MAT 4002 - Geometry and Topology

Forward

This book is taken notes from the MAT4002 in spring semester, 2019. These lecture notes were taken and compiled in LATEX by Jie Wang, an undergraduate student in spring 2019. The tex writter would like to thank Prof. Daniel Wong and some students for their detailed and valuable comments and suggestions, which significantly improved the quality of this notebook. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Contents

Notations and Conventions

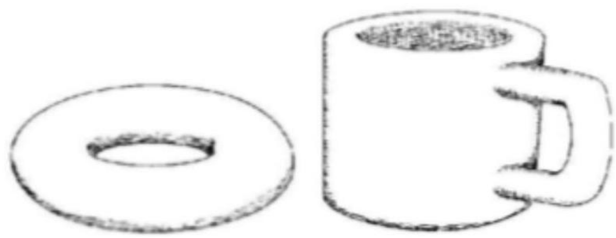
$(X, \mathcal{T})$	Topological space
$X \cong Y$	The space X is homeomorphic to space Y
$E^\circ, \partial E, \bar{E}$	The interior, boundary, closure of E
$p_X : X \times Y \to X$	Projection mapping
$X \times Y$	Product Topology
$X/\sim$	Quotient space of X by the equivalence relation $\sim$
S^n	The n-sphere $\{\mathbf{x} \in \mathbb{R}^{n+1} \mid \| \mathbf{x} \| = 1\}$
D^n	The n -disk $\{\mathbf{x} \in \mathbb{R}^n \mid \| \mathbf{x} \| \leq 1\}$
$\mathbb{T}^2$	The 2-torus in $\mathbb{R}^3$
Δ^n	The n-simplex
$i : A \hookrightarrow X$	Inclusion mapping from $A \subseteq X$ to X
$K = (V, \Sigma)$	(Abstract) Simplicial Complex
$\|K\|$	Topological realization of the simplicial complex K
$\langle X \mid R \rangle$	Presentation of a group with generators X and relations R
$H : f \simeq g$	f and g are homotopic
$X \simeq Y$	The space X and Y are homotopy equivalent
$G \cong H$	The group G is isomorphic to group H
$\pi_1(X, x)$	The fundamental group of X w.r.t. the base point $x \in X$
$E(K, b)$	The edge loop group of the space K w.r.t. the base point b
$f_* : \pi_1(X, x) \to \pi_1(Y, y)$	The induced homomorphism of $f : X \to Y$

Chapter 1

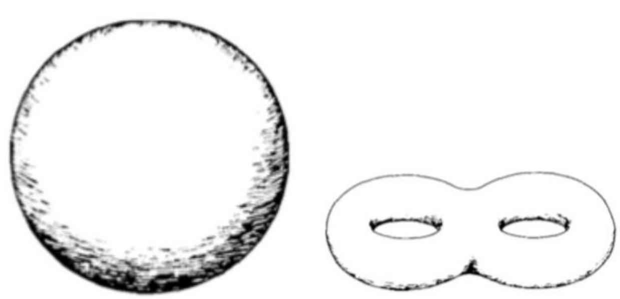
From Metric to Topology

We will study global properties of a geometric object, i.e., the distance between 2 points in an object is totally ignored. For example, the objects shown below are essentially invariant under a certain kind of transformation:

Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.1 Metric Spaces

In this section, we will study a special kind of topological space.

Definition 1.1 (Metric Space)**.** A metric space (X, d) is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$ for $\forall \mathbf{x}, \mathbf{y} \in X$ with equality iff $\mathbf{x} = \mathbf{y}$
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
3. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ (triangle inequality)

Example 1.2. 1. Let $X = \mathbb{R}^n$. Then the distance functions:

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \quad , \quad d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

define two different metric spaces of the same set X.

2. Let X be any set, and define the discrete metric

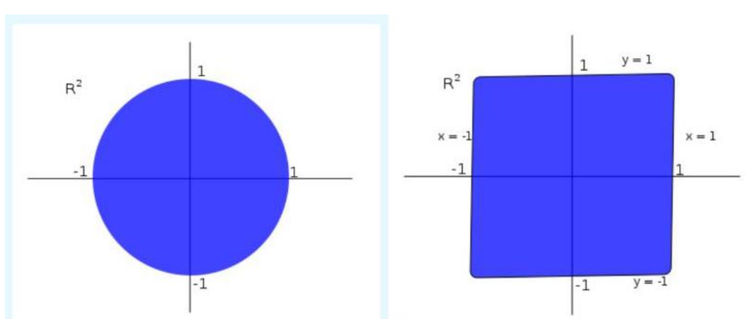
$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

(Homework Show that (1) and (2) defines a metric.)

Definition 1.3 (Open Ball)**.** Let (X, d) be a metric space. An open ball of radius r centered at $\mathbf{x} \in X$ is the set

$$B_r(\mathbf{x}) = \{\mathbf{y} \in X \mid d(\mathbf{x}, \mathbf{y}) < r\}$$

Example 1.4. 1. The set $B_1(0, 0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:



2. Under the discrete metric $(X = \mathbb{R}^2, d_{discrete})$, the open ball $B_1(\mathbf{x}, 0)$ is a single point.

Definition 1.5 (Open Set)**.** Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$.

Now we get our first taste of topology- the main subject we are studying in this course:

Definition 1.6. The *topology* $(X, \mathcal{T})$ induced from a metric space (X, d) is the collection of all open sets

$$\mathcal{T} := \{U \subset X \mid U \text{ is open}\}$$

in (X, d) (by convention, we decree that the empty set $\varnothing \in \mathcal{T}$ is open).

Proposition 1.7. *All open balls $B_r(\mathbf{x})$ are open in (X, d), i.e. $B_r(\mathbf{x}) \in \mathcal{T}$.*

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x - r, x + r)$. Take $\mathbf{y} \in B_r(\mathbf{x})$ such that $d(\mathbf{x}, \mathbf{y}) = q < r$ and consider $B_{(r-q)/2}(\mathbf{y})$: for all $z \in B_{(r-q)/2}(\mathbf{y})$, we have

$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) < q + \frac{r - q}{2} < r,$$

which implies $z \in B_r(x)$. □

Proposition 1.8. *Let (X, d) be a metric space, and $\mathcal{T}$ is the topology induced from (X, d), then*

1. *Let $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be such that $G_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then*

$$\bigcup_{\alpha \in \mathcal{A}} G_\alpha \in \mathcal{T}$$

.

2. *let $G_1, \ldots, G_n \in \mathcal{T}$, then*

$$\bigcap_{i=1}^{n} G_i \in \mathcal{T}$$

that is, the finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ such that

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^{n} G_i$, i.e., $x \in G_i$ for $i = 1, \ldots, n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1, \ldots, n$. Take $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, which implies

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^{n} G_i$

□

Exercise 1.1. *1. Let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrics d_2, d_∞ in $\mathbb{R}^2$. Show that $\mathcal{T}_2 = \mathcal{T}_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.*

2. *Let $\mathcal{T}$ be the topology induced from the discrete metric $(X, d_{\text{ discrete }})$. What is $\mathcal{T}$?*

(*Hint:* for (1), show that an open ball in d_2-metric is open in d_∞; any open set in d_2-metric is open in d_∞; then switch the roles of d_2 and d_∞).

1.2 Understanding Metric Spaces via Topology

In a standard second course of analysis, one studies various properties of metric spaces (X, d) such as closedness, compactness, continuous maps, etc. In this section, we will re-define these familiar notions by just using the topology $\mathcal{T}$ of X.

1.2.1 Closed sets

Definition 1.9 (Closed)**.** A subset $V \subseteq X$ is closed if $X \setminus V$ is open.

Example 1.10. Under the metric space $(\mathbb{R}, d_1)$,

$$\mathbb{R} \setminus [b, a] = (a, \infty) \bigcup (-\infty, b) \text{ is open } \Rightarrow [b, a] \text{ is closed}$$

1. $\varnothing, X$ is closed in X
2. If F_α is closed in X, so is $\bigcap\limits_{\alpha \in A} F_\alpha$.
3. If $F_1, \ldots, F_k$ is closed, so is $\bigcup\limits_{i=1}^{k} F_i$.

Proof. 1. Note that X is open in X, which implies $\varnothing = X \setminus X$ is closed in X; Similarly, $\varnothing$ is open in X, which implies $X = X \setminus \varnothing$ is closed in X;

2. The set F_α is closed implies there exists open $U_\alpha \subseteq X$ such that $F_\alpha = X \setminus U_\alpha$. By De Morgan's Law,

$$\bigcap_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha) = X \setminus \left(\bigcup_{\alpha \in A} U_\alpha \right).$$

By part (1) in Proposition 1.8, the set $\bigcup\limits_{\alpha \in A} U_\alpha$ is open, which implies $\bigcap\limits_{\alpha \in A} F_\alpha$ is closed.

3. The result follows from part (2) in Proposition 1.8 by taking complements. We illustrate examples where open set is used to define convergence and continuity. □

1.2.2 Convergence of sequences

Recall that in the case of metric space (X, d), a sequence of X $\{x_n\} \to x$ converges to $x \in X$ means

$$\forall \varepsilon > 0, \exists N \text{ such that } d(x_n, x) < \varepsilon, \forall n \geq N.$$

We will study the convergence by using $\mathcal{T}$ instead of d.

Proposition 1.11. *Let (X, d) be a metric space, then $\{x_n\} \to x$ if and only if for $\forall$ open set $U \ni x$, there exists N such that $x_n \in U$ for $\forall n \geq N$.*

Proof. Necessity: Since $U \ni x$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Since $\{x_n\} \to x$, there exists N such that $d(x_n, x) < \varepsilon$, i.e., $x_n \in B_\varepsilon(x) \subseteq U$ for $\forall n \geq N$.

Sufficiency: Let $\varepsilon > 0$ be given. Take the open set $U = B_\varepsilon(x) \ni x$, then there exists N □

1.2.3 Continuity

Definition 1.12 (Continuity)**.** Let (X, d) and (Y, ρ) be given metric spaces. Then $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \varepsilon.$$

The function f is continuous on X if f is continuous for all $x_0 \in X$.

As before, we now get rid of metrics to study continuity:

Proposition 1.13. *Let X, Y be metric spaces, and $f : X \to Y$ is a function.*

1. *The function f is continuous at x if and only if for all open $U \ni f(x)$, there exists $\delta > 0$ such that the set $B(x, \delta) \subseteq f^{-1}(U)$.*
2. *The function f is continuous on X if and only if $f^{-1}(U)$ is open in X for each open set $U \subseteq Y$.*

During the proof we will apply a small lemma, whose proof will be left as an exercise to the readers:

Lemma 1.14. *f is continuous at x if and only if for all $\{x_n\} \to x$, we have $\{f(x_n)\} \to f(x)$.*

Proof.

1. Necessity: Due to the openness of $U \ni f(x)$, there exists a ball $B(f(x), \varepsilon) \subseteq U$. Due to the continuity of f at x, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$, which implies:
$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U,$$
and hence $B(x, \delta) \subseteq f^{-1}(U)$.
Sufficiency: Let $\{x_n\} \to x$. By Lemma 1.14, it suffices to show $\{f(x_n)\} \to f(x)$.
By hypothesis, for each open $U \ni f(x)$, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$. Since $\{x_n\} \to x$, there exists N such that
$$x_n \in B_\delta(x) \subseteq f^{-1}(U), \forall n \geq N \Rightarrow f(x_n) \in U, \forall n \geq N$$
We now show that f is continuous in the conventional sense - let $\varepsilon > 0$ be given, and then construct the $U = B_\varepsilon(f(x))$. The argument above shows that $f(x_n) \in B_\varepsilon(f(x))$ for $\forall n \geq N$, which implies $\rho(f(x_n), f(x)) < \varepsilon$, i.e., $\{f(x_n)\} \to f(x)$.
2. For the forward direction, it suffices to show that each point x of $f^{-1}(U)$ is an interior point of $f^{-1}(U)$, which is shown by (1); the converse follows trivially by applying (1).

□

As illustrated above, convergence, continuity, (and compactness) can be defined by using open sets $\mathcal{T}$ only.

Chapter 2

Foundations of Topology

Now we can generalize our observations on metric spaces to topological spaces:

Definition 2.1 (Topological Space)**.** A topological space $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets $\mathcal{T}$ of X such that:

1. $\varnothing, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \cap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup\limits_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

The elements in $\mathcal{T}$ are called *open subsets* of X. The $\mathcal{T}$ is called a topology on X.

Obviously, for any metric space (X, d), the set

$$\mathcal{T} = \{\text{all open subsets of} X\}$$

defines a topology on X.

Definition 2.2. For any set X, define the discrete topology by

$$\mathcal{T}_{\text{dis}} := \{\text{all subsets of } X\}$$

It's clear that $\mathcal{T}_{\text{dis}}$ is a topology on X

Note that if we use the discrete metric (X, d_{dis}), then all subsets are open sets. So we say $(X, \mathcal{T}_{\text{dis}})$ is **induced** from the discrete metric (X, d_{dis}). More generally, we say $(X, \mathcal{T})$ is metrizable if $\mathcal{T}$ is precisely the family of open subsets of some metric d of X.

Here is another extreme of topology with the smallest possible number of open sets:

Example 2.3. Let X be a set containing more than one element. Consider the indiscrete topology $(X, \mathcal{T}_{\text{indis}})$, where:

$$\mathcal{T}_{\text{indis}} = \{\varnothing, X\}.$$

Once again, it is easy to check that it defines a topology of X.

Now we explore whether $(X, \mathcal{T}_{\text{indis}})$ is metrizable or not. Firstly, note that for any metric d defined on X, let x, y be distinct points in X. Then $\varepsilon := d(x, y) > 0$, hence $B_{\frac{1}{2}\varepsilon}(x)$ is a open set belonging to the corresponding induced topology. Since $x \in B_{\frac{1}{2}\varepsilon}(x)$ and $y \notin B_{\frac{1}{2}\varepsilon}(y)$, we conclude that $B_{\frac{1}{2}\varepsilon}(x)$ is neither $\varnothing$ nor X, i.e., the topology induced by any metric d is not the indiscrete topology.

Example 2.4. For any set X, consider the cofinite topology $(X, \mathcal{T}_{\text{cofin}})$:

$$\mathcal{T}_{\text{cofin}} = \{U \mid X \setminus U \text{is a finite set}\} \bigcup \{\varnothing\}$$

(Question - is $(X, \mathcal{T}_{\text{cofin}})$ metrizable?)

Definition 2.5 (Equivalence)**.** Two metrics d_1, d_2 of X are topologically equivalent if they give rise to the same topology.

For instance, the metrics d_1, d_2, d_∞ in $\mathbb{R}^n$ are topologically equivalent.

Definition 2.6 (Closed set)**.** Let $(X, \mathcal{T})$ be a topological space. Then $V \subseteq X$ is closed if $X \setminus V \in \mathcal{T}$

Example 2.7. Under the usual topology $(\mathbb{R}, \mathcal{T}_{\text{usual}})$, $(b, \infty) \cup (-\infty, a) \in \mathcal{T}$. Therefore,

$$[a, b] = \mathbb{R} \setminus \Big((b, \infty) \bigcup (-\infty, a)\Big)$$

is closed in $\mathbb{R}$ under usual topology.

It is **important** to say that V is closed in X. You need to specify the underlying the space X.

Proposition 2.8. *Let $(X, \mathcal{T})$ be a topological space.*

1. *$\varnothing, X$ are closed in X*
2. *V_1, V_2 closed in X implies that $V_1 \cup V_2$ closed in X*
3. *$\{V_\alpha \mid \alpha \in \mathcal{A}\}$ closed in X implies that $\bigcap_{\alpha \in \mathcal{A}} V_\alpha$ closed in X*

Proof. (1) is obvious. For (2) and (3), one can apply De Morgan's Law $\left(X \setminus \bigcup_{i \in I} U_i\right) = \bigcap_{i \in I} (X \setminus U_i)$ □

2.1 Convergence

Inspired by the discussions in Chapter 1 on metric spaces, we have:

Definition 2.9 (Convergence)**.** A sequence $\{x_n\}$ of a topological space $(X, \mathcal{T})$ converges to $x \in X$ if for all open $U \ni x$, there exists N such that $x_n \in U, \forall n \geq N$.

Example 2.10. 1. For the usual topology $(X = \mathbb{R}^n, d_2) \to (X, \mathcal{T})$, convergence of sequence in $(\mathbb{R}^n, \mathcal{T})$ is the usual convergence in analysis.

Note that for $\mathbb{R}^n$ or metric space, the limit of sequence (if exists) is unique. However, this is no longer true if the topology is not metrizable.

2. Consider the topological space $(X, \mathcal{T}_{\text{indis}})$. Take any sequence $\{x_n\}$ in X, it is convergent to any $x \in X$. Indeed, For all $U \ni x$ open, the only possibility is $U = X$. Therefore,
$$x_n \in U\,(= X)\,, \forall n \geq 1.$$
3. Consider the topological space $(X, \mathcal{T}_{\text{cofin}})$, where X is infinite. Consider $\{x_n\}$ is a sequence satisfying $m \neq n$ implies $X_m \neq x_n$. Then $\{x_n\}$ is convergent to any $x \in X$.
4. Consider the topological space $(X, \mathcal{T}_{\text{discrete}})$, the sequence $\{x_n\} \to x$ is equivalent to saying $x_n = x$ for all sufficiently large n.

One lesson we learned from the previous example is that the limit of sequences may not be unique. In such a case, the reason is that $\mathcal{T}$ is not "big enough". We will give a criterion to make sure the limit is unique in the future. (Hausdorff)

Proposition 2.11. *If $F \subseteq (X, \mathcal{T})$ is closed, then for any convergent sequence $\{x_n\}$ in F, the limit(s) are also in F.*

Proof. Let $\{x_n\}$ be a sequence in F with limit $x \in X$. Suppose on the contrary that $x \notin F$ (i.e., $x \in X \setminus F$ that is open). There exists N such that
$$x_n \in X \setminus F, \forall n \geq N$$
i.e., $x_n \notin F$, which is a contradiction. □

If the $(X, \mathcal{T})$ is metrizable, the converse holds. Otherwise, the converse may not be true.

Example 2.12. Consider the co-countable topological space $(X = \mathbb{R}, \mathcal{T}_{\text{co-count}})$, where
$$\mathcal{T}_{\text{co}-\text{count}} = \{U \mid X \setminus U \text{is a countable set}\} \bigcup \{\varnothing\},$$
and X is uncountable. Then note that $F = [0, 1] \subsetneqq X$ is an un-countable set, and under co-countable topology, $F \supseteq \{x_n\} \to x$ implies $x_n = x \in F$ for all n. It is clear that $X \setminus F \notin \mathcal{T}_{\text{co}-\text{count}}$, that is, F is not closed.

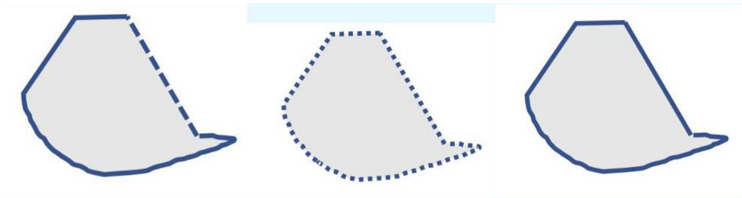
2.2 Interior, Closure, Boundary

Definition 2.13. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.

1. The interior of A is
$$A^{\circ} = \bigcup_{U \subseteq A, U \text{ is open}} U$$
2. The closure of A is
$$\bar{A} = \bigcap_{A \subseteq V, V \text{ is closed}} V$$

If $\bar{A} = X$, we say that A is dense in X.

The above definition can be illustrated as follows:



(a) Illustration of A (b) Illustration of A° (c) Illustration of $\overline{A}$

Example 2.14.

- For $[a, b) \subseteq \mathbb{R}$, we have:

$$[a, b)^\circ = (a, b)\,,\ \overline{[a, b)} = [a, b]$$

- For $X = \mathbb{R}, \mathbb{Q}^\circ = \varnothing$ and $\overline{\mathbb{Q}} = \mathbb{R}$.
- Consider the discrete topology $(X, \mathcal{T}_{\text{discrete}})$, we have

$$S^\circ = S,\ \bar{S} = S$$

The insights behind Definition 2.13 is as follows:

Proposition 2.15.

1. *A° is the largest open subset of X contained in A ;*
2. *$\bar{A}$ is the smallest closed subset of X containing A.*
3. *If $A \subseteq B$, then $A^\circ \subseteq B$ and $\bar{A} \subseteq \bar{B}$*
4. *A is open in X is equivalent to say $A^\circ = A$; A is closed in X is equivalent to say $\bar{A} = A$.*

Example 2.16. Let (X, d) be a metric space. What's the closure of an open ball $B_r(x)$? The direct intuition is to define the closed ball

$$\bar{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}.$$

One would like to know whether $\bar{B}_r(x) = \overline{B_r(x)}$. Indeed, since $\bar{B}_r(x)$ is a closed subset of X, and $B_r(x) \subseteq \bar{B}_r(x)$, we imply that

$$\overline{B_r(x)} \subseteq \bar{B}_r(x)$$

However, we may find an example such that $\overline{B_r(x)} \subsetneq \bar{B}_r(x)$ - consider the discrete metric space (X, d_{dis}). Then for all $x \in X$,

$$B_1(x) = \{x\} \Rightarrow \overline{B_1(x)} = \{x\},\ \bar{B}_1(x) = X$$

The equality $\bar{B}_r(x) = \overline{B_r(x)}$ holds when (X, d) is a normed space.

Here is another characterization of $\bar{A}$:

Proposition 2.17.

$$\bar{A} = \{x \in X \mid \text{for all open } U \ni x,\ U \cap A \neq \varnothing\}$$

Proof. Define

$$S = \{x \in X \mid \text{for all open } U \ni x, U \cap A \neq \varnothing\}$$

It suffices to show that $\bar{A} = S$:

Claim: S is closed, or equivalently $X \setminus S$ is open: Note that

$$X \setminus S = \{x \in X \mid \exists\ U_x \ni x \text{ open such that } U_x \cap A = \varnothing\}$$

Take $x \in X \setminus S$, i.e. there exists open $U_x \ni x$ such that $U_x \cap A = \varnothing$. Then our claim follows if one can prove that

$$U_x \subseteq X \setminus S.$$

Indeed, for all $y \in U_x$, since $U_x \ni y$ and $U_x \cap A = \varnothing$ by definition, hence $y \in X \setminus S$. Therefore,

$$X \setminus S = \bigcup_{x \in X \setminus S} \{x\} \subseteq \bigcup_{x \in X \setminus S} U_x \subseteq X \setminus S,$$

which implies $X \setminus S = \bigcup_{x \in X \setminus S} U_x$ is open, i.e., S is closed in X and the claim is proved.

