Area of circle segment

Describe a circle centre *O* and radius *r*; then construct a chord *QQ'* bisected by radial *OP* at midpoint *X*, and normal to *QQ'*. The *depth* or thickness of the segment *PQQ'* is distance *PX*, designated *t* (see diagram below).

Determine the area of segment *PQQ'* given depth *t*.

Consider the half-segment *PQX* – this is contained within the circle sector *PQO*, complemented by right triangle *OQX*. It can be seen that from this construction we have:

$$s^2 + (r - t)^2 = r^2 \quad \rightarrow \quad s = \sqrt{(2rt - t^2)}$$

and

$$\alpha = acos\left(\frac{r-t}{r}\right)$$

Now the area of sector *PQO* is in proportion to the circle area as α is to 2π; that is –

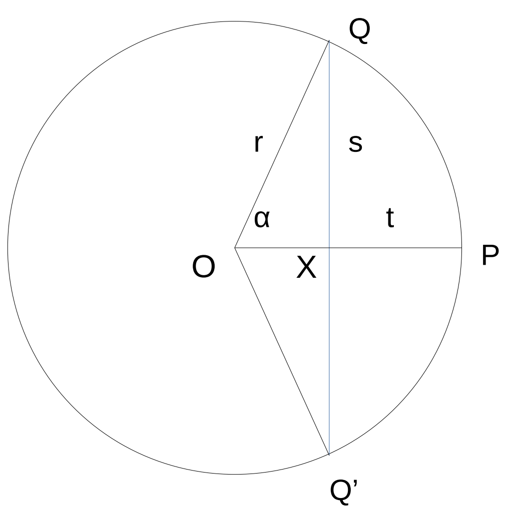
$$A_C = \frac{\alpha r^2}{2}$$

while the area of triangle *OQX* will be –

$$A_T = \frac{s(r-t)}{2}$$

with the difference yielding the half-segment area –

$$A_H = ½\,\alpha r^2 + ½\,s(t-r) \quad = \quad \frac{r^2}{2} acos\left(1-\frac{t}{r}\right) \quad + \quad \frac{t-r}{2}\sqrt{2rt-t^2}$$



Now this half-segment area should be given by integrating *s* over *t* thus –

$$A_H = \int_0^t (s)\,dt \quad \rightarrow \quad \int_0^t \sqrt{2rt - t^2}\,dt$$

It follows that the the solution to this intractable looking integral must be equivalent to the expression derived trigonometrically above. So generally we can state:

$$\int \sqrt{2rt - t^2}\,dt \quad = \quad \frac{r^2}{2} acos\left(1 - \frac{t}{r}\right) \quad + \quad \frac{t-r}{2}\sqrt{2rt - t^2} \quad + C$$

As such, this can be found in amongst published Tables of Integrals (e.g. see E.J.Purcell #55).