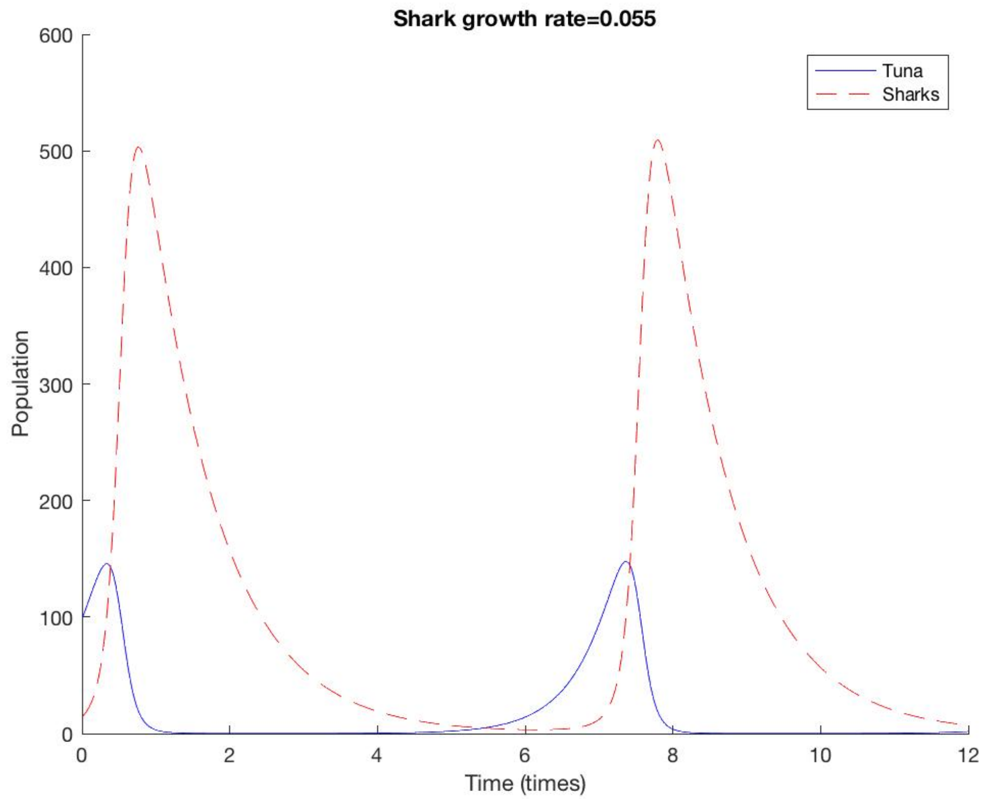

Figure 6: Plot using shark growth constant of 0.055 and base values of other constants.

6), I observed the much higher and more rapid peaks in the shark population and much lower peaks in the tuna populations.

The graph I generated with a shark growth rate constant of 0.1 (Figure 7) also appears very similar to the graph with a shark growth rate constant of 0.055 (Figure 6) but the peaks of the shark population are much higher and the peaks for the tuna population are slightly lower.

With a shark growth constant of 0.505, the massive rapid growth in the shark population dwarfs the peak of the tuna population, leading to a peak of the shark population more than 27 times the size of the peak of the tuna population (see Figure 8). Then, both populations rapidly fall. If I were to plot for more time, I would predict that both populations remain near 0 since the sharks consume the tuna so quickly that any tuna growth is negated, also negating shark growth.

Next, I played with the tuna death rate constant. With a tuna death rate reduced to 0.01 (see Figure 9) from 0.02, the peaks of the shark population became higher than those of the tuna population and both populations peaked at higher values.

When I raised the tuna death rate to 0.505, the shark population and tuna population immediately dropped off instead of beginning the first peak. Then, the tuna population regrew much faster than the shark population. When the shark population finally regrew to around 15 again, the tuna population quickly fell down again, leading to a similar fall in the shark population (see Figure 10).

With a tuna death rate of 1, the tuna population fell in size more quickly, but so did the shark population. The tuna population reached greater heights, but fell even more rapidly (see Figure 11).

I also tried different values of the shark death constant. With a reduced shark death constant of 0.01, the shark population grew larger than the tuna population then stayed there for a while, slowly dropping off. If I continued this plot for more months, I would expect the shark population to continue to fall until the rate at which tuna were born

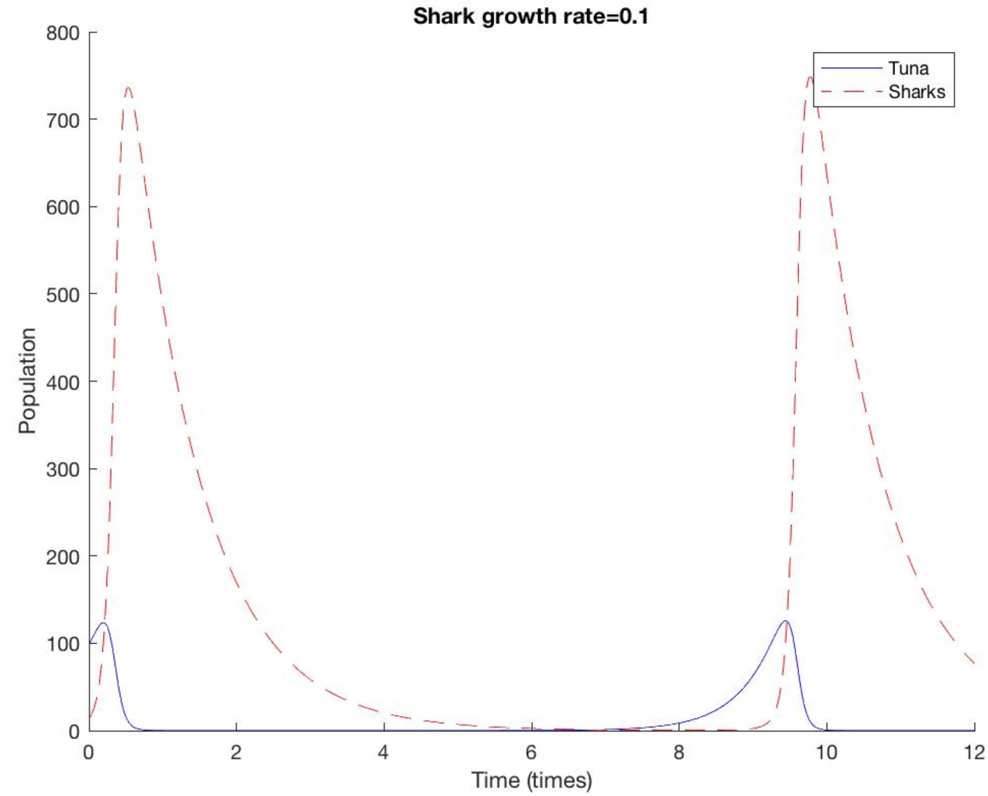

Figure 7: Plot using shark growth constant of 0.1 and base values of other constants.

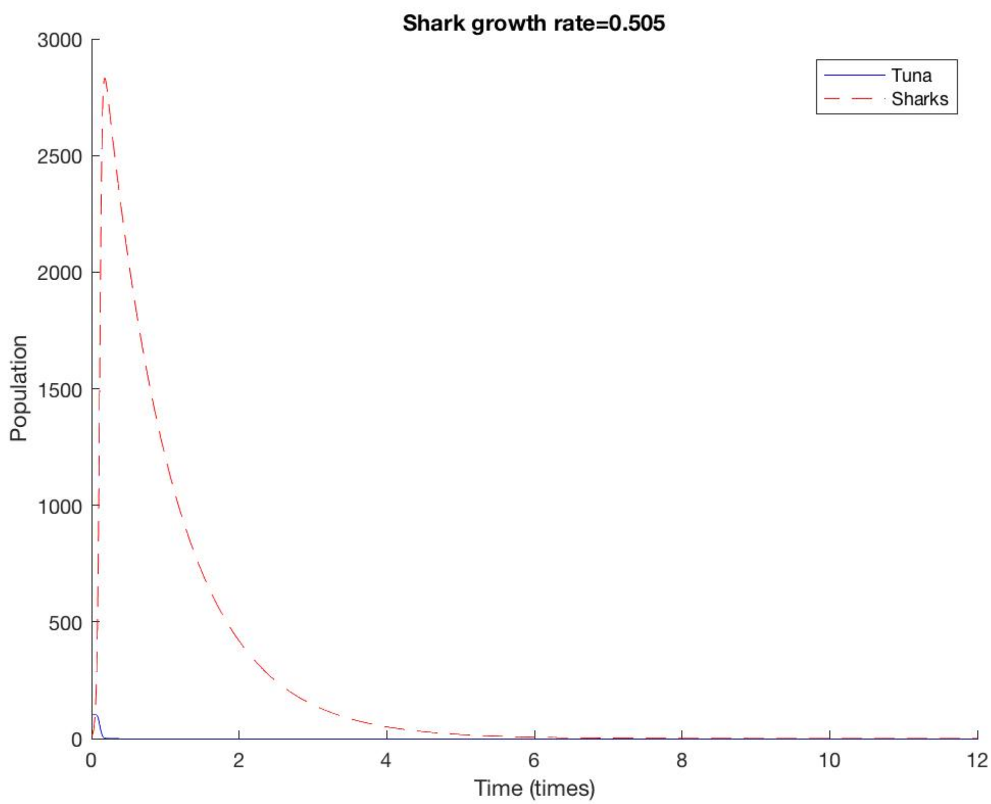

Figure 8: Plot using shark growth constant of 0.505 and base values of other constants.

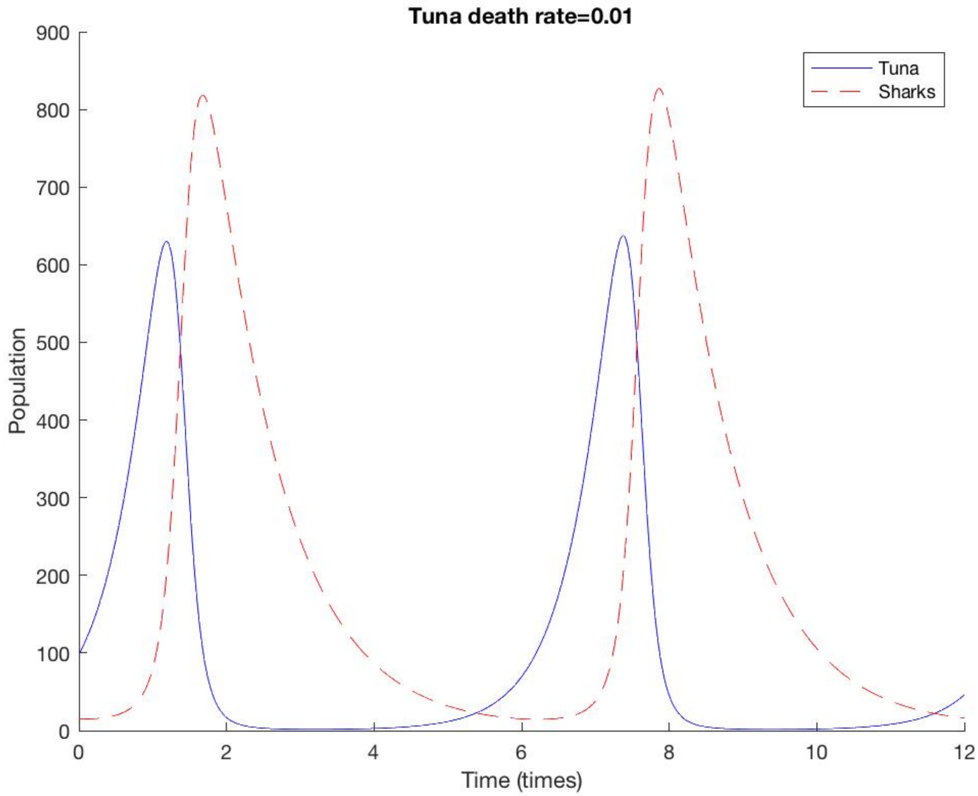

Figure 9: Plot using tuna death constant of 0.01 and base values of other constants.

