DERIVED V-FILTRATIONS AND THE KONTSEVICH–SABBAH–SAITO THEOREM

KENDRIC SCHEFERS

Abstract. Let $f : X \to \mathbb{A}^1$ be a regular function on a smooth complex algebraic variety X. We formulate and prove an equivalence between the algebraic formal twisted de Rham complex of f and the vanishing cycles with respect to f as objects in the category of sheaves valued in the derived ∞-category of modules over $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$, the ring of germs of polynomial formal microdifferential operators. This is a direct generalization of Kontsevich's conjecture, proven in work by Sabbah and then Sabbah–Saito, of an algebraic formula computing vanishing cohomology.

The novelty in our approach is the introduction of a canonical V-filtration on the derived ∞-category of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules, and the use of various techniques from the theory of higher categories and higher algebra in the context of the subject of microdifferential calculus.

Contents

1. Introduction

The Kontsevich–Sabbah–Saito theorem is a formula that computes the vanishing cohomology of a regular function $f : X \to \mathbb{A}^1$ on a smooth complex variety X algebraically in terms of the algebraic formal twisted de Rham complex. The formula was originally conjectured by Maxim Kontsevich in connection with a question appearing in [KS11] about how to define the notion of vanishing cycles of a regular function on a smooth formal scheme. Kontsevich's conjecture was initially proven by Claude Sabbah in [Sab10], whose arguments and results were refined and improved upon in joint work by Sabbah and Morihiko Saito in [SS14].

The exact statement of the Kontsevich–Sabbah–Saito theorem may be found, of course, in [Sab10] as Theorem 1.1, or later in this present work as Theorem 8.15. The formula proven therein is an isomorphism of the k th vanishing cohomology of f with the k th hypercohomology of the algebraic formal twisted de Rham complex as ordinary $\mathbb{C}(\!(u)\!)$-modules with connection. In other words, it determines an isomorphism of ordinary modules over $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}}$, the ring of germs of polynomial formal microdifferential operators on $\mathbb{C}$ at $(0;1) \in T^*\mathbb{C}$.[1] In this work, we formulate and prove a sheaf-theoretic generalization of the Kontsevich–Sabbah–Saito theorem. Namely, our generalization establishes an equivalence of Zariski sheaves valued in the bounded derived ∞-category of $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}}$-modules. The following is the precise statement of our main theorem.

Theorem 1.1 (Theorem 8.19)**.** *Let X be a smooth complex algebraic variety, and $f : X \to \mathbb{A}^1$ a regular function on X. Let $\mathscr{M}$ be a locally free $\mathcal{O}_X$-module of finite rank equipped with a flat connection ∇ having regular singularity at infinity, and let $\mathscr{L}$ denote its local system of analytic flat sections. Let $\tau : X^{\mathrm{an}} \to X^{\mathrm{Zar}}$ denote the canonical continuous functor from the analytic topology on X to the underlying Zariski topology on X. Then there is an equivalence of objects in* $\mathrm{Shv}^b(X^{\mathrm{Zar}}; \widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}})$,

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}) \simeq \bigoplus_{c \in \mathbb{C}} \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}(\!(u)\!)} \tau_* \mu \mathcal{RH}_{\mathrm{Shv}} \left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right), \tag{1.2}$$

which recovers the original statement of the Kontsevich–Sabbah–Saito theorem upon taking hypercohomology.

[1]We provide a description of $\widehat{\mathscr{E}}_{\mathbb{C},0}$ and its subring $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}}$ later in Section 6. We indicate how a $\mathbb{C}(\!(u)\!)$-module with connection determines a $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}}$-module in Remark 8.3.

The object appearing on the left-hand side of (1.2) is the algebraic formal twisted de Rham complex on X, while $\mu\mathcal{RH}_{\mathrm{Shv}}$ is a functor from the ∞-category of constructible $\mathcal{D}^b(\mathcal{L}\mathrm{oc}(\mathbb{C}\setminus 0))$-valued sheaves to the ∞-category of constructible[2] $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$-valued sheaves. Precise definitions of all these terms are given later in the text.

A novelty in our approach is the introduction of a derived V-filtration (also called the Kashiwara-Malgrange filtration), on objects of $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$, the bounded derived ∞-category of regular holonomic modules over $\mathscr{D}_{\mathbb{C},0}$, the ring of germs of differential operators at $0\in\mathbb{C}$. This novelty, combined with a suitable interpretation of objects appearing in the Kontsevich–Sabbah–Saito theorem, allow us to bootstrap Sabbah's original arguments in [Sab10] to the setting of sheaves valued in higher categories.

1.1. **A confusing point.** The original Kontsevich–Sabbah–Saito theorem is formulated as an isomorphism of various $\mathbb{C}(\!(u)\!)$-modules with connection, but the proof found in [Sab10] at some points uses the formalism of $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules. At various points throughout this work, we work with $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules, $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-modules, and $\mathbb{C}(\!(u)\!)$-modules with connection. We wish here to clarify the relationship between these three types of objects, and explain the sometimes subtle jump from one type to another.

A $\mathbb{C}(\!(u)\!)$-module with connection is a module $\mathbb{C}(\!(u)\!)$-module E equipped an operator $\nabla:\mathbb{C}(\!(u)\!)\partial_u\times E\to E$ which is $\mathbb{C}(\!(u)\!)$-linear in the first variable and which satisfies the Leibniz rule with respect to the $\mathbb{C}(\!(u)\!)$-action in the second. As explained in Remark 8.3, the connection on E allows us to define an action of $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$, that is, Laurent series in the formal variable u whose coefficients are polynomials in the affine coordinate t. Conversely, given a $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-module M, its underlying $\mathbb{C}(\!(\xi)\!)$-module E is a $\mathbb{C}(\!(u)\!)$-module with a canonical connection after relabeling $u:=\xi$ and setting $\nabla_{\partial_u}(e):=-t\cdot e\xi^{-2}$ for $e\in E$. By means of the resultant equivalence of categories, $\mathrm{Conn}_{\mathbb{C}(\!(u)\!)}\simeq\mathcal{A}_{\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}}$, we use $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-modules and $\mathbb{C}(\!(u)\!)$-modules with connection interchangeably throughout the text.

As its notation suggests, $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$ is a subring of $\widehat{\mathscr{E}}_{\mathbb{C},0}$, the ring of germs at $(0;1)\in T^*\mathbb{C}$ of formal microdifferential operators on $\mathbb{C}$. The abelian categories of modules over these two rings are not equivalent, but because we would like to use existing results in the literature, it will be convenient to work with $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules at times. This is facilitated by the fact that the categories of *regular holonomic* modules over $\widehat{\mathscr{E}}_{\mathbb{C},0}$ and $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$, respectively, *are* equivalent (see Lemma 6.12). Because the central objects of our proof are constructible sheaves with regular holonomic stalks, this equivalence provides enough leverage to put the theory of $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules to use in proving the main theorem.

1.2. **Structure of the paper.** In Section 2, we list a few conventions that hold throughout the body of this work, and we include a glossary of commonly used categories with possibly nonstandard notation, for the benefit of the reader.

In Section 3, we recall some preliminaries definitions and notions in the theory of analytic $\mathscr{D}$-modules, including the V-filtration on holonomic modules and the statement of the Riemann-Hilbert correspondence. We end the section with a particular formulation of Deligne's correspondence that we use throughout the body of the paper.

In Section 4 we establish the basic notion and facts regarding modules over the ring of germs at 0 of differential operators on $\mathbb{C}$.

In Section 5, we introduce a V-filtration functor on $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$ taking values in the category of $V^\bullet\mathscr{D}_{\mathbb{C},0}$-module objects in the category of filtered spectra.

[2]Constructibility for us will always include the condition that stalks belong to some nice subcategory. See Appendix A for details. In this case, the stalks of a constructible $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$-valued sheaf are required have regular holonomic cohomology $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules.

In Section 6, we recall the ring of germs of formal microdifferential operator at $(0;1) \in T^*\mathbb{C}$ and establish an equivalence of the formal microlocalization and vanishing cycles of objects in $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$.

In Section 7, we show that the constructions of the previous sections induce corresponding constructions on the ∞-categories of constructible sheaves valued of $\mathscr{D}_{\mathbb{C},0}$-modules and $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules. In particular, we show the existence and uniqueness of V-filtration on objects of $\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$.

In Section 8, we recall the precise statement of the Kontsevich–Sabbah–Saito theorem, and formulate our main theorem. We also state the proof of the main theorem, which is very short if one assumes the results of the subsequent section.

In Section 9, we state and prove the components used in the proof of the Main Theorem of the previous section. Our results and presentation in this section follow Sabbah's in [Sab10] very closely. The only originality we contribute here is the application of the foundational results of the previous sections that allows us to adapt Sabbah's arguments to the setting of sheaves in higher categories.

In Appendix A, we establish our notion and conventions for constructible sheaves taking values in a wide variety of ∞-categories, as well as prove some results needed to prove our Main Theorem and show the existence of a V-filtration on sheaves. This section is a mix of known and (as far as we know) new results, and we have tried to indicate throughout which is which.

In Appendix B, we collect some results on derived local systems on the punctured complex plane for use in stating a derived version of Deligne's correspondence.

In Appendix C, we establish our notation and conventions for the category of filtered objects in an ∞-category admitting sequential limits, which are taken directly from the paper of Gwilliam and Pavlov on the subject. We also prove a few results on filtered objects in the context of categories of sheaves.

1.3. **Acknowledgments.** I thank my advisor Sam Raskin for his constant willingness to answer technical questions about nearly everything, but especially about the finer points of higher algebra and derived categories. I also thank my other advisor David Ben-Zvi for suggesting a major simplification of my original approach to this problem.

I would also like to thank Tom Gannon, Rok Gregoric, and Sam Gunningham for helpful discussions relating to this work.

I am grateful to Mads Bach Villadsen for patiently explaining aspects of V-filtrations to me over several days during the Summer School on Motivic Integration theory hosted at the HHU Düsseldorf, where we met.

Lastly, I warmly thank Claude Sabbah for answering questions about his own work, and Pierre Schapira for answering my questions about microdifferential operators.

2. Notation and Conventions

2.0.1. All functors are implicitly derived.

2.0.2. Our grading conventions (homological vs. cohomological) in this paper are at least locally constant, but possibly globally nonconstant.

2.0.3. We freely use the language of ∞-categories. Our main reference for higher categories and higher algebra will be [HTT] and [HA], respectively. Our main reference for sheaves in the higher categorical context, in addition to the two already mentioned, will be [SAG].

2.0.4. pt will always denote the terminal map from a topological space to a point. When the domain of pt is not clear from context, we denote its domain using a subscript, e.g. pt_X.

2.0.5. We use $\mathbb{A}^1$ to denote the algebraic affine line, and $\mathbb{C}$ to denote the analytic affine line.

2.0.6. f_* and f^* denote the sheaf-theoretic pushforward and pullback along the map f (as opposed to potentially the $\mathscr{D}$-module pushforward and pullback).

2.1. Glossary of categories.

General notation for categories.

- $\mathrm{Ch}(\mathcal{A})$ denotes the ordinary category of chain complexes of objects in the abelian category $\mathcal{A}$.
- $\mathcal{D}(\mathcal{A})$ and its bounded variations denotes the derived ∞-categories of the abelian category $\mathcal{A}$.
- If $\mathcal{D}$ is a stable ∞-category with a t-structure:
  - $\mathcal{D}_{\geq 0}$ denotes the connective part of the t-structure.
  - $\mathcal{D}_{\leq 0}$ denotes the coconnective part of the t-structure.
  - $\mathcal{D}^{\heartsuit} := \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\geq 0}$ denotes the heart of the t-structure.
- $\mathrm{Seq}(\mathscr{C})$ the category of sequences in the stable ∞-category $\mathscr{C}$). See Definition C.4.
  - $\mathsf{Fil}(\mathscr{C})$ is the filtered category of the ∞-category $\mathscr{C}$ admitting sequential limits. It is a localization of $\mathrm{Seq}(\mathscr{C})$ by graded equivalences. See Definition C.6.
- $\mathrm{Seq}(\mathcal{A})$ is the category of sequences in the abelian category $\mathcal{A}$. See Definition C.1.
  - $\mathsf{Fil}(\mathcal{A})$ is the filtered category of the abelian category $\mathcal{A}$. Its objects consists of sequences in $\mathcal{A}$ whose structure maps are monomorphisms. See Definition C.2.

Special categories.

- Sp denotes the stable ∞-category of spectra.
  - Sp^*, where $* = +, -, b$, denote the category of left-bounded, right-bounded, and bounded spectra, respectively.
- Mod_R, for R a classical ring, denotes either the stable ∞-category of R-module spectra if R is a commutative ring, or left R-module spectra otherwise.
- $\mathcal{A}_R$, for R a classical ring, denotes the abelian category of R-modules if R is commutative, left R-modules otherwise. Note that $\mathcal{A}_R \simeq \mathrm{Mod}_R^{\heartsuit}$ with the canonical t-structure coming from spectra on the latter.
- $\mathcal{D}(R)$ and its bounded variations denote the derived ∞-categories of $\mathcal{A}_R$. They are equivalent Mod_R and its bounded variations.
- Vect is special notation for $\mathrm{Mod}_{\mathbb{C}}$ (or $\mathcal{D}(\mathbb{C})$).
  - Perf denotes the ∞-subcategory of compact objects in Vect, also know as perfect complexes.

Categories of sheaves.

- $\mathrm{Shv}(\mathcal{X};\mathscr{C})$ denotes the ∞-category of $\mathscr{C}$-valued sheaves on an ∞-topos $\mathcal{X}$.
  - $\mathrm{Shv}(X;\mathscr{C})$ denotes the ∞-category of sheaves on the ∞-topos associated to the topological space X.
- $\mathrm{Shv}(\mathcal{X};R)$ denotes the ∞-category of Mod_R-valued sheaves on the ∞-topos $\mathcal{X}$.
  - $\mathrm{Shv}^*(\mathcal{X};R)$, where $* = -, +, b$, denotes the ∞-category of Mod_R^*-valued sheaves on $\mathcal{X}$.
- $\mathrm{Shv}_{\mathrm{w.c.}}(X;\mathscr{C})$ denotes the ∞-category of weakly constructible sheaves. See Definition A.18.
  - $\mathrm{Shv}_c(X;\mathscr{C})$ denotes the full ∞-subcategory of constructible sheaves with respect to some coefficient pair $(\mathscr{C},\mathscr{N})$. See Definition A.20.
  - $\mathrm{Shv}_c^{f.s}(X;\mathscr{C})$ denotes the full ∞-category spanned by sheaves that are constructible with respect to some finite stratification of X.
  - $\mathrm{Shv}^A(X;\mathscr{C})$ denotes the full ∞-subcategory of sheaves that are A-constructible, for the stratification $X \to A$. See definition Definition A.15.
- $\mathcal{A}(X;R)$ denotes the abelian category of sheaves of R-modules on X, for R a ring.

  - $\mathrm{Loc}(X)$ denotes the full abelian subcategory of $\mathcal{A}(X;R)$ spanned by locally constant sheaves with projective stalks.
- $\mathcal{D}(X;R)$ and its bounded variations denote the derived ∞-categories of $\mathcal{A}(X;R)$. Note that $\mathcal{D}(X;R)$ (and its bounded variations) is equivalent to $\mathrm{Shv}(X;R)$ (and its bounded variations).
- $\mathrm{Perv}(X)$ denotes the abelian category of perverse sheaves on the complex analytic space X. It is the heart of the middle perverse t-structure on $\mathrm{Shv}_c(X;\mathbb{C})$. $\mathcal{L}\mathrm{oc}(X)$ denotes the full subcategory of $\mathrm{Perv}(X)$ spanned by local systems of finite rank, shifted by the dimension of their supports, so as to be perverse.
- $\mathcal{D}^b_{\mathrm{Loc}}(X;\mathbb{C})$ denotes the full subcategory of $\mathcal{D}^b(X;\mathbb{C})$ spanned by objects whose cohomology sheaves belong to $\mathrm{Loc}(X)$.
- $\mathcal{D}^b_{\mathcal{L}\mathrm{oc}}(X;\mathbb{C})$ denotes the full subcategory of $\mathrm{Shv}_c(X;\mathbb{C})$ spanned by objects whose perverse cohomology sheaves belong to $\mathcal{L}\mathrm{oc}(X)$.

3. Preliminaries on $\mathscr{D}_X$-modules

3.1. **Definitions.** Let X be a complex manifold, and let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, which we regard as the structure sheaf on X. Let $\mathscr{D}_X$ denote the sheaf of differential operators on X. By definition, $\mathscr{D}_X$ is the subsheaf of $\mathbb{C}$-algebras of $\mathcal{E}nd(\mathcal{O}_X)$, the sheaf of $\mathbb{C}$-linear endomorphisms of the $\mathcal{O}_X$, generated by $\mathcal{O}_X$, which acts by multiplication, and the sheaf of $\mathbb{C}$-linear derivations $\mathcal{D}er(X) \subset \mathcal{E}nd(\mathcal{O}_X)$.

Definition 3.1. A sheaf of $\mathcal{O}_X$-modules M is a (left) $\mathscr{D}_X$-module if for every open set $U \subseteq X$, $M(U)$ has a left $\mathscr{D}_X(U)$-module structure, compatible with restrictions.

There is a definition of right $\mathscr{D}_X$-module in which the sections of M over U are required to have a right $\mathscr{D}_X(U)$-module structure. We will work principally with left $\mathscr{D}_X$-modules in this note, and in the sequel we will refer to them simply as "$\mathscr{D}_X$-modules."

We let $\mathcal{A}_{\mathscr{D}_X}$ denote the category of $\mathscr{D}_X$-modules. The category $\mathcal{A}_{\mathscr{D}_X}$ is abelian, and we denote by $\mathcal{D}^b(\mathscr{D}_X)$ its bounded derived ∞-category.

Definition 3.2. We let $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X}$ denote the full subcategory of $\mathcal{A}_{\mathscr{D}_X}$ spanned by regular holonomic $\mathscr{D}_X$-modules.

The precise definitions of regularity and holonomicity are not important to us, so we do not recall them. On the other hand, it is important to note that the category $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X}$ is abelian, and, by a theorem of Beilinson ([Bei87, Theorem 1.3]), the natural map,

$$\mathcal{D}^b(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X}) \to \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_X)$$

is an equivalence, where the left-hand side denotes the bounded derived ∞-category of $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X}$ and the right-hand side denotes the full ∞-subcategory of $\mathcal{D}^b(\mathscr{D}_X)$ spanned by objects with regular holonomic cohomology.

3.2. **The V-filtration.** Given a complex manifold X of dimension n, and a smooth hypersurface $H \subset X$, there is a canonical decreasing filtration on $\mathscr{D}_X$ called the *V-filtration* (also called the *Kashiwara-Malgrange filtration*). We recall its definition below.

Definition 3.3. Let $I_H \subset \mathcal{O}_X$ denote the ideal of definition of the smooth hypersurface $H \subset X$. The V-filtration along H on $\mathscr{D}_X$, denoted $V^\bullet\mathscr{D}_X$ is a decreasing $\mathbb{Z}$-filtration given by,

$$V^\ell\mathscr{D}_X := \left\{ P \in \mathscr{D}_X | P \cdot {I_H}^j \subset {I_H}^{j+\ell} \text{ for all } j \in \mathbb{Z} \right\}.$$

In local coordinates $(x,t) := (x_1, \dots x_{n-1}, t)$ on X, where locally $H = \{t = 0\}$, the V-filtration takes the following form,

$$V^0\mathscr{D}_X = \left\{ \sum_{\alpha,k} a_{\alpha,k}(x,t){\partial_x}^{\alpha}(t\partial_t)^k \middle| \alpha \in \mathbb{N}^n, k \in \mathbb{N}, a_{\alpha,k} \in \mathcal{O}_X \right\},$$

$$V^k\mathscr{D}_X = \begin{cases} t^kV^0\mathscr{D}_X & \text{if } k \geq 0, \\ \sum_{j=0}^{-k} {\partial_t}^jV^0\mathscr{D}_X & \text{if } k \leq 0. \end{cases}$$

Remark 3.4. Observe that $tV^k\mathscr{D}_X \subset V^{k+1}\mathscr{D}_X$, and $\partial_tV^k\mathscr{D}_X \subset V^{k-1}\mathscr{D}_X$. On the other hand, all operators in the x-direction (i.e. of the form $\sum_{j\in\mathbb{N}^{n-1}}^m a_j(x){\partial_x}^j$, for $a_j(x) \in \mathcal{O}_X$) preserve the pieces of the filtration.

There is also a related notion of V-filtration along H for $\mathscr{D}_X$-modules.

Definition 3.5. Given a $\mathscr{D}_X$-modules M, a *Kashiwara-Malgrange* or *V-filtration along H* on M is any exhaustive, decreasing $\mathbb{Z}$-filtration on M that satisfies the following properties:

(i) Each V^kM is a coherent module over $V^0\mathscr{D}_X$.
(ii) $V^kM = t^kV^0M$ for $k \geq 0$.
(iii) $V^kM = \sum_{j=0}^{-k-1} {\partial_t}^jV^{-1}M$ for $k > 0$.
(iv) Locally on X, there exists a polynomial $b(s) \in \mathbb{C}[s]$ with roots in $\mathbb{Q} \cap [0,1)$ such that $b(t\partial_t - k)$ vanishes identically on $\mathrm{gr}_V^k M$ for each k.

A well-known result of Kashiwara ([Kas83, Theorem 1]) states that if a V-filtration exists, it is unique. It turns out that holonomic $\mathscr{D}_X$-modules always admit Kashiwara-Malgrange filtrations along any smooth hypersurface. In fact, any morphism $f : M \to N$ between $\mathscr{D}_X$-modules that possess Kashiwara-Malgrange filtrations is strictly compatible with the filtrations in the sense that $f(V^iM) = f(M) \cap V^iN$.[3] We note that the V-filtration is compatible with the standard V-filtration on $\mathscr{D}_X$

Using the notation established in Appendix C, we might summarize the above discussion by saying that the V-filtration is a functor on regular holonomic $\mathscr{D}_X$-modules taking values in modules over the filtered sheaf of algebras $V^\bullet\mathscr{D}_X$,

$$\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X} \xrightarrow{V^\bullet} \mathrm{Mod}_{V^\bullet\mathscr{D}_X}(\mathsf{Fil}(\mathrm{Coh}(X))),$$

where $\mathrm{Coh}(X)$ denotes the abelian category of coherent sheaves on X. As discussed in Appendix C, $\mathsf{Fil}(\mathrm{Coh}(X))$ is not abelian, though it is additive and admits kernels and cokernels. In general, however, the the natural map from the coimage of a morphism to the image is not an isomorphism. Nonetheless, we may still talk about short exact sequences in $\mathrm{Mod}_{V^\bullet\mathscr{D}_X}(\mathsf{Fil}(\mathrm{Coh}(X)))$ and exact functors to and from $\mathrm{Mod}_{V^\bullet\mathscr{D}_X}(\mathsf{Fil}(\mathrm{Coh}(X)))$.

Lemma 3.6. *The V-filtration functor, $V^\bullet$ is an exact functor.*

Proof. It is clear that $V^\bullet$ is an additive functor. That it preserves short exact sequences follows from the strict compatibility of maps of $\mathscr{D}_X$-modules with the V-filtration. □

3.3. **de Rham functors.** Let X be a complex manifold of dimension n. For $0 \leq p \leq n$, we let Ω_X^p denote the sheaf of degree p holomorphic differential forms on X. There is a canonical right $\mathscr{D}_X$-module structure on Ω_X^n. We will use Ω_X to denote Ω_X^n as a $\mathscr{D}_X$-module.

Definition 3.7. The *de Rham functor* on $\mathcal{D}^b(\mathscr{D}_X)$, denoted DR_X, is defined to be

$$\Omega_X \otimes_{\mathscr{D}_X} - : \mathcal{D}^b(\mathscr{D}_X) \to \mathrm{Shv}(X;\mathbb{C}).$$

[3] In his online notes, Mihnea Popa leaves this as an exercise for the reader, as will we.

The right $\mathscr{D}_X$-module Ω_X is also the transfer bimodule $\mathscr{D}_{*\leftarrow X}$. There is a canonical locally free resolution of Ω_X as a right $\mathscr{D}_X$-module given by the complex,

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} \mathscr{D}_X \to \cdots \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathscr{D}_X \to \Omega_X \to 0.$$

More generally, given a decomposition $X = Y \times Z$, we may consider the transfer bimodule, $\mathscr{D}_{Y\leftarrow X}$.

Definition 3.8. The *relative de Rham* functor for $\mathrm{pr} : X \to Y$ on $\mathcal{D}^b(\mathscr{D}_X)$, denoted $\mathrm{DR}_{X/Y}$, is defined to be,

$$\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} - : \mathscr{D}^b(\mathscr{D}_X) \to \mathcal{D}^b(\mathrm{pr}^* \mathscr{D}_Y),$$

where pr^* denote the pullback of sheaves (rather than the pullback of $\mathscr{D}_X$-modules).

The bimodule $\mathscr{D}_{Y\leftarrow X}$ has a resolution as a right $\mathscr{D}_X$-module, similarly given by tensoring with the *relative* differential forms on X. Set $d = \dim Z$, and let $\Omega^k_{X/Y}$ denote the sheaf of relative differential forms on X, for $0 \le k \le d$. Then the complex,

$$0 \to \Omega^0_{X/Y} \otimes_{\mathcal{O}_X} \mathscr{D}_X \to \cdots \to \Omega^d_{X/Y} \otimes_{\mathcal{O}_X} \mathscr{D}_X \to \mathscr{D}_{Y\leftarrow X} \to 0$$

is a locally free resolution of $\mathscr{D}_{Y\leftarrow X}$. Using this locally free resolution, we have, for any $M \in \mathcal{A}_{\mathscr{D}_X}$,

$$\mathrm{DR}_{X/Y}(M) \simeq \Big[0 \to \Omega^0_{X/Y} \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^d_{X/Y} \otimes_{\mathcal{O}_X} M \to 0\Big] \tag{3.9}$$

as objects of the derived category $\mathcal{D}^b(\mathrm{pr}^* \mathscr{D}_Y)$.

3.4. **The Riemann-Hilbert correspondence.** Let X be a complex manifold of dimension n. Kashiwara's constructibility theorem ([HTT08, Theorem 4.6.3]) states that the image of a holonomic $\mathscr{D}_X$-module under the de Rham functor DR_X lies in the subcategory $\mathrm{Shv}_c(X;\mathbb{C}) \subset \mathrm{Shv}(X;\mathbb{C})$. In fact, DR_X induces an equivalence between the $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_X)$ and the stable ∞-category of constructible sheaves.

Theorem 3.10 (The Riemann-Hilbert correspondence)**.** *The de Rham functor induces a t-exact equivalence of stable ∞-categories,*

$$\mathrm{DR}_X : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_X) \xrightarrow{\simeq} \mathrm{Shv}_c(X;\mathbb{C})$$

with respect to the standard t-structure on the left-hand category and the middle perverse t-structure on the right-hand category.

3.4.1. *Deligne's correspondence.* The Riemann-Hilbert correspondence generalizes the classical correspondence of Deligne between integrable connections and local systems.

Definition 3.11. We say $M \in \mathcal{A}_{\mathscr{D}_X}$ is an integrable connection if its underlying $\mathcal{O}_X$-module is locally free of finite rank. We denote the full subcategory of $\mathcal{A}_{\mathscr{D}_X}$ spanned by integrable connections by $\mathrm{Conn}(X)$.

Definition 3.12. A local system on X is a perverse sheaf on X of the form $\mathscr{L}[n]$, where $\mathscr{L}$ is a locally constant sheaf of $\mathbb{C}$-vector spaces of finite rank. We denote by $\mathcal{L}\mathrm{oc}(X)$ the full subcategory of $\mathrm{Perv}(X)$ spanned by local systems.

Let $D \subset X$ be a divisor (complex hypersurface) on X. We briefly recall the notion of *regular meromorphic connection on X along D*. A meromorphic connection on X along D is a coherent $\mathcal{O}_X(D)$-module with a $\mathbb{C}$-linear connection. Heuristically, a meromorphic connection M on the unit open disk $B \subset \mathbb{C}$ along 0 is regular if its associated connection matrix has poles of at most order 1 at 0. A meromorphic connection M on X along D is regular if for any map $u : B \to X$ such that $u^{-1}(D) = \{0\}$, the germ of i^*M is regular along $0 \subset B$. We refer the reader to §5.1.2 and §5.2 in [HTT08] for precise definitions.

Let $\mathrm{Conn}(X;D)$ denote the category of meromorphic connections on X along D, and let $\mathrm{Conn}^{\mathrm{reg}}(X;D)$ denote the full subcategory of $\mathrm{Conn}(X;D)$ spanned by regular meromorphic connections. Restriction of a meromorphic connection M on X along D to the complement of the divisor defines a functor, $\mathrm{Del}_{(X,D)} : \mathrm{Conn}^{\mathrm{reg}}(X;D) \to \mathrm{Conn}(X \setminus D)$, i.e. the restriction $M|_{X\setminus D}$ is locally free. A landmark result of Deligne ([Del70]) states that the restriction of $\mathrm{Del}_{(X;D)}$ to the subcategory of regular connections induces an equivalence,

$$\mathrm{Del}_{(X;D)} : \mathrm{Conn}^{\mathrm{reg}}(X;D) \xrightarrow{\simeq} \mathrm{Conn}(X \setminus D).$$

Remark 3.13. Regular meromorphic connections on X along D define regular holonomic $\mathscr{D}_X$-modules. In fact, $\mathrm{Conn}^{\mathrm{reg}}(X;D)$ is an abelian subcategory of $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_X}$.