Now go back to the proof. By definition, it is clear that $A \subseteq S$, since

$$\forall a \in A, \forall \text{ open } U \ni a,\ U \cap A \supseteq \{a\} \neq \varnothing$$

which implies that $a \in S$. Therefore,

$$\bar{A} \subseteq \bar{S} = S$$

by the above claim. And one needs to show it is an equality.

Suppose on contrary that there exists $y \in S \setminus \bar{A}$. Since $y \notin \bar{A}$, by definition, there exists $F \supseteq A$ closed such that $y \notin F$. Therefore, $y \in X \setminus F$ that is open, and

$$(X \setminus F) \cap A \subseteq (X \setminus A) \cap A = \varnothing \Rightarrow y \notin S,$$

which is a contradiction. Therefore, $S = \bar{A}$. $\square$

Definition 2.18 (Accumulation point)**.** Let $A \subseteq X$ be a subset in a topological space. We call $x \in X$ an *accumulation point (limit point)* of A if

$$\text{for all } U \subseteq X \text{ open such that } U \ni x, (U \setminus \{x\}) \cap A \neq \varnothing.$$

Example 2.19. 1. There exists some point in A but not in A' :

$$A = \{1, 2, 3, \ldots, n, \ldots\}$$

Then any point in A is not in A'.

2. There also exists some point in A' but not in A :

$$A = \left\{ \frac{1}{n} \middle| \; n \geq 1 \right\}$$

Then the point 0 is in A' but not in A.

Proposition 2.20. $\bar{A} = A \bigcup A'$.

Proof. This proposition directly follows from Proposition 2.17 and the definition of A′. □

Definition 2.21 (Sequential Closure)**.** Let A_S be the set of limits of any convergent sequence in A, then A_S is called the sequential closure of A.

Lemma 2.22. *Retain the above notations. Then*

$$A \subseteq A_S \subseteq \bar{A}.$$

Proof. It is clear that $A \subseteq A_S$, since the sequence $\{a_n := a\}$ for all $n \in \mathbb{N}$ is convergent to a for all $a \in A$.

For the other inclusion, suppose $a \in A_S$. Then we have $\{a_n\} \to a$ for some sequence $\{a_n\}$. Therefore, for any open $U \ni a$, there exists N such that $\{a_N, a_{N+1}, \ldots\} \subseteq U \cap A \neq \varnothing$. Therefore, $a \in \bar{A}$, i.e., $A_S \subseteq \bar{A}$. □

We would like to know if the sequential closure is equal to the closure. And the answer is yes for metric spaces:

Proposition 2.23. *Let* (X, d) *be a metric space, then* $A_S = \bar{A}$.

Proof. In view of the lemma above, one only needs to prove the inclusion $\bar{A} \subseteq A_S$.

Let $a \in \bar{A}$, then there exists $a_n \in B_{1/n}(a) \cap A$, which implies $\{a_n\} \to a$, i.e., $a \in A_S$. □

Indeed, the same result goes for first countable topological spaces. However, A_S is a proper subset of $\bar{A}$ in general: Let $A \subseteq X$ be the set of continuous functions, where $X = \mathbb{R}^{\mathbb{R}}$ denotes the set of all real-valued functions on $\mathbb{R}$, with the topology of pointwise convergence. Then $A_S = B_1$, the set of all functions of first Baire-Category on $\mathbb{R}$; and $[A_S]_S = B_2$, the set of all functions of second Baire-Category on $\mathbb{R}$. Since $B_1 \neq B_2$, we have $[A_S]_S = A_S$. Note that $\bar{\bar{A}} = \bar{A}$. We conclude that A_S cannot equal to $\bar{A}$, since the sequential closure operator cannot be idempotent.

Definition 2.24 (Boundary)**.** The boundary of A is defined as

$$\partial A = \bar{A} \setminus A^{\circ}$$

Proposition 2.25. *Let $(X, \mathcal{T})$ be a topological space with $A, B \subseteq X$. Then*

1. $\overline{X \setminus A} = X \setminus A^\circ$,
2. $(X \setminus B)^\circ = X \setminus \bar{B}$,
3. $\partial A = \bar{A} \cap \left(\overline{X \setminus A}\right)$.

Proof. 1. $X \setminus A^\circ = X \setminus \left(\bigcup_{U \text{is open}, U \subseteq A} U \right) = \bigcap_{U \text{is open}, U \subseteq A} (X \setminus U) = \bigcap_{V \text{is closed}, F \supseteq X \setminus A} F = \overline{X \setminus A}$.

2. Denote $B := X \setminus A$ by B, we obtain:

$$(X \setminus B)^\circ = A^\circ = X \setminus (X \setminus A^\circ) = X \setminus \overline{X \setminus A} = X \setminus \bar{B}.$$

3. By definition of ∂A,

$$\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap (X \setminus A^\circ) = \bar{A} \cap \left(\overline{X \setminus A}\right)$$

□

2.3 Functions on Topological Space

We now study maps $f : X \to Y$ between topological spaces. As in linear algebra/abstract algebra, we would like our map to preserve certain properties of the spaces (e.g. we study **linear transformations** for linear algebra, and **group homomorphisms** for groups).

In the case of topological spaces, we have:

Definition 2.26 (Continuous Map)**.** Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. Then the function f is continuous, if

$$U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$$

Example 2.27. 1. The identity map $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined as $x \mapsto x$ is continuous

2. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{indiscrete}})$ defined as $x \mapsto x$ is continuous. Since $\mathrm{id}^{-1}(\varnothing) = \varnothing$ and $\mathrm{id}^{-1}(X) = X$

3. The identity map $\mathrm{id} : (X, \mathcal{T}_{\text{indiscrete}}) \to (X, \mathcal{T}_{\text{discrete}})$ defined as $x \mapsto x$ is not continuous.

Proposition 2.28. *If $f : X \to Y$, and $g : Y \to Z$ be continuous, then $g \circ f$ is continuous*

Proof. For given $U \in \mathcal{T}_Z$, we imply

$$g^{-1}(U) \in \mathcal{T}_Y \Rightarrow f^{-1}\left(g^{-1}(U)\right) \in \mathcal{T}_X,$$

i.e., $(g \circ f)^{-1}(U) \in \mathcal{T}_X$ □

The following proposition is well-known in calculus and metric spaces. This remains to be true for topological spaces:

Proposition 2.29. *Suppose $f : X \to Y$ is continuous between two topological spaces. Then $\{x_n\} \to x$ implies $\{f(x_n)\} \to f(x)$.*

Proof. Take open $U \ni f(x)$, which implies $f^{-1}(U) \ni x$. Since $f^{-1}(U)$ is open, we imply there exists N such that

$$\{x_n \mid n \geq N\} \subseteq f^{-1}(U)$$

i.e., $\{f(x_n) \mid n \geq N\} \subseteq U$ $\square$

Here is the definition in topology analogous to isomorphism in algebra:

Definition 2.30 (Homeomorphism)**.** A homeomorphism between spaces topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

such that both f and f^{-1} are continuous.

WARNING: In linear algebra and abstract algebra, if $f : X \to Y$ is a bijective linear transformation or homomorphism, then its inverse $f^{-1} : Y \to X$ is automatically a linear transformation or homomorphism. However, this is **NOT TRUE** for topological spaces! Therefore, we need to decree that both f and f^{-1} are continuous in our definition above.

Example 2.31. The sets $(0, \infty)$ and $\mathbb{R}$ are homeomorphic, since

$$\exp : \mathbb{R} \to (0, \infty)$$

is bijective and continuous, with inverse

$$\log : (0, \infty) \to \mathbb{R}$$

also being continuous.

2.4 New Spaces From Old

2.4.1 Subspace Topology and Basis

Let $A \subseteq X$ be a subset, we define a topology of A from that of X:

Definition 2.32. Let $A \subseteq X$ be a non-empty set. The subspace topology of A is defined by any of the equivalent ways:

1. $\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}_X\}$.
2. $\mathcal{T}_A$ is the coarsest topology on A such that the inclusion map

$$i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X), \; i(x) = x$$

is continuous. (We say the topology $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, or $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ e.g., $\mathcal{T}_{\text{discrete}}$ is the finest topology, and $\mathcal{T}_{\text{indiscrete}}$ is coarsest topology.)

3. $\mathcal{T}_A$ is the unique topology such that for any topological spaces $(Y, \mathcal{T}_Y)$,

$$f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A) \text{ is continuous iff } i \circ f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X) \text{ is continuous.}$$

Exercise 2.1.

1. *The definition (1) in Definition 2.32 does define a topology of A.*
2. *The closed sets of A under subspace topology are of the form $V \cap A$, where V is closed in X*
3. *Show that (1) and (2) are equivalent (Hint: one can make sure of the easy fact that $i^{-1}(S) = S \cap A$ for all subsets S).*

Example 2.33. Let all English and numerical letters be subset of $\mathbb{R}^2$. Then

P, 6

are homeomorphic using the subspace topology.

Proposition 2.34. *The statements in Definition 2.32 are equivalent.*

Proof. We only need to prove (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1):

(**1**) $\Rightarrow$ (**3**): Suppose $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}_X\}$. Let $f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$, then by (1) $\Leftrightarrow$ (2), $i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X)$ is continuous. So

$$f \text{ continuous } \Rightarrow \ i \circ f \text{ continuous}$$

follows immediately from Proposition 2.28.

For the other direction, suppose $i \circ f$ is continuous. Then for all $U \in \mathcal{T}_X$, $f^{-1}(i^{-1}(U)) = f^{-1}(U \cap A)$ is open in Y. Therefore, for all $V \in \mathcal{T}_A$, it is of the form $V := U \cap A$ for some $U \in \mathcal{T}_X$, and hence we have

$$\text{For every } V = U \cap A \in \mathcal{T}_A,\ f^{-1}(V) \text{ is open in } Y.$$

which is precisely the condition for f to be continuous.

(**3**) $\Rightarrow$ (**1**): Suppose $\mathcal{T}_A$ is a topology on A with the property: for every space Y and map $f : Y \to A$,

$$f \text{ is continuous } \Leftrightarrow \ i \circ f \text{ is continuous.}$$

Then we need to show $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}_X\}$:

Suppose $V \in \mathcal{T}_A$. Consider $Y = (A, \mathcal{T}_A)$, $f = \mathrm{id} : A \to A$. Then $i \circ f = i$ is continuous by hypothesis. In other words,

$$U \in \mathcal{T}_X \ \Rightarrow \ i^{-1}(U) = U \cap A \in \mathcal{T}_A.$$

and hence

$$\{U \cap A \mid U \in \mathcal{T}_X\} \subseteq \mathcal{T}_A.$$

But by (1) $\Leftrightarrow$ (2), $\mathcal{T}_A$ is the coarsest topology making i a continuous map. By Exercise 2.1(1), this implies the inclusion is an equality, i.e. $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}_X\}$. □

Proposition 2.35. *Suppose $(A, \mathcal{T}_A) \subseteq (X, \mathcal{T}_X)$ is a subspace topology, and $B \subseteq A \subseteq X$. Then*

1. $\bar{B}^A = \bar{B}^X \cap A$.
2. $B^{\circ A} \supseteq B^{\circ X}$.

Proof.

1. By Proposition 2.34, $\bar{B}^X \cap A$ is closed in A, and $\bar{B}^X \cap A \supseteq B$, which implies

   $$\bar{B}^A \subseteq \bar{B}^X \cap A$$

   Note that $\bar{B}^A \supseteq B$ is closed in A, which implies $\bar{B}^A = V \cap A \subseteq V$, where V is closed in X. Therefore,

   $$\bar{B}^X \subseteq V \Rightarrow \bar{B}^X \cap A \subseteq V \cap A = \bar{B}^A$$

   Therefore, $\bar{B}^A = \bar{B}^X \cap A \subseteq V$
2. Let $x \in B^{\circ X}$. Then, there exists $U_x \in \mathcal{T}_X$ such that $x \in U_x \subseteq B$. This implies

   $$U_x \cap A \in \mathcal{T}_A \text{ and } x \in U_x \cap A \subseteq B.$$

   By Definition 2.13, $x \in B^{\circ A} = \bigcup_{U \subseteq B, U \in \mathcal{T}_A} U$.

□

Remark 2.36. The inclusion may be strict on 2. A simple example is when $X = \mathbb{R}$, $A = [0, 2]$ and $B = [0, 1)$.

As in vector spaces, we would like to look at the a subset of $\mathcal{T}$ that 'generates' the whole topology $\mathcal{T}$.

Definition 2.37 (Basis)**.** Let $(X, \mathcal{T})$ be a topological space. A family of subsets $\mathcal{B}$ in X is a basis (or base) for $\mathcal{T}$ if

1. $\mathcal{B} \subseteq \mathcal{T}$, i.e., everything in $\mathcal{B}$ is open
2. Every $U \in \mathcal{T}$ can be written as union of elements in $\mathcal{B}$.

Example 2.38.

1. $\mathcal{B} = \mathcal{T}$ is a basis.
2. For $X = \mathbb{R}^n$,

   $$\mathcal{B} = \{B_r(\mathbf{x}) \mid \mathbf{x} \in \mathbb{Q}^n, r \in \mathbb{Q} \cap (0, \infty)\}$$

   is a basis of the usual topology of X (note that $\mathcal{B}$ is countable but $\mathcal{T}$ is uncountable.)

Proposition 2.39. *Let X, Y be topological spaces, and $\mathcal{B}$ a basis for topology on Y. Then*

$$f : X \to Y \textit{ is continuous} \Leftrightarrow f^{-1}(B) \textit{ is open in } X, \forall B \in \mathcal{B}.$$

In other words, to check whether f is continuous, it suffices to check $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

Proof. The forward direction follows from the fact $B \subseteq \mathcal{T}_Y$.

To show the reverse direction, let $U \in \mathcal{T}_Y$, then $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$, which implies

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

which is open in X since it is union of opens. □

Corollary 2.40. *Let $f : X \to Y$ be a bijection. Suppose there is a basis $\mathcal{B}_X$ of $\mathcal{T}_X$ such that $\{f(B) \mid B \in \mathcal{B}_X\}$ forms a basis of $\mathcal{T}_Y$. Then $X \cong Y$ is homeomorphic.*

Proof. Suppose $W \in \mathcal{T}_Y$, then by our hypothesis,

$$W = \bigcup_{i \in I} f(B_i), B_i \in \mathcal{B}_X \Rightarrow f^{-1}(W) = \bigcup_{i \in I} B_i \in \mathcal{T}_X,$$

which implies f is continuous.

On the other hand, suppose $U \in \mathcal{T}_X$, then

$$U = \bigcup_{i \in I} B_i \Rightarrow f(U) = \bigcup_{i \in I} f(B_i) \in \mathcal{T}_Y \Rightarrow \left[f^{-1}\right]^{-1}(U) \in \mathcal{T}_Y,$$

i.e., f^{-1} is continuous. □

To recognize whether a family of subsets is a basis for some given topology, one can use the following:

Proposition 2.41. *Let X be a set, $\mathcal{B}$ is a collection of subsets satisfying*

1. *X is a union of sets in $\mathcal{B}$, i.e., every $x \in X$ lies in some $B_x \in \mathcal{B}$*
2. *The intersection $B_1 \cap B_2$ For all $B_1, B_2 \in \mathcal{B}$ is a union of sets in $\mathcal{B}$, i.e., for each $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Then the collection of subsets $\mathcal{T}_{\mathcal{B}}$, formed by taking any union of sets in $\mathcal{B}$, is a topology, and $\mathcal{B}$ is a basis for $\mathcal{T}_B$.

Proof. Firstly, for $x \in X, B_x \in \mathcal{B}$, by hypothesis (1),

$$X = \bigcup_{x \in X} B_x \in \mathcal{T}_{\mathcal{B}}$$

Now we show $\mathcal{T}_{\mathcal{B}}$ is closed under intersection: Let $T_1, T_2 \in \mathcal{T}_{\mathcal{B}}$. Let $x \in T_1 \cap T_2$, where T_i is a union of subsets in $\mathcal{B}$. Therefore,

$$\begin{cases} x \in B_1 \subseteq T_1, & B_1 \in \mathcal{B} \\ x \in B_2 \subseteq T_2, & B_2 \in \mathcal{B} \end{cases}$$

which implies $x \in B_1 \cap B_2$, i.e., $x \in B_x \subseteq B_1 \cap B_2$ for some $B_x \in \mathcal{B}$. Therefore,

$$\bigcup_{x \in B_1 \cap B_2} \{x\} \subseteq \bigcup_{x \in B_1 \cap B_2} B_x \subseteq B_1 \cap B_2$$

it is easy to see $\bigcup_{x \in B_1 \cap B_2} B_x \supseteq B_1 \cap B_2$, $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x$. Hence, $B_1 \cap B_2 \in \mathcal{T}_{\mathcal{B}}$.

Finally, the property that $\mathcal{T}_{\mathcal{B}}$ is closed under union operations can be checked directly, which completes the proof. □

2.4.2 Product Space and Disjoint Union

Now we discuss how to construct new topological spaces out of given ones is by taking Cartesian products:

Definition 2.42. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Consider the family of subsets in $X \times Y$:

$$\mathcal{B}_{X \times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

This $\mathcal{B}_{X \times Y}$ forms a basis of a topology on $X \times Y$. The induced topology from $\mathcal{B}_{X \times Y}$ is called product topology.

For example, for $X = \mathbb{R}, Y = \mathbb{R}$, the elements in $\mathcal{B}_{X \times Y}$ are rectangles.

Proposition 2.43. *Definition 2.42 does form a basis of a topology of $X \times Y$.*

Proof. We apply Proposition 2.41 to check whether $B_{X \times Y}$ forms a basis:

1. For any $(x, y) \in X \times Y$, we imply $x \in X, y \in Y$. Note that $X \in \mathcal{T}_X, Y \in \mathcal{T}_Y$, we imply $(x, y) \in X \times Y \in \mathcal{B}_{X \times Y}$.

2. Suppose $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}_{X \times Y}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

where $U_1 \cap U_2 \in \mathcal{T}_X, V_1 \cap V_2 \in \mathcal{T}_Y$. Therefore, $(U_1 \times V_1) \cap (U_2 \times V_2) \in \mathcal{B}_{X \times Y}$. □

Note that we only define the product topology $\mathcal{T}_{X \times Y}$ of $X \times Y$ by specifying its basis $\mathcal{B}_{X \times Y} \subset \mathcal{T}_{X \times Y}$.

Example 2.44. The space $\mathbb{R} \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^2$, where the product topology is defined on $\mathbb{R} \times \mathbb{R}$ and the standard topology is defined on $\mathbb{R}^2$.

To see so, construct the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ with $f(a,b):=(a,b)$. Obviously, $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ is a bijection.

Take the basis of the topology on $\mathbb{R}$ as open intervals, i.e. $\mathcal{B}_{\mathbb{R}}=\{(a,b)\mid a<b \text{ in } \mathbb{R}\}$ Then the set

$$\mathcal{B}:=\{(a,b)\times(c,d)\mid a<b,c<d\}$$

forms a basis for the product topology of $\mathbb{R}\times\mathbb{R}$. On the other hand, one can easily check that the rectangles in $\mathbb{R}^2$:

$$\{f(B)\mid B\in\mathcal{B}\}=\{(a,b)\times(c,d)\mid a<b,c<d\}$$

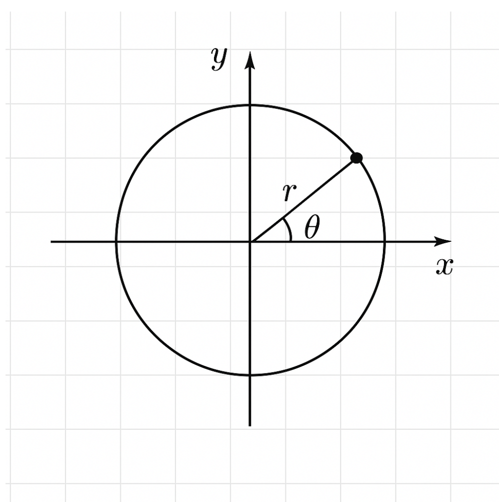
forms a basis of the usual topology in $\mathbb{R}^2$. By Corollary 2.40, we imply $\mathbb{R}\times\mathbb{R}\cong\mathbb{R}^2$.

Example 2.45. Let $S^1=\{(\cos x,\sin x)\mid x\in[0,2\pi)\}$ be a unit circle on $\mathbb{R}^2$ (you can treat S^1 to be equipped with subspace topology of $\mathbb{R}^2$). Consider

$$f:S^1\times(0,\infty)\to\mathbb{R}^2\setminus\{\mathbf{0}\}$$

(where $S^1\times(0,\infty)$ is equipped with product topology) given by

$$f(\cos x,\sin x,r):=(r\cos x,r\sin x).$$



It's clear that f is a bijection, and f is continuous. Moreover, the inverse $g:=f^{-1}$ is defined as

$$g(a,b)=\left(\frac{a}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}},\sqrt{a^2+b^2}\right)$$

which is continuous as well. Therefore, the $f:\mathcal{S}^1\times(0,\infty)\to\mathbb{R}^2\setminus\{\mathbf{0}\}$ is a homeomorphism.

Proposition 2.46. *a ring torus is homeomorphic to the Cartesian product of two circles, say* $S^1\times S^1\cong T$.

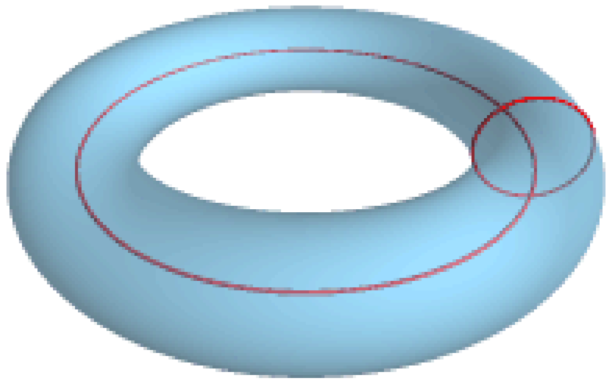


Figure 2.1: Ring torus degenerating into a spindle torus.(The animation only works in Adobe.)

