LIQUID FUNCTIONAL CALCULUS

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ABSTRACT. We develop a formalism of functional calculus for entire holomorphic functions in the setting of Clausen and Scholze's p-liquid vector spaces.

CONTENTS

1. INTRODUCTION

1.1. **Classical functional calculus.** Let k be a field, and let V be a vector space over k. The data of a linear map $T : V \to V$ is equivalent to a choice of element in the set $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, \mathrm{End}(V))$, where $\mathrm{End}(V)$ denotes the set of linear endomorphisms of V. Of course $\mathrm{End}(V)$ has additional structure:

it is canonically a k-vector space (in fact a k-*algebra*). There is an adjunction between $\mathcal{S}\mathrm{et}$ and the category Vect_k[1] of k-vector spaces given by the forgetful functor from k-vector spaces to $\mathcal{S}\mathrm{et}$ and the free k-vector space construction on sets which furnishes an equivalence,

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, \mathrm{End}(V)) \simeq \mathrm{Hom}_{\mathrm{Vect}_k}(k[x], \mathrm{End}(V)).$$

This equivalence gives a functional calculus for polynomials—given a linear operator, you can act by polynomials in this operator.

If $k = \mathbb{C}$, we might consider topological vector spaces over $\mathbb{C}$. If the topology on such a vector space V is induced by a complete metric, such as when V is a Banach space, we can go further and make sense of things everywhere convergent power series, i.e. entire holomorphic functions $\mathcal{O}(\mathbb{C})$, in this operator. In other words, if V is Banach, the equivalence above lifts as follows,

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathcal{O}(\mathbb{C}), \mathrm{End}(V)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(*, \mathrm{End}(V)) \arrow[r, "\simeq"] \arrow[ur, dashed] & \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathbb{C}[x], \mathrm{End}(V)), \end{tikzcd} \tag{1.1}$$

where the vertical arrow is restriction along the inclusion of $\mathbb{C}$-vector spaces $\mathbb{C}[x] \to \mathcal{O}(\mathbb{C})$. This lift defines what we call the *entire holomorphic functional calculus.*

1.2. **Functional calculus for liquid vector spaces.**

1.2.1. *Condensed vector spaces.* Condensed sets are sheaves of sets defined on the site of profinite sets equipped with the effective epimorphism topology. Condensed sets (nearly) form a topos, so one may consider all manner of algebraic objects, such as rings and modules, defined internally to this topos. A beautiful feature of this category is that it contains the category of compactly generated topological spaces as a full subcategory. As a consequence, the category of compactly generated topological $\mathbb{C}$-vector spaces embeds fully faithfully into the category of condensed $\mathbb{C}$-vector spaces, and the latter is shown by Clausen and Scholze ([CS19]) to be an extremely well-behaved abelian category.

If we try to repeat the procedure above for obtaining a polynomial functional calculus, but now for linear operators in the condensed topos Cond, we find something similar. The endomorphisms of a condensed $\mathbb{C}$-vector space V are a condensed set $\mathrm{End}(V)$ equivalent to the internal mapping object $\mathrm{Hom}_{\mathrm{Cond}}(*, \mathrm{End}(V))$, and by adjunction, we obtain an equivalence,

$$\mathrm{Hom}_{\mathrm{Cond}}(*, \mathrm{End}(V)) \simeq \mathrm{Hom}_{\mathrm{Cond}(\mathbb{C})}(\mathbb{C}[*], \mathrm{End}(V)),$$

where $\mathbb{C}[*]$ denotes the free condensed $\mathbb{C}$-vector space on the point and $\mathrm{Cond}(\mathbb{C})$ denotes the category of condensed $\mathbb{C}$-vector spaces. If V is in the image of the embedding of topological vector spaces into condensed ones, this equivalence reduces the polynomial functional calculus of the previous section.

1.2.2. *Liquid vector spaces.* Recall that, in order to build a functional calculus for entire functions in the previous section, we restricted our attention to complete vector spaces. In the condensed setting, the role of complete vector spaces is played by *liquid vector spaces.* The category of p-liquid vector spaces, for fixed $0 < p \leq 1$, is an abelian category which serves as a well-behaved enlargement of the category of Banach spaces over $\mathbb{C}$. Roughly, it is defined as follows. To each (extremally disconnected[2]) profinite set S is assigned a space of "measures," denoted $\mathcal{M}_{<p}(S)$. One can imagine that an element of $\mathcal{M}_{<p}(S)$ assigns a weight to each point in S. The points of S may themselves be considered a measure by sending $s \in S$ to the Dirac measure assigning weight 1 to

[1]This notation conflicts with our conventions in the rest of the paper where Vect_k will denote the derived ∞-category k-vector spaces, or, equivalently, the ∞-category of k-module spectra.

[2]See the discussion at the end of §3.2.

s and 0 to all other points of S; this gives a map $S \to \mathcal{M}_{<p}(S)$. A p-liquid vector space V is one for which every map of condensed sets $\varphi : S \to V$ extends to a map $\widetilde{\varphi} : \mathcal{M}_{<p}(S) \to V$ of condensed vector spaces. This extension can be thought of as sending $\mu \in \mathcal{M}_{<p}(S)$ to the S-indexed sum of the images of the points $s \in S$ with coefficients given by the weights of μ—in other words, it can be thought of as sending a "measure" μ to the integral $\int_S \varphi d\mu$.

1.2.3. *The main idea.* A choice of endomorphism $T : V \to V$ determines a map of condensed sets $\mathbb{N} \to \mathrm{End}(V)$ by the assignment $T \mapsto T^n$, where $\mathrm{End}(V)$ is the condensed $\mathbb{C}$-algebra given by the internal mapping object of $\mathrm{Cond}(\mathbb{C})$. Assume furthermore that $\mathrm{End}(V)$ is p-liquid. If we pretend for a moment that $\mathbb{N}$ is a profinite set, the property that $\mathrm{End}(V)$ is p-liquid implies the existence of the dotted line in the following diagram,

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{Cond}(\mathbb{C})}(\mathcal{M}_{<p}(\mathbb{N}), \mathrm{End}(V)) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{\mathrm{Cond}}(*, \mathrm{End}(V)) \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_{\mathrm{Cond}}(\mathbb{N}, \mathrm{End}(V)), \end{tikzcd} \tag{1.2}$$

because the vertical arrow is an equivalence. The classical functional calculus was defined by the lift depicted pictorially as the dotted line in diagram (1.1).

Main Idea. Our strategy for defining a functional calculus for liquid vector spaces analogously will be to *identify entire functions as a subspace of* $\mathcal{M}_{<p}(\mathbb{N})$. Using this approach, for a given operator $T : * \to \mathrm{End}(V)$, the operator $f(T)$ is obtained by evaluating the lift $\widetilde{T} : \mathcal{M}_{<p}(\mathbb{N}) \to \mathrm{End}(V)$ on the measure $\mu_f \in \mathcal{M}_{<p}(\mathbb{N})$ associated to the entire function f.

Remark 1.3. Of course, $\mathbb{N}$ is not a profinite set, but this problem is easily solved by considering the space $\mathbb{N} \cup \{\infty\}$, which is (though it is not extremally disconnected).

1.3. **Results.** We define the measure $\mu_f \in \mathcal{M}_{<p}(\mathbb{N} \cup \{\infty\})$ associated to an entire function f in Definition 5.4. Our definition is actually shown to given an element in the space $\mathcal{M}_{<p}(\mathbb{N} \cup \{\infty\})$ in Proposition 5.5. Using μ_f, we define in Definition 5.6 the operator $f(T) \in \mathrm{End}(V)$ for an endomorphism T of an object V of the derived ∞-category $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$ of p-liquid vector spaces with Banach space cohomologies. By restricting our attention to objects of this category, we guarantee that its space of endomorphisms is p-liquid.

The entire functional calculus defined in this way turns out to be compatible with the entire classical functional calculus on Banach spaces. More precisely, in Corollary 5.10 we show that if V is an object in the heart of $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$ (i.e. a Banach space viewed as complex concentrated in degree 0) the operator $f(T)$ defined in Definition 5.6 coincides with the image of the bounded linear operator obtained from the classical functional calculus on Banach spaces under the embedding of Banach spaces into p-liquid vector spaces. Additionally, we show in Proposition 5.11 that the induced maps on homology spaces given by $f(T)$ agrees with the maps obtained via the classical functional calculus on Banach spaces by applying f to the induced maps on homology spaces given by T.

As a trivial application of our results, we deduce a functional calculus for perfect complexes of $\mathbb{C}$-vector spaces in §6.

1.3.1. *Significance.* The results described above allow one to make sense of the make sense of things like the exponential of an automorphism of complexes of Banach spaces, up to p-liquid quasi-isomorphism. The author has no immediate application in mind for such a device, but examples where one considers complexes of infinite-dimensional vector spaces with non-trivial topologies abound in mathematical physics.

1.4. **Remarks on general holomorphic functional calculus.** Classically, there is a more general theory of functional calculus for holomorphic functions defined only on a open neighborhood $\Omega \subset \mathbb{C}$ of the spectrum $\sigma(a)$ of a given element a in a Banach algebra A. More precisely (following [Rud91, Chapter 10]), fix an open subset $\Omega \subset \mathbb{C}$ and let $A_\Omega := \{a \in A | \sigma(a) \subset \Omega\}$. Let $H(\Omega)$ denote the algebra of holomorphic functions on Ω, and let $C(A_\Omega)$ denote the algebra of A-valued functions on A_Ω. Then the holomorphic functional calculus is an assignment

$$f \mapsto \widetilde{f}$$

which is an algebra isomorphism onto its image, as well as continuous in the sense that if $f_n \to f$ uniformly on compact subsets of Ω, then $\widetilde{f}_n(a) \to \widetilde{f}(a) \in A$. The operators $\widetilde{f}(a)$ are defined using contour integration around a contour containing the spectrum $\sigma(a)$.

We suspect that our functional calculus should extend to a full holomorphic functional calculus for an endomorphism T of an object $V \in \mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$ and holomorphic functions defined on an open set containing the spectrum of T, using as our definition of spectrum of $T \in \mathrm{End}(V)$ the support of V as a section of the category $\mathcal{D}(\mathrm{Liq}_p(\mathbb{C}[T]))$ considered as a sheaf over $\mathbb{C}$. Doing so, however, would require a systemic way of treating integrals of functions valued in liquid vector spaces (and complexes thereof), which is outside the scope of this work.

2. Notation

Notation 2.1. If R is a classical associative ring, we may view it as an $\mathbb{E}_1$-algebra in Sp. We let LMod_R denote the ∞-category of left R-module objects in Sp. This category has a canonical t-structure, with respect to which $\mathrm{LMod}_R^\heartsuit$ is the abelian category of left R-modules. Note that LMod_R is equivalent to the unbounded derived ∞-category of its heart, $\mathcal{D}(\mathrm{LMod}_R^\heartsuit)$.

When R is commutative, it has a natural structure of an $\mathbb{E}_\infty$-algebra, and we let Mod_R denote the category of R-modules in spectra.

2.1. **Morphisms.** We use Prof to denote the category of profinite sets, i.e. compact Hausdorff totally disconnected topological spaces.

We use the notation "$\underline{\mathrm{Hom}}_{\mathscr{C}}(-,-)$" to denote the internal hom of a category $\mathscr{C}$; that is, $\underline{\mathrm{Hom}}_{\mathscr{C}}(-,-)$ is always itself an object of $\mathscr{C}$.

On the other hand, we use the notation "$\mathrm{Hom}_{\mathscr{C}}(-,-)$" to denote the ∞-groupoid (resp. set) of morphisms in the ∞-category (resp. 1-category) $\mathscr{C}$.

2.2. **Non-abelian derived categories.** We use Ani to denote the ∞-category of spaces, the non-abelian derived category of finite sets.

2.3. **Grothendieck universes.** It is inevitable when working with higher categories that one will encounter "large" collections of objects, i.e. collections that do not form sets. We adopt the same approach these objects as Lurie outlines in §1.2.15 of [HTT], and which we recall below.

2.3.1. *Strongly inaccessible cardinals.*

Assumption 2.2. We assume that for each cardinal α_0, there exists a strongly inaccessible cardinal $\alpha \geq \alpha_0$.

Let $\mathcal{U}(\alpha)$ denote the collection of all sets having rank $< \alpha$. Then $\mathcal{U}(\alpha)$ is a Grothendieck universe; it satisfies all of the usual axioms of set theory.

Definition 2.3. A mathematical object is *α-small* if it belongs to $\mathcal{U}(\alpha)$. It is *essentially α-small* if it is equivalent (in whatever relevant sense) to an α-small object. We let $\mathcal{S}\mathrm{et}_\alpha$ denote the category of α-small sets.

Outside of foundational work, mathematics takes place in a fixed Grothendieck universe, so when an author like Lurie writes "small," he has implicitly chosen a Grothendieck universe.