On the other hand, a simple computation[4]shows that the de Rham functor induces an equivalence of abelian categories,

$$\mathrm{DR}_X : \mathrm{Conn}(X) \xrightarrow{\simeq} \mathcal{L}\mathrm{oc}(X).$$

This observation allows us to state what is known as *Deligne's correspondence.*

Theorem 3.14 (Deligne's correspondence)**.** *Let X be a complex manifold, and let $D \subset X$ be a divisor. Then there is an equivalence,*

$$\mathrm{DR}_X \circ \mathrm{Del}_{(X;D)} : \mathrm{Conn}^{\mathrm{reg}}(X;D) \xrightarrow{\simeq} \mathcal{L}\mathrm{oc}(X \setminus D).$$

We will be interested in the special case when $X = \mathbb{C}$ and $D = \{0\}$, in which case Deligne's correspondence produces an equivalence, $\mathrm{Conn}^{\mathrm{reg}}(\mathbb{C};0) \xrightarrow{\simeq} \mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0)$. Taking bounded derived ∞-categories, this equivalence induces a t-exact equivalence,

$$\mathcal{D}^b(\mathrm{Conn}^{\mathrm{reg}}(\mathbb{C};0)) \xrightarrow{\simeq} \mathcal{D}^b(\mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0)).$$

By Corollary B.6, the right-hand category is t-exact equivalent to $\mathcal{D}^b_{\mathcal{L}\mathrm{oc}}(\mathbb{C}\setminus 0;\mathbb{C})$, where the latter is the full subcategory of $\mathrm{Shv}_c(\mathbb{C}\setminus 0;\mathbb{C})$ spanned by objects whose perverse cohomology sheaves belong to $\mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0)$. This fact combined with the Riemann-Hilbert correspondence yields the following lemma.

Lemma 3.15. *Let $\mathcal{D}^b_{\mathrm{reg.\,conn.}}(\mathscr{D}_{\mathbb{C}})$ denote the full subcategory of $\mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C}})$ spanned by objects whose cohomology lies in* $\mathrm{Conn}^{\mathrm{reg}}(\mathbb{C};0)$*. Then there is a t-exact equivalence*

$$\mathcal{D}^b(\mathrm{Conn}^{\mathrm{reg}}(\mathbb{C};0)) \xrightarrow{\simeq} \mathcal{D}^b_{\mathrm{reg.\,conn.}}(\mathscr{D}_{\mathbb{C}}).$$

Combining Lemma 3.15 above with Lemma B.9 and Corollary B.6 in Appendix B, we obtain the following t-exact equivalence, which is a kind of derived version of Deligne's correspondence.

Corollary 3.16 (Derived Deligne's correspondence)**.** *There is a t-exact equivalence of stable ∞-categories,*

$$\mathcal{D}^b_{\mathrm{reg.\,conn.}}(\mathscr{D}_{\mathbb{C}}) \xrightarrow{\simeq} \mathrm{Fun}(S^1, \mathrm{Perf}), \tag{3.17}$$

where the t-structure on the right-hand category is the one coming from the perverse t-structure on $\mathrm{Shv}_{\mathcal{L}\mathrm{oc}}(\mathbb{C} \setminus 0;\mathbb{C})$ *under the equivalence obtained in Lemma B.9.*

Remark 3.18. Note that the above discussion holds for *any* subset of $\mathbb{C}$ whose homotopy type is $K(\mathbb{Z},1)$. In particular, (3.17) holds for the complex analytic disk $D_\varepsilon(0) \subset \mathbb{C}$ of radius $\varepsilon > 0$ in place of $\mathbb{C}$, and $\mathring{D}_\varepsilon(0)$ in place of $\mathbb{C} \setminus 0$.

[4]See the discussion preceding [HTT08, Theorem 4.2.4] for details.

4. $\mathscr{D}_{\mathbb{C},0}$-MODULES

Definition 4.1. Let $\mathscr{D}_{\mathbb{C},0}$ denote the stalk at 0 of the sheaf of differential operators $\mathscr{D}_{\mathbb{C}}$. As a module, it is $\mathbb{C}\{t\}\langle\partial_t\rangle$, where $\mathbb{C}\{t\}$ denotes the ring of germs of holomorphic functions in the variable t; its product structure is given by the Leibniz rule.

Since $\mathscr{D}_{\mathbb{C},0}$ is an ordinary associative ring, it can be regarded as a discrete $\mathbb{E}_1$-ring. Moreover, the ∞-category of left $\mathscr{D}_{\mathbb{C},0}$-module spectra is equivalent to the unbounded derived ∞-category, $\mathcal{D}(\mathscr{D}_{\mathbb{C},0})$, of the abelian category of left $\mathscr{D}_{\mathbb{C},0}$-modules, $\mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$ (see [HA, Remark 7.1.1.16]).

Remark 4.2. Both $\mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$ and its derived category $\mathcal{D}^b(\mathscr{D}_{\mathbb{C},0})$ inherit symmetric monoidal products (the latter the derived functor of the former) coming from the canonical tensor product[5]of $\mathscr{D}_{\mathbb{C}}$-modules, since every $\mathscr{D}_{\mathbb{C},0}$-module $\mathcal{M} \in \mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$ a representative in $\mathcal{A}_{\mathscr{D}_{\mathbb{C}}}$ of which it is the germ at $0 \in \mathbb{C}$.

4.1. **Regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules.**

Given an ordinary $\mathscr{D}_{\mathbb{C}}$-module M, its germ at $0 \in \mathbb{C}$ is naturally a $\mathscr{D}_{\mathbb{C},0}$-module. This feature provides a convenient way of defining the notion of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-module.

Definition 4.3. Given an ordinary $\mathscr{D}_{\mathbb{C},0}$-module, $M \in \mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$, we say that M is *regular holonomic* if there a regular holonomic $\mathscr{D}_{\mathbb{C}}$-module $\widetilde{M}$ such that the germ of $\widetilde{M}$ at $0 \in \mathbb{C}$ is M.

We denote the abelian category of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules by $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$. This category turns out to be equivalent to the category of integrable connections on a punctured disc around the origin that are regular at $0 \in \mathbb{C}$. Let $D_\varepsilon \subset \mathbb{C}$ denote the disc of radius ε around 0 as above.

Proposition 4.4 (e.g. [Dim04, §5.3])**.** *The following categories are naturally equivalent,*

(i) $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$
(ii) The category of holonomic $\mathscr{D}_{D_\varepsilon}$-modules M such that $M|_{\mathring{D}_\varepsilon}$ is an integrable connection that is regular along $0 \in D_\varepsilon$ (i.e. $\mathrm{Conn}^{\mathrm{reg}}(D_\varepsilon; 0)$).
(iii) The category of holonomic $\mathscr{D}_{\mathbb{P}^1}$-modules M (either algebraic or analytic) such that $M|_{\mathbb{P}^1\setminus\{0,\infty\}}$ is an integrable connection regular at 0 and ∞.
(iv) The category of holonomic A_t-modules M, regular at 0 and at ∞ such that $M|_{\mathbb{A}^1\setminus 0}$ is an integrable connection. Here $A_t = \mathbb{C}[t]\langle\partial_t\rangle$ is the Weyl algebra of polynomial linear differential operators.

The equivalences between (iii) and (i), and (ii) and (i), are induced by taking germs; the equivalence between (iii) and (ii) is induced by restriction. The equivalence between (i) and (iv) is induced either by restriction along the ring inclusion $A_t \hookrightarrow \mathscr{D}_{\mathbb{C},0}$, or base change $\mathscr{D}_{\mathbb{C},0} \otimes_{A_t} -$, which adjoint equivalences.

4.2. **Riemann-Hilbert for $\mathscr{D}_{\mathbb{C},0}$-modules.**

Combining Proposition 4.4 and Deligne's correspondence we obtain an equivalence,

$$\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}} \xrightarrow{\simeq} \mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0). \tag{4.5}$$

Remark 4.6. Let $(V, T) \in \mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0)$ be a finite-dimensional $\mathbb{C}$-linear representation of $\pi_1(\mathbb{C} \setminus 0)$. Under the equivalence (4.5), (V, T) is sent to the $\mathscr{D}_{\mathbb{C},0}$-module $\mathbb{C}\{t\} \otimes_{\mathbb{C}} V$, with the action of ∂_t given by $\nabla_{\partial_t}(f \otimes v) = df(t) \otimes v + \frac{f}{t} \otimes Mv$, where $M : V \to V$ is such that $\exp(-2\pi i M) = T$. In order words, under Deligne's correspondence, (V, T) corresponds to the $\mathbb{C}\{t\}$-module $\mathbb{C}\{t\} \otimes_{\mathbb{C}} V$ equipped with connection $\nabla = d + \frac{M}{t} dt$. It is easy to see that the monodromy of this connection is T (see, e.g. [Sab07, §4.b]).

[5] See e.g. [HTT08, Proposition 1.2.9] and [HTT08, pg. 38-40] for the definition of tensor products of $\mathscr{D}$-modules.

Definition 4.7. We denote by $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \subset \mathcal{D}^b(\mathscr{D}_{\mathbb{C},0})$ the full subcategory of $\mathcal{D}^b(\mathscr{D}_{\mathbb{C},0})$ on objects whose coholomogies are regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules.

The following lemma is a direct consequence of Proposition 4.4 and Lemma 3.15. It can be thought of a local version on $\mathbb{C}$ of Beilinson's famous [Bei87, Theorem 1.3] showing that the derived category of regular holonomic modules is equivalent to the category of complexes with regular holonomic coholomogies.

Lemma 4.8. *There is a t-exact equivalence,* $\mathcal{D}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \xrightarrow{\simeq} \mathcal{D}_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$.

Combined with Lemma 4.8, the derived Deligne's correspondence correspondence (3.17) induces the following equivalence, which we shall call Deligne's correspondence for $\mathscr{D}_{\mathbb{C},0}$-modules.

Lemma 4.9 (Deligne's correspondence for $\mathscr{D}_{\mathbb{C},0}$-modules)**.** *There is a t-exact equivalence of stable ∞-categories,*

$$\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \xrightarrow{\simeq} \mathrm{Fun}(S^1, \mathrm{Perf}), \tag{4.10}$$

where the t-structure on the right-hand category is the one used in the statement of Corollary 3.16.

Notation 4.11. Later on we will be interested in the inverse functor to the equivalence given in Lemma 4.9, so we will denote it by

$$\mathcal{RH} : \mathrm{Fun}(S^1, \mathrm{Perf}) \to \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}).$$

We have chosen this notation to allude to the explicit inverse to the de Rham functor constructed by Kashiwara in his original proof of the Riemann-Hilbert correspondence, which is denoted by "RH" (see [Kas84]).

4.3. **Fourier transform.** The category of $\mathscr{D}_{\mathbb{C},0}$-modules has a Fourier transform coming from its description as holonomic modules over the Weyl algebra in Proposition 4.4.

Definition 4.12 (e.g. [Gar04])**.** The Fourier anti-involution is the map of $\mathbb{C}$-algebras $A_t \to A_\eta$ given by the assignment on generators,

$$t \mapsto -\partial_\eta, \qquad\qquad \partial_t \mapsto \eta.$$

The *Fourier transform* of an object $M \in \mathrm{Mod}_{A_t}$ is defined as the object,

$$\mathscr{F}(M) := A_\eta \otimes_{A_t} M$$

in Mod_{A_η}, where A_η is regarded as a right A_t-module via the Fourier anti-involution.

Remark 4.13. We will regard the Fourier transform as an involution of the category of A_t-module spectra, $\mathscr{F} : \mathrm{Mod}_{A_t} \to \mathrm{Mod}_{A_t}$.

By [HTT08, Proposition 3.2.7], a coherent, ordinary A_t-module is holonomic if and only if its Fourier transform is holonomic, as well. On the other hand, the Fourier transform of a regular A_t-module is not necessarily regular. Fortunately, the Fourier transforms of the highly structured sort of A_t-modules that correspond to regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules *are* regular, by the following lemma.

Lemma 4.14. *Suppose that M is holonomic A_t-module that is regular at 0 and ∞ and such that $M|_{\mathbb{A}^1\setminus 0}$ is an integrable connection. Then $\mathscr{F}(M)$ is also a holonomic A_t-module that is regular at 0 and ∞ and such that $\mathscr{F}(M)|_{\mathbb{A}^1\setminus 0}$ is an integrable connection.*

Proof. As mentioned, $\mathscr{F}(M)$ is holonomic if and only if M is. It remains to show that $\mathscr{F}(M)$ is regular at 0 and ∞.

By a theorem of Brylinski ([Bry86]), $\mathscr{F}(M)$ is regular if M is a *monodromic*[6] A_t-module.[7] On the other hand, by [Dim04, Proposition 5.3.9],[8] an A_t-module M is monodromic if and only if it is regular holonomic and its restriction to $\mathbb{A}^1 \setminus 0$ is an integrable connection. Thus, under our hypotheses, $\mathscr{F}(M)$ is regular holonomic.

We note that $\mathscr{F}(M)$ is also monodromic by [IT20, Theorem 1.2], so it follows from another application of [Dim04, Proposition 5.3.9] that $\mathscr{F}(M)$ is an integrable connection on $\mathbb{A}^1 \setminus 0$. It follows that $\mathscr{F}(M)$ has regular singularities contained in the complement of $\mathbb{A}^1 \setminus 0 \subset \mathbb{P}^1$, at 0 and ∞. □

It follows from Lemma 4.14 that the Fourier transform restricts to an involution,[9]

$$\mathscr{F} : \mathcal{D}_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathcal{D}_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}).$$

4.4. **Vanishing cycles.** We recall the definition of of vanishing cycles for regular holonomic[10] $\mathscr{D}_{\mathbb{C}}$-modules. Let $M \in \mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C}}}$, and consider the Kashiwara-Malgrange filtration with respect to the hypersurface $\{0\} \subset \mathbb{C}$. The vanishing cycles of M, denoted $\varphi_{\mathscr{D}}(M)$ is defined to be the -1 associated graded piece of this filtration. The result is a local system on the punctured plane: a vector space equipped with a monodromy automorphism. It is equivalent to $\varphi_t \operatorname{DR}_{\mathbb{C}}(\widetilde{M})$, where $\widetilde{M}$ is a $\mathscr{D}_{\mathbb{C}}$-module representative for M and φ_t is the usual vanishing cycles functor[11] on perverse sheaves. The vanishing cycles functor on $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$ induces the vanishing cycles functor on the derived category, $\varphi_{\mathscr{D}} : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \operatorname{Fun}(S^1, \operatorname{Perf})$, which, by abuse of notation, we also denote by $\varphi_{\mathscr{D}}$.

5. The V-filtration on $\mathscr{D}_{\mathbb{C},0}$-modules

The standard V-filtration on $\mathscr{D}_{\mathbb{C}}$ along the hypersurface $\{0\} \subset \mathbb{C}$ induces a filtration on $\mathscr{D}_{\mathbb{C},0}$, also called the standard V-filtration, which we we denote by $V^{\bullet}\mathscr{D}_{\mathbb{C},0}$. Concretely, the filtration is given by

$$\begin{aligned} V^0\mathscr{D}_{\mathbb{C},0} &= \mathbb{C}\{t\}\langle t\partial_t \rangle, \\ V^k\mathscr{D}_{\mathbb{C},0} &= \begin{cases} t^k V^0 \mathscr{D}_{\mathbb{C},0} & \text{if } k \geq 0, \\ \sum_{j=0}^{-k} \partial_t{}^j V^0\mathscr{D}_{\mathbb{C},0} & \text{if } k \leq 0. \end{cases} \end{aligned}$$

There is a corresponding notion of V-filtration on $\mathscr{D}_{\mathbb{C},0}$-modules as well, which differs slightly from what one might expect by comparison with Definition 3.5.

Definition 5.1. Given a $\mathscr{D}_{\mathbb{C},0}$-module M, a *Kashiwara-Malgrange* or *V-filtration* on M is any exhaustive, decreasing $\mathbb{Z}$-filtration on M that satisfies the following properties:

(i) Each V^kM is a coherent module over $\mathbb{C}\{t\}$.
(ii) $V^kM = t^kV^0M$ for $k \geq 0$.
(iii) $V^kM = \sum_{j=0}^{-k-1} \partial_t{}^j V^{-1}M$ for $k > 0$.
(iv) There exists a polynomial $b(s) \in \mathbb{C}[s]$ with roots in $\mathbb{Q} \cap [0, 1)$ such that $b(t\partial_t - k)$ vanishes identically on $\operatorname{gr}^k_V M$ for each k.

[6] See [Dim04, Definition 5.3.8]. Alternatively, a A_t-module is monodromic if its image under Riemann-Hilbert is locally constant along $\mathbb{C}^*$-orbits.

[7] In fact the converse is true. See [IT20, Theorem 1.2].

[8] Also [DS93].

[9] See also the definition in the Introduction of [Sab08] of what the author calls the local formal Laplace transform.

[10] The definition we give is valid more generally for all $\mathscr{D}_{\mathbb{C}}$-modules that are specializable along $0 \in \mathbb{C}$, but we will not need this.

[11] See [Dim04, §4.2] for the definition. We use the convention that φ_t preserves perversity.

Just as for the V-filtration on $\mathscr{D}_{\mathbb{C}}$-modules, if a V-filtration on $M \in \mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$ exists, then it is unique, so we speak of *the* Kashiwara-Malgrange filtration. Crucially, the V-filtration on M is compatible with the standard V-filtration on $\mathscr{D}_{\mathbb{C},0}$, making $V^\bullet M$ a module over $V^\bullet \mathscr{D}_{\mathbb{C},0}$.

If $M \in \mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$, then the V-filtration on M exists, so we obtain an additive functor,

$$V^\bullet : \mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}} \to \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Ab})).$$

The additive category $\mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Ab}))$ is in fact quasi-abelian, so it makes sense the state the following lemma.

Lemma 5.2. *The above functor $V^\bullet$ is fully faithful, exact, and preserves filtered colimits.*

Proof. We recall that any morphism $f : M \to N$ of $\mathscr{D}_{\mathbb{C},0}$-modules induces a strict map of filtered $V^\bullet \mathscr{D}_{\mathbb{C},0}$-modules, $V^\bullet M \to V^\bullet N$, if the Kashiwara-Malgrange filtration exists on both. It follows that $V^\bullet$ is fully faithful. Because $V^\bullet$ takes morphisms of $\mathscr{D}_{\mathbb{C},0}$-modules to strict morphisms, it suffices to show that it preserves injections and surjections in order to show it is exact. But this is clear from the fact that, given $f : M \to N$, $f(V^i M) = f(M) \subset V^i N$. Finally, we show that $V^\bullet$ preserves filtered—in fact, all—colimits. On the one hand, it follows from the definition of the V-filtration that $V^\bullet$ preserves coproducts; indeed, one can check that setting $V^k(M \oplus N) := V^k M \oplus V^k N$ defines a filtration which satisfies all the properties of the Kashiwara-Malgrange filtration. On the other hand, a functor preserves all colimits if and only if it preserves coequalizers and coproducts. Since $V^\bullet$ is additive, it preserves coequalizers if and only if it preserves cokernels, which was shown above. $\square$

Taking $R = \mathscr{D}_{\mathbb{C},0}$ in Proposition C.15, we obtain a fully faithful functor $i_\heartsuit : \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Ab})) \hookrightarrow \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ which takes exact sequences to fiber sequences and preserves filtered colimits. Since $\mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ admits filtered (in fact, all) colimits, the composition $i_\heartsuit \circ V^\bullet$ extends to a functor,

$$\mathrm{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})),$$

that preserves filtered colimits, which by abuse of notation, we denote by $V^\bullet$.

If C is an ordinary 1-category, then $\mathrm{Ind}(C)$ is also a 1-category, and coincides with the standard ind-category of C. Moreover, it is well-known that if $\mathcal{A}$ is an abelian category, $\mathrm{Ind}(\mathcal{A})$ is a *Grothendieck* abelian category. In particular, $\mathrm{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}})$ is a Grothendieck abelian category.

Lemma 5.3. *The functor $V^\bullet : \mathrm{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ takes exact sequences to fiber sequences.*

Proof. This is a direct consequence of the fact that $V^\bullet$ is induced by an exact functor out of $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$. $\square$

We recall a proposition of Lurie that characterizes that will allows us to extend $V^\bullet$ to a functor on the unseparated derived ∞-category of $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$.

Proposition 5.4. *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathscr{C}$ be a stable presentable ∞-category. Then then there is an equivalence between colimit-preserving functors $\check{\mathcal{D}}(\mathcal{A}) \to \mathscr{C}$, on the one hand, and functors $\mathcal{A} \to \mathscr{C}$ that preserve filtered colimits and send exact sequences to fiber sequences, on the other.*

Proof. See [Lur17]. $\square$

By the above lemma and proposition, we obtain a colimit-preserving functor,

$$\check{\mathsf{V}}^\bullet : \check{\mathcal{D}}(\mathrm{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}})) \to \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})).$$

Restricting $\check{\mathsf{V}}^\bullet$ to the full subcategory of bounded complexes, $\check{\mathcal{D}}^b(\mathrm{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}))$, we obtain another colimit-preserving functor, which we also denote by $\check{\mathsf{V}}^\bullet$.

Lemma 5.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there is a t-exact equivalence $\check{\mathcal{D}}^+(\mathcal{A}) \xrightarrow{\simeq} \mathcal{D}^+(\mathcal{A})$.*

Proof. The following argument is due to Sam Raskin. More generally, let $\mathscr{C}$ be a presentable stable ∞-category with a t-structure compatible with filtered colimits, and let $\mathcal{A}$ denote its heart. In order to check that the induced functor $\mathcal{D}^+(\mathcal{A}) \to \mathscr{C}^+$ is an equivalence it suffices to check that for every pair of objects $F, I \in A$ with I injective, $H^i \operatorname{Hom}_{\mathscr{C}}(F, I) = 0$ for all $i > 0$, where $\mathcal{A}$ is considered as a subcategory of $\mathscr{C}$. Since this is obviously true taking $\mathscr{C}$ to be either $\check{\mathcal{D}}(\mathcal{A})$ or $\mathcal{D}(\mathcal{A})$ itself, we are done. $\square$

Using Lemma 5.5, we view $\check{\mathsf{V}}^\bullet$ as a functor on $\mathcal{D}^b(\operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}))$. Finally, composition with the functor $\mathcal{D}^b(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \mathcal{D}^b(\operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}))$, induced by the exact embedding $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}} \hookrightarrow \operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}})$, gives the functor

$$\mathsf{V}^\bullet : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \simeq \mathcal{D}^b(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \operatorname{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})), \tag{5.6}$$

which we call the *derived Kashiwara-Malgrange filtration*. Given an object $M \in \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$, we will refer to $\mathsf{V}^\bullet M$ as the Kashiwara-Malgrange filtration on M.

Definition 5.7. A *filtered derived $\mathscr{D}_{\mathbb{C},0}$-module* is an object of $\operatorname{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$.

Remark 5.8. Note that the structure of a left module over $V^\bullet \mathscr{D}_{\mathbb{C},0}$ on an object $M^\bullet \in \mathsf{Fil}(\mathrm{Sp})$ includes the data of a factorization of the multiplication map as follows,

$$\begin{tikzcd} & M^{i+j} \arrow[dr] & \\ V^i \mathscr{D}_{\mathbb{C},0} \times M^j \arrow[ur, dashed] \arrow[rr, "\text{mult}"] & & M^\bullet \end{tikzcd}$$

We let $\mathsf{V}^k : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \operatorname{Mod}_{V^0 \mathscr{D}_{\mathbb{C},0}}$ denote the functor obtained obtained similarly from the universal property of the unseparated derived ∞-category using the composition of $(i_\heartsuit \circ V^\bullet)$: $\operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \operatorname{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ with with the evaluation functor $\operatorname{ev}_k : \operatorname{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})) \to \operatorname{Mod}_{V^0 \mathscr{D}_{\mathbb{C},0}}$ (see Remark C.16), noting that the latter functor also commutes with filtered colimits and takes fiber sequences to fiber sequences. By definition, V^k is a t-exact functor with respect to the canonical t-structures on both the source and the target.

Lemma 5.9. *The functors V^k and $\operatorname{ev}_k \circ \mathsf{V}^\bullet$ are naturally equivalent.*

Proof. Note that the functors,

$$\check{\mathsf{V}}^k, \operatorname{ev}_k \circ \check{\mathsf{V}}^\bullet : \check{\mathcal{D}}(\operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}})) \to \operatorname{Mod}_{V^0 \mathscr{D}_{\mathbb{C},0}},$$

both preserve colimits, and both restrict to the functor, $\operatorname{ev}_k \circ V^\bullet : \operatorname{Ind}(\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}) \to \operatorname{Mod}_{V^0 \mathscr{D}_{\mathbb{C},0}}$, which preserves filtered colimits and takes exact sequences to fiber sequences. By the universal property of $\check{\mathcal{D}}$, it follows that $\check{\mathsf{V}}^k \simeq \operatorname{ev}_k \circ \check{\mathsf{V}}^\bullet$. Restriction the full subcategory $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$ yields the result. $\square$

In the same way as we obtain V^k, we may obtain a t-exact functor $\operatorname{gr}^k_{\mathsf{V}} : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathrm{Vect}_{\mathbb{C}}$.

Lemma 5.10. *The functors $\operatorname{gr}^k_{\mathsf{V}}$ and $\operatorname{pr}_k \circ \operatorname{gr}^\bullet_{\mathsf{V}}$ are naturally equivalent.*

Proof. Proven in the same manner as Lemma 5.9. $\square$

We show in the following proposition that the derived V-filtration produces a filtered derived $\mathscr{D}_{\mathbb{C},0}$-module which satisfies a list of properties analogous to those satisfied by the V-filtration on the abelian category of regular holonomic modules.

Proposition 5.11. *Suppose that $M \in \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$. Then $\mathsf{V}^\bullet M$ satisfies the following properties:*

(i) *$\mathsf{V}^k M$ is a perfect $\mathbb{C}\{t\}$-module.*
(ii) *The map, $t\cdot : \mathsf{V}^k M \to \mathsf{V}^{k+1} M$ is an equivalence for each $k \geq 0$.*
(iii) *The induced map $\sum_{j=0}^{-k-1} \partial_t{}^j \mathsf{V}^{-1} M \to \mathsf{V}^k M$ is an equivalence for each $k < 0$.*
(iv) *For each $k \in \mathbb{Z}$, $\operatorname{gr}^k_{\mathsf{V}} M$ is a perfect $\mathbb{C}$-module, and there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ whose roots have real parts in $[0,1)$ and a factorization,*

$$\begin{tikzcd} & \mathsf{V}^{k+1} M \arrow[dr, "i_{k+1}"] & \\ \mathsf{V}^k M \arrow[ur, dashed] \arrow[rr, "b(t\partial_t - k)"] & & M. \end{tikzcd}$$

Proof. We show that $\mathsf{V}^\bullet$ satisfies the enumerated properties one-by-one.

(i) Since V^k is t-exact by the discussion preceding the proposition, we have an natural equivalence $\pi_i(\mathsf{V}^k M) \simeq V^k(\pi_i M)$, for $M \in \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$, where V^k denote the kth piece of the ordinary Kashiwara-Malgrange filtration on the discrete $\mathscr{D}_{\mathbb{C},0}$-module $\pi_i M$. It follows from the properties of the ordinary V-filtration that $\pi_i(\mathsf{V}^k M)$ is a finite type $\mathbb{C}\{t\}$-module. Moreover, t-exactness also implies that $\mathsf{V}^k M \in \operatorname{Mod}^b_{V^0\mathscr{D}_{\mathbb{C},0}}$. Therefore, noting that $\mathbb{C}\{t\}$ is a regular ring, we find that $\mathsf{V}^k M \in \operatorname{Mod}^{\text{perf}}_{\mathbb{C}\{t\}}$ under the forgetful functor $\operatorname{Mod}^b_{V^0\mathscr{D}_{\mathbb{C},0}} \to \operatorname{Mod}^b_{\mathbb{C}\{t\}}$.

(ii) We show that, each $k \geq 0$, the morphism $t\cdot : \mathsf{V}^k M \to \mathsf{V}^{k+1} M$ induced isomorphisms of $V^0\mathscr{D}_{\mathbb{C},0}$-modules on all homotopy groups. Let $i \in \mathbb{Z}$, and consider the induced map, $\pi_i(\mathsf{V}^k M) \to \pi_i(\mathsf{V}_{k+1} M)$. Since V^n is t-exact for any $n \in \mathbb{Z}$, this is the map $t\cdot : V^k(\pi_i M) \to V^{k+1}(\pi_i M)$, which is an isomorphism by the properties of the ordinary Kashiwara-Malgrange filtration.

(iii) This is proven in the exact same manner as the item above, except we note additionally that π_i commutes with finite coproducts.

