10-601: Homework 1

Due: 18 September 2014 11:59pm (Autolab)

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Please answer to the point, and do not spend time/space giving irrelevant details. You should not require more space than is provided for each question. If you do, please think whether you can make your argument more pithy, an exercise that can often lead to more insight into the problem. Please state any additional assumptions you make while answering the questions. You need to submit a single PDF file on autolab. Please make sure you write legibly for grading.

You can work in groups. However, no written notes can be shared, or taken during group discussions. You may ask clarifying questions on Piazza. However, under no circumstances should you reveal any part of the answer publicly on Piazza or any other public website. The intention of this policy is to facilitate learning, not circumvent it. Any incidents of plagiarism will be handled in accordance with CMU's Policy on Academic Integrity.

⋆: Code of Conduct Declaration

- Did you receive any help whatsoever from anyone in solving this assignment? (Yes) / No.
- If you answered *yes*, give full details: Graeme Rock & Dawei Wang explained a few questions to me. (e.g. *Jane explained to me what is asked in Question 3.4*)
- Did you give any help whatsoever to anyone in solving this assignment? (Yes) / No.
- If you answered *yes*, give full details: I provided pointers to Graeme (e.g. *I pointed Joe to section 2.3 to help him with Question 2*). Rock & Dawei Wang on a couple questions.

1: The truth will set you free. (TA:- Abhinav Maurya)

State whether true (with a brief reason) or false (with a contradictory example). Credit will be granted only if your reasons/counterexamples are correct.

(a) During decision tree construction, if you reach a node where the maximum information gain for a node split using any attribute is zero, then all training examples at that node have the same label.

False, if the remaining attributes provides the same amount of entropy as the first node, the information gain would be 0 even though the data may have different labels. [2 points]

(b) Whenever a set S of labeled instances is split into two sets S_1 and S_2, the average entropy will not increase, irrespective of the split attribute or the split point.

False. If S_1 & S_2 are split in a way where $p(S_1)$ is 100% for a label and $P(S_2)$ is 100% for another label, each set with have an entropy of 0 ⇒ hence the avg. entropy is 0 as well. However, if considering S as a whole, its entropy would be 1. [2 points]

(c) A decision tree can be represented as a decision list and vice versa. (Hint: A decision list is a sequentially applied list of decision rules of the form: *If condition$_1$ and condition$_2$ and ... condition$_n$, then output is y_i.* Each condition is a test on a single feature similar to the nodes of a decison tree.)

False. ex. If round and flat, it is a pizza.
If moves and red, it is a clock. [2 points]

This list cannot be converted to a tree as no common attribute is used to determine labels. (we do not know whether clock is round or not).

(d) If X_1 and X_2 are independent gaussian random variables, $X = \frac{1}{4}(X_1 - X_2)$ is a gaussian random variable.

True. Subtracting two independent gaussian variables lead to $U = \frac{1}{4}(u_1 - u_2)$ and $\frac{1}{4}(\sigma_1^2 - \sigma_2^2)$, which is still a gaussian distribution. [2 points]

(e) If $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are the probability density functions of independent gaussian random variables, $f(X) = \frac{1}{2}\{f_{X_1}(x_1) + f_{X_2}(x_2)\}$ is a probability density function corresponding to a gaussian random variable.

False. Adding the function of two gaussian ~~variables~~ functions do not result in another gaussian function as the exponential terms of the functions cannot be summed directly. [2 points]

2: Maximum Likelihood Estimation. (TA:- Jingwei Shen)

(a) $X_1, X_2, \cdots, X_n$ are random variables that are uniformly distributed between $[-\theta/2, \theta/2]$, $\theta \in \mathbb{R}$. Write down the MLE for the parameter θ and explain it. (You do not have to derive it.)

0. The MLE is 0 as there is an equal distribution of dataset on both sides of the center point. [3 points]

$$\frac{\partial f(x)}{\partial \theta} = \frac{1}{\theta - a} \text{ for } [-\theta/2, \theta/2]$$

$$\theta = 0$$

(b) We have two coins - an unbiased one with probability $p_1 = 1/2$ of showing heads on a toss, and a biased one with probability $p_2 = 1/3$ for showing heads. We do 100 tosses. Each time we choose one of the two coins. With an unknown probability p, we choose the biased coin, and with probability $1 - p$, we choose the unbiased one. And we observe 40 heads during the 100 tosses. Write down the MLE estimate of parameter p and explain it. (You do not have to derive it.)

$P(\text{heads}) = p(p_1) + (1-p)(p_2) = \frac{40}{100}$

$p(0.5) + (1-p)(0.33) = 0.4$

$\boxed{p = 0.41}$

[3 points]

This means that a given probability of ~~getting~~ ~~heads~~ using the unbiased coin of 41% best matches the observed 40/100 heads scenario with the information provided.