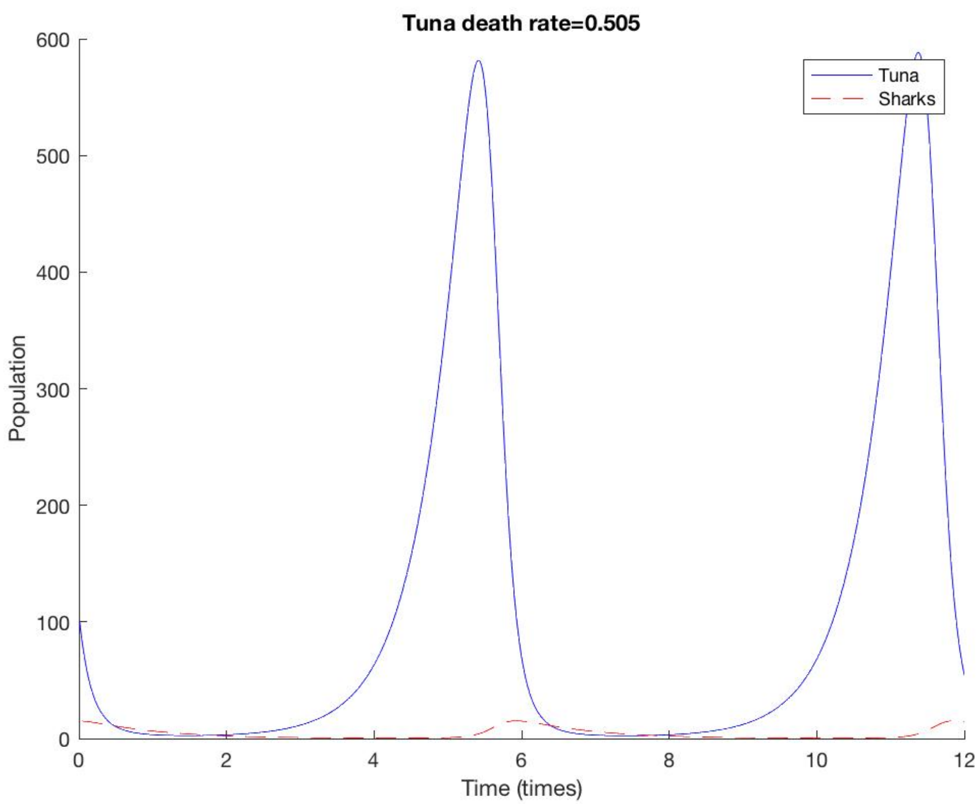

Figure 10: Plot using tuna death constant of 0.505 and base values of other constants.

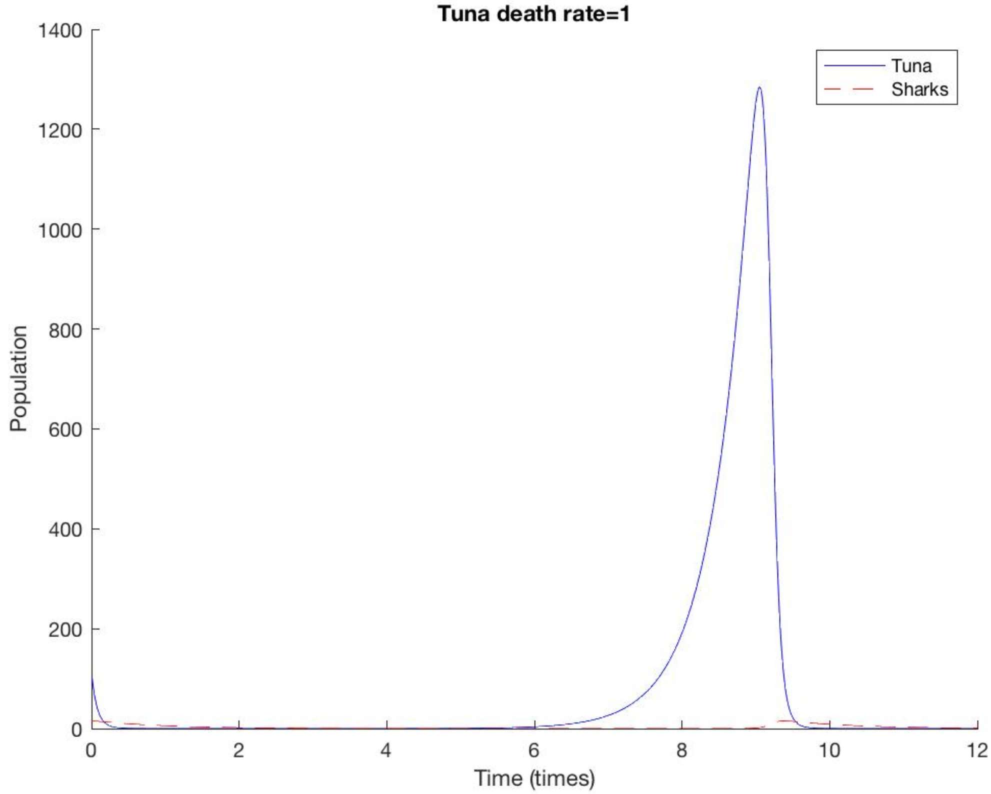

Figure 11: Plot using tuna death constant of 1 and base values of other constants.

reached or exceeded the rate at which they died, continuing with cyclic behavior (see Figure 12).

When I increased the shark death constant further to 0.505 (see Figure 13), I found that the peaks in the shark population appeared in a similar fashion (and reached a similar height) to those in Figure 1 but lasted longer and happened less frequently. The peaks of the shark death population were also lower in general.

Figure 1 shows a fishing rate of 0 with all of the base constant values for comparison. When I increased the fishing rate constant to 0.01 (see Figure 14), each of the first peaks appeared to happen slightly later from the start of the simulation and the peaks for the shark population were lower than they were with a fishing rate of 0.

With a fishing rate of 1.005, the troughs of the tuna population appeared to have minimums of higher population values and the maximums of the shark population appeared to be lower than with the lower fishing rate. Otherwise, the tuna population appeared mainly unchanged (see Figure 15).

When I increased the fishing rate to 2, which is equal to the tuna growth constant of 2, the tuna population remains constant once the shark population has died out (obviously, based on the architecture of the model from the differential equations) (see Figure 16).

(c) To summarize, as the fishing rate changes, the tuna population stays relatively the same, perhaps growing slightly slower, yet the shark population grows slower and reaches lesser and lesser peaks. Predator H then seems to more greatly affect the shark population, P. While I didn't graph this combination specifically, mathematically, the shark population should die out when $d_P P + h_P PH = g_P YP$ which is some point between the fishing rates of 1 and 2.

(d) In this exercise, I set $H(t)$ equal to $a * cos(p * t + pi) + f$ with base values of constants $a = 1.5$, $p = \pi/6$, and $f = a$. I first noticed that allowing the fishing rate to increase up to 3 has a profound effect on the simulation. Fishing then becomes the dominant form of death and thereby constraint on the peaks of the tuna and shark populations. The

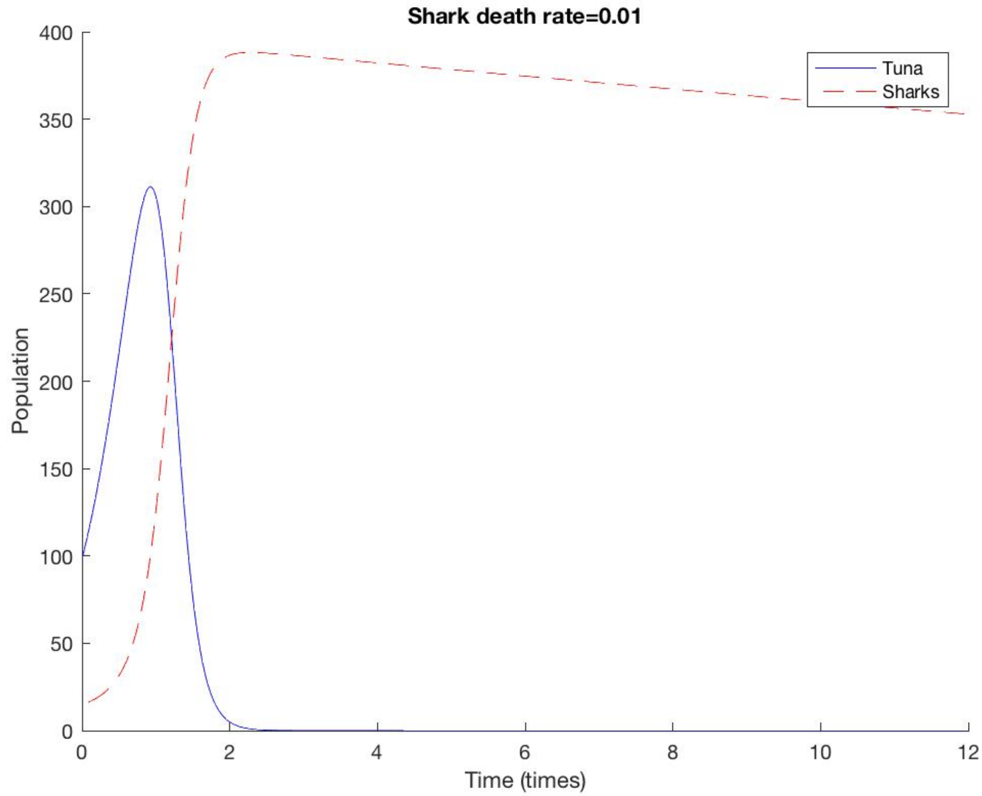

Figure 12: Plot using shark death constant of 0.01 and base values of other constants.

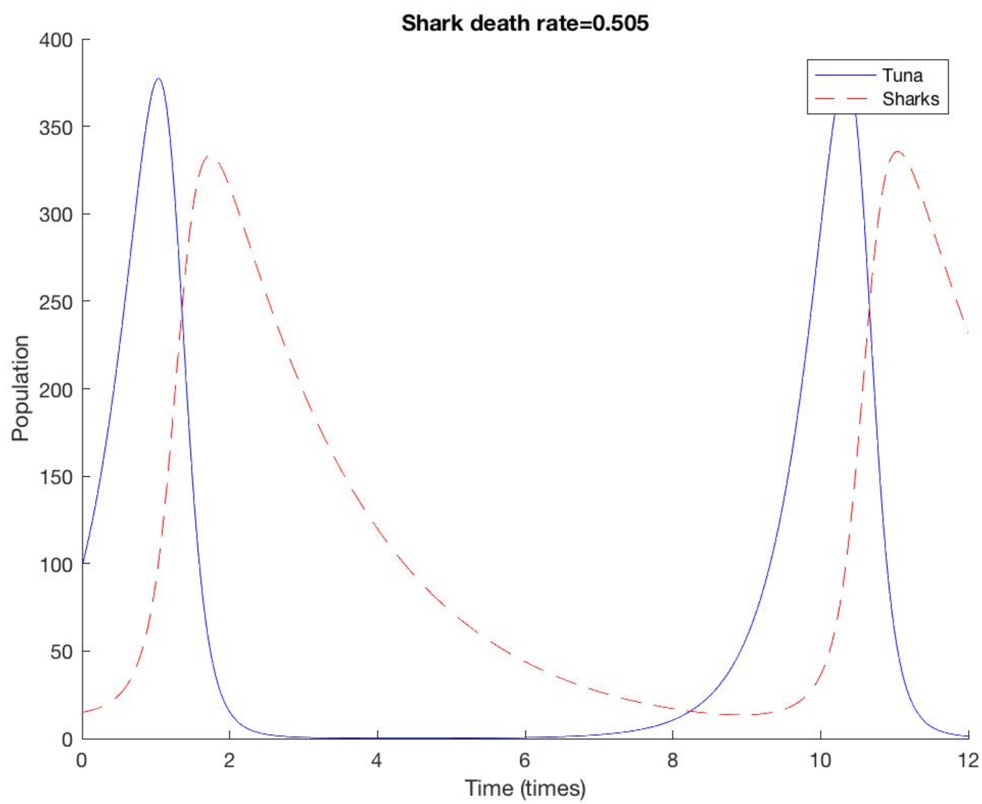

Figure 13: Plot using shark death constant of 0.0505 and base values of other constants.