(iv) We show that $\operatorname{gr}^k_{\mathsf{V}} M$ is a perfect complex of $\mathbb{C}$-vector spaces in the same manner as item (i), using that $\operatorname{gr}^k_{\mathsf{V}}$ is exact. Since $\operatorname{gr}^k_{\mathsf{V}} M$ is perfect, it has finitely many nonzero homotopy groups. Let $I \subset \mathbb{Z}$ be the collection of indices i such that $\pi_i(\operatorname{gr}^k_{\mathsf{V}} M) \not\simeq 0$. By the properties of the ordinary Kashiwara-Malgrange filtration, for each $i \in I$ there exists a nonzero polynomial $b_i(s)$ with roots in $\mathbb{Q} \cap [0,1)$ such that $b_i(t\partial_t - k)$ vanishes identically on $\operatorname{gr}^k_V(\pi_i M)$. Let $b(s) := \prod_{i \in I} b_i(s)$. Since $\operatorname{gr}^k_{\mathsf{V}}$ is t-exact, $\operatorname{gr}^k_V(\pi_i M) \simeq \pi_i(\operatorname{gr}^k_{\mathsf{V}} M)$, so we obtain that the morphism,

$$b(s)\cdot : \operatorname{gr}^k_{\mathsf{V}} M \to \operatorname{gr}^k_{\mathsf{V}} M,$$

induces the zero map on all homotopy groups. Thus, the action of $b(s)$ on $\operatorname{gr}^k_{\mathsf{V}} M$ is nullhomotopic. The choice of nullhomotopy furnishes a factorization as desired.

□

Moreover, the properties enumerated in Proposition 5.11 uniquely characterize the filtered derived $\mathscr{D}_{\mathbb{C},0}$-module $\mathsf{V}^\bullet M$.

Proposition 5.12. *Suppose that $M \in \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$, and suppose that $\mathsf{U}^\bullet M$ and $\widetilde{\mathsf{U}}^\bullet M$ are two objects in $\operatorname{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ whose underlying derived $\mathscr{D}_{\mathbb{C},0}$-module is M. Suppose additionally that each of $\mathsf{U}^\bullet M$ and $\widetilde{\mathsf{U}}^\bullet M$ satisfies the enumerated properties in Proposition 5.11. Then there is an equivalence $\mathsf{U}^\bullet M \xrightarrow{\simeq} \widetilde{\mathsf{U}}^\bullet M$.*

Proof. We begin by producing, for each k, a factorization

$$\begin{tikzcd} & \widetilde{\mathsf{U}}^j M \arrow[dr, "\imath_j"] & \\ \mathsf{U}^k M \arrow[ur, dashed] \arrow[rr, "i_k"] & & M, \end{tikzcd} \tag{5.13}$$

for some $j_k \leq k$. By assumption, $\mathsf{U}^k M$ is a perfect $\mathbb{C}\{t\}$-module. By [HA, Proposition 7.2.4.11], any object $M \in \mathrm{Mod}^{\mathrm{perf}}_{\mathbb{C}\{t\}}$ can be obtained by as a colimit over a finite diagram $P : I \to \mathrm{Mod}_{\mathbb{C}\{t\}}$ such that each P_i is a finitely generated free $\mathbb{C}\{t\}$-module. Since we have a map $\mathsf{U}^k M \to M$, we also obtain maps $P_i \to M$ that exhibit M as a cocone over P. Since P_i is finitely generated free, say of rank n_i, such a map is determined by an n_i-tuple $(x_1, \dots, x_{n_i})$ of maps $x_\ell : * \to \pi_0(M)$. Since $M \simeq \varinjlim_j \widetilde{\mathsf{U}}^j M$, the image of x_ℓ lies in the image of $\pi_0(\widetilde{\mathsf{U}}^{j_\ell} M) \to \pi_0(M)$ for some j_ℓ; hence x_ℓ factors through $\pi_0(\widetilde{\mathsf{U}}^{j_\ell} M) \to \pi_0(M)$. Set $j_k := \max_\ell j_\ell$. Then by the universal property of free modules, $P_i \to M$ factors through $\widetilde{\mathsf{U}}^{j_k} M$. It is easy to check that this exhibits $\widetilde{\mathsf{U}}^{j_k} M$ as a cocone over P, so by the universal property, we obtain a map $\mathsf{U}^k M \to \widetilde{\mathsf{U}}^{j_k} M$ that factorizes $\mathsf{U}^k M \to M$.

We proceed to show that there exists an $N \geq 0$ such that we can take j_k to be $k - N$ for *any* k. In fact, we may take $N = \max(-j_0, -(1 + j_1))$. Indeed, if $k \geq 0$, $\mathsf{U}^k M \to M$ factors through $\widetilde{\mathsf{U}}^{k-j_0} M \to M$, and, if $k \leq 0$, $\mathsf{U}^k M \to M$ factors through $\widetilde{\mathsf{U}}^{k-j_1} M \to M$, using the equivalences furnished by properties (ii) and (iii).

Continuing, we use the existence of the b-functions and factorizations guaranteed by property (iv). Indeed, let $b(s)$ and $\widetilde{b}(s)$ be the functions, and

$$\begin{tikzcd} & \mathsf{U}^{k+1} M \arrow[dr, "i_{k+1}"] & \\ \mathsf{U}^k \mathcal{F} \arrow[ur, dashed] \arrow[rr, "b(t\partial_t - k)"'] & & M \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\mathsf{U}}^{k+1} M \arrow[dr, "\imath_{k+1}"] & \\ \widetilde{\mathsf{U}}^k M \arrow[ur, dashed] \arrow[rr, "\widetilde{b}(t\partial_t - k)"'] & & M \end{tikzcd}$$

the factorizations, for $\mathsf{U}^\bullet$ and $\widetilde{\mathsf{U}}^\bullet$, respectively, guaranteed by property (iv). Combining the left-hand diagram above with (5.13) we obtain the commutative diagram,

$$\begin{tikzcd} & \mathsf{U}^{k+1} M \arrow[dr, "i_{k+1}"] \arrow[rr, dashed, "(2)"] & & \widetilde{\mathsf{U}}^{k-N+1} M \arrow[dl, "\imath_{k-N+1}"] \\ \mathsf{U}^k M \arrow[ur, dashed, "(1)"] \arrow[rr, "b(t\partial_t - k)\cdot"'] & & M. & \end{tikzcd}$$

For similar reasons, we also obtain the commutative diagram,

$$\begin{tikzcd} & \widetilde{\mathsf{U}}^{k-N} M \arrow[dr, "\widetilde{b}(t\partial_t - k + N)\cdot"] \arrow[rr, dashed, "(2')"] & & \widetilde{\mathsf{U}}^{k-N+1} M \arrow[dl, "\imath_{k-N+1}"] \\ \mathsf{U}^k M \arrow[ur, dashed, "(1')"] \arrow[rr, "\widetilde{b}(t\partial_t - k + N)\cdot"'] & & M. & \end{tikzcd}$$

Since $b(s-k)$ and $\widetilde{b}(s-k+N)$ have no common root, there exist $p(s), q(s) \in \mathbb{C}[s]$ such that $p(s)b(s-k) + q(s)\widetilde{b}(s-k+N) = 1$ by Bézout's lemma. Hence, using the diagrams above, we obtain a chain of homotopies,

$$\begin{aligned} i_k &= p(t\partial_t)b(t\partial_t - k) + q(t\partial_t)\widetilde{b}(t\partial_t - k + N)\cdot \\ &\simeq p(t\partial_t)\cdot \circ \imath_{k-N+1} \circ (2) \circ (1) + q(t\partial_t)\cdot \circ \imath_{k-N+1} \circ (2') \circ (1') \\ &\simeq \imath_{k-N+1} \circ \left(p(t\partial_t)\cdot \circ (2) \circ (1) + q(t\partial_t)\cdot \circ (2') \circ (1')\right), \end{aligned}$$

whose composition gives a factorization $i \simeq \imath \circ F$, where we have let $F := p(t\partial_t)\cdot \circ (2) \circ (1) + q(t\partial_t)\cdot \circ (2') \circ (1')$. Repeating this process, we obtain a map $\mathsf{U}^k M \to \widetilde{\mathsf{U}}^k M$ and a factorization,

$$\begin{tikzcd} & \widetilde{\mathsf{U}}^k M \arrow[dr, "\imath_k"] & \\ \mathsf{U}^k M \arrow[ur, dashed] \arrow[rr, "i_k"'] & & M. \end{tikzcd}$$

Tracing through the constructions, one sees that the maps obtained in this way are natural in the index k, so it obtains a map of filtered objects, $\mathsf{U}^\bullet M \to \widetilde{\mathsf{U}}^\bullet M$. In order to show that this map is an equivalence, we shows that it induces an equivalence on associated graded objects. Let $n \in \mathbb{Z}$. By the exactness of gr^n, the induced map on homotopy groups, for each $i \in \mathbb{Z}$, $\pi_i \, \mathrm{gr}^n_{\mathsf{U}} M \to \pi_i \, \mathrm{gr}^n_{\widetilde{\mathsf{U}}} M$ equivalently gives a map,

$$\mathrm{gr}^n_U(\pi_i M) \to \mathrm{gr}^n_{\widetilde{U}}(\pi_i M), \tag{5.14}$$

where $U^\bullet$ and $\widetilde{U}^\bullet$ are the filtrations on $\pi_i M$ induced by $\mathsf{U}^\bullet$ and $\widetilde{\mathsf{U}}^\bullet$. It is easy to see that both $U^\bullet$ and $\widetilde{U}^\bullet$ satisfy the enumerated properties of the usual V-filtration in Definition 5.1. It is also clear that the application of $\pi_i(\mathrm{gr}^n(-))$ to the objects and maps in each step of the above proof yields an argument identical to Kashiwara's original proof of the uniqueness of the V-filtration (c.f. [Kas83, Theorem 1.1]). It follows that (5.14) is an equivalence, which completes the proof. □

6. $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules

Heuristically, the sheaf of formal microdifferential operators $\widehat{\mathscr{E}}_X$ on a complex manifold X is a sheaf on T^*X containing differential operators on X, such that if the principal symbol of a differential operator P is invertible on an open set $U \subset T^*X$, then P itself is invertible in $\widehat{\mathscr{E}}_X$. By construction there is a canonical map of sheaves,

$$\pi^*\mathscr{D}_X \to \widehat{\mathscr{E}}_X,$$

where $\pi : T^*X \to X$ denotes the canonical projection. When $X = \mathbb{C}$, taking the stalk at $(0;1) \in T^*\mathbb{C}$ induces a canonical inclusion of rings,

$$\mathscr{D}_{\mathbb{C},0} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}, \tag{6.1}$$

where $\widehat{\mathscr{E}}_{\mathbb{C},0}$ denotes the ring of germs at $(0;1)$ of formal microdifferential operators on $\mathbb{C}$. The inclusion $\mathscr{D}_{C,0} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}$ makes $\widehat{\mathscr{E}}_{\mathbb{C},0}$ into both a left and a right $\mathscr{D}_{\mathbb{C},0}$-module.

Definition 6.2. We let $\widehat{(-)} : \mathcal{D}_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathcal{D}_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ denote the functor given by $N \mapsto \widehat{\mathscr{E}}_{\mathbb{C},0} \otimes_{\mathscr{D}_{\mathbb{C},0}} N$. We will call $\widehat{N}$ the *formal microlocalization* of N.

Remark 6.3. The inclusion $\mathscr{D}_{\mathbb{C},0} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}$ is a flat map of rings.

Concretely, $\widehat{\mathscr{E}}_{\mathbb{C},0}$ is given as a $\mathbb{C}$-vector space by the subspace of $\mathbb{C}\{t\}(\!(\xi)\!)$ spanned by elements of form $\sum_{k \geq k_0} a_k(t)\xi^k$, where there is some fixed $r > 0$ such that the power series $a_k(t)$ all have radius of convergence greater than or equal to r. That is, elements of $\widehat{\mathscr{E}}_{\mathbb{C},0}$ are Laurent series in the formal variable ξ whose coefficients all converge on some fixed disc of radius r, independent of k. The product structure on $\widehat{\mathscr{E}}_{\mathbb{C},0}$ is not the standard commutative one, but rather is noncommutative. It is written explicitly, for example, in [Sch85, §1.2]. Using this concrete description of $\widehat{\mathscr{E}}_{\mathbb{C},0}$, the canonical inclusion $\mathscr{D}_{\mathbb{C},0} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}$ is given by sending $\partial_t \mapsto \xi^{-1}$.

Remark 6.4. We recommend either [Sch85] or [Kas03] as introductions to the theory of microdifferential operators and microdifferential systems.

6.0.1. *Microdifferential operators with polynomial coefficients.* Elements of $\mathbb{C}\{t\}$ that are polynomials trivially converge on a disc of any positive radius, so elements of $\mathbb{C}\{t\}(\!(\xi)\!)$ whose coefficients are all polynomials in t form a subring of $\widehat{\mathscr{E}}_{\mathbb{C},0}$.

Definition 6.5. Let $\widehat{\mathscr{E}}^{\text{poly}}_{\mathbb{C},0}$ denote the subring of $\widehat{\mathscr{E}}_{\mathbb{C},0}$ spanned by elements of $\mathbb{C}\{t\}(\!(\xi)\!)$ whose coefficients are polynomials in t.

The inclusion of rings $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}$ induces a canonical restriction functor,

$$\text{res} : \text{Mod}_{\widehat{\mathscr{E}}_{\mathbb{C},0}} \to \text{Mod}_{\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}},$$

as well as a functor on the hearts, $\text{res} : \mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}} \to \mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$.

6.0.2. *Regular holonomic modules.* Just as for $\mathscr{D}_X$-modules, there is a notion of regular holonomic $\widehat{\mathscr{E}}_X$-module. We will not recall the definition, which can be found in [Kas03] as Definition 8.27.

Regular holonomic $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules are defined as those objects in $\mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$ which are germs of regular holonomic $\widehat{\mathscr{E}}_{\mathbb{C}}$-modules defined in a neighborhood of $(0;1) \in T^*\mathbb{C}$. We denote by $\mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$ the full abelian subcategory of $\mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$ spanned by such objects. Fortunately, regular holonomic $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules admit an alternative description as those which are formal microlocalizations of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules, by the following variant of a theorem of Kawai and Kashiwara.

Theorem 6.6 (c.f. [KK81, Theorem 5.1.1])**.** *The formal microlocalization induces an equivalence of abelian categories,*

$$\mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\mathscr{D}_{\mathbb{C},0}} \xrightarrow[\simeq]{\widehat{(-)}} \mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}},$$

whose inverse is its right adjoint of restriction of modules along the inclusion $\mathscr{D}_{\mathbb{C},0} \hookrightarrow \widehat{\mathscr{E}}_{\mathbb{C},0}$.

Proof sketch. The precise statement of [KK81, Theorem 5.1.1] implies an equivalence of categories between the abelian category of regular holonomic $\mathscr{E}_{\mathbb{C},0}$-modules with characteristic variety in generic position at $(0;1) \in T^*\mathbb{C}$ and the abelian category of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules also with characteristic variety in generic position at $(0;1)$. The proof of [KK81, Theorem 5.1.1] goes through verbatim for formal microdifferential operators by decorating every "$\mathscr{E}_{C,0}$" with a hat. The condition that the characteristic varieties be in generic position at $(0;1)$ can be eased by applying a quantum contact transformation. $\square$

Because of Kashiwara and Kawai's theorem, we are able to make the following our working definition of regular holonomic $\widehat{\mathscr{E}}_{\mathbb{C},0}$-module.

Definition 6.7. An $\widehat{\mathscr{E}}_{\mathbb{C},0}$-module $M \in \mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$ is *regular holonomic* if there exists an object $N \in \mathcal{A}_{\mathscr{D}_{\mathbb{C},0}}$ such that $M = \widehat{N}$. We define $\mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ to be the full subcategory of $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$ on objects whose cohomology modules are regular holonomic.

Remark 6.8. It follows from Lemma 4.8 that $\mathcal{D}^b(\mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}) \xrightarrow{\simeq} \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$.

We will cheat and define the notion of regular holonomic module over $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}$ as follows.

Definition 6.9. $\mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}}$ is defined to be the essential image of $\mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}$ under the functor res. The ∞-category $\mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}})$ is defined to be the full subcategory of $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}})$ spanned by objects whose cohomology modules are regular holonomic.

Definition 6.10. We denote by $\mu\mathcal{RH} : \text{Fun}(S^1, \text{Perf}) \to \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}})$ the functor obtained as the composition of the following chain of functors,

$$\text{Fun}(S^1, \text{Perf}) \xrightarrow{\mathcal{RH}} \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \xrightarrow{\mathscr{F}} \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \xrightarrow{\widehat{\mathscr{E}}_{\mathbb{C},0} \otimes -} \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0}). \xrightarrow{\text{res}} \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}).$$

Proposition 6.11. *The restriction of $\mu\mathcal{RH}$ to $\mathcal{L}\text{oc}(\mathbb{C} \setminus 0)$, viewed as a functor $\mathcal{L}\text{oc}(\mathbb{C} \setminus 0) \to \mathcal{R}\text{eg}\mathcal{H}\text{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}}$, is naturally isomorphic to the functor $\mu\,\text{RH}$ defined in Definition 8.12.*

Proof. Let $(V,T) \in \mathcal{L}oc(\mathbb{C}\setminus 0)$ be a pair of a finite dimensional vector space and an automorphism representing a local system on $\mathbb{C}\setminus 0$. By Remark 4.6, the image of (V,T) under Deligne's correspondence is the $\mathscr{D}_{\mathbb{C},0}$-module given by $\mathbb{C}\{t\}\otimes_{\mathbb{C}} V$ with the action of ∂_t given by $\partial_t\cdot(f\otimes v) := \partial_t f\otimes v + \frac{f}{t}\otimes\frac{\log(T)v}{-2\pi i}$. Let us denote this $\mathscr{D}_{\mathbb{C},0}$-module by $M_{V,T}$. The Fourier transform of $M_{V,T}$ is computed, essentially by definition, as follows. First, consider $M_{V,T}$ as a $\mathbb{C}[t]\langle\partial_t\rangle$-module by restricting along the ring inclusion $\mathbb{C}[t]\langle\partial_t\rangle \hookrightarrow \mathbb{C}\{t\}\langle\partial_t\rangle$, which we denote by $M^{\mathrm{alg}}_{V,T}$. Compute the Fourier transform of $M^{\mathrm{alg}}_{V,T}$ as an A_t-module to obtain an A_η-module, $\mathscr{F}(M^{\mathrm{alg}}_{V,T})$. Now $\mathscr{F}(M_{V,T}) = \mathbb{C}\{\eta\}\langle\partial_\eta\rangle\otimes_{\mathbb{C}[\eta]\langle\partial_\eta\rangle}\mathscr{F}(M^{\mathrm{alg}}_{V,T})$. The object $\mathscr{F}\circ\mathcal{RH}(M) \in \mathcal{A}_{\mathbb{C}\{\eta\}((\partial_\eta^{-1}))}$ is therefore equivalent to $\mathbb{C}\{\eta\}((\partial_\eta^{-1}))\otimes_{\mathbb{C}[\eta]\langle\partial_\eta\rangle}\mathscr{F}(M^{\mathrm{alg}}_{V,T})$. Unraveling the definitions, it is clear that this the restriction of the module structure of the latter object to the subring $\mathbb{C}[\eta]((\partial_\eta^{-1})) \subset \mathbb{C}\{\eta\}((\partial_\eta^{-1}))$ is precisely the $\mathbb{C}(\!(u)\!)$-module $V(\!(u)\!)$ with connection $\nabla = d + \frac{\log(T)}{-2\pi i u}du$, once we set $u := \partial_\eta^{-1}$. □

Recall from Proposition 4.4 that the abelian category of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules is equivalent to a certain subcategory of the category of regular holonomic modules over the Weyl algebra. In other words, when a $\mathscr{D}_{\mathbb{C},0}$-module is regular holonomic it is actually determined by its module structure over the much smaller subring of polynomial differential operators inside $\mathscr{D}_{\mathbb{C},0}$. This observation motivates the following lemma.

Lemma 6.12. *The functor* res *induces an equivalence of categories,*

$$\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\widehat{\mathscr{E}}_{\mathbb{C},0}} \xrightarrow[\mathrm{res}]{\simeq} \mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}}.$$

Proof sketch. The lemma can be deduced by a GAGA argument similar to the one showing the equivalence between $\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}$ and the abelian category of holonomic A_t-modules M regular at 0 and at infinity such that $M|_{\mathbb{A}^1\setminus 0}$ of Proposition 4.4. Details of the argument showing the latter are found in [Mal91, §I.4]. □

Remark 6.13. In light of Lemma 6.12, we regard $\mu\mathcal{RH}$ as a functor from $\mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0})$ $\mathcal{D}^b_{\mathrm{reg.\,hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$. It follows that $\mu\mathcal{RH}$ may be defined alternatively as $\mu\mathcal{RH} := (\widehat{\mathscr{E}}_{\mathbb{C},0}\otimes -)\circ\mathscr{F}\circ\mathcal{RH}$.

6.1. **Vanishing cycles and microlocalization.**

Definition 6.14. Let $\mu : \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ denote the composition $\mu\mathcal{RH}\circ\varphi_{\mathscr{D}}$.

The following proposition can be viewed as describing the relationship between vanishing cycles and the formal microlocalization. It is the derived version of a result appearing in [Sab10].

Proposition 6.15. *There is a natural isomorphism,* $\widehat{(-)} \simeq \mu(-)$.

Proof. By [Sab10, Corollary 3.6], we have a natural isomorphism,

$$\widehat{(-)}|_{\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}} \simeq \mu|_{\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}}. \tag{6.16}$$

It follows that, since $\widehat{(-)}|_{\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}}$ is right exact, $\mu|_{\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}}$ is too. Thus, μ is the left derived functor of its restriction $\mu|_{\mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{\mathbb{C},0}}}$ by [HA, Theorem 1.3.3.2]. and the natural isomorphism (6.16) induces a natural equivalence of derived functors, $\widehat{(-)} = \mu$. □

6.2. **Monoidal structure on $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules.**

Although $\widehat{\mathscr{E}}_{\mathbb{C},0}$ is a noncommutative ring, there is a canonical symmetric monoidal structure on $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$ in the same way that there is a canonical symmetric monoidal structure on $\mathscr{D}_{\mathbb{C},0}$-modules.

Example 6.17. The tensor product of two $\mathbb{C}((u))$-modules with connection[12] in $\mathcal{A}_{\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}}$ is easy to write down explicitly. Let $\mathcal{M} = (V, \nabla)$ and $\mathcal{M}' = (V', \nabla')$ denote two such objects, which we view as $\mathbb{C}((u))$-modules V, V' equipped with u-connections ∇, ∇'. Then

$$\mathcal{M} \otimes \mathcal{M}' = (V \otimes_{\mathbb{C}((u))} V', \nabla + \nabla').$$

Notation 6.18. We denote the symmetric monoidal product on $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$ by $\otimes$.

We denote the restricted symmetric monoidal product on $\mathcal{D}^b(\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0})$ by $\otimes_{\mathbb{C}((u))}$ in order to emphasize that $\mathbb{C}((u))$ with the trivial $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-module structure is the monoidal unit for this structure and to more closely match the notation in [Sab10].

Remark 6.19. The subcategory $\mathcal{D}^b_{\mathrm{reg.\,hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ is stable under the monoidal product $\otimes$.

7. Sheaves of $\mathscr{D}_{\mathbb{C},0}$-modules

Constructible sheaves of $\mathscr{D}_{\mathbb{C},0}$-modules play a key role in Sabbah's proof of Kontsevich's conjecture. In this section, we collect some results about these objects in the ∞-categorical setting that will allow us to formulate our main theorem.

7.1. **Constructible sheaves of $\mathscr{D}_{\mathbb{C},0}$-modules.**

The subcategory $\mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0}) \subset \mathcal{D}(\mathscr{D}_{\mathbb{C},0})$ determines a coefficient pair in the sense of Definition A.17. As such, given a complex analytic space X, there is the associated category $\mathrm{Shv}_c(X; \mathcal{D}(\mathscr{D}_{\mathbb{C},0}))$ of constructible sheaves on X with respect to $(\mathcal{D}(\mathscr{D}_{\mathbb{C},0}), \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0}))$ as in Definition A.20. Likewise, $(\mathcal{D}(\widehat{\mathscr{E}}_{\mathbb{C},0}), \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0}))$ is a coefficient pair, and we might consider the category $\mathrm{Shv}_c(X; \mathcal{D}(\widehat{\mathscr{E}}_{\mathbb{C},0}))$.

Notation 7.1. We will use $\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$ and $\mathrm{Shv}_c(X; \widehat{\mathscr{E}}_{\mathbb{C},0})$, respectively, to denote $\mathrm{Shv}_c(X; \mathcal{D}(\mathscr{D}_{\mathbb{C},0}))$ and $\mathrm{Shv}_c(X; \mathcal{D}(\widehat{\mathscr{E}}_{\mathbb{C},0}))$, respectively.

Remark 7.2. The triangulated category $\pi_0(\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0}))$ is equivalent to $D^b_{\mathbb{C}\text{-c,rh}}(X; \mathscr{D}_{\mathbb{C},0})$ the bounded derived category of sheaves of $\mathscr{D}_{\mathbb{C},0}$-modules whose cohomology sheaves are constructible sheaves of regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules, defined in [Sab10, §3.a].

Example 7.3. Let $\mathcal{F} \in \mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$. Although the stalks of $\mathcal{F}$ are bounded with regular holonomic cohomology, for a general open U, it is not the case that $\mathcal{F}(U) \in \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0})$. Indeed, consider the complex analytic space, $\mathbb{C} \setminus \mathbb{Z}$. We have that $\mathbb{H}^1(\mathbb{C} \setminus \mathbb{Z}; \underline{\mathbb{Z}}) \simeq H^1(\mathbb{C} \setminus \mathbb{Z}; \mathbb{Z}) \simeq \mathbb{Z}^{\oplus_{i \in \mathbb{Z}}}$. Since $\mathscr{D}_{\mathbb{C},0}$ is torsion-free, it is flat over $\mathbb{Z}$, so this implies that $\mathbb{H}^1(\mathbb{C} \setminus \mathbb{Z}; \underline{\mathscr{D}_{\mathbb{C},0}}) \simeq \mathscr{D}_{\mathbb{C},0}^{\oplus_{i \in \mathbb{Z}}}$ Thus the derived global sections of the constant sheaf $\underline{\mathscr{D}_{\mathbb{C},0}}_{\mathbb{C}\setminus\mathbb{Z}}$ is not coherent,[13]so in particular not regular holonomic.

In light of Example 7.3, we see that the category $\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$ is not equivalent to $\mathrm{Shv}_c(X; \mathscr{D}^{b,\mathrm{reg.\,hol.}}_{\mathbb{C},0})$; i.e. the sections assigned to any open $U \subset X$ do not necessarily have regular holonomic cohomology. As such, a limit-preserving functor $F : \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathcal{D}^b_{\mathrm{reg.\,hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ does not naively define a functor,

$$\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0}) \to \mathrm{Shv}_c(X; \widehat{\mathscr{E}}_{\mathbb{C},0})$$

via composition of functors.

Observe, however, that, as a complex analytic space, X admits a basis of open subsets which are homotopy equivalent to finite CW complexes (finite homotopy type). For each such open U

[12]See Remark 8.3.

[13]Recall that a $\mathscr{D}_{\mathbb{C},0}$-module M is regular holonomic if and only if there exists a $\mathscr{D}_{\mathbb{C}}$-module $\widetilde{M}$ whose germ at 0 is M. If a $\mathscr{D}_{\mathbb{C}}$-module $\widetilde{M}$ is coherent, there exists a neighborhood U around each point $z \in \mathbb{C}$ such that the restriction $\widetilde{M}|_U$ admits a surjection $\mathscr{D}^n_{\mathbb{C}}|_U \to \widetilde{M}|_U$ for some natural number n. This clearly implies that the germ of $\widetilde{M}$ at any point must be finitely generated.

and each constructible sheaf $\mathcal{F} \in \mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$, $\mathcal{F}(U)$ in fact *is* an object of $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$. This observation suggests that, if X admits a basis of opens (including itself) that have finite homotopy type, we might instead consider working with sheaves defined on this basis.

In Appendix A, we have collected a number of results that allow us to obtain functors on $\mathrm{Shv}_c(X; \mathscr{D}_{\mathbb{C},0})$ from functors on $\mathcal{D}_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$ when X has finite homotopy type. In particular, Proposition A.26 shows that, by restriction, working with basis sheaves on X, as suggested above, is equivalent to working with constructible sheaves on the entirety of $\mathcal{U}(X)$. In the examples that follow, let X and $\mathcal{B}(X)$ be as in Convention A.23.

Example 7.4. By Corollary A.30, the functors $\mu, \widehat{(-)} : \mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0}) \to \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ induce functors,

$$\mu_{\mathrm{Shv}}, \widehat{(-)}_{\mathrm{Shv}} : \mathrm{Shv}^{f.s.}_c(X; \mathscr{D}_{\mathbb{C},0}) \to \mathrm{Shv}^{f.s.}_c(X; \widehat{\mathscr{E}}_{\mathbb{C},0}),$$

which are naturally isomorphic by Proposition 6.15 and Corollary A.32.