3: Three Prisoners and a Warden (TA:- Jingwei Shen)

Three prisoners - A, B, and C - are on death row. The governor decides to pardon one of the three and chooses the prisoner to pardon at random. He informs the warden of his choice but requests the name to be kept as a secret.

Having heard of the pardon rumor through grapevine, A tries to get the warden to tell him his fate. The warden refuses. Then A asks which of B or C will be executed. The warden thinks a while and tells A that B is to be executed. (Assume that the warden picks a random legal answer for A's question).

(a) Let A, B, C denote the event that A, B, C will be pardoned respectively. Let $!B$ denote the event that the warden says B will die. Compute $P(A\,|!B)$. Does the chance of A's survival increase with the additional information about B's death? (Hint: compare $P(A|!B)$ and $P(A)$).

$P(A|!B) = \frac{P(!B|A)P(A)}{P(!B)}$ $\quad P(A) = \frac{1}{3} = 33\%$

$P(!B) = 1$

$P(A|!B) = \boxed{33\%}$

[3 points]

(b) Suppose A reveals all of the above to C. Show the probability of C surviving at this time is 2/3. (Hint: Prove $P(C|!B) = 2/3$).

$P(C|!B) = \frac{P(!B|C)P(C)}{P(!B)}$

$= \frac{P(!B|C)P(C)}{P(!B|P_a)P(P_a) + P(!B|P_b)P(P_b) + P(!B|P_c)P(P_c)}$

$= \frac{1(0.33)}{\frac{1}{2}(0.33) + 0 + 1(0.33)} = \boxed{\frac{2}{3}}$

[3 points]

4: Probability Theory (TA:- Jingwei Shen)

(a) Let A, B, C be three discrete random variables. Show that

1. $P(A \mid B, C) = \frac{P(A,B|C)}{P(B|C)}$
2. $P(A \mid C) = \sum_B P(A, B \mid C)$
3. $P(A \mid C) = \sum_B P(A \mid B, C) \cdot P(B \mid C)$

[3 points]

1) $P(A|BC) = \frac{P(ABC)}{P(BC)}$

$\frac{P(BC|A)P(A)}{P(BC)} = \frac{P(C|AB)P(AB)}{P(C) \cdot P(BC)}$

$\frac{P(ABC)}{P(BC)} = \frac{P(ABC)}{P(BC)}$ ✓

2) $P(A|C) = \sum_B P(ABC)$

$= \sum_B P(A|C) P(B|C)$ ~~$P(C)$~~

$= P(A|C) \sum_B \frac{P(C|B) P(B)}{P(C)}$

$= P(A|C)$ ✓

3) $P(A|C) = \sum_B P(A|BC) P(B|C)$

$= \frac{P(ABC)}{P(BC)} \sum_B P(BC)$

$= \frac{P(A|C) P(BC)}{P(BC)} \sum_B \frac{P(C|B) P(B)}{P(C)}$

$= P(A|C)$ ✓

(b) Suppose that 0.5% men and 0.25% women are color-blind. A person is chosen randomly at the university where the number of men is twice of that of women. The chosen person is color-blind. What is the probability that the person is male?

[2 points]

$P(Y|M) = 0.005$

$P(Y|F) = 0.0025$

$P(M) = 0.66$

$P(F) = 0.33$

$P(M|Y) = \frac{P(Y|M)P(M)}{P(Y)}$

$= \frac{0.005(0.66)}{0.004125}$

$= 0.8 = 80\%$

$P(Y) = P(Y|M) P(M) + P(Y|F) P(F)$

$= 0.05 \times 0.66 + 0.0025 \times 0.33$

$= 0.4125\%$

(c) Consider the probability density function $f_{X,Y}(x, y)$ over a 2-dimensional random variable $[X, Y]$.

$$f_{X,Y}(x, y) = \begin{cases} c(x + y^2) & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, c is a constant appropriate for $f_{X,Y}(x, y)$ to be a density function. Find $P(X < \frac{1}{2} | Y = \frac{1}{2})$

[3 points]

$$\int_0^{1/2} f(x) = \int_0^{1/2} C(x+1/2)^2$$

$$= \frac{(x+1/2)^3}{3}\left(\frac{2}{x^2}\right)\Big]_0^{1/2}$$

$$= \frac{1}{12}(1/2)\left((1)+3+3\right)$$

$$= \boxed{\frac{7}{24}}$$

5: Nearest neighbors to the rescue. (TA:- Jingwei Shen)

(a) Consider two classes C_1, C_2 in the two-dimensional space. The data from class C_1 are uniformly distributed in a circle of radius r. The data from class C_2 are uniformly distributed in another circle of radius r. The centers of two circles are at a distance greater than $4r$. Show that the accuracy of 1-NN is greater than or equal to the accuracy of k-NN, where k is an odd integer and $k \geq 3$.

