MAT 4002 Homework 1

1. Show that the topologies induced from the metrics d_1, d_2, d_∞ in $\mathbf{R}^2$ are equivalent.

2. Compute the topology τ_{dis} of the discrete metric space $\left(\mathbf{R}^2, d_{dis}\right)$.

3. Let (X, d) be a metric space, and $A \subset X$ be a subset. The closure of $A, \bar{A}$, is defined as the smallest closed subset containing A , i.e. $\bar{A}$ satisfies the following:

   (i) $A \subset \bar{A}$;

   (ii) $\bar{A} \subset X$ is closed; and

   (iii) if B is a closed subset containing A , then $\bar{A} \subset B$.

   (a) Show that $\bar{A}$ is well-defined. In other words, if both $\bar{A}, \bar{A}'$ satisfy (i)-(iii) above, then
   $$\bar{A} = \bar{A}'.$$

   (b) Prove that a closed ball is a closed set in the topology given by(X, d). Hence show that the closure of an open ball is contained in the closed ball.

   (c) Draw the picture of the closed ball with center $\left(\frac{1}{2}, 0\right)$ and of radius $\frac{1}{2}$ in the metric space $\left(\mathbf{R}^2, d_\infty\right)$.

   (d) Consider the subset $E = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$ of $\mathbf{R}^2$ endowed with the metric d_∞ . What are the open and closed ball of center $\left(\frac{1}{2}, 0\right)$ and of radius $\frac{1}{2}$ in (E, d_∞) ? Show that the closure of the open ball is a proper subset of the closed ball.