Example 7.5. The functor $\mu\mathcal{RH} : \mathrm{Fun}(S^1, \mathrm{Vect}) \to \mathcal{D}^b_{\text{reg. hol.}}(\widehat{\mathscr{E}}_{\mathbb{C},0})$ similarly defines a functor,

$$\mu\mathcal{RH}_{\mathrm{Shv}} : \mathrm{Shv}^{f.s.}_c(X; \mathrm{Fun}(S^1, \mathrm{Vect})) \to \mathrm{Shv}^{f.s.}_c(X; \widehat{\mathscr{E}}_{\mathbb{C},0}).$$

Note that, using the equivalence (A.33), we may also regard $\mu\mathcal{RH}_{\mathrm{Shv}}$ as a functor $\mathrm{Fun}(S^1, \mathrm{Shv}^{f.s.}_c(X; \mathbb{C})) \to \mathrm{Shv}^{f.s.}_c(X; \widehat{\mathscr{E}}_{\mathbb{C},0})$.

Example 7.6. Using Corollary A.29, the derived functor of the V-filtration on regular holonomic $\mathscr{D}_{\mathbb{C},0}$-modules, $\mathsf{V}^\bullet$, induces a functor,

$$\mathsf{V}^\bullet : \mathrm{Shv}^{f.s.}_c(X; \mathscr{D}_{\mathbb{C},0}) \to \mathrm{Shv}(X; \mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))).$$

Sheaves valued in $\mathrm{Mod}_{V^\bullet \mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ are equivalently left modules in $\mathsf{Fil}(\mathrm{Sp})$-valued sheaves over the constant sheaf, $\underline{V^\bullet \mathscr{D}_{\mathbb{C},0}}$. At the same time, $\mathrm{Shv}(X; \mathsf{Fil}(\mathrm{Sp})) \simeq \mathsf{Fil}(\mathrm{Shv}(X; \mathrm{Sp}))$ by Corollary C.19. These facts taken all together, we may regard $\mathsf{V}^\bullet$ as a functor $\mathrm{Shv}^{f.s.}_c(X; \mathscr{D}_{\mathbb{C},0}) \to \mathrm{Mod}_{V^\bullet \underline{\mathscr{D}_{\mathbb{C},0}}}(\mathsf{Fil}(\mathrm{Shv}(X; \mathrm{Sp}))$, where $V^\bullet \underline{\mathscr{D}_{\mathbb{C},0}}$ denotes the filtration on the constant sheaf induced by the standard V-filtration on $\mathscr{D}_{\mathbb{C},0}$.

7.2. **The V-filtration on sheaves.** In Example 7.6, we built a filtration on constructible sheaves $\mathrm{Shv}^{f.s.}_c(X; \mathscr{D}_{\mathbb{C},0})$ using the V-filtration on objects in $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$. Just as the classical V-filtration is unique, we might hope that the $\mathsf{V}^\bullet$-filtration on constructible sheaves is unique, as well. In this section, we show that this is indeed the case by imitating Kashiwara's original proof of the uniqueness of the $V^\bullet$-filtration on regular holonomic $\mathscr{D}$-modules.

Kashiwara's proof relies on the structure maps in the $V^\bullet$-filtration being inclusions, so in order to imitate his proof, we establish the following lemma showing that the structure morphisms of the $\mathsf{V}^\bullet$-filtration are monomorphisms.

Proposition 7.7. *Let $\mathcal{F} \in \mathrm{Shv}^{f.s.}_c(X; \mathscr{D}_{\mathbb{C},0})$, and let $X \to A$ denote a finite Whitney stratification with respect to which $\mathcal{F}$ is constructible. Suppose that $\mathsf{U}^\bullet\mathcal{F}$ and $\widetilde{\mathsf{U}}^\bullet\mathcal{F}$ are two objects of $\mathrm{Mod}_{V^\bullet \underline{\mathscr{D}_{\mathbb{C},0}}} \mathsf{Fil}(\mathrm{Shv}^A(X; \mathrm{Sp}))$ whose underlying $\underline{\mathscr{D}_{\mathbb{C},0}}$-module each is $\mathcal{F}$, that each satisfy the following properties:*

(i) For each $a \in A$, $\mathsf{U}^k\mathcal{F}|_{X_a}$ is a locally constant sheaf of perfect $\mathbb{C}\{t\}$-modules.
(ii) The map, $t\cdot : \mathsf{U}^k\mathcal{F} \to \mathsf{U}^{k+1}\mathcal{F}$ is an equivalence for each $k \geq 0$.
(iii) The induced map $\sum_{j=0}^{-k-1} \partial_t{}^j \mathsf{U}^{-1}\mathcal{F} \to \mathsf{U}^k\mathcal{F}$ is an equivalence for each $k < 0$.
(iv) For each $a \in A$ and each $k \in \mathbb{Z}$, $\mathrm{gr}^k_{\mathsf{U}} \mathcal{F}|_{X_a}$ is a locally constant sheaf of perfect $\mathbb{C}$-modules, and, locally on X, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ whose roots have real parts

in $[0,1)$ *and a factorization,*

$$\begin{tikzcd} & \mathsf{U}^{k+1}\mathcal{F} \arrow[dr, "i_{k+1}"] & \\ \mathsf{U}^k\mathcal{F} \arrow[ur, dashed] \arrow[rr, "b(t\partial_t - k)"] & & \mathcal{F}. \end{tikzcd}$$

Then there is a natural map $\mathsf{U}^\bullet\mathcal{F} \to \widetilde{\mathsf{U}}^\bullet\mathcal{F}$ *which is an equivalence of objects in* $\mathrm{Mod}_{V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}}\,\mathsf{Fil}(\mathrm{Shv}^A(X;\mathrm{Sp}))$.

Proof. Suppose that $\mathsf{U}^\bullet\mathcal{F}$ and $\widetilde{\mathsf{U}}^\bullet\mathcal{F}$ are two such filtrations of $\mathcal{F}$ satisfying the enumerated properties. Using recollement for sheaves and induction on the strata, we demonstrate that it suffices to produce an equivalence $\mathsf{U}^\bullet\mathcal{F}|_{X_a} \xrightarrow{\simeq} \widetilde{\mathsf{U}}^\bullet\mathcal{F}|_{X_a}$, for each $a \in A$.

Without loss of generality, we assume that each X_a is connected. Let X_{a_0} be one of the closed strata in X, and denote by $U_{a_0} = X \setminus X_{a_0}$ its complement. We denote the inclusion into X of the former by i and of the latter by j. Since $\mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ is a closed symmetric monoidal presentable stable ∞-category, $\mathrm{Shv}(X;\mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})))$ is a recollement of $\mathrm{Shv}(X_{a_0};\mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})))$ and $\mathrm{Shv}(U_{a_0};\mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})))$ in the sense of [HA, Definition A.8.1] (see [Vol21, Remark 4.12]). This means, in particular, that we obtain a fiber sequence,

$$i_*i^*\mathcal{G} \to \mathcal{G} \to j_*j^*\mathcal{G},$$

for any $\mathcal{G} \in \mathrm{Shv}(X;\mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp})))$, and that, moreover, a morphism u is an equivalence if and only if i^*u and j^*u are. Thus, to obtain an equivalence $\mathsf{U}^\bullet\mathcal{F} \xrightarrow{\simeq} \widetilde{\mathsf{U}}^\bullet\mathcal{F}$, it suffices to exhibit equivalences $i^*\mathsf{U}^\bullet\mathcal{F} \simeq i^*\widetilde{\mathsf{U}}^\bullet\mathcal{F}$ and $j^*\mathsf{U}^\bullet\mathcal{F} \simeq j^*\widetilde{\mathsf{U}}^\bullet\mathcal{F}$. Repeating this argument for a stratum $X_{a_1} \subset X$, contained in and closed in U_{a_0} and its complement $U_{a_1} := U_{a_0} \setminus X_{a_1}$, we find that in order to obtain an equivalence, $j^*\mathsf{U}^\bullet\mathcal{F} \simeq j^*\widetilde{\mathsf{U}}^\bullet\mathcal{F}$, it suffices to exhibit equivalences $(j^*\mathsf{U}^\bullet\mathcal{F})|_{X_{a_1}} \simeq (j^*\widetilde{\mathsf{U}}^\bullet\mathcal{F})|_{X_{a_1}}$ and $(j^*\mathsf{U}^\bullet\mathcal{F})|_{U_{a_1}} \simeq (j^*\widetilde{\mathsf{U}}^\bullet\mathcal{F})|_{U_{a_1}}$. Repeating this process, we see by induction that to produce an equivalence $\mathsf{U}^\bullet\mathcal{F} \xrightarrow{\simeq} \widetilde{\mathsf{U}}^\bullet\mathcal{F}$, it suffices to exhibit equivalences $\mathsf{U}^\bullet\mathcal{F}|_{X_a} \xrightarrow{\simeq} \widetilde{\mathsf{U}}^\bullet\mathcal{F}|_{X_a}$ for each $a \in A$.

Since $\mathcal{F}$ is assumed constructible with respect to $X \to A$, $\mathcal{F}|_{X_a}$ is the constant sheaf given by $\mathrm{pt}^*(\mathcal{F}_x)$, for any fixed $x \in X_a$. It therefore suffices to exhibit a factorization,

$$\begin{tikzcd} & \widetilde{\mathsf{U}}^k\mathcal{F}_x \arrow[dr, "i_k"] & \\ \mathsf{U}^k\mathcal{F}_x \arrow[ur, dashed] \arrow[rr, "i_k"] & & \mathcal{F}_x, \end{tikzcd}$$

for each k, where the notation $\mathsf{U}^k\mathcal{F}_x$ is unambiguous because $(\mathrm{ev}_k \circ \mathsf{U}^\bullet\mathcal{F})_x \simeq \mathrm{ev}_k \circ (\mathsf{U}^\bullet\mathcal{F})_x$, as ev_k commutes with filtered colimits. To exhibit such a factorization, we note that because $\mathsf{U}^\bullet\mathcal{F}$ and $\widetilde{\mathsf{U}}^\bullet\mathcal{F}$ satisfy properties enumerated in the statement of the proposition above, the stalks $\mathsf{U}^\bullet\mathcal{F}_x, \widetilde{\mathsf{U}}^\bullet\mathcal{F})_x \in \mathrm{Mod}_{V^\bullet\mathscr{D}_{\mathbb{C},0}}(\mathsf{Fil}(\mathrm{Sp}))$ satisfy the properties for the derived Kashiwara-Malgrange filtration enumerated in Proposition 5.11. The result now follows. $\square$

We now verify that $\mathsf{V}^\bullet\mathcal{F}$ satisfies all the enumerated properties in Proposition 7.7.

Proposition 7.8. *The V-filtration on sheaves,* $\mathsf{V}^\bullet$, *satisfies all of the properties enumerated in Proposition 7.7.*

Proof. Let $\mathcal{F} \in \mathrm{Shv}_c^{f.s.}(X;\mathscr{D}_{\mathbb{C},0})$, and let $X \to A$ be a finite Whitney stratification of X with respect to which $\mathcal{F}$ is constructible. It is clear from its construction that $\mathsf{V}^\bullet\mathcal{F} \in \mathrm{Mod}_{V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}}\,\mathsf{Fil}(\mathrm{Shv}^A(X;\mathrm{Sp}))$.

We begin by showing that $\mathsf{V}^\bullet\mathcal{F}$ satisfies properties (ii) and (iii). Observe that, by Corollary A.29, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Shv}_c^{f.s.}(X;\mathscr{D}_{\mathbb{C},0}) \arrow[r, "\mathsf{V}^n"] \arrow[d, "\theta_1"', "\simeq"] & \mathrm{Shv}(X;\mathscr{D}_{\mathbb{C},0}) \arrow[d, "\theta_2"', "\simeq"] \\
\mathrm{Shv}_c^{f.s.}(\mathcal{B}(X);\mathscr{D}_{\mathbb{C},0}) \arrow[r, "(\mathrm{ev}_n)_{\mathcal{B}(X)}\circ \mathsf{V}^\bullet{}_{\mathcal{B}(X)}"] & \mathrm{Shv}(\mathcal{B}(X);\mathscr{D}_{\mathbb{C},0}),
\end{tikzcd}$$

where we have used the notation therein. Note that $(\mathrm{ev}_n)_{\mathcal{B}(X)}\circ \mathsf{V}^\bullet{}_{\mathcal{B}(X)} \simeq (\mathrm{ev}_n\circ \mathsf{V}^\bullet)_{\mathcal{B}(X)}$ is given by pointwise composition with the the nth piece of the $\mathsf{V}^\bullet$-filtration on $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$. It follows from the properties of the $\mathsf{V}^\bullet$-filtration that $t : (\mathsf{V}^\bullet{}_{\mathcal{B}(X)}\theta_1(\mathcal{F}))^k \to (\mathsf{V}^\bullet{}_{\mathcal{B}(X)}\theta_1(\mathcal{F}))^{k+1}$ is an equivalence for each $k \geq 0$, and that $(\mathsf{V}^\bullet{}_{\mathcal{B}(X)}\theta_1(\mathcal{F}))^{-k-1} \simeq \sum_{j=0}^k \partial_t^j(\mathsf{V}^\bullet{}_{\mathcal{B}(X)}\theta_1(\mathcal{F}))^{-1}$ for each $k \geq 0$. It now follows that $\mathsf{V}^\bullet\mathcal{F}$ satisfies properties (ii) and (iii) by the commutative square.

By Corollary A.31, $\mathsf{V}^\bullet$ is compatible with restriction along the inclusion of a complex analytic subset $Z \subset X$ in the sense that, $\mathsf{V}^k(\mathcal{F}|_Z) \simeq (\mathsf{V}^k\mathcal{F})|_Z$. Thus, in order to show that it satisfies properties (i) and (iv), we may assume that $\mathcal{F}$ is a local system. Without loss of generality, we may also assume X is connected. In this case, two more applications of Corollary A.31 yields an equivalence $\mathsf{V}^\bullet\mathcal{F} \simeq \mathrm{pt}^*(\mathsf{V}^\bullet\mathcal{F}_x) \simeq \mathrm{pt}^*(\mathsf{V}^\bullet(\mathcal{F}_x))$, for any $x \in X$. But the latter sheaf is the constant sheaf on X with value $\mathsf{V}^\bullet(\mathcal{F}_x)$, so properties (i) and (iv) follow from the analogous properties for $\mathsf{V}^k(\mathcal{F}_x)$ and $\mathrm{gr}^k_{\mathsf{V}}(\mathcal{F}_x)$ coming from $\mathsf{V}^\bullet$-filtration on objects of $\mathcal{D}^b_{\text{reg. hol.}}(\mathscr{D}_{\mathbb{C},0})$. □

8. The Main Theorem

8.1. **The Kontsevich–Sabbah–Saito theorem.** We recall the set-up of [Sab10] in order to state the Kontsevich–Sabbah–Saito theorem. For clarification or details, we refer the reader to the introduction of *loc. cit.*

8.1.1. Let X be either a smooth complex algebraic variety or a complex manifold. In each case, we let $\mathcal{O}_X$ denote the structure sheaf of X.

Notation 8.1. We use the following notation: given any $\mathbb{C}$-vector space E, we denote by $E[\![u]\!]$ the $\mathbb{C}[\![u]\!]$-module of formal power series in the formal variable u with coefficients in E and by $E(\!(u)\!)$ the $\mathbb{C}(\!(u)\!)$-vector space of formal Laurent series with coefficient in E. For a sheaf $\mathcal{F}$ on X, $\mathcal{F}(\!(u)\!)$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)(\!(u)\!)$. Note that the sheaf $\mathcal{O}_X(\!(u)\!)$ is flat over $\mathcal{O}_X$.

Definition 8.2. Given a $\mathbb{C}(\!(u)\!)$-vector space E, a u-connection on E is a map $\nabla : \mathbb{C}(\!(u)\!)\partial_u \times E \to E$ which is $\mathbb{C}(\!(u)\!)$-linear in the first factor and satisfies the Leibniz rule with respect to scalar multiplication by $\mathbb{C}(\!(u)\!)$ in the following sense: if $\ell(u) \in \mathbb{C}(\!(u)\!)$ and $v \in E$, then $\nabla_{\partial_u}(v \cdot \ell(u)) = \nabla_{\partial_u}(v)\cdot \ell(u) + v \cdot \partial_u(\ell)$. We denote the category of $\mathbb{C}(\!(u)\!)$-modules with connection by $\mathrm{Conn}_{\mathbb{C}(\!(u)\!)}$.

Remark 8.3. A $\mathbb{C}(\!(u)\!)$-module E with connection has a canonical $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}$-module structure given by setting $\xi \cdot e := eu$ and $t \cdot e := \nabla_{\partial_u}(e)u^2$, for $e \in E$. One may check that the action so-defined satsifies the ring relations for $\widehat{\mathscr{E}}_{\mathbb{C},0}$. Conversely, given a $\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}$-module M, its underlying $\mathbb{C}(\!(\xi)\!)$-module E is a $\mathbb{C}((u)$-module with a canonical connection after relabeling $u := \xi$ and setting $\nabla_{\partial_u}(e) := t \cdot e\xi^{-2}$. There two constructions are obviously inverse to each other, and given an equivalence of categories,

$$\mathrm{Conn}_{\mathbb{C}(\!(u)\!)} \simeq \mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}^{\text{poly}}}.$$

Definition 8.4. Given a topological space Y and a sheaf of $\mathbb{C}$-vector spaces on $\mathcal{F}$ on Y, a u-connection on $\mathcal{F}(\!(u)\!)$ is a map of sheaves $\nabla_{\partial_u} : \mathcal{F}(\!(u)\!) \to \mathcal{F}(\!(u)\!)$ which is $\mathbb{C}(\!(u)\!)$-linear and satisfies the Leibniz rule on sections.

More generally, given a complex of $\mathbb{C}(\!(u)\!)$-vector spaces $E^\bullet$, a u-connection on $E^\bullet$ is defined in the same way as above. Given a complex $\mathcal{F}^\bullet$ of sheaves of $\mathbb{C}$-vector spaces on Y, a u-connection on $\mathcal{F}(\!(u)\!)$ is defined similarly.

Remark 8.5. A $\mathbb{C}(\!(u)\!)$-module with connection is the same thing as a $\widehat{\mathscr{E}}_{\mathbb{C},0}$-module. Given a bounded complex of $E^\bullet$ of $\mathbb{C}(\!(u)\!)$-vector spaces with connection, we will often identify $E^\bullet$ with its image under the localization map $\mathrm{Ch}^b(\mathcal{A}_{\widehat{\mathscr{E}}_{\mathbb{C},0}}) \to \mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$. Similarly, given a sheaf $\mathcal{F}$ of $\mathbb{C}$-vector spaces, a u-connection on $\mathcal{F}(\!(u)\!)$ determines an object of $\mathcal{A}(Y;\widehat{\mathscr{E}}_{\mathbb{C},0})$. We will often identify a bounded complex $\mathcal{F}^\bullet(\!(u)\!)$ of such sheaves with u-connection with its image under the localization map $\mathrm{Ch}^b(\mathcal{A}(Y;\widehat{\mathscr{E}}_{\mathbb{C},0})) \to \mathrm{Shv}^b(Y;\widehat{\mathscr{E}}_{\mathbb{C},0})$.

Remark 8.6. When it is clear from context that we are working with a u-connection on an object with a $\mathbb{C}(\!(u)\!)$- (or $\underline{\mathbb{C}(\!(u)\!)}$-) action, we simply talk about "the connection" on that object.

Let $f \in \Gamma(X;\mathcal{O}_X)$ be a globally defined function on X (i.e. either an algebraic function $X \to \mathbb{A}^1$ or holomorphic function $X \to \mathbb{C}$. In this case, we denote by $\widehat{\mathscr{E}}_X^{-f/u}$ the sheaf $\mathcal{O}_X(\!(u)\!)$ equipped with the $\mathbb{C}(\!(u)\!)$-connection given by $d - df/u$. That is, given a section $s \in \mathcal{O}_X(U)$, $U \subset X$, and a formal Laurent series $\ell(u) \in \mathbb{C}(\!(u)\!)$,

$$\nabla_{\partial_u}(s \otimes \ell(u)) := ds \otimes \ell(u) - df(s) \otimes \frac{\ell(u)}{u}.$$

8.1.2. *The formal twisted de Rham complex.* Let $\mathscr{M}$ be a locally free $\mathcal{O}_X$-module of finite rank equipped with a flat connection ∇ having regular singularity at infinity. Then $\mathscr{M} \otimes_{\mathcal{O}_X} \widehat{\mathscr{E}}_X^{-f/u}$ is a locally free $\mathcal{O}_X(\!(u)\!)$-module with a canonical connection given by $u\cdot\nabla - df \otimes \mathrm{id}_{\mathscr{M}}$. Sabbah's theorem concerns the *formal twisted de Rham complex* of $\mathscr{M}$.

Definition 8.7. The formal twisted de Rham complex of $\mathscr{M}$ by f is defined to be the complex of $\mathcal{O}_X(\!(u)\!)$-modules given by

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}) := (\Omega_X^\bullet(\!(u)\!) \otimes_{\mathcal{O}_X} \mathscr{M}, u\cdot\nabla - df \otimes \mathrm{id}_{\mathscr{M}}).$$

This complex comes with a connection, defined by $\nabla_{\partial_u} := \partial_u + f/u^2$, that commutes with the differential.

Remark 8.8. The global sections, $\Gamma(X;\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}))$, give a complex of $\mathbb{C}(\!(u)\!)$-vector spaces with connection ∇_{∂_u}. Its cohomology groups are $\mathbb{C}(\!(u)\!)$-vector spaces with connection.

Remark 8.9. We note that, because $\mathscr{M}$ is a coherent $\mathcal{O}_X$-module, there is an isomorphism $\mathscr{M}(\!(u)\!) = \mathscr{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\!(u)\!)$, as well as quasi-isomorphisms, $\Omega_X^\bullet(\!(u)\!) \otimes_{\mathcal{O}_X} \mathscr{M} \simeq \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M}(\!(u)\!) \simeq (\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M})(\!(u)\!) \simeq \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\!(u)\!)$, of complexes in $\mathrm{Ch}^b(\mathrm{QCoh}^\heartsuit(X))$.

8.1.3. *Vanishing cycles and microlocal formal Riemann-Hilbert.* If X is a complex algebraic variety, it has an associated complex analytic variety called the *analytification* of X, which is a complex manifold in the case that X is smooth. All of the algebraic objects that we have discussed above therefore have analytic counterparts. Denote the analytification of X by X^{an}. Given an algebraic vector bundle with connection, $(\mathscr{M},\nabla)$, its analytification is analytic vector bundle with flat connection $(\mathscr{M}^{\mathrm{an}},\nabla^{\mathrm{an}})$ on X^{an}. Consider the analytic function $f^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{C}$ associated to the regular function f above.

Let $\mathscr{L} = \ker \nabla^{\mathrm{an}}$ denote the local system on X^{an} of flat sections of $\mathscr{M}^{\mathrm{an}}$. This is a locally constant sheaf (in degree 0) on X^{an} with coefficients in $\mathbb{C}$.

Definition 8.10 (e.g. [Dim04, Definition 4.2.4])**.** For each $c \in \mathbb{C}$, let $\varphi_{f-c}\mathscr{L}$ denote the vanishing cycles of $\mathscr{L}$ with respect to the function $f^{\mathrm{an}} - c$. It is an object of $\mathrm{Shv}_c(X_c^{\mathrm{an}};\mathbb{C})$, where $X_c^{\mathrm{an}} = (f^{\mathrm{an}})^{-1}(c)$, and it is constructible with respect to an *algebraic* stratification of X^{an}. Moreover, we may assume this stratification is finite (c.f. [Dim04, Corollary 4.1.8]).

Remark 8.11. In this work, the shift in our definition for the vanishing cycles functor is normalized so that the functor preserves perversity.

The complex of vanishing cycles, $\varphi_{f-c}(\mathscr{L})$, has a canonical monodromy automorphism that we denote by T_c. The hypercohomology groups,

$$H^k(f^{-1}(c); \varphi_{f-c}(\mathscr{L}))$$

are finite dimensional $\mathbb{C}$-vector spaces equipped with an automorphism T_{f-c}.

Definition 8.12. Let E be a finite dimensional $\mathbb{C}$-vector space equipped with an automorphism T. Writing $T = \exp(-2\pi i M)$ for some $M : E \to E$, define

$$\mu\,\mathrm{RH}(E, T) := (E(\!(u)\!), d + M du/u),$$

where the right-hand side is a $\mathbb{C}(\!(u)\!)$-vector with connection.[14]

Definition 8.13. We define the following $\mathbb{C}(\!(u)\!)$-vector space with connection,

$$\widehat{\mathscr{E}}^{-c/u} := (\mathbb{C}(\!(u)\!), d + c du/u^2).$$

Important Remark 8.14. While $\mu\,\mathrm{RH}$ is a u-connection that determines a regular holonomic $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-module, hence a $\widehat{\mathscr{E}}_{\mathbb{C},0}$-module by Lemma 6.12, $\widehat{\mathscr{E}}^{-c/u}$ is an *irregular* connection, so determines just a $\widehat{\mathscr{E}}_{\mathbb{C},0}$-module. The Kontsevich–Sabbah–Saito theorem therefore obtains an isomorphism of $\widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0}$-modules (or $\mathbb{C}(\!(u)\!)$-modules with connection) rather than an isomorphism of $\widehat{\mathscr{E}}_{\mathbb{C},0}$-modules as one might hope.

8.1.4. *The Kontsevich–Sabbah–Saito theorem.* The following theorem was conjectured by Maxim Kontsevich as a possible answer to a question raised in [KS11, §7.4] about how to define the vanishing cycles of a function on a smooth formal scheme. It was proven by Claude Sabbah in the following form in [Sab10].

Theorem 8.15 ([Sab10, Theorem 1.1])**.** *Let X be a smooth complex algebraic variety, and $f : X \to \mathbb{A}^1$ a regular function on X. Let $\mathscr{M}$ be a locally free $\mathcal{O}_X$-module of finite rank equipped with a flat connection ∇ having regular singularity at infinity, and let $\mathscr{L} : -\ker \nabla^{\mathrm{an}}$ denote its locally constant sheaf of flat sections. Then for each k, we have an isomorphism of $\mathbb{C}(\!(u)\!)$-vector spaces with connection,*

$$\Big(H^k(X; \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M})), \nabla_{\partial_u}\Big) \simeq \bigoplus_{c \in \mathbb{C}} \widehat{\mathscr{E}}^{-c/u} \otimes_{\mathbb{C}(\!(u)\!)} \mu\,\mathrm{RH}\Big(H^k(f^{-1}(c); \varphi_{f-c}(\mathscr{L})), T_c\Big), \quad (8.16)$$

where we have used the notation established above.

Remark 8.17. Note that the direct sum on the right-hand side of (8.16) is finite since $\varphi_{f-c}(\mathscr{L}) \simeq 0$ when c is a non-critical value of f^{an}. Since f^{an} is holomorphic, it has only finitely many critical values.

Remark 8.18. Since X^{an} is a complex manifold, the underlying $\mathcal{O}_{X^{\mathrm{an}}}$-module of $\mathscr{M}$ is given by $\mathscr{L} \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}$, so we may also write $\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}) \simeq \mathscr{L} \otimes_{\mathbb{C}} (\Omega^\bullet_{X^{\mathrm{an}}}(\!(u)\!), u \cdot \nabla - df \otimes \mathrm{id}_{\mathscr{M}^{\mathrm{an}}})$.

[14] This functor is denoted $\widehat{\mathrm{RH}}^{-1}$ in [Sab10].

8.2. **Refinement of the Kontsevich–Sabbah–Saito theorem.** Let X be a complex algebraic variety. There is a continuous map of topological spaces $\tau : X^{\mathrm{an}} \to X^{\mathrm{Zar}}$, where X^{Zar} denotes the Zariski topology on X. Furthermore, X^{an} is of finite homotopy type,[15] and we can complete $\{X^{\mathrm{an}}\}$ to a basis $\mathcal{B}(X)$ as in Convention A.23 by virtue of the local topology of X^{an}.

Theorem 8.19 (Main Theorem)**.** *Let X be a smooth complex algebraic variety, and $f : X \to \mathbb{A}^1$ a regular function on X. Let $\mathscr{M}$ be a locally free $\mathcal{O}_X$-module of finite rank equipped with a flat connection ∇ having regular singularity at infinity, and let $\mathscr{L} := \ker \nabla^{\mathrm{an}}$ denote its locally constant sheaf of flat sections. Then there is an equivalence of objects in $\mathrm{Shv}^b(X^{\mathrm{Zar}}; \widehat{\mathscr{E}}^{\mathrm{poly}}_{\mathbb{C},0})$,*

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}) \simeq \bigoplus_{c\in\mathbb{C}} \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \tau_* \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right), \tag{8.20}$$

which recovers the isomorphisms (8.16) *upon taking hypercohomology.*

Proof. By taking a covering of X by quasi-projective Zariski open sets, we reduce to the case where X is quasi-projective. By Corollary 9.22 below, there is an equivalence,

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})|_{X_c^{\mathrm{an}}} \simeq \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \mu\mathcal{RH}_{\mathrm{Shv}}(\varphi_{f-c}(\mathscr{L}), T_{f-c}),$$

of objects in $\mathrm{Shv}_c(X_c^{\mathrm{an}}; \widehat{\mathscr{E}}_{\mathbb{C},0})$. Since, $\varphi_{f-c}(\mathscr{L}) \simeq 0$ for all but finitely many $c \in \mathbb{C}$, we find that

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}) \simeq \bigoplus_{c\in\mathbb{C}} \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right).$$

Apply τ_* to the above equivalence. Observing that τ_* commutes with finite direct sums and using the projection formula, we obtain

$$\begin{aligned}
\tau_*\left(\bigoplus_{c\in\mathbb{C}} \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right)\right) &\simeq \bigoplus_{c\in\mathbb{C}} \tau_*\left(\underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right)\right) \\
&\simeq \bigoplus_{c\in\mathbb{C}} \tau_*\left(\tau^* \underline{\widehat{\mathscr{E}}}^{-c/u}_{X_c^{\mathrm{Zar}}} \otimes_{\mathbb{C}((u))} \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right)\right) \\
&\simeq \bigoplus_{c\in\mathbb{C}} \underline{\widehat{\mathscr{E}}}^{-c/u} \otimes_{\mathbb{C}((u))} \tau_* \mu\mathcal{RH}_{\mathrm{Shv}}\left(\varphi_{f-c}(\mathscr{L}), T_{f-c}\right).
\end{aligned}$$

On the other hand, since we have reduced to the case where X is quasi-projective, Proposition 9.8 below establishes a natural algebraic-analytic comparison equivalence, $\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}) \xrightarrow{\simeq} \tau_{X*}\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$, from which the result follows. $\square$

9. Proof of the Main Theorem

The proof of Theorem 8.19 we have given above closely imitates Sabbah's proof of [Sab10, Theorem 1.1]. Namely, it proceeds in two broad parts:

(i) A comparison of the *algebraic* formal twisted de Rham complex $\mathrm{DR}(\widehat{\mathscr{E}}^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M})$ to the *analytic* formal twisted de Rham complex, $\mathrm{DR}(\widehat{\mathscr{E}}^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$ (Proposition 9.8).
(ii) A comparison of the analytic formal twisted de Rham complex to the right-hand side of (8.20) (Proposition 9.20 and Corollary 9.22).

