I. Area of circle segment

Describe a circle centre *O* and radius *r*; then construct a chord *QQ'* bisected by radial *OP* at midpoint *X*, and normal to *QQ'*. The *depth* or thickness of the segment *PQQ'* is distance *PX*, designated *t* (see diagram below).

Determine the area of segment *PQQ'* given depth *t*.

Consider the half-segment *PQX* – this is contained within the circle sector *PQO*, complemented by right triangle *OQX*. It can be seen that from this construction we have:

$$s^2 + (r - t)^2 = r^2 \quad \rightarrow \quad s=\sqrt{2rt-t^2}$$

and

$$\alpha=acos\left(\frac{r-t}{r}\right)$$

Now the area of sector *PQO* is in proportion to the circle area as α is to 2π; that is –

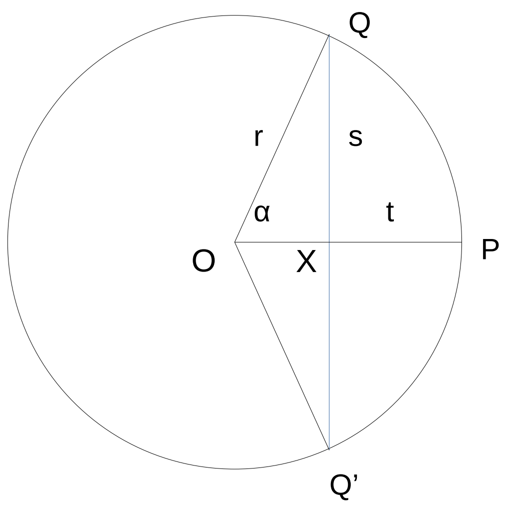
$$A_C = \frac{\alpha r^2}{2}$$

while the area of triangle *OQX* will be –

$$A_T = \frac{s(r-t)}{2}$$

with the difference yielding the half-segment area –

$$A_H = ½\ \alpha r^2 + ½\ s(t-r) \quad = \quad \frac{r^2}{2}acos\left(1-\frac{t}{r}\right) \quad + \quad \frac{t-r}{2}\sqrt{2rt-t^2}$$



Now this half-segment area should be given by integrating *s* over *t* thus –

$$A_H = \int_0^t (s)\,dt \quad \rightarrow \quad \int_0^t \sqrt{2rt - t^2}\,dt$$

It follows that the the solution to this intractable looking integral must be equivalent to the expression derived trigonometrically above. So generally we can state:

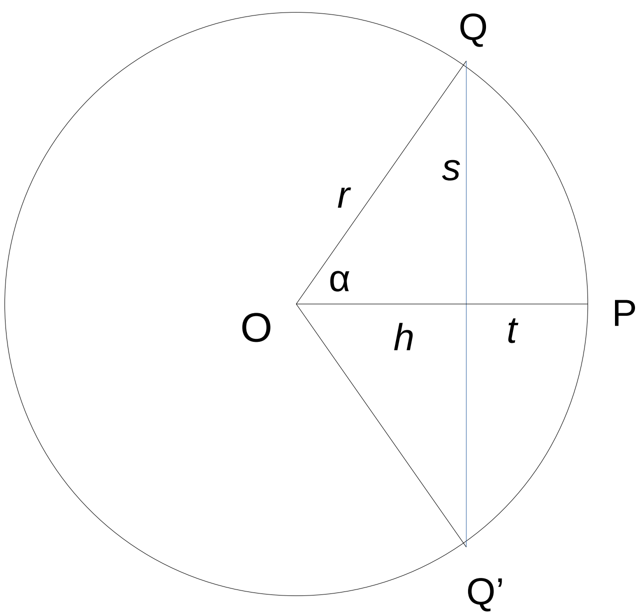
$$\int \sqrt{2rt - t^2}\,dt \;=\; \frac{r^2}{2} acos\left(1 - \frac{t}{r}\right) \;+\; \frac{t-r}{2}\sqrt{2rt - t^2} \;+ C$$

As such, this can be found in amongst published Tables of Integrals (e.g. see E.J.Purcell #55).

Furthermore, we can readily verify this result by finding its first derivative, using product and chain rules of differentiation.

II. Area of segment of sphere

Now consider a segment of a sphere centre *O* and radius *r* – refer diagram below, understanding that *QQ'* represents the plane of segmentation, subtended at *O* at an angle equal to 2α. We note that $s = r.\sin(\alpha)$; and that the interval $h = r.\cos(\alpha)$ – and therefore $t = r - r.\cos(\alpha)$.



$s = r \sin \alpha$
$h = r \cos \alpha$
$t = r - r \cos \alpha$

SEGMENTAREA

The surface area of this segment A_S (not including the segmental plane) can be derived by integrating, over α, circular bands on the sphere's surface whose circumference equals $2\pi s = 2\pi \sin(\alpha)$, and whose width is equal to $r.d\alpha$. That is to say-

$$\int 2\pi \ \sin\alpha . r . d\alpha$$

or...

$$-2\pi r \cos\alpha \ + \ C$$

Since at $\alpha = 0$ we should have zero value, C needs to equal to $2\pi r$. So our expression becomes:

$$2\pi r - 2\pi r \cos\alpha$$

or…

$$2\pi r(r - r\cos\alpha)$$

Which is to say, the segment surface area $A_s = 2\pi rt$.

III. Brinell Hardness

If a load P is applied onto a flat surface by a sphere (with diameter D), we can gauge the deformation caused by determining the area of the depression (diameter d). Then, provided that the strength and hardness of the impinging piece is greater than that of the receiving surface, expressing the ratio of the force to the resulting indentation area (P/A_s) can be regarded as a property of the material being tested – which is to say, a measure of its hardness.

Per the diagram above, let the indentation be represented by the segment PQQ'. We note that $h = \sqrt{r^2 - s^2}$, and therefore $t = r - \sqrt{r^2 - s^2}$.

From the formula derived derived previously it follows that –

$$A_s = 2\pi r(r - \sqrt{r^2 - s^2})$$

Now, letting the the diameter of the sphere be $D = 2r$, and that of the depression be $d = 2s$, we have –

$$A_s = \pi D\left(\frac{D}{2} - \frac{1}{2}\sqrt{D^2 - d^2}\right) \quad \text{or} \quad \frac{\pi D}{2}\left(D - \sqrt{D^2 - d^2}\right)$$

So the aforementioned hardness ratio P/A_s becomes –

$$H = \frac{2P}{\pi D(D - \sqrt{D^2 - d^2})}$$

which accords with the formula for the Brinell hardness value.