2.4. **Glossary of categories.** This work involves many different categories whose definitions and differences are often subtle. This work also contains many results which may be of independent interest to the reader. To facilitate the reader trying to read this work in a piecemeal fashion, we furnish a glossary of categories below. We have tried to group the categories by theme. In what follows, κ is an uncountable strong limit cardinal, and $\alpha > \kappa$ is a strongly inaccessible cardinal.

2.4.1. *Glossary of categories.*

Categories of purely algebraic objects.

- Ring is the ordinary category of rings.
  - Ring_α is the full subcategory of Ring on α-small rings.
- Grp is the ordinary category of groups.
  - Grp_α is the full subcategory of Grp on α-small groups.
- Ab is the ordinary category of abelian groups.
  - Ab_α is the full subcategory of Ab on α-small abelian groups.
- $\mathcal{D}_{\geq 0}(\text{Ab})$ is the ∞-category of non-negative homologically graded chain complexes in Ab. It is equivalent to the animation of Ab, Ani(Ab).
  - $\mathcal{D}(\text{Ab})$ is the stabilization of $\mathcal{D}_{\geq 0}(\text{Ab})$, also known as the unbounded derived ∞-category of Ab.
- $\text{Vect}_\mathbb{C}$ is the unbounded derived ∞-category of $\mathbb{C}$-vector spaces, also know as the ∞-category of chain complexes of $\mathbb{C}$-vector spaces with quasi-isomorphisms inverted.
  - $\text{Vect}_\mathbb{C}^{f.d.}$ is ordinary category of finite dimensional $\mathbb{C}$-vector spaces. They are the compact projective generators of $\text{Vect}_\mathbb{C}$.

Categories of categories

- $\widehat{\mathcal{C}\text{at}}_\infty$ is the category of *all* (i.e. not necessarily small) ∞-categories.
  - $\mathcal{C}\text{at}_\infty$ is the category of small ∞-categories.

Categories of topological objects.

- $\mathcal{T}\text{op}$ is the ordinary category of topological spaces, whose morphisms are continuous functions.
  - $\mathcal{T}\text{op}_\kappa$ is the full subcategory of $\mathcal{T}\text{op}$ on κ-compactly generated topological spaces.
- $\mathcal{T}\text{Ring}$ is the ordinary category of topological rings, whose morphisms are continuous ring homomorphisms.
  - $\mathcal{T}\text{Ring}_\kappa$ is the full subcategory of $\mathcal{T}\text{Ring}$ on objects whose underlying topological space is κ-compactly generated.
- $\mathcal{T}\text{Grp}$ is the ordinary category of topological groups, whose morphisms are continuous group homomorphisms.
  - $\mathcal{T}\text{Grp}_\kappa$ is the full subcategory of $\mathcal{T}\text{Grp}$ on objects whose underlying topological space is κ-compactly generated.
- $\mathcal{T}\mathcal{V}\text{ect}$ is the ordinary category of topological $\mathbb{C}$-vector spaces, whose morphisms are continuous linear maps.
  - $\mathcal{T}\mathcal{V}\text{ect}_{\text{l.c.}}^{\text{comp}}$ is the full subcategory of $\mathcal{T}\mathcal{V}\text{ect}$ on locally compact topological vector spaces whose topology is induced by a complete metric.
- **Ban** is the ordinary category of complex Banach spaces, whose morphisms are bounded linear operators.

Categories coming from condensed mathematics.

- Cond is the category of κ-condensed sets.
  - Cond(Ring) is the category of κ-condensed rings.
  - Cond(Grp) is the category of κ-condensed groups.
  - Cond(Ab) is the category of κ-condensed abelian groups.

- $\mathrm{Cond}(\mathscr{C})$ is the category of κ-condensed objects in the ∞-category $\mathscr{C}$.
- $\mathrm{Cond}(\mathcal{A})$ is the category of modules in $\mathrm{Cond}(\mathrm{Ab})$ over the condensed commutative ring $\mathcal{A}$.
  - $\heartsuit(\mathcal{A}, \mathcal{M})$ is the full subcategory of $\mathrm{Cond}(\mathcal{A})$ determined by the analytic ring structure $(\mathcal{A}, \mathcal{M})$, defined in Definition 4.9.
- $\mathcal{D}_{\geq 0}(\mathcal{A})$ is the ∞-category of modules in $\mathrm{Cond}(\mathcal{D}_{\geq 0}(\mathrm{Ab}))$ over the condensed animated ring $\mathcal{A}$.
  - $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ is the full subcategory of $\mathcal{D}_{\geq 0}(\mathcal{A})$ determined by the analytic animated ring structure $(\mathcal{A}, \mathcal{M})$, defined in Definition 4.8.
- $\mathcal{D}(\mathcal{A})$ is the stable ∞-category given by the stabilization of $\mathcal{D}_{\geq 0}(\mathcal{A})$.
  - $\mathcal{D}(\mathcal{A}, \mathcal{M})$ is the stabilization of $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$.
- $\mathrm{Liq}_p(\mathbb{C})$ is alternative notation for $\heartsuit(\mathbb{C}, \mathcal{M}_{<p})$, where $(\mathbb{C}, \mathcal{M}_{<p})$ denotes the p-liquid analytic ring structure on $\underline{\mathbb{C}}$.
- $\mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$ is the category $\mathcal{D}(\mathcal{A}, \mathcal{M})$ listed above for the analytic animated ring $(\mathbb{C}, \mathcal{M}_{<p})$.
  - $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$ is the full subcategory of $\mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$ on objects whose homology spaces lie in the essential image of the embedding $\mathbf{Ban} \hookrightarrow \mathrm{Liq}_p(\mathbb{C})$.

3. Condensed mathematics

3.1. **Condensed sets.** Roughly speaking, a condensed set is a sheaf of sets on the pro-étale site of a geometric point, denoted $*_{\text{proét}}$. Explicitly, $*_{\text{proét}}$ is the site given by the category of profinite sets, denoted Prof, with open covers given by finite families of jointly surjective maps. For the sake of clarity, we recall the precise definition.

Definition 3.1. Let $\mathcal{U}(\alpha)$ be a fixed Grothendieck universe. For any α-small, ordinary category $\mathscr{C}$, the category $\mathsf{Pro}^{\alpha}(\mathscr{C})$ is defined as the full subcategory of $\mathrm{Fun}(\mathscr{C}, \mathcal{S}\mathrm{et}_{\alpha})^{\mathrm{op}}$ on those functors which are limits of cofiltered diagrams of functors representable under the Yoneda embedding $\mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}, \mathcal{S}\mathrm{et}_{\alpha})^{\mathrm{op}}$. We set $\mathrm{Prof}^{\alpha} := \mathsf{Pro}^{\alpha}(\mathcal{F}\mathrm{in})$.

Remark 3.2. As a category, Prof^{α} is equivalent to the category of compact Hausdorff totally disconnected topological spaces whose underlying sets are α-small.

There is a problem, however, naively defining condensed sets as sheaves on this site. Given any choice of Grothendieck universe $\mathcal{U}(\alpha)$, the category Prof^{α} is large, so it is not a good idea to work with the category of sheaves on it since such a category would not be a topos. Clausen-Scholze circumvent this problem by working with a modification of Prof^{α} obtained as follows. Choose an uncountable strong limit cardinal $\kappa < \alpha$, and instead consider the category of κ-small profinite sets, denoted Prof_{κ}, rather than Prof^{α}.

Definition 3.3 ([CS19, Definition 2.1])**.** The site $*_{\kappa-\text{proét}}$ is the site of κ-small profinite sets S with covers given by finite families of jointly surjective maps.

Remark 3.4. Note that Prof_{κ} is an α-small category; this follows from the definition of strongly inaccessible cardinal.

Clausen-Scholze define the category of κ-condensed sets as the category of $\mathcal{S}\mathrm{et}_{\alpha}$-valued (resp. Grp_{α}-valued, Ring_{α}-valued, Ab_{α}-valued) sheaves on $*_{\kappa-\text{proét}}$, which they denote by Cond_{κ} (resp. $\mathrm{Cond}_{\kappa}(\mathrm{Grp})$, $\mathrm{Cond}_{\kappa}(\mathrm{Ring})$, $\mathrm{Cond}_{\kappa}(\mathrm{Ab})$). For any two choices of uncountable strong limit cardinal $\kappa' > \kappa$, they show that there is a fully faithful embedding of κ-condensed sets into κ'-condensed sets ([CS19, Proposition 2.9]). They then define the category of condensed sets to be the filtered colimit,

$$\varinjlim_{\kappa} \mathrm{Cond}_{\kappa}$$

(resp. $\varinjlim_{\kappa} \mathrm{Cond}_{\kappa}(\mathrm{Grp})$, $\varinjlim_{\kappa} \mathrm{Cond}_{\kappa}(\mathrm{Ring})$, $\varinjlim_{\kappa} \mathrm{Cond}_{\kappa}(\mathrm{Ab})$), taken in some suitable category of categories.

Convention 3.5. In the sequel, we fix a Grothendieck universe $\mathcal{U}(\alpha)$ by fixing a strongly inaccessible cardinal α. Having fixed a Grothendieck universe, we omit any reference to α in our notation. For example, $\mathcal{S}\mathrm{et}_\alpha$, Pro^α, and Prof^α, respectively, will be denoted by $\mathcal{S}\mathrm{et}$, Pro, and Prof, respectively.

3.1.1. *Condensed sets as replacements for topological spaces.* Clausen and Scholze defined condensed sets as replacements for topological spaces that have better behavior when considered with algebraic structures. Strong evidence for their suitability as a replacement is given by the following proposition of Clausen-Scholze.

Proposition 3.6 ([CS19, Proposition 1.7])**.** *Let X be a topological space, and denoted by $\underline{X}$ the condensed set given by the assignment*

$$S \mapsto \mathrm{Cont}(S, X),$$

for $S \in \mathrm{Prof}_\kappa$, where $\mathrm{Cont}(S, X)$ denotes the set of continuous functions $S \to X$. Then $\underline{(-)} : \mathcal{T}\mathrm{op}_\kappa \to \mathrm{Cond}_\kappa$ is a fully faithful functor from the category of κ-compactly generated topological spaces to κ-condensed sets.

Remark 3.7. The functor $\underline{(-)}$ also induces fully faithful embeddings of $\mathcal{T}\mathrm{Grp}_\kappa$ and $\mathcal{T}\mathrm{Ring}_\kappa$ into $\mathrm{Cond}_\kappa(\mathrm{Grp})$ and $\mathrm{Cond}_\kappa(\mathrm{Ring})$, respectively.

Remark 3.8. If R is a topological ring with the discrete topology, $\underline{R}$ is a constant sheaf on Prof_κ. Note that the image, for example, of the topological field $\mathbb{R}$ under $\underline{(-)}$, however, is *not* a constant sheaf. In particular, modules in $\mathrm{Cond}_\kappa(\mathrm{Ab})$ over each of $\underline{\mathbb{R}}$ and $\underline{\mathbb{R}_{\mathrm{disc}}}$ are different.

Unlike the category of topological abelian groups, the category of condensed abelian groups forms an abelian category with extremely nice properties.

Theorem 3.9 ([CS19, Theorem 2.2])**.** *The category of κ-condensed abelian groups is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*), to wit: all limits (AB3*) and colimits (AB3) exist, arbitrary products (AB4*), arbitrary direct sums (AB4) and filtered colimits (AB5) are exact, and (AB6) for any index set J and filtered categories I_j, $j \in J$, with functors $i \mapsto M_i$ from I_j to κ-condensed abelian groups, the natural map*

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism. Moreover, the category of κ-condensed abelian groups is generated by compact projective objects.

Remark 3.10. A variant of the above theorem holds for the category of *all* condensed abelian groups, $\varinjlim_\kappa \mathrm{Cond}_\kappa(\mathrm{Ab})$.

3.2. **Condensed objects.** As a means of proving the various properties of Cond, Clausen-Scholze take advantage of the fact that condensed sets are determined by their values on very special types of profinite sets, called *extremally disconnected* profinite sets.

Proposition 3.11 ([CS19, Proposition 2.7])**.** *Consider the site of κ-small extremally disconnected profinite sets, denoted $\mathrm{ExtProf}_\kappa$, with covers given by finite families of jointly surjective maps. Its category of sheaves is equivalent to κ-condensed sets via restriction from profinite sets.*

Motivated by the proposition, Clausen-Scholze define κ-condensed objects of a category $\mathscr{C}$, more generally, as follows.

Definition 3.12 ([CS20, Definition 11.7])**.** Let $\mathscr{C}$ be an ∞-category that admits all small colimits. The category of κ-condensed objects of $\mathscr{C}$, denoted $\mathrm{Cond}_\kappa(\mathscr{C})$, is the category of contravariant sheaves from $\mathrm{ExtProf}_\kappa$ to $\mathscr{C}$ that take finite coproducts to products.