In the remaining sections of this paper, before the appendices, we establish each of these component parts.

[15] See [Cis] for justification of this claim.

9.1. **Algebraic-to-analytic comparison.** Let $F : Y \to \mathbb{P}^1$ be regular function from a smooth projective variety Y to $\mathbb{P}^1$ extending f, meaning that there exists a commutative diagram,

$$\begin{array}{ccc} X & \xhookrightarrow{j} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1, \end{array}$$

and such that $D := Y \setminus X$ is a (not necessarily normal crossings) divisor in Y. Then $(\mathscr{M}, \nabla)$ extends as a coherent $\mathcal{O}_Y(*D)$-module[16]with connection having regular singularity along D, that we continue to denote by $(\mathscr{M}, \nabla)$. In particular, $\mathscr{M}$ is $\mathscr{D}_Y$-holonomic, hence $\mathscr{D}_Y$-coherent. There are several sheaves we define on Y:

- $\widehat{\mathcal{O}}_{(Y,D)} := \mathcal{O}_Y(*D)(\!(u)\!)$.
- $\widehat{\mathscr{D}}_{(Y,D)} := \widehat{\mathcal{O}}_{(Y,D)} \otimes_{\mathcal{O}_Y} \mathscr{D}_Y$.
- $\widehat{\mathscr{E}}^{-F/u}_{(Y,D)}$ denotes the $\widehat{\mathscr{D}}_{(Y,D)}$-module, $\mathcal{O}_Y(*D)(\!(u)\!)$, equipped with the connection $(-dF \wedge -) + ud$.

With the above set-up and notation, we recall the *algebraic formal twisted de Rham complex* associated to $(\mathscr{M}, D)$:

$$(\Omega^\bullet_Y(*D)(\!(u)\!) \otimes_{\mathcal{O}_Y} \mathscr{M}, u \cdot \nabla + (-dF \wedge -) \otimes \mathrm{id}_{\mathscr{M}}) \tag{9.1}$$

Similarly, recall the *analytic formal twisted de Rham complex* associated to $(\mathscr{M}, D)$:

$$\left(\Omega^\bullet_{Y^{\mathrm{an}}}(*D)(\!(u)\!) \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}, u \cdot \nabla + (-dF \wedge -) \otimes \mathrm{id}_{\mathscr{M}^{\mathrm{an}}}\right). \tag{9.2}$$

We denote (9.1) and (9.2) by $\mathrm{DR}(\widehat{\mathscr{E}}^{-F/u}_{Y,D} \otimes_{\mathcal{O}_Y} \mathscr{M})$ and $\mathrm{DR}(\widehat{\mathscr{E}}^{-F/u}_{Y,D} \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$, respectively, and regard them as objects of $\mathrm{Shv}(Y; \widehat{\mathscr{E}}_{\mathbb{C},0})$ and $\mathrm{Shv}(Y^{\mathrm{an}}; \widehat{\mathscr{E}}_{\mathbb{C},0})$, respectively.

9.1.1. *Reminder on GAGA.* We have already introduced the associated analytic space X^{an} to a complex algebraic variety X. The definition of this associated analytic space may be found in §1 of [Ser56]. As alluded to above, there is a map on underlying topological spaces,

$$\tau : X^{\mathrm{an}} \to X^{\mathrm{Zar}},$$

which is continuous. The pullback of the structure sheaf $\mathcal{O}_X$ on X is a subsheaf of the sheaf of holomorphic functions,

$$\tau^*\mathcal{O}_X \subset \mathcal{O}_{X^{\mathrm{an}}}.$$

Definition 9.3 ([Ser56, §3.9 Definition 2])**.** The *analytification* of an $\mathcal{O}_X$-module $\mathcal{F}$ is defined to be

$$\mathcal{F}^{\mathrm{an}} := \tau^*\mathcal{F} \otimes_{\tau^*\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}.$$

There is a canonical morphism

$$\tau^*\mathcal{F} := \tau^*\mathcal{F} \otimes_{\tau^*\mathcal{O}_X} \tau^*\mathcal{O}_X \to \tau^*\mathcal{F} \otimes_{\tau^*\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}} =: \mathcal{F}^{\mathrm{an}}$$

induced by the inclusion $\tau^*\mathcal{O}_X \subset \mathcal{O}_{X^{\mathrm{an}}}$. Composition of this morphism with the unit of the (τ^*, τ_*) adjunction yields a natural map,

$$\mathcal{F} \to \tau_*\mathcal{F}^{\mathrm{an}}. \tag{9.4}$$

Proposition 9.5 ([SS14, Proposition 1])**.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module. Then the natural morphism* (9.4) *induces an equivalence on global sections*

$$\Gamma(Y; \mathrm{DR}(\widehat{\mathscr{E}}^{-F/u}_{Y,D} \otimes_{\mathcal{O}_Y} \mathscr{M})) \xrightarrow{\simeq} \Gamma(Y^{\mathrm{an}}; \mathrm{DR}(\widehat{\mathscr{E}}^{-F/u}_{Y,D} \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})), \tag{9.6}$$

as objects in $\mathcal{D}(\widehat{\mathscr{E}}_{\mathbb{C},0})$.

[16]$\mathcal{O}_Y(*D)$ denotes the sheaf of rational functions on Y that are regular on $Y \setminus D$.

Proof. This follows immediately from the proof of Proposition 1 in [SS14]. $\square$

We record the following proposition of Sabbah, which relates the twisted de Rham complex of X^{an} and the twisted de Rham complex of its compactification Y^{an}.

Proposition 9.7 ([Sab10, Proposition 4.1])**.** *Let X be a smooth complex variety, and $f : X \to \mathbb{A}^1$ a regular function. Let $F : Y \to \mathbb{P}^1$ be a map from a smooth projective variety to $\mathbb{P}^1$, extending f. We denote by $j^{\mathrm{an}} : X^{\mathrm{an}} \hookrightarrow Y^{\mathrm{an}}$ the open inclusion, and by $D = Y^{\mathrm{an}} \setminus X^{\mathrm{an}}$ the divisor given by the complement of X^{an}. Let $\mathscr{M}$ be a coherent $\mathcal{O}_{X^{\mathrm{an}}}(*D)$-module equipped with a flat connection ∇ with regular singularities along D, where $\mathcal{O}_{X^{\mathrm{an}}}(*D)$ denotes the sheaf of meromorphic functions with poles at most along D. Then the natural morphism of sheaves,*

$$\mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_X(*D)} \mathscr{M}) \xrightarrow{\simeq} Rj_* \,\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}|_X)$$

is an equivalence.

Although our standing assumption is that all functors are implicitly derived, we have written "Rj_*" in the proposition above to emphasize that Sabbah's result is an equivalence of objects in the derived category.

Proposition 9.8. *Let X be a smooth complex algebraic variety, and let $f : X \to \mathbb{A}^1$ be a regular function. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and let $\tau_X : X^{\mathrm{an}} \to X$ denote the canonical map on topological spaces. Then the natural morphism* (9.4) *induces an equivalence*

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M}) \xrightarrow{\simeq} \tau_{X*}\,\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}), \tag{9.9}$$

between objects in $\mathrm{Shv}(X; \widehat{\mathscr{E}}_{\mathbb{C},0})$.

Proof. It suffices to show that the natural morphism (9.9) induces an equivalence on sections over each $U \subset X$,

$$\Gamma(U; \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M})) \xrightarrow{\simeq} \Gamma(U; \tau_*\,\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})). \tag{9.10}$$

The left-hand side of (9.10) is clearly equivalent to global sections of the sheaf, $\mathrm{DR}(\widehat{\mathscr{E}}_U^{-f|_U/u} \otimes_{\mathcal{O}_U} \mathscr{M}|_U)$. On the other hand, by smooth base change[17] using the following pullback square of topological spaces,

$$\begin{array}{ccc} U^{\mathrm{an}} & \xhookrightarrow{j^{\mathrm{an}}} & X^{\mathrm{an}} \\ {\scriptstyle \tau_U}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_X} \\ U & \xhookrightarrow{j} & X, \end{array}$$

the right-hand side of (9.10) is equivalent to global sections of the sheaf $\mathrm{DR}(\widehat{\mathscr{E}}_U^{-f|_U/u} \otimes_{\mathcal{O}_{U^{\mathrm{an}}}} (\mathscr{M}|_U)^{\mathrm{an}})$. Observe that, as an open subset of X, U is also a smooth complex variety. Thus, without loss of generality, it suffices to prove that (9.10) is an equivalence for $U = X$.

In order to show the equivalence for global sections on X, we invoke [Sab10, Proposition 4.1], which we have reproduced above as Proposition 9.7. Consider the following pullback square

$$\begin{array}{ccc} X^{\mathrm{an}} & \xhookrightarrow{j^{\mathrm{an}}} & Y^{\mathrm{an}} \\ {\scriptstyle \tau_X}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_Y} \\ X & \xrightarrow{j} & Y \end{array}$$

[17] See [Vol21, Lemma 3.3] for a reference for smooth base change in the setting of sheaves valued in presentable ∞-categories.

of topological spaces. It follows that

$$\begin{aligned}\Gamma(Y^{\mathrm{an}}; \mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}(*D)} j_+^{\mathrm{an}} \mathscr{M}^{\mathrm{an}})) &\simeq \mathrm{pt}_* \, \tau_{Y*} \, \mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}(*D)} j_+^{\mathrm{an}} \mathscr{M}^{\mathrm{an}}) && (9.11)\\
&\simeq \mathrm{pt}_* \, \tau_{Y*} j_*^{\mathrm{an}} \, \mathrm{DR}(\widehat{\mathscr{E}}_{X}^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}) && (9.12)\\
&\simeq \mathrm{pt}_* \, j_* \tau_{X*} \, \mathrm{DR}(\widehat{\mathscr{E}}_{X}^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}) && (9.13)\\
&\simeq \Gamma(X^{\mathrm{an}}; \tau_{X*} \, \mathrm{DR}(\widehat{\mathscr{E}}_{X}^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})), && (9.14)\end{aligned}$$

where (9.12) follows from Proposition 9.7. On the other hand, by [Sab10, Remark 2.8], we have the equivalence,

$$\mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_Y} j_+ \mathscr{M}) \xrightarrow{\simeq} (\Omega_Y^\bullet \otimes_{\mathcal{O}_Y} j_* \mathscr{M}(\!(u)\!), u \cdot \nabla + (-dF \wedge -))\,.$$

Because we are in the algebraic setting, $j_* \mathscr{M}(\!(u)\!) \simeq (j_* \mathscr{M})(\!(u)\!)$. It follows that

$$\begin{aligned}\Omega_Y^\bullet \otimes_{\mathcal{O}_Y} (j_* \mathscr{M})(\!(u)\!) &\simeq (\Omega_Y^\bullet \otimes_{\mathcal{O}_Y} j_* \mathscr{M})(\!(u)\!)\\
&\simeq (j_*(j^* \Omega_Y^\bullet \otimes_{\mathcal{O}_X} \mathscr{M})(\!(u)\!)\\
&\simeq j_*(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M})(\!(u)\!),\end{aligned}$$

where we have used the projection formula to obtain the penultimate equivalence. Thus, we obtain

$$\begin{aligned}\Gamma(Y; \mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_Y(*D)} j_+ \mathscr{M})) &\simeq \Gamma\left(Y; (j_*(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M})(\!(u)\!), u \cdot \nabla + (-dF \wedge -))\right)\\
&\simeq \Gamma\left(X; (\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{M})(\!(u)\!), u \cdot \nabla + (-dF \wedge -))\right)\\
&= \Gamma(X; \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M})).\end{aligned}$$

The result now follows from [Sab10, Proposition 2.6], as illustrated in the following commutative diagram,

$$\begin{tikzcd}
\Gamma(Y; \mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_Y} j_+ \mathscr{M})) \arrow[r, bend left, "{[\text{Sab10, Proposition 2.6}]}", "\simeq"'] \arrow[d, "\simeq"'] & \Gamma(Y^{\mathrm{an}}; \mathrm{DR}(\widehat{\mathscr{E}}_{Y,D}^{-F/u} \otimes_{\mathcal{O}_{Y^{\mathrm{an}}}} j_+^{\mathrm{an}} \mathscr{M}^{\mathrm{an}})) \arrow[d, "\simeq"'] \\
\Gamma(X; \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_X} \mathscr{M})) \arrow[r, "(9.10)"] & \Gamma(X^{\mathrm{an}}; \tau_{X*} \, \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})).
\end{tikzcd}$$

□

9.2. **Analytic-to-vanishing cycles comparison.** Now let X be a complex analytic manifold of dimension n, and $f : X \to \mathbb{C}$ a holomorphic function on X. Let $\mathscr{M}$ be a locally free $\mathcal{O}_X$-module of finite rank d equipped with a flat holomorphic connection ∇, whose associated local system of flat sections, $\ker \nabla$ is denoted by $\mathscr{L}$.

Let $\mathrm{pr}_2 : X \times \mathbb{C} \to \mathbb{C}$ denote the canonical projection. Using $\mathscr{M}$ and its connection ∇, Sabbah defines the following object of $\mathcal{D}^b(\mathrm{pr}_2^* \mathscr{D}_{\mathbb{C}})$.

Definition 9.15. Denote by $i_f : X \to X \times \mathbb{C}$ the graph of f, and let $\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}$ denote the relative de Rham functor. Then $\mathscr{K}_f$ is defined to be $\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(\mathscr{M}_f)$, where $\mathscr{M}_f := i_{f+}\mathscr{M}$.[18]

Let $X_0 := f^{-1}(0)$. We will be interested in the restriction of $\mathscr{K}_f$ to the analytic subspace $X_0 \subset X$, and denote this restriction by $\mathscr{K}_{f,0} := \mathscr{K}_f|_{X_0}$. It follows from the definition that $\mathscr{K}_{f,0}$ is an object of $\mathrm{Shv}^b(X_0; \mathscr{D}_{\mathbb{C},0})$.

[18]Here, i_{f+} denotes the pushforward of $\mathscr{D}$-modules. Because i_f is a closed immersion, i_{f+} is exact on ordinary $\mathscr{D}_X$-modules, and therefore $\mathscr{M}_f \in \mathcal{R}\mathrm{eg}\mathcal{H}\mathrm{ol}_{\mathscr{D}_{X\times\mathbb{C}}}$.

Using the locally free resolution of the relative de Rham complex recalled earlier in (3.9), Sabbah works with an explicit model for $\mathscr{K}_{f,0}$ in $\mathrm{Ch}^b(X_0;\mathscr{D}_{\mathbb{C},0})$, the category of bounded complexes of $\underline{\mathscr{D}_{\mathbb{C},0}}$-modules on X_0, in order to prove the results of [Sab10]. We denote the complex in $\mathrm{Ch}^b(X_0;\underline{\mathscr{D}_{\mathbb{C},0}})$ representing $\mathscr{K}_{f,0}$ by $\mathscr{K}^\bullet_{f,0}$. The complex $\mathscr{K}^\bullet_{f,0}$ admits a filtration which is compatible with the filtration, $V^\bullet\mathscr{D}_{\mathbb{C},0}$, making it into a "$V$-filtered complex" in the words of [Sab10].

Definition 9.16. We set $U^k\mathscr{K}^\bullet_{f,0} := \mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(V^k\mathscr{M}_f)|_{X_0}$, where $V^k\mathscr{M}_f$ denotes the Kashiwara-Malgrange filtration along the hypersurface $X\times 0\subset X\times\mathbb{C}$.

Remark 9.17. As noted in [Sab10], it makes sense to apply $\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}$ to $V^k\mathscr{M}_f$ because it is a coherent $V^0\mathscr{D}_{X\times\mathbb{C}}$-module, and hence a $\mathscr{D}_{X\times\mathbb{C}/\mathbb{C}}$-module.

The main properties of the filtration $U^\bullet\mathscr{K}^\bullet_{f,0}$ are established in [Sab10, Theorem 3.9]:

- The complex, $\mathrm{gr}_U^{-1}\mathscr{K}^\bullet_{f,0}$ equipped with the operator $\exp(-2\pi it\partial_t)$ is quasi-isomorphic to the complex $\varphi_f(\mathscr{L})$ equipped with its monodromy operator, T_f.
- $\mathscr{K}^\bullet_{f,0}$ satisfies the assumptions of [Sab10, Corollary 3.4]. Namely,
  - (i) the image of $\mathscr{K}^\bullet_{f,0}$ under the canonical localization functor, $\mathrm{Ch}^b(X_0;\mathscr{D}_{\mathbb{C},0})\to\mathrm{Shv}^b(X;\mathscr{D}_{\mathbb{C},0})$, where $K^b(X_0;\mathscr{D}_{\mathbb{C},0})$ denotes the ordinary category of bounded complexes of $\underline{\mathscr{D}_{\mathbb{C},0}}_{X_0}$-modules, lies in $\mathrm{Shv}_c(X_0;\mathscr{D}_{\mathbb{C},0})$,
  - (ii) $t: U^k(\mathscr{K}^i_{f,0})\to U^{k+1}(\mathscr{K}^i_{f,0})$ is an isomorphism of $\mathbb{C}\{t\}$-modules for each $k>0$ and each i,
  - (iii) $U^{-k-1}(\mathscr{K}^i_{f,0})=\sum_{\ell=0}^k\partial_t^\ell U^{-1}(\mathscr{K}^i_{f,0})$ for each $k>0$ and each i,
  - (iv) there exists a nonzero polynomial $b(s)\in\mathbb{C}[s]$ with roots having their real part in $[0,1)$ such that $b(t\partial_t-k)$ vanishes on each $\mathrm{gr}^k_U(\mathscr{K}^i_{f,0})$ for each i, and
  - (v) for each i, each germ $H^i(U^{-1}\mathscr{K}^\bullet_{f,0})_y$ is finite type over $\mathbb{C}\{t\}$.

It is clear from the properties listed above that the V-filtered complex $U^\bullet\mathscr{K}^\bullet_{f,0}$ endows $\mathscr{K}_{f,0}$ with the structure of a module in $\mathsf{Fil}(\mathrm{Shv}(X_0;\mathrm{Sp})$ over $V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}$ satisfying the properties enumerated in Proposition 7.7. Moreover, it is clear from the definition of $\mathscr{K}^\bullet_{f,0}$ that it is constructible with respect to a finite stratification of X_0. Altogether, we have the following corollary.

Corollary 9.18. *The object $\mathscr{K}_{f,0}$ is an object of $\mathrm{Shv}_c^{f.s.}(X_0;\mathscr{D}_{\mathbb{C},0})$ and has the canonical enhancement to an object of $\mathrm{Mod}_{V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}}(\mathsf{Fil}(\mathrm{Shv}_c^{f.s.}(X_0;\mathrm{Sp}))$ satisfying the properties enumerated in Proposition 7.7, coming from the V-filtered complex, $U^\bullet\mathscr{K}^\bullet_{f,0}$.*

Proposition 9.19. *There is a natural equivalence,*

$$\varphi_{\mathrm{Shv}}(\mathscr{K}_{f,0})\simeq(\varphi_f(\mathscr{L}),T_f)$$

of objects in $\mathrm{Fun}(S^1,\mathrm{Shv}_c^{f.s}(X_0;\mathbb{C}))$.

Proof. Recall that $\varphi_f(\mathcal{L})\simeq\mathrm{DR}_X(\mathrm{gr}_V^{-1}\mathscr{M}_f)$ by a theorem independently obtained by Kashiwara and Malgrange. On the other hand, $U^i\mathscr{K}^\bullet_{f,0}$ is by definition the image of $V^i(\mathscr{M}_f|_{X_0\times 0})$ under the relative de Rham functor $\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(-)$. Since $\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(-)$ commutes with colimits (given by tensor with relative differentials), we obtain,

$$\begin{aligned}\mathrm{gr}_U^{-1}(\mathscr{K}^\bullet_{f,0})&\simeq\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(V^{-1}\mathscr{M}_f)/\,\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(V^0\mathscr{M}_f)\\&\simeq\mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}(\mathrm{gr}_V^{-1}\mathscr{M}_f)\\&\simeq\mathrm{DR}_X(\mathrm{gr}_V^{-1}\mathscr{M}_f)\\&\simeq\varphi_f(\mathscr{L}),\end{aligned}$$

where we have used the fact that $\mathrm{gr}_V^{-1}\mathscr{M}_f$ is supported on $X \times 0$, and $\mathrm{DR}_X \simeq \mathrm{DR}_{X\times\mathbb{C}/\mathbb{C}}$ on $\mathscr{D}_{X\times\mathbb{C}}$-modules supported on $X \times 0$.

Let $U^\bullet\mathscr{K}_{f,0}$ denote the filtered $V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}$-module of Corollary 9.18. Since $\mathrm{gr}_{\mathsf{V}}^{-1}(-) \simeq \varphi_{\mathrm{Shv}}(-)$, it suffices to show that $U^\bullet\mathscr{K}_{f,0}$ and $\mathsf{V}^\bullet\overline{\mathscr{K}_{f,0}}$ are equivalent objects of $\mathrm{Mod}_{V^\bullet\underline{\mathscr{D}_{\mathbb{C},0}}}\mathsf{Fil}(\mathrm{Shv}(X_0;\mathrm{Sp}))$. By Proposition 7.7, it suffices to show that $U^\bullet\mathscr{K}_{f,0}$ and $\mathsf{V}^\bullet\mathscr{K}_{f,0}$ each satisfy the properties enumerated therein. For $\mathsf{V}^\bullet\mathscr{K}_{f,0}$ this is the content of Proposition 7.8. For $U^\bullet\mathscr{K}_{f,0}$, this is the content of Corollary 9.18. □

Proposition 9.20. *There is a natural equivalence,*

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u}\otimes_{\mathcal{O}_X}\mathscr{M})|_{X_0} \simeq \mu\mathcal{RH}_{\mathrm{Shv}}(\varphi_f(\mathscr{L}),T_f)$$

as objects of $\mathrm{Shv}_c(X_0;\widehat{\mathscr{E}}_{\mathbb{C},0})$.

Proof. There is a natural morphism

$$\widehat{\mathscr{K}_f^\bullet} := \widehat{\mathscr{E}}_{\mathbb{C},0}\otimes_{\mathscr{D}_{\mathbb{C},0}}\mathscr{K}_f^\bullet \to \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u}\otimes_{\mathcal{O}_X}\mathscr{M}) \simeq \mathscr{L}\otimes_{\mathbb{C}}(\Omega_X^\bullet(\!(u)\!), ud + (-df\wedge -))), \tag{9.21}$$

obtained by identifying u with u, which is in fact a quasi-isomorphism (after restriction to X_0) by [Sab10, Lemma 3.15]. As such, we obtain the equivalence,

$$\begin{aligned}
\mu\mathcal{RH}(\varphi_f(\mathscr{L}),T_f) &\simeq \mu\mathcal{RH}(\mathrm{gr}_U^{-1}\mathscr{K}_{f,0},\exp(-2\pi i t\partial_t)) && \text{(a)}\\
&\simeq \mu\mathcal{RH}(\varphi_{\mathrm{Shv}}\mathscr{K}_{f,0},\exp(-2\pi i t\partial_t)) && \text{(b)}\\
&\simeq \widehat{\mathscr{K}_{f,0}} && \text{(c)}\\
&\xrightarrow{\simeq} \mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u}\otimes_{\mathcal{O}_X}\mathscr{M})|_{X_0} && \text{(d)}
\end{aligned}$$

where the equivalence (b) is the content of Proposition 9.19 above, and the equivalence (c) is shown in Example 7.4. □

Corollary 9.22. *More generally, for any $c\in\mathbb{C}$, there is a natural equivalence,*

$$\mathrm{DR}(\widehat{\mathscr{E}}_X^{-f/u}\otimes_{\mathcal{O}_X}\mathscr{M})|_{X_c} \simeq \underline{\widehat{\mathscr{E}}}_{X_c}^{-c/u}\otimes_{\mathbb{C}(\!(u)\!)}\mu\mathcal{RH}_{\mathrm{Shv}}(\varphi_{f-c}(\mathscr{L}),T_{f-c})$$

as objects of $\mathrm{Shv}_c(X_c;\widehat{\mathscr{E}}_{\mathbb{C},0}^{\mathrm{poly}})$.

Proof. Let $g := f-c$. Then by Proposition 9.20, $\mathrm{DR}(\widehat{\mathscr{E}}_X^{-g/u}\otimes_{\mathcal{O}_X}\mathscr{M})|_{g^{-1}(0)} \simeq \mu\mathcal{RH}_{\mathrm{Shv}}(\varphi_g(\mathscr{L}),T_g)$. Now observe that the connection on $\mathrm{DR}(\widehat{\mathscr{E}}_X^{-g/u}\otimes_{\mathcal{O}_X}\mathscr{M})$ is given by $\nabla_{\partial_u} = \partial_u + g/u^2 = \partial_u + f/u^2 - c/u^2$. Thus, we find that $\mathrm{DR}(\widehat{\mathscr{E}}_X^{-g/u}\otimes_{\mathcal{O}_X}\mathscr{M}) \simeq \underline{\widehat{\mathscr{E}}}^{-c/u}\otimes_{\mathbb{C}(\!(u)\!)}\mathrm{DR}(\widehat{\mathscr{E}}_X^{-g/u}\otimes_{\mathcal{O}_X}\mathscr{M})$ as objects of $\mathrm{Shv}_c(X;\widehat{\mathscr{E}}_{\mathbb{C},0})$. The constant sheaf $\underline{\widehat{\mathscr{E}}}^{c/u}$ has a multiplicative inverse given by $\underline{\widehat{\mathscr{E}}}^{-c/u}$ under the tensor product of sheaves valued in $\mathcal{D}^b(\widehat{\mathscr{E}}_{\mathbb{C},0})$, since $\widehat{\mathscr{E}}^{-c/u}\otimes_{\mathbb{C}(\!(u)\!)}\widehat{\mathscr{E}}^{c/u}\simeq\mathbb{C}(\!(u)\!)$, where the latter denotes the $\mathbb{C}(\!(u)\!)$-module with trivial ∂_u action. The corollary now follows. □

Appendix A. Constructible sheaves

In this section, we recall the basics of constructible sheaves in the higher categorical setting. The main reference for this section is [HA, Appendix A]. Whenever possible, however, we have used the notation found there.

A.1. **Locally constant sheaves.** Constructible sheaves are defined in terms of locally constant sheaves, whose definition we recall below.

Definition A.1 ([HA, Definition A.1.12])**.** Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{F}$ be an object of $\mathcal{X}$. We will say that $\mathcal{F}$ is constant if it lies in the essential image of the geometric morphism $\pi^*:\mathcal{S}\to\mathcal{X}$. We will say that $\mathcal{F}$ is locally constant if there exists a small collection of objects $\{U_\alpha\in\mathcal{X}\}_{\alpha\in S}$ such that the following conditions are satisfied:

(i) The objects U_α cover $\mathcal{X}$: that is, there is an effective epimorphism $\coprod U_\alpha \to \mathbf{1}$, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.
(ii) For each $\alpha \in S$, the product $\mathcal{F} \times U_\alpha$ is a constant object of the ∞-topos $\mathcal{X}_{/U_\alpha}$.

Under certain conditions on $\mathcal{X}$, the full subcategory of locally constant objects of $\mathcal{X}$ can be described as functors out of a particular ∞-groupoid, called the *shape* of $\mathcal{X}$.

Definition A.2 ([HA, Definition A.1.1])**.** Let $\mathcal{X}$ be an ∞-topos, let $\pi_* : \mathcal{X} \to \mathcal{S}$ be the functor corepresented by the final object of $\mathcal{X}$, and let π^* be the left adjoint to π_*. We will say that $\mathcal{X}$ has constant shape if the composition $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$ is corepresentable.

Definition A.3 ([HA, Definition A.1.5])**.** Let $\mathcal{X}$ be an ∞-topos. We will say that an object $U \in \mathcal{X}$ has constant shape if the ∞-topos $\mathcal{X}_{/U}$ has constant shape. We will say that $\mathcal{X}$ is locally of constant shape if every object $U \in \mathcal{X}$ has constant shape.