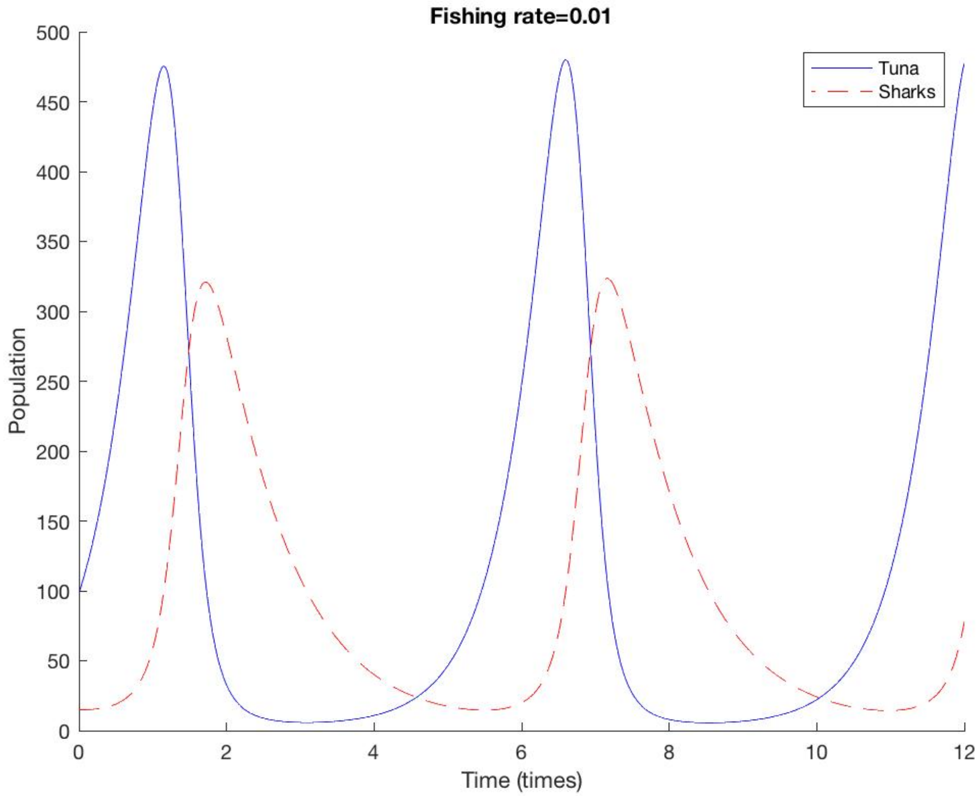

Figure 14: Plot using fishing rate of 0.01 and base values of other constants.

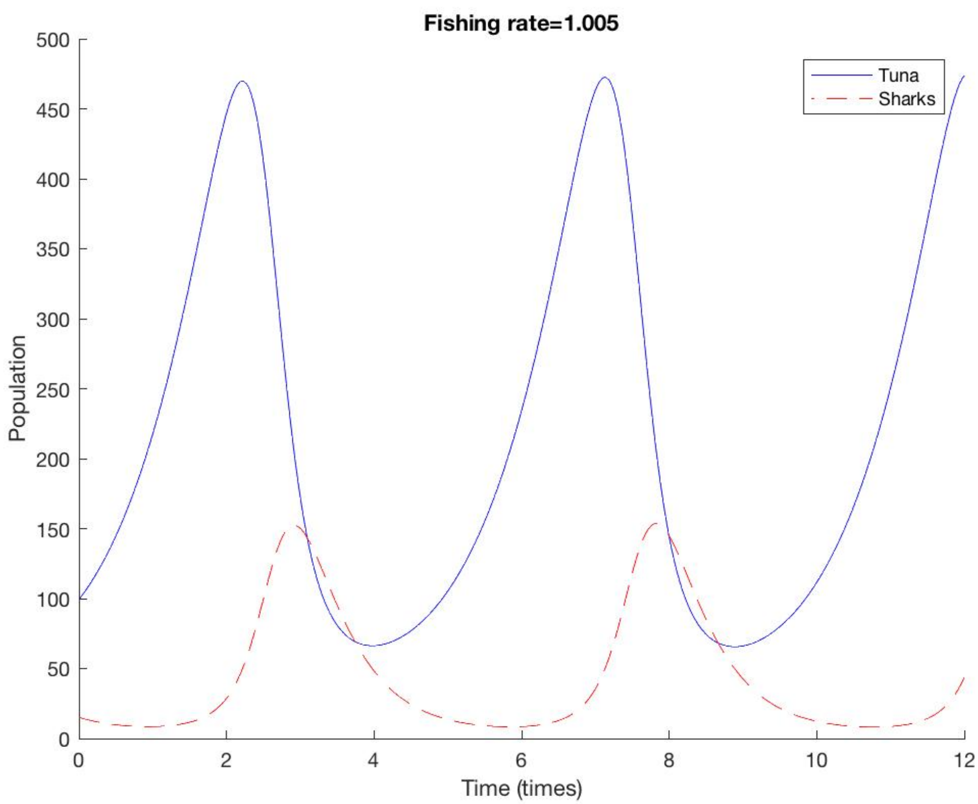

Figure 15: Plot using fishing rate of 1.005 and base values of other constants.

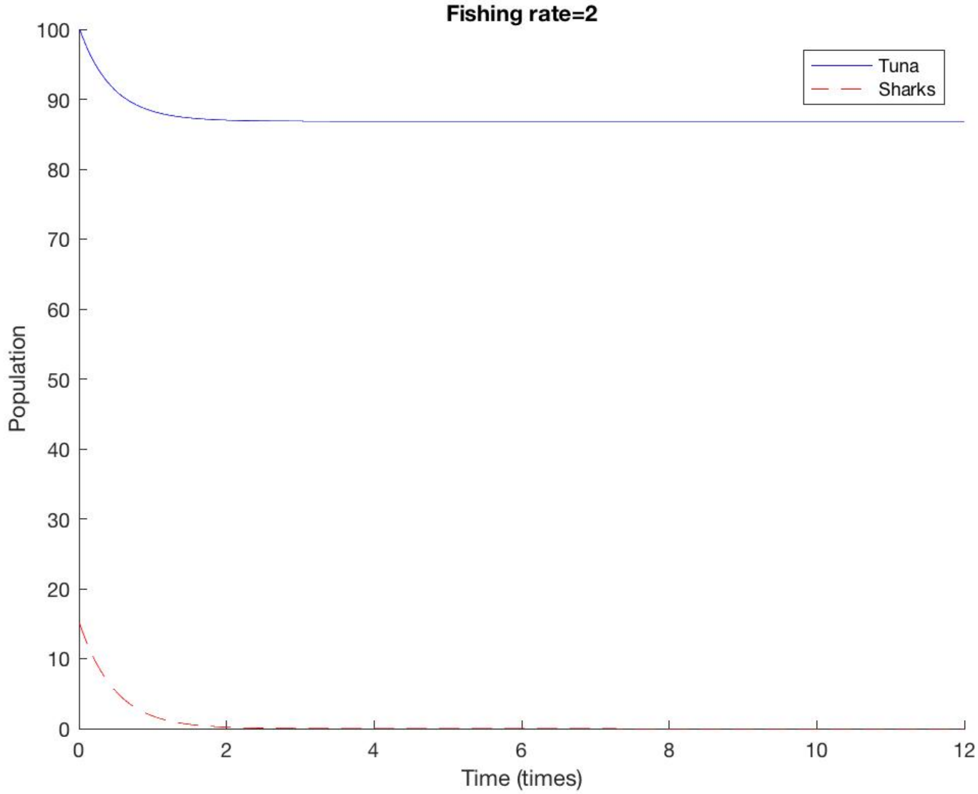

Figure 16: Plot using fishing rate of 2 and base values of other constants.

cyclic nature of each of these populations then reflects the 6 month period of the fishing season.

Figures 17, 18, 19, and 20 show the effect of change the a amplitude constant. When a grows, the peaks of both populations become more frequent and a pattern in which there are smaller peaks building up to a larger peak then resetting occurs.

Figures 21, 22, and 23 reflect the effect of changing the f base fishing rate constant. Essentially, the effect of increasing this constant was to reduce how long it took before the tuna population reached its peak population from a sequence of smaller peaks.

Figures 24, 25, and 26 reflect the effect of changing the p period-related constant. Increasing this constant meant that the fishing season began and was over in a shorter amount of time but happened more frequently. The result shown in the graphs was that there was a sequences of peaks of each population growing in height that reset after some time that happened more frequently for larger p.

2. **Change It Up!**

(a) In this exercise, I chose to model the number of fishermen based on the number of tuna and sharks in the oceans since after a few bad fishing seasons (due to a decreased number of fish in the population) a fisherman might have to sell his boat and pursue something else. Similarly, when the number of fish in the ocean appears to be plentiful, more people might become fishermen and go out on trips. Therefore, I will model the population of fishermen similar to that of the population of sharks, involving a growth constant g_H and a death constant d_H. I also use a coefficient of 0.05 times HY or HP respectively at a given time to reduce either the population of Y or P during a given time step.

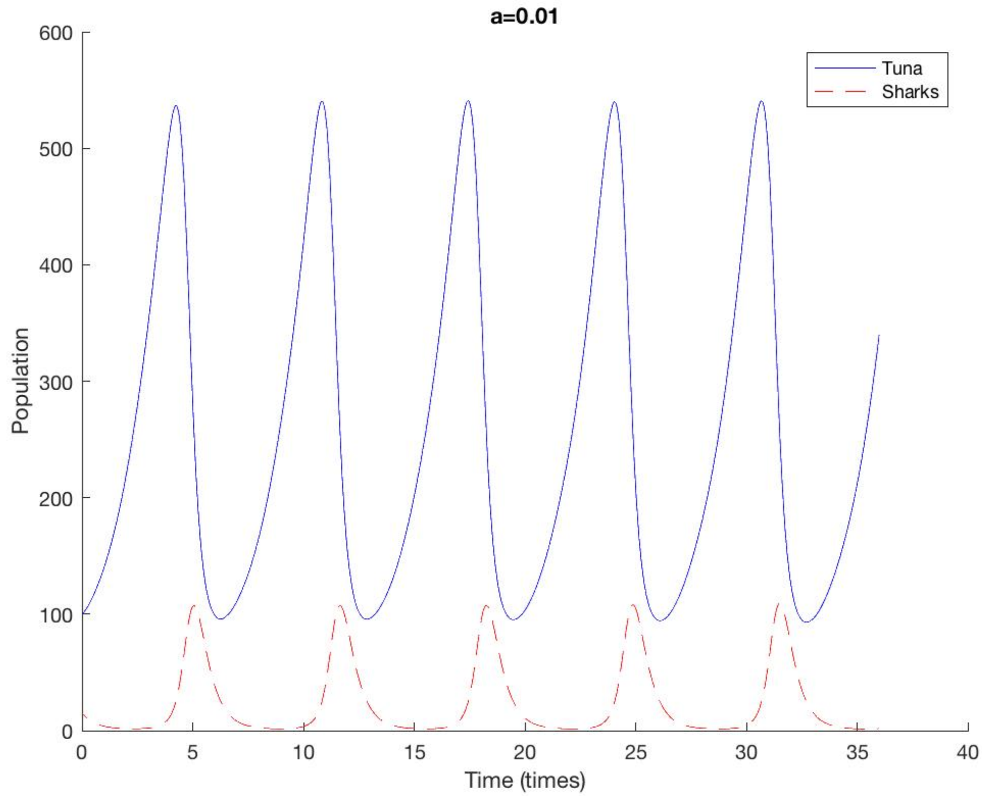

Figure 17: Plot using $a = 0.01$ and base values of other constants.

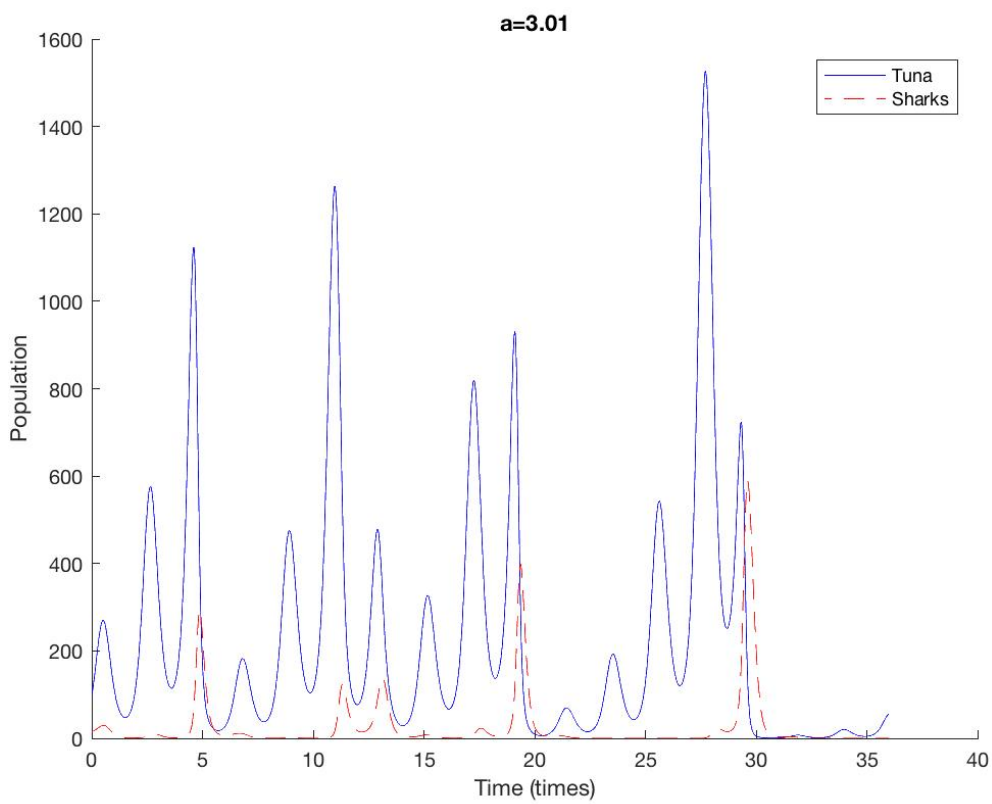

Figure 18: Plot using $a = 3.01$ and base values of other constants.