Remark 3.13. If $\mathscr{C}$, in addition, admits all small limits, then $\mathrm{Cond}_\kappa(\mathscr{C})$ can be identified with the category of $\mathscr{C}$-valued sheaves on $\mathrm{ExtProf}_\kappa$.

As above, for any two choices of cardinals, $\kappa' > \kappa$, there is a fully faithful functor,

$$\mathrm{Cond}_\kappa(\mathscr{C}) \to \mathrm{Cond}_{\kappa'}(\mathscr{C}),$$

given as the left adjoint to the forgetful functor from $\mathrm{Cond}_{\kappa'}(\mathscr{C})$ to $\mathrm{Cond}_\kappa(\mathscr{C})$. We may likewise form the category of condensed objects in $\mathscr{C}$ as the filtered colimit,

$$\varinjlim_{\kappa} \mathrm{Cond}_\kappa(\mathscr{C}),$$

taken in, $\widehat{\mathcal{C}\mathrm{at}}_\infty$, the ∞-category of all (e.g. not necessarily small) ∞-categories.

3.2.1. *Pyknotic objects.* While Clausen and Scholze were developing their theory of condensed sets, Barwick and Haine had been studying essentially the same notion, which they call *pyknotic sets*. The difference between pyknotic sets and condensed sets is purely set-theoretic in nature. We refer the reader to §0.3 of [BH19] a discussion on the differences between the two theories.

For our purposes, it will be useful to deploy some results from theory of pyknotic objects in the current setting of condensed objects. In order to do so, we briefly recall the notion of pyknotic set. In the notation of the previous section: assume the existence of a smallest strongly inaccessible cardinal $\alpha' > \alpha$. The category Prof_α is small in the universe $\mathcal{U}(\alpha')$, so sheaves on it form a topos in this larger universe.

Definition 3.14. The category of pyknotic sets is defined to be the category of $\mathcal{S}\mathrm{et}_{\alpha'}$-valued sheaves on Prof^α. More generally, given an ∞-category $\mathscr{C}$, the category of pyknotic objects of $\mathscr{C}$, denoted $\mathrm{Pyk}_\alpha(\mathscr{C})$, is subcategory of contravariant functors from $\mathrm{ExtProf}^\alpha$ to $\mathscr{C}$ that take finite coproducts to products.

Remark 3.15. As with condensed objects, it $\mathscr{C}$ admits all small limits, then

$$\mathrm{Pyk}_\alpha(\mathscr{C}) = \mathrm{Shv}_{\mathscr{C}}(\mathrm{ExtProf}^\alpha).$$

Though we have fixed α and α' both to be strongly inaccessible cardinals, the results in the pyknotic literature hold equally well if we take α to be an uncountable strong limit cardinal and $\alpha' > \alpha$ a strongly inaccessible cardinal. In other words, we may take α to be κ of the previous section, and α' to be α of the previous section. With these choices, it is clear that

$$\mathrm{Cond}_\kappa(\mathscr{C}) = \mathrm{Pyk}_\kappa(\mathscr{C}).$$

As such, we freely use the results of [BH19] below when working with κ-condensed objects.

3.2.2. *Condensed objects as sheaves on* Prof_κ*.* We would like to prove an analogue of [CS19, Proposition 2.7] for sheaves on Prof_κ with values in categories other than $\mathcal{S}\mathrm{et}$. As observed in Remark 3.4, $\mathrm{ExtProf}_\kappa$ and Prof_κ are small sites, so each determines an ∞-topos. By abuse of notation, we let Prof_κ and $\mathrm{ExtProf}_\kappa$ denote the ∞-topoi determined by each site.

Proposition 3.16. *Let* $\widehat{\mathrm{Prof}}_\kappa$ *denote the hypercompletion of* Prof_κ*. Then* $\widehat{\mathrm{Prof}}_\kappa$ *and* $\mathrm{ExtProf}_\kappa$ *are equivalent* ∞*-topoi.*

Proof. This is [BH19, Warning 2.2.2] and [BH19, Corollary 2.4.4]. □

Corollary 3.17. *Let* $\mathscr{C}$ *be a presentable* ∞*-category. Restriction to extremally disconnected profinite sets determines an equivalence of categories,*

$$\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa) \xrightarrow{\simeq} \mathrm{Shv}_{\mathscr{C}}(\mathrm{ExtProf}_\kappa).$$

In particular, if $\mathscr{C}$ *additionally admits all small limits, then*

$$\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa) \simeq \mathrm{Cond}_\kappa(\mathscr{C}).$$

Proof. Not that the category of $\mathscr{C}$-valued sheaves on *any* ∞-topos $\mathfrak{X}$ is given by the Lurie tensor product of presentable ∞-categories, $\mathfrak{X} \otimes \mathscr{C}$. Since, $\widehat{\mathrm{Prof}}_\kappa \xrightarrow{\simeq} \mathrm{ExtProf}_\kappa$ via restriction by the above proposition, it follows that

$$\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa) \simeq \widehat{\mathrm{Prof}}_\kappa \otimes \mathscr{C} \xrightarrow{\simeq} \mathrm{ExtProf}_\kappa \otimes \mathscr{C} \simeq \mathrm{Shv}_{\mathscr{C}}(\mathrm{ExtProf}_\kappa),$$

as desired. □

Note that since $\widehat{\mathrm{Prof}}_\kappa \to \mathrm{Prof}_\kappa$ is fully faithful, the functor $\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa) \to \mathrm{Shv}_{\mathscr{C}}(\mathrm{Prof}_\kappa)$ is also fully faithful. The significance of Corollary 3.17 is that it allows us to take sections of condensed objects of $\mathscr{C}$ on arbitrary κ-small profinite sets though condensed objects are a priori defined only on extremally disconnected profinite sets, as shown in the following lemma.

Lemma 3.18. *Suppose that $\mathscr{C}$ is a complete presentable ∞-category. Then there is a fully faithful embedding,*

$$\mathrm{Cond}_\kappa(\mathscr{C}) \hookrightarrow \mathrm{Shv}_{\mathscr{C}}(\mathrm{Prof}_\kappa).$$

Proof. By Corollary 3.17, it suffices to show that $\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa)$ embeds fully faithfully into $\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa)$. First note that the inclusion $i_* : \widehat{\mathrm{Prof}}_\kappa \subset \mathrm{Prof}_\kappa$ is a fully faithful geometric morphism (this is true for the inclusion of the hypercomplete objects of any ∞-topos). As such, it admits a left exact left adjoint $i^* : \mathrm{Prof}_\kappa \to \widehat{\mathrm{Prof}}_\kappa$. Pointwise composition with $(i^*)^{\mathrm{op}}$ determines a functor,

$$\mathrm{Fun}(\widehat{\mathrm{Prof}}_\kappa^{\mathrm{op}}, \mathscr{C}) \xrightarrow{i_*} \mathrm{Fun}(\mathrm{Prof}_\kappa^{\mathrm{op}}, \mathscr{C})$$

which is fully faithful because i^* is fully faithful. Moreover, since i^* is left adjoint, $(i^*)^{\mathrm{op}}$ preserves small limits, meaning i_* restricts to a functor,

$$\mathrm{Shv}_{\mathscr{C}}(\widehat{\mathrm{Prof}}_\kappa) \xhookrightarrow{i_*} \mathrm{Shv}_{\mathscr{C}}(\mathrm{Prof}_\kappa),$$

as desired. □

3.3. **Important convention.** Throughout the remainder of this paper, we fix an uncountable strong limit cardinal κ and work with objects in $\mathrm{Cond}_\kappa(\mathscr{C})$. In doing so, we will omit κ from all of our notation. In particular, we will use $\mathrm{Cond}(\mathscr{C})$, Prof, and ExtProf to denote $\mathrm{Cond}_\kappa(\mathscr{C})$, Prof_κ, and $\mathrm{ExtProf}_\kappa$, respectively. When we say "condensed" we mean "κ-condensed."

Warning 3.19. This is a significant, and potentially confusing, departure from the conventions of Clausen-Scholze and the rest of the condensed mathematics literature; $\mathrm{Cond}(\mathscr{C})$ universally denotes the colimit $\varinjlim_\kappa \mathrm{Cond}_\kappa(\mathscr{C})$. Thus, for the sake of clarity, we reiterate: $\mathrm{Cond}(\mathscr{C})$ for us denotes the category of κ-condensed objects of $\mathscr{C}$, for some fixed κ.

3.4. **Miscellanea.** We state and prove below a handful of results that will be useful for our purposes.

3.4.1. $\mathcal{T}\mathrm{op}$ *embeds fully faithfully into condensed anima.* In the next section, we would like to view the topological fields $\mathbb{R}$ and $\mathbb{C}$ not only as condensed rings under the embedding of Remark 3.7, but also as condensed *animated* (meaning simplicial) rings.

Lemma 3.20. *There is a fully faithful functor,*

$$\mathcal{T}\mathrm{op} \to \mathrm{Cond}(\mathrm{Ani}).$$

Proof. Given any cocomplete, compactly generated category $\mathscr{C}$, there is an inclusion $\mathscr{C} \to \mathrm{Ani}(\mathscr{C})$. Letting $\mathscr{C} = \mathrm{Cond}$, we obtain a fully faithful functor $\mathrm{Cond} \to \mathrm{Ani}(\mathrm{Cond})$. Using [CS20, Proposition 11.8], this gives a fully faithful functor $\mathrm{Cond} \to \mathrm{Cond}(\mathrm{Ani})$. The desired functor is then obtained by composing this functor with the embedding $\mathcal{T}\mathrm{op} \hookrightarrow \mathrm{Cond}$ from Proposition 3.6. □

Remark 3.21. Obviously, similar results hold for $\mathcal{T}\mathrm{Grp}$ and $\mathcal{T}\mathrm{Ring}$.

3.4.2. *Stabilization commutes with taking condensed objects.*

Lemma 3.22. *Let $\mathscr{C}$ be a presentable ∞-category. Then there is a natural equivalence of ∞-categories,*

$$\mathrm{Sp}(\mathrm{Cond}(\mathscr{C})) \simeq \mathrm{Cond}(\mathrm{Sp}(\mathscr{C})). \tag{3.23}$$

Proof. Recall that, for any presentable ∞-category $\mathscr{D}$, the category of $\mathscr{D}$-valued sheaves on any ∞-topos $\mathcal{X}$ is given by the Lurie tensor product[3],

$$\mathrm{Shv}_{\mathscr{C}}(\mathcal{X}) \simeq \mathscr{C} \otimes \mathcal{X}$$

(see [SAG, Remark I.1.3.1.6]). In particular, Cond(Ani) is the ∞-topos of sheaves on the small site $\{*\}_{\text{proét}_\kappa}$, so $\mathrm{Shv}_{\mathrm{Sp}(\mathscr{C})}(\mathrm{Cond}(\mathrm{Ani})) \simeq \mathrm{Sp}(\mathscr{C}) \otimes \mathrm{Cond}(\mathrm{Ani})$. On the other hand, the stabilization of a presentable ∞-category $\mathscr{C}$ admits a characterization as the Lurie tensor product of $\mathscr{C}$ with Sp[4],

$$\mathrm{Sp}(\mathscr{C}) \simeq \mathrm{Sp} \otimes \mathscr{C}.$$

Together, these two facts reduce (3.23) to the claim that,

$$\mathrm{Sp} \otimes (\mathrm{Cond}(\mathrm{Ani}) \otimes \mathscr{C}) \simeq \mathrm{Cond}(\mathrm{Ani}) \otimes (\mathrm{Sp} \otimes \mathscr{C}).$$

But this is clear from the fact that the ∞-category of presentable ∞-categories, $\mathcal{P}\mathrm{r}^L$, is symmetric monoidal under the Lurie tensor product. □

3.4.3. *Stabilization commutes with taking module objects.*

Lemma 3.24. *Suppose that $\mathcal{X}$ is an ∞-topos, and let $\mathcal{O} \in \mathcal{X}$ be a grouplike commutative algebra object. Then there is an equivalence of stable ∞-categories,*

$$\mathrm{Sp}(\mathrm{Mod}_{\mathcal{O}}(\mathcal{X})) \simeq \mathrm{Mod}_{\mathcal{O}_\Sigma}(\mathrm{Sp}(\mathcal{X})),$$

where $\mathcal{O}_\Sigma$ denotes the image of $\mathcal{O}$ in $\mathrm{Sp}(\mathcal{X})$.