If $\mathcal{X}$ is locally of constant shape, the shape of $\mathcal{X}$ may be identified with the object $\pi_!\mathbf{1} \in \mathcal{S}$, where $\pi_!$ is the left adjoint to π^* (see [HA, Proposition A.1.8]).

Theorem A.4 ([HA, Theorem A.1.15])**.** *Let $\mathcal{X}$ be an ∞-topos which is locally of constant shape, and let $\psi^* : \mathcal{S}_{\pi_!\mathbf{1}} \to \mathcal{X}$ be the functor of [HA, Proposition A.1.11]. Then ψ^* is a fully faithful embedding, whose essential image is the full subcategory of $\mathcal{X}$ spanned by the locally constant objects of $\mathcal{X}$.*

A.1.1. *Monodromy equivalence.* The classical monodromy equivalence states that, for a topological space X, there is an equivalence,

$$\mathrm{Fun}(\pi_{\leq 1}X, \mathcal{S}\mathrm{et}) \simeq \mathcal{L}\mathrm{oc}(X)$$

between representations of the fundamental groupoid of X and locally constant sheaves of sets on X. This classical equivalence generalizes to an equivalence between the *full* ∞-groupoid of X and locally constant sheaves of *spaces* on X, as we now recall.

To every topological space X, the associated ∞-groupoid is denoted by the $\mathrm{Sing}(X)$. The functor $\mathrm{Sing} : \mathcal{T}\mathrm{op} \to \mathcal{S}$ has a left adjoint called *geometric realization*, denoted by $|-| : \mathcal{S} \to \mathcal{T}\mathrm{op}$.

Definition A.5 ([HA, Definition A.4.9])**.** Let X be a topological space. We will say that X has singular shape if the counit map $|\mathrm{Sing}(X)| \to X$ is a shape equivalence.

Definition A.6 ([HA, Definition A.4.15])**.** We will say that topological space X is locally of singular shape if every open set $U \subseteq X$ has singular shape.

If X is locally of singular shape, the ∞-topos $\mathrm{Shv}(X; \mathcal{S})$ is locally of constant shape. It follows from Theorem A.4 that the locally constant sheaves on X are identified with representations of the shape of $\mathcal{X}$. Moreover the the shape of $\mathcal{X}$ is identified with the homotopy type $\mathrm{Sing}(X)$ of X itself (see [HA, §A.4]).

A.1.2. *Presentable stable coefficients.* By imitating Definition A.1, we obtain a notion of locally constant sheaves for more general coefficients in presentable stable ∞-categories.

Indeed, by [SAG, Remark I.1.3.2.8], any geometric morphism of ∞-topoi $g^* : \mathcal{X} \to \mathcal{Y}$ induces a functor of spectral sheaves, $\mathrm{Shv}(\mathcal{X}; \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{Y}; \mathrm{Sp})$, which by abuse of notation we also denote by g^*. As also noted in the remark, g^* is left adjoint to the pushforward functor $g_* : \mathrm{Shv}(\mathcal{Y}; \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{X}; \mathrm{Sp})$, given by pointwise composition with $g^* : \mathcal{X} \to \mathcal{Y}$.

More generally, given any presentable stable ∞-category $\mathscr{C}$, $\mathrm{Shv}(\mathcal{X}; \mathscr{C}) \simeq \mathrm{Shv}(\mathcal{X}; \mathrm{Sp}) \otimes \mathscr{C}$. By functoriality, we obtain morphisms $g^* : \mathrm{Shv}(\mathcal{X}; \mathscr{C}) \to \mathrm{Shv}(\mathcal{Y}; \mathscr{C})$ and $g_* : \mathrm{Shv}(\mathcal{Y}; \mathscr{C}) \to \mathrm{Shv}(calX; \mathscr{C})$ associated to any geometric morphism g^*. In particular, given $U \in \mathcal{X}$, we may talk about the restriction of $\mathcal{F} \in \mathrm{Shv}(\mathcal{X}; \mathscr{C})$ to $\mathcal{X}_{/U}$, denoted $\mathcal{F}|_U$.

Definition A.7. Let $\mathcal{X}$ be an ∞-topos, and $\mathscr{C}$ a presentable stable ∞-category. Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathcal{X};\mathscr{C})$. We will say that $\mathcal{F}$ is constant if it lies in the essential image of a functor $\pi^* : \mathscr{C} \to \mathrm{Shv}(\mathcal{X};\mathscr{C})$ induced by the geometric morphism $\pi^* : \mathcal{S} \to \mathcal{X}$. We will say that $\mathcal{F}$ is locally constant if there exists a small collection of objects $\{U_\alpha \in \mathcal{X}\}_{\alpha\in S}$ such that the following conditions are satisfied:

(i) The objects U_α cover $\mathcal{X}$: that is, there is an effective epimorphism $\coprod U_\alpha \to \mathbf{1}$, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.
(ii) For each $\alpha \in S$, the restriction $\mathcal{F}|_{U_\alpha}$ is a constant object of $\mathrm{Shv}(\mathcal{X}_{/U_\alpha};\mathscr{C})$.

A.2. **Constructible sheaves.**

Definition A.8 ([HA, Definition A.5.1])**.** Let A be a partially ordered set. We will regard A as a topological space, where a subset $U \subseteq A$ is open if it is closed upwards: that is, if $x \le y$ and $x \in U$ implies that $y \in U$. Let X be a topological space. An A-stratification of X is a continuous map $f : X \to A$. Given an A-stratification of a space X and an element $a \in A$, we let X_a, $X_{\le a}$, $X_{<a}$, $X_{\ge a}$, and $X > a$ denote the subsets of X consisting of those points $x \in X$ such that $f(x) = a$, $f(x) \le a$, $f(x) < a$, $f(x) \ge a$, and $f(x) > a$, respectively.

For a given $a \in A$, the subset X_a is called a *stratum*. A collection of such subsets for varying $a \in A$ are called *strata*.

Definition A.9 ([HA, Definition A.5.2])**.** Let A be a partially ordered set and let X be a topological space equipped with an A-stratification. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X;\mathcal{S})$ is A-constructible if, for every element $a \in A$, the restriction $F|_{X_a}$ is a locally constant object of $\mathrm{Shv}(X_a;\mathcal{S})$. Here $\mathcal{F}|_{X_a}$ denotes the image of $\mathcal{F}$ under the left adjoint to the pushforward functor $\mathrm{Shv}(X_a;\mathcal{S}) \to \mathrm{Shv}(X;\mathcal{S})$.

We let $\mathrm{Shv}^A(X;\mathcal{S})$ denote the full subcategory of $\mathrm{Shv}(X;\mathcal{S})$ spanned by A-constructible objects.

A.2.1. *Conically stratified spaces.* To ensure the theory of A-constructible sheaves is well-behaved, Lurie introduces a regularity condition on the stratification $f : X \to A$. Given such an A-stratified space, we may define an associated $A^{\triangleleft}$-stratified space, denoted $C(X)$, called the open cone of X. The definition of $C(X)$ is given in [HA, Definition A.5.3]. The regularity condition on stratifications is formulated using the notion of open cone.

Definition A.10 ([HA, Definition A.5.5])**.** Let A be a partially ordered set, let X be an A-stratified space, and let $x \in X_a \subseteq X$ be a point of X. We will say that X is conically stratified at the point x if there exists an $A_{>a}$-stratified topological space Y, a topological space Z, and an open embedding $Z \times C(Y) \hookrightarrow X$ of A-stratified spaces whose image U_x contains x. Here we regard $Z \times C(Y)$ as endowed with the A-stratification determined by the $A^{\triangleleft}_{>a} \simeq A_{\ge a}$-stratification of $C(Y)$.

We say X is conically stratified if it is conically stratified at every point $x \in X$.

A.2.2. *Whitney stratified spaces.* This are other regularity conditions on stratifications available to us when X is a complex analytic space which comprise the notion of a Whitney stratification.

Definition A.11 (e.g. [Mac90, §7.1.2])**.** Let X be a complex analytic space. Then an A-stratification $f : X \to A$ is said to be a Whitney stratification if

(i) The strata X_a are smooth manifolds.
(ii) The stratification is locally finite.
(iii) The closure of a stratum $\overline{X_a}$ is a union of strata.
(iv) Any two strata satisfy Whitney's conditions[19] A and B.

The precise definition of a Whitney stratification is not terribly relevant for our purposes. It is important, however, to note that a Whitney A-stratified space is conically stratified stratified.

[19] These are easily searchable terms.

Indeed, around each point in a Whitney stratified space, there exist a neighborhood homeomorphic to the product of a contractible space with a cone over a stratified sphere (by, for example, [Mac90, Theorem 7.3]).

A.2.3. *Exodromy equivalence.* In the previous section, we recalled that if a topological space X is locally of singular shape, the locally constant objects of $\mathrm{Shv}(X;\mathcal{S})$ are equivalent to the functor category $\mathrm{Fun}(\mathrm{Sing}(X),\mathcal{S})$. There is a similar description of A-constructible sheaves as functors from a subcategory of $\mathrm{Sing}(X)$.

Definition A.12. Let X be a paracompact topological space, and let $f : X \to A$ be a conical stratification of X. Then $\mathrm{Sing}^A(X)$ is an ∞-category called the *exit-path category* of X with respect to $f : X \to A$.

Theorem A.13 ([HA, Theorem A.9.3])**.** *Let X be a paracompact topological space which is locally of singular shape and is equipped with a conical A-stratification, where A is a partially ordered set satisfying the ascending chain condition.*[20] *Then there is a functor $\Psi_X : \mathrm{Fun}(\mathrm{Sing}^A(X),\mathcal{S}) \to \mathrm{Shv}^A(X;\mathcal{S})$ which induces an equivalence of categories,*

$$\mathrm{Fun}(\mathrm{Sing}^A(X),\mathcal{S}) \xrightarrow{\simeq} \mathrm{Shv}^A(X;\mathcal{S}).$$

The equivalence proven in the above theorem is usually called the (topological) *exodromy equivalence.*

Remark A.14. Any complex analytic space X is paracompact and locally of singular shape, and any Whitney stratification $X \to A$ of it is automatically conical and satisfies the ascending chain condition.

A.2.4. *Coefficients in presentable stable ∞-categories.* Given an arbitrary ∞-category $\mathscr{C}$, the category of A-constructible $\mathscr{C}$-valued sheaves on X is defined in much the same way as above.

Definition A.15. Let X be an A-stratified topological space. An object $\mathcal{F} \in \mathrm{Shv}(X;\mathscr{C})$ is defined to be A-constructible if, for every $a \in A$, the restriction $\mathcal{F}|_{X_a}$ is a locally constant object of $\mathrm{Shv}(X_a;\mathscr{C})$.

We will be interested in the case when $\mathscr{C}$ is a presentable stable ∞-category. In particular, we would like an analogue of Theorem A.13 to hold for $\mathscr{C}$-valued sheaves. Bootstrapping from Theorem A.13 to such an analogue is difficult. In their recent preprint, [PT22], Mauro Porta and Jean-Baptiste Teyssier prove a very general exodromy equivalence for stable presentable coefficients, of which the following is a special case of interest to us.

Theorem A.16 ([PT22, Theorem 5.17])**.** *Let X be a paracompact topological space locally of singular shape, and let $X \to A$ be a conical stratification satisfying the ascending chain condition. Then,*

$$\mathrm{Shv}^A(X;\mathscr{C}) \simeq \mathrm{Fun}(\mathrm{Sing}^A(X),\mathscr{C}).$$

Throughout this work, we will consider A-constructible sheaves valued in certain presentable ∞-categories with a distinguished subcategory of objects. In these cases, we would like to consider only those A-constructible sheaves whose stalk at each point of X belongs to this distinguished subcategory. In order to handle these cases uniformly, we introduce the following formalism, which is a variant of the formalism found in [Nor02].

Definition A.17. A *coefficient pair* $(\mathscr{N},\mathscr{C})$ consists of a stable presentable ∞-category $\mathscr{C}$, and a subcategory $\mathscr{N} \subset \mathscr{C}$ that is stable under finite limits.

[20] We say that a partially ordered set A satisfies the ascending chain condition if every nonempty subset of A has a maximal element.

To each coefficient pair, there are two distinct categories one might consider, with somewhat counter-intuitive names and notation.

Definition A.18. Suppose that X is a complex analytic space and that $(\mathscr{C}, \mathscr{N})$ is a coefficient pair. We denote by $\mathrm{Shv}_{\mathrm{w.c.}}(X;\mathscr{C})$ the full subcategory of $\mathrm{Shv}(X;\mathscr{C})$ spanned by sheaves that are A-constructible with respect to *some* Whitney stratification $X \to A$.

We call objects of $\mathrm{Shv}_{\mathrm{w.c.}}(X;\mathscr{C})$ *weakly constructible sheaves*, and the category itself the category of *weakly constructible $\mathscr{C}$-valued sheaves on X*.

Remark A.19. Though an object of $\mathrm{Shv}_{\mathrm{w.c.}}(X;\mathscr{C})$ is called weakly constructible, it is, by definition, A-constructible for some Whitney stratification $X \to A$.

Definition A.20. We denote by $\mathrm{Shv}_c(X;\mathscr{C})$ the full subcategory of $\mathrm{Shv}_{\mathrm{w.c.}}(X;\mathscr{C})$ spanned by objects $\mathcal{F}$ such that $\mathcal{F}_x \in \mathscr{N}$ for every $x \in X$.

We call objects of $\mathrm{Shv}_c(X;\mathscr{C})$ *constructible sheaves* (without reference to any stratification $X \to A$) with respect to the coefficient pair $(\mathscr{C}, \mathscr{N})$, and the category itself the category of *constructible $\mathscr{C}$-valued sheaves on X with respect to $(\mathscr{C}, \mathscr{N})$*.

We introduce one more category of sheaves, which is not typically considered in the literature.

Definition A.21. We denote by $\mathrm{Shv}_c^{f.s.}(X;\mathscr{C})$ the full subcategory of $\mathrm{Shv}_c(X;\mathscr{C})$ spanned by objects $\mathcal{F}$ which are constructible with respect to some Whitney stratification $X \to A$ where A is a *finite* partially ordered set. We will sometimes call such objects *finitely constructible sheaves.*

A.3. **Constructible sheaves on finite CW complexes.** Let $(\mathscr{C}, \mathscr{N})$ be a coefficient pair, and let X be a topological space. While the stalks of a constructible sheaf $\mathcal{F} \in \mathrm{Shv}_c(X;\mathscr{C})$ lie in $\mathscr{N}$, it is not the case that all sections of $\mathcal{F}$ take values in $\mathscr{N}$. Nonetheless, sections of $\mathcal{F}$ over open subsets $U \subseteq X$ of finite homotopy type (i.e. homotopy equivalent to a finite CW complex) *do* lie in $\mathscr{N}$, as shown in the following proposition.

Proposition A.22. *Suppose that $\mathcal{F} \in \mathrm{Shv}_c(X;\mathscr{C})$ is A-constructible with respect to some Whitney stratification $X \to A$, and that $U \subseteq X$ is an open subset of finite homotopy type such that the induced stratification on U is finite. Then $\mathcal{F}(U) \in \mathscr{N}$.*

Proof. By imitating the proof of [Dim04, Corollary 4.1.8], we see that there exists a filtration $\mathcal{F}_0 \to \cdots \to \mathcal{F}_n = \mathcal{F}|_U$ of length $n = \{\#\,\text{of strata in}\,U\}$ of $\mathcal{F}|_U$ by objects of $\mathrm{Shv}_c(U;\mathscr{C})$, such that the ith associated graded piece is the extension-by-zero of some local system $\mathcal{L}_i$ on the stratum U_i. The object $\mathcal{F}_0$ is locally constant with stalks in $\mathscr{N}$. Assume that $\mathcal{F}_0(U) \in \mathscr{N}$. By induction, it then suffices to show that if $\mathcal{F}_i(U) \in \mathscr{N}$, then $\mathcal{F}_{i+1}(U) \in \mathscr{N}$. But $\mathcal{F}_{i+1}(U)$ is an extension of $\mathcal{F}_i(U)$ by $\mathcal{L}_i(U)$ and $\mathscr{N}$ is closed under extensions, so we win.

Thus, it remains to show that $\mathcal{F}_0(U) \in \mathscr{N}$. This is shown by the classical proof that the singular cohomology of a finite CW complex is bounded with finite dimensional cohomology groups, which generalizes immediately to the setting of local systems with stalks in $\mathscr{N}$. ☐

This observation suggests that, if X has a basis of open subsets of finite homotopy type, it might be possible to produce functors on $\mathrm{Shv}_c(X;\mathscr{C})$ from functors on $\mathscr{N}$. With this in mind, we formulate the following convention.

Convention A.23. Assume that X has finite homotopy type. Let $\mathcal{B}(X)$ denote the partially ordered set associated to a fixed basis of open subsets $B \subset X$ of finite homotopy type which includes X itself and is stable under finite intersections.

Definition A.24. The partially ordered set $\mathcal{B}(X)$ has a natural Grothendieck topology on it. Let $\mathrm{Shv}(\mathcal{B}(X);\mathscr{C})$ denote the ∞-category of $\mathscr{C}$-valued sheaves on $\mathcal{B}(X)$, where $\mathscr{C}$ is an arbitrary ∞-category.

Definition A.25. If $(\mathscr{C}, \mathscr{N})$ is a coefficient pair, we define the ∞-subcategory, $\mathrm{Shv}_c(\mathcal{B}(X); \mathscr{C})$ (resp. $\mathrm{Shv}_{\mathrm{w.c.}}(\mathcal{B}(X); \mathscr{C})$) (resp. $\mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C})$) $\subset \mathrm{Shv}(\mathcal{B}(X); \mathscr{C})$ to be the essential image of $\mathrm{Shv}_c(X; \mathscr{C})$ (resp. $\mathrm{Shv}_{\mathrm{w.c.}}(X; \mathscr{C})$) (resp. $\mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C})$) under the restriction functor $\theta : \mathrm{Shv}(X; \mathscr{C}) \to \mathrm{Shv}(\mathcal{B}(X); \mathscr{C})$.

Let X and $\mathcal{B}(X)$ be as in Convention A.23, and let $(\mathscr{C}, \mathscr{N})$ and $(\mathscr{C}', \mathscr{N}')$ be coefficient pairs. Then any functor $\mathscr{N} \to \mathscr{N}'$ induces a functor on basis sheaves,

$$F_{\mathcal{B}(X)} : \mathrm{Shv}_c(\mathcal{B}(X); \mathscr{C}) \to \mathrm{Shv}_c(\mathcal{B}(X); \mathscr{C}')$$

by pointwise composition.

Proposition A.26. *Let X and $\mathcal{B}(X)$ be as in Convention A.23, and let $\mathscr{C}$ be an arbitrary category admitting limits. Then the canonical restriction map,*

$$\theta : \mathrm{Shv}(X; \mathscr{C}) \to \mathrm{Shv}(\mathcal{B}(X); \mathscr{C})$$

is an equivalence of ∞-categories. If, furthermore, $(\mathscr{C}, \mathscr{N})$ is a coefficient pair, then θ restricts to equivalences,

$$\mathrm{Shv}_c(X; \mathscr{C}) \xrightarrow{\simeq} \mathrm{Shv}_c(\mathcal{B}(X); \mathscr{C}) \tag{A.27}$$

$$\mathrm{Shv}_c^{f.s.}(X; \mathscr{C}) \xrightarrow{\simeq} \mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C}). \tag{A.28}$$

Proof. We begin by showing that the restriction of ∞-topoi,

$$\theta_{\mathcal{S}} : \mathrm{Shv}(X; \mathcal{S}) \to \mathrm{Shv}(\mathcal{B}(X); \mathcal{S}),$$

is an equivalence.

The following argument is based on the discussion right before [HTT, Warning 7.1.1.4]. Observe that the category $\mathcal{B}(X)$ has all finite products by [Stacks, Lemma 002O] since it is closed under fiber products (intersections) and because it has a final object, (X). Moreover, it is a small 0-category. The proof of [HA, Proposition 6.4.5.7] shows that the ∞-topos, $\mathrm{Shv}(\mathcal{B}(X); \mathcal{S})$, is a 0-localic. As such, it is determined by the locale[21] of subobjects of the final object $\underline{*} \in \mathrm{Shv}(\mathcal{B}(X); \mathcal{S})$, which in turn is equivalent to the locale given by the partially ordered set $\mathcal{B}(X)$ itself. For the same reasons, the ∞-topos $\mathrm{Shv}(X; \mathcal{S})$ is determined by the partially ordered set, $\mathcal{U}(X)$, of opens in X. Now observe that $\mathcal{B}(X)$ being a basis means precisely that the inclusion of partially ordered sets, $\mathcal{B}(X) \subset \mathcal{U}(X)$ induces an isomorphism of locales. It follows that the restriction map $\theta_{\mathcal{S}}$ is an equivalence of ∞-topoi.

Now recall [SAG, Proposition I.1.3.1.7], which states that, for a small ∞-category $\mathcal{T}$ equipped with a Grothendieck topology and $\mathscr{E}$ an arbitrary ∞-category admitting limits,

$$\mathrm{Shv}_{\mathscr{E}}(\mathrm{Shv}(\mathcal{T}; \mathcal{S})) \xrightarrow{\simeq} \mathrm{Shv}(\mathcal{T}; \mathscr{E}),$$

Thus, in light of the equivalence $\theta_{\mathcal{S}}$, the restriction map, $\mathrm{Shv}(X; \mathscr{C}) \xrightarrow{\theta'} \mathrm{Shv}(\mathcal{B}(X); \mathscr{C})$, is an equivalence.

Finally, by definition, the restriction of a weakly constructible sheaf X to a sheaf on $\mathcal{B}(X)$ belongs to $\mathrm{Shv}_{\mathrm{w.c.}}(\mathcal{B}(X); \mathscr{C})$ and the restriction of sheaves to $\mathcal{B}(X)$ preserves stalks because $\mathcal{B}(X) \to \mathcal{U}(X)$ is cofinal. It follows that the equivalence θ' restricts to a functor $\theta : \mathrm{Shv}_c(X; \mathscr{C}) \to \mathrm{Shv}_c(\mathcal{B}(X); \mathscr{C})$ which is also an equivalence (e.g. by fully faithfulness and essential surjectivity). $\square$

Proposition A.26 has several immediate corollaries, which we list below.

Corollary A.29. *Let X and $\mathcal{B}(X)$ be as in Convention A.23. Let $(\mathscr{C}, \mathscr{N})$ be a coefficient pair, and $\mathscr{E}$ an arbitrary ∞-category that admits limits. Suppose that $F : \mathscr{N} \to \mathscr{E}$ is an arbitrary functor. Then F induces a functor*

$$F_{\mathrm{Shv}} : \mathrm{Shv}_c^{f.s.}(X; \mathscr{C}) \to \mathrm{Shv}(X; \mathscr{E}).$$

[21] [HTT, Definition 6.4.2.3]

Moreover, if $\mathscr{E}'$ is another ∞-category that admits limits and $G : \mathscr{E} \to \mathscr{E}'$ is another functor, then $(G \circ F)_{\mathrm{Shv}}$ is naturally isomorphic to $G_{\mathrm{Shv}} \circ F_{\mathrm{Shv}}$, where $G_{\mathrm{Shv}} : \mathrm{Shv}(X; \mathscr{E}) \to \mathrm{Shv}(X; \mathscr{E}')$ is the functor given by the sheafification of the $\mathscr{E}'$-valued presheaf given by pointwise composition with G.

Proof. By composition with F, we have an induced functor $F_{\mathcal{B}(X)} : \mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C}) \to \mathcal{P}\mathrm{rShv}(\mathcal{B}(X); \mathscr{E})$. Composition with the sheafification functor $L : \mathcal{P}\mathrm{rShv}(\mathcal{B}(X); \mathscr{E}) \to \mathrm{Shv}(\mathcal{B}(X); \mathscr{E})$ obtains the functor $L \circ F_{\mathcal{B}(X)}$, which we also denote by $F_{\mathcal{B}(X)}$. By Proposition A.26, the restriction functors $\theta_{\mathscr{C}}$ and $\theta_{\mathscr{E}}$ are equivalences. The composition $\theta_{\mathscr{E}}^{-1} \circ F_{\mathcal{B}(X)} \circ \theta_{\mathscr{C}}$ furnishes the promised functor, F_{Shv}. Now let $G_{\mathcal{B}(X)} : \mathrm{Shv}(\mathcal{B}(X); \mathscr{E}) \to \mathcal{P}\mathrm{rShv}(\mathcal{B}(X); \mathscr{E}') \xrightarrow{L} \mathrm{Shv}(\mathcal{B}(X); \mathscr{E}')$ be the functor given by pointwise composition with G, followed by sheafification. Clearly, $(G \circ F)_{\mathcal{B}(X)}$ is given by the composition $G_{\mathcal{B}(X)} \circ F_{\mathcal{B}(X)}$, so we have

$$\begin{aligned}(G \circ F)_{\mathrm{Shv}} &= \theta_{\mathscr{E}'}^{-1} \circ (G \circ F)_{\mathcal{B}(X)} \circ \theta_{\mathscr{C}} \\ &\simeq \theta_{\mathscr{E}'}^{-1} \circ G_{\mathcal{B}(X)} \circ F_{\mathcal{B}(X)} \circ \theta_{\mathscr{C}} \\ &\simeq \theta_{\mathscr{E}'}^{-1} \circ G_{\mathcal{B}(X)} \circ \theta_{\mathscr{E}} \circ \theta_{\mathscr{E}}^{-1} \circ F_{\mathcal{B}(X)} \circ \theta_{\mathscr{C}} \\ &= G_{\mathrm{Shv}} \circ F_{\mathrm{Shv}}.\end{aligned}$$

□

Corollary A.30. *Let X and $\mathcal{B}(X)$ be as in Convention A.23. Suppose that $(\mathscr{C}, \mathscr{N})$ and $(\mathscr{C}', \mathscr{N}')$ are coefficient pairs, and let $F : \mathscr{N} \to \mathscr{N}'$ be a functor that preserves limits. Then F induces a functor,*

$$F_{\mathrm{Shv}} : \mathrm{Shv}_c^{f.s.}(X; \mathscr{C}) \to \mathrm{Shv}_c^{f.s.}(X; \mathscr{C}').$$

Proof. The proof is almost identical to that of Corollary A.29 above. By composition with F, we have an induced functor $F_{\mathcal{B}(X)} : \mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C}) \to \mathrm{Shv}_c^{f.s.}(\mathcal{B}(X); \mathscr{C}')$. By Proposition A.26, the restriction functors $\theta_{\mathscr{C}}$ and $\theta_{\mathscr{C}'}$ are equivalences. The composition $\theta_{\mathscr{C}'}^{-1} \circ F_{\mathcal{B}(X)} \circ \theta_{\mathscr{C}}$ furnishes the promised functor, F_{Shv}. □

Corollary A.31. *Let $(\mathscr{C}, \mathscr{N})$ be a coefficient pair, and let $F : \mathscr{N} \to \mathscr{E}$ be a functor to an ∞-category that admits limits. Let X and Y be two complex analytic spaces with bases $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ as in Convention A.23, and let $f : X \to Y$ be a map of analytic spaces. Then f^* preserves constructibility, and F_{Shv} commutes with f^* in the sense that the following diagram commutes,*

$$\begin{tikzcd} \mathrm{Shv}_c^{f.s.}(Y; \mathscr{C}) \arrow[r, "F_{\mathrm{Shv}}"] \arrow[d, "f^*"] & \mathrm{Shv}(Y; \mathscr{E}) \arrow[d, "f^*"] \\ \mathrm{Shv}_c^{f.s.}(X; \mathscr{C}) \arrow[r, "F_{\mathrm{Shv}}"] & \mathrm{Shv}(X; \mathscr{E}) \end{tikzcd}$$

Proof. Given any finite stratification of $Y \to B$, there exists a finite stratification of $X \to A$ such that $f^{-1}(Y_b)$ is a union of strata in X, for all $b \in B$. It is clear that if $\mathcal{F} \in \mathrm{Shv}(Y; \mathscr{C})$ is B-constructible, then $f^*\mathcal{F}$ is A-constructible. Moreover, f^* preserves stalks. Thus, f^* indeed restricts to a functor $f^* : \mathrm{Shv}_c(Y; \mathscr{C}) \to \mathrm{Shv}_c(X; \mathscr{C})$.