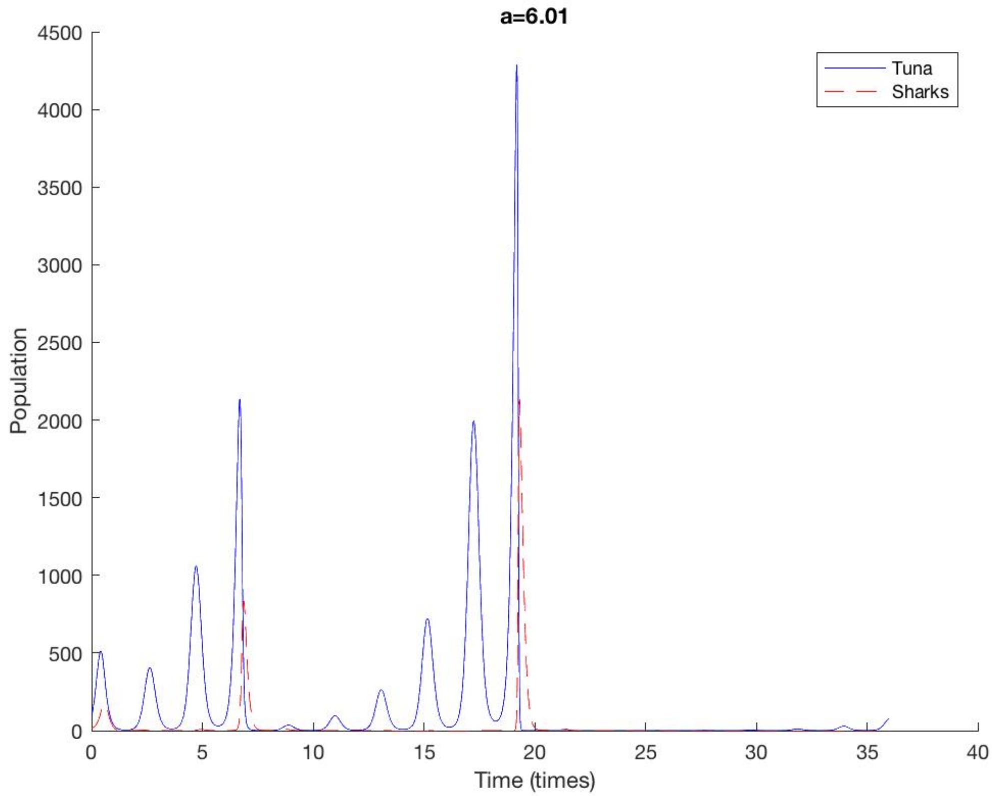

Figure 19: Plot using $a = 6.01$ and base values of other constants.

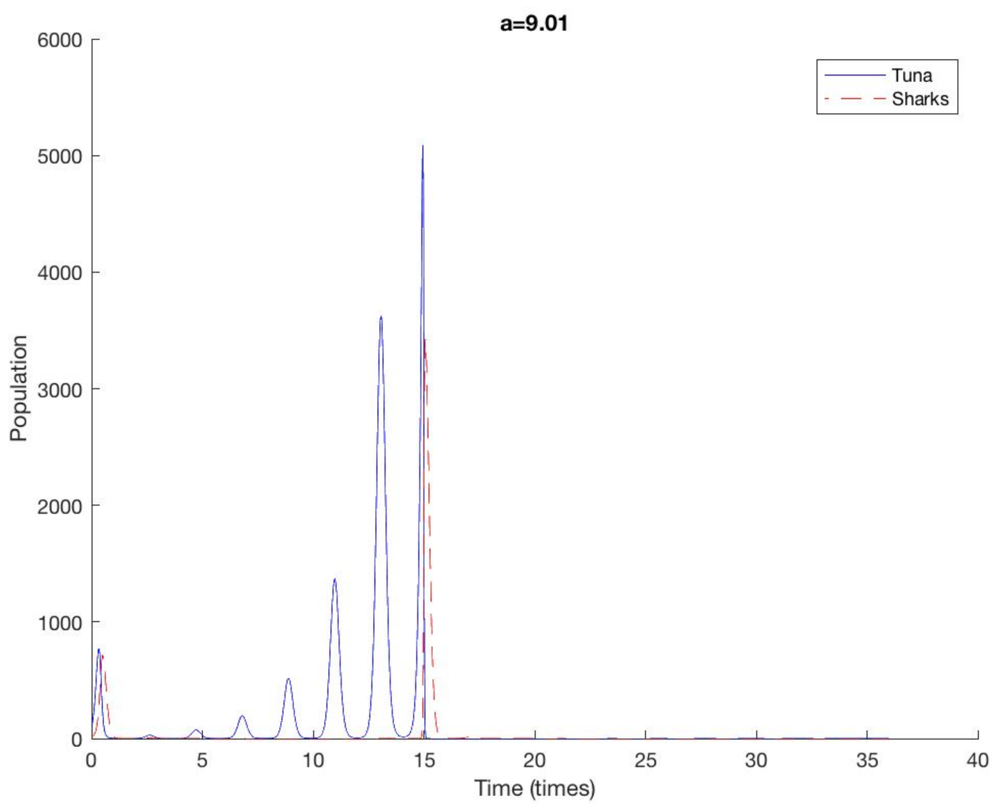

Figure 20: Plot using $a = 9.01$ and base values of other constants.

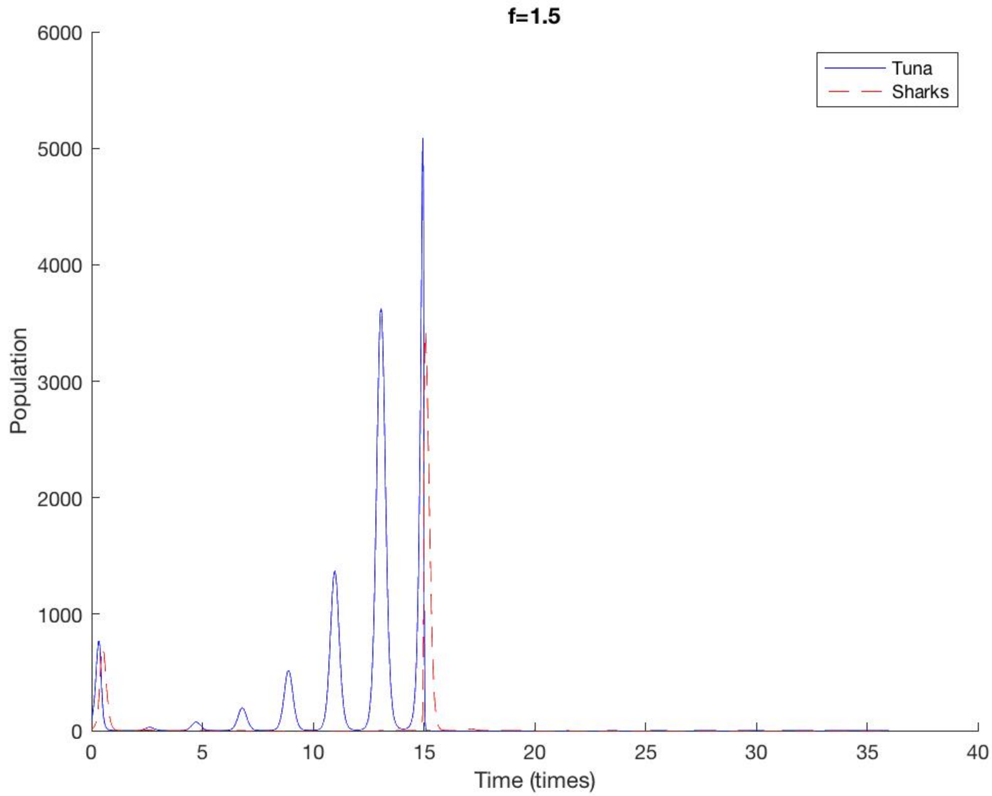

Figure 21: Plot using $f = 1.5$ and base values of other constants.

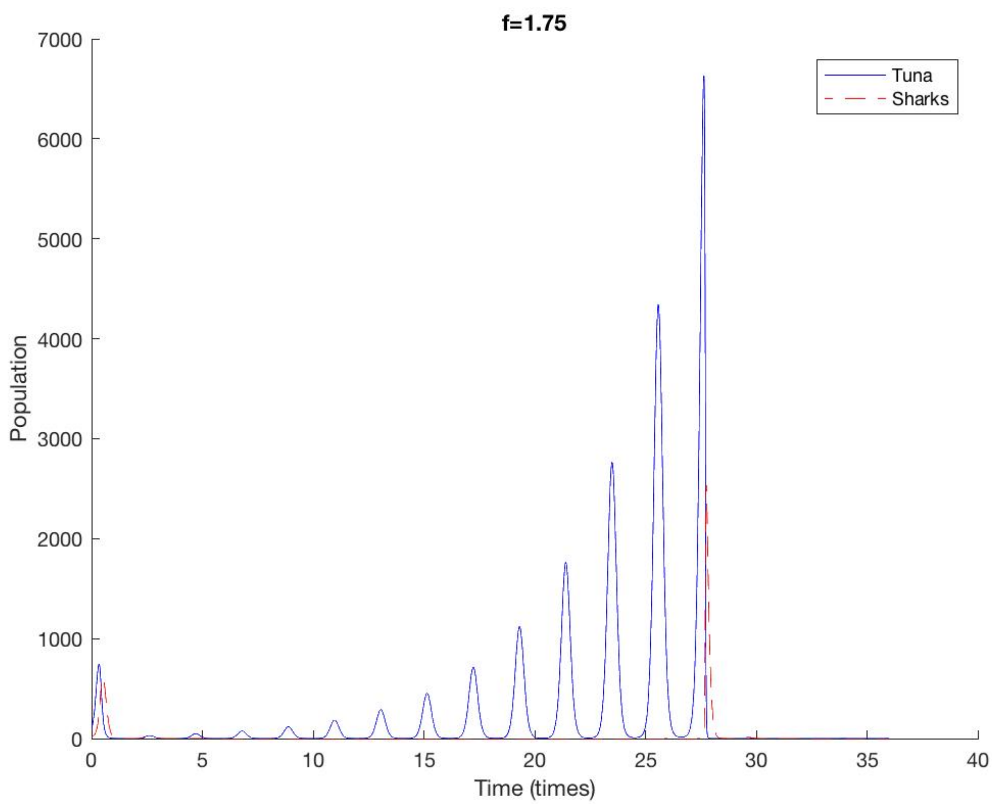

Figure 22: Plot using $f = 1.75$ and base values of other constants.

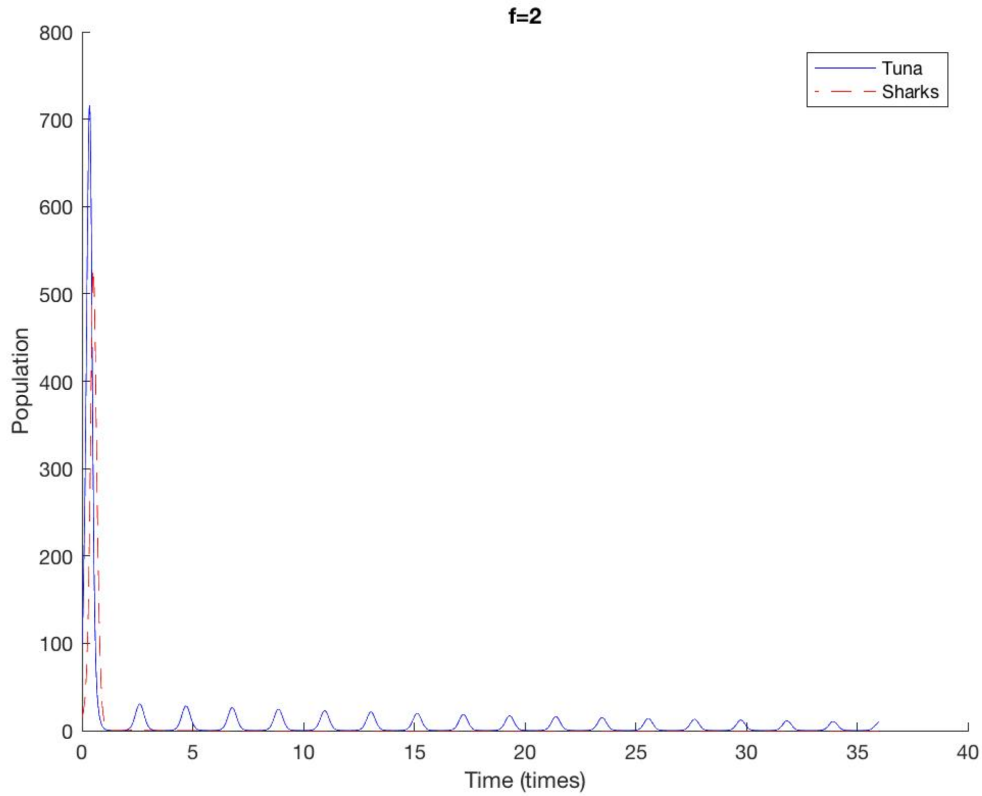

Figure 23: Plot using $f = 2$ and base values of other constants.