Test data is in C_1

a) If training data is such that C_1 - 1 data & C_2 - 2 data
1NN → C_1 3NN → C_2

b) If training data is such that C_1 - 2 data & C_2 - 1 data
1NN → C_1 3NN → C_1

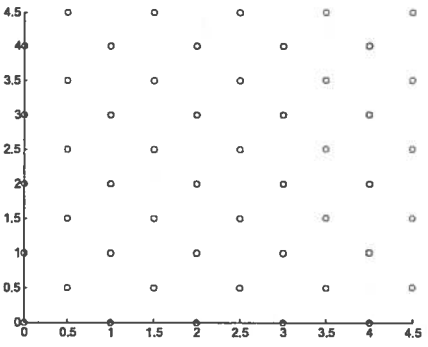
c) If training data is such that C_1 - 3 data & C_2 - 0 data
1NN → C_1 3NN → C_1

d) If training data is such that C_1 - 0 data & C_2 - 3 data
1NN → C_2 3NN → C_2

[3 points]

1NN	3NN
✓	X
✓	✓
✓	✓
X	X

Figure 1: Q4 Dataset



(b) In the dataset shown in figure 1, what is the leave-one-out accuracy of the k-NN method when $k = 2$? Remember that a data point cannot be considered its own neighbor since it is left out. (Ignore the datapoints that have an output tie for $k = 2$ nearest neighbors.)

0%. k=2 will always yield the wrong label. [2 points]

(c) In this problem, explain briefly why you think k-NN performs worse than randomly guessing, which has an accuracy near 50%?

k-NN is worse since there will always be more incorrect labels than correct ones as the test data is always surrounded by the wrong labels. [2 points]

6: A tree about the important things in life. (TA:- Abhinav Maurya)

The following dataset will be used to learn a decision tree for predicting whether a person is Happy (H) or Sad (S) based on the color of their shoes, whether they wear a wig and the number of ears they have.

Color	Wig	Num. Ears	Emotion
G	Y	2	S
G	N	2	S
G	Y	2	S
B	N	2	S
B	N	2	H
R	N	2	H
R	N	2	H
R	N	2	H
R	Y	3	H

(a) What is Entropy(Emotion | Wig=Y)? [1 points]

$$H(E|W=Y) = \sum_{w=y} -P(E=S)\log_2 P(E=S) - P(E=H)\log_2 P(E=H)$$
$$= -0.66\log_2(0.66) - 0.33\log_2(0.33)$$
$$= 0.92$$

(b) Which attribute would the decision-tree building algorithm choose to use for the root of the tree (assume no pruning)? [2 points]

$H(E) = -4/9\log_2(4/9) - 5/9\log_2(5/9) = 0.99$

$H(E|ears) = (-1/2\log_2(1/2) - 1/2\log_2(1/2))(8/9)$
$+ (-1\log_2(1) - 0\log_2(0))(1/9)$
$= 0.89$

$H(E|wig) = 0.92(1/3) + 0.92(2/3) = 0.92$

$H(E|color) = (-1\log_2(1) - 0\log_2(0))(3/9)$
$+ (-1/2\log_2(1/2) - 1/2\log_2(1/2))(2/9)$
$+ (-1\log_2(1) - 0\log_2(0))(4/9)$
$= 0.22$ color!

(c) Draw the full decision tree that would be learned for this data (assume no pruning).

[3 points]

(d) What would be the training set error for this dataset? Express your answer as the percentage of records that would be misclassified.

1/9

[2 points]

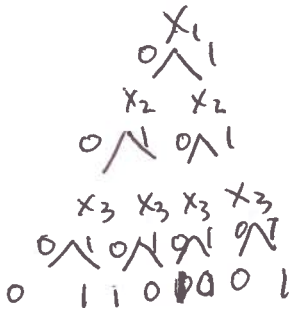
7: Digging up the dense binary tree. (TA:- Abhinav Maurya)

Consider the following data with three binary attributes, where x^i denotes the i^{th} datapoint, x_j denotes the j^{th} feature of the datapoint, and y denotes the class label:-

	x_1	x_2	x_3	y
x^0	0	0	0	0
x^1	0	0	1	1
x^2	0	1	0	1
x^3	0	1	1	0
x^4	1	0	0	1
x^5	1	0	1	0
x^6	1	1	0	0
x^7	1	1	1	1

(a) Draw the decision tree for the above dataset using the entropy criterion to decide node splits (assume no pruning).