Proof. Let $S^1 \times S^1 = \{(\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)) \mid 0 \leq \theta, \phi < 2\pi\}$. The product topology of $S^1 \times S^1$ is just the subspace topology of $\mathbb{R}^4$. Define a mapping $f : S^1 \times S^1 \to T$ by

$$f(\theta, \phi) = ((R + r\cos\theta)\cos\phi,\ (R + r\cos\theta)\sin\phi,\ r\sin\theta)$$

for $R > r$. Denote $i : T \to \mathbb{R}^3$ as the usual inclusion, then we imply

$$i \circ f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3 \text{ is continuous}$$

Therefore, by Proposition 2.34 we imply $f : [0, 2\pi) \times [0, 2\pi) \to T$ is continuous. We can also show it is bijective, and f^{-1} is continuous as in the previous example which are left as exercises. $\square$

Proposition 2.47. *Let $X \times Y$ be endowed with product topology. Then the projection mappings defined as*

$$p_X : X \times Y \to X, \text{ with } p_X(x, y) = x, \qquad p_Y : X \times Y \to Y, \text{ with } p_Y(x, y) = y$$

are continuous. Moreover:

1. *The product topology is the coarsest topology on $X \times Y$ such that p_X and p_Y are both continuous.*
2. *Let Z be a topological space, then the product topology is the unique topology that the red and the blue line in the diagram commutes:*

$$\begin{tikzcd} & Z \arrow[dl, "p_X \circ F"'] \arrow[d, "F"] \arrow[dr, "p_Y \circ F"] & \\ X & X \times Y \arrow[l, "p_X"'] \arrow[r, "p_Y"] & Y \end{tikzcd}$$

namely, the mapping $F : Z \to X \times Y$ is continuous iff both $p_X \circ F : Z \to X$ and $p_Y \circ F : Z \to Y$ are continuous.

Proof. The first statement of the proposition is easy - for any open U, we imply $p_X^{-1}(U) = U \times Y \in \mathcal{B}_{X\times Y} \subseteq \mathcal{T}_{X\times Y}$, i.e., $p_X^{-1}(U)$ is open. The same goes for p_Y.

1. It suffices to show any topology $\mathcal{T}$ that meets the condition in (2) must contain $\mathcal{T}_{\text{product}}$. We imply that For all $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$,

$$\begin{cases} p_X^{-1}(U) = U \times Y \in \mathcal{T} \\ p_Y^{-1}(V) = X \times V \in \mathcal{T} \end{cases} \Rightarrow (U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V \in \mathcal{T},$$

   which implies $\mathcal{B}_{X\times Y} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is closed for union operation on subsets, we imply $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}$.

2. First, we show that $\mathcal{T}_{\text{product}}$ satisfies the commutative diagram.

   - For the forward direction, by the first statement of the proposition, we imply both $p_X \circ F$ and $p_Y \circ F$ are continuous, since the composition of continuous functions are continuous as well.
   - For the reverse direction, For all $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$,

   $$F^{-1}(U \times V) = (p_X \circ F)^{-1}(U) \cap (p_Y \circ F)^{-1}(V),$$

     which is open due to the continuity of $p_X \circ F$ and $p_Y \circ F$.

   Next, we show the uniqueness of $\mathcal{T}_{\text{product}}$. Let $\mathcal{T}$ be another topology $X \times Y$ satisfying the commutative diagram:

   - Take $Z = (X \times Y, \mathcal{T})$, and consider the identity mapping $F = \text{id} : Z \to Z$, which is continuous. Therefore $p_X \circ \text{id}$ and $p_Y \circ \text{id}$ are continuous, i.e., p_X and p_Y are continuous. By (1), we imply $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}$.
   - Take $Z = (X \times Y, \mathcal{T}_{\text{product}})$, and consider the identity mapping $F = \text{id} : Z \to Z$. Note that $p_X \circ F = p_X$ and $p_Y \circ F = p_Y$, which is continuous. Therefore, the identity mapping $F : (X \times Y, \mathcal{T}_{\text{product}}) \to (X \times Y, \mathcal{T})$ is continuous, which implies

   $$U = \text{id}^{-1}(U) \subseteq \mathcal{T}_{\text{product}} \text{ for all } U \in \mathcal{T},$$

     i.e., $\mathcal{T} \subseteq \mathcal{T}_{\text{product}}$.

Consequently, the proof is complete. $\square$

Definition 2.48 (Disjoint Union)**.** Let $X \times Y$ be two topological spaces, then the *disjoint union* of X and Y is

$$X \coprod Y := (X \times \{0\}) \cup (Y \times \{1\}) \,,$$

where $\mathcal{T}_{X \coprod Y}$ consists of all $U \subseteq X \coprod Y$ satisfying:

- $U \cap (X \times \{0\})$ is open in $X \times \{0\}$; and
- $U \cap (Y \times \{1\})$ is open in $Y \times \{1\}$.

(Exercise: show $\mathcal{T}_{X \coprod Y}$ defines a topology)

In other words, S is open in $X \coprod Y$ iff S can be expressed as

$$S = (U \times \{0\}) \cup (V \times \{1\})$$

where $U \subseteq X$ is open and $V \subseteq Y$ is open.

Chapter 3

Properties of Topological Spaces

3.1 Hausdorff Property

Definition 3.1 (First Separation Axiom, T_1)**.** A topological space X is said to satisfy the *first separation axiom* (or to be a T_1 *space*) if for every pair of distinct points $x, y \in X$, there exists an open set $U \subseteq X$ such that

$$x \in U \quad \text{and} \quad y \notin U.$$

Proposition 3.2. *A topological space X satisfies the first separation axiom (T_1) if and only if every singleton set $\{x\}$ is closed in X.*

Proof. *($\Rightarrow$)* Assume X has the first separation property. Fix $x \in X$. For each $y \in X \setminus \{x\}$, there exists an open set U_y with $y \in U_y$ and $x \notin U_y$. Then

$$X \setminus \{x\} = \bigcup_{y \neq x} \{y\} \subseteq \bigcup_{y \neq x} U_y \subseteq X \setminus \{x\}.$$

Hence $X \setminus \{x\} = \bigcup_{y \neq x} U_y$ is open, so $\{x\}$ is closed.

($\Leftarrow$) Suppose every singleton is closed. Given $x \neq y$, the set

$$U := X \setminus \{y\}$$

is open, contains x, and does not contain y. Thus X has the first separation property. $\square$

Definition 3.3 (Hausdorff Space)**.** A topological space X is called *Hausdorff* (or said to satisfy the second separation axiom, T_2) if for every pair of distinct points $x, y \in X$ there exist open neighborhoods U of x and V of y such that

$$U \cap V = \varnothing.$$

Example 3.4. Every metrizable topological space is Hausdorff. Indeed, let (X, d) be a metric space and take two distinct points $x, y \in X$. Since $d(x, y) = r > 0$, consider the open balls

$$U := B_{r/2}(x), \qquad V := B_{r/2}(y).$$

Then U and V are disjoint open neighborhoods of x and y, respectively. Hence (X, d) is Hausdorff.

Note that a topological space that is first separable may not necessarily be second separable:

Example 3.5. Consider $\mathcal{T}_{\text{co-finite}}$, then X is first separable but not Hausdorff if X is infinite: Suppose on the contrary that for given $x \neq y$, there exists open sets U, V such that $x \in U, y \in V$, and

$$U \cap V = \varnothing \Rightarrow X = X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V),$$

implying that the union of two finite sets equals X which is infinite, so we have a contradiction.

We now explore some nice properties of a Hausdorff topological space:

Proposition 3.6. *If the topological space $(X, \mathcal{T})$ is Hausdorff, then all sequences $\{x_n\}$ in X has at most one limit.*

Proof. Suppose on the contrary that

$$\{x_n\} \to a, \ \{x_n\} \to b, \text{ with } a \neq b$$

By Hausdorff-ness, there exists $U, V \in \mathcal{T}$ and $U \cap V = \varnothing$ such that $U \ni a$ and $V \ni b$.

By tie openness of U, there exists N such that $\{x_N, x_{N+1}, \ldots\} \subseteq U$, since $\{x_n\} \to a \in U$. Similarly, there exists M such that $\{x_M, x_{M+1}, \ldots\} \subseteq V$. Let $K = \max\{M, N\} + 1$, then $\varnothing \neq U \cap V \ni x_K$, which is a contradiction. □

Proposition 3.7. *Let X and Y be Hausdorff spaces. Then the product space $X \times Y$, equipped with the product topology, is Hausdorff.*

Proof. Suppose $(x_1, y_1) \neq (x_2, y_2)$ in $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$.

Without loss of generality, assume $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets $U, V \subseteq X$ such that $x_1 \in U$ and $x_2 \in V$. Then

$$(x_1, y_1) \in U \times Y, \quad (x_2, y_2) \in V \times Y, \quad (U \times Y) \cap (V \times Y) = \varnothing.$$

Thus (x_1, y_1) and (x_2, y_2) have disjoint open neighborhoods. Therefore, $X \times Y$ is Hausdorff. □

The same argument applies if the second separation property is replaced by first separation property.

Definition 3.8 (Topological embedding)**.** A continuous injective map $f : X \to Y$ between topological spaces is called a *topological embedding* if

$$f : X \longrightarrow f(X)$$

is a homeomorphism onto its image, where $f(X)$ is equipped with the subspace topology from Y.

Remark 3.9. Equivalently, f is an embedding if it is continuous, injective, and for every open set $U \subseteq X$, there exists an open set $V \subseteq Y$ such that

$$U = f^{-1}(V) \cap X.$$

Proposition 3.10. *Let $f : X \to Y$ be a topological embedding. If Y is Hausdorff, then X is Hausdorff.*

Proof. Since f is an embedding, $f : X \to f(X)$ is a homeomorphism onto its image, where $f(X)$ is given the subspace topology from Y.

Take distinct points $a, b \in X$. Then $f(a) \neq f(b)$ in Y. As Y is Hausdorff, there exist disjoint open sets $U, V \subseteq Y$ with

$$f(a) \in U, \qquad f(b) \in V, \qquad U \cap V = \varnothing.$$

It follows that $U \cap f(X)$ and $V \cap f(X)$ are disjoint open neighborhoods of $f(a)$ and $f(b)$ in $f(X)$.

By the homeomorphism $f : X \to f(X)$, the preimages $f^{-1}(U \cap f(X))$ and $f^{-1}(V \cap f(X))$ are disjoint open neighborhoods of a and b in X. Thus X is Hausdorff. □

Remark 3.11. If one only assumes that f is continuous and injective, the conclusion may fail. For example, let X be any non-Hausdorff space and let $f : X \to \{*\}$ be the constant injective map into a one-point Hausdorff space. Then f is continuous and injective, but X is not Hausdorff. The embedding condition ensures that the topology of X is faithfully represented inside Y.

Corollary 3.12. *If $f : X \to Y$ is a homeomorphism, then X is Hausdorff if and only if Y is Hausdorff. In particular, Hausdorff-ness is a* topological property*, i.e., a property preserved under homeomorphism.*

3.2 Connectedness

Definition 3.13 (Connected space)**.** A topological space $(X, \mathcal{T})$ is said to be *disconnected* if there exist non-empty open sets $U, V \in \mathcal{T}$ such that

$$U \cap V = \varnothing, \qquad U \cup V = X.$$

If no such pair exists, then X is called *connected*.

Proposition 3.14. *For a topological space $(X, \mathcal{T})$, the following are equivalent:*

1. *X is connected.*
2. *The only subsets of X which are both open and closed are $\varnothing$ and X.*
3. *Every continuous function $f : X \to \{0, 1\}$ (where $\{0, 1\}$ is equipped with the discrete topology) is constant.*

Proof. (1) $\Rightarrow$ (2): Suppose $U \subseteq X$ is both open and closed. Then U and $X \setminus U$ are disjoint open sets with

$$U \cup (X \setminus U) = X.$$

By connectedness, either $U = \varnothing$ or $X \setminus U = \varnothing$. Hence $U = \varnothing$ or $U = X$.

(2) $\Rightarrow$ (3): Let $f : X \to \{0, 1\}$ be continuous. Define

$$U = f^{-1}(\{0\}), \qquad V = f^{-1}(\{1\}).$$

Then U, V are open, disjoint, and $U \cup V = X$. By (2), either $(U, V) = (X, \varnothing)$ or $(U, V) = (\varnothing, X)$. Thus f is constant.

(3) $\Rightarrow$ (2): Suppose $U \subseteq X$ is both open and closed. Define

$$f(x) = \begin{cases} 0, & x \in U, \\ 1, & x \in X \setminus U. \end{cases}$$

This map $f : X \to \{0, 1\}$ is continuous. By (3), f is constant, hence $U = \varnothing$ or $U = X$.

(2) $\Rightarrow$ (1): Suppose, for contradiction, that X is disconnected. Then there exist non-empty disjoint open sets $U, V \subseteq X$ with $U \cup V = X$. In this case, $U = X \setminus V$ is both open and closed, contradicting (2). Hence X must be connected. □

Corollary 3.15. *The interval $[a, b] \subseteq \mathbb{R}$ is connnected*

Proof. Suppose on the contrary that there exists continuous function $f : [a, b] \to \{0, 1\}$ that takes 2 values. Construct the mapping $\widetilde{f} : [a, b] \to \mathbb{R}$

$$\widetilde{f} : [a, b] \xrightarrow{f} \{0, 1\} \xrightarrow{i} \mathbb{R}$$

$$\text{with } \widetilde{f} = i \circ f.$$

Note that $\{0, 1\} \subseteq \mathbb{R}$ denotes the subspace topology, we imply the inclusion mapping $i : \{0, 1\} \to \mathbb{R}$ with $s \mapsto s$ is continuous. The composition of continuous mappings is continuous as well, i.e., $\widetilde{f}$ is continuous.

Since the function f can take two values, there exists $p, q \in [a, b]$ such that $\widetilde{f}(p) = i \circ f(p) = 0$ and $\widetilde{f}(q) = i \circ f(q) = 1$. By intermediate value theorem, there exists $r \in [a, b]$ such that $\widetilde{f}(r) = i \circ f(r) = 1/2$, which implies $f(r) = \frac{1}{2}$, which is a contradiction. □

Definition 3.16 (Connected subset)**.** A non-empty subset $S \subseteq X$ is connected if $(S, \mathcal{T}_S)$, where $\mathcal{T}_S$ denotes the subspace topology on S, is a connected topological space.

Equivalently, $S \subseteq X$ is connected if, whenever $U, V \subseteq X$ are open sets such that

$$S \subseteq U \cup V, \qquad (U \cap V) \cap S = \varnothing,$$

it follows that either $U \cap S = \varnothing$ or $V \cap S = \varnothing$.

Proposition 3.17. *If $f : X \to Y$ is continuous and $A \subseteq X$ is connected, then $f(A)$ is connected. In other words, the continuous image of a connected set is connected.*

Proof. Suppose that $U, V \subseteq Y$ are open sets such that

$$f(A) \subseteq U \cup V, \qquad (U \cap V) \cap f(A) = \varnothing.$$

Then

$$A \subseteq f^{-1}(U) \cup f^{-1}(V), \qquad (f^{-1}(U) \cap A) \cap (f^{-1}(V) \cap A) = \varnothing.$$

By the connectedness of A, either $f^{-1}(U) \cap A = \varnothing$ or $f^{-1}(V) \cap A = \varnothing$. Therefore, either $f(A) \cap U = \varnothing$ or $f(A) \cap V = \varnothing$. Hence $f(A)$ is connected. □

Proposition 3.18. *If $\{A_i\}_{i \in I}$ is a family of connected subsets of X such that*

$$A_i \cap A_j \neq \varnothing \quad \text{for all } i, j \in I,$$

then the union $\bigcup_{i \in I} A_i$ is connected.

Proof. Suppose the function $f : \bigcup\limits_{i \in I} A_i \longrightarrow \{0, 1\}$ is a continuous map. Then we imply that its restriction $f|_{A_i} = f \circ i : A_i \to \{0, 1\}$ is continuous for all $i \in I$. Thus $f|_{A_i}$ is a constant for all $i \in I$. Due to the non-empty intersection of A_i, A_j for $\forall i, j \in I$, we imply f is constant. By Proposition 3.14, $\bigcup\limits_{i \in I} A_i$ is connected. □

Exercise 3.1. *Use Proposition 3.17 to give another proof.* Hint. *Fix $j \in I$. Construct a continuous map $f_j : \bigcup_{i \in I} A_i \longrightarrow A_j$.*

Proposition 3.19. *If X and Y are connected spaces, then the product space $X \times Y$, equipped with the product topology, is connected.*

Proof. Fix $y_0 \in Y$. The subset

$$B := X \times \{y_0\} \subseteq X \times Y$$

is homeomorphic to X, hence connected. Similarly, for each $x \in X$, the subset

$$C_x := \{x\} \times Y \subseteq X \times Y$$

is homeomorphic to Y, hence connected.

Moreover, for each $x \in X$,

$$B \cap C_x = \{(x, y_0)\} \neq \varnothing.$$

Thus, by Proposition 3.18, the union

$$B \cup \bigcup_{x \in X} C_x = X \times Y$$

is connected. $\square$

Definition 3.20 (Path-Connectedness). Let $(X, \mathcal{T})$ be a topological space.

1. A *path* between two points $x, y \in X$ is a continuous map

$$\tau : [0,1] \to X, \qquad \tau(0) = x,\ \tau(1) = y.$$

2. The space X is said to be *path-connected* if any two points in X can be joined by a path.
3. A subset $A \subseteq X$ is *path-connected* if it is path-connected with respect to the subspace topology. Equivalently, A is path-connected if for every pair of points $x, y \in A$, there exists a continuous map

$$\tau : [0,1] \to A, \qquad \tau(0) = x,\ \tau(1) = y.$$

Proposition 3.21. *All path-connected spaces are connected.*

Proof. Fix $x \in X$. For each $y \in X$, by path-connectedness there exists a continuous map

$$p_y : [0,1] \to X, \qquad p_y(0) = x,\ p_y(1) = y.$$

Define

$$C_y := p_y([0,1]) \subseteq X.$$

By Proposition 3.17, each C_y is connected, since it is the continuous image of the connected set $[0,1]$.

Moreover, for any $y, y' \in X$,

$$C_y \cap C_{y'} \supseteq \{x\} \neq \varnothing.$$

Thus the family $\{C_y\}_{y \in X}$ is a collection of connected sets with non-empty mutual intersection. By Proposition 3.18, we conclude that

$$X = \bigcup_{y \in X} C_y$$

is connected. $\square$

Exercise 3.2. *If $A \subset B \subset \overline{A}$, then A connected implies B connected. (Hint: for every open $U \subseteq X$, if $U \cap A = \varnothing$, then also $U \cap \overline{A} = \varnothing$.)*

Example 3.22 (Topologist's comb). The converse of Proposition 3.21 is false. Consider the subset $X \subseteq \mathbb{R}^2$ defined by

$$X = ([0,1] \times \{0\}) \cup \bigcup_{n \in \mathbb{N}_{\geq 1}} (\{1/n\} \times [0,1]) \cup \{(0,1)\},$$

where we take $\mathbb{N}_{\geq 1} = \{1, 2, 3, \dots\}$.

This space, known as the *Topologist's comb*, is connected but not path-connected (see Figure 3.1).

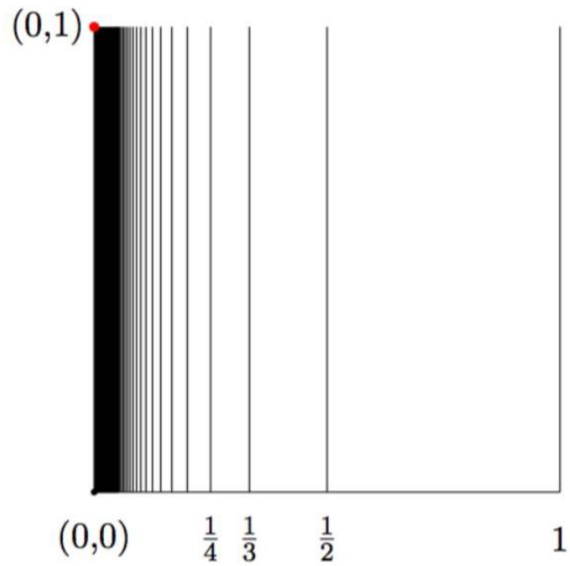


Figure 3.1: A connected but not path-connected space X.

(a) X is not path-connected. Suppose there exists a path $p : [0,1] \to X$ with

$$p(0) = (0,1), \qquad p(1) = (x,y) \in X \setminus \{(0,1)\}.$$

Define

$$A := \{t \in [0,1] \mid p(t) = (0,1)\}.$$

We will show $A = [0,1]$, i.e. the path must be constant.

(i) A is closed since it is the preimage of the closed singleton $\{(0,1)\}$.

(ii) A is open: fix $t_0 \in A$. By continuity of p, there exists $\delta > 0$ such that

$$\|p(t) - (0,1)\| < \tfrac{1}{2}, \qquad t \in [0,1] \cap (t_0 - \delta, t_0 + \delta).$$

Thus $p(t)$ cannot lie on the x-axis. Its x-coordinate must therefore belong to $\{0\} \cup \{1/n : n \geq 1\}$. Let $I := [0,1] \cap (t_0 - \delta, t_0 + \delta)$. Consider the continuous function

$$f := \pi_x \circ p : I \to \mathbb{R},$$

where $\pi_x(a,b) = a$ is the x-coordinate. Since I is connected, $f(I)$ is connected. But $f(I) \subseteq \{0\} \cup \{1/n : n \geq 1\}$, whose only connected subsets are singletons. Since $f(t_0) = 0$, we deduce $f(I) = \{0\}$. Hence $p(t) = (0,1)$ for all $t \in I$, i.e. $I \subseteq A$. Thus A is open.

Therefore A is nonempty, closed, and open in the connected interval $[0,1]$, so $A = [0,1]$. Hence any such path is constant, and no path connects $(0,1)$ to another point of X.

(b) Nonetheless, X is connected. (Proof omitted here, but it relies on showing that any separation would disconnect the horizontal interval from the vertical combs, which is impossible.)

3.3 Compactness

Compactness generalizes the Euclidean notion of "closed and bounded" subsets of $\mathbb{R}^n$.