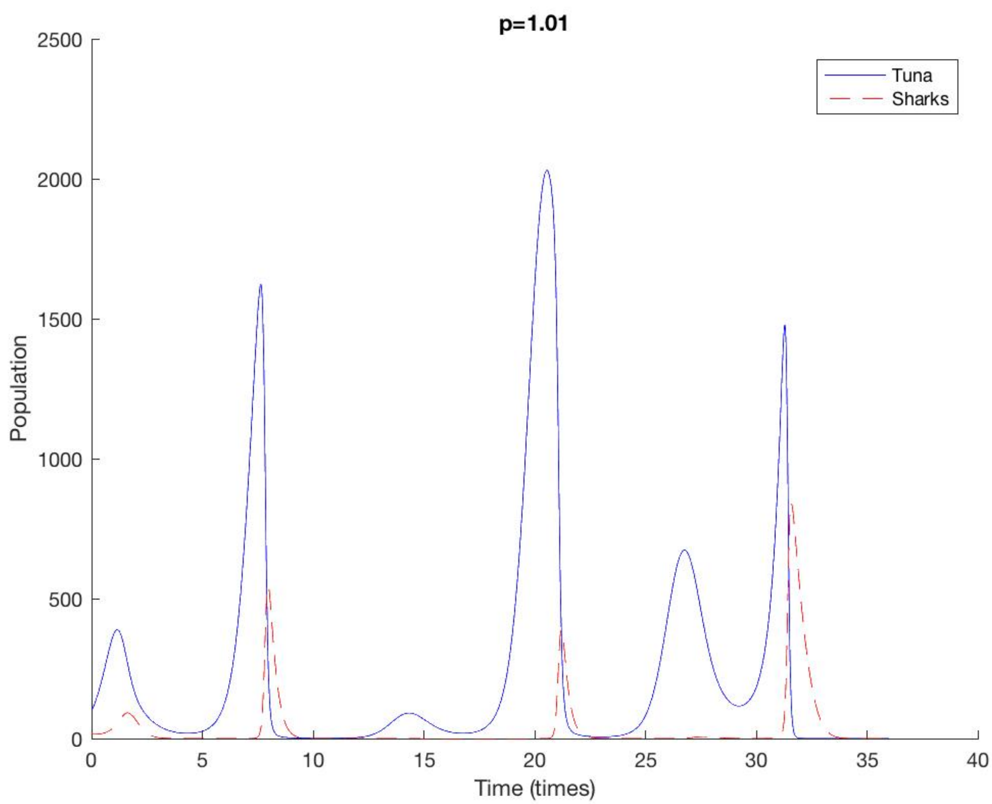

Figure 24: Plot using $p = 1.01$ and base values of other constants.

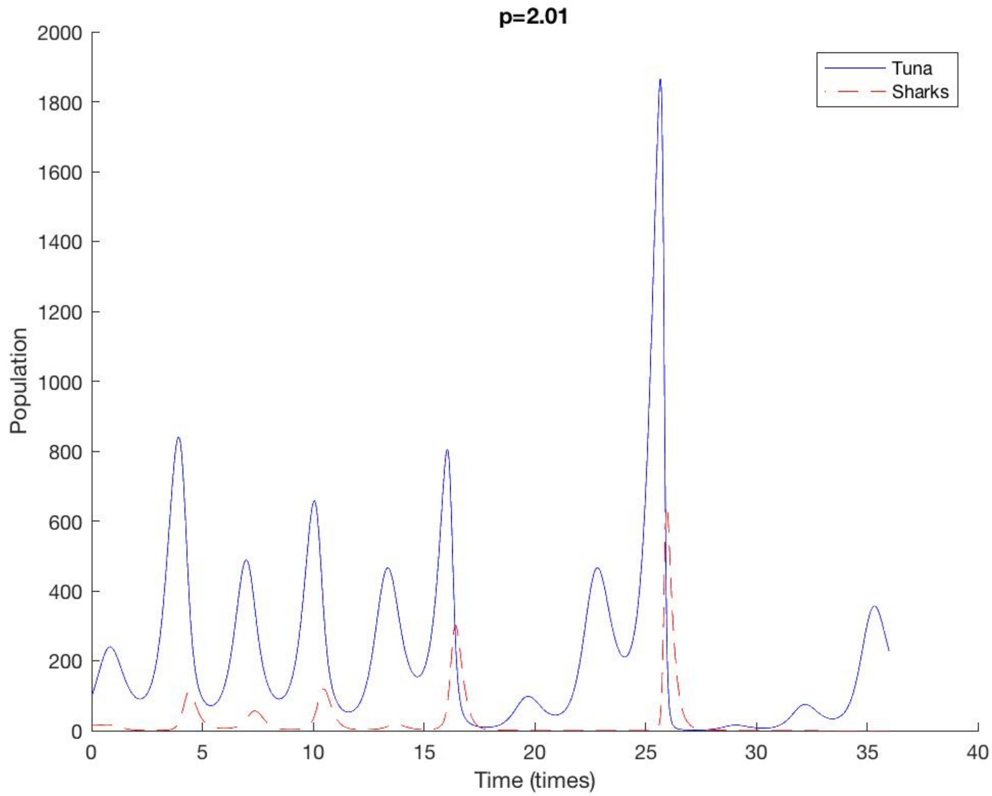

Figure 25: Plot using $p = 2.01$ and base values of other constants.

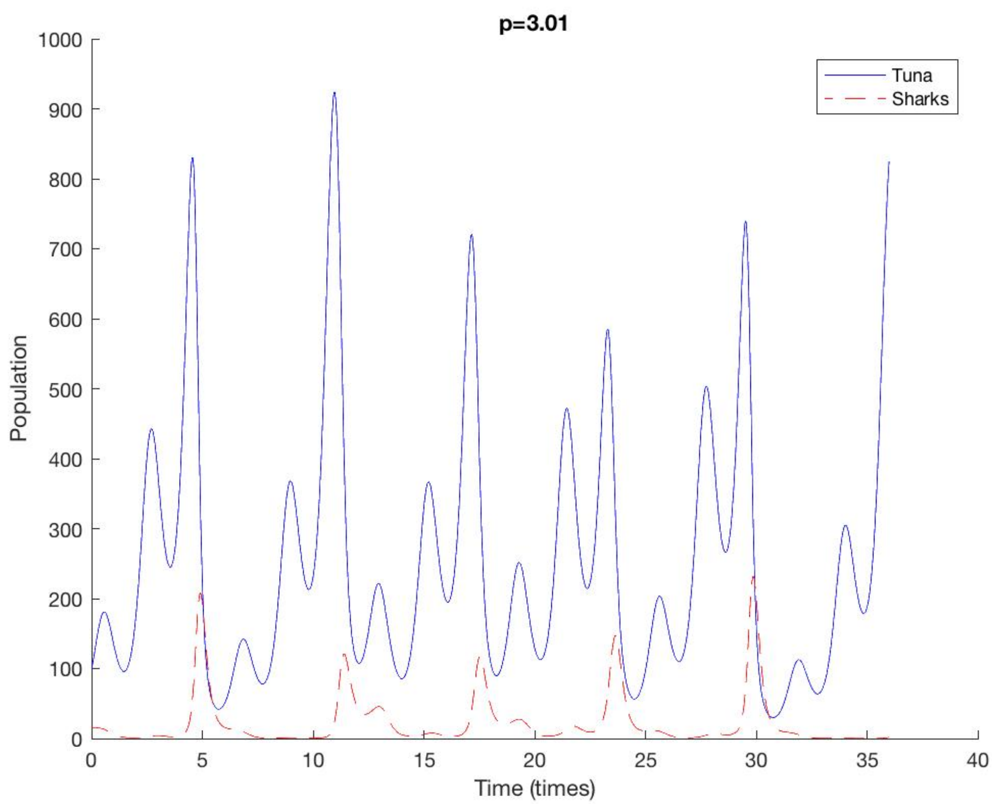

Figure 26: Plot using $p = 3.01$ and base values of other constants.

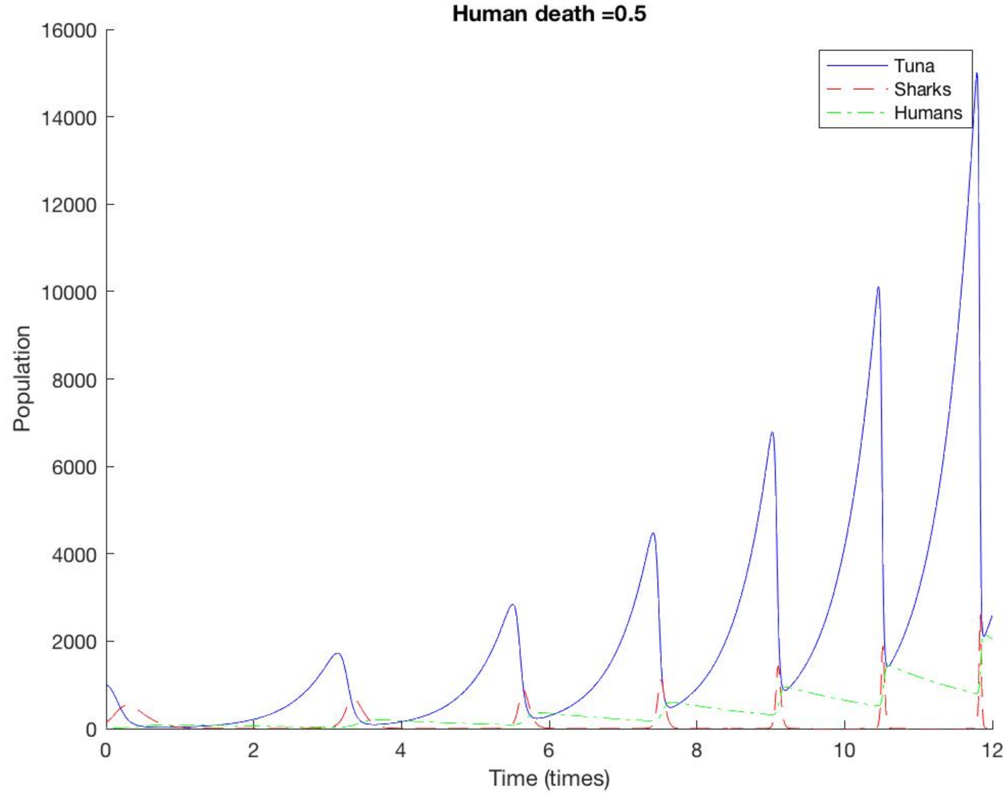

Figure 27: Plot using $h = 0.5$ and base values of other constants.

$$\begin{aligned}
Y' &= g_Y Y - d_Y Y P \\
P' &= -d_P P + g_P Y P - h_P P H \\
H' &= d_H * H + g_H H P
\end{aligned}$$

With a human death rate (e.g. fishermen leaving) of 0.5 (see Figure 27, all the populations appear with increasing peaks and then a reset. The populations all reach significantly higher levels eventually (tuna is close to 16000). The peaks of the populations are in the order of tuna, sharks, then humans. With a human death rate of 1.0 (see 28), the pattern is far more regular with peaks in the same order. However, the peaks reach much lower maximums.

