Solutions for $2^a + 2^{-a} = k$

We are presented with the algebraic expression: $2^a + 2^{-a} = 4$, and asked to solve for *a*. This we can do as follows:

Let $x = 2^a$, then…

$$x - 4 + 1/x = 0$$

so…

$$x^2 - 4x + 1 = 0$$

and solving for roots of quadratic…

$$x = \frac{4 \pm \sqrt{16-4}}{2}$$

so...

$$x = 2 \pm \sqrt{3}$$

since...

$$a = \log x / \log 2$$

yields…

$$a \approx \pm 1.9$$

* * *

Now we could repeat this exercise for various values for the constant on the right hand side other than 4, which is to say for: $2^a + 2^{-a} = k$. Using the same approach as above:

$$x = \frac{k \pm \sqrt{k^2-4}}{2} \qquad (A)$$

which will yield us real positive roots for all values of $k >= 2$. Some examples follow.

k	x	~x	~a
5	½(5±√21)	4.791, 0.209	2.260, -2.260
4.5	½(4.5±√16.25)	4.266, 0.234	2.093, -2.093
4	2±√3	3.732, 0.268	1.900, -1.900
3.5	½(3.5±√8.25)	3.186, 0.314	1.672, -1.672
3	½(3±√5)	2.618, 0.382	1.388, -1.388
2.5	½(2.5±√2.25)	2, 0.5	1, -1
2	1±0	1, 1	0, 0

* * *

Next let us consider values for k between 2 and -2. Here, in the equation $^{(A)}$ above for x the discriminant assumes a negative value – clearly solutions will involve complex values.

roots of quadratic…

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

or...

$$x = \frac{k \pm \sqrt{4 - k^2}\, i}{2}$$

Now it can be shown that the logarithm of a complex number, when expressed in the polar form $r \cdot cis(\theta)$, is equal to: $\log(r) + i\theta$.

given...

$$x = \frac{k}{2} \pm \frac{\sqrt{4 - k^2}\, i}{2}$$

polarising…

$$r^2 = \frac{k^2}{4} + \frac{4 - k^2}{4} \quad ==> \quad r = 1$$

$$\theta = \pm atan \frac{\sqrt{4 - k^2}}{k}$$

and because...

$$a = \log x / \log 2$$

we have…

$$a = \frac{\log 1 \pm i\theta}{\log 2} \quad ==> \quad a = \frac{\pm i\theta}{\log 2}$$

Consequently, the solution set for a in this domain is always a pure imagenary number. E.g.:

k	*x.real*	*x.imag*	*a.real*	*a.imag*
1.999'	~ 1	0	0	~ 0
1.5	0.75	0.661	0	±1.043
1	0.5	0.866	0	±1.511
0.5	0.25	0.968	0	±1.902
0	0	1	0	±2.266
-0.5	-0.25	-0.968	0	±2.631
-1	-0.5	0.866	0	±3.022
-1.5	-0.75	0.661	0	±3.490
-1.999'	~ -1	0	0	~ ±4.532†

† that is π/log 2

* * *

2^A Solutions

It remains to ask what happens when k < -2. Here the equation [(A)] above will always give a negative real result for x. However, this can be treated as a complex number with an imaginary component equal to zero; then we can use the same method as in the previous section to find our roots.

Or, using the Euler identity $e^{\pi i} = -1$ (thus log -1 = πi) we can reason:

roots of quadratic…

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2} \quad ==> \quad x < 0 \ \text{ where } k < -2$$

and...

$$\log x = \log (|x| \cdot -1) = \log |x| + \pi i$$

and because...

$$a = \log x / \log 2$$

we have…

$$a = \frac{\log|x| + \pi i}{\log 2} \quad ==> \quad a = \frac{\log|x|}{\log 2} + \sim 4.532 i$$

Tabulating results for selected values of $k \leq -2$:

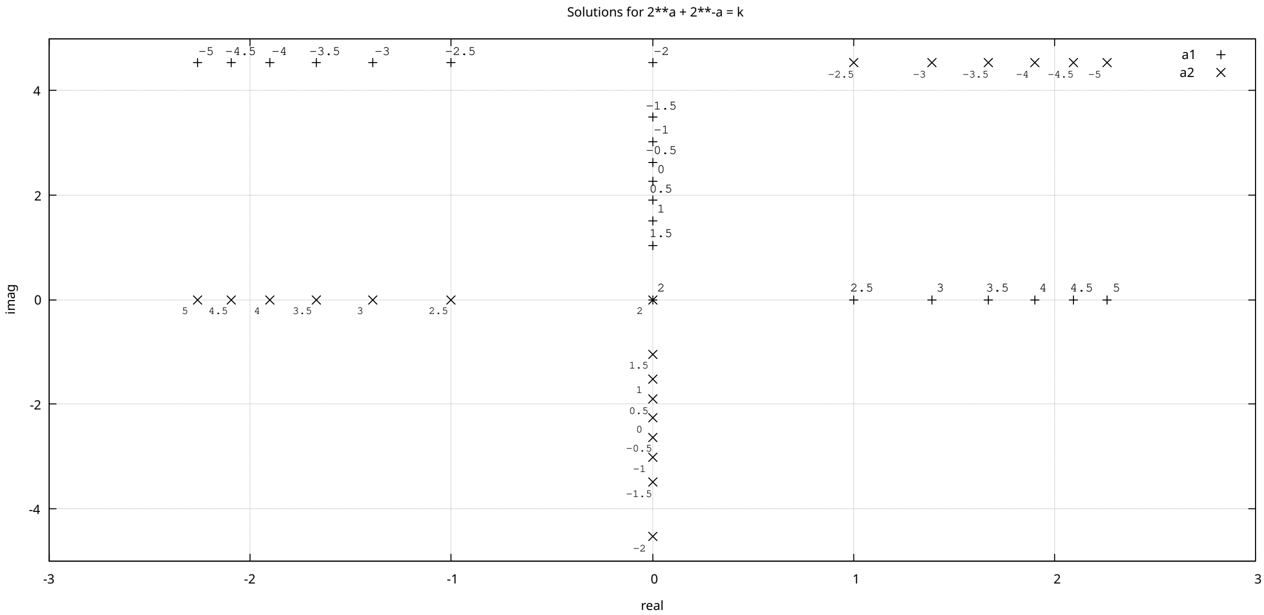
k	*x*	*log |x|*	*a.real*	*a.imag*
-2	-1	0	0	4.532
-2.5	-0.5, -2	±0.693	±1	4.532
-3	-0.382, -2.618	±0.962	±1.388	4.532
3.5	-0.314, -3.186	±1.159	±1.672	4.532
-4	-0.268, -3732	±1.317	±1.900	4.532
-4.5	-0.234, -4.266	±1.451	±2.093	4.532
-5	-0.209, -4.791	±1.567	±2.260	4.532

* * *

The chart below plots the solutions for selected values of k, between +5 and -5. The roots $a1$ and $a2$ are displayed with the k value beside the marker.

We see a symmetrical pattern for the ranges of solutions, intersecting at zero for k = 2.

2^A Solutions



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We have shown that for any value of any value of *k*, there is a pair of solutions for the equation $2^a + 2^{-a} = k$.[1] Let us call these the canonical roots of this equation.

Now consider the b^z where *b* is a real number > 0…

$$b^z = b^z \times e^{2\pi i}$$

and…

$$b^z = b^{z + 2\pi i / \log b}$$

or in our case…

$$2^a = 2^{a + 2\pi i / \log 2}$$

Which is to say, for any given solution *a*, there is another $a + 2\pi i/\log 2$; and by the same reasoning others $a \pm 2N\pi i/\log 2$, where *N* is a natural number. So, while for any *k* their exists a pair of complex roots whose imaginary part lies in the range $-\pi <$ a.imag $<= \pi$; there also exist infinite pairs of non-canonical roots displaced by $2N\pi i/\log 2$ (approx. N x $9.0647i$) above or below the former on the complex number plane.[2]

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1 For the special cases where $k = 2$ and $k = -2$, there are double roots.
2 Or, where we stated that the logarithm lz of a complex number z expressed in polar form equals $\log(r) + i\theta$, there are any number of alternative values $\log(r) + i(\theta \pm 2N\pi)$, such that e ^ $lz = z$.