Definition 3.23 (Compactness)**.** Let $(X, \mathcal{T})$ be a topological space.

1. A collection $\mathcal{U} = \{U_i \mid i \in I\}$ of open sets is called an *open cover* of X if

$$X = \bigcup_{i \in I} U_i.$$

2. A *subcover* of $\mathcal{U}$ is a subfamily

$$\mathcal{U}' = \{U_j \mid j \in J\}, \qquad J \subseteq I,$$

such that

$$X = \bigcup_{j \in J} U_j.$$

3. If J is finite, then $\mathcal{U}'$ is called a *finite subcover*.

We say that X is *compact* if every open cover of X admits a finite subcover i.e. $X = \bigcup\limits_{j=1}^{n} U_j$.

If $A \subseteq X$ is equipped with the subspace topology, then A is compact if and only if, for every collection of open sets $\{U_i\}_{i \in I}$ in X such that

$$A \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subcollection

$$A \subseteq \bigcup_{k=1}^{n} U_{i_k}.$$

Proposition 3.24 (Finite Intersection Property)**.** *Let X be a topological space. The following are equivalent:*

1. *X is compact.*
2. *If $\{V_i \mid i \in I\}$ is a collection of closed subsets of X with the property that*

$$\bigcap_{j \in J} V_j \neq \varnothing, \qquad \textit{for all finite } J \subseteq I,$$

then

$$\bigcap_{i \in I} V_i \neq \varnothing.$$

Compactness is an *intrinsic* property of a space: it depends only on the topology of X, not on how X may be embedded into some larger space.

Example 3.25.

1. If $X \subseteq \mathbb{R}^n$, then X is compact if and only if X is closed and bounded. (This is the classical *Heine–Borel theorem*.)
2. Let $K \subseteq \mathbb{R}^n$ be compact, and define

   $$\mathcal{C}(K) = \{ f : K \to \mathbb{R} \mid f \text{ continuous} \}.$$

   Equip $\mathcal{C}(K)$ with the ∞-norm

   $$\|f\|_\infty = \sup_{k \in K} |f(k)|.$$

   This induces the metric space $(\mathcal{C}(K), d_\infty)$, where

   $$d_\infty(f, g) = \|f - g\|_\infty,$$

   which is in fact a Banach space (a complete normed vector space).

   A subset $\mathcal{J} \subseteq \mathcal{C}(K)$ is compact if and only if it is

   $$[\text{closed, bounded, and equicontinuous}]$$

   (where *equicontinuous* means: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{J}$ whenever $\|x - y\| < \delta$). (This is the celebrated *Arzelà–Ascoli theorem*.)

   This shows that in infinite-dimensional spaces, compactness is no longer the same as closedness plus boundedness.

Proposition 3.26. *Let X be a compact space. Then every closed subset $A \subseteq X$ is compact.*

Proof. Let $\{V_i \mid i \in I\}$ be a collection of closed subsets of A such that

$$\bigcap_{j \in J} V_j \neq \varnothing \quad \text{for all finite } J \subseteq I.$$

Since A is closed in X, each V_i can be written as $V_i = A \cap W_i$ for some closed set $W_i \subseteq X$. Thus the family $\{W_i \mid i \in I\}$ has the finite intersection property in X.

By compactness of X, we deduce

$$\bigcap_{i \in I} W_i \neq \varnothing.$$

Intersecting with A, we get

$$\bigcap_{i\in I} V_i = A \cap \bigcap_{i\in I} W_i \neq \varnothing.$$

Hence A is compact by the finite intersection property. $\square$

Now consider the reverse direction of Proposition 3.26, i.e., are all compact subsets $K \subseteq X$ closed in X ?

In general, the converse does not hold. Note that $K = \{x\}$ is compact for any topology X. However, there are some topologies such that a singleton is not closed.

In order to obtain the converse of Proposition 3.26, we impose the second separation axiom.

Proposition 3.27 (Point–compact set separation in Hausdorff spaces)**.** *Let X be Hausdorff, let $K \subseteq X$ be compact, and let $x \in X \setminus K$. Then there exist open sets $U, V \subseteq X$ such that*

$$K \subseteq U, \qquad x \in V, \qquad U \cap V = \varnothing.$$

Proof. Let $k \in K$. Since X is Hausdorff, there exist open sets $U_k \ni k$ and $V_k \ni x$ with $U_k \cap V_k = \varnothing$. Then $\{U_k\}_{k\in K}$ is an open cover of K. By compactness of K, there exist $k_1, \ldots, k_n \in K$ such that $K \subseteq \bigcup_{i=1}^n U_{k_i}$. Define

$$U := \bigcup_{i=1}^n U_{k_i} \qquad \text{and} \qquad V := \bigcap_{i=1}^n V_{k_i}.$$

Both U and V are open; clearly $K \subseteq U$ and $x \in V$. Moreover,

$$U \cap V \subseteq \bigcup_{i=1}^n (U_{k_i} \cap V_{k_i}) = \varnothing,$$

so U and V are disjoint. This proves the claim. $\square$

By making use of this separation axiom, we obtain the converse of Proposition 3.26:

Corollary 3.28. *If X is Hausdorff and $K \subseteq X$ is compact, then K is closed in X.*

Proof. Fix $x \in X \setminus K$. By Proposition 3.27, there exist open sets $U, V \subseteq X$ such that $K \subseteq U$, $x \in V$, and $U \cap V = \varnothing$. Hence $V \subseteq X \setminus U \subseteq X \setminus K$, so x has an open neighborhood contained in $X \setminus K$. Since this holds for every $x \in X \setminus K$, the complement $X \setminus K$ is open, and thus K is closed. $\square$

3.3.1 Continuous Functions on Compact Space

Proposition 3.29. *Let $f : X \to Y$ be a continuous map between topological spaces, and let $A \subseteq X$ be compact. Then $f(A) \subseteq Y$ is compact.*

Proof. Let $\{U_i \mid i \in I\}$ be an open cover of $f(A)$, i.e.

$$f(A) \subseteq \bigcup_{i\in I} U_i, \qquad U_i \in \mathcal{T}_Y.$$

Then the preimages $\{f^{-1}(U_i) \mid i \in I\}$ form an open cover of A, since

$$A \subseteq f^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} f^{-1}(U_i).$$

By compactness of A, there exists a finite subcover

$$A \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}).$$

Applying f gives

$$f(A) \subseteq f\Big(\bigcup_{k=1}^{n} f^{-1}(U_{i_k})\Big) \subseteq \bigcup_{k=1}^{n} U_{i_k}.$$

Thus $f(A)$ has a finite subcover, and hence is compact. □

Corollary 3.30. ***1.*** *Suppose X is compact, and $f : X \to \mathbb{R}$ is continuous. Then $f(X)$ is closed and bounded. In particular, there exist $m, M \in X$ such that*

$$f(m) \leq f(x) \leq f(M), \qquad \forall x \in X.$$

2. *Suppose moreover that X is connected. Then*

$$f(X) = [\,f(m), f(M)\,].$$

Proof. (1) By Proposition 3.29, $f(X)$ is compact in $\mathbb{R}$. By the Heine–Borel theorem, compact subsets of $\mathbb{R}$ are closed and bounded. Hence $f(X)$ attains both its minimum and maximum values at some points $m, M \in X$.

(2) If X is also connected, then by continuity of f, the image $f(X)$ is connected. But the only connected subsets of $\mathbb{R}$ are intervals. Since $f(X)$ is compact, it must be a closed interval. Thus

$$f(X) = [\,f(m), f(M)\,].$$ □

Lemma 3.31 (Tube Lemma)**.** *Let X, Y be topological spaces, and assume Y is compact. If $W \subseteq X \times Y$ is open and*

$$\{x_0\} \times Y \subseteq W,$$

then there exists an open neighborhood $U \subseteq X$ of x_0 such that

$$U \times Y \subseteq W.$$

Proof. For each $y \in Y$, since $(x_0, y) \in W$ and W is open, there exist neighborhoods $U_y \subseteq X$, $V_y \subseteq Y$ such that

$$(x_0, y) \in U_y \times V_y \subseteq W.$$

Then $\{V_y : y \in Y\}$ is an open cover of Y. By compactness, there exist finitely many points $y_1, \ldots, y_n \in Y$ such that

$$Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n}.$$

Define

$$U := U_{y_1} \cap \cdots \cap U_{y_n}.$$

Then U is an open neighborhood of x_0, and we have

$$U \times Y \subseteq (U_{y_1} \times V_{y_1}) \cup \cdots \cup (U_{y_n} \times V_{y_n}) \subseteq W.$$

Thus the claim follows. □

Theorem 3.32 (Tychonoff for finite products)**.** *Let X, Y be topological spaces. Then $X \times Y$ is compact under the product topology if and only if both X and Y are compact.*

Proof. ($\Rightarrow$) Suppose $X \times Y$ is compact. Consider the projection maps

$$p_X : X \times Y \to X, \qquad p_Y : X \times Y \to Y.$$

These are continuous, so by Proposition 3.29 the images $p_X(X \times Y) = X$ and $p_Y(X \times Y) = Y$ are compact.

($\Leftarrow$) Suppose X and Y are compact. Let $\{W_i\}_{i \in I}$ be an open cover of $X \times Y$. Fix $x_0 \in X$. Then $\{x_0\} \times Y$ is compact, so it is covered by finitely many $W_{i_1}, \ldots, W_{i_n}$. Define

$$W := W_{i_1} \cup \cdots \cup W_{i_n},$$

which is an open neighborhood of $\{x_0\} \times Y$. By the tube lemma, there exists an open neighborhood $U_{x_0} \subseteq X$ of x_0 such that

$$U_{x_0} \times Y \subseteq W.$$

Thus $\{U_{x_0} : x_0 \in X\}$ is an open cover of X. By compactness of X, choose finitely many $x_1, \ldots, x_m \in X$ such that

$$X = U_{x_1} \cup \cdots \cup U_{x_m}.$$

It follows that

$$X \times Y \subseteq \bigcup_{\ell=1}^{m} (U_{x_\ell} \times Y) \subseteq \bigcup_{\ell=1}^{m} (W_{i_1} \cup \cdots \cup W_{i_n}),$$

a finite union of members of the original cover. Therefore $X \times Y$ is compact. □

Theorem 3.33. *Suppose that X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection. Then f is a homeomorphism.*

Proof. Since f is bijective, to show that f^{-1} is continuous it suffices to prove that $(f^{-1})^{-1}(V) = f(V)$ is closed in Y for every closed set $V \subseteq X$.

Let $V \subseteq X$ be closed. Then V is compact since closed subsets of a compact space are compact, by Proposition 3.26. By continuity of f and Proposition 3.29. , the image $f(V)$ is compact in Y. Since Y is Hausdorff, compact subsets are closed (see Corollary 3.28). Hence $f(V)$ is closed in Y.

Thus, for every closed set $V \subseteq X$, the image $f(V)$ is closed in Y. This is exactly the statement that f^{-1} is continuous. Therefore f is a homeomorphism. □

Corollary 3.34. *If X is compact, Y is Hausdorff, and $f : X \to Y$ is injective and continuous, then*

$$f : X \longrightarrow f(X)$$

is a homeomorphism (where $f(X)$ carries the subspace topology from Y).

Example 3.35 (Second proof of $S^1 \times S^1$ is isomorphic to a torus)**.** Identify $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\} \subset \mathbb{C}$. Define

$$f : S^1 \times S^1 \longrightarrow \mathbb{R}^3, \qquad (e^{i\theta}, e^{i\phi}) \longmapsto \big((R + r\cos\theta)\cos\phi,\ (R + r\cos\theta)\sin\phi,\ r\sin\theta\big),$$

with parameters $R > r > 0$. Let $T := f(S^1 \times S^1) \subset \mathbb{R}^3$.

- $X := S^1 \times S^1$ is compact, and $\mathbb{R}^3$ is Hausdorff.
- f is continuous and injective (this is the standard parametrization of a ring torus).
- The image T is the (embedded) *ring torus* in $\mathbb{R}^3$ with major radius R and minor radius r.

By Corollary 3.34, a continuous injective map from a compact space into a Hausdorff space is a homeomorphism onto its image. Hence $f : S^1 \times S^1 \to T$ is a homeomorphism, i.e.

$$S^1 \times S^1 \ \cong\ T\ \subset \mathbb{R}^3.$$

Definition 3.36 (Sequential Compactness)**.** A topological space X is *sequentially compact* if every sequence in X admits a convergent subsequence, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and a point $x \in X$ such that

$$x_{n_k} \to x \quad \text{in } X.$$

In a metric space (X, d), compactness and sequential compactness are equivalent. In particular, for $\mathbb{R}^n$ with the Euclidean metric, a subset is compact if and only if it is sequentially compact. (Check notes for MAT3006)

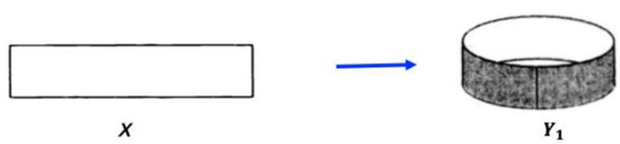
Remark 3.37. For general topological spaces, compactness and sequential compactness need not coincide. For example, the first uncountable ordinal ω_1 with the order topology is compact but not sequentially compact.

Chapter 4

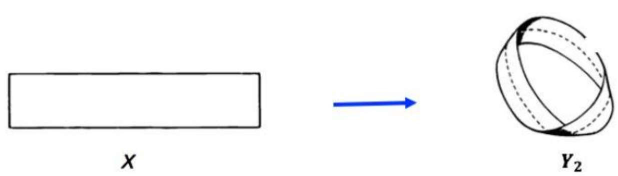
Quotient Space and Simplicial Complex

In the end of Chapter 2, we introduced product space and disjoint union. In this Chapter, we give another way to construct new topological spaces from some old ones. This new way of construction is by gluing some special pieces from old topological spaces together.

The rough idea is as follows: Let $X = [0,1] \times [0,1]$ (just like a piece of paper on a plane), we want to glue the leftmost edge with the rightmost edge to form a cylinder Y_1, as shown below:



If we give a half-twist to the strip before glue the ends together, we will get the Moebius strip Y_2 shown below:



Interestingly, the first topology Y_1 has two sides, while the second has only one side.

4.1 Equivalence Relations and Equivalence Classes

Definition 4.1 (Equivalence Relation)**.** The equivalence relation on a set X is a relation šuch that

1. (Reflexive): $x \sim x, \forall x \in X$

2. (Symmetric): $x \sim y$ implies $y \sim x$
3. (Transitive): $x \sim y$ and $y \sim z$ implies $x \sim z$.

Example 4.2. 1. Let $X = V$ be a vector space, and $W \leq V$ be a vector subspace. Define $\mathbf{v}_1 \sim \mathbf{v}_2$ if $\mathbf{v}_1 - \mathbf{v}_2 \in W$. (The well-definedness is left as exercise).

2. (Mobius Strip): Let $X = [0,1] \times [0,1]$. We define $(x_1, y_1) \sim (x_2, y_2)$ if

- $x_1 = x_2, y_1 = y_2$; (e.g., $(0.5, 0.6) \sim (0.5, 0.6)$) or
- $x_1 = 0, x_2 = 1$, and $y_1 = 1 - y_2$ (e.g., $(0, 1/4) \sim (1, 3/4)$)
- $x_1 = 1, x_2 = 0$, and $y_1 = 1 - y_2$ (e.g., $(1, 3/4) \sim (0, 1/4)$)

Another way to define an equivalence relation is to use partitions:

Definition 4.3 (Partition)**.** Let X be a nonempty set. A partition $\mathcal{P} = \{p_i \mid i \in I\}$ of X is a collection of subsets such that

1. $P_i \subseteq X$ is non-empty
2. $P_i \cap P_j = \varnothing$ if $i \neq j$
3. $\bigcup_{i \in I} P_i = X$.

Given a partition $\mathcal{P} = \{p_i \mid i \in I\}$, we can define an equivalence relation $\sim$ on X by setting

$$x \sim y \ \text{ whenever } x, y \in p_i, \text{ for some } i \in I$$

Example 4.4. Let $X = [0,1] \times [0,1]$. Then the partition

$$X = \{(x,y)\}_{x \in (0,1), y \in [0,1]} \cup \{(1,y), (0, 1-y)\}_{y \in [0,1]}$$

gives the same equivalence relation as in part (2) in Example 4.2.

Conversely, for any equivalence relation $\sim$ of X, we could form a corresponding partition of X. This kind of partition is called the equivalence class:

Definition 4.5 (Equivalence Class)**.** Let X be a set with equivalence relation $\sim$. The **equivalence class** of an element $x \in X$ is

$$[x] := \{y \in X \mid x \sim y\}.$$

The collection of all equivalence classes is called the **quotient space**:

$$X/\sim = \{[x] \mid x \in X\}.$$

Example 4.6. 1. Consider the equivalence class defined in part (1) in Example 4.2. The equivalence class has the form

$$[\mathbf{v}] = \{\mathbf{u} \in V \mid \mathbf{v} - \mathbf{u} \in W\} := \mathbf{v} + W.$$

Therefore, the equivalence class is a generalization of the coset in linear algebra. Similarly, we define the set of generalized cosets as quotient space. The quotient space $V/\sim$ reduces to the V/W in linear algebra:

$$V/\sim = \{[\mathbf{v}] \mid \mathbf{v} \in V\} = \{\mathbf{v} + W \mid \mathbf{v} \in V\} = V/W.$$

2. Consider part (2) in Example 4.2 again. Then $X/\sim$ essentially forms the Mobius band. For instance, two 'points' $[(1/2, 1/2)], [(1, 3/4)] \in X/\sim$ are:

$$[(1/2,1/2)] = \{(1/2,1/2)\}$$
$$[(1,3/4)] = \{(1,3/4), (0,1/4)\} = [(0,1/4)].$$

Therefore, we have 'glued' the two points $(1,3/4), (0,1/4)$ together into a single point in $X/\sim$.

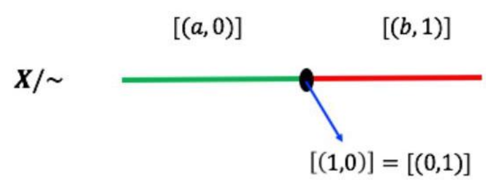
3. Consider $X = [0,1] \sqcup [0,1]$, i.e.,

$$X = ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$$

Take a partition on X by

$$\{(a,0)\}_{0 \le a < 1} \cup \{(b,1)\}_{0 < b \le 1} \cup \{(1,0), (0,1)\}$$

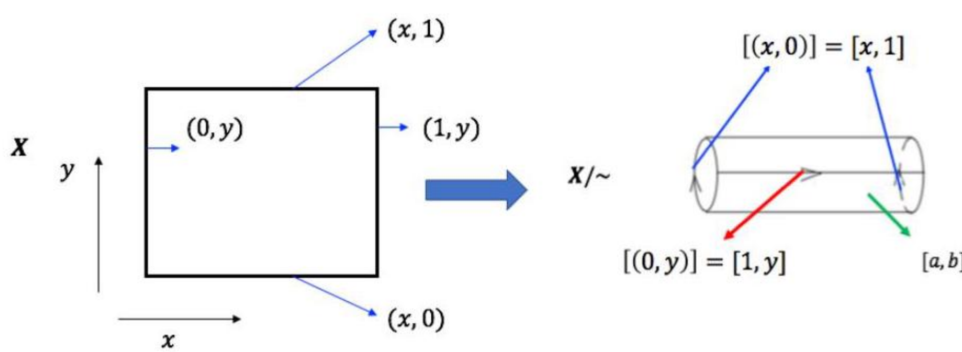
In other words, we 'glue' the right end $(1,0)$ of the first segment to the left end $(0,1)$ of the second segment. As a result, the corresponding quotient space is plotted below:



4. Let $X = [0,1] \times [0,1]$. Consider the equivalence class with partition

$$\{(a,b)\}_{0<a<1; 0<b<1} \cup \{(x,0), (x,1)\}_{0 \le x \le 1} \cup \{(0,y), (1,y)\}_{0<y<1}$$

The corresponding quotient space is plotted below:



4.2 Quotient Topology

Now given a topologcal space X and an equivalence relation $\sim$ on it, our goal is to construct a topology on the space $X/\sim$.

Proposition 4.7 (Quotient Topology)**.** *Suppose $(X, \mathcal{T})$ is a topological space, and $\sim$ is an equivalence relation on X. Define the canonical projection map:*

$$p : X \to X/\sim \qquad \textit{with} \qquad x \mapsto [x]$$

which assigns each point $x \in X$ into the equivalence class $[x]$. Define a family of subsets $\widetilde{\mathcal{T}}$ on $X/\sim$ by:

$$\widetilde{U} \subseteq X/\sim \;\; \textit{is in } \widetilde{\mathcal{T}} \textit{ if } p^{-1}\left(\widetilde{U}\right) \;\; \textit{is in } \mathcal{T}$$

Then $\widetilde{\mathcal{T}}$ is a topology for $X/\sim$.

We say $(X/\sim \widetilde{\mathcal{T}})$ the **quotient topology** *of the quotient space, and $X/\sim$.*

Proof. 1. $p^{-1}(X/\sim) = X \in \mathcal{T}$ and $p^{-1}(\varnothing) = \varnothing \in \mathcal{T}$, which implies $X/\sim \in \widetilde{\mathcal{T}}$ and $\varnothing \in \widetilde{\mathcal{T}}$.

2. Suppose that $\widetilde{U}, \widetilde{V} \in \widetilde{\mathcal{T}}$, then we imply

$$p^{-1}\left(\widetilde{U}\right), p^{-1}\left(\widetilde{V}\right) \in \mathcal{T} \Rightarrow p^{-1}\left(\widetilde{U} \cap \widetilde{V}\right) \in \mathcal{T},$$

i.e., $\widetilde{U} \cap \widetilde{V} \in \widetilde{\mathcal{T}}$.

3. Following the similar argument in (2), and the relation

$$p^{-1}\left(\bigcup \widetilde{U}_i\right) = \bigcup p^{-1}\left(\widetilde{U}_i\right),$$

we conclude that $\widetilde{T}$ is closed under countably union, and the proof is complete. $\square$

Remark 4.8. 1. Proposition 4.7 claims that $\widetilde{U}$ is open in $X/\sim$ iff $p^{-1}\left(\widetilde{U}\right)$ is open in X. The general question is that, does $p(U)$ is open in $X/\sim$, given that U is open in X ? This may not necessarily hold. (See example (6.4)) In general $p^{-1}(p(U))$ is strictly larger than U, and may not be necessarily open in X, even when U is open.

