MAT 4002 Homework 6

1. (a) Let X be a compact space, and let $(V_n)_{n\in\mathbb{N}}$ be a nested sequence of non-empty closed subsets of X (nested means that $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$). Prove that $\bigcap\limits_{n=1}^{\infty} V_n \neq \varnothing$.

   (b) Now suppose that X is Hausdorff as well as compact, and let $f : X \to X$ be a continuous map. Let $X_0 = X, X_1 = f(X_0)$ and inductively define $X_{n+1} = f(X_n)$ for $n \geq 1$. Show that $A = \bigcap\limits_{n} X_n$ is non-empty.

   (c) Prove that $f(A) = A$.

2. Let $(X, \mathcal{T})$ be a topological space and let $C = X \cup \{\infty\}$ where ∞ denotes some extra point not in X . Let $\mathcal{T}'$ denote the union of $\mathcal{T}$ with all subsets of C of the form $V \cup \{\infty\}$ where $V \subseteq X$ and $X \setminus V$ is compact and closed in X .

   Prove that $(C, \mathcal{T}')$ is a compact topological space containing $(X, \mathcal{T})$ as a subspace. This is called the one-point (or the Alexandrov) compactification of X .

3. Recall that the integer part (or integral part) of a real number x is the unique integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. We denote it by $I(x)$.

   On $\mathbb{R}$ we define the relation $x\mathcal{R}y \Leftrightarrow I(x) = I(y)$.

   (a) Prove that $\mathcal{R}$ is an equivalence relation.

   (b) Let $p : \mathbb{R} \to \mathbb{R}/\mathcal{R}$ be the quotient map, let $\mathbb{R}/\mathcal{R}$ be endowed with the quotient topology, and let U be an open set in $\mathbb{R}/\mathcal{R}$. Prove that if $n \in \mathbb{Z}$ is such that $p(n) \in U$ then $p(n-1) \in U$.

   (c) Deduce that the open sets in $\mathbb{R}/\mathcal{R}$ are $\varnothing, \mathbb{R}/\mathcal{R}$ and the image sets $p(-\infty, n]$, where $n \in \mathbb{Z}$.

   (d) Consider the map $I : \mathbb{R} \to \mathbb{Z}, x \mapsto I(x)$. Is the map I continuous (when $\mathbb{Z}$ is endowed with the subspace topology)?

Prove that I defines a bijection $\widetilde{I} : \mathbb{R}/\mathcal{R} \to \mathbb{Z}$. What is the topology on $\mathbb{Z}$ making $\widetilde{I}$ a homeomorphism ?