(b) In this exercise, I simply modified the differential equation for tuna back to its state in the first model (essentially reincorporating the $-h_Y Y H$ term).

$$\begin{aligned}
Y' &= g_Y Y - d_Y Y P - h_Y Y H \\
P' &= -d_P P + g_P Y P - h_P P H \\
H' &= d_H * H + g_H H P
\end{aligned}$$

Figures 29, 30, 31, and 32 show this model for varying human death rates. Higher human death rates reduces the size of the peaks considerably for all three populations. The effect is most noticeable in the maximum value of the tuna population. However, the frequency of the peaks first increases with increasing human death (as humans die off faster) in Figures 30and 31 then decreases in 32 as the human population dies of nearly immediately.

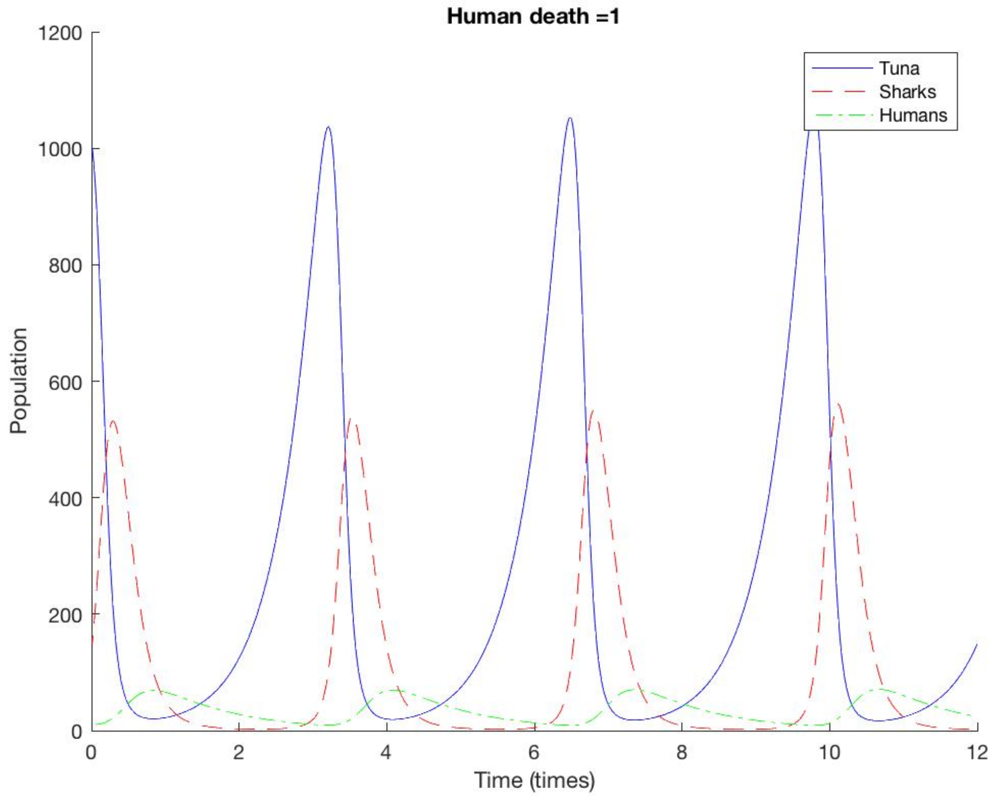

Figure 28: Plot using $h = 1$ and base values of other constants.

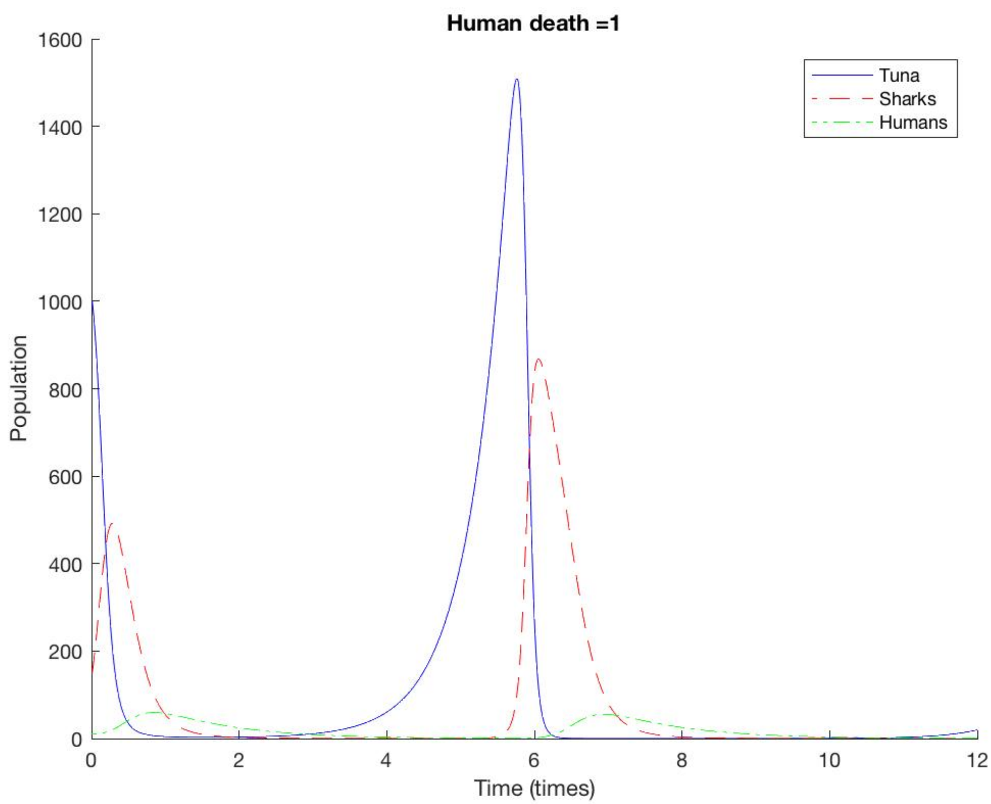

Figure 29: Plot using $h = 1$ and base values of other constants.

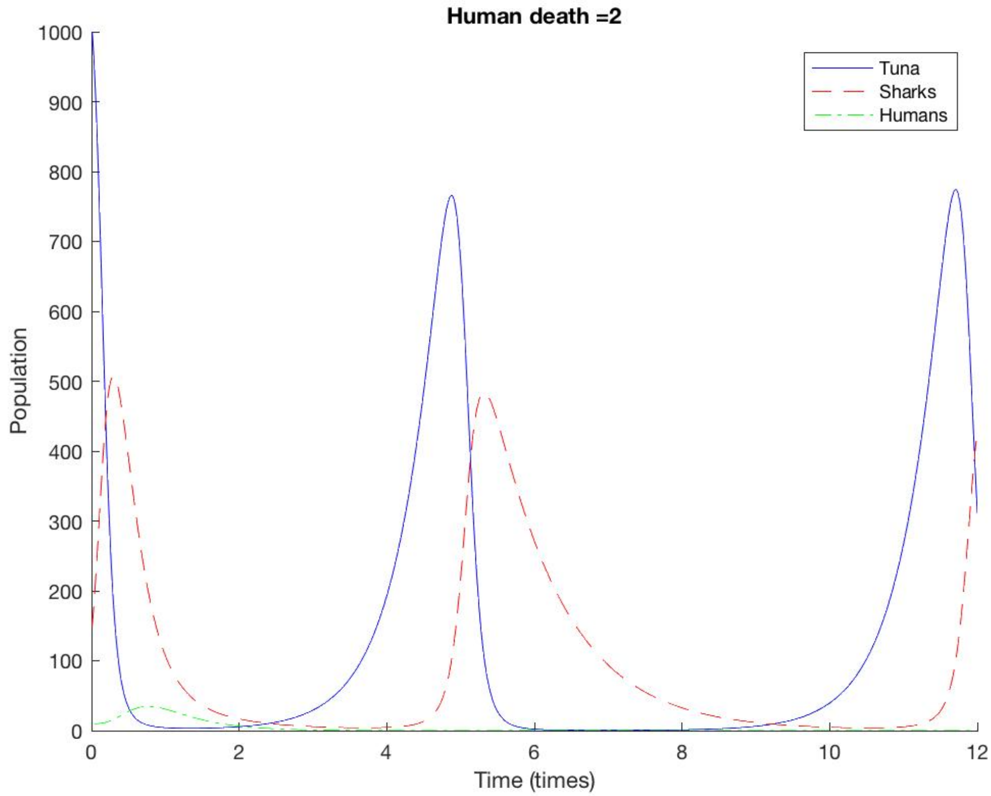

Figure 30: Plot using $h = 2$ and base values of other constants.

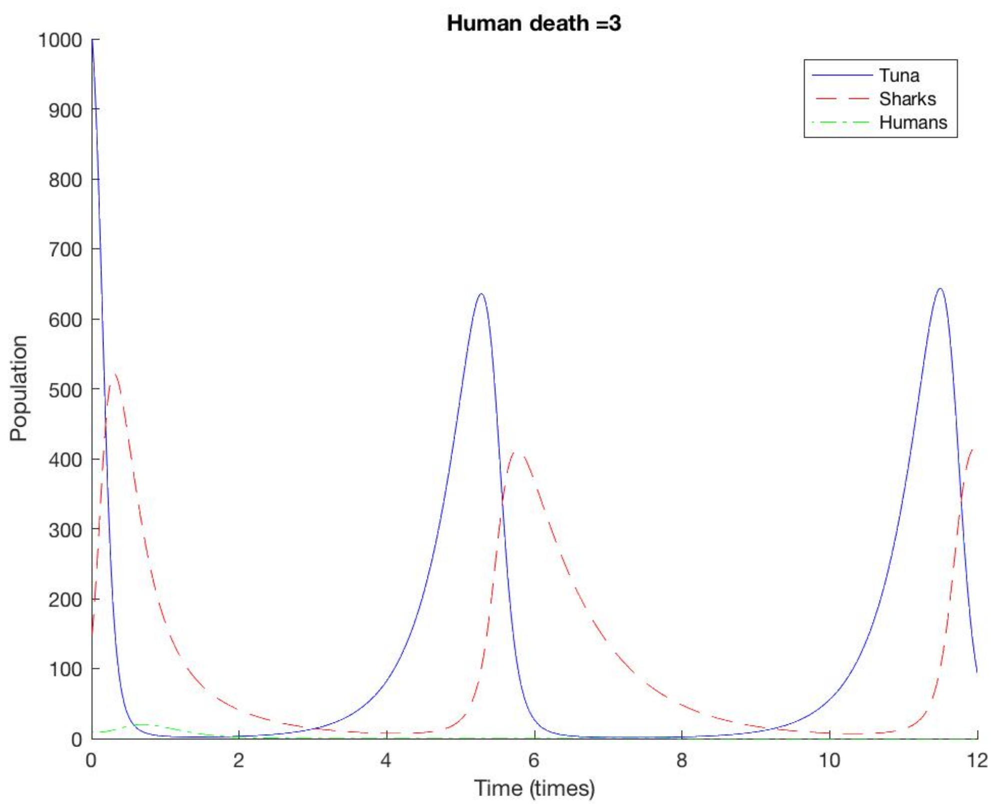

Figure 31: Plot using $h = 3$ and base values of other constants.

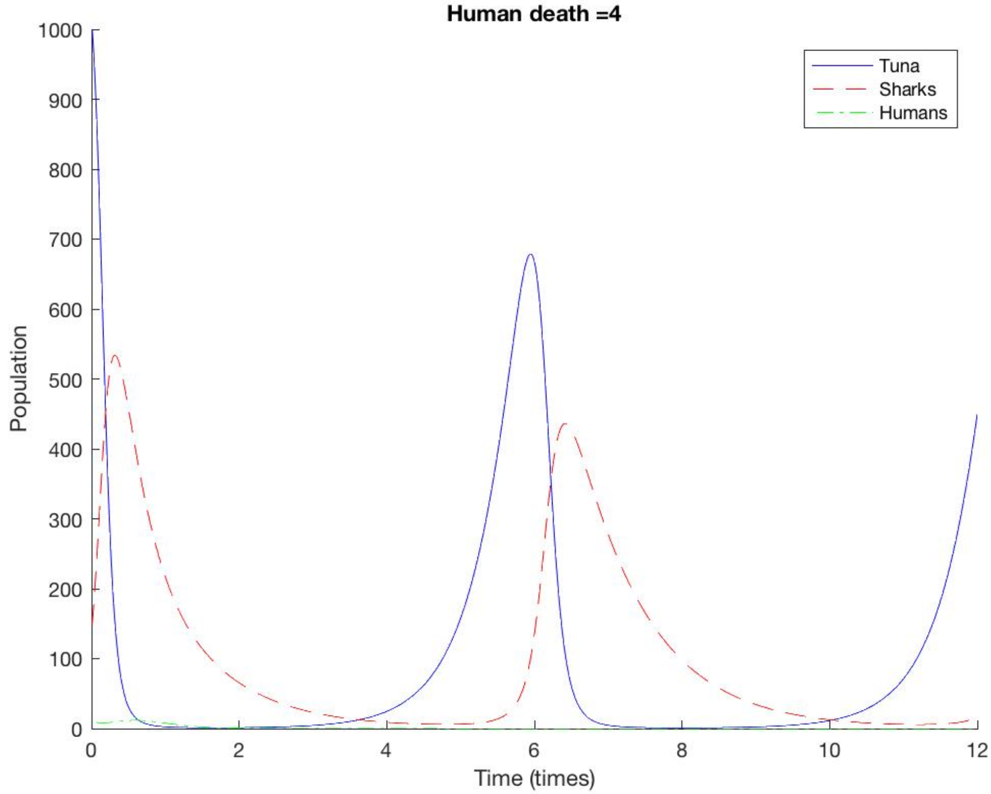

Figure 32: Plot using $h = 4$ and base values of other constants.

