MAT4002 Spring 2020 Mid-Term Homework

Problem 1

For metric spaces, many properties can be characterized by sequences (e.g., compact $\iff$ every sequence has a convergent subsequence). This is no longer true for topological spaces in general. Let X be an uncountable set equipped with the co-countable topology τ_{cc}, i.e.,

$$U \subset X \text{ is open } \iff U = \emptyset \text{ or } X \setminus U \text{ is a countable set.}$$

(You do not need to show τ_{cc} forms a topology.)

1. Show that (X, τ_{cc}) is not metrizable. (Hint: Is (X, τ_{cc}) Hausdorff?)
2. Suppose $(x_n) \to x$ under the co-countable topology of X. Show that there exists N such that $x_n = x$ for all $n \geq N$.

Definition: We say that the function $f : (X, \tau_X) \to (Y, \tau_Y)$ is sequentially continuous if for any sequence $(x_n) \to x$, we have $f(x_n) \to f(x)$.

1. By considering the co-countable topology, give an example such that f is sequentially continuous, yet f is not continuous.

Problem 2

Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ iff $x = y = 0$ or $xy > 0$.

1. How many elements are there in $\mathbb{R}/\sim$? Write down a representative for every point.
2. Write down all the open sets of the quotient topology on $\mathbb{R}/\sim$.
3. Is $\mathbb{R}/\sim$ Hausdorff? Justify your claim.

Problem 3

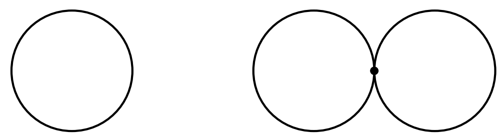
Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective, continuous function between topological spaces, and $D \subset X$ be a dense subset, i.e., $\bar{D} = X$.

1. Show that $f(D)$ is dense in Y.
2. If $E \subset Y$ is dense, is the preimage $f^{-1}(E)$ dense? Prove it, or give a counterexample.

Problem 4

Let X be a topological space.

1. Write down the definition for a subset $A \subset X$ to be connected.
2. Show that if $A \subset X$ is connected, and $f : X \to Y$ is a continuous map between topological spaces, then $f(A) \subset Y$ is connected.
3. Explain why the circle $X = S^1$ and the "∞-shaped space" $Y = S^1 \vee S^1$ are not homeomorphic:



$X = S^1$ $\qquad\qquad$ $Y = S^1 \vee S^1$

where the two circles of ∞ intersect at a single point. You can justify your arguments using pictures, but you need to express your reasoning clearly.

Problem 5

Let $X \subset Y$ be an open subset of a compact, Hausdorff topological space Y.

1. Show that a subset $K \subset X$ is compact iff $K \subset Y$ is compact. That is, compactness is an intrinsic property.
2. Suppose $x \in X$. Show that there is an open set $U \subset Y$ containing x such that its closure $\bar{U}$ in Y satisfies $\bar{U} \subset X$.
3. Let $\tilde{X} := X \cup \{\infty\}$ be the one-point compactification defined in Homework 6. Show that $\tilde{X}$ is Hausdorff.
4. By constructing a continuous, surjective map $f : Y \to \tilde{X}$, show that the spaces

$$Y/\sim \,\cong \tilde{X}$$

are homeomorphic, where the equivalence relation $\sim$ in Y is given by the partition

$$P := \left(\bigcup_{x \in X} \{x\} \right) \cup (Y \setminus X).$$

Hint for (a) and (b): You can use the results in Homework sets without proving them.