In order to prove that F_{Shv} commutes with f^*, it suffices to show the commutativity of

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{B}(Y); \mathscr{N}) \arrow[r, "F_{\mathcal{B}(Y)}"] \arrow[d, "{f^*}_{\mathscr{N}}"'] & \mathcal{P}\mathrm{rShv}(\mathcal{B}(Y); \mathscr{E}) \arrow[d, "{f^*}_{\mathscr{E}}"'] \\ \mathrm{Shv}(\mathcal{B}(X); \mathscr{N}) \arrow[r, "F_{\mathcal{B}(X)}"] & \mathcal{P}\mathrm{rShv}(\mathcal{B}(X); \mathscr{E}). \end{tikzcd}$$

Both ${f^*}_{\mathscr{N}}$ and ${f^*}_{\mathscr{E}}$ are induced by the underlying geometric morphism of ∞-topoi, $f^* : \mathcal{B}_Y := \mathrm{Shv}(\mathcal{B}(Y); \mathcal{S}) \to \mathrm{Shv}(\mathcal{B}(X); \mathcal{S}) =: \mathcal{B}_X$ via composition with the limit-preserving functor ${f_*}^{\mathrm{op}}$:

$\mathcal{B}_X^{\mathrm{op}} \to \mathcal{B}_Y^{\mathrm{op}}$. As such, the commutativity of the above diagram comes from its identification with the following one, which is canonically commutative:

$$\begin{tikzcd} \mathrm{RFun}(\mathcal{B}_Y^{\mathrm{op}}, \mathscr{N}) \arrow[r, "F\circ-"] \arrow[d, "-\circ f_*{}^{\mathrm{op}}"'] & \mathrm{Fun}(\mathcal{B}_Y^{\mathrm{op}}, \mathscr{E}) \arrow[d, "-\circ f_*{}^{\mathrm{op}}"] \\ \mathrm{RFun}(\mathcal{B}_X^{\mathrm{op}}, \mathscr{N}) \arrow[r, "F\circ-"] & \mathrm{Fun}(\mathcal{B}_X^{\mathrm{op}}, \mathscr{E}). \end{tikzcd}$$

□

Corollary A.32. *Let X and $\mathcal{B}(X)$ be as in Convention A.23. Suppose that $(\mathscr{C}, \mathscr{N})$ and $(\mathscr{C}', \mathscr{N}')$ are coefficient pairs, and that $F, \widetilde{F} : \mathscr{N} \to \mathscr{N}'$ are limit-preserving functors between distinguished subcategories. If F and $\widetilde{F}$ are naturally isomorphic, then $F_{\mathrm{Shv}} \simeq \widetilde{F}_{\mathrm{Shv}}$.*

Proof. Suppose that $\mathcal{N}_{F\to\widetilde{F}} : F \xrightarrow{\simeq} \widetilde{F}$ is a natural isomorphism. Then $\theta_{\mathscr{C}'}^{-1} \circ \mathcal{N}_{F\to\widetilde{F}} \circ \theta_{\mathscr{C}}$ is a natural isomorphism $F_{\mathrm{Shv}} \xrightarrow{\simeq} \widetilde{F}_{\mathrm{Shv}}$. □

A.4. **Constructible sheaves of local systems.** As noted earlier in this work, the pair (Vect, Perf) is a coefficient pair. Moreover, $\big(\mathrm{Fun}(S^1, \mathrm{Vect}), \mathrm{Fun}(S^1, \mathrm{Perf})\big)$ is clearly a coefficient pair, as well.[22] The goal of this section is to produce an equivalence,

$$\mathrm{Fun}(S^1, \mathrm{Shv}_c^{f.s.}(X; \mathbb{C})) \simeq \mathrm{Shv}_c^{f.s.}(X; \mathrm{Fun}(S^1, \mathrm{Vect})). \tag{A.33}$$

Lemma A.34. *Let I be a small category, and suppose that $F : I \to \mathcal{P}\mathrm{r}^R$ is an I-shaped diagram of presentable categories $\mathscr{C}_{i\in I}$ such that its structure morphisms reflect limits. Let $\mathcal{X}$ be an arbitrary ∞-topos. Then there is an equivalence $\mathrm{Shv}(\mathcal{X}; \lim_I \mathscr{C}_i) \xrightarrow{\simeq} \lim_I \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i)$.*

Proof. Let $\mathscr{C}$ be an arbitrary ∞-category. Recall that $\mathrm{Shv}(\mathcal{X}; \mathscr{C})$ is defined as the fully subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathscr{C})$ on functors which preserve limits. Since small limits in $\mathcal{P}\mathrm{r}^R$ exist, all of the structure morphisms $F_i : \lim_I \mathscr{C}_i \to \mathscr{C}_i$ preserve limits, as well. Composition with F_i therefore induces morphisms $\mathrm{Shv}(\mathcal{X}; \lim_I \mathscr{C}_i) \to \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i)$, so we obtain, by the universal property, a functor $\mathrm{Shv}(\mathcal{X}; \lim_I \mathscr{C}_i) \to \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i)$. On the other hand, $\lim_I \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i)$ is a full subcategory of $\mathrm{Fun}(\mathcal{X}; \lim_I \mathscr{C}_i)$.[23] Because the structure morphisms of F reflect limits, functors in the subcategory $\lim_I \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i)$ necessarily preserve limits. Thus, $\lim_I \mathrm{Shv}(\mathcal{X}; \mathscr{C}_i) \subset \mathrm{Shv}(\mathcal{X}; \lim_I \mathscr{C}_i)$. It is clear that this inclusion is inverse to the universal morphism above. □

Lemma A.35. *Let I be a finite category, and suppose that $F : I \to \mathcal{P}\mathrm{r}^R_{st}$ is an I-shaped diagram of presentable stable ∞-categories $\mathscr{C}_{i\in I}$ such that its structure morphisms reflect limits. Let X be a complex analytic space. Then there is an equivalence $\mathrm{Shv}_{\mathrm{w.c.}}(X; \lim_I \mathscr{C}_i) \xrightarrow{\simeq} \lim_I \mathrm{Shv}_{\mathrm{w.c.}}(X; \mathscr{C}_i)$.*

Proof. By Lemma A.34, we have an equivalence, $\mathrm{Shv}(X; \lim_i \mathscr{C}_i) \xrightarrow{\simeq} \lim_I \mathrm{Shv}(X; \mathscr{C}_i)$. It therefore suffices to show that the restriction of this equivalence to $\mathrm{Shv}_{\mathrm{w.c.}}(X; \lim_i \mathscr{C}_i)$ is an equivalence. The essential image of this restriction is obviously contained in $\lim_I \mathrm{Shv}(X; \mathscr{C}_i)$.[24]

It remains to show that an object of $\lim_I \mathrm{Shv}_{\mathrm{w.c.}}(X; \mathscr{C}_i)$ is weakly constructible as an object of $\mathrm{Shv}(X; \lim_I \mathscr{C}_i)$ under the obvious inclusion. For each $i \in I$, the sheaf $\mathcal{F}_i$ is constructible with respect to some Whitney stratification $f_i : X \to A_i$, where A_i is a partially ordered set. Because I is finite, this collection of stratifications admits a common refinement $f : X \to A$ with respect to which $\mathcal{F}_i$ is constructible. Thus, it suffices to show that an object of $\lim_I \mathrm{Shv}^A(X; \mathscr{C}_i)$ is A-constructible under the inclusion $\lim_I \mathrm{Shv}^A(X; \mathscr{C}_i) \hookrightarrow \mathrm{Shv}(X; \lim_I \mathscr{C}_i)$. By Theorem A.16 (i.e. [PT22, Theorem

[22] Note, for example, that $\mathrm{Fun}(S^1, \mathrm{Vect}) \simeq \mathcal{D}(\mathbb{C}[t, t^{-1}])$.

[23] Recall that $\mathrm{Fun}(-,-)$ preserves limits in the second variable.

[24] The composition of a sheaf $\mathcal{U}(X)^{\mathrm{op}} \to \mathscr{C}$, constructible with respect to a particular stratification of X, with a functor $\mathscr{C} \to \mathscr{D}$ is constructible with respect that same stratification.

5.17]), $\mathrm{Shv}^A(X; \lim_I \mathscr{C}_i) \simeq \mathrm{Fun}(\mathrm{Sing}^A(X), \lim_I \mathscr{C}_i)$, where $\mathrm{Sing}^A(X)$ denotes the exit-path category. It follows that

$$\begin{aligned}\lim_I \mathrm{Shv}^A(X; \mathscr{C}_i) &\simeq \lim_I \mathrm{Fun}(\mathrm{Sing}^A(X), \mathscr{C}_i)\\ &\simeq \mathrm{Fun}(\mathrm{Sing}^A(X), \lim_I \mathscr{C}_i)\\ &\simeq \mathrm{Shv}^A(X; \lim_I \mathscr{C}_i)\\ &\subset \mathrm{Shv}(X; \lim_I \mathscr{C}_i).\end{aligned}$$

Moreover, after unraveling the definitions of the functors involved, it is not hard to see that the composition of the chain of functors above is the naturally the embedding $\lim_I \mathrm{Shv}^A(X; \mathscr{C}_i) \hookrightarrow \mathrm{Shv}(X; \lim_I \mathscr{C}_i)$, so we are done. □

Before stating the next lemma, we introduce the following definition.

Definition A.36. Let $(\mathscr{C}, \mathscr{N})$ and $(\mathscr{C}', \mathscr{N}')$ be coefficient pairs. Any functor $F : \mathscr{C} \to \mathscr{C}'$ such that the image of $\mathscr{N}$ is contained in $\mathscr{N}'$ is called a *functor of coefficient pairs.*

Lemma A.37. *Suppose that I is a finite category, and let $F : I \to \mathcal{P}\mathrm{r}^R_{st}$ is an I-shaped diagram such that the structure morphisms of F are functors of coefficient pairs $(\mathscr{C}_i, \mathscr{N}_i)_{i\in I}$. Then the pair $(\lim_I \mathscr{C}_i, \lim_I \mathscr{N}_i)$ is a coefficient pair.*

Proof. Since $\mathcal{P}\mathrm{r}^R_{st}$ admits limits, $\lim_I \mathscr{C}_i$ is a presentable stable ∞-category. It suffices therefore to show that $\lim_I \mathscr{N}_i$ is a subcategory of $\mathscr{C}_i$ which is stable under finite limits and colimits. By universality, we have a functor $\lim_I \mathscr{N}_i \to \lim_I \mathscr{C}_i$. Note that the mapping space between two objects a, b in an arbitrary ∞-category $\mathscr{C}$ is given by $\mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C})} (a, b)$. The functor $\lim_I \mathscr{N}_i \to \lim_I \mathscr{C}_i$ induces a functor $\mathrm{Fun}(\Delta^1, \lim_I \mathscr{N}_i) \to \mathrm{Fun}(\Delta^1, \lim_I \mathscr{C}_i)$ by composition. Since $\mathrm{Fun}(-,-)$ preserves limits in the second variable, we obtain a functor $\lim_I \mathrm{Fun}(\Delta^1, \mathscr{N}_i) \to \lim_I \mathrm{Fun}(\Delta^1, \mathscr{C}_i)$. This functor is the universal one induced by the morphism of I-shaped diagrams whose terms are $\mathrm{Fun}(\Delta^1, \mathscr{N}_i)$ and $\mathrm{Fun}(\Delta^1, \mathscr{C}_i)$, respectively. Since the functors $\mathrm{Fun}(\Delta^1, \mathscr{N}_i) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{N}_i)} (a_i, b_i) \to \mathrm{Fun}(\Delta^1, \mathscr{C}_i) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C}_i)} (a_i, b_i)$ induced by the structure morphisms are equivalences by the assumption that $\mathscr{N}_i \subset \mathscr{C}_i$ is fully faithful, we obtain that

$$\mathrm{Fun}(\Delta^1, \lim_I \mathscr{N}_i) \times_{\mathrm{Fun}(\partial\Delta^1, \lim_I \mathscr{N}_i)} (a, b) \to \mathrm{Fun}(\Delta^1, \lim_I \mathscr{C}_i) \times_{\mathrm{Fun}(\partial\Delta^1, \lim_I \mathscr{C}_i)} (a, b),$$

for $a, b \in \lim_I \mathscr{N}_i$, is an equivalence as well, showing that the universal functor $\lim_I \mathscr{N}_i \to \lim_I \mathscr{C}_i$ is fully faithful, as well. □

Lemma A.38. *Let I be a finite category, and suppose that $F : I \to \mathcal{P}\mathrm{r}^R_{st}$ is an I-shaped diagram of presentable stable ∞-categories $\mathscr{C}_{i\in I}$ with subcategories $\mathscr{N}_i \subset \mathscr{C}_i$ such that $(\mathscr{C}_i, \mathscr{N}_i)$ is a coefficient pair. Assume that the structure morphisms of F are functors of coefficient pairs and reflect limits. Denote by $(\mathscr{C}, \mathscr{N})$ the coefficient pair $(\lim_I \mathscr{C}_i, \lim_I \mathscr{N}_i)$. Then if X is a complex analytic space, there is an equivalence $\mathrm{Shv}_c(X; \mathscr{C}) \xrightarrow{\simeq} \lim_I \mathrm{Shv}_c(X; \mathscr{C}_i)$.*

Corollary A.39. *Let $(\mathscr{C}, \mathscr{N})$ be a coefficient pair. There is an equivalence of categories,*

$$\mathrm{Fun}(S^1, \mathrm{Shv}_c^{f.s.}(X; \mathscr{C})) \simeq \mathrm{Shv}_c^{f.s.}(X; \mathrm{Fun}(S^1, \mathscr{C})),$$

where the latter category denotes constructible sheaves on X with respect to the coefficient pair $(\mathrm{Fun}(S^1, \mathscr{C}), \mathrm{Fun}(S^1, \mathscr{N}))$.

Proof. Recall that $S^1 \simeq \mathrm{colim}_{S^1} *$. Because $\mathrm{Fun}(-,-)$ takes colimits in the first variable to limits, we have $\mathrm{Fun}(S^1, \mathrm{Shv}_c(X; \mathbb{C})) \simeq \lim_{S^1} \mathrm{Shv}_c(X; \mathbb{C})$. On the other hand, we have

$$\begin{aligned}\mathrm{Shv}_c(X; \mathrm{Fun}(S^1, \mathrm{Vect})) &\simeq \mathrm{Shv}_c(X; \lim_{S^1} \mathrm{Vect})\\ &\simeq \lim_{S^1} \mathrm{Shv}_c(X; \mathrm{Vect}),\end{aligned}$$

where we have used Lemma A.38. Restricting the equivalence

$$\mathrm{Shv}_c(X; \mathrm{Fun}(S^1, \mathscr{C})) \xrightarrow{\simeq} \mathrm{Fun}(S^1, \mathrm{Shv}_c(X; \mathscr{C}))$$

obtained in this manner to the subcategory $\mathrm{Shv}_c^{f.s.}(X; \mathrm{Fun}(S^1, \mathrm{Vect}))$ furnishes the desired equivalence. □

The desired equivalence (A.33) now follows from Corollary A.39 by taking the coefficient pair $(\mathscr{C}, \mathscr{N})$ to be $(\mathrm{Vect}, \mathrm{Vect}^b)$.

Appendix B. Derived local systems on $\mathbb{C} \setminus 0$

We gather some results on derived local systems on $\mathbb{C} \setminus 0$, i.e. objects of $\mathcal{D}^b(\mathcal{L}\mathrm{oc}(\mathbb{C} \setminus 0))$.

Definition B.1. Let X be a complex algebraic variety or complex analytic space. We denote by $\mathrm{Loc}(X)$ the full subcategory of $\mathcal{A}(X; \mathbb{C})$ spanned by locally constant sheaves of finite rank.

Remark B.2. If $\dim X = n$, shift the cohomological degree by n gives an equivalence of categories $(-)[n] : \mathrm{Loc}(X) \xrightarrow{\simeq} \mathcal{L}\mathrm{oc}(X)$.

Let $\mathcal{D}^b_{\mathrm{Loc}}(X; \mathbb{C})$ denote the full subcategory of $\mathrm{Shv}_c(X; \mathbb{C})$ spanned by objects whose cohomology sheaves are objects of $\mathrm{Loc}(X)$. The fully faithful embedding $\mathrm{Loc}(X) \hookrightarrow \mathcal{A}(X; \mathbb{C})$ induces a functor $\mathcal{D}^b(\mathrm{Loc}(X)) \to \mathcal{D}^b_{\mathrm{Loc}}(X; \mathbb{C})$.

Lemma B.3. *The functor $\mathcal{D}^b(\mathrm{Loc}(\mathbb{A}^1 \setminus 0)) \to \mathcal{D}^b_{\mathrm{Loc}}(\mathbb{A}^1 \setminus 0; \mathbb{C})$ is a t-exact equivalence of stable ∞-categories.*

Proof. The image of the functor is clearly generated by local systems, so it suffices to show that the functor is fully faithful. In order to show that it is fully faithful, it suffices to show that for any $\mathcal{L} \in \mathrm{Loc}(\mathbb{A}^1 \setminus 0)$, the functor $\mathrm{Ext}^q_{\mathrm{Shv}(\mathbb{A}^1 \setminus 0; \mathbb{C})}(\mathcal{L}, -)|_{\mathrm{Loc}(\mathbb{A}^1 \setminus 0)}$ is effaceable for every $q > 0$. In order to do so, we follow the strategy employed by Nori in his proof of [Nor02, Theorem 3].

We introduce the following notation. Let $\Delta : \mathbb{A}^1 \to \mathbb{A}^2$ be the diagonal map, $\mathrm{pr}_i : \mathbb{A}^2 \to \mathbb{A}^1$ be the canonical projections, and $j : \mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1$ and $i : 0 \hookrightarrow \mathbb{A}^1$ be the canonical inclusions. For the remainder of the proof, all functors are underived unless otherwise indicated. Now let $\mathscr{L} \in \mathrm{Loc}(\mathbb{A}^1 \setminus 0)$, and define $\mathcal{G}$ by the short exact sequence,

$$0 \to \mathcal{G} \to \mathrm{pr}_1^* j_! \mathscr{L} \to \Delta_* j_! \mathscr{L} \to 0. \tag{B.4}$$

The derived functor of pr_{2*} induces a morphism, $j_!\mathscr{L} = \mathrm{pr}_{2*}\Delta_* j_! \mathscr{L} \to \mathcal{F} := R^1\mathrm{pr}_{2*}\mathcal{G}$, which an monomorphism of sheaves by [Nor02, Proposition 2.2]. Applying the exact functor j^* to this morphism, we obtain a morphism,

$$u : \mathscr{L} \simeq j^* j_! \mathscr{L} \to j^* \mathcal{F}.$$

We claim that $j^*\mathcal{F} \in \mathrm{Loc}(\mathbb{A}^1 \setminus 0)$, u is a monomorphism, and that $H^q(\mathbb{A}^1 \setminus 0; j^*\mathcal{F}) = 0$ for all $q \geq 0$.

In order to show that $j^*\mathcal{F}$ is a local system, we use [Nor02, Remark 1.5]. Let $V := \mathrm{pr}_1^{-1}(0) \cup \Delta(\mathbb{A}^1) \subset \mathbb{A}^2$. The restriction $\mathrm{pr}_2|_V$ is clearly a finite surjective morphism. From the definition of $\mathcal{G}$, we see that the restriction of $\mathcal{G}$ to the complement of V is locally constant, and $\mathcal{G}|_V = 0$. We may also express $\mathbb{A}^1$ as the disjoint union of $X_0 = 0$ and $X_1 = \mathbb{A}^1 \setminus 0$, so that $\mathrm{pr}_2 : (\mathrm{pr}_1^{-1} X_i \cap V)_{\mathrm{red}} \to X_i$, for $i = 0, 1$, is a finite étale morphism. Then by *loc. cit.*, $j^*\mathcal{F} = R^1\mathrm{pr}_{2*}\mathcal{G}|_{X_1}$ is locally constant. By [Nor02, Proposition 2.2 (3)], $\mathcal{F}$ is constructible, so its stalks are finite dimensional. Since j^* preserves stalks, it follows that $j^*\mathcal{F}$ has finite dimensional stalks and therefore belongs to $\mathrm{Loc}(\mathbb{A}^1 \setminus 0)$, as desired.

By [Nor02, Proposition 2.2 (1)], the morphism $j_!\mathscr{L} \to \mathcal{F}$ is a monomorphism of sheaves. It follows from the exactness of the functor j^* that u is a monomorphism.

In order to prove that $H^q(\mathbb{A}^1\backslash 0; j^*\mathcal{F}) = 0$ for all $q \geq 0$, we imitate the proof of [Nor02, Proposition 2.2 (2)]. Let $\jmath : \mathbb{A}^2 \setminus \mathrm{pr}_2^{-1}(0) \hookrightarrow \mathbb{A}^2$ be the canonical inclusion. Applying the exact functor $\jmath^*$ to (B.4) and using smooth base change, we obtain another short exact sequence,

$$0 \to \jmath^*\mathcal{G} \to \mathrm{pr}_1'^{\,*}\mathscr{L} \to \Delta'_*\mathscr{L} \to 0, \tag{B.5}$$

where pr_1' denotes $\mathrm{pr}_1|_{\mathbb{A}^2\setminus \mathrm{pr}_2^{-1}(0)}$ and Δ' denotes $\Delta|_{\mathbb{A}^1\setminus 0}$. From the long exact sequence obtained by Applying the derived functor $R\mathrm{pr}'_{1*}$ to (B.5), we obtain the long exact sequence,

$$0 \to \mathrm{pr}'_{1*}\jmath^*\mathcal{G} \to \cdots \to R^q\mathrm{pr}'_{1*}\mathrm{pr}_1'^{\,*}\mathscr{L} \to R^q\mathrm{pr}'_{1*}\Delta'_*\mathscr{L} \to R^{q+1}\mathrm{pr}'_{1*}\jmath^*\mathscr{L} \to \cdots$$

We note, however, that $R\mathrm{pr}'_{1*}\Delta'_* = R(\mathrm{pr}'_1 \circ \Delta')_* = \mathrm{id}_{\mathbb{A}^1\setminus 0_*}$, and that $\mathrm{pr}'_{1*}\mathrm{pr}_1'^{\,*}$ is exact because there exists a homotopical isomorphism over $\mathbb{A}^1$ from $\mathbb{A}^2\setminus \mathrm{pr}_2^{-1}(0)$ to a trivial circle bundle over $\mathbb{A}^1$, whose projection map is clearly proper. Consequently, we see that $R^q\mathrm{pr}_{1*}(\jmath^*\mathcal{G}) = 0$ for all $q > 0$. From the Leray spectral sequence applied to the terminal map, $\mathrm{pt} = \mathrm{pt} \circ \mathrm{pr}'_1$, we see that $R\,\mathrm{pt}_*(\jmath^*\mathcal{G}) = 0$ for all $q > 0$.

On the other hand, by [Nor02, Proposition 1.3A], we have that $R^q\mathrm{pr}'_{2*}(\jmath^*\mathcal{G}) = 0$ for $q > 1$. Another application of the Leray spectral sequence yields an equality, $R^q\,\mathrm{pt}_*\,R^1\mathrm{pr}'_{2*}(\jmath^*\mathcal{G}) = R^{q+1}\,\mathrm{pt}_*(\jmath^*\mathcal{G}) = 0$ for all $q \geq 0$. Now, using [Nor02, Corollary 1.3B], we note that $R^1\mathrm{pr}'_{2*}(\jmath^*\mathcal{G}) \simeq \jmath^*R^1\mathrm{pr}_{2*}\mathcal{G} =: j^*\mathcal{F}$, so we have shown that $H^q(\mathbb{A}^1 \setminus 0; j^*\mathcal{F}) = 0$ for $q \geq 0$ as desired.

Altogether, this shows that the functor $H^q(\mathbb{A}^1 \setminus 0; -)|_{\mathrm{Loc}(\mathbb{A}^1\setminus 0)}$ is effaceable for all $q \geq 0$. By [Nor02, Remark 3.8], we see that the functor $H^q(\mathbb{A}^1 \setminus 0; \mathcal{E}\mathrm{xt}^p(\mathcal{L}, -))|_{\mathrm{Loc}(\mathbb{A}^1\setminus 0)}$ is also effaceable for all $p, q \geq 0$. Because $\mathcal{L}$ is locally constant sheaf with projective (i.e. finite dimensional) stalks, $\mathcal{E}\mathrm{xt}^q(\mathcal{L}, \mathcal{H}) = 0$ for all $q > 0$ and for any sheaf $\mathcal{H}$. Consequently, the local-to-global spectral sequence relating Ext and $\mathcal{E}$xt gives a natural isomorphism,

$$\mathrm{Ext}^q_{\mathrm{Shv}(\mathbb{A}^1\setminus 0;\mathbb{C})}(\mathcal{L}, -)|_{\mathrm{Loc}(\mathbb{A}^1\setminus 0)} \simeq H^q(\mathbb{A}^1 \setminus 0; \mathcal{H}\mathrm{om}(\mathcal{L}, -)|_{\mathrm{Loc}(\mathbb{A}^1\setminus 0)}).$$

The proof is concluded by observing that $\mathcal{H}\mathrm{om}(\mathcal{L}, -) = \mathcal{E}\mathrm{xt}^0(\mathcal{L}, -)$. □

Note that the embedding $\mathcal{L}\mathrm{oc}(\mathbb{A}^1 \setminus 0) \hookrightarrow \mathrm{Perv}(\mathbb{A}^1 \setminus 0)$ similarly induces a functor $\mathcal{D}^b(\mathcal{L}\mathrm{oc}(\mathbb{A}^1 \setminus 0)) \to \mathrm{Shv}_{\mathcal{L}\mathrm{oc}}(\mathbb{A}^1 \setminus 0; \mathbb{C})$,. where now $\mathrm{Shv}_{\mathcal{L}\mathrm{oc}}(\mathbb{A}^1 \setminus 0; \mathbb{C})$ denotes the full subcategory of $\mathrm{Shv}_c(\mathbb{A}^1 \setminus 0; \mathbb{C})$ spanned by objects whose *perverse* cohomology sheaves belong to $\mathcal{L}\mathrm{oc}(\mathbb{A}^1 \setminus 0)$. The following corollary follows easily from Lemma B.3 and Remark B.2 above.

Corollary B.6. *The functor $\mathcal{D}^b(\mathcal{L}\mathrm{oc}(\mathbb{A}^1\setminus 0)) \to \mathcal{D}^b_{\mathcal{L}\mathrm{oc}}(\mathbb{A}^1\setminus 0; \mathbb{C})$ is a t-exact equivalence of stable ∞-categories, where the t-structure on the right-hand category is induced from the perverse t-structure on $\mathrm{Shv}_c(\mathbb{A}^1 \setminus 0; \mathbb{C})$.*

Remark B.7. Of course, if X is a complex algebraic variety, then $\mathcal{L}\mathrm{oc}(X) \simeq \mathcal{L}\mathrm{oc}(X^{\mathrm{an}})$ canonically, so the results above hold *mutatis mutandis* for $\mathbb{C} \setminus 0$ in place of $\mathbb{A}^1 \setminus 0$.

Lemma B.8. *Let X be a complex analytic space. The stable ∞-category $\mathcal{D}^b_{\mathrm{Loc}}(X; \mathbb{C})$ is equivalent to the ∞-category of locally constant* Perf*-valued sheaves, as defined by Lurie in [HA, Appendix A.1].*

Proof. The category of locally constant Perf-valued sheaves is a full subcategory of $\mathcal{D}^b(X; \mathbb{C})$. Moreover, it is clear that the cohomology objects of a locally constant Perf-valued sheaf are local systems, since the pullback of sheaves is left exact. Thus, the category of locally constant Perf-valued sheaves embeds fully faithfully into $\mathcal{D}^b_{\mathrm{Loc}}(X; \mathbb{C})$. It remains to show that this embedding is essentially surjective. Let $\mathcal{F} \in \mathcal{D}^b_{\mathrm{Loc}}(X; \mathbb{C})$, for any point $x \in X$, there exists a neighborhood of x, $U_i(x) \subset X$ such that $(H^i\mathcal{F})|_{U_i(x)}$ is constant. If $H^i\mathcal{F} \simeq 0$, we let $U_i(x)$ be X itself. Let $U(x) := \cap_{i\in\mathbb{Z}} U_i(x)$. Since $H^i\mathcal{F}$ is non-zero for only finitely many $i \in \mathbb{Z}$, $U(x)$ is an open subset of X. Since any object in $V \in \mathrm{Vect}$ is equivalent to the direct such of its cohomologies $\oplus_i H^iV[i]$, it follows that $\mathcal{F}|_{U(x)}$ is constant, as well. Since x was arbitrary, we conclude that $\mathcal{F}$ is locally constant, which concludes the proof. □

Lemma B.9. *Suppose that X is a topological space locally of singular shape. Then $\mathcal{D}^b_{\mathrm{Loc}}(X;\mathbb{C})$ is equivalent as a stable ∞-category to the category of functors* $\mathrm{Fun}(\mathrm{Sing}(X), \mathrm{Perf})$.

Proof. Since X is locally of singular shape, the ∞-topos associated to X, $\mathrm{Shv}_{\mathcal{S}}(X)$, is locally of constant shape, and the shape of X can be identified with the space $\mathrm{Sing}(X)$. By [HA, Theorem A.1.15], the fully ∞-subcategory of locally constant objects of $\mathrm{Shv}_{\mathcal{S}}(X)$ is equivalent to the overcategory $\mathcal{S}_{/\,\mathrm{Sing}(X)}$. Locally constant sheaves valued in Perf are given by Perf-valued sheaves on the ∞-topos $\mathcal{S}_{/\,\mathrm{Sing}(X)}$, i.e. the full subcategory of $\mathrm{Fun}(\mathcal{S}_{/\,\mathrm{Sing}(X)}{}^{\mathrm{op}}, \mathrm{Perf})$ spanned by functors which preserve limits. On the other hand, $\mathcal{S}_{/\,\mathrm{Sing}(X)}$ is generated under small colimits by $\mathrm{Sing}(X)$, so $\mathrm{Shv}_{\mathrm{Perf}}(\mathcal{S}_{/\,\mathrm{Sing}(X)}) \simeq \mathrm{Fun}(\mathrm{Sing}(X), \mathrm{Perf})$. Now use Lemma B.8 to conclude. □

In particular, if X is an Eilenberg-MacLane space, $K(\mathbb{Z}, 1)$, $\mathrm{Fun}(\mathrm{Sing}(X), \mathrm{Perf})$ identifies with $\mathrm{Fun}(S^1, \mathrm{Perf})$, where S^1 here is the space S^1 treated as an ∞-category with a single object. Informally, objects of $\mathrm{Fun}(S^1, \mathrm{Perf})$ are pairs of an object $E \in \mathrm{Perf}$ and an automorphism $T \in \mathrm{End}_{\mathrm{Vect}_{\mathbb{C}}}(E)$.