[3 points]



(b) Decision trees are often pruned so that they can better generalize for prediction on the test set. Do you think you could prune any of the lower levels of the above decision tree used to predict the XOR of 3 binary digits? Give reasons for your decision.

No, since every level is a factor into the decision, if any layer is prune, the error will greatly increase. [2 points]

(c) Considering a generalization of the above problem, let's say that we train a decision tree without any pruning to output the XOR function using *all* possible binary strings of length n. Out of the decision tree and KNN classifier (using l_1 distance and $k = 1$), which one would be more accurate when the test samples are also binary strings of length n?

Decision tree. KNN classifier will always yield the wrong answer as its closes neighbors are all of opposite label. [2 points]

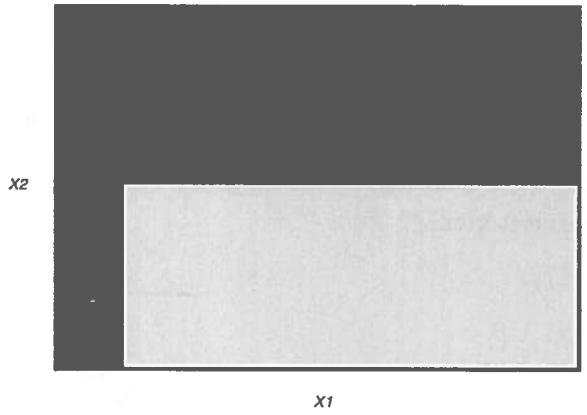
(d) Out of the decision tree and KNN classifiers considered in the previous question, which one will take lesser time to predict the output label of a new test datapoint? Why? (Hint: Note that there are 2^n possible datapoints due to n binary input features. Consider the number of nodes traversed by the decision tree and the number of distance computations performed by the KNN classifier to predict the label of a test datapoint with n binary input features.)

Decision tree. Decision tree has a runtime of n
KNN has a runtime of $2^n(n)$ [2 points]

8: On the hardness of learning optimal binary decision trees (TA:- Abhinav Maurya)

In figure 2, assume that the rectangular region consisting of two features x_1 and x_2 is densely packed with points. The red, green, and yellow subrectangles represent the three classes C_1, C_2, and C_3 of datapoints. The x_1 X x_2 dimensions of the red, green, and yellow rectangles are 1 X 6, 7 X 3, and 7 X 3 respectively. The red rectangle is uniformly populated with 6,000 datapoints of class C_1. The green rectangle is uniformly populated with 42,000 datapoints of class C_2. The yellow rectangle is uniformly populated with 42,000 datapoints of class C_3.

Figure 2: A 2D dataset with three classes



(a) What is the minimum number of nodes that a decision tree needs to have in order to classify the above dataset correctly?

2 X_1 then X_2

[2 points]

(b) What is the number of nodes in the decision tree trained on the above dataset using the entropy criterion?

3 X_2 then X_1

[2 points]

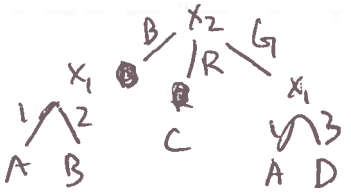
(c) Are the number of nodes in the two cases identical or different? Why do you think that is?

Different. Since the entropy case will always split along the line where it could categorize the largest amount of datapoints.

[3 points]

(d) Construct another toy dataset where the entropy gain criterion leads to a suboptimal decision tree i.e. one with more nodes than another tree of comparable accuracy. Your dataset should have at least four labels and be sufficiently different from the given toy dataset.

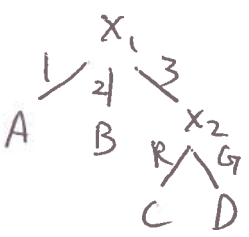
X_1	X_2	label
1	B	A
2	B	B
3	R	C
3	R	C
3	R	C
3	G	D
3	G	D
3	G	D
3	G	D
1	G	A



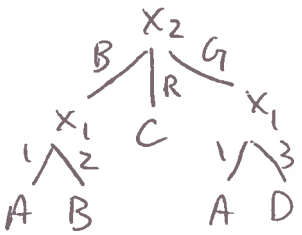
[3 points]

(e) For your suggested dataset, draw the optimal decision tree as well as the decision tree obtained using the entropy minimization criterion.

optimal



$H(t)$



[3 points]

(f) A decision tree can classify the dataset in figure 2 with 100% test accuracy (assuming that there is no label noise). What are the general conditions on a dataset under which a decision tree can provide 100% test accuracy? (Hint: Each internal node of a decision tree performs a split based on a single feature. Think about the class of separation functions such a decision tree entails.)

for functions to be 100% test accurate, there must be enough features such that no two dataset with different labels are categorized as the same label.

[3 points]

Total: 70