Proof. We note that, by [SAG, Remark I.1.3.5.1], there is a canonical equivalence,

$$\mathrm{Shv}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathcal{X}) \simeq \mathrm{CAlg}(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})).$$

The reasoning in that remark applies to prove to a similar statement for $\mathrm{CAlg}(\mathcal{S})$. Namely the forgetful functor $\mathrm{CAlg}(\mathcal{S}) \to \mathcal{S}$ is conservative and preserves small limits by [HA, Lemma 3.2.2.6 and Corollary 3.2.2.5]. It follows that we have a canonical equivalence,

$$\mathrm{Shv}_{\mathrm{CAlg}(\mathcal{S})}(\mathcal{X}) \simeq \mathrm{CAlg}(\mathcal{X}).$$

We now note that the pointwise application of the suspension functor, $\Sigma^\infty : \mathcal{S} \to \mathrm{Sp}$, induces a functor, $F_{\Sigma^\infty} : \mathcal{P}\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}) \to \mathcal{P}\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, which descends to a functor of sheaves,

$$\mathcal{X} \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}),$$

given by $L \circ F_{\Sigma^\infty}$, where L is the sheafification functor, $\mathcal{P}\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. This functor is symmetric monoidal with respect to the Cartesian monoidal structure on $\mathcal{X}$ and the smash product monoidal structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, so it induces a functor,

$$\mathrm{CAlg}(\mathcal{X}) \xrightarrow{\Sigma} \mathrm{CAlg}(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})).$$

[3]See [HA, §4.8] for the definition of the Lurie tensor product of ∞-categories.
[4]See [HA, Example 4.8.1.23] for a proof of this fact.

Let $\mathcal{O}_\Sigma$ denote the image of $\mathcal{O} \in \mathrm{CAlg}(\mathcal{X})$ under the above functor. Now we have the following chain of equivalences:

$$\begin{aligned}\mathrm{Sp}(\mathrm{Mod}_{\mathcal{O}}(\mathcal{X})) &\simeq \mathrm{Sp}(\mathcal{X}_{\mathrm{Bar}(\mathcal{O})/}) \\ &\simeq \mathrm{Sp}\otimes(\mathcal{X}_{\mathrm{Bar}(\mathcal{O})/}) \\ &\simeq \mathrm{Sp}(\mathcal{X})_{\Sigma(\mathrm{Bar}(\mathcal{O}))} \\ &\simeq \mathrm{Sp}(\mathcal{X})_{\mathrm{Bar}(\mathcal{O}_\Sigma)} \\ &\simeq \mathrm{Mod}_{\mathcal{O}}(\mathrm{Sp}(\mathcal{X})),\end{aligned}$$

where we have used the equivalence $\mathrm{Mod}_{\mathcal{O}}(\mathcal{X}) \simeq \mathcal{X}_{\mathrm{Bar}(\mathcal{O})/}$ (see [HA, Remark 5.2.6.28]); the fact that $\mathrm{Bar}(\mathcal{O})$ is again an object of $\mathrm{CAlg}(\mathcal{X})$ under the equivalence given by the forgetful functor $\mathrm{CAlg}(\mathcal{X}) \xrightarrow{\simeq} \mathcal{X}$; and the fact that Σ commutes with geometric realizations as a left adjoint, so $\mathrm{Bar}(\mathcal{O}_\Sigma) \simeq \Sigma(\mathrm{Bar}(\mathcal{O}))$. $\square$

4. Analytic and analytic animated rings

4.1. **Analytic rings.** Let us recall the notion of analytic ring found in [CS19, Lecture VII].

Definition 4.1. A pre-analytic ring $(\mathcal{A}, \mathcal{M})$ is a condensed ring $\mathcal{A}$ together with a functor,

$$\mathcal{M}[-] : \mathrm{ExtProf} \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$$

taking finite disjoint unions to products, and a natural transformation of functors, $\mathrm{ExtProf} \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$,

$$\mathcal{A}[S] \to \mathcal{M}[S].$$

An analytic ring is a pre-analytic ring $(\mathcal{A}, \mathcal{M})$ such that for any complex,

$$C : \cdots C_i \to \cdots \to C_1 \to C_0 \to 0,$$

of $\mathcal{A}$-modules in condensed abelian groups, such that all C_i are direct sums of objects of the form $\mathcal{M}[T]$ for varying extremally disconnected T, the map

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

of complexes of condensed abelian groups is a quasi-isomorphism for all extremally disconnected sets S.

Remark 4.2. Heuristically, a pre-analytic ring is supposed to be a ring equipped with a notion in condensed abelian groups of “free” module over that ring, given by the functor in the definition. The condition in the definition of analytic ring specifies that this notion of “free” module should be well-behaved: maps into particular kinds of $\mathcal{A}$-modules cannot distinguish between the “free” modules $\mathcal{M}[-]$, and free $\mathcal{A}$-modules in the category of condensed abelian groups.

4.2. **Derived category of condensed modules.** There is a related notion of analytic animated ring, in which both the ring $\mathcal{A}$ and its space of measures $\mathcal{M}$ are allowed to be objects in condensed anima.

Notation 4.3. Let $\mathcal{A}$ be an animated condensed ring[5]. Let $\mathcal{D}_{\geq 0}(\mathcal{A})$ denote the prestable ∞-category, $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathcal{D}_{\geq 0}(\mathrm{Ab})))$, the category of $\mathcal{A}$-modules in animated condensed abelian groups.

We denote the stabilization of $\mathcal{D}_{\geq 0}(\mathcal{A})$ by $\mathcal{D}(\mathcal{A}) := \mathrm{Sp}(\mathcal{D}_{\geq 0}(\mathcal{A}))$. It is a stable ∞-category with a natural t-structure whose connective part is $\mathcal{D}_{\geq 0}(\mathcal{A})$. We observe that $\mathcal{D}(\mathcal{A})$ admits a forgetful

[5]By [CS20, Lemma 11.8], we may permute the adjectives “animated” and “condensed” with impunity.

functor $\mathrm{oblv}_{\mathrm{Sp}} : \mathcal{D}(\mathcal{A}) \to \mathrm{Cond}(\mathrm{Sp})$ which we call taking *the underlying condensed spectrum.* Indeed, we have the following chain of equivalences,

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &:= \mathrm{Sp}(\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathcal{D}_{\geq 0}(\mathrm{Ab})))) \\ &\simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{Sp}(\mathrm{Cond}(\mathcal{D}_{\geq 0}(\mathrm{Ab})))) && \text{(Lemma 3.24)} \\ &\simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathcal{D}(\mathrm{Ab}))) && \text{(Lemma 3.22)} \\ &\simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Sp})). \end{aligned}$$

Under this equivalence, $\mathrm{oblv}_{\mathrm{Sp}}$ is simply the functor on $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Sp}))$ of forgetting the $\mathcal{A}$-module structure.

When $\mathcal{A}$ is commutative, both $\mathcal{D}_{\geq 0}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are canonically symmetric monoidal categories. whose symmetric monoidal structure is induced by the commutative algebra $\mathcal{A}$ (see [HA, Theorem 3.3.3.9]).

Lemma 4.4. *The canonical symmetric monoidal structure on $\mathcal{D}(\mathcal{A})$ (resp. $\mathcal{D}_{\geq 0}(\mathcal{A})$), induced by the commutative algebra $\mathcal{A}$, is closed. That is, the symmetric monoidal product admits a right adjoint, which we denote by $\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(-,-)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,-)$), called the* internal hom.

Proof. We note that $\mathcal{A}$ is a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos ExtProf, and $\mathcal{D}(\mathcal{A})$ is the category of modules[6]over $\mathcal{A}$ in $\mathrm{Shv}_{\mathrm{Sp}}(\mathrm{ExtProf})$. As mentioned, this category has a canonical symmetric monoidal structure, whose product we denote by $\otimes_{\mathcal{A}}$. By [SAG, Proposition I.2.1.0.3], $\otimes_{\mathcal{A}}$ preserves small colimits in both variables, so by the Adjoint Functor Theorem, it admits a right adjoint, which is $\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(-,-)$.

Recall that $\mathcal{D}_{\geq 0}(\mathcal{A})$ is tautologically the connective part of the t-structure on $\mathcal{D}(\mathcal{A})$, and that $\mathcal{A}$ is a connective sheaf of $\mathbb{E}_\infty$-rings on ExtProf. By [SAG, Proposition I.2.1.1.1(b)], $\mathcal{D}_{\geq 0}(\mathcal{A})$ is closed under the tensor product $\otimes_{\mathcal{A}}$ and contains the unit object, so $\mathcal{D}_{\geq 0}(\mathcal{A})$ is symmetric monoidal under $\otimes_{\mathcal{A}}$. Moreover, $\mathcal{D}_{\geq 0}(\mathcal{A})$ is closed under small colimits, and reflects all colimits in $\mathcal{D}(\mathcal{A})$, so $\otimes_{\mathcal{A}}$ preserves small colimits in $\mathcal{D}_{\geq 0}(\mathcal{A})$, as well. Thus, $\otimes_{\mathcal{A}}$ admits a right adjoint in $\mathcal{D}_{\geq 0}(\mathcal{A})$ called $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,-)$, which, by uniqueness of the right adjoint, must be the restriction of $\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(-,-)$. □

Remark 4.5. A similar argument as presented in the proof of Lemma 4.4 proves the existence of an internal hom for $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ for $\mathcal{A}$ a discrete condensed commutative ring.

4.3. **Analytic animated rings.**

Definition 4.6. A pre-analytic animated ring $(\mathcal{A}, \mathcal{M})$ is an animated condensed ring $\mathcal{A}$ together with a functor,

$$\mathcal{M}[-] : \mathrm{ExtProf} \to \mathcal{D}(\mathcal{A})$$

that preserves finite coproducts, and a natural transformation,

$$\mathcal{A}[S] \to \mathcal{M}[S],$$

of condensed anima.

An analytic animated ring is a pre-analytic animated ring $(\mathcal{A}, \mathcal{M})$ with the property that for any object $C \in \mathcal{D}_{\geq 0}(\mathcal{A})$ that is a sifted colimits of objects of the form $\mathcal{M}[T]$ for varying extremally disconnected T, the natural map,

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C), \tag{4.7}$$

is an equivalence of condensed anima for all extremally disconnected profinite sets S.

[6] $\mathrm{Mod}_{\mathcal{A}}$ in the notation of Lurie's [SAG].

To any analytic animated ring $(\mathcal{A}, \mathcal{M})$ we may associate the subcategory of $\mathcal{D}_{\geq 0}(\mathcal{A})$ of *all* such objects C satisfying the condition (4.7) for all profinite sets S. More precisely, we have the following definition found in [CS20], whose notation we have modified for our purposes.

Definition 4.8. Let $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M}) \subset \mathcal{D}_{\geq 0}(\mathcal{A})$ denote the full ∞-subcategory spanned by all objects $\mathcal{C} \in \mathcal{D}_{\geq 0}(\mathcal{A})$ such that the natural map (4.7) is an equivalence of condensed anima for all extremally disconnected profinite S.

One may similarly define an abelian subcategory of $\mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ for a given analytic ring.

Definition 4.9. Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring as defined in Definition 4.1. We denote by $\heartsuit(\mathcal{A}, \mathcal{M}) \subset \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ the full subcategory of all objects $C \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ such that the map,

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is an isomorphism for all extremally disconnected profinite S.

The following example of an analytic ring will be the only one we use in this paper.

Example 4.10. Fix $0 < p \leq 1$, and consider the pair $(\mathbb{R}, \mathcal{M}_{<p})$[7]. This is an analytic ring by [CS20, Theorem 6.5], and the abelian category $\mathrm{Liq}_p(\mathbb{R}) := \heartsuit(\mathbb{R}, \mathcal{M}_{<p})$ is called the category of *p-liquid* $\mathbb{R}$-vector spaces. Likewise, the pair $(\mathbb{C}, \mathcal{M}_{<p})$ is also an analytic ring, whose category $\heartsuit(\mathbb{C}, \mathcal{M}_{<p})$ we also denote $\mathrm{Liq}_p(\mathbb{C})$.

Warning 4.11. Note that $\mathbb{R}_{\mathrm{disc}}$ is *not* p-liquid as a condensed vector space.

The following lemma shows that $(\mathbb{C}, \mathcal{M}_{<p})$ is also an analytic animated ring, under the inclusion of condensed rings into condensed animated rings (see Lemma 3.20).

Lemma 4.12. *Suppose that* $(\mathcal{A}, \mathcal{M})$ *is an analytic ring. Then* $(\mathcal{A}, \mathcal{M})$ *is also an analytic animated ring.*

As such, to each analytic ring $(\mathcal{A}, \mathcal{M})$ we may associate two categories:
- the abelian category $\heartsuit(\mathcal{A}, \mathcal{M})$ associated to $(\mathcal{A}, \mathcal{M})$ as an analytic ring,
- and the prestable ∞-category $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ associated to $(\mathcal{A}, \mathcal{M})$ as an analytic *animated* ring.

We will use following proposition from [CS20] to relate these two categories to each other.