Remark B.10. Both Lemma B.8 and Lemma B.9 hold with $\mathcal{D}^b_{\mathcal{L}\mathrm{oc}}(X;\mathbb{C})$ in place of $\mathcal{D}^b_{\mathrm{Loc}}(X;\mathbb{C})$.

Appendix C. Filtered objects

C.1. **Filtered objects in abelian categories.** Our presentation in this section and the next cleaves very closely to that found in [GP18], except that we modify some of their definitions to account for the fact that we work exclusively with decreasing filtrations in this work. To this point, the presentation for filtered objects in abelian categories is separate from presentation for filtered objects in stable ∞-categories. Briefly: the former are defined as sequences of monomorphisms in an abelian category, while the latter are defined as the localization of the category of sequences in a stable ∞-category at a certain collection of morphisms. In particular, the maps of a filtration in the stable ∞-categorical setting are not required to be monomorphisms.

C.1.1. Let $\mathcal{J} := \mathbb{Z}^{\mathrm{op}}$, where $\mathbb{Z}$ is the linearly ordered set of integers regarded as an ordinary category. Let $\mathcal{A}$ denote an abelian category.

Definition C.1 ([GP18, Definition 1.2])**.** The category of *sequences in* $\mathcal{A}$ is the functor category $\mathrm{Fun}(\mathcal{J}, \mathcal{A})$, which we denote by $\mathrm{Seq}(\mathcal{A})$.

Definition C.2 ([GP18, Definition 1.3])**.** The *filtered category of* $\mathcal{A}$, denoted $\mathsf{Fil}(\mathcal{A})$ is the full subcategory of $\mathrm{Seq}(\mathcal{A})$ spanned by functors $F : \mathcal{J} \to \mathcal{A}$ satisfying that condition that $F(m \to n)$ is a monomorphism for every $n \leq m$. Given an filtered object $F : \mathcal{J} \to \mathcal{A}$, we will say that F is *a filtration* of $F(\infty) := \mathrm{colim}_{\mathcal{J}} F(n)$.

Thus, the only filtrations we consider here are "exhaustive" filtrations in usual terminology. We will often denote $F(n)$ by F_n and call it the nth filtered piece of F_∞. For each $n \in \mathcal{J}$ and $F \in \mathsf{Fil}(\mathcal{A})$, we let $\mathrm{gr}_n F := \mathrm{coker}\, F(n+1) \to F(n)$. We let $\mathrm{gr} : \mathsf{Fil}(\mathcal{A}) \to \prod_{n\in\mathcal{J}} \mathrm{gr}_n$ denote the *associated graded* functor which sends a filtered object $F \in \mathsf{Fil}(\mathcal{A})$ to the graded object $\prod_{n\in\mathcal{J}} \mathrm{gr}_n F$ in $\mathcal{A}$.

The filtered category $\mathsf{Fil}(\mathcal{A})$, is additive and admits kernels and cokernels, but it is famously not an abelian category in general. Though derived categories of such categories can still be considered (see e.g. [SS16]), it is more natural in this context to consider the so-called "filtered" derived category. Let $\mathrm{Ch}(\mathcal{A})$ denote the abelian category of unbounded chain complexes in $\mathcal{A}$.

Definition C.3 ([GP18, Definition 1.7])**.** The *filtered derived category* of $\mathcal{A}$, denoted $D^{\mathsf{fil}}(\mathcal{A})$, is the localization of $\mathrm{Ch}(\mathsf{Fil}(\mathcal{A}))$ ($\simeq \mathsf{Fil}(\mathrm{Ch}(\mathcal{A}))$) with respect to the collection filtered weak equivalences, which are maps of sequences $f : F \to G$ such that $\mathrm{gr}(f) : \mathrm{gr}\, F \to \mathrm{gr}\, G$ is an indexwise quasi-isomorphism.

C.2. **Filtered objects in stable ∞-categories.** By abuse of notation, we let $\mathcal{J} := \mathbb{Z}^{\mathrm{op}}$, where $\mathbb{Z}$ is the linearly ordered set of integers regarded as an ∞-category.

Definition C.4 (c.f. [GP18, §2.2])**.** Let $\mathscr{C}$ be a stable ∞-category. We define $\mathrm{Seq}(\mathscr{C})$ to be the functor category $\mathrm{Fun}(\mathcal{J}, \mathscr{C})$. An object of $\mathrm{Seq}(\mathscr{C})$ is called a *sequence* in $\mathscr{C}$.

To each $n \in \mathcal{J}$, there is a functor $\mathrm{ev}_n : \mathrm{Seq}(\mathscr{C}) \to \mathscr{C}$ given by evaluation at n. For $F \in \mathrm{Seq}(\mathscr{C})$, we denote the composition $\mathrm{ev}_n \circ F$ either by F_n or $F(n)$. For each $n \in \mathcal{J}$ and $F \in \mathrm{Seq}(\mathscr{C})$, we let $\mathrm{gr}^n F$ denote the cofiber of the map $F_{n+1} \to F_n$.

Definition C.5 ([GP18, Definition 2.1])**.** The *associated graded functor* $\mathrm{gr} : \mathrm{Seq}(\mathscr{C}) \to \prod_{\mathbb{Z}} \mathscr{C}$ is the functor sending a sequence F to the graded object $\{\mathrm{gr}^n F\}_{n \in \mathbb{Z}}$. We denote by $\mathrm{pr}_n : \prod_{\mathbb{Z}} \mathscr{C} \to \mathscr{C}$ the canonical projection onto the nth factor. Clearly, $\mathrm{gr}^n \simeq \mathrm{pr}_n \circ \mathrm{gr}^\bullet$.

We say that a morphism of sequences, $f : F \to G \in \mathrm{Mor}(\mathrm{Seq}(\mathscr{C}))$, is a graded equivalence if $\mathrm{gr}(f)$ is an equivalence in $\mathscr{C}$. Let $\mathcal{W}_{\mathrm{gr}}$ denote the collection of graded equivalences in $\mathrm{Seq}(\mathscr{C})$.

Definition C.6 ([GP18, Definition 2.3])**.** Let $\mathscr{C}$ be a stable ∞-category. The *filtered ∞-category* of $\mathscr{C}$ is defined to be the localization $\mathrm{Seq}(\mathscr{C})[\mathcal{W}_{\mathrm{gr}}^{-1}]$, which we denote by $\mathsf{Fil}(\mathscr{C})$. We will say that an object $F \in \mathsf{Fil}(\mathscr{C})$ is a filtration of the object $\lim_{\mathcal{J}} F \in \mathscr{C}$.

As shown in [GP18, Lemma 2.15], if it exists, $\mathsf{Fil}(\mathscr{C})$ may be identified with the full subcategory of $\mathcal{W}_{\mathrm{gr}}$-local objects in $\mathrm{Seq}(\mathscr{C})$, meaning that any functor on $\mathrm{Seq}(\mathscr{C})$ restricts to a functor on $\mathsf{Fil}(\mathscr{C})$. In particular, we if a sequence F lies in $\mathsf{Fil}(\mathscr{C})$, we call F_n or $F(n)$, the *nth filtered piece* of F.

We have reason in this work to consider sequences whose morphisms are monomorphisms,[25]like in the case of an abelian category.

Definition C.7. We denote by $\mathsf{Fil}^{\mathrm{mon.}}(\mathscr{C})$ the full subcategory of $\mathsf{Fil}(\mathscr{C})$ spanned by sequences whose structure morphisms are monomorphisms.

The following two theorems of Gwilliam–Pavlov guarantee the existence of the localization at $\mathcal{W}_{\mathrm{gr}}$ for a large class of stable ∞-categories, and illuminate the connection of their construction with the classical filtered derived category.

Theorem C.8 ([GP18, Theorem 2.5])**.** *If $\mathscr{C}$ is a stable ∞-category that admits sequential limits, then $\mathsf{Fil}(\mathscr{C})$ exists and is a stable ∞-category.*

Theorem C.9 ([GP18, Theorem 2.6])**.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then the homotopy category of $\mathsf{Fil}(\mathcal{D}(\mathcal{A}))$ is equivalent to the classical filtered derived category $D^{\mathsf{fil}}(\mathcal{A})$.*

With this in mind, we called $\mathsf{Fil}(\mathcal{D}(\mathcal{A}))$ the filtered derived *∞-category* of $\mathcal{A}$.

C.2.1. *Filtered categories of presentable categories.* Suppose that $\mathscr{C}$ is additionally presentable. Then $\mathrm{Seq}(\mathscr{C})$ is a presentable as well, by [HTT, Proposition 5.5.3.6]. Moreover, $\mathsf{Fil}(\mathscr{C})$ is an accessible left exact localization of $\mathrm{Seq}(\mathscr{C})$ by [GP18, Proposition 2.14] by the parenthetical remark contained in the proof of [GP18, Theorem 3.9]. Altogether, this shows the following lemma.

Lemma C.10. *Suppose that $\mathscr{C}$ is a presentable stable ∞-category. Then $\mathsf{Fil}(\mathscr{C})$ is also a presentable stable ∞-category.*

[25]In the sense of [HTT, §5.5.6]. Taken from *loc. cit.*: a monomorphism is a morphism $f : C \to D$ which is (-1)-truncated as an object of the mapping space; this is equivalent to the assertion that the functor $\mathscr{C}_{/f} \to \mathscr{C}_{/D}$ is fully faithful.

C.2.2. *Symmetric monoidal structure on filtered categories.* It turns out that if $\mathscr{C}$ is a presentable, closed symmetric monoidal stable ∞-category, whose monoidal product we denote by $\otimes$, then $\mathrm{Seq}(\mathscr{C})$ is as well under the Day convolution product, which we denote by $\star$. Concretely, given $F, G \in \mathrm{Seq}(\mathscr{C})$, the nth pieces of the Day convolution is given by

$$F \star G(n) \simeq \operatorname*{colim}_{p+q \geq n} F(p) \otimes G(q).$$

We direct the reader to §2.23 of [GP18] for more details.

The symmetric monoidal structure on $\mathrm{Seq}(\mathscr{C})$ induces a closed symmetric monoidal structure on $\mathsf{Fil}(\mathscr{C})$ by completion. Let $\mathrm{oblv} : \mathsf{Fil}(\mathscr{C}) \to \mathrm{Seq}(\mathscr{C})$ denote the forgetful functor. This has a left adjoint called the *completion* functor which we denote by comp, which is the homotopical counterpart of the classical completion of filtrations. We recall the following definition/theorem.

Definition C.11 ([GP18, Theorem 2.25] and)**.** The filtered ∞-category $\mathsf{Fil}(\mathscr{C})$ is closed symmetric monoidal ∞-category under the *completed convolution product* $\widehat{\star}$, defined by

$$F \mathbin{\widehat{\star}} G := \mathrm{comp}(\mathrm{oblv}(F) \star \mathrm{oblv}(G)),$$

for any $F, G \in \mathsf{Fil}(\mathscr{C})$.

Moreover, the associated graded functor is strong monoidal by [GP18, Proposition 2.26], intertwining the completed convolution product on $\mathsf{Fil}(\mathscr{C})$ with the Day convolution product on $\prod_{n \in \mathcal{J}} \mathscr{C}$, which we denote by $\star_{\mathrm{gr}}$.

C.3. **Modules over a filtered ring.** As an application of the formalism of Gwilliam–Pavlov, we consider the case of filtered modules over a filtered ring. We first state the definition of a filtered algebra in the higher categorical context.

Definition C.12. Let $\mathcal{O}^{\otimes}$ be an ∞-operad in the sense of [HA], and let $\mathscr{C}^{\otimes}$ be a symmetric monoidal presentable stable ∞-category. A *filtered $\mathcal{O}$-algebra of $\mathscr{C}$* is a map of ∞-operads, $\mathcal{O}^{\otimes} \to \mathsf{Fil}(\mathscr{C})^{\otimes}$, where the ∞-operad $\mathsf{Fil}(\mathscr{C})^{\otimes}$ is given by Day convolution.

Remark C.13. If $\mathcal{O}^{\otimes}$ is the associative operad Assoc (i.e. $\mathbb{E}_1^{\otimes}$), we will say that the map $\mathcal{O}^{\otimes} \to \mathsf{Fil}(\mathscr{C})^{\otimes}$ is a filtered associative algebra in $\mathscr{C}$.

Interlude C.14. At this point, we recall the "left module" ∞-operad $\mathcal{LM}^{\otimes}$ introduced by Lurie in §4.2.1 of [HA]. In what follows, we try to use the notation of that section. One of the basic features of $\mathcal{LM}^{\otimes}$ is that is contains $\mathrm{Assoc}^{\otimes}$ as a subcategory and also admits a map of ∞-operads $\mathcal{LM}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. Let $\mathscr{C}$ denote a monoidal ∞-category, given by a fibration of ∞-operads $\mathscr{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. The category $\mathrm{LMod}(\mathscr{C})$ is defined to be the category $\mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathscr{C})$. An object $M \in \mathrm{LMod}(\mathscr{C})$ is given by a functor $M : \mathcal{LM}^{\otimes} \to \mathscr{C}^{\otimes}$, and the restriction $M|_{\mathrm{Assoc}^{\otimes}}$ is an associative algebra of $\mathscr{C}$. There is a categorical fibration $\mathrm{LMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$, and the category of left modules over a given algebra $A \in \mathrm{Alg}(\mathscr{C})$ is defined to be the fiber product $\mathrm{LMod}(\mathscr{C}) \times_{\mathrm{Alg}(\mathscr{C})} \{A\}$, which we denote by $\mathrm{LMod}_A(\mathscr{C})$. We call $\mathrm{LMod}_A(\mathscr{C})$ *the category of left A-modules in $\mathscr{C}$.*

Suppose that $R^{\bullet}$ is a classical filtered associative $\mathbb{C}$-algebra in the sense that $R^i \cdot R^j \subset R^{i+j}$. Then R can be considered as a filtered associative algebra in the category of spectra, $\mathsf{Fil}(\mathrm{Sp})$. As such, we may consider its category of modules, $\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp}))$. On the other hand, since $R^{\bullet}$ is classical, we may consider module in $\mathsf{Fil}(\mathrm{Ab})$ over it.

Proposition C.15. *There is an embedding,*

$$\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Ab})) \hookrightarrow \mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp})),$$

which preserves filtered colimits and sends exact sequences in $\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Ab}))$ *to fiber sequences in* $\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp}))$.

Proof. We identify Ab with the heart of the canonical t-structure on Sp and let $i : \mathrm{Ab} \to \mathrm{Sp}$ denote its inclusion. We note that this inclusion is lax symmetric monoidal. Composition with i induces a functor on the category of sequences, $\mathrm{Seq}(\mathrm{Ab}) \to \mathrm{Seq}(\mathrm{Sp})$. Restriction of this functor to the subcategory $\mathsf{Fil}(\mathrm{Ab}) \subset \mathrm{Seq}(\mathrm{Ab})$ followed by composition with the localization functor $\mathrm{Seq}(\mathrm{Sp}) \to \mathsf{Fil}(\mathrm{Sp})$ obtains a functor,

$$\mathsf{Fil}(\mathrm{Ab}) \to \mathsf{Fil}(\mathrm{Sp}).$$

Since i is lax symmetric monoidal, so is the functor $\mathsf{Fil}(\mathrm{Ab}) \to \mathsf{Fil}(\mathrm{Sp})$ with respect to the symmetric monoidal structures on each induced by Day convolution. As noted above in the special case of R, if $\mathrm{Assoc}^{\otimes} \to \mathsf{Fil}(\mathrm{Ab})^{\otimes}$ is a filtered associative algebra in Ab, we obtain a filtered associative algebra in Sp by composition with the above functor. In this case, we obtain a commutative diagram of ∞-operads,

$$\begin{tikzcd} & \mathsf{Fil}(\mathrm{Sp})^{\otimes} \\ \mathrm{Assoc}^{\otimes} \arrow[ur] \arrow[r] & \mathsf{Fil}(\mathrm{Ab})^{\otimes}. \arrow[u] \end{tikzcd}$$

Now, by the general formalism of ∞-operads, we obtain a functor of left module categories,

$$\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Ab})) \to \mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp})),$$

given heuristically by taking a left R-module, $\mathcal{LM}^{\otimes} \to \mathsf{Fil}(\mathrm{Ab}))^{\otimes}$ and composing it with the map of ∞-operads $\mathsf{Fil}(\mathrm{Ab})^{\otimes} \to \mathsf{Fil}(\mathrm{Sp})^{\otimes}$. It remains to show that the functor $\mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Ab})) \to \mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp}))$ preserves limits and take exact sequences to fiber sequences. This follows immediately from the corresponding statement for i. □

Remark C.16. Given an integer $k \in \mathbb{Z}$, the evaluation functor ev_k lifts to a functor $\mathrm{ev}_k : \mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp})) \to \mathrm{Mod}_{R^0}(\mathrm{Sp})$, which we denote by the same.

Remark C.17. Similarly, the associated graded functor $\mathrm{gr}^{\bullet} : \mathsf{Fil}(\mathrm{Sp}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$ lifts to a functor $\mathrm{gr}^{\bullet} : \mathrm{Mod}_R(\mathsf{Fil}(\mathrm{Sp})) \to \prod_{\mathbb{Z}} \mathrm{Mod}_{R^0}$, which we denote by the same.

C.4. **Sheaves valued in the filtered category.** We end this appendix with a useful proposition, which may be known, but is not present in [GP18].

Proposition C.18. *Let $\mathfrak{X}$ be an ∞-topos. Given a stable ∞-category $\mathscr{C}$ that admits sequential limits, there is an equivalence of categories,*

$$\mathrm{Shv}(\mathfrak{X}; \mathrm{Seq}(\mathscr{C})) \xrightarrow{\simeq} \mathrm{Seq}(\mathrm{Shv}(\mathfrak{X}; \mathscr{C})).$$

Proof. Recall that $\mathrm{Shv}(\mathfrak{X}; \mathrm{Seq}(\mathscr{C}))$ is the full subcategory of $\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathrm{Seq}(\mathscr{C}))$ that preserves limits. Since $\mathrm{Seq}(\mathscr{C}) := \mathrm{Fun}(\mathfrak{J}, \mathscr{C})$, we obtain by adjunction an equivalence

$$\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathrm{Seq}(\mathscr{C})) \xrightarrow[\simeq]{\mathcal{A}d} \mathrm{Fun}(\mathfrak{J}, \mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathscr{C}))$$

Note that, under $\mathcal{A}d$, an object $F \in \mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathrm{Seq}(\mathscr{C}))$ corresponds to a sequence in $\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathscr{C})$ whose nth piece is the functor, $\mathrm{ev}_n \circ F$. We claim that $\mathcal{A}d$ sends limit-preserving functors in $\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathrm{Seq}(\mathscr{C}))$ to sequences in $\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathscr{C})$ whose pieces are limit-preserving functors.

Suppose we are given a small diagram $D : I \to \mathfrak{X}^{\mathrm{op}}$, and suppose that $F \in \mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}, \mathrm{Seq}(\mathscr{C}))$ preserves the limit over D. Composition with D gives a diagram of sequences $F \circ D \in \mathrm{Fun}(I, \mathrm{Seq}(\mathscr{C}))$. Since limits in functor categories are computed pointwise, $\lim_I(F \circ D)$ is equivalent to the functor $\mathfrak{J} \to \mathscr{C}$ given by $n \mapsto \lim_I(\mathrm{ev}_n \circ F \circ D)$. On the other hand, because F is assumed to preserve the limit over D, $\lim_I(F\circ D)\simeq F(\lim_I D)$. Altogether, we have

$$\begin{aligned}\mathcal{A}d(F)(\lim D)_n &\simeq (\mathrm{ev}_n\circ F)(\lim D)\\ &\simeq \mathrm{ev}_n(F(\lim D))\\ &\simeq \mathrm{ev}_n(\lim(F\circ D))\\ &\simeq \lim(\mathrm{ev}_n\circ F\circ D)\\ &\simeq \lim(\mathcal{A}d(F)_n\circ D).\end{aligned}$$

Since D was an arbitrary small diagram, the lemma follows. □

The following corollary follows easily from the proof of the above proposition by restricting to the subcategory of limit-preserving functors $\mathcal{X}^{\mathrm{op}}\to \mathrm{Seq}(\mathscr{C})$ which factor through the fully faithful embedding $i:\mathsf{Fil}(\mathscr{C})\hookrightarrow \mathrm{Seq}(\mathscr{C})$.

Corollary C.19. *Let $\mathcal{X}$ be an ∞-topos. Given a stable ∞-category $\mathscr{C}$ that admits sequential limits, the equivalence established in Proposition C.18 restricts to a fully faithful functor,*

$$\imath:\mathrm{Shv}(\mathcal{X};\mathsf{Fil}(\mathscr{C}))\to \mathsf{Fil}(\mathrm{Shv}(\mathcal{X};\mathscr{C})).$$

Proof. Note that, since $i:\mathsf{Fil}(\mathscr{C})\hookrightarrow \mathrm{Seq}(\mathscr{C})$ preserves limits, composition with i induces a functor $\mathrm{Shv}(\mathcal{X};\mathsf{Fil}(\mathscr{C}))\to \mathrm{Shv}(\mathcal{X};\mathrm{Seq}(\mathscr{C}))$. It is clear that this functor is fully faithful, and by Proposition C.18, we obtain a fully faithful functor embedding $\mathrm{Shv}(\mathcal{X};\mathsf{Fil}(\mathscr{C}))\to \mathrm{Seq}(\mathrm{Shv}(\mathcal{X};\mathscr{C}))$. It remains to show that this functor sends equivalences in $\mathrm{Shv}(\mathcal{X};\mathsf{Fil}(\mathscr{C}))$ to graded equivalences in $\mathrm{Seq}(\mathrm{Shv}(\mathcal{X};\mathscr{C}))$. But this follows from the corresponding statement for the embedding $i:\mathrm{Shv}(X;\mathsf{Fil}(\mathscr{C}))\to \mathrm{Shv}(X;\mathrm{Seq}(\mathscr{C}))$ and the following commutative diagram of functor categories,

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathrm{Seq}(\mathscr{C})) \arrow[r, "\mathcal{A}d"] \arrow[d, "\mathrm{gr}^\bullet\circ-"] & \mathrm{Fun}(\mathcal{J},\mathrm{Fun}(\mathcal{X}^{\mathrm{op}},\mathscr{C})) \arrow[d, "\mathrm{gr}^\bullet"] \\ \mathrm{Fun}(\mathcal{X}^{\mathrm{op}};\prod_{\mathbb{Z}}\mathscr{C}) \arrow[r, "\simeq"] & \prod_{\mathbb{Z}}\mathrm{Fun}(\mathcal{X}^{\mathrm{op}};\mathscr{C}). \end{tikzcd}$$

□

References

[Bei87] A. A. Beilinson. "On the derived category of perverse sheaves". In: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*. Vol. 1289. Lecture Notes in Math. Springer, Berlin, 1987, pp. 27–41. DOI: 10.1007/BFb0078365.

[Bry86] Jean-Luc Brylinski. "Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques". In: 140-141. Géométrie et analyse microlocales. 1986, pp. 3–134, 251.

[Cis] D.-C. Cisinski. *Homotopy types of schemes*. MathOverflow. (version: 2015-10-12). eprint: https://mathoverflow.net/q/220653. URL: https://mathoverflow.net/q/220653.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970, pp. iii+133.

[Dim04] Alexandru Dimca. *Sheaves in Topology*. Springer-Verlag Berlin Heidelberg, 2004. ISBN: 978-3-642-18868-8.

[DS93] Alexandru Dimca and Morihiko Saito. "On the cohomology of a general fiber of a polynomial map". In: *Compositio Math.* 85.3 (1993), pp. 299–309. ISSN: 0010-437X.

[Gar04] Ricardo Garcia Lopez. "Microlocalization and stationary phase". In: *Asian J. Math.* 8.4 (2004), pp. 747–768. ISSN: 1093-6106.

[GP18] Owen Gwilliam and Dmitri Pavlov. "Enhancing the filtered derived category". In: *J. Pure Appl. Algebra* 222.11 (2018), pp. 3621–3674. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2018.01.004.

[HA] Jacob Lurie. *Higher Algebra*. URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[HTT] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: `10.1515/9781400830558`.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*. Vol. 236. Progress in Mathematics. Translated from the 1995 Japanese edition by Takeuchi. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. xii+407. ISBN: 978-0-8176-4363-8. DOI: `10.1007/978-0-8176-4523-6`.

[IT20] Yohei Ito and Kiyoshi Takeuchi. "On some topological properties of Fourier transforms of regular holonomic D-modules". In: *Canad. Math. Bull.* 63.2 (2020), pp. 454–468. ISSN: 0008-4395. DOI: `10.4153/s0008439519000559`.

[Kas03] Masaki Kashiwara. *D-modules and microlocal calculus*. Vol. 217. Translations of Mathematical Monographs. Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+254. ISBN: 0-8218-2766-9. DOI: `10.1090/mmono/217`.

[Kas83] M. Kashiwara. "Vanishing cycle sheaves and holonomic systems of differential equations". In: *Algebraic geometry (Tokyo/Kyoto, 1982)*. Vol. 1016. Lecture Notes in Math. Springer, Berlin, 1983, pp. 134–142. DOI: `10.1007/BFb0099962`.

[Kas84] Masaki Kashiwara. "The Riemann-Hilbert problem for holonomic systems". In: *Publ. Res. Inst. Math. Sci.* 20.2 (1984), pp. 319–365. ISSN: 0034-5318. DOI: `10.2977/prims/1195181610`.

[KK81] Masaki Kashiwara and Takahiro Kawai. "On holonomic systems of microdifferential equations. III. Systems with regular singularities". In: *Publ. Res. Inst. Math. Sci.* 17.3 (1981), pp. 813–979. ISSN: 0034-5318. DOI: `10.2977/prims/1195184396`.

[KS11] Maxim Kontsevich and Yan Soibelman. "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants". In: *Commun. Number Theory Phys.* 5.2 (2011), pp. 231–352. ISSN: 1931-4523. DOI: `10.4310/CNTP.2011.v5.n2.a1`.

[Lur17] Jacob Lurie (https://mathoverflow.net/users/7721/jacob-lurie). *The universal property of the unseparated derived category*. MathOverflow. (version: 2017-10-07). URL: `https://mathoverflow.net/q/282893`.

[Mac90] Robert MacPherson. *Intersection Homology and Perverse Sheaves*. Dec. 1990. URL: `https://www.math.ru.nl/~bmoonen/Seminars/IntersectionHomology.pdf`.

[Mal91] B. Malgrange. *Équations différentielles à coefficients polynomiaux*. Vol. 96. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1991, pp. vi+232. ISBN: 0-8176-3556-4.

[Nor02] Madhav V. Nori. "Constructible sheaves". In: *Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000)*. Vol. 16. Tata Inst. Fund. Res. Stud. Math. Tata Inst. Fund. Res., Bombay, 2002, pp. 471–491.

[PT22] Mauro Porta and Jean-Baptiste Teyssier. *Topological exodromy with coefficients*. 2022. DOI: `10.48550/ARXIV.2211.05004`. URL: `https://arxiv.org/abs/2211.05004`.

[Sab07] Claude Sabbah. *Isomonodromic deformations and Frobenius manifolds*. French. Universitext. An introduction. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2007, pp. xiv+279. ISBN: 978-1-84800-053-7; 978-2-7598-0047-6.

[Sab08] Claude Sabbah. "An explicit stationary phase formula for the local formal Fourier-Laplace transform". In: *Singularities I*. Vol. 474. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 309–330. DOI: `10.1090/conm/474/09262`.

[Sab10] Claude Sabbah. *On a twisted de Rham complex, II*. 2010. DOI: `10.48550/ARXIV.1012.3818`. URL: `https://arxiv.org/abs/1012.3818`.

[SAG] Jacob Lurie. *Spectral Algebraic Geometry*. URL: http://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[Sch85] Pierre Schapira. *Microdifferential systems in the complex domain*. Vol. 269. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985, pp. x+214. ISBN: 3-540-13672-X. DOI: 10.1007/978-3-642-61665-5.

[Ser56] Jean-Pierre Serre. “Géométrie algébrique et géométrie analytique”. In: *Ann. Inst. Fourier (Grenoble)* 6 (1955/56), pp. 1–42. ISSN: 0373-0956.

[SS14] Claude Sabbah and Morihiko Saito. “Kontsevich's conjecture on an algebraic formula for vanishing cycles of local systems”. In: *Algebr. Geom.* 1.1 (2014), pp. 107–130. ISSN: 2313-1691. DOI: 10.14231/AG-2014-006.

[SS16] Pierre Schapira and Jean-Pierre Schneiders. “Derived categories of filtered objects”. In: *Astérisque* 383 (2016), pp. 103–120. ISSN: 0303-1179.

[Stacks] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

[Vol21] Marco Volpe. *Six functor formalism for sheaves with non-presentable coefficients*. 2021. DOI: 10.48550/ARXIV.2110.10212. URL: https://arxiv.org/abs/2110.10212.