Antiderivatives of ~1/x

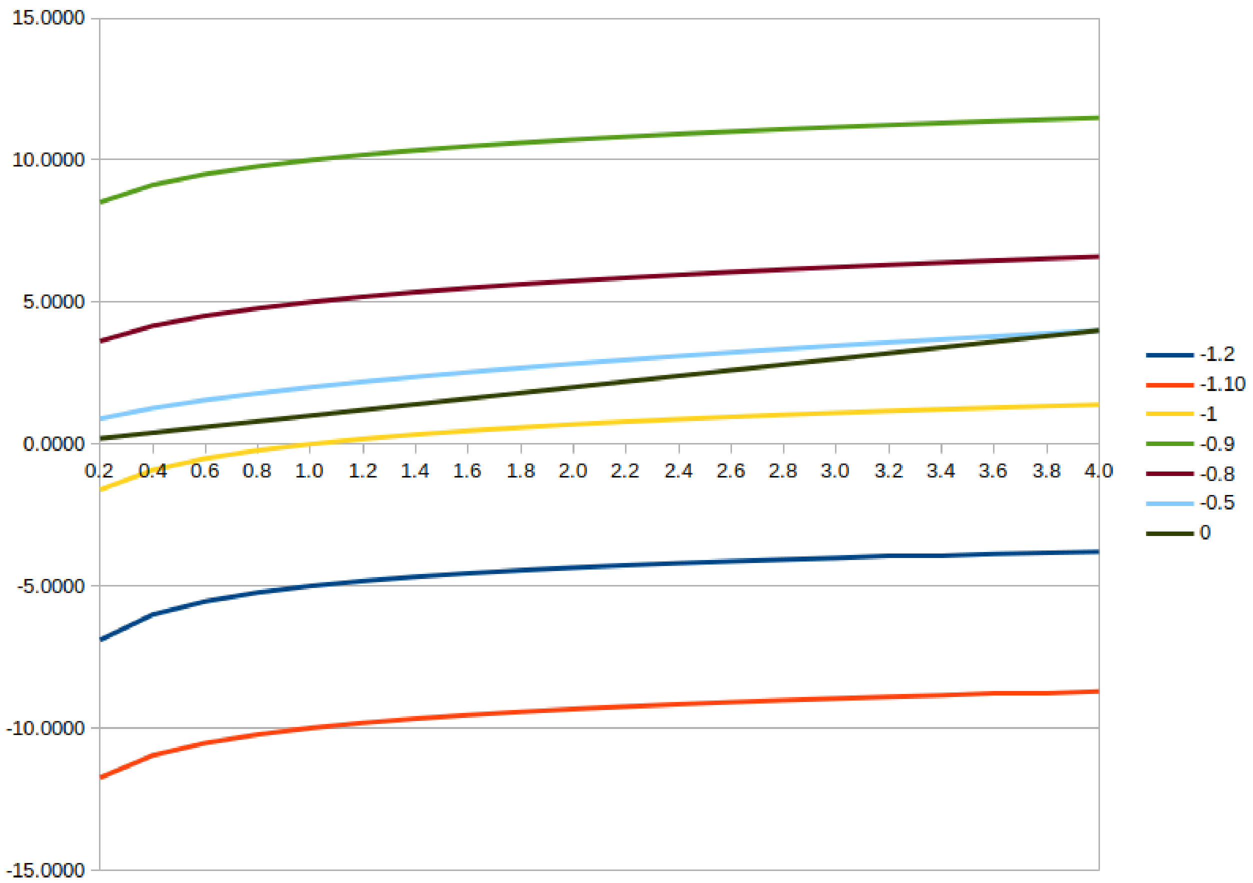
Consider the antiderivatives of functions on x^a where a is equal to values close to -1, comparing with that of a = -1. Now basic calculus gives us:

$$\int x^{a} \, . \, dx = 1/(a+1) \, . \, x^{a+1} + c \quad \text{(where } a \neq -1\text{)}$$

and of course:

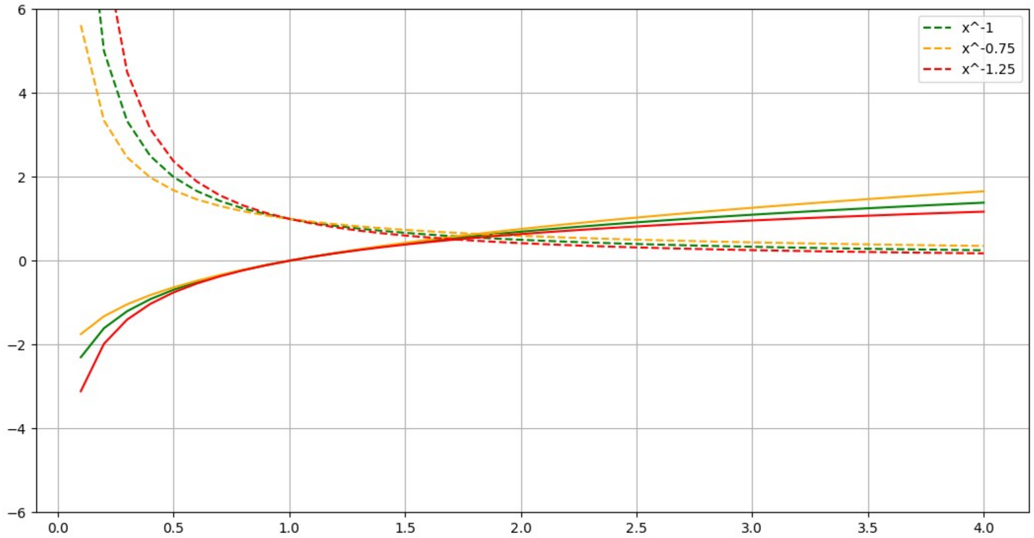
$$\int x^{-1} \, . \, dx = \ln x + c$$

Which leads us to wonder how the first expression compares with the second as a approaches -1. Plotting some values produces the following graphs –