Proposition 4.13 ([CS20, Proposition 12.4])**.** *Let* $(\mathcal{A}, \mathcal{M})$ *be an analytic animated ring. The* ∞*-category* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ *is generated under sifted colimits by the objects* $\mathcal{M}[S]$ *for varying extremally disconnected profinite sets* S*, which are compact projective objects of* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$*. The full* ∞*-subcategory*

$$\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M}) \subset \mathcal{D}_{\geq 0}(\mathcal{A})$$

is stable under all limits and colimits and admits a left adjoint

$$- \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M}) : \mathcal{D}_{\geq 0}(\mathcal{A}) \to \mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$$

characterized as the unique functor commuting with colimits that sends $\mathcal{A}[S]$ *to* $\mathcal{M}[S]$*.*

The ∞*-category* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ *is a prestable. Its heart is the full abelian subcategory of* $\mathrm{Mod}_{\pi_0 \mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ *generated under colimits by* $\pi_0 \mathcal{M}[S]$ *for varying* S*. An object* $\mathcal{C} \in \mathcal{D}_{\geq 0}(\mathcal{A})$ *lies in* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ *if and only if all* $H_i(\mathcal{C})$ *lie in* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})^{\heartsuit}$*.*

If $\mathcal{A}$ *has the structure of an animated condensed commutative ring so that* $\mathcal{D}_{\geq 0}(\mathcal{A})$ *is naturally a symmetric monoidal* ∞*-category, there is a unique symmetric monoidal structure on* $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ *making* $- \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$ *symmetric monoidal.*

[7] See Definition 6.3 and page 35 of [CS20, Lecture VI] for the definition of $\mathcal{M}_{<p}$.

Corollary 4.14. *Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring. The heart of the prestable ∞-category $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ is the abelian category of, $\heartsuit(\mathcal{A}, \mathcal{M})$. Moreover, $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ is the connective part of a t-structure, compatible with filtered colimits, on a stable presentable ∞-category which we denote by $\mathcal{D}(\mathcal{A}, \mathcal{M})$.*

Proof. By Proposition 4.13, the heart of $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ is the full subcategory of condensed $\mathcal{A}$-modules generated under colimits by $\mathcal{M}[S]$ for varying extremally disconnected profinite S. But by [CS19, Proposition 7.5], the collection $\mathcal{M}[S]$ for S extremally disconnected are a family of compact projective generators for $\heartsuit(\mathcal{A}, \mathcal{M})$. Thus, $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})^{\heartsuit}$ and $\heartsuit(\mathcal{A}, \mathcal{M})$ are two abelian categories generated under colimits by the same full subcategory $\mathscr{C}^{\mathrm{c.p.}} \subset \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ on objects, $\{\mathcal{M}[S]\}_{S \in \mathrm{ExtProf}}$. Moreover, objects of $\mathscr{C}^{\mathrm{c.p.}}$ are compact projectives in both $\heartsuit(\mathcal{A}, \mathcal{M})$ and $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})^{\heartsuit}$. Thus, $\heartsuit(\mathcal{A}, \mathcal{M}) \simeq \mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})^{\heartsuit}$. The remainder of the claim follows from the fact that $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ is prestable. □

The following corollary is immediate from Proposition 4.13 and Corollary 4.14.

Corollary 4.15. *Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring. An object $C \in \mathcal{D}(\mathcal{A})$ lies in $\mathcal{D}(\mathcal{A}, \mathcal{M})$ if and only if $H_i(C) \in \heartsuit(\mathcal{A}, \mathcal{M})$ for all i.*

Remark 4.16. Clearly, the results of this section involving $\mathcal{D}(\mathcal{A}, \mathcal{M})$ are equally valid for its bounded variations: $\mathcal{D}^b(\mathcal{A}, \mathcal{M})$, $\mathcal{D}^-(\mathcal{A}, \mathcal{M})$, and $\mathcal{D}^-(\mathcal{A}, \mathcal{M})$.

4.4. **The derived category of $\heartsuit(\mathcal{A}, \mathcal{M})$.**

4.4.1. We fix an analytic ring $(\mathcal{A}, \mathcal{M})$ for the remainder of this section. We let $\mathrm{Cond}(\mathcal{A}) := \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$.

4.4.2.

Proposition 4.17. *The category $\heartsuit(\mathcal{A}, \mathcal{M})$ is a Grothendieck abelian category.*

Proof. Recall that a Grothendieck abelian category is an abelian category that possesses arbitrary coproducts, in which filtered colimits are exact, and which has a single generator[8]. Observe that $\mathrm{Cond}(\mathcal{A})$ is a complete and cocomplete abelian category with exact filtered colimits by [Stacks, Lemma 18.14.2] because it is the category of modules over a sheaf of rings on the topos, Cond(Ab).

By [CS19, Proposition 7.5], $\heartsuit(\mathcal{A}, \mathcal{M})$ is a full subcategory of $\mathrm{Cond}(\mathcal{A})$ that is stable under arbitrary limits and colimits, meaning that limit and colimits over arbitrary diagrams in Cond(Ab) whose terms lie in $\heartsuit(\mathcal{A}, \mathcal{M})$ also lie in $\heartsuit(\mathcal{A}, \mathcal{M})$. As mentioned above, $\mathrm{Cond}(\mathcal{A})$ admits all colimits and limits, so $\heartsuit(\mathcal{A}, \mathcal{M})$ possesses all colimits, arbitrary coproduct in particular, and limits, as well. Since filtered colimits in $\mathrm{Cond}(\mathcal{A})$ are exact, it suffices to show that the the natural inclusion $\heartsuit(\mathcal{A}, \mathcal{M}) \hookrightarrow \mathrm{Cond}(\mathcal{A})$ is an exact functor. But this is clear as the inclusion functor preserves arbitrary limits and colimits[9].

It remains to show that $\heartsuit(\mathcal{A}, \mathcal{M})$ has a generator. By [CS20, Theorem 6.5], it is generated by compact projective objects, $\{\mathcal{M}[S]\}_{S \in \mathrm{Prof}}$. We claim that $\bigoplus_{S \in \mathrm{Prof}} \mathcal{M}[S]$ is such a generator. To show that it generates $\heartsuit(\mathcal{A}, \mathcal{M})$, we must show that for any two distinct morphisms $f, f' : X \to Y$, there exists a morphism $g : \bigoplus_{S \in \mathrm{Prof}} \mathcal{M}[S] \to X$, such that $f \circ g \neq f' \circ g$. Since $\{\mathcal{M}[S]\}_{S \in \mathrm{Prof}}$ are a collection of generators for $\heartsuit(\mathcal{A}, \mathcal{M})$, there exists such a morphism $g_0 : \mathcal{M}[S_0] \to X$ for some $S_0 \in \mathrm{Prof}$. Now let g be the morphism induced by g_0 and the zero morphism $0 : S \to X$ for all $S \neq S_0$ via the universal property of the direct sum. It is clear that this satisfies $f \circ g \neq f' \circ g$, so we are done. □

[8]More succinctly: an AB5 category with a generator.

[9]Indeed, since $\heartsuit(\mathcal{A}, \mathcal{M}) \subset \mathrm{Cond}(\mathrm{Ab})$ is a full subcategory, any limit cone in $\heartsuit(\mathcal{A}, \mathcal{M})$ is also a limit cone in Cond(Ab); and since $\heartsuit(\mathcal{A}, \mathcal{M})$ is stable under limits, the limit formed in Cond(Ab) is also the limit formed in $\heartsuit(\mathcal{A}, \mathcal{M})$. Ditto for colimits and co-cones.

As a Grothendieck abelian category, the unbounded derived ∞-category of $\heartsuit(\mathcal{A},\mathcal{M})$ is a presentable stable ∞-category with a well-behaved t-structure[10].

Proposition 4.18. *There is an equivalence of categories,*

$$\mathcal{D}(\heartsuit(\mathcal{A},\mathcal{M})) \simeq \mathcal{D}(\mathcal{A},\mathcal{M}).$$

Proof. It suffices to furnish an equivalence of Grothendieck prestable ∞-categories, $\mathcal{D}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0} \simeq \mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$.

Let $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})^{\text{c.p.}}$ denote the full subcategory of compact projective objects of $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$. Since $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$ is projectively generated and admits all small colimits, it is equivalent to the non-abelian derived category of a minimal model[11] of $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})^{\text{c.p.}}$, which is closed under finite coproducts by the definition of analytic animated ring. The latter category is a 1-category, which are always trivially minimal. Thus, there is an equivalence,

$$\mathcal{P}_\Sigma(\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})^{\text{c.p.}}) \simeq \mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M}).$$

We remark that $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})^{\text{c.p.}}$ is an additive ∞-category because it contains the zero object in condensed animated $\mathcal{A}$-modules, $\mathcal{M}[\emptyset] \simeq 0$, and is closed under finite finite biproducts; it is also small because it is a full subcategory of a locally small category ($\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$) whose collection of objects is indexed by the small category ExtProf. As such, $\mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$ is the non-abelian derived category of a small additive ∞-category, so by [SAG, Remark C.1.5.9] and [SAG, Proposition C.5.3.4], it is a 0-complicial complete Grothendieck prestable ∞-category. This combination of adjectives implies, by [SAG, Corollary C.5.9.7], that the inclusion of the heart, $\heartsuit(\mathcal{A},\mathcal{M}) \hookrightarrow \mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M})$, extends to an equivalence,

$$\widehat{\mathcal{D}}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0} \xrightarrow{\simeq} \mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M}), \tag{4.19}$$

of ∞-categories.

By [SAG, Proposition C.5.9.2], the completion functor on $\mathsf{Groth}_\infty^{\text{lex}}$ restricts to an equivalence of categories,

$$\mathsf{Groth}_\infty^{\text{ch,lex}} \xrightarrow[\simeq]{\widehat{(-)}} \mathsf{Groth}_\infty^{\text{comp,lex}},$$

between anticomplete Grothendieck prestable ∞-categories and complete Grothendieck prestable ∞-categories. By the universal property of the unseparated derived prestable ∞-category ([SAG, Corollary C.5.8.9]), there exists a map

$$\check{\mathcal{D}}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0} \to \mathcal{D}_{\geq 0}(\mathcal{A},\mathcal{M}) \tag{4.20}$$

extending the inclusion of the heart. By unraveling the definitions, we see that this map (which is also the map in [SAG, Corollary C.5.8.11]) is sent under $\widehat{(-)}$ to precisely the functor (4.19). Since $\widehat{(-)}$ is an equivalence of categories, it follows that (4.20) is also an equivalence.

Thus, we have that $\check{\mathcal{D}}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0} \simeq \widehat{\mathcal{D}}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0}$ by (4.19) and (4.20). We claim that this implies that $\check{\mathcal{D}}(\heartsuit(\mathcal{A},\mathcal{M}))_{\geq 0} \simeq \mathcal{D}(\heartsuit(\mathcal{A},\mathcal{M}))$. By [SAG, Theorem C.5.4.9], $\mathcal{D}(-)_{\geq 0} : \mathsf{Groth}_{\text{ab}}^{\text{lex}} \to \mathsf{Groth}_\infty^{\text{lex,sep}}$ is the left adjoint to the functor of restriction to the heart. That is,

$$\mathrm{LFun}^{\text{lex}}(\mathcal{D}(\mathscr{A})_{\geq 0}, \mathscr{C}) \simeq \mathrm{LFun}^{\text{lex}}(\mathscr{A}, \mathscr{C}^{\heartsuit}),$$

where $\mathrm{LFun}^{\text{lex}}(-,-)$ denotes the morphisms of $\mathsf{Groth}_\infty^{\text{lex,sep}}$ and $\mathsf{Groth}_{\text{ab}}^{\text{lex}}$[12]. On the other hand, $\check{\mathcal{D}}(-)_{\geq 0}$ is the left adjoint in $\mathsf{Groth}_\infty^{\text{lex}}$ to the functor of restriction to the heart by [SAG, Corollary

[10] See [HA, Proposition 1.3.5.9 and Proposition 1.3.5.21].

[11] See [HTT, Definition 2.3.3.1] for the definition of a minimal ∞-category. A minimal model of an ∞-category $\mathscr{C}$ is a subcategory of $\mathscr{C}$ which is both minimal and equivalent to $\mathscr{C}$.

[12] This is Lurie's notation in [SAG, Appendix C]. When both arguments of $\mathrm{LFun}(-,-)$ are presentable categories (such as is the case for Grothendieck prestable categories), LFun denotes functors that preserve small colimits. The superscript "lex" denotes those functors which further preserve finite limits.

C.5.8.9]. The functor $\mathscr{C} \mapsto \mathscr{C}^{\mathrm{sep}}$ taking a prestable category to its separable quotient is left adjoint to the inclusion, $\mathsf{Groth}_\infty^{\mathrm{lex,sep}} \to \mathsf{Groth}_\infty^{\mathrm{lex}}$, by [SAG, Corollary C.3.6.2]. By the essential uniqueness of adjoint functors, we therefore have

$$(\check{\mathcal{D}}(-)_{\geq 0})^{\mathrm{sep}} \xrightarrow{\simeq} \mathcal{D}(-)_{\geq 0}.$$

But now observe that $\check{\mathcal{D}}(-)_{\geq 0}$ is complete, and therefore already separated, so $\check{\mathcal{D}}(-)_{\geq 0} \simeq (\check{\mathcal{D}}(-)_{\geq 0})^{\mathrm{sep}}$.