2. By definition, we can show that p is continuous.

To fill the gap on the question shown in the remark, we consider the notion of the open mapping:

Definition 4.9. [Open Mapping] A function $f : X \to Y$ between two topological spaces is an open mapping if for each open U in X, $f(U)$ is open in Y.

Example 4.10. The mapping $p : [0,1] \times [0,1] \to ([0,1] \times [0,1])/\sim$ sending the square to the Mobius band M is not an open mapping: Consider the open ball $U = B_{1/2}((0,0))$ in $[0,1] \times [0,1]$. Note that $p(U)$ is open in M iff $p^{-1}(p(U))$ is open in $[0,1] \times [0,1]$. We can calculate $p^{-1}(p(U))$ explicitly:

$$p^{-1}(p(U)) = U \cup \{(1, y) \mid 1/2 \leq y \leq 1\},$$

which is not open.

Proposition 4.11. *A subset $\widetilde{V}$ is closed in the quotient space $X/\sim$ iff $p^1\left(\widetilde{V}\right)$ is closed in X, where $p : X \to X/\sim$ denotes the canonical projection mapping.*

Proof. It follows from the fact that

$$p^{-1}\left((X/\sim) \setminus \widetilde{V}\right) = X \setminus p^{-1}\left(\widetilde{V}\right)$$

□

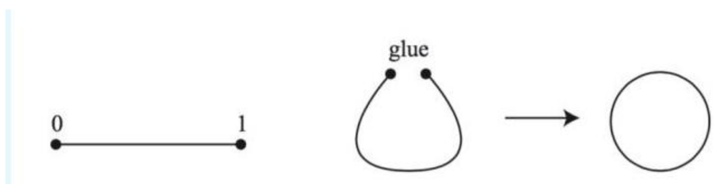
Example 4.12. Consider $X = [0,1]$. We define $x_1 \sim x_2$ if:

$$x_1 = 0, x_2 = 1, \text{ or } x_1 = 1, x_2 = 0$$

In other words, the partition on X is given by:

$$X = \{0,1\} \cup \left(\bigcup_{x\in(0,1)} \{x\}\right)$$

The quotient space "glues" the endpoints of the interval $[0,1]$ together, shown in the figure below:



It is intuitive that the constructed quotient space should be homeomorphic to a circle S^1. We will give a formal proof on this fact.

Proposition 4.13. *Let X and Z be topological spaces, and $\sim$ an equivalence relation on X. Let $g : X/\sim \to Z$ be a function, and $p : X \to X/\sim$ is a projection mapping The mapping g is continuous if and only if $g \circ p : X \to Z$ is continuous.*

Proof. 1. Necessity. Suppose that g is continuous. It's clear that p is continuous, i.e, $g \circ p : X \to Z$ is continuous.

2. Sufficiency. Suppose that $g \circ p : X \to Z$ is continuous. Given any open U in Z, we imply $(g \circ p)^{-1}(U) = p^{-1}g^{-1}(U)$ is open in X. By definition of the quotient topology, we imply $g^{-1}(U)$ is open in $X/\sim$. Therefore, g is continuous. □

This useful lemma can be generalized into the case for generalized canonical projection mapping, called quotient mapping.

Definition 4.14. A map $p : X \to Y$ between topological spaces is a quotient mapping if

1. p is surjective; and
2. p is continuous;
3. For any $U \subseteq Y$ such that $p^{-1}(U)$ is open in X, we imply U is open in Y.

Obviously, the canonical projection map $p : X \to X/\sim$ is a quotient map. The advantage of quotient map is as given below:

Proposition 4.15. *Suppose that $p : X \to Y$ is a quotient map and that $g : Y \to Z$ is any mapping to another space Z. Then g is continuous iff $g \circ p$ is continuous.*

Proof. The proof follows similarly as in Proposition 4.13. □

Now we give a formal proof of the conclusion in Example 4.12:

Proof. Define the mapping

$$f : [0,1] \to S^1 \quad \text{with} \quad t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Since $f(0) = f(1)$, the function f induces a well-defined function

$$g : [0,1]/\sim \to S^1 \quad \text{with} \quad [t] \mapsto f(t)$$

such that $f = g \circ p$, where p denotes the canonical projection mapping. Note that f is continuous. By Proposition 4.15, we imply g is continuous. Furthermore,

1. Since $[0,1]$ is compact and p is continuous, we imply $p([0,1]) = [0,1]/\sim$ is compact
2. S^1 is Hausdorff
3. g is a bijection

By applying Theorem 3.33, we conclude that g is a homeomorphism, i.e., $[0,1]/\sim$ and S^1 are homeomorphic. □

The argument in the proof can be generalized into the proposition below:

Proposition 4.16. *Let $f : X \to Y$ be a surjective continuous mapping between topologcial spaces, and $\sim$ be an equivalence relation on X defined by the partition $\{f^{-1}(y) \mid y \in Y\}$ (i.e., $x \sim x'$ iff $f(x) = f(x')$). If X is compact and Y is Hausdorff, then $X/\sim$ and Y are homeomorphic.*

The above proposition allows us to apply the following argument, which we shall use several times: To show

$$X/\sim \cong Y$$

are homeomorphic, construct a surjective continuous mapping $f : X \to Y$ such that

$$f(x_1) = f(x_2) \text{ whenever } x_1 \sim x_2.$$

Therefore, f will induce a well-defined function $g : X/\sim \to Y$ such that $f = g \circ p$. Then checking the conditions in Theorem 3.33 leads to the desired results.

Example 4.17. Consider $X = [0,1] \times [0,1]$ and define $(s_1, t_1) \sim (s_2, t_2)$ if one of the following holds:

- $s_1 = s_2$ and $t_1 = t_2$;
- $\{s_1, s_2\} = \{0,1\}, t_1 = t_2$;
- $\{t_1, t_2\} = \{0,1\}$ and $s_1 = s_2$;
- $\{s_1, s_2\} = \{0,1\}, \{t_1, t_2\} = \{0,1\}$

We now show the corresponding quotient space $([0,1] \times [0,1]) / \sim$ is hoemomorhpic to the torus $\mathbb{T}^2$: Define the mapping

$$f : [0,1] \times [0,1] \to \mathbb{T}^2 \qquad (t_1, t_2) \mapsto \left(e^{2\pi i t_1}, e^{2\pi i t_2}\right).$$

Then

- f is surjective, which also implies $\mathbb{T}^2 = f\left([0,1] \times [0,1]\right)$ is compact.
- $\mathbb{T}^2$ is Hausdorff.
- It's clear that $(s_1, t_1) \sim (s_2, t_2)$ implies $f(s_1, t_1) = f(s_2, t_2)$. Conversely, suppose

  $$e^{2\pi i s_1} = e^{2\pi i s_2}, \; e^{2\pi i t_1} = e^{2\pi i t_2}$$

  By the familiar property of e^{ix}, we imply either $t_1 = t_2$ or $\{t_1, t_2\} = \{0, 1\}$; and either $s_1 = s_2$ or $\{s_1, s_2\} = \{0,1\}$

By applying Proposition 4.16, we conclude that $([0,1] \times [0,1]) / \sim$ is homeomorphic to $\mathbb{T}^2$.

Example 4.18. Consider the closed disk $\mathbb{D}^2 = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\right\}$, and define $(x_1, y_1) \sim (x_2, y_2)$ by:

- $x_1 = x_2$ and $y_1 = y_2$;
- (x_1, y_1) and (x_2, y_2) are in the boundary circle $\mathbb{S}^1$

Then the corresponding quotient space $\mathbb{D}^2/\sim$ is hoemomorhpic to the 2-dimension sphere $\mathbb{S}^2 = \left\{(x,y,z) \mid x^2 + y^2 + z^2 = 1\right\}$: Define the mapping

$$f : \; \mathbb{D}^2 \to \mathbb{S}^2$$

by

$$\begin{aligned}
&(0,0) \mapsto (0,0,1) \\
&(x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}} \sin\left(\pi\sqrt{x^2+y^2}\right), \frac{y}{\sqrt{x^2+y^2}} \sin\left(\pi\sqrt{x^2+y^2}\right), \cos\left(\pi\sqrt{x^2+y^2}\right)\right)
\end{aligned}$$

It's easy to check the conditions in Proposition 4.16, and we conclude that $\mathbb{D}^2/\sim$ is hoemomorhpic to $\mathbb{S}^2$.

In Proposition 4.16, we show the homeomorphism between $X/\sim$ and Y given the compactness of X and Hausdorffness of Y. Now we show the generalize the proposition by replacing these conditions with the quotient mapping q:

Proposition 4.19. *Suppose $q : X \to Y$ is a quotient map, and that $\sim$ is an equivalence relation on X given by the partition $\{q^{-1}(y) \mid y \in Y\}$. Then $X/\sim$ and Y are homeomorphic.*

Proof. Construct the mapping

$$h: \ X/\sim \to Y \quad \text{with} \quad h([x]) = q(x)$$

Note that the mapping h is well-defined and bijective. And the quotient mapping $q := h \circ p$ is continuous by definition. By applying Proposition 4.13, h is continuous, and we are only left to show that h^{-1} is continuous, i.e. for any open $\widetilde{U} \subseteq X/\sim$, $h\left(\widetilde{U}\right)$ is open in Y.

To see so, note that

$$q^{-1}\left(h\left(\widetilde{U}\right)\right) = p^{-1}h^{-1}\left(h\left(\widetilde{U}\right)\right) = p^{-1}\left(\widetilde{U}\right),$$

which is open by the definition of quotient topology. Therefore, $h\left(\widetilde{U}\right)$ is open by (2) in Definition 4.14. ☐

Example 4.20. $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle S^1: Indeed, consider the mapping

$$q : \mathbb{R} \to S^1 \qquad x \mapsto e^{2\pi i x}$$

It is clear that

1. q is a continuous open mapping (why?)
2. q is surjective

Therefore, $\mathbb{R}/\sim \cong S^1$, provided that $x \sim y$ iff $q(x) = q(y)$, i.e., $x - y \in \mathbb{Z}$. Therefore,

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

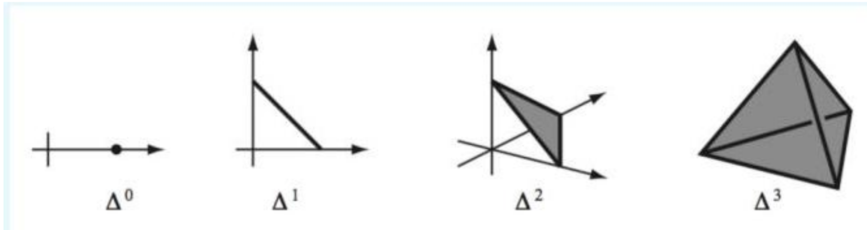
4.3 Simplicial Complex

The idea of simplicial complex is to build some new spaces from some "fundamental" objects. The combinatorists often study topology by the combinatorics of these fundamental objects. First we define what are the "fundamental" objects:

Definition 4.21 (n -simplex)**.** The standard n -simplex is the set

$$\Delta^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \forall i \text{ and } \sum_{i=1}^{n+1} x_i = 1\right\}$$

The 0, 1, 2 and 3 simplices can be visualized as follows:



1. The non-negative integer n is the *dimension* of this simplex;
2. Its *vertices*, denoted as $V(\Delta^n)$, are those points $(x_1, \ldots, x_{n+1})$ in Δ^n such that $x_i = 1$ for some i.
3. For each given non-empty $\mathcal{A} \subseteq \{1, \ldots, n+1\}$, its *facet* is defined as

$$\{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i = 0, \forall i \notin \mathcal{A}\}$$

In particular, Δ^n is a face of itself

4. The *inside* of Δ^n is

$$\text{inside}(\Delta^n) := \{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i > 0, \forall i\}$$

In particular, the inside of Δ^0 is Δ^0.

Definition 4.22 (Face Inclusion)**.** A *face inclusion* of Δ^m into Δ^n $(m < n)$ is a function $\Delta^m \to \Delta^n$ which comes from the restriction of an injective linear map $f : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ that maps vertices in Δ^m into vertices in Δ^n.

For example, the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined below is a face inclusion:

$$f(1,0) = (0,1,0), \ f(0,1) = (0,0,1).$$

Remark 4.23. Any injection mapping from $\{1, \ldots, m+1\} \to \{1, \ldots, n+1\}$ gives a face inclusion $\Delta^m \to \Delta^n$, and vice versa.

Now we build new spaces by gluing simplices together. This new space is called the simplicial complex. If a simplex is a part of the complex, so are all its faces.

Definition 4.24 (Abstract Simplicial Complex)**.** An (abstract) simplicial complex is a pair $K = (V, \Sigma)$, where V is a set of vertices and Σ is a collection of non-empty finite subsets of V (simplices) such that

1. For any $v \in V$, the 1-element set $\{v\}$ is in Σ
2. If σ is an element of Σ, then so is any non-empty subset of σ.

For example, if $V = \{1, 2, 3, 4\}$, then one can take:

$$\Sigma = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,3,4\}, \{2,4\}, \{1,3\}, \{3,4\}, \{1,4\}.\} \tag{4.1}$$

We can associate to an abstract simplicial complex K a topological space $|K|$, which is called its geometric realization:

Definition 4.25 (Topological Realization)**.** The topological realization of $K = (V, \Sigma)$ is a topological space $|K|$ (or denoted as $|(V, \Sigma)|$), where

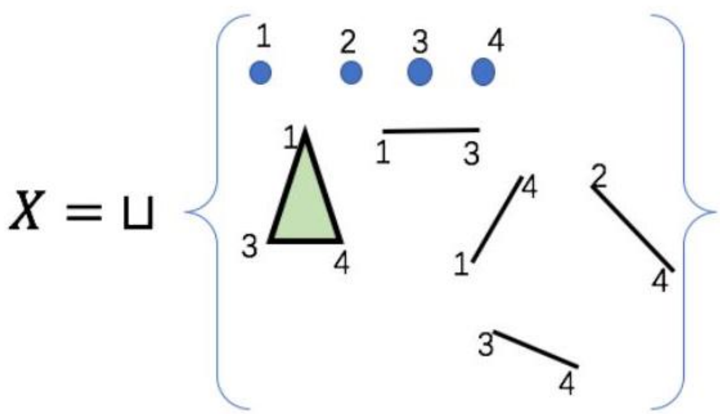
1. For each $\sigma \in \Sigma$ with $|\sigma| = n + 1$, take a copy of n -simplex and denote it as Δ_σ
2. Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a face of Δ_τ through face inclusion.

Or equivalently, $|K|$ is a quotient space of the disjoint union

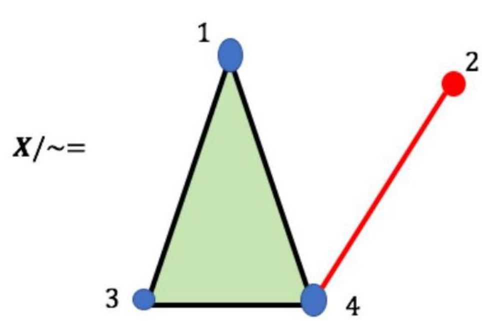
$$\coprod_{\sigma \in \Sigma} \sigma$$

by the equivalence relation which identifies a point $y \in \sigma$ with its image under the face inclusion $\sigma \to \tau$, for any $\sigma \subset \tau$.

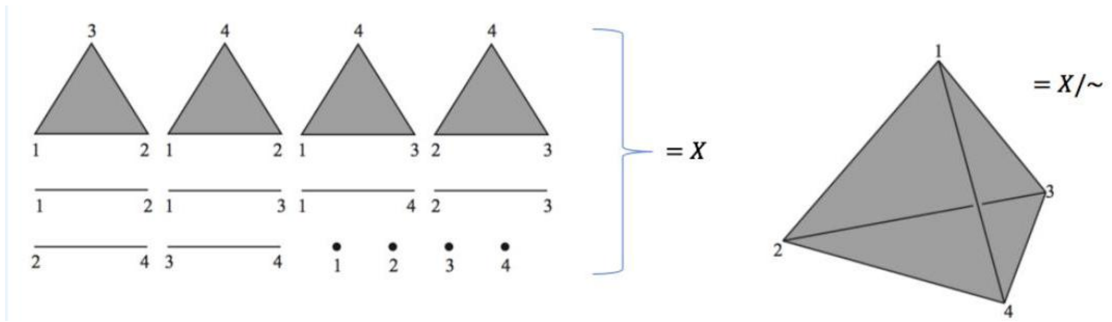
Example 4.26. Take (V, Σ) as in Equation 4.1, so that



Then its topological realization is:



Example 4.27. Take $V = \{1, 2, 3, 4\}$ and $\Sigma = \{$ all subsets of V except $V\}$. Then its topological realization is $|(V, \Sigma)| = \Delta^3$ as shown in the figure below:

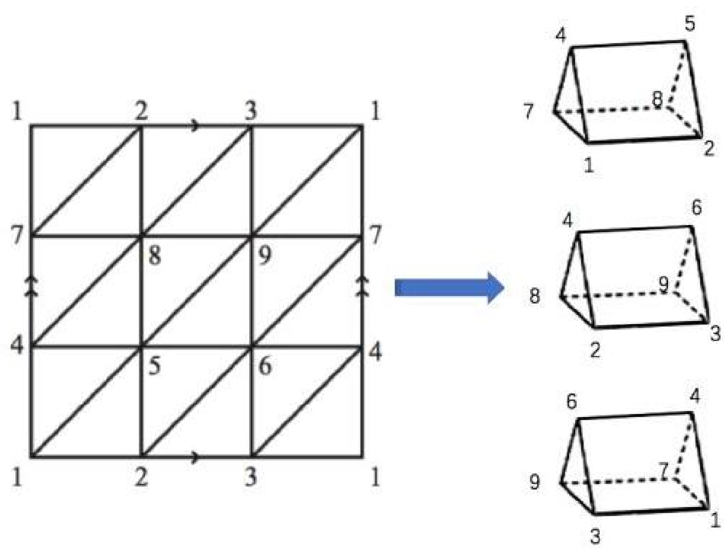


Definition 4.28 (Triangulation)**.** A *triangulation* of a topological space X is a simplicial complex $K = (V, \Sigma)$ together with a choice of homeomorphism $|K| \to X$.

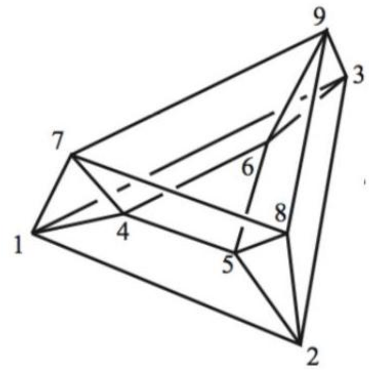
Example 4.29. Consider the simplical complex $K = (V, \Sigma)$ with

$$V = \{1, 2, 3, 4, \ldots, 9\},\ \Sigma = \begin{cases} \text{9 subsets with 1 element} \\ \text{27 subsets with 2 elements} \\ \text{18 subsets with 3 elements} \end{cases}$$

as given below. We start to build the topological realization of K with 9 0-simplicies, 27 1-simplicies, and 18 2-simplicies. The identification of them is as follows:

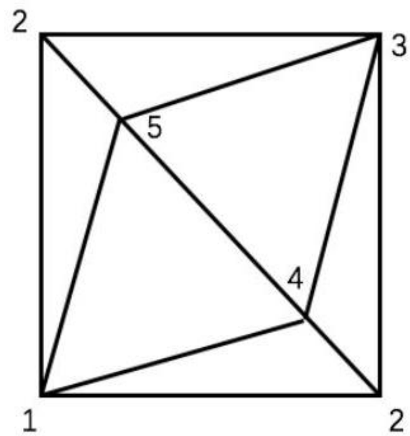


Step 1: Identify 3 columns separately, i.e., identify $\{1, 7, 4, 1, 2, 8, 5, 2\}$, $\{2, 8, 5, 2, 3, 9, 6, 3\}$, and $\{3, 9, 6, 3, 1, 7, 4, 1\}$.

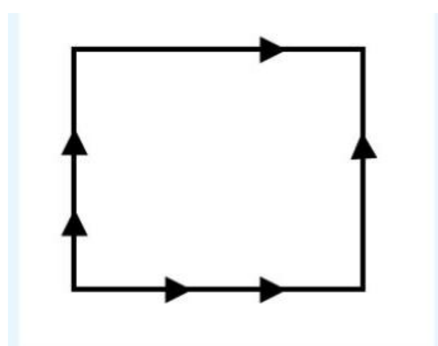


Step 2: "glue" these three prisms in the figure above together. Then it will give us a 'torus'. We will see in a moment that this $K = (V, \Sigma)$ is indeed a triangulation of $\mathbb{T}$.

Example 4.30. The simplicial complex below is a topological realization of S^2:

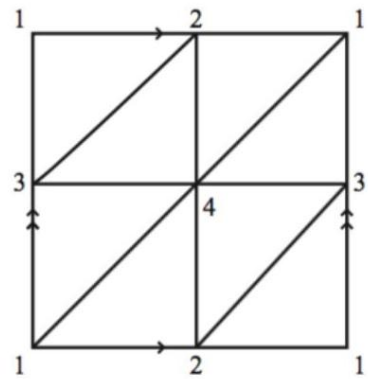


Indeed, S^2 is homeomorphic to the quotient space of $[0, 1] \times [0, 1]$ with the following identification:



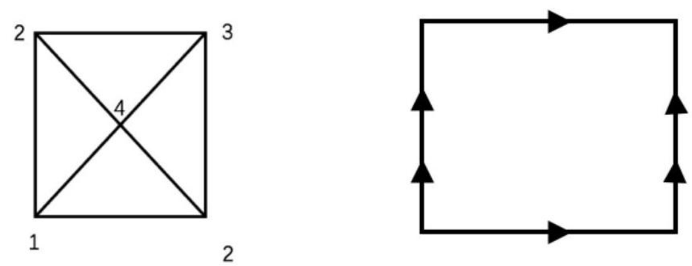
Example 4.31. One may ask whether one we build a triangulation of the tours using fewer simplices as in Example 4.29? The answer is no.

For instance, one may think the number of triangles can be reduced as in the figure below:



However, if one pays attention to the bottom edge of the square, there are two 1-simplicies labeled $\{1, 2\}$, which means we have to 'glue the two bottom edges together, which cannot happen in the construction of torus.