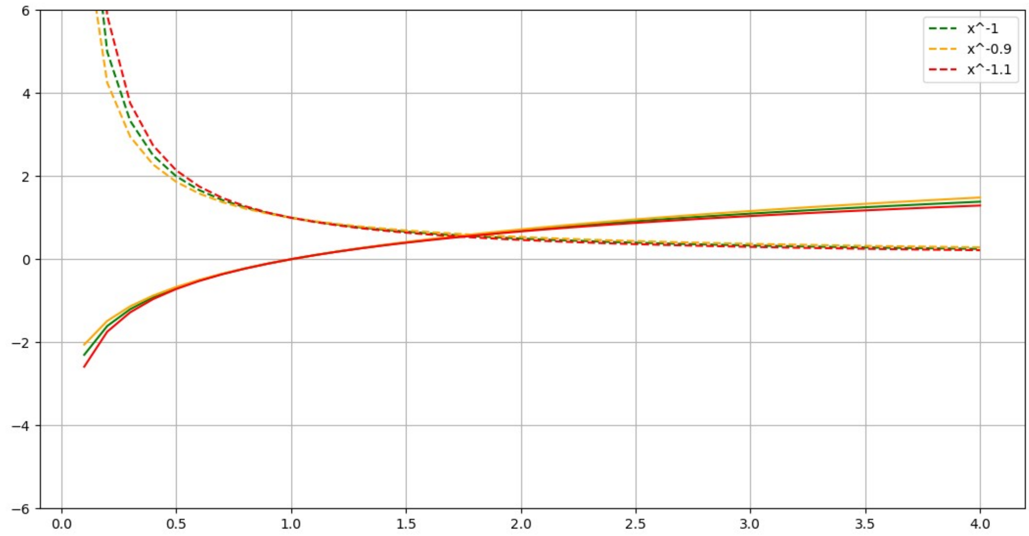


Here the yellow curve is the iconic natural logarithm and the others represent various antiderivatives for exponents about the -1 value. Now these all represent the case where c in the intergrand above is zero – as can be seen in the graphs, where the exponent is close to -1 the *shape* of the curve closely resembles the ln(x) graph; whereas its *position* is greatly displaced.

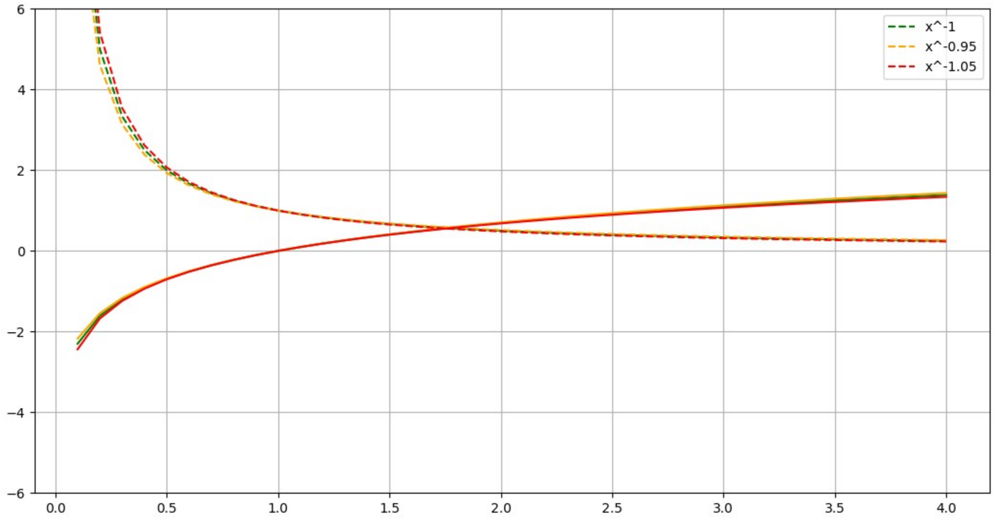
Now this displacement can be countered by choosing an appropriate value for c. We can normalise antiderivative function $1/(a+1) \,.\, x^{a+1} + c$ at the $x = 1$ value by setting $c = 1/(a \pm 1)$; the sign depending on whether a is less than -1 (plus) or greater than -1 (minus). This produces –



Here we have the antiderivative functions (solid lines) plotted against their unintegrated counterparts (dashed lines) where the exponent a varies from 1 by ±0.25, as well as the comparable ln(x) and $1/x$ functions (green). Now, closing in to 1 ± 0.1 gives –



And for 1 ± 0.05 –



So, as we would expect, the antiderivative of the x^a function closely aligns with the ln(x) function as a approaches -1, provided we adjust the integral constant by the appropriate amount, namely –

$$c = 1/(a \pm 1)$$

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Taking the antiderivative function arrived at above:

$$1/(a+1)\ .\ x^{a+1} + 1/(a \pm 1)$$

Re-expressing this by letting $\alpha = a + 1$, and considering just the case where α is positive (i.e. where a approaches -1 from the high side), we get:

$$(1/\alpha)\ .\ x^{\alpha} - 1/\alpha$$

$$= \frac{x^{\alpha}-1}{\alpha}$$

Which leads to the *conjecture* –

$$\mathrm{Lim}_{\alpha \to 0+} \frac{x^{\alpha}-1}{\alpha} = \ln x$$