All together, we obtain:

$$\begin{aligned} \mathcal{D}(\heartsuit(\mathcal{A}, \mathcal{M}))_{\geq 0} &\simeq \check{\mathcal{D}}(\heartsuit(\mathcal{A}, \mathcal{M}))_{\geq 0} \\ &\simeq \widehat{\mathcal{D}}(\heartsuit(\mathcal{A}, \mathcal{M}))_{\geq 0} \\ &\simeq \mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M}), \end{aligned}$$

which completes the proof. □

Proposition 4.21. *Both $\heartsuit(\mathcal{A}, \mathcal{M})$ and $\mathcal{D}(\mathcal{A}, \mathcal{M})$ are closed symmetric monoidal categories with respect to the symmetric monoidal structures specified in [CS19, Proposition 7.5] and [CS20, Proposition 12.4], respectively. Moreover, $\otimes_{\mathcal{D}^-(\mathcal{A},\mathcal{M})}$ is the left derived functor of $\otimes_{\heartsuit(\mathcal{A},\mathcal{M})}$, and $\underline{\mathrm{Hom}}_{\mathcal{D}^+(\mathcal{A},\mathcal{M})}$ is the right derived functor of $\underline{\mathrm{Hom}}_{\heartsuit(\mathcal{A},\mathcal{M})}$.*

Proof. By Proposition 4.13, the left adjoint, $- \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$ is symmetric monoidal. Thus, the tensor product in $\mathcal{D}(\mathcal{A})$ has a right adjoint, denoted $\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(-,-)$, the tensor product in $\mathcal{D}(\mathcal{A}, \mathcal{M})$ also has a right adjoint, given by $\underline{\mathrm{Hom}}_{\mathcal{D}(\mathcal{A})}(-,-) \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$. The same argument shows that the application of the left adjoint to the inclusion $\heartsuit(\mathcal{A}, \mathcal{M}) \hookrightarrow \mathrm{Mod}_{\mathcal{A}}(\mathrm{Cond}(\mathrm{Ab}))$ (the liquidification if $(\mathcal{A}, \mathcal{M})$ were $(\mathbb{C}, \mathcal{M}_{<p})$) to $\underline{\mathrm{Hom}}_{\mathrm{Cond}(\mathcal{A})}(-,-)$ gives the internal hom in $\heartsuit(\mathcal{A}, \mathcal{M})$.

The claim that $\otimes_{\mathcal{D}^-(\mathcal{A},\mathcal{M})}$ is the left derived functor of $\otimes_{\heartsuit(\mathcal{A},\mathcal{M})}$ follows immediately from Proposition 4.18 and the uniqueness of left derived functors, Indeed, if $P \in \heartsuit(\mathcal{A}, \mathcal{M})$, then $P \otimes -$ determines a right t-exact functor[13],

$$P \otimes - : \mathcal{D}^-(\heartsuit(\mathcal{A}, \mathcal{M})) \to \mathcal{D}^-(\mathcal{A}, \mathcal{M}),$$

that restricts to $P \otimes_{\heartsuit(\mathcal{A},\mathcal{M})} -$ on the hearts. This functor is right exact as it commutes with colimits; so, by [HA, Theorem 1.3.3.2], it is the left derived functor of $\otimes_{\heartsuit(\mathcal{A},\mathcal{M})}$ (noting that $\mathcal{D}^-(\mathcal{A}, \mathcal{M})$ is left-complete).

The claim that $\underline{\mathrm{Hom}}_{\mathcal{D}^+(\mathcal{A},\mathcal{M})}$ is the right derived functor of $\underline{\mathrm{Hom}}_{\heartsuit(\mathcal{A},\mathcal{M})}$ now follows immediately. □

5. Functional calculus via condensed mathematics

5.0.1. Fix $0 < p \leq 1$.

5.0.2. We now specialize the results of the previous section to the analytic ring $(\mathbb{C}, \mathcal{M}_{<p})$ recalled in Example 4.10.

The theory of p-liquid vector spaces, i.e. objects of $\mathrm{Liq}_p(\mathbb{C})$, is an excellent framework in which to do functional analysis. Many of the most commonly encountered types of topological vector spaces are p-liquid vector spaces, which, as seen above, enjoy great homological properties as a category. Banach spaces, viewed as p-liquid vector spaces, will be our main objects of interest.

5.1. **The category of Banach spaces.** We consider the following two categories of topological vector spaces:

(i) Let $\mathcal{TV}\mathrm{ect}_{\mathrm{l.c.}}^{\mathrm{comp}}$ denote the category of complete[14]locally convex topological $\mathbb{C}$-vector spaces with morphisms given by continuous linear operators.

[13] Because it tautologically restricts to a symmetric monoidal functor on $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$.

[14] i.e. the underlying topological space is completely metrizable.

(ii) Let **Ban** denote the category of complex Banach spaces with morphisms given by bounded linear maps

There is an obvious forgetful functor oblv : $\mathbf{Ban} \to \mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$ sending a Banach space to its underlying locally convex topological vector space. The functor oblv is fully faithful; every continuous linear operator between Banach spaces is bounded.

It is shown in [CS20] that **Ban** embeds fully faithfully into p-liquid vector spaces as follows. Complete locally convex topological spaces are compactly generated, so they embed into condensed sets (see discussion after [CS20, Definition 3.1]). Moreover they are $\mathcal{M}$-complete in the sense of [CS20, Definition 4.1] by [CS20, Proposition 3.4], so they are p-liquid by the discussion at the start of [CS20, §VI]. The embedding of Banach spaces into p-liquid spaces factors as

$$\mathbf{Ban} \hookrightarrow \mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}} \hookrightarrow \mathrm{Liq}_p(\mathbb{C}). \tag{5.1}$$

Using this embedding, we make the following definition.

Definition 5.2. Let $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$ denote the full subcategory of $\mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$ spanned by objects whose homology spaces lie in the essential image of the embedding $\mathbf{Ban} \hookrightarrow \mathrm{Liq}_p(\mathbb{C})$.

5.2. **Classical functional calculus.** We briefly review the classical theory of functional calculus on Banach spaces. Our reference for the content of this section is [Rud91, Chapter 10].

5.2.1. *Motivation of functional calculus.* Given a Banach space $V \in \mathbf{Ban}$, and a bounded linear operator $T : V \to V$, the symbol "T^n" has a clear, unambiguous meaning for each non-negative integer n. For $n > 0$, T^n is the n-fold composition of T with itself, and $T^0 = I$. Likewise, given a polynomial $f(z)$ with complex coefficients, the symbol "$f(T)$" has a clear meaning as well. The purpose of functional calculus is to try and extend the definition of symbols "$f(T)$" to include functions f which are holomorphic on an open neighborhood of $\mathbb{C}$ containing the spectrum of T.

5.2.2. This is easily achieved in the classical setting by observing that the space of bounded endomorphisms, $\mathrm{End}_{\mathbf{Ban}}(V)$, is itself a Banach space under the strong operator norm. In fact, it is a Banach *algebra* under the composition of bounded operators.

The following is an easy result in classical functional analysis that uses the fact that polynomials are dense inside the space the space of holomorphic functions.

Proposition 5.3. *Let A be a Banach algebra; $x \in A$ an element; and $f(z)$ an entire function of one complex variable. Then there is a unique element $f(x)$ extending the definition of $f(x)$ from polynomials to entire functions in a continuous way.*

In the sequel, we will often refer to the endomorphism $f(T) \in \mathrm{End}_{\mathbf{Ban}}(V)$ obtained "from classical functional calculus." By this, we mean the endomorphism obtained from the application of f to T as an element of the Banach algebra $\mathrm{End}_{\mathbf{Ban}}(V)$ in the sense of Proposition 5.3.

5.3. **Liquid functional calculus.** Suppose given $E \in \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$ such that the internal hom, $\mathrm{End}(E)$, lies in the subcategory $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$. The goal of this section is to make sense of $f(T)$ for any $T \in \mathrm{End}(E)$ and any entire function f in such a way that the maps $H_i(-)$ induced by $f(T)$ coincide with the operators obtained from classical functional calculus.

5.3.1. Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be an entire function of a single variable. Since $\mathrm{End}(E) \in \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$, the following morphism of condensed $\underline{\mathbb{C}}$-module mapping spectra[15] is equivalent,

$$\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{C})}(\mathcal{M}_{<p}(S), \mathrm{End}(E)) \to \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{C})}(\mathbb{C}[S], \mathrm{End}(E)),$$

[15] See Lemma 4.4 for proof of the existence of internal hom in $\mathcal{D}(\mathbb{C})$ and the discussion above for remarks about the underlying condensed spectra of an object in $\mathcal{D}(\mathbb{C})$.

for any profinite set S. The latter, in turn, is equivalent as a condensed spectrum to $\underline{\mathrm{Hom}}_{\mathrm{Cond(Sp)}}(S, \mathrm{End}(E))$, because $\mathbb{C}[S]$ is the free object on S in $\mathcal{D}(\mathbb{C})$ (i.e. is the left adjoint to the forgetful functor $\mathcal{D}(\mathbb{C}) \to \mathrm{Cond(Sp)}$).

Definition 5.4. Observe that $S := \mathbb{N} \cup \{\infty\}$ is a profinite set. Let f be as above. The element $\mu_f \in \mathbb{C}[S]$ is defined by the assignment

$$\begin{aligned} n &\mapsto \sqrt{a_n} \\ \infty &\mapsto 0. \end{aligned}$$

Proposition 5.5. *The measure μ_f is an element of $\mathcal{M}_{<p}(S)$.*

Proof. Consider S as the projective limit of the (totally ordered) finite sets $S_i := \{0, \dots, i, \infty\}$, where the transition maps $f_{j>i} : S_j \to S_i$ for the directed diagram are given by

$$f_{j>i}(x) = \begin{cases} x & \text{if } 0 \le x \le i \\ \infty & \text{if } x > i \end{cases}.$$

Recall that for any finite set T,

$$\mathbb{R}[T]_{\ell^p \le c} := \{(a_t)_{t \in T} \in \mathbb{R}[T] | \sum_t |a_t|^p \le c\},$$

where $c > 0$, and that for a profinite set, $T = \varprojlim_i T_i$,

$$\mathcal{M}_p(T) := \bigcup_{c>0} \varprojlim_i \mathbb{R}[T_i]_{\ell^p \le c}.$$

We claim that $\mu_f \in \mathcal{M}_p(S)$ for any $0 < p \le 1$. Indeed, observe that $\mathbb{R}[S] \simeq \varprojlim_i \mathbb{R}[S_i]$ and consider the element $\mu_i \in \mathbb{R}[S_i]$ given by the assignment $n \mapsto \sqrt{a_n}$, $\infty \mapsto 0$. Clearly, $\mu_f = (\dots, \mu_i, \dots, \mu_0) \in \varprojlim_i \mathbb{R}[S_i]$ under the identification of $\mathbb{R}[S]$ with the projective limit $\varprojlim_i \mathbb{R}[S_i]$.

It now suffices to show that $\mu_i \in \mathbb{R}[S_i]_{\ell^p \le c_0}$, for some $c_0 > 0$ independent of i. Since $f(z)$ is holomorphic, its power series representation $\sum_{i=0}^\infty a_i z^i$ is absolutely convergent for any value of z. In particular, by Lemma 5.12 and Lemma 5.14 stated at the end of this section, we obtain that the series $\sum_{i=0}^\infty |a_i|^p$ is convergent. Applying Lemma 5.14 a second time to this latter series, we see that $\sum_{i=0}^\infty \left|\sqrt{a_i}\right|^p$ is convergent. Since $\sum_{i=0}^m \left|\sqrt{a_i}\right|^p \le \sum_{i=0}^\infty \left|\sqrt{a_i}\right|^p$ for any choice of m, we may take $c_0 := \sum_{i=0}^\infty |\sqrt{a_i}|^p$. Thus, $\mu_f \in \mathcal{M}_p(S)$.

Finally, recall that $\mathcal{M}_{<p}(T) := \bigcup_{p'<p} \mathcal{M}_{p'}(T)$ for any profinite set T. Since $\mu_f \in \mathcal{M}_p(\mathrm{S})$ for any fixed $0 < p' \le 1$, it belongs in particular to $\mathcal{M}_{p'}(S)$ for some $p' < p$, and therefore belongs to $\mathcal{M}_{<p}(S)$. ☐

Proposition 5.5 allows us to make sense of the application of f to an endomorphism of E. Let $T \in \mathrm{End}(E)$ be an endomorphism of E, and denote by $\varphi_{T,f} : S \to \mathrm{End}(E)$ the map of condensed spectra given by the assignment,

$$\begin{aligned} n &\mapsto \sqrt{a_n} \cdot T^n \\ \infty &\mapsto 0, \end{aligned}$$

where we note that this indeed defines a map of condensed spectra by the fully faithful embedding of spectra into condensed spectra. Since $\mathrm{End}(E)$ is p-liquid for any $0 < p \le 1$,

$$\underline{\mathrm{Hom}}_{\mathbb{C}}(\mathcal{M}_{<p}(S), \mathrm{End}(E)) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathrm{Cond(Sp)}}(S, \mathrm{End}(E))$$

is a equivalence of condensed spectra. Thus, $\varphi_{T,f}$ extends to a map $\widetilde{\varphi_{T,f}} : \mathcal{M}_{<p}(S) \to \mathrm{End}(E)$.