As another example, the following diagram **DOES NOT** give a triangulation of S^2:



(7.1a): The simplicial complex (V, Σ) (7.1b): Quotient Space of S^2

Note that the 2-simplex $\Delta_{\{2,3,4\}}$ appears twice in the left hand side of the figure. This means that we need to stick the top triangle and the right triangle together, which contradicts to the structure of the quotient space S^2 shown on the right hand side.

Simplicial complex gives us a combinatorial way to study X, i.e. it suffices to study (V, Σ) such that $|(V, \Sigma)| \cong X$. For example, if we want to distinguish $\mathbb{T} = S^1 \times S^1$ and S^2, we just need distinguish between their topological realizations. One way to do so is the following:

Theorem 4.32 (Euler Characteristic)**.** *For any simplicial complex (V, Σ), the Euler characteristic is defined by*

$$\mathcal{X}(V, \Sigma) := \sum_{i=1}^{\infty} (-1)^i \text{ \textit{(number of subsets in} } \Sigma \text{ \textit{with} } (i+1) \text{ \textit{-element)}}$$

Suppose that $|(V_1, \Sigma_1)| \cong |(V_2, \Sigma_2)|$, then

$$\mathcal{X}\left(V_1, \Sigma_1\right) = \mathcal{X}\left(V_2, \Sigma_2\right).$$

In particular, for two triangulations of the same topological space X, it has the same Euler characteristic. So we can define

$$\mathcal{X}(X) := \mathcal{X}(V, \Sigma)$$

for any *triangulation* $|(V, \Sigma)| \cong X$ *of* X*.*

From previous examples, we can see that $\mathcal{X}(S^2) = 5 - 9 + 6 = 2$ and $\mathcal{X}(\mathbb{T}) = 9 - 27 + 18 = 0$, which implies

$$S^2 \ncong \mathbb{T}.$$

4.4 Properties of Simplicial Complex

Definition 4.33 (Simplicial Subcomplex)**.** A subcomplex of a simplicial complex $K = (V, \Sigma)$ is a simplicial complex $K' = (V', \Sigma')$ such that

$$V' \subseteq V,\ \Sigma' \subseteq \Sigma$$

Proposition 4.34. *Suppose* K' *is subcomplex of* K*, then* $|K'|$ *is closed in* $|K|$*.*

Proof. Suppose that D is the disjoint union of all the simplicial complex forming $|K|$ (note that the number of component in D is $|\Sigma|$) Consider the canonical projection mapping

$$p : D \to |K|\,.$$

Observe that $p^{-1}(|K'|)$ precisely equals to $\coprod_{\sigma' \in \Sigma'} \sigma'$, which is closed in D. By definition of quotient topology, $|K'|$ is also closed. □

Definition 4.35. Let $K = (V, \Sigma)$ be a simplicial complex and $V' \subseteq V$. Then the subcomplex spanned by V' is (V', Σ') such that

- V' denotes the vertex set.
- the simplices Σ' is given by
$$\{\sigma \in \Sigma \mid \sigma \subseteq V'\}$$

Definition 4.36 (Link and Star)**.** Let $(V, \Sigma) = K$ be a simplicial complex.

- The *link* of $v \in V$, denoted as $\mathrm{lk}\,(v)$ is the sub-complex with vertex set:
$$\{w \in V \setminus \{v\} \mid \{v, w\} \in \Sigma\}$$
and simplicies:
$$\{\sigma \in \Sigma \mid \mathbf{v} \notin \sigma \text{ and } \sigma \cup \{\mathbf{v}\} \in \Sigma\}$$
- The *star* of v (denoted as $\mathrm{st}\,(v)$) is
$$\bigcup_{\sigma \in \Sigma,\ v \in \sigma} \mathrm{inside}\,(\sigma)$$
(c.f. Definition 4.21 for the definition of inside of an n-simplex)

Proposition 4.37. $\mathrm{st}\,(v)$ *is open and* $v \in \mathrm{st}\,(v)$.

Proof. Omitted - In fact, $|K| \setminus \mathrm{st}\,(v)$ is the simplicial subcomplex spanned by V. □

Proposition 4.38. *Suppose that* $K = (V, \Sigma)$, *where* V *is finite. Then* $|K|$ *is compact.*

Proof. The mapping $p : D \to |K|$ is a canonical projection mapping, which is continuous; and D (the finite disjoint union of Δ_σ 's) is compact. Therefore, $p\,(D) = |K|$ is compact. □

Proposition 4.39. *For any simplicial complex* $K = (V, \Sigma)$, *where* V *is finite, there is a continuous injection*

$$f : |K| \to \mathbb{R}^n \text{ for some } n$$

Proof. Let $K^\Pi = \left(V, \Sigma^\Pi\right)$, where Σ^Π = power set of V. Then

$$\left|K^\Pi\right| = \Delta^{|V|-1} \subseteq \mathbb{R}^{|V|}$$

Consider the inclusion

$$i : |K| \to \left|K^\Pi\right|$$

which comes from the following:

1. Consider the $D := \coprod_{\sigma\in\Sigma} \Delta_\sigma$ and $D' = \coprod_{\Sigma^\Pi\in\Sigma^\Pi} \Delta_{\Sigma^\Pi}$ in (V, Σ) and $\left(V, \Sigma^\Pi\right)$
2. Construct the mapping $\widetilde{i} : D \hookrightarrow D' \xrightarrow{p'} |K|$.
3. The mapping $\widetilde{i}$ descends to $i : D/\sim \to \left|K^\Pi\right|$ (try to write down the detailed mapping), which is continuous and injective.

Therefore, $|K| \hookrightarrow \left|K^\Pi\right| (\hookrightarrow \mathbb{R}^n)$, and the proof is complete. □

Remark 4.40. More generally, if $K = (V, \Sigma)$ is a simplicial subcomplex of $\widetilde{K} = \left(\widetilde{V}, \widetilde{\Sigma}\right)$, we can construct a continuous injection from $|K|$ to $\left|\widetilde{K}\right|$.

Let $D_\Sigma := \coprod_{\sigma\in\Sigma} \sigma$ and $D_{\widetilde{\Sigma}} := \coprod_{\widetilde{\sigma}\in\widetilde{\Sigma}} \widetilde{\sigma}$, then $|K| = D_\Sigma/{\sim_\Sigma}$ and $\left|\widetilde{K}\right| = D_{\widetilde{\Sigma}}/{\sim_{\widetilde{\Sigma}}}$. It follows that

$$f : D_\Sigma \to D_{\widetilde{\Sigma}} \xrightarrow{p} D_\Sigma/{\sim_\Sigma},$$

where p denotes the canonical projection mapping. Then f descends to a continuous mapping

$$\widetilde{f} : D_\Sigma/{\sim_\Sigma} \to D_{\widetilde{\Sigma}}/{\sim_{\widetilde{\Sigma}}}$$

Note that $\widetilde{f}$ is injective since for all $x, y \in D_\Sigma$,

$$x {\sim_{\widetilde{\Sigma}}} y \quad \Leftrightarrow \quad i\,(x) \sim_{\widetilde{\Sigma}} i\,(y)\,,$$

where $i : D_\Sigma \hookrightarrow D_{\widetilde{\Sigma}}$ denotes the inclusion mapping.

Proposition 4.41. *If $K = (V, \Sigma)$ with finite V, then $|K|$ is Hausdorff.*

Proof. Let $g : |K| \hookrightarrow \mathbb{R}^n$. Consider the bijective $g : |K| \to g(|K|)$, which is continuous. Since $|K|$ is compact, and $g(|K|) \subseteq \mathbb{R}^n$ is Hausdorff, we imply that $|K|$ and $g(|K|)$ are homeomorphic, i.e., $|K|$ is Hausdorff. □

Definition 4.42 (Edge Path)**.** An edge path of $K = (V, \Sigma)$ is a sequence of vertices $(v_1, \ldots, v_n), v_i \in V$ such that $\{v_i, v_{i+1}\} \in \Sigma, \forall i$.

Proposition 4.43. *Let $K = (V, \Sigma)$ be a simplicial complex. The following are equivalent:*

1. $|K|$ is connected.

2. $|K|$ is path-connected.

3. Any 2 vertices in (V, Σ) can be joined by an edge path, i.e., for $\forall u, v \in V$, there exists $v_1, \ldots, v_k \in V$ such that $(u, v_1, \ldots, v_k, v)$ is an edge path.

Sketch of Proof (to be revised). 1. (3) implies (2): For every $x, y \in |K|$,

$$\begin{cases} x \in \Delta_{\sigma_1} \text{ for some } \sigma_1 \in \Sigma. \\ y \in \Delta_{\sigma_2} \text{ for some } \sigma_2 \in \Sigma. \end{cases}$$

Take a path joining x to a vertex $v_1 \in \sigma_1$ and a path joining y to a vertex $v_2 \in \sigma_2$. By (3), we have a path joining v_1 and v_2.

2. (1) implies (3): Suppose on the contrary that there is a vertex v not satisfying (3). Take V' as the set of vertexs that can be joined with v ; and V'' as the set of vertexs that cannot be joinied with v.

Then $V', V'' \neq \varnothing$. Consider K', K'' be simplicial subcomplexes of K, spanned by V' and V''. Then $|K'|, |K''|$ are disjoint, closed in $|K|$.

$|K| = |K'| \cup |K''|$. If there exists $x \in |K| \setminus (|K'| \cup |K''|)$, then for any $\sigma \in \Sigma$ such that $x \in \Delta_\sigma$, we imply $\Delta_\sigma \nsubseteq |K'|$ or $|K''|$.

Therefore, σ consists of vertices in both V' and V''. Then there is $v', v'' \in \sigma$ joining V' and V''.

Therefore, there is no such x and hence $|K| = |K'| \cup |K''|$ is a disjoint union of two closed sets, i.e., not connected.

Chapter 5

Homotopy

To understand an object X (in our focus, X denotes topological space), one may try to understand functions

$$f : A \to X, \quad \text{or} \quad g : X \to B$$

One special example is to let $B = \mathbb{R}$. As for two topological spaces, there are many continuous mappings from X to Y. We will group all these mappings into equivalence classes by checking whether f can be 'continuously deformed' to g.

Definition 5.1. [Homotopy] A *homotopy* between two continuous maps $f, g : X \to Y$ is a continuous map

$$H : X \times [0, 1] \to Y$$

such that

$$H(x, 0) = f(x), \; H(x, 1) = g(x)$$

If such H exists, we say f and g are homotopic, denoted as

$$f \simeq g.$$

Example 5.2. Let $Y \subseteq \mathbb{R}^2$ be a convex subset. Consider two continuous maps $f : X \to Y$ and $g : X \to Y$. They are always homotopic since we can define the homotopy

$$H(x, t) = tg(x) + (1 - t) f(x).$$

Example 5.3. In the picture below, one can take

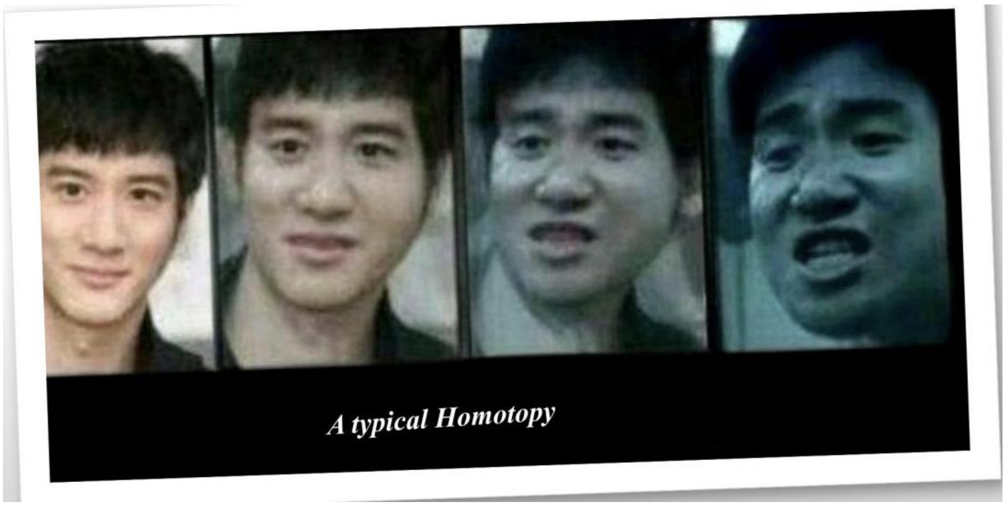
$$X = [0, H] \times [0, W],\; Y = [0, 255] \times [0, 255] \times [0, 255] \quad \text{and} \quad f, g : X \to Y$$

$f(h, w) := (r_0, g_0, b_0)$ and $g(h, w) := (r_1, g_1, b_1)$ are the pixels of the leftmost and rightmost picture at coordinate point (h, w) respectively.

The homotopy $H : X \times [0, 1] \to Y$ is a continuous 'deformation' of the pixels on the picture.

For instance, the pixels of the pictures are continuously deformed by

$$H\left((h,w),0\right)=f(h,w)\quad\rightarrow\quad H\left((h,w),\frac{1}{3}\right)\quad\rightarrow\quad H\left((h,w),\frac{2}{3}\right)\quad\rightarrow\quad H\left((h,w),1\right)=g(h,w).$$



Proposition 5.4. *Homotopy is an equivalence relation.*

Proof. 1. Let $f:X\rightarrow Y$ be any continuous map. Then $f\simeq f$: we can define a homotopy $H\left(x,t\right)=f\left(x\right),\forall 0\leq t\leq 1$.

2. Suppose $f\simeq g$, i.e., H is a homotopy between f and g, then $g\simeq f$: Define the mapping $H'\left(x,t\right)=H\left(x,1-t\right)$, then

$$H'\left(x,0\right)=g\left(x\right),\ H'\left(x,1\right)=f\left(x\right)$$

3. Let $f,g,h:X\rightarrow Y$ be three continuous maps. If f and g are homotopic and g and h are homotopic, then f and h are homotopic: Let $H:X\times[0,1]\rightarrow Y$ be a continuous map such that

$$H\left(x,0\right)=f\left(x\right),H\left(x,1\right)=g\left(x\right);$$

$K:X\times[0,1]\rightarrow Y$ be a continuous map such that

$$K\left(x,0\right)=g\left(x\right),K\left(x,1\right)=h\left(x\right).$$

Define a function $J:X\times[0,1]\rightarrow Y$ by

$$J\left(x,t\right)=\begin{cases}H\left(x,2t\right), & 0\leq t\leq 1/2\\ K\left(x,2t-1\right), & 1/2\leq t\leq 1\end{cases}$$

Then J is continuous, since for all closed $V\subseteq Y$,

$$J^{-1}\left(V\right)=\left(J^{-1}\left(V\right)\cap\left(X\times[0,1/2]\right)\right)\cup\left(J^{-1}\left(V\right)\cap\left(X\times[1/2,1]\right)\right)=H^{-1}\left(V\right)\cup K^{-1}\left(V\right),$$

and the closedness of $H^{-1}(V)$ and $K^{-1}(V)$ implies the closedness of $J^{-1}(V)$.

Moreover, J has the property that $J(x,0) = H(x,0) = f(x)$, while $J(x,1) = K(x,1) = h(x)$. $\square$

Back to Example 5.2: If $Y \subseteq \mathbb{R}^n$ is convex, then the set of continuous functions $f : X \to Y$ form a single equivalence class. In other words, all such maps are homotopic to each other.

Proposition 5.5. *Consider four continuous mappings*

$$W \xrightarrow{f} X,\ X \xrightarrow{g} Y,\ X \xrightarrow{h} Y,\ Y \xrightarrow{k} Z.$$

If $g \simeq h$, then

$$g \circ f \simeq h \circ f,\ k \circ g \simeq k \circ h$$

Proof. Suppose there exists the homotopy $H : g \simeq h$, then $k \circ H : X \times I \to Z$ gives the homotopy between $k \circ g$ and $k \circ h$.

Similarly, $H \circ (f \times \mathrm{id}_I) : W \times I \to Y$ gives the homotopy $g \circ f \simeq h \circ f$. $\square$

Definition 5.6. [Homotopy Equivalence] Two topological spaces X and Y are homotopy equivalent if there are continuous maps $f : X \to Y$, and $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_{X \to X}$$
$$f \circ g \simeq \mathrm{id}_{Y \to Y},$$

which is denoted as $X \simeq Y$.

Proposition 5.7. *Let X and Y be topological spaces.*

1. *If $X \cong Y$ are homeomorphic, then they are homotopic equivalent.*
2. *The homotopy equivalence $X \simeq Y$ gives a bijection between $\{\phi :$ continuous $W \to X\}/\sim$ and $\{\phi :$ continuous $W \to Y\}/\sim$, for any given topological space W.*
3. *The homotopy equivalence $X \simeq Y$ forms an equivalence relation between topological spaces.*

Proof. Since $X \simeq Y$, we can find $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We construct a mapping

$$\alpha :\ \{\phi : W \to X \mid \phi \text{ continuous}\}/\sim \longrightarrow \{\phi : W \to Y \mid \phi \text{ continuous}\}/\sim$$

by $[\phi] \mapsto [f \circ \phi]$. Then one can check α is well-defined, since $\phi_1 \sim \phi_2$ implies $f \circ \phi_1 \sim f \circ \phi_2$.

Also, we can construct a mapping

$$\beta :\ \{\phi : W \to Y \mid \phi \text{ continuous}\}/\sim \longrightarrow \{\phi : W \to X \mid \phi \text{ continuous}\}/\sim$$

by $[\psi] \mapsto [g \circ \phi]$. Similarly, β is well-defined.

Now one can check that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. For example,

$$\alpha \circ \beta [\psi] = [f \circ g \circ \psi] = [\psi],$$

where the last equality is due to the fact that $f \circ g \simeq \mathrm{id}_Y$.

3. is left as an exercise. $\square$

Compared with homeomorphism, some properties are lost when we consider homotopy equivalence.

Definition 5.8. [Contractible] The topological space X is *contractible* if it is homotopy equivalent to any point $\{\mathbf{c}\}$.

In other words, there exists continuous mappings f, g such that

$$\{\mathbf{c}\} \xrightarrow{f} X \xrightarrow{g} \{\mathbf{c}\}, g \circ f \simeq \mathrm{id}_{\{\mathbf{c}\}}$$

$$X \xrightarrow{g} \{\mathbf{c}\} \xrightarrow{f} X, f \circ g \simeq \mathrm{id}_X$$

Note that $g \circ f \simeq \mathrm{id}_{\{c\}}$ follows naturally; and since $X \cong X$, we can find f, g such that $f \circ g = c_y$ for some $y \in X$, where $c_y : X \to X$ is a constant function $c_y(x) = y, \forall x \in X$. Therefore, to check X is contractible, it suffices to check $c_y \simeq \mathrm{id}_X, \forall y \in X$ Therefore, X is contractible if its identity map id_X is homotopic to any constant map $c_y, \forall y \in X$.

Proposition 5.9. *The definition for X being contractible can be simplified further:*

1. *X is contractible if it is homotopy equivalent to some point $\{c\}$*
2. *X is contractible if the identity map id_X is homotopic to some constant map*

$$c_y(x) = y.$$

Proof. The only thing is to show that $c_y \simeq c_{y'}, \forall y, y' \in X$. By homework 3, X is path-connected, and therefore there exists continuous $p(t)$ such that

$$p(0) = y,\ p(1) = y'$$

Therefore, we construct the homotopy between c_y and $c_{y'}$ as follows:

$$H(x,t) = p(t).$$

$\square$

Example 5.10. $X = \mathbb{R}^2$ is contractible: It suffices to show that the mapping $f(\mathbf{x}) = \mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^2$ is homotopic to the constant function $g(x) = (0,0), \forall x \in \mathbb{R}^2$, i.e., $g = c_{(0,0)}$.

Consider the continuous mapping $H(\mathbf{x}, t) = tf(\mathbf{x})$, with

$$H(\mathbf{x}, 0) = c_{(0,0)}, \; H(\mathbf{x}, 1) = \mathrm{id}_X$$

Therefore, $c_{(0,0)} \simeq \mathrm{id}_X$. Since $c_{(0,0)} \simeq c_{\mathbf{y}}, \forall \mathbf{y} \in \mathbb{R}^2$, we imply $c_{\mathbf{y}} \simeq \mathrm{id}_X$ for any $\mathbf{y} \in \mathbb{R}^2$. Therefore, X is contractible. More generally, any convex $X \subseteq \mathbb{R}^n$ is contractible.

Remark 5.11. S^1 is not contractible, and we will see later. In particular, we are not able to construct the continuous mapping

$$H : S^1 \times [0,1] \to S^1$$

such that

$$H\left(e^{2\pi ix}, 0\right) = e^{2\pi ix}, \; H\left(e^{2\pi ix}, 1\right) = e^{2\pi i(0)} = 1$$

One may ask - how about the mapping $H\left(e^{2\pi ix}, t\right) = e^{2\pi ixt}$? Unfortunately, it is not well-defined, since

$$H\left(e^{2\pi i(1)}, t\right) = e^{2\pi it} = H\left(e^{2\pi i(0)}, t\right) = 1$$

and the equality is not true for $t \neq 0, 1$.

Definition 5.12. [Homotopy Retract] Let $A \subseteq X$ and $i : A \hookrightarrow X$ be an inclusion. We say A is a *homotopy retract* of X if there exists continuous mapping $r : X \to A$ such that

$$r \circ i : A \hookrightarrow X \xrightarrow{r} A = \mathrm{id}_A$$

$$i \circ r : X \xrightarrow{r} A \hookrightarrow X \simeq \mathrm{id}_X$$

In particular, $A \simeq X$.