(c) My primary hypothesis is that if the human growth rate is increased enough, the shark population will nearly entirely die out but the tuna population will continue at significant levels. My competing alternative hypothesis is that both the shark and the tuna populations will die out.

To test my hypothesis, I created a loop to run logarithmically spaced values of g_H for each model.

Figures 33 through 41 show that there is no such point and act as contradictory evidence to the first hypothesis but supporting evidence to the second hypothesis. What appears to happen is that the peaks get less and less frequent and higher and higher in maximum as an effect of increasing human growth rate.

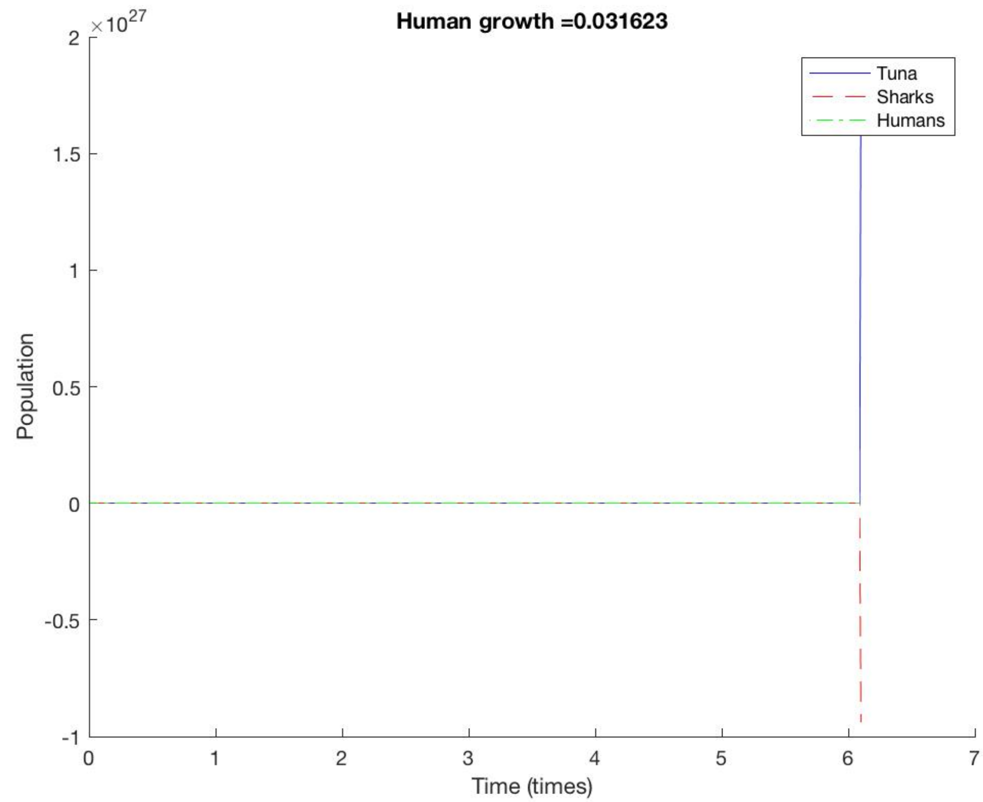

Figure 33: Plot using $h = 0.03$ and base values of other constants for model from 2a.

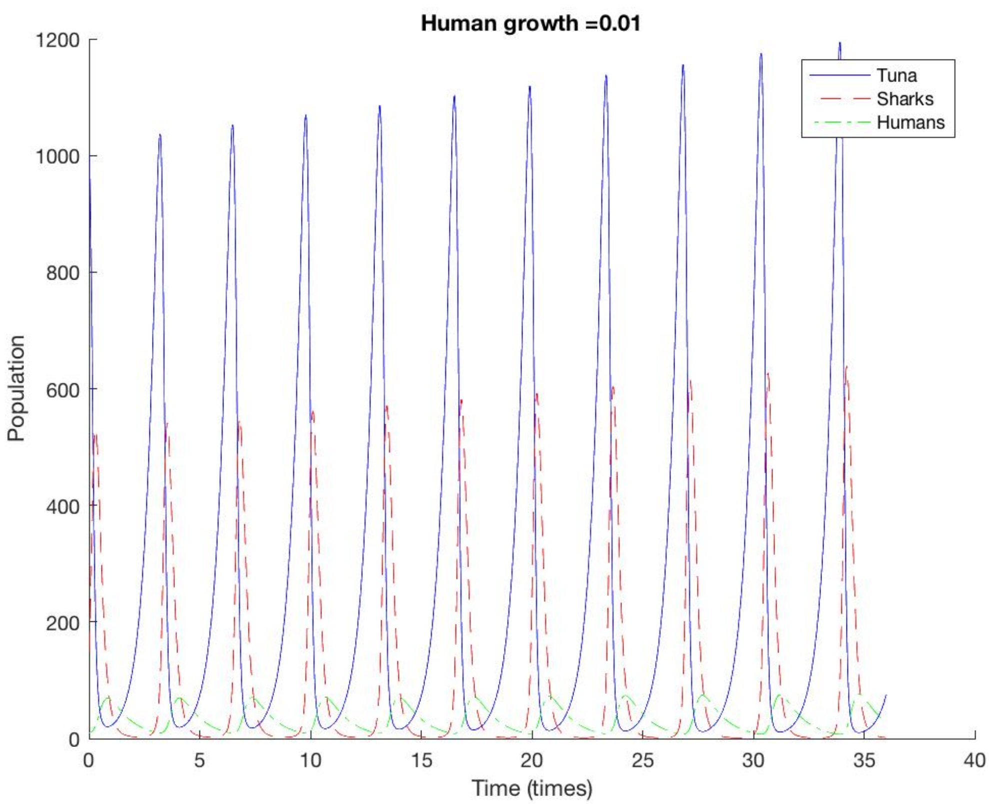

Figure 34: Plot using $h = 0.01$ and base values of other constants for model from 2a.

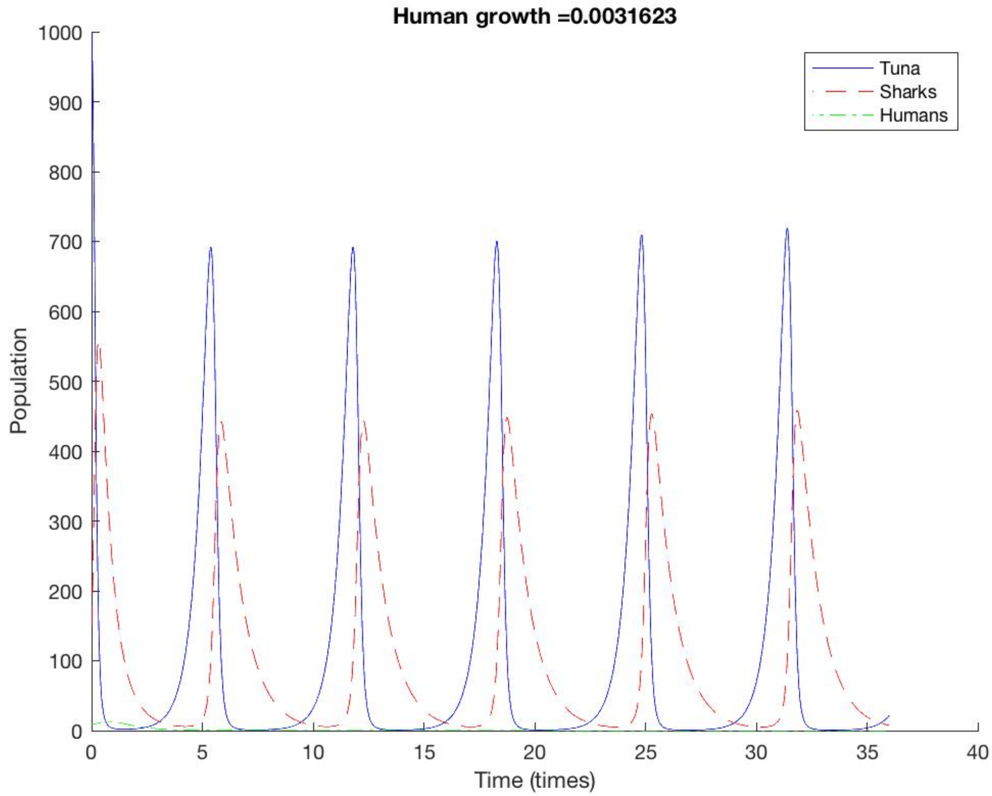

Figure 35: Plot using $h = 0.003$ and base values of other constants for model from 2a.

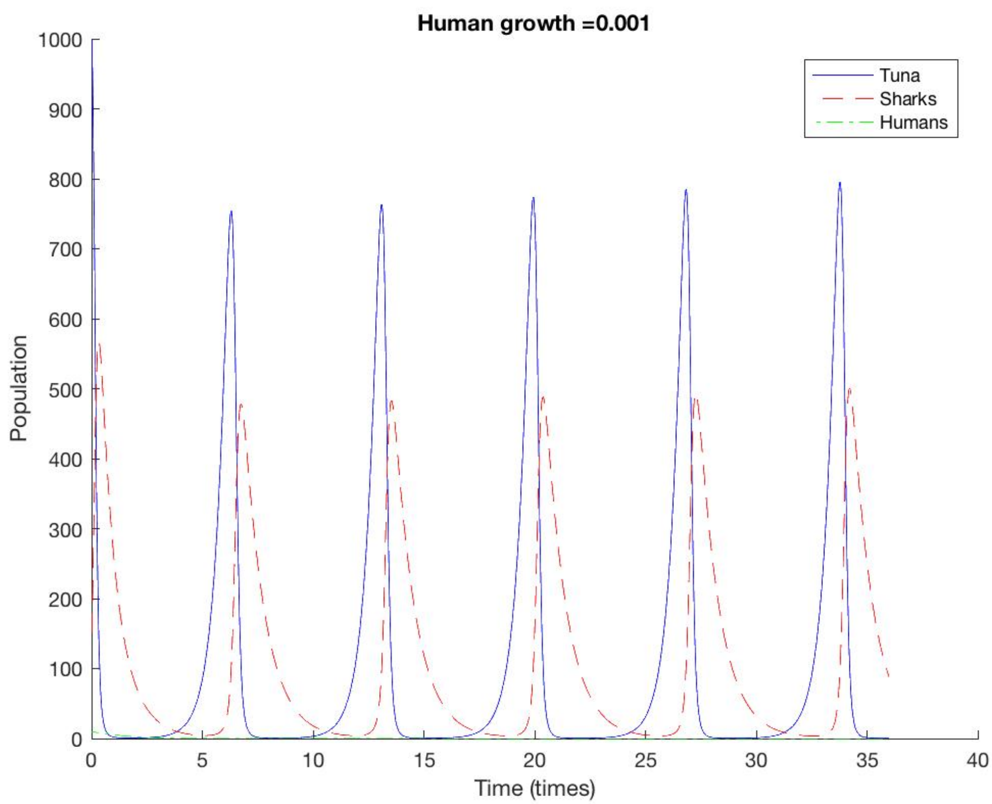

Figure 36: Plot using $h = 0.001$ and base values of other constants for model from 2a.