Definition 5.6. Define $f(T) \in \mathrm{End}(E)$ to be the endomorphism,

$$f(T) := \widetilde{\varphi_{T,f}}(\mu_f),$$

given by evaluating $\widetilde{\varphi_{T,f}}$ on the measure $\mu_f \in \mathcal{M}_{<p}(S)$.

Remark 5.7. Observe that if $E \in \mathrm{Liq}_p(\mathbb{C}) \subset \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$, the spaces of morphisms, $\mathrm{Hom}_{\mathcal{D}(\mathbb{C})}(\mathcal{M}_{<p}(S), E)$ and $\mathrm{Hom}_{\mathrm{Cond(Sp)}}(S, E)$ are discrete. Thus, if E is p-liquid as a $\mathbb{C}$-module spectrum, it is p-liquid as a $\mathbb{C}$-vector space in the sense of [CS20, Theorem 6.5]. Moreover, the extension $\widetilde{f} : \mathcal{M}_{<p}(S) \to E$ of a map $f : S \to E$ is the unique extension guaranteed by [CS20, Theorem 6.5], as well.

5.4. **Comparison of liquid to classical functional calculus.** It remains to show that, if E is a Banach space viewed a condensed $\mathbb{C}$-vector space, the notation $f(T)$ is compatible with its meaning coming from classical functional analysis.

Given a profinite set S, let $\mathcal{M}(S)$ be the real vector space of signed Radon measures on S[16]. Let W be an arbitrary Banach space. By [CS19, Proposition 3.4], W is $\mathcal{M}$-complete in the sense that any map $f : S \to W$ from a profinite set S extends uniquely to a map of topological vector spaces,

$$\begin{aligned} \overline{f} : \mathcal{M}(S) &\to W \\ \mu &\mapsto \int_S f d\mu \end{aligned}$$

where the right-hand side is the integral of a Banach space-valued function on a compact measure space (see [Rud91, Theorem 3.27]).

Every $\mathcal{M}$-complete vector space is p-liquid for any fixed $0 < p \leq 1$; indeed, $\mathcal{M}_{<p}(S) \subset \mathcal{M}(S)$ for any such p. Moreover, since the extension of f to a function $\widetilde{f} : \mathcal{M}_{<p}(S) \to W$ is unique by the p-liquid property of W, it follows that $\widetilde{f} = \overline{f}|_{\mathcal{M}_{<p}(S)}$.

With the preceding discussion in mind, we state and prove the following lemma, which claims that a convergent series (equivalently sequence) of terms in W can be viewed as the evaluation of $\widetilde{f}$ on a particular measure.

Lemma 5.8. *Suppose that W is a complex Banach space, and let $\sum_{i=0}^{\infty} a_i x_i$, $x_i \in W$, be a convergent series in W such that $\sum_{i=0}^{\infty} a_i$ passes the ratio test.*

Let $f : S := \mathbb{N} \cup \{\infty\} \to W$ be the continuous function given by the assignment $n \mapsto \sqrt{a_n} \cdot x_n$, $\infty \mapsto 0$, and let μ be the signed Radon measure on S given by $\mu(n) = \sqrt{a_n}$. Then,

$$\sum_{i=0}^{\infty} a_i x_i = \widetilde{f}(\mu). \tag{5.9}$$

Proof. Firstly, we note that, by Lemma 5.14, $\mu \in \mathcal{M}_{<p}(S)$ for any fixed p, so the right-hand side of (5.9) is well-defined.

By the preceding discussion, we have that $\widetilde{f} = \overline{f}|_{\mathcal{M}_{<p}(S)}$, so

$$\begin{aligned} \widetilde{f}(\mu) &= \overline{f}(\mu) \\ &= \int_S f d\mu \\ &= \sum_{i=0}^{\infty} a_i x_i, \end{aligned}$$

by the definition of the integral, $\int_S f d\mu$. □

In particular, suppose $W := \mathrm{End}(V)$ is the Banach algebra of bounded linear operators on a given Banach space V. As a Banach space, $\mathrm{End}(V)$ is p-liquid, so the map $\varphi_{T,f} : S \to \mathrm{End}(V)$ given by $n \mapsto \sqrt{a_n} \cdot T^n$, $\infty \mapsto 0$ for $T \in \mathrm{End}(V)$ extends uniquely to a map,

$$\widetilde{\varphi_{T,f}} : \mathcal{M}_{<p}(S) \to \mathrm{End}(V).$$

The following corollary follows immediately from Lemma 5.8 and Remark 5.7.

[16] See [CS19, Exercise 3.3] for more context.

Corollary 5.10. *Let $\mu_f \in \mathcal{M}_{<p}(S)$ be as in Proposition 5.5. Then*

$$\widetilde{\varphi_{T,f}}(\mu_f) = f(T),$$

where the right-hand side denotes the operator given by the convergent sum $\sum_{i=0}^{\infty} a_i T^i$, in the strong operator norm.

5.5. **Passing to homology.** Let $E \in \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$ and $T \in \mathrm{End}(E)$ be as before. Then T induces endomorphisms of the homology groups of E (with respect to the t-structure on $\mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})$), via the homology functor, $H^i : \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p}) \to \mathrm{Liq}_p(\mathbb{C})$. Namely, for each $i \in \mathbb{Z}$, we obtain a map of condensed liquid vector spaces

$$H_i(-) : \mathrm{End}(E) \to \mathrm{End}(H_i(E))$$

by functoriality. Since it was assumed $H^i(E)$ lies in the image of Banach spaces inside of p-liquid vector spaces, these endomorphisms are bounded operators on complex Banach spaces, so $f(H_i(T))$ for an entire function f is easily defined classically. The following proposition asserts that the condensed functional calculus we have defined recovers the classical functional calculus upon taking homology.

Proposition 5.11. *The induced map on homology,*

$$H_i(f(T)) : H_i(E) \to H_i(E),$$

are given by the Banach space endomorphism obtained via classical functional calculus by applying f to

$$f(H_i(T)) : H_i(E) \to H_i(E).$$

Proof. Consider the map $\varphi_{T,f,i} : S := \mathbb{N} \cup \{\infty\} \to \mathrm{End}(H_i(E))$ given by the assignment,

$$\begin{aligned} n &\mapsto \sqrt{a_n} \cdot H_i(T)^n \\ \infty &\mapsto 0. \end{aligned}$$

The target of this map is a p-liquid vector space, so $\varphi_{T,f,i}$ extends uniquely to a map $\widetilde{\varphi_{T,f,i}} : \mathcal{M}_{<p}(S) \to \mathrm{End}(H_i(E))$. By Corollary 5.10, $\widetilde{\varphi_{T,f,i}}(\mu_f)$ is the endomorphism $f(H_i(T))$ given by classical functional calculus.

Thus, it remains to show that

$$H_i(\widetilde{\varphi_{T,f}}(\mu_f)) = \widetilde{\varphi_{T,f,i}}(\mu_f).$$

In order to see this, we show that $H_i \circ \widetilde{\varphi_{T,f}}$ is an extension to $\mathcal{M}_{<p}(S)$ of the map $\varphi_{T,f,i}$, since such an extension is unique by the definition of p-liquid vector space. Pictorially, we have

$$\begin{tikzcd} S \arrow[r, "\varphi_{T,f}"] \arrow[d, hook, "\widetilde{\varphi_{T,f}}"] & \mathrm{End}(E) \arrow[r, "H_i"] & \mathrm{End}(H_i(E)) \\ \mathcal{M}_{<p}(S) \arrow[ur] \arrow[urr, "H_i(\widetilde{\varphi_{T,f}})"'] & & \end{tikzcd}$$

from which we see that $(H_i \circ \widetilde{\varphi_{T,f}})|_S = H_i \circ \varphi_{T,f}$.

Since $H_i : \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p}) \to \mathcal{D}(\mathbb{C}, \mathcal{M}_{<p})^{\heartsuit} \simeq \mathrm{Liq}_p(\mathbb{C})$ is an additive functor of categories enriched over condensed $\mathbb{C}$-vector spaces, $H_i : \mathrm{End}(E) \to \mathrm{End}(H_i(E))$ is $\mathbb{C}$-linear. As such, we have the equality,

$$H_i(c \cdot T^n) = c \cdot H_i(T)^n$$

for any natural number $n \in \mathbb{N}$ and complex scalar $c \in \mathbb{C}$. Computing, we obtain

$$\begin{aligned} (H_i \circ \varphi_{T,f})(n) &= H_i(\sqrt{a_n} \cdot T^n) \\ &= \sqrt{a_n} \cdot H_i(T)^n \\ &= \varphi_{T,f,i}(n). \end{aligned}$$

To conclude, we observe that $(H_i \circ \varphi_{T,f})(\infty) = 0$. $\square$

5.6. **Postponed lemmas.** We state and prove two lemmas on the ratio test for convergence of series that were used in the proof of Proposition 5.5.

Lemma 5.12. *Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be any entire function of a single complex variable. Then the series $\sum_{i=0}^{\infty} |a_i|$ passes the ratio test for convergence.*

Proof. For fixed $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have the following equality,

$$\left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z| .$$

Taking the limits as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = \left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) |z|. \tag{5.13}$$

Since $\sum_{i=0}^{\infty} a_i z^i$ is absolutely convergent for all $z \in \mathbb{C}$, the left-hand side of (5.13) converges to a value ≤ 1 for any value of z, for otherwise $\sum_{i=0}^{\infty} a_i z^i$ would diverge by the ratio test. Now, take any $z = z_0$ such that $|z_0| > 1$ to obtain that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

which was to be shown. $\square$

Lemma 5.14. *If $\sum_{i=0}^{\infty} a_i$ passes the ratio test, then $\sum_{i=0}^{\infty} a_i{}^r$ does too, for any fixed $0 < r$.*

Proof. We compute,

$$\left| \frac{a_{n+1}{}^r}{a_n{}^r} \right| = \left| \left(\frac{a_{n+1}}{a_n} \right)^r \right| = \left| \frac{a_{n+1}}{a_n} \right|^r .$$

Since $(-)^r$ is a continuous function of a complex variable,

$$\lim_{n\to\infty} \left(\left| \frac{a_{n+1}}{a_n} \right|^r \right) = \left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right)^r . \tag{5.15}$$

By assumption, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, so the right-hand side of (5.15) is also < 1, proving the lemma. $\square$

6. Functional calculus in Perf

The results of the previous section allow us to define things such as the exponential of an endomorphism in $\mathrm{Vect}^b(\mathbb{C})$ of an object with finite dimensional homology spaces (i.e. a perfect complex), as we show in this section.

Lemma 6.1. Ban *is an additive category.*

Proof. Clearly, the hom-sets in **Ban** are abelian groups: the point-wise sum of any two bounded linear operators is again a bounded linear operator. The composition of morphism in **Ban** is also clearly bilinear, so **Ban** is enriched over Ab. It remains to check that **Ban** admits finite products. The product of two Banach spaces V_1 and V_2 is canonical a Banach space under several equivalent norms, such as the max norm: $\| \cdot \|_{V_1 \times V_2} := \max(\| \cdot \|_{V_1}, \| \cdot \|_{V_2})$. $\square$

Since **Ban** is additive, we can consider its category of chain complexes, Ch(**Ban**), which is also additive.

Remark 6.2. Neither **Ban** nor Ch(**Ban**) are abelian. In fact, one can see the theory of p-liquid vector spaces as a remedial solution to this fact.

Notably, this embedding is *additive*. The functor $\mathbf{Ban} \hookrightarrow \mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$ obviously preserves finite products; we claim that $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}} \hookrightarrow \mathrm{Liq}_p(\mathbb{C})$ does too. In fact, we prove a stronger claim.

Lemma 6.3. *The fully faithful embedding* $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}} \hookrightarrow \mathrm{Liq}_p(\mathbb{C})$ *preserves limits*[17].

Proof. By [CS19, Proposition 1.7], the embedding $\mathcal{T}\mathrm{op}_\kappa \hookrightarrow \mathrm{Cond}_\kappa$ of κ-compactly generated topological spaces into κ-condensed sets is a right adjoint functor, so preserves limits. Thus, it suffices to show that the forgetful functors, $\mathrm{oblv}_1 : \mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}} \to \mathcal{T}\mathrm{op}_\kappa$ and $\mathrm{oblv}_2 : \mathrm{Cond}_\kappa(\mathbb{C}) \to \mathrm{Cond}_\kappa$, reflect limits.