Example 5.13. The 1-sphere S^1 is a homotopy retract of Mobius band M. More explicitly, let $M = [0,1]^2 / \sim$ and $S^1 = [0,1] / \sim$ by the appropriate equivalence relations. Then the inclusion i and the retraction r are given by:

$$i([x]) := \left[\left(x, \frac{1}{2}\right)\right]$$

$$r\left[(x,y)\right] := [x]$$

As a result,

$$r \circ i = \mathrm{id}_{S^1}, \; i \circ r\left(\left[(x,y)\right]\right) = \left[(x, 1/2)\right]$$

It suffices to show $i \circ r \simeq \mathrm{id}_M$, where $\mathrm{id}_M\left(\left[(x,y)\right]\right) = \left[(x,y)\right]$. To do so, construct the continuous mapping $H : M \times I \to M$ with

$$H\left(\left[(x,y)\right], t\right) \; := \; \left[(x, (1-t)\,y + t/2)\right]$$

To show the well-definedness of H, we need to check

$$H\left(\left[(0,y)\right],t\right) = H\left(\left[(1,1-y)\right],t\right),\ \forall y \in [0,1]$$

It's clear that H gives a homotopy between $i \circ r$ and id_M, i.e., $i \circ r \simeq \mathrm{id}_M$

Example 5.14. The $n-1$ -sphere S^{n-1} is a homotopy retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$: Consider the usual inclusion $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ and

$$r:\ \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$$

with $r(x) = \frac{x}{\|x\|}$. Therefore, $r \circ i = \mathrm{id}_{S^{n-1}}$ and $i \circ r\,(x) = \frac{x}{\|x\|}$.

It suffices to show that $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$. To see so, consider the homotopy

$$H\left(x,t\right) = tx + (1-t)\mathbf{x}/\parallel \mathbf{x} \parallel.$$

Then H is well-defined, since $H\left(x,t\right) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$.

Note that

$$H\left(\mathbf{x},0\right) = i \circ r\left(\mathbf{x}\right),\ H\left(\mathbf{x},1\right) = \mathbf{x} = \mathrm{id}\left(\mathbf{x}\right)$$

so the result follows.

Definition 5.15. [Homotopic Relative] Let $A \subseteq X$ be topological spaces. We say $f, g : X \to Y$ are homotopic relative to A if there exists $H : X \times I \to Y$ such that

$$\begin{cases} H\left(x,0\right) = f\left(x\right) \\ H\left(x,1\right) = g\left(x\right) \end{cases} \text{and } H\left(a,t\right) = f\left(a\right) = g\left(a\right), \forall a \in A$$

5.1 Simplicial Approximation Theorem

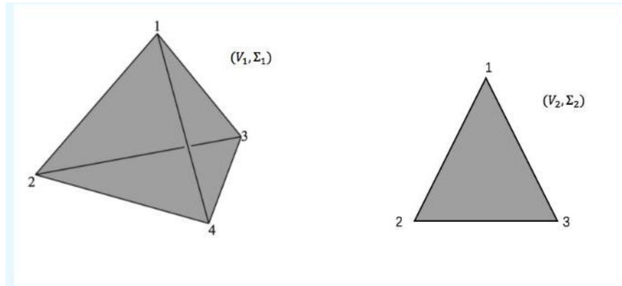
In this section, we wish to understand homotopy between simplicial complexes $f, g : |K| \to |L|$

Definition 5.16 (Simplicial Map)**.** A simplicial map between $K_1 = (V_1, \sum_1)$ and $K_2 = (V_2, \sum_2)$ is a mapping $f : K_1 \to K_2$ such that

1. It maps vertices to vertices
2. It maps simplicies to simplicies, i.e.,

$$f\left(\sigma_1\right) \in \sum\nolimits_2, \forall \sigma_1 \in \sum\nolimits_1,$$

Example 5.17. Consider the simplicial complexes defined as follows:



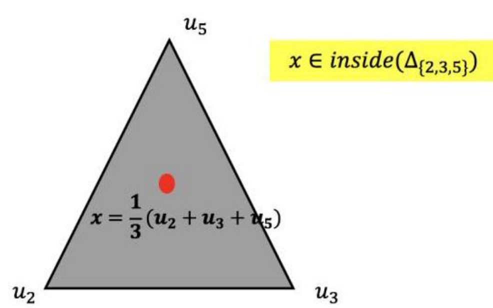
In particular, $\{1,2,3,4\} \notin \sum_1$ and $\{1,2,3\} \in \sum_2$. Then we can define the simplicial map as:

$$f(1) = 1,\ f(2) = 2,\ f(3) = 3,\ f(4) = 3$$

In particular, $f(\{1,2,4\}) = \{1,2,3\} \in \sum_2$.

Now we want to define the simplicial map between the topological realizations. There are several observations:

1. We have seen that each $|K| \subseteq \mathbb{R}^m$ for some m. In particular, $m = \#V - 1$.
2. Each point $x \in |K|$ lies uniquely on an inside of some Δ_σ, , where $\sigma \in \sum$.
3. Suppose that the vertices of K_1 are $V_1 = \{u_1, \ldots, u_n\} \subseteq \mathbb{R}^m$. Then every $\mathbf{x} \in K_1$ can be uniquely written as
$$\mathbf{x} = \sum_{i=1}^{k} \alpha_i U_{\sigma_i}$$
with $\alpha_i > 0, \sum \alpha_i = 1$ and $\sigma = \{U_{\sigma_1}, \ldots, U_{\sigma_k}\}$ is the unique simplex where $x \in$ inside (Δ_σ).



4. Our simplicial map f maps V_1 to $V_2 = \{w_1, \ldots, w_p\} \subseteq \mathbb{R}^m$, so for each i, we have $f(\mathbf{u}_i) = \mathbf{w}_j$ for some $j \in \{1, \ldots, p\}$.

Definition 5.18 (Mapping induced from Simplicial Mapping)**.** The simplicial map $f : K_1 \to K_2$ induces a mapping $|f| : |K_1| \to |K_2|$ between the topological realizations such that

1. It maps vertexes to vertexes, i.e., $|f|(v_1) = f(v_1), \forall v_1 \in V(K_1)$.
2. it is affine, i.e.,
$$|f| \left(\sum_{i=1}^{k} \alpha_i v_i \right) = \sum_{i=1}^{k} \alpha_i |f|(v_i)$$

(note that $|f| : |K_1| \to |K_2|$ is continuous).

Here is our more refined goal: Suppose we are given a continuous map $|g| : |K| \to |L|$, we want to approximate $|g|$ by $|f|$, where $f : K \to L$ is a simplicial map. It is obvious that f is an easier object to study compared with $|g|$.

However, we cannot achieve this goal unless we subdivide K into smaller pieces:

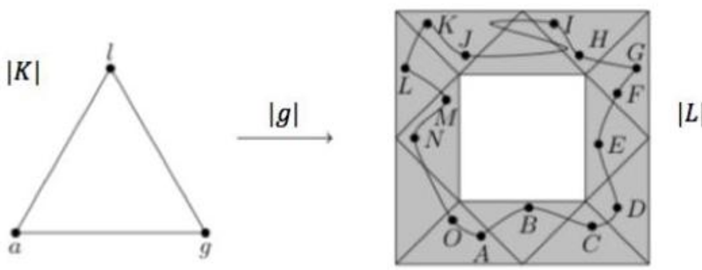
Definition 5.19 (Subdivision)**.** Let K be a simplicial complex. A simplicial complex K' is called a *subdivision* of K if

1. Each simplex of K' is contained in a simplex of K
2. Each simplex of K equals the union of finitely many simplices of K'

As a result, we can form an homeomorphism $h : |K'| \to |K|$ such that for each $\sigma' \in \sum_{K'}$, there exists $\sigma \in \sum_K$ satisfying

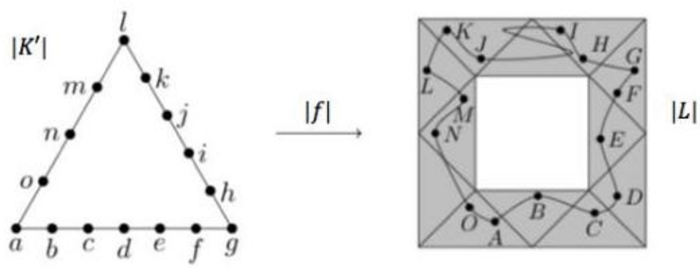
$$f\left(\Delta_{\sigma'}\right) \in \Delta_\sigma$$

Example 5.20. Consider the map $|g| : |K| \to |L|$ given in the figure below:



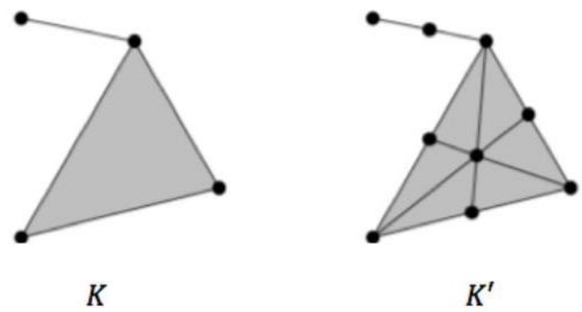
where $|g|(a) = A$, $|g|(g) = G$ and $|g|(l) = L$. It is clear that we cannot find a simplicial map $\gamma : K \to L$ such that $|\gamma| = |g|$, since (for instance) the 1-simplex $\Delta_{\{a,l\}}$ of K does not lie in a single simplex Δ_σ of L.

To remedy this, we subdivide K into smaller pieces as follows:



In this case, it is clear that one can define a simplicial map $f : K' \to L$ such that $|f| : |K'| \to |L|$ is our original map on the topological spaces.

Example 5.21 (Barycentric Subdivision)**.** One typical subdivision is the *barycentric subdivision.* Namely, for each simplicial complex K, we add extra vertices, which corresponds to the barycenters of the topological realization of K. One example of such subdivision is given below:



Remark 5.22. Suppose we have a metric on $|K|$. By subdivision, we can consider $|K'|$ such that for any $\sigma' \in \sum_{K'}$, any two points in $\Delta_{\sigma'}$ has a smaller distance.

The following result gives a criterion for the existence of a simplicial approximation for a mapping between topological realizations. For this we recall the notion of star at a vertex v of K:

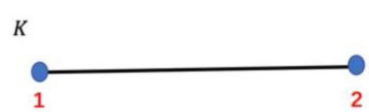
$$\mathrm{star}\,(v) = \bigcup_{v \in \sigma} \sigma^{\circ}.$$

Proposition 5.23. *Let $f : |K| \to |L|$ be a continuous mapping. Suppose that for each $v \in V_K$, there exists $g\,(v) \in V_L$ such that*

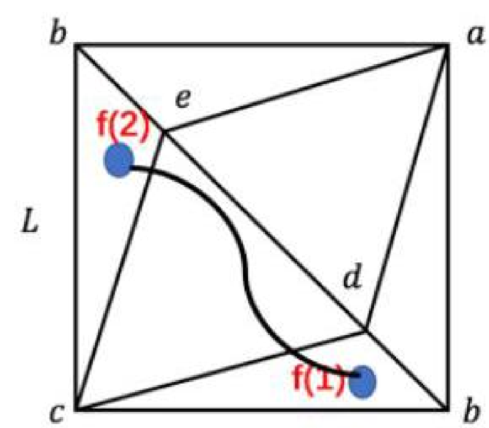
$$f\,(\mathrm{st}_K\,(v)) \subseteq \mathrm{st}_L\,(g\,(v))$$

then the mapping $g : V_K \to V_K$ gives $|g| \simeq f$. In such a case, g is called a **simplicial approximation** *to f.*

Example 5.24. Let K is as given below:

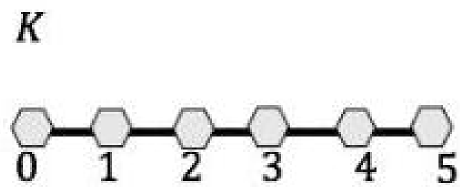


If the map $f : |K| \to |L|$ is given by:

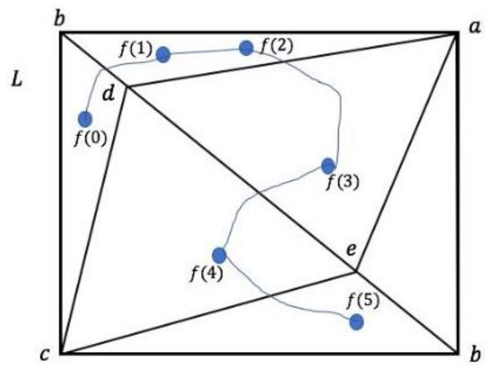


The hypothesis of Proposition 5.23 is not satisfied, so we cannot apply this proposition to construct a simplicial map $g : K \to L$ such that $|g| \simeq f$ (and indeed there is no such g).

However, if we subdivide K into smaller parts (5 in the picture below):



and $f : |K| \to |L|$ is given by:



then Proposition 5.23 is satisfied. In such a case, we can take a simplicial approximation g by

$$g(0) = b, g(1) = g(2) = d, g(3) = e, g(4) = c, g(5) = b.$$

Proof. We first show a statement: Suppose that $\sigma = \{v_0, \ldots, v_n\} \in \sum(K)$, and $x \in \text{inside}(\sigma) \subseteq |K|$. If $f(x) \in |L|$ lies in the inside of the (unique) simplex $\tau \in \sum_L$, (i.e., $f(x)$ can uniquely be expressed as $\sum\limits_{u_i \in \tau} \beta_i u_i$, such that $\beta_i > 0, \forall i$ and $\sum\limits_i \beta_i = 1\Big)$ then $g(v_0), \ldots, g(v_n)$ are vertices of τ.

By definition of inside (σ), $x = \sum\limits_{i=0}^{n} \alpha_i v_i$ with $\alpha_i > 0$ and $\sum\limits_{i=1}^{n} \alpha_i = 1$. Therefore, $x \in \text{st}_K(v_i)$ for $i = 1, \ldots, n$, where

$$\text{st}_K(v_i) := \left\{ a v_i + \sum_{j=1}^{m} b_j w_j \mid a > 0, b_j > 0, a + \sum_{j=1}^{m} b_j = 1, \{v_i, w_1, \ldots, w_m\} \in \sum\nolimits_K \right\}.$$

Therefore, $f(x) \in \text{int}(\text{st}_K(v_i)) \subseteq \text{st}_L(g(v_i))$, which follows that

$$f(x) = a g(v_i) + \sum_{j=1}^{m} b_j u_j, \text{ where } a > 0, b_j > 0, a + \sum_{j=1}^{m} b_j = 1, \{g(v_i), u_1, \ldots, u_m\} \in \sum\nolimits_L$$

Comparing the above formula with our hypothesis on $f(x), g(v_i)$ is a vertex of the simplex $\tau, i = 1, \ldots, n$. Moreover, $\{g(v_0), \ldots, g(v_n)\}$ is a subset of τ, which is a face of τ, and therefore $\{g(v_0), \ldots, g(v_n)\} \in \sum_L$.

- Therefore, the mapping $g : K \to L$ maps simplicies to simplicies, which is a simplicial mapping. We can construct a homotopy between f and $|g|$ as follows: Consider any $x \in |K|$, and let $\tau \in \sum_L$ be such that $f(x) \in \text{inside}(\tau)$. We write $x = \sum_{i=0}^{n} \lambda_i v_i$ for some $\{v_0, \ldots, v_n\} \in \sum_K$ and $\lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1$. Applying our claim,

$$|g|(x) = \sum_{i=0}^{n} \lambda_i g(v_i),$$

where $g(v_0), \ldots, g(v_n)$ are all vertices of τ.

We can directly construct a homotopy between f and $|g|$. Before that, we need some reformulations. Since $f(x) \in \text{inside}(\tau)$, we let $f(x) = \sum_{i=0}^{m} \mu_i \tau_i$. Since $|g|(x) = \sum_{i=0}^{n} \lambda_i g(v_i) \in \text{inside}(\tau)$, we rewrite $|g|(x) = \sum_{i=0}^{m} \lambda'_i \tau_i$. (by adding some $\lambda'_i := 0$ if necessary) We define the map

$$H : |K| \times I \to |L|$$

$$\text{with } (x, t) \mapsto \sum_{i=0}^{m} t\lambda'_i + (1-t)\mu_i$$

which follows that $f \simeq |g|$. □

Theorem 5.25. *[Simplicial Approximation Theorem] Let K, L be simplicial complexes with V_K finite, and $f : |K| \to |L|$ be continuous. Then there exists a subdivision $|K'|$ of $|K|$ together with a simplicial map g such that $|g| \simeq f$.*

Here the way for constructing subdivision $|K'|$ is as follows. There exists a constant $\delta > 0$. As long as the coarseness of K' is less than δ, our constructed subdivision satisfies the condition.

Proof. The sets $\{\text{st}_L(w) \mid w \in V(L)\}$ forms an open cover of $|L|$, which implies $\{f^{-1}(\text{st}_L(w))\}$ forms an open cover of $|K|$. By compactness, there exists a finite subcover of $|K|$, denoted as

$$|K| \subseteq \bigcup_{i=1}^{n} f^{-1}(\text{st}_L(w_i))$$

There exists a small number $\delta > 0$ such that for any $x, y \in |K|$ with $d(x, y) < \delta$, $x, y \in f^{-1}(\text{st}_L(w_i))$ for some i. Then we construct a simplicial subdivision $|K'|$ of $|K|$ with coarseness less than δ, i.e., $\forall x, y \in \text{st}_{K'}(v), d(x, y) < \delta$.

Therefore, $\text{st}_{K'}(v) \subseteq f^{-1}(\text{st}_L(w_i))$ for any $v \in V(K)$ and some $w_i \in V(L)$, i.e., $f(\text{st}_{K'}(v)) \subseteq \text{st}_L(w_i)$.

Chapter 6

Group Theory

In order to have a more systemetic study on topological spaces, we need to use some abstract algebra to 'extract' some properties of these spaces. In this chapter, we will focus on some aspects of group theory, which is covered in any standard abstract algebra course (e.g. MAT3004).

6.1 Basic Group Theory

When one talks about algebraic structure, we would think of addition $a+b$ and multiplication $a \cdot b$. In general, we make the following definition:

Definition 6.1. Let S be a set. A **binary operation** S is a map

$$* : S \times S \to S.$$

A subset $T \subseteq S$ is **closed under** $*$ if for all $a, b \in T$, $a * b \in T$.

Definition 6.2. A group G is a set along with a binary operation $* : G \times G \to G$ satisfying:

- $(a * b) * c = a * (b * c)$;
- There exists $e \in G$ such that $e * g = g * e = g$ for all $g \in G$.
- For all $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

Example 6.3.

1. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are groups.
2. $(\mathbb{R}[x], +)$ is a group.
3. (Modular arithmetic) $(\mathbb{Z}_n, +)$ is a group.
4. As for multiplication, $(R, \cdot)$ is not a group for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}[x]$ or $\mathbb{Z}_n$, since 0^{-1} does not exist in all cases.
5. $\mathbb{Z}\backslash\{0\}$ is still not a group, since 3^{-1} does not exist. But $(\mathbb{Q}\backslash\{0\}, \cdot)$ is a group.
6. $(GL_n(\mathbb{R}), \cdot)$ is a group under matrix multiplication.

7. Consider the regular n-gon $P_n \subseteq \mathbb{R}^2$ centered at the origin. Let D_{2n} be the symmetries of P_n, for instance $r \in D_{2n}$ is the rotation of P_n by $2\pi/n$ anti-clockwise about the origin, and $s \in D_{2n}$ be any reflection of P_n. Then obviously

$$r^n = e, \quad s^2 = e$$

in D_n. Indeed, all elements of D_{2n} can be obtained by $r^i s^j, 0 \leq i \leq r-1,\ 0 \leq j \leq 1$.

Definition 6.4. A group $(G, *)$ is called **abelian/commutative** if

$$a * b = b * a$$

for all $a, b \in G$.

Example 6.5. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{R}[x], +)$, $(\mathbb{Z}_n, +)$, $(\mathbb{Q}\backslash\{0\}, \cdot)$ are commutative, but $(GL_n(\mathbb{R}), \cdot)$, $(D_{2n}, \cdot)$ are not commutative.

Definition 6.6. Let G, H be two groups. The product group $(G \times H, *)$ is defined as

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

with $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

For example, $(\mathbb{R} \times \mathbb{R}, +) = \{(x, y) \mid x, y \in \mathbb{R}\}$ coincides with the usual $\mathbb{R}^2$, where

$$(x, y) * (x', y') = (x + x', y + y')$$

Here is the analogue of 'linear transformation' between two groups G and H in group theory:

Definition 6.7. A map between two groups $\phi : G \to H$ is a *homomorphism* if

$$\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$$

In other words, a homomorphism is a map preserving multiplications of groups.

Following the similar idea as in linear algebra, if $\phi : G \to H$ is a homomorphism, then $\phi(e_G) = e_H$.

Example 6.8. Let $G = (\mathbb{R}, +, 0)$, and $H = \{H_2, *, I_2\}$, with H_2 of the form

$$H_2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$$

Define a mapping

$$\phi : \; G \to H$$

by $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then ϕ is a homomorphism, i.e.:

$$\phi\left(x *_{\mathbb{R}} y\right) = \phi\left(x+y\right) = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \phi\left(x\right) *_{H_2} \phi\left(y\right)$$

Definition 6.9 (Isomorphism)**.** A homomorphism $\phi : G \to H$ is an isomorphism if ϕ is bijective. The isomorphism between G and H is denoted as $G \cong H$.

6.2 Free Groups

Definition 6.10. Let S be a set (the "alphabet").

- Define $S^{-1} := \left\{s^{-1} \in s \in S\right\}$ such that
  - $S \cap S^{-1} = \varnothing$; and
  - the elements $(s^{-1})^{-1} \in (S^{-1})^{-1}$ satisfy $\left(s^{-1}\right)^{-1} = s$. In particular $(S^{-1})^{-1} = S$.
- A *word* in S is a finite sequence $w = w_1 \cdots w_m$, where $m \in \mathbb{N}^{+} \cup \{0\}$, and each $w_i \in S \cup S^{-1}$. In particular, when $m = 0$, we view w as the empty sequence, denoted as $\varnothing$.
- The *concatenation* of two words $x_1 \cdots x_m$ and $y_1 \cdots y_n$ is the word $x_1 \cdots x_m y_1 \cdots y_n$
- Two words w, w' are *equivalent*, denoted as $w \sim w'$, if there are words $w_1, \ldots, w_n$ and $w = w_1, w' = w_n$ such that w_i and w_{i+1} differ from each other by

  $$w_i = \cdots y_1 x x^{-1} y_2 \cdots, \quad w_{i+1} = \cdots y_1 y_2 \cdots$$

  or

  $$w_i = \cdots y_1 y_2 \cdots, \quad w_{i+1} = \cdots y_1 x x^{-1} y_2 \cdots$$

  for some $x \in S \cup S^{-1}$.

Example 6.11. Let $S = \{a, b\}$ $S^{-1} = \left\{a^{-1}, b^{-1}\right\}$ and

$$w = aabab^{-1}b^{-1}a^{-1}abaabb^{-1}a, \quad w' = aabab^{-1}b^{-1}a^{-1}abaaa$$

Here w and w' differ by bb^{-1}. Therefore, $w \sim w'$, and w is said to be an elementary expansion of w'.

Moreover, for

$$w = aabab^{-1}b^{-1}a^{-1}abaabb^{-1}a, \quad w'' = aabab^{-1}b^{-1}baabb^{-1}a,$$

w and w'' differ by $a^{-1}a$, i.e., $a^{-1}\left(a^{-1}\right)^{-1}$, and therefore $w \sim w''$.

Definition 6.12 (Free Group)**.** Let S be a set. The *free group* $F\left(S\right)$ is defined to be the equivalence class of words, i.e.,

$$[w] \ := \ \{w' \text{ is a word in } S \mid w \sim w'\} \in F(S)$$

Note that $F(S)$ is indeed a group:

- $[w] * [w'] = [ww']$ (concatenation) check $w_1 \sim w_2, u_1 \sim u_2$ implies $w_1 u_1 \sim w_2 u_2$
- Identity element: $e = [\varnothing]$
- Inverse element: $[x_1 \cdots x_n]^{-1} = \left[x_n^{-1} \cdots x_1^{-1}\right]$

Example 6.13. Let $S = \{a\}$ and $S^{-1} = \{a^{-1}\}$. Any word w has the form

$$w = a \cdots a a^{-1} \cdots a^{-1} a \cdots a a^{-1} \cdots a^{-1} \cdots$$

In shorthand, we denote w as $w = \cdots a^p (a^{-1})^q a^r (a^{-1})^s \cdots$, and

$$[w] = \left[\cdots a^p \left(a^{-1}\right)^q a^r \left(a^{-1}\right)^s \cdots\right] = \left[\cdots a^{p-1} \left(a^{-1}\right)^{q-1} a^r \left(a^{-1}\right)^s \cdots\right] = \left[\cdots a^{p-1} \left(a^{-1}\right)^{q-2} a^{r-1} \left(a^{-1}\right)^s \cdots\right]$$

For instance, we can always eliminate the adjacent terms a and a^{-1} up to equivalence class. Therefore,

$$F(S) = \left\{\cdots, \left[a^{-2}\right], \left[a^{-1}\right], [\varnothing], [a], \left[a^2\right], \cdots\right\}.$$

It is clear that $F(S) \cong \mathbb{Z}$, where the isomorphism $\phi : \mathbb{Z} \to F(S)$ is $\phi(n) = [a^n]$.

Example 6.14. Let $S = \{a, b\}$ and $S^{-1} = \{a^{-1}, b^{-1}\}$. In this case, $[ab] \neq [ba]$, and $[ab^{-1}a^2b^2a^{-2}b]$ cannot be reduced further.

Since $F(S)$ is not an abelian group, we imply $F(S) \ncong \mathbb{Z} \times \mathbb{Z}$.

Definition 6.15 (Group With Relations)**.** Let S be a set. A group with relations is written as

$$G = \langle S \mid R(S) \rangle$$

where $R(S)$ consists of elements in $F(S)$, and every element in G can be written as the form $[w]$ for some $w \in F(S)$. We insist that $[w] = [w']$ in G iff

- w and w' differ by some $xx^{-1}, x \in S \cup S^{-1}$, or
- w and w' differ by some element $z \in R(S)$, or its inverse.

(One can easily check the above two conditions defines an equivalence relationship $\sim$ on $F(S)$, and hence $[w] \in G := F(S)/\sim$ can be seen as the equivalence class with representative w.

Example 6.16. Let

$$G = \left\langle a, b \mid a^2, b^2, abab^{-1}a^{-1}b^{-1} \right\rangle.$$

We want to enumerate all possible elements in G. To do so, observe that

$$\left[b^{-1}\right] = \left[b^{-1}b^2\right] = [b], \quad \left[a^{-1}\right] = [a], \quad [bab] = \left[abab^{-1}a^{-1}b^{-1}bab\right] = \left[abab^{-1}b\right] = [aba]$$

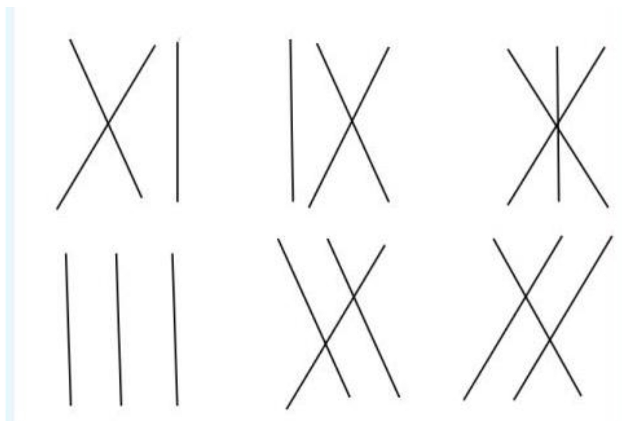
As a result,

- $[a^{-n}] = [a^n]$ and $[b^{-n}] = [b^n]$
- $[a^{2n+1}] = [a]\,, [b^{2n+1}] = [b]\,, [a^{2n}] = [\varnothing]\,, [b^{2n}] = [\varnothing]$
- All elements of G are of the form $[\cdots abababab \cdots]$.

Moreover, since each aba can be changed into bab, the elements in G are

$$[\varnothing]\,, [a]\,, [b]\,, [ab]\,, [ba]\,, [aba]$$

In fact, $G \cong D_6(\cong S_3)$, where $S_3 := \{\mu : \{1,2,3\} \to \{1,2,3\} \mid \mu \text{ is bijective}\}$ is the symmetric (or permutation) group of 3 elements. There are $6 = 3!$ elements of S_3:



for instance, $\times \mid$ is the bijection

$$1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 1$$

and $\mid \times$ is the bijection

$$1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2.$$

Then the isomorphism $\phi : S_3 \to G$ is given by: where $\phi(\times \mid) = a$, $\phi(\mid \times) = b$.

Example 6.17. 1. The dihedral group D_{2n} is isomorphic to (or one can simply take this as a definition):

$$D_{2n} := \left\langle a, b \mid a^n = e, b^2 = e, bab = a^{-1} \right\rangle$$

2. Consider $G =< a, b \mid ab = ba >$. Then any element of G can be expressed as $[\cdots a^s b^t a^u b^v \cdots]$. Using the relations $ab = ba$ and $ba^{-1} = a^{-1}b$, we can always push all powers of a to the left. Therefore, all elements in G are of the form $[a^p b^q],\ p, q \in \mathbb{Z}$, and we have the relation

$$[a^{p_1} b^{q_1}]\,[a^{p_2} b^{q_2}] = [a^{p_1+p_2} b^{q_1+q_2}].$$

Therefore, $G \cong \mathbb{Z} \times \mathbb{Z}$, where the isomorphism is given by:

$$\phi :\ \mathbb{Z} \times \mathbb{Z} \to G_2$$

with $(p,q) \mapsto [a^p b^q]$.

3. Let
$$H = \left\langle a \mid a^5 \right\rangle = \left\{1, a, a^2, \ldots, a^4\right\}$$
It's clear that $H \cong \mathbb{Z}/5\mathbb{Z}$, where the isomorphism is given by:
$$\phi : \ \mathbb{Z}/5\mathbb{Z} \to H$$
with $m + 5\mathbb{Z} \mapsto [a^m]$.

6.3 Cayley Graph for finitely presented groups

Graphs have strong connection with groups. Here we introduce a way of building graphs using groups, and the graphs are known as Cayley graphs. They describe many properties of the group in a topological way.

Definition 6.18 (Oriented Graph)**.** An oriented graph T is specified by

1. A countable or finite set V, known as vertices
2. A countable or finite set E, known as edges
3. A function $\delta : E \to V \times V$ given by

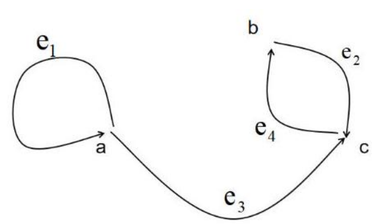
$$\delta(e) = (\ell(e), \tau(e))$$

where $\ell(e)$ denotes the initial vertex and $\tau(e)$ denotes the terminal vertex.

For example, let

- $V = \{a, b, c\}$
- $E = \{e_1, e_2, e_3, e_4\}$
- $\delta(e_1) = (a,a), \delta(e_2) = (b,c), \delta(e_3) = (a,c), \delta(e_4) = (b,c)$

Then its oriented graph looks like:



Definition 6.19 (Cayley graph)**.** Let $G = \langle S \mid R(S)\rangle$ with $|S| < \infty$. The Cayley graph associated to G is an oriented graph with

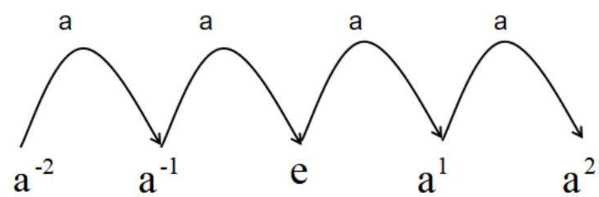
1. The vertex set G
2. The edge set $E \ := \ G \times S$

3. The function $\ell : E \to V \times V$ is given by:

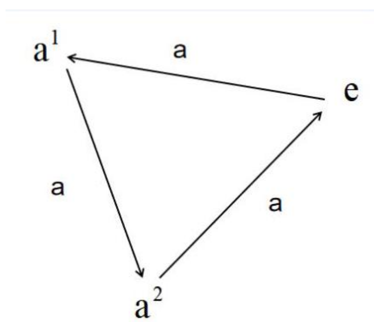
$$\ell : \ G \times S \to G \times G$$

with $(g, s) \mapsto (g, g \cdot s)$. In particular, we link two elements in G if they differ by a generator multiplied on the right.

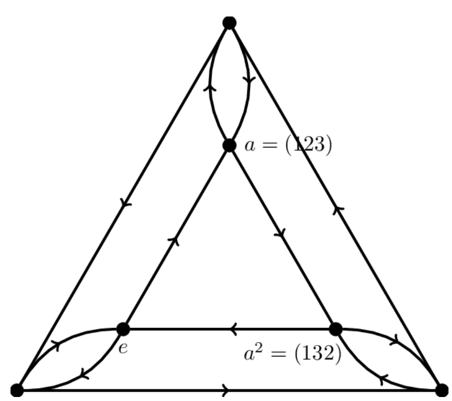
Example 6.20. 1. The Cayley graph for $G = \langle a \rangle \, (\cong \mathbb{Z})$ is shown below:



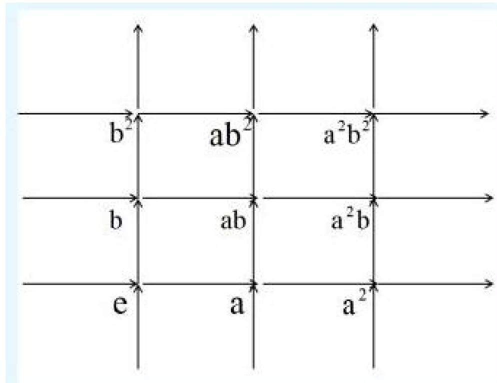
2. The Cayley graph for $G = \langle a \mid a^3 \rangle$ is:



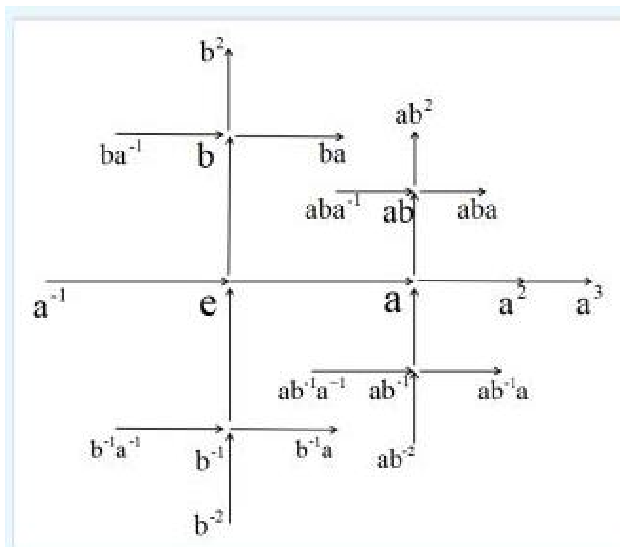
3. The Cayley graph for $G = \langle a, b \mid a^2, b^2, aba = bab \rangle$ is:



4. The Cayley graph for $G = \langle a, b \mid ab = ba \rangle$ is:



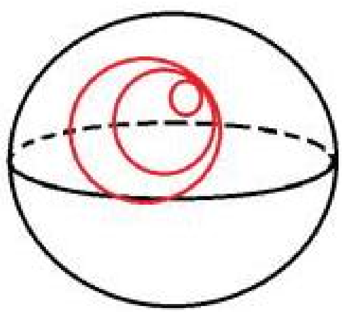
5. The Cayley graph for $G = \langle a, b \rangle$ is:



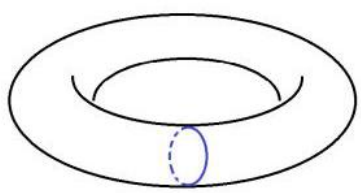
Chapter 7

Fundamental Group

The fundamental group connects topology and algebra together. To get an intuition about the fundamental group of a topological space, consider the example S^2 (2-sphere) and $\mathbb{T} = S^1 \times S^1$ (torus). Pictorially, all loops in S^2 be contracted into a point:



while some loops in the torus cannot be contracted into a point:



In this chapter, we will use groups to describe this phenomenon formally.

Definition 7.1 (loop)**.** Let X be a topological space. A *loop* on X is a constant map $\ell : [0,1] \to X$ such that $\ell(0) = \ell(1)$.

We say ℓ is *based at* $b \in X$ if $\ell(0) = \ell(1) = b$.

Definition 7.2 (composite loop)**.** Suppose that $\mathbf{u}, \mathbf{v}$ are loops on X based at $b \in X$. The composite loop $u \cdot v$ is given by

$$u \cdot v = \begin{cases} u(2t), & \text{if } 0 \le t \le 1/2 \\ v(2t-1), & \text{if } 1/2 \le t \le 1 \end{cases}$$

Definition 7.3. [fundamental group] The homotopy class of loops relative to $\{0,1\}$ based at $b \in X$ forms a group. It is called the fundamental group of X based at b, denoted as $\pi_1(X,b)$.

More precisely, let

$$[\ell] = \{m \mid m \text{ is a loop based at } b \text{ that is homotopic to } \ell, \text{ relative to } \{0,1\}\}$$

Then

$$\pi_1(X,b) := \{[\ell] \mid \ell \text{ are loops based at } b\}.$$

The multiplication operation in $\pi_1(X,b)$ is defined as:

$$[\ell] * [\ell'] \;:=\; [\ell \cdot \ell'] \,,\ \forall [\ell], [\ell'] \in \pi_1(X,b) \,.$$

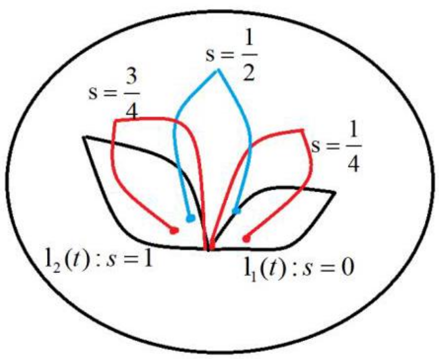
Recall that two paths $\ell_1, \ell_2 : [0,1] \to X$ are homotopic **relative to** $\{0,1\}$ if we can find $H : [0,1] \times [0,1] \to X$ such that

$$H(t,0) = \ell_1(t) \,,\ H(t,1) = \ell_2(t)$$

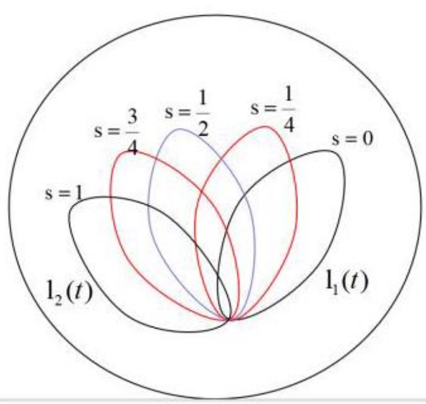
and

$$H(0,s) = \ell_1(0) = \ell_2(0) \,, \forall 0 \le s \le 1,\ H(1,s) = \ell_1(1) = \ell_2(1) \,, \forall 0 \le s \le 1.$$

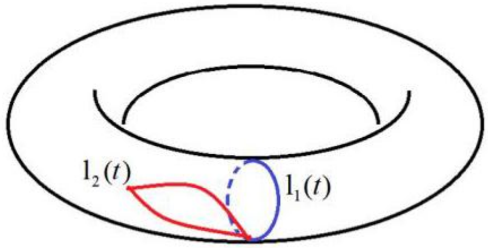
For instance, the following picture denotes a homotopy of ℓ_1 and ℓ_2 **not** relative to $\{0,1\}$:



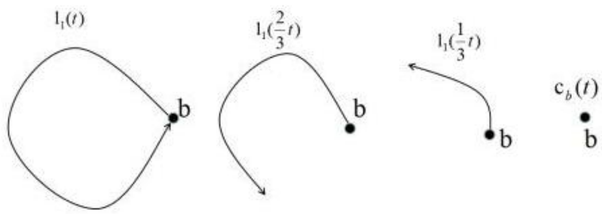
And below is a homotopy relative to $\{0,1\}$:



It is essential to study homotopy relative to $\{0,1\}$. For example, given a torus with a loop $\ell_1(t)$ and a base point b. We want to distinguish $\ell_1(t)$ and $\ell_2(t)$:



Obviously, there should be something different between $\ell_1(t)$ and $\ell_2(t)$. However, if we get rid of this 'relative' condition, all loops are homotopic to the constant map $c_b(t) = b$:



In this case, $\ell \simeq c_b$ for any loop ℓ. As a result, there is only one trivial element $\{[c_b]\}$ in $\pi_1(X,b)$.

That's the reason why we define $\pi_1(X,b)$ as the collection of homotopy classes relative to $\{0,1\}$ based at b in X, so that the continuous deformations of ℓ_1 are all loops based at b.

Now we need to show that the multiplication we defined in Definition 7.3 does give a group structure of $\pi_1(X,b)$:

Theorem 7.4. *Let $[\cdot]$ denote the homotopy class of loops relative to $\{0,1\}$ based at b, and define the operation*

$$[\ell] * [\ell'] = [\ell \cdot \ell']$$

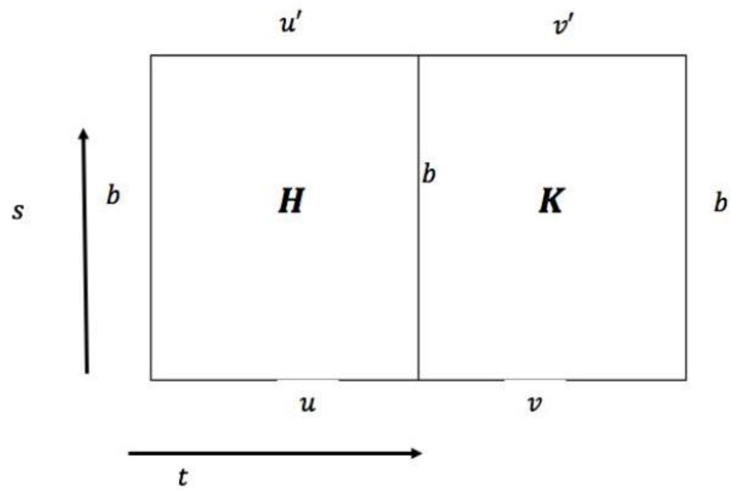
Then $(\pi_1(X,b), *)$ forms a group, where

$$\pi_1(X,b) \ := \ \{[\ell] \mid \ell : [0,1] \to X \ \textit{denotes loops based at } b\}$$

Proof. 1. Well-definedness: Suppose that $u \sim u'$ and $v \sim v'$, it suffices to show $u \cdot v \simeq u' \cdot v'$. Consider the given homotopies $H : u \simeq u', K : v \simeq v'$. Construct a new homotopy $L : I \times I \to X$ by

$$L(t,s) = \begin{cases} H(2t,s), & 0 \le t \le 1/2 \\ K(2t-1,s), & 1/2 \le t \le 1 \end{cases}$$

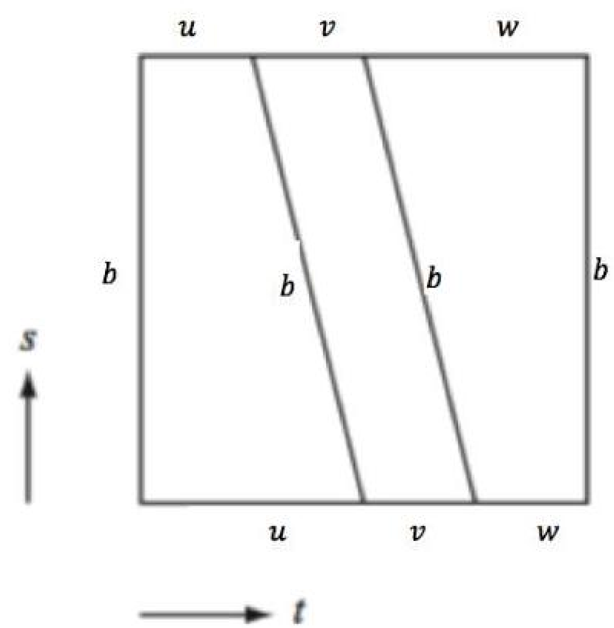
The diagram below explains the ideas for constructing L. The plane denote the set $I \times I$, and the labels characterize the images of each point of $I \times I$ under L.



Therefore, $u \cdot v \simeq u' \cdot v'$.

2. Associativity: Note that $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are essentially different loops. Although they go with the same path, they are with different speeds. More explicitly, the loop $(u \cdot v) \cdot w$ travels u, v using 1/4 seconds, and w in 1/2 seconds; but the loop $u \cdot (v \cdot w)$ travels u in 1/2 seconds, and then v, w in 1/4 seconds.

We want to construct a homotopy that describes the