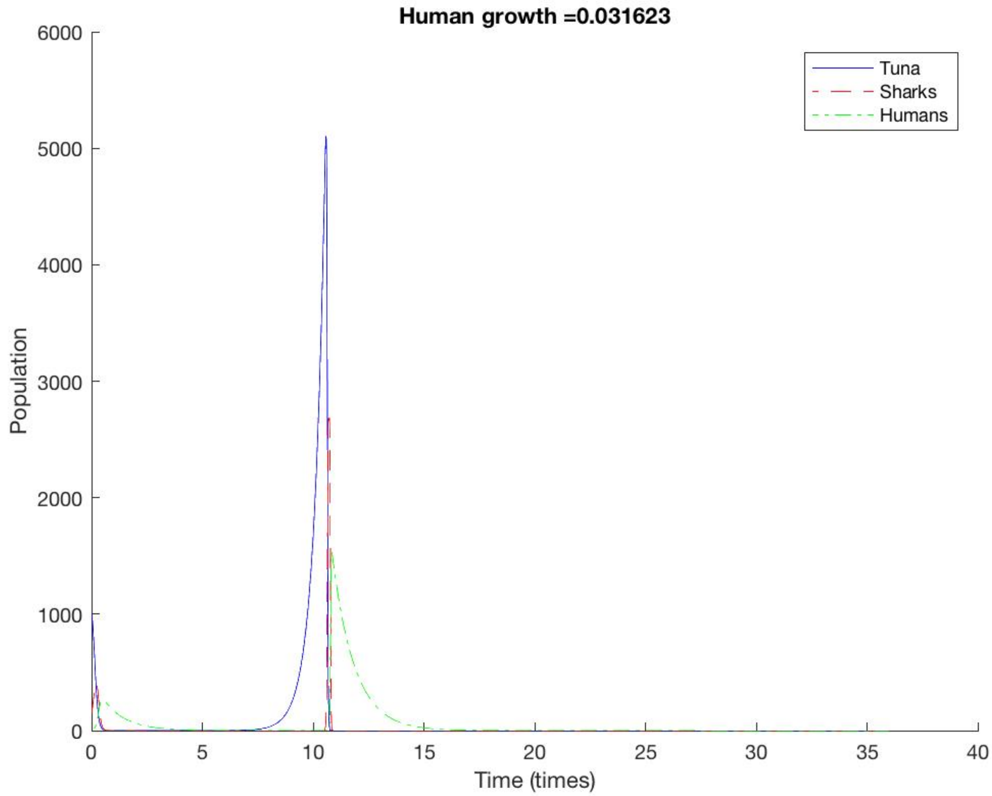

Figure 37: Plot using $h = 0.3$ and base values of other constants for model from 2b.

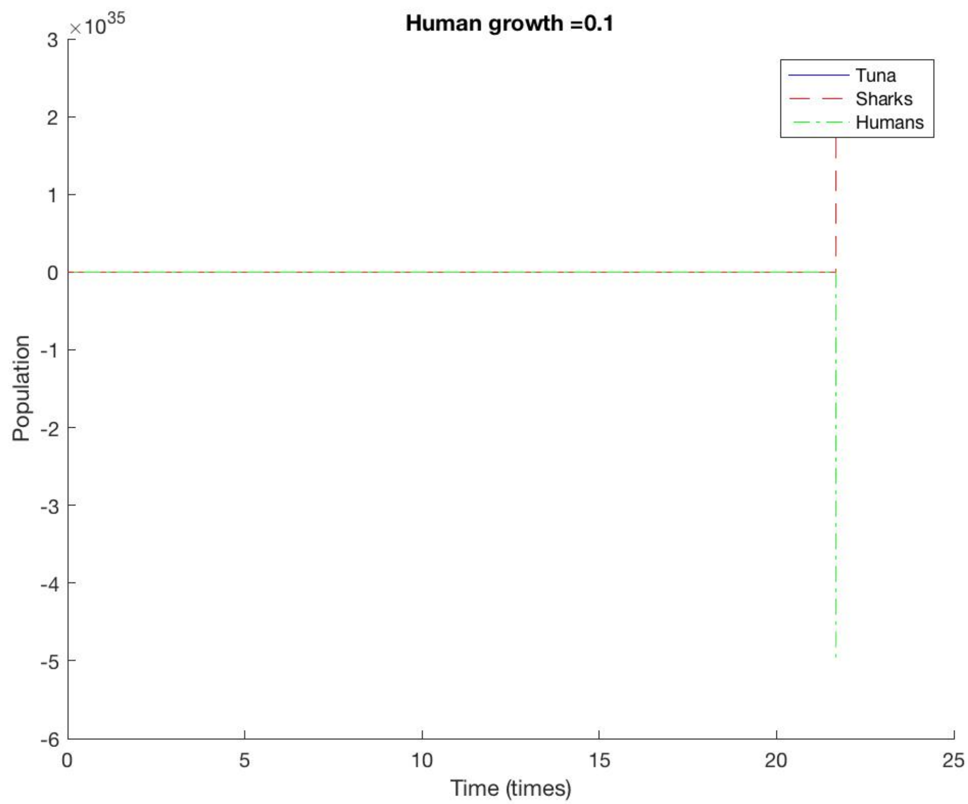

Figure 38: Plot using $h = 0.1$ and base values of other constants for model from 2b.

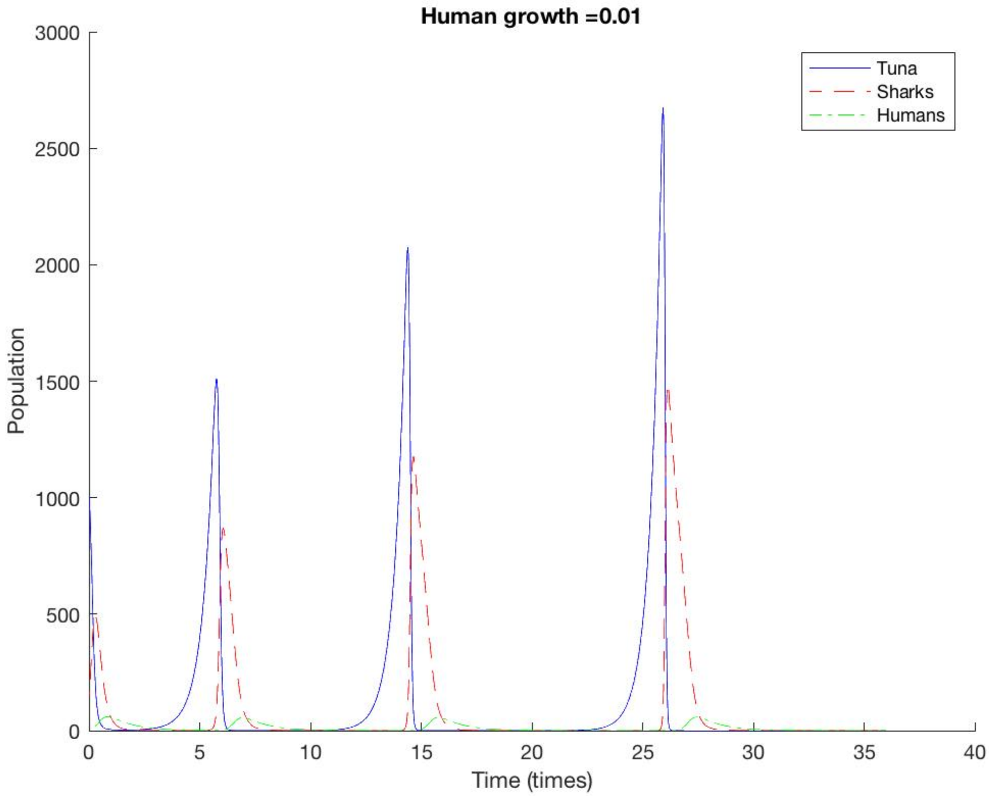

Figure 39: Plot using $h = 0.01$ and base values of other constants for model from 2b.

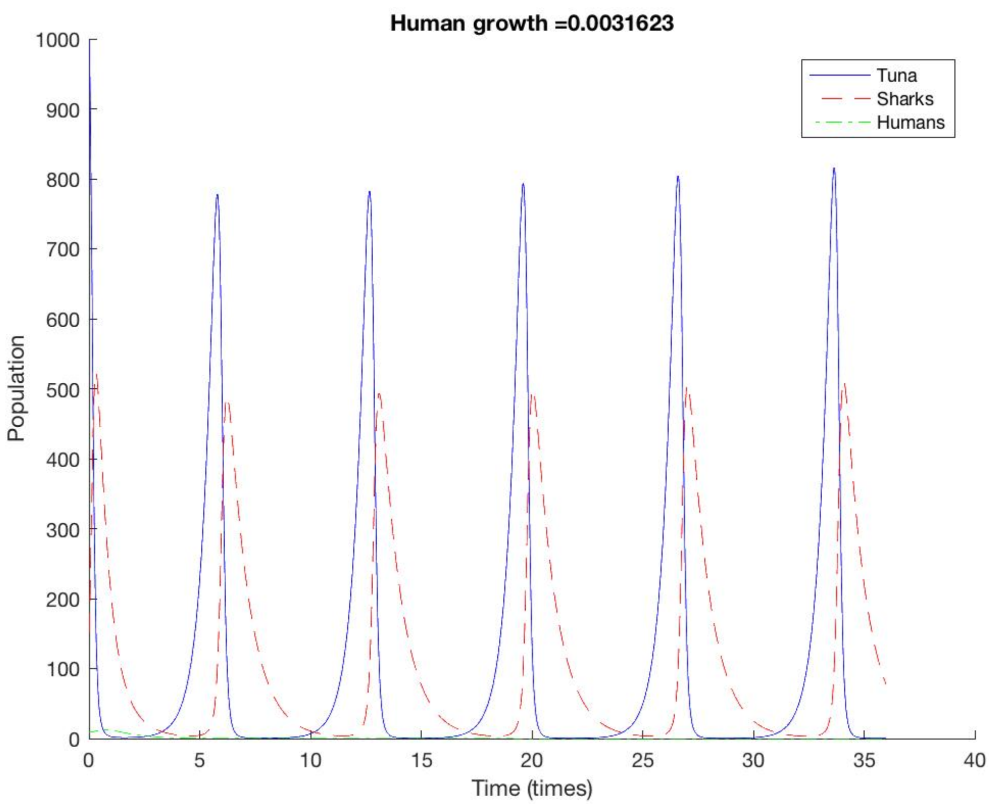

Figure 40: Plot using $h = 0.003$ and base values of other constants for model from 2b.

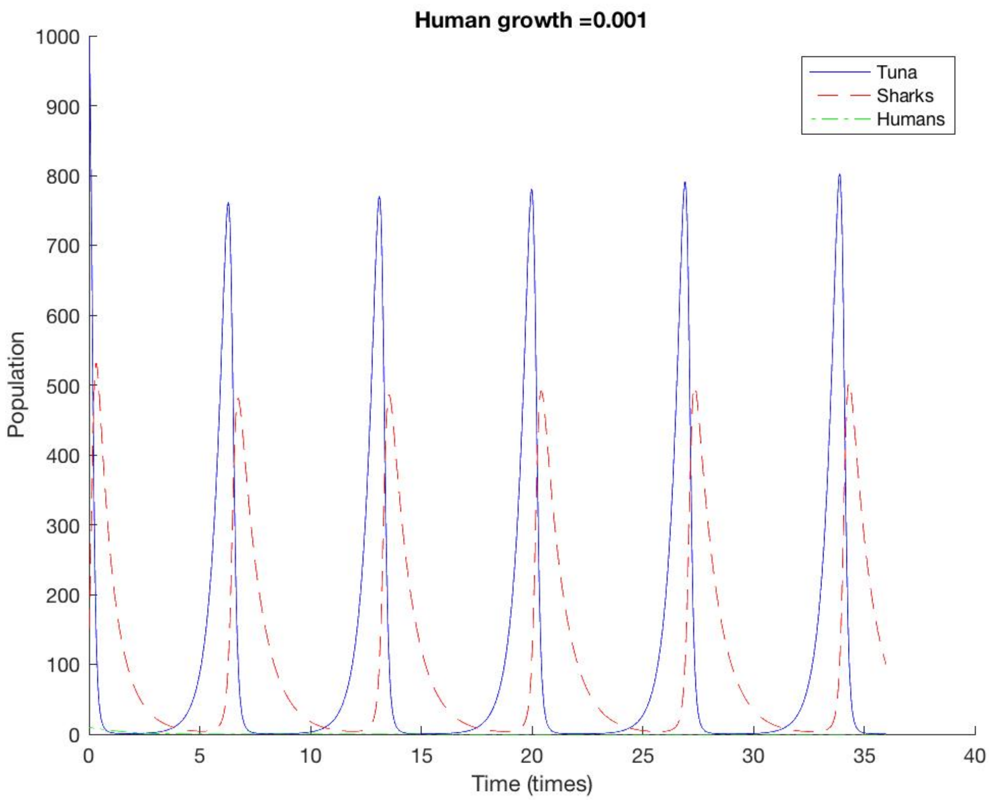

Figure 41: Plot using $h = 0.001$ and base values of other constants for model from 2b.