We begin with oblv_1. By [nLa22, Proposition 3.3], a conservative functor reflects any limits that exist in the domain and which it preserves. The functor oblv_1 is conservative by [Rud91, 2.12 Corollaries (a) and (b)], so it suffices to show that oblv_1 preserves all limits. By [Stacks, Lemma 4.14.11], it suffices to check that oblv_1 preserves products and equalizers. Products are preserved under the forgetful functor, since the underlying topological space of a product in $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$ has underlying topological space given by the Cartesian product of the underlying topological spaces of each of the factors. Since $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$ is an additive category, it suffices just to check that equalizers of the form,

$$V \underset{0}{\overset{f}{\rightrightarrows}} W,$$

i.e. kernels, are preserved. But the kernel of f in $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$ is given by the vector subspace $f^{-1}(0)$, since this is automatically closed, and therefore complete. We now conclude by noting that the underlying topological space of $f^{-1}(0)$ is precisely the fiber product $*\,{}_0\!\times_f V$ taken in $\mathcal{T}\mathrm{op}_\kappa$.

We employ the same strategy to show that oblv_2 reflects limits. The forgetful functor oblv_2 is conservative and preserves limits for general sheaf-theoretic reasons: $\mathrm{Cond}_\kappa(\mathbb{C})$ is the category of modules over a sheaf of rings on a small site (Prof_κ), so the forgetful functor to the category of $\mathcal{S}$et-valued sheaves is conservative by [HA, Corollary 3.4.4.6], and preserves limits by [HA, Corollary 3.4.3.2]. Thus, oblv_2 reflects limits, which concludes the proof. □

Since all norms on a finite dimensional complex vector space are equivalent and complete, and any linear map between such objects is bounded, there is an inclusion $\mathrm{Vect}^{f.d.}_{\mathbb{C}} \hookrightarrow \mathbf{Ban}$ where we view $\mathbf{Ban}$ as a full subcategory of $\mathcal{TV}\mathrm{ect}^{\mathrm{comp}}_{\mathrm{l.c.}}$. It is not hard to see that this inclusion is an additive functor, so composing with (5.1), we obtain the fully faithful, additive functor,

$$\underline{(-)} : \mathrm{Vect}^{f.d.}_{\mathbb{C}} \hookrightarrow \mathrm{Liq}_p(\mathbb{C}), \tag{6.4}$$

of abelian categories.

Warning 6.5. The embedding (6.4) is *not* the restriction of the embedding $\mathrm{Vect}_{\mathbb{C}} \hookrightarrow \mathrm{Cond(Ab)}$ given by viewing $\mathbb{C}$ as a discrete topological ring and sending an object in $\mathrm{Vect}_{\mathbb{C}}$ to a $\underline{\mathbb{C}_{\mathrm{disc}}}$-module in condensed abelian groups.

Lemma 6.6. *The embedding* $(\mathrm{Vect}^{f.d.}_{\mathbb{C}}, \otimes) \xhookrightarrow{\underline{(-)}} (\mathrm{Liq}_p(\mathbb{C}), \otimes_{\mathrm{Liq}_p})$ *is exact and symmetric monoidal.*

Proof. To show that $\underline{(-)}$ is exact, we must show that it preserve finite limits and colimits. We have that $\underline{(-)}$ preserves finite limits by Lemma 6.3, so we need only show that it preserves finite colimits. For this, it suffices by [Stacks, Lemma 4.14.12], to show that $\underline{(-)}$ preserves cokernels and finite coproducts.

Recall that $\underline{(-)}$ sends an object $V \in \mathrm{Vect}^{f.d.}_{\mathbb{C}}$ to the sheaf given by the assignment

$$S \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(S, V),$$

[17] cf. [CS19, Remark 1.8]

for S an extremally disconnected profinite set, where V is considered with its canonical Euclidean topology. For objects $V, W \in \mathrm{Vect}_{\mathbb{C}}^{f.d.}$, the presheaf given by

$$S \mapsto \underline{V}(S) \oplus \underline{W}(S)$$

is actually a sheaf because sheafification commutes with colimits as a left adjoint functor. This sheaf is the coproduct $\underline{V} \oplus \underline{W}$. Then we have

$$\begin{aligned}
\underline{V} \oplus \underline{W}(S) &= \underline{V}(S) \oplus \underline{W}(S) \\
&= \underline{V}(S) \times \underline{W}(S) \\
&= \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(S, V) \times \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(S, W) \\
&= \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(S, V \times W) \\
&= \underline{(V \times W)}(S) \\
&= \underline{(V \oplus W)}(S),
\end{aligned}$$

where $\underline{V}(S) \oplus \underline{W}(S) = \underline{V}(S) \times \underline{W}(S)$ because $\oplus$ and $\times$ here are taken in Ab, where they coincide. Thus, $\underline{(-)}$ preserves coproducts. The proof that $\underline{(-)}$ preserves cokernels is similar, using that cokernels are colimits.

Finally, to see that $\underline{(-)}$ is symmetric monoidal, we note that $\mathrm{Vect}_{\mathbb{C}}^{f.d.}$ is a full subcategory of the category of *dual nuclear Fréchet* spaces, which embed in the obvious way into $\mathrm{Liq}_p(\mathbb{C})$ as a subset of *nuclear* spaces[18]. Now we turn to Scholze's initial post on the Xena Project blog ([Sch20]) putting forth the mathematical formalization challenge known as the "Liquid Tensor Experiment," in which he mentions that $\otimes_{\mathrm{Liq}_p}$ agrees with the usual completed tensor product on nuclear spaces, so we are done. $\square$

Observe that both $\mathrm{Vect}_{\mathbb{C}}^{f.d.}$ and $\mathrm{Liq}_p(\mathbb{C})$ have enough projectives. Indeed, all objects in $\mathrm{Vect}_{\mathbb{C}}^{\heartsuit}$ are projective, and since $\mathrm{Vect}_{\mathbb{C}}^{f.d.}$ is a full subcategory of $\mathrm{Vect}_{\mathbb{C}}^{\heartsuit}$, all of its objects are projective as well. On the other hand, $\mathrm{Liq}_p(\mathbb{C})$ is generated by compact projectives ([CS20, Theorem 6.5]). By [HA, Proposition 1.3.3.2] in conjunction with Lemma 6.6, we therefore obtain a canonical right t-exact functor of right-bounded derived ∞-categories,

$$\underline{\mathcal{D}} : \mathcal{D}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.}) \to \mathcal{D}^-(\mathrm{Liq}_p(\mathbb{C})).$$

Lemma 6.7. *The functor $\underline{\mathcal{D}}$ is t-exact, fully faithful, and symmetric monoidal.*

Proof. We first show that $\underline{\mathcal{D}}$ is fully faithful. The categories $\mathcal{D}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.})$ and $\mathcal{D}^-(\mathrm{Liq}_p(\mathbb{C}))$ admit descriptions as $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.}))$ and $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathrm{Liq}_p(\mathbb{C})_{\mathrm{proj}}))$, respectively, where $\mathrm{Ch}^-(\mathcal{A})$ denotes the right-bounded chain complexes in the additive category $\mathcal{A}$ and N_{dg} is the differential graded nerve of a differential graded category[19].

Let $V_n \in \mathrm{Vect}_{\mathbb{C}}^{f.d.} \subset \mathcal{T}\mathcal{V}\mathrm{ect}_{\mathrm{l.c.}}^{\mathrm{comp}}$ be a vector space of dimension n, and choose a basis $\{v_i\}_{i \in S_n}$ for V_n, where S_n is an index set of cardinality n. Then $\underline{V_n} \simeq \mathcal{M}_{<p}(S_n)$, so $\underline{V_n}$ is a projective p-liquid vector space. This shows that $\mathrm{Vect}_{\mathbb{C}}^{f.d.}$ actually embeds as a full subcategory of $\mathrm{Liq}_p(\mathbb{C})_{\mathrm{proj}}$. This embedding induces a functor of differential graded categories,

$$\mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.}) \to \mathrm{Ch}^-(\mathrm{Liq}_p(\mathbb{C})_{\mathrm{proj}}). \tag{6.8}$$

We claim that the functor given by the application of N_{dg} to (6.8) is $\underline{\mathcal{D}}$. To see this, note that by [HA, Theorem 1.3.3.2], $\underline{\mathcal{D}}$ is the (essentially) unique extension F of $\underline{(-)} : \mathrm{Vect}_{\mathbb{C}}^{f.d.} \to \mathrm{Liq}_p(\mathbb{C})$ to

[18]See [CS22, Lecture VIII] for the definitions of dual nuclear Fréchet space and nuclear space in this context.

[19]See [HA, Definition 1.3.1.1] for the definition of differential graded category, [HA, Construction 1.3.1.6] for the definition of differential graded nerve, and [HA, Definition 1.3.2.1] the differential graded category structure on $\mathrm{Ch}^-(\mathcal{A})$.

derived categories such that $\underline{(-)} = \tau_{\geq 0} \circ (F|_{\mathcal{D}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.})^{\heartsuit}})$. Now conclude by observing that $\mathrm{N}_{\mathrm{dg}}((6.8))$ satisfies this property.

Using [HA, Remark 1.3.3.6], we have that $\underline{\mathcal{D}}$ is t-exact because the underlying functor on hearts, $\mathrm{Vect}_{\mathbb{C}}^{f.d.} \to \mathrm{Liq}_p(\mathbb{C})$, is exact by Lemma 6.6. Finally, the symmetrical monoidality of $\underline{\mathcal{D}}$ also follows immediately from Lemma 6.6. □

6.1. **The main theorem.** Before we state the main theorem of the first part, we state and prove the following elementary lemma.

Lemma 6.9. *There is a t-exact equivalence of stable ∞-categories,*

$$\mathrm{Perf} \simeq \mathcal{D}^b(\mathrm{Vect}_{\mathbb{C}}^{f.d.}).$$

Proof. The inclusion $\mathrm{Vect}_{\mathbb{C}}^{f.d.} \hookrightarrow \mathrm{Vect}_{\mathbb{C}}^{\heartsuit}$ induces by functoriality a t-exact functor

$$\mathcal{D}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.})) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{\heartsuit})) =: \mathrm{Vect}_{\mathbb{C}}^-.$$

The restriction of this functor to the subcategory of bounded complexes induces a map,

$$\mathcal{D}^b(\mathrm{Vect}_{\mathbb{C}}^{f.d.}) \to \mathrm{Perf}$$

by the definition of Perf. This functor is essentially surjective by the definition of Perf, and is fully faithful because the differential graded nerve preserves fully faithfulness (essentially because the Dold-Kan functor preserves equivalences) and the map of differential graded categories, $\mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{f.d.}) \to \mathrm{Ch}^-(\mathrm{Vect}_{\mathbb{C}}^{\heartsuit})$ is fully faithful. □

We are now finally able to state and prove our main theorem.

Theorem 6.10. *Suppose given $X \in \mathrm{Perf}$ and $T \in \mathrm{End}_{\mathrm{Perf}}(X)$. Then for any entire function f, there exists an endomorphism $f(T) \in \mathrm{End}_{\mathrm{Perf}}(X)$ such that $H_i(f(T))$ is the linear map $f(H_i(T))$ obtained from classical functional calculus by applying f to the induced map $H_i(T) : H_i(X) \to H_i(X)$.*

Proof. Lemma 6.9 in conjunction with Lemma 6.7 tells us that Perf admits a t-exact fully faithful embedding into $\mathcal{D}^b(\mathrm{Liq}_p(\mathbb{C}))$. On the other hand, the latter category is equivalent to $\mathcal{D}^b(\mathbb{C}, \mathcal{M}_{<p})$ by Proposition 4.18, so we have a fully faithful embedding, $\mathrm{Perf} \hookrightarrow \mathcal{D}^b(\mathbb{C}, \mathcal{M}_{<p})$ whose essential image clearly lies in the full subcategory, $\mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p})$, of objects with homology objects belonging to the essential image of $\mathbf{Ban} \hookrightarrow \mathrm{Liq}_p(\mathbb{C})$. We denote this embedding by

$$\underline{\mathcal{D}} : \mathrm{Perf} \hookrightarrow \mathcal{D}_{\mathbf{Ban}}(\mathbb{C}, \mathcal{M}_{<p}).$$

By the discussion in Section 5.3 combined with Proposition 5.11, $f(\underline{\mathcal{D}}(T))$ exists, and induces the endomorphisms $f(H_*(\underline{\mathcal{D}}(T)))$ upon taking homology. Since $\underline{\mathcal{D}}$ is fully faithful, $f(\underline{\mathcal{D}}(T))$ corresponds to an endomorphism in Perf which we denote $f(T) \in \mathrm{End}_{\mathrm{Perf}}(X)$. Finally, since $\underline{\mathcal{D}}$ is t-exact, $H_*(f(T))$ corresponds under $\underline{\mathcal{D}}$ to the liquid endomorphism $f(H_*(\underline{\mathcal{D}}(T)))$. □

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