

Let (\mathbf{x}_i, y_i) denote the training data pair of feature vectors and labels. The regularized logistic regression max log-likelihood can be written as

$$\min_{\mathbf{w}, b} E = \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_i g(-y_i(\mathbf{w}^T \mathbf{x}_i + b)),$$

$$g(\psi) = \log(1 + e^\psi).$$

Note that $g(\psi)$ is the negative of the log-likelihood associated with the probability model,

$$p(y|\mathbf{x}) = \frac{1}{1 + \exp(-y\mathbf{w}^T \mathbf{x} + b)}.$$

We can rewrite E as the constrained minimization

$$\min E = \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_i g(\psi_i),$$

$$\psi_i = -y_i(\mathbf{w}^T \mathbf{x}_i + b), \quad \forall i.$$

Thus, the Lagrangian for this problem is

$$L = \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_i g(\psi_i) + \sum_i \alpha_i [-\psi_i - y_i(\mathbf{w}^T \mathbf{x}_i + b)],$$

which has the KKT optimality conditions given by

$$\nabla_{\mathbf{w}} L = \lambda \mathbf{w} - \sum_i \alpha_i y_i \mathbf{x}_i = 0;$$

$$\frac{\partial L}{\partial b} = \sum_i \alpha_i y_i = 0;$$

$$\frac{\partial L}{\partial \psi_i} = g'(\psi_i) - \alpha_i = 0, \quad \forall i.$$

Hence, these conditions give

$$\mathbf{w}(\alpha) = \frac{1}{\lambda} \sum_i \alpha_i y_i \mathbf{x}_i; \quad \psi_i(\alpha_i) = (g')^{-1}(\alpha_i),$$

and the constraint $\sum_i \alpha_i y_i = 0$.

To continue, let $G(\alpha_i) = g(\psi_i) - \alpha_i \psi_i$, where this is part of L , then

$$\frac{\partial G}{\partial \alpha_i} = \frac{\partial \psi_i}{\partial \alpha_i} g'(\psi_i) - \alpha_i \frac{\partial \psi_i}{\partial \alpha_i} - \psi_i = -\psi_i = -(g')^{-1}(\alpha_i).$$

Therefore, $G'(\alpha_i) = -(g')^{-1}(\alpha_i)$. We can now substitute in the logistic regression

probability model and find $(g')^{-1}(z)$,

$$\begin{aligned} g(z) &= \log(1 + e^z), \\ g'(z) &= e^z / (1 + e^z), \\ g'(z)(1 + e^z) &= e^z, \\ g'(z) &= e^z(1 - g'(z)) \\ g'(z)/(1 - g'(z)) &= e^z \\ \log(g'(z)/(1 - g'(z))) &= z = (g')^{-1}(g'(z)). \end{aligned}$$

Thus,

$$\begin{aligned} (g')^{-1}(u) &= \log(u/(1 - u)), \\ G(\alpha_i) &= -\alpha_i \log \alpha_i - (1 - \alpha_i) \log(1 - \alpha_i). \end{aligned}$$

This gives us the new objective

$$\begin{aligned} \min_{\alpha, b} L &= \frac{\lambda}{2} \|\mathbf{w}(\alpha)\|^2 + \sum_i G(\alpha_i) - \sum_i \alpha_i y_i (\mathbf{w}(\alpha)^T \mathbf{x}_i + b), \\ \text{s.t. } \sum_i \alpha_i y_i &= 0. \end{aligned}$$

Due to the constraint, the term $\sum_i \alpha_i y_i b = 0$. Note that this is also why you cannot just append a column of ones to the original feature vector to get the bias implicitly—this weight will just go to zero. Then, we can substitute in $\mathbf{w}(\alpha)$ and get

$$L = \frac{1}{2\lambda} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \sum_i G(\alpha_i) - \frac{1}{\lambda} \sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j.$$

Combining terms and putting it all together with $K = [\phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)]^{n \times n}$, we get

$$\min_{\alpha} E(\alpha) = -\frac{1}{2\lambda} (\alpha \circ \mathbf{y})^T K (\alpha \circ \mathbf{y}) + \sum_i G(\alpha_i),$$

where \circ is the Hadamard product (\cdot in Matlab). This provides the optimal α , which is used to compute $\mathbf{w}(\alpha)$, but does not compute b , which is essential. However, remember that we are maximizing the log-likelihood of the logistic probability model,

$$p(y|\mathbf{x}) = \frac{1}{1 + \exp(-y(\mathbf{w}(\alpha)^T \mathbf{x} + b))}.$$

Hence, we can substitute in $\mathbf{w}(\alpha)^T \mathbf{x}_j = \sum_i \alpha_i y_i \mathbf{x}_i^T \mathbf{x}_j = (\alpha \circ \mathbf{y})^T K_j$, where K_j is the j th column of K . Then, we can maximize the log-likelihood wrt b ,

$$\max_b \log \left(\prod_j p(y_j | \mathbf{x}_j) \right) = \min_b \sum_j \log (1 + \exp(-y_j((\alpha \circ \mathbf{y})^T K_j + b))).$$

Note that optimization objective can be vectorized in Matlab as

$$\text{sum} \log(1 + \exp(-\mathbf{y} \cdot (K(\alpha \circ \mathbf{y}) + b))).$$

This is an unconstrained optimization; hence, you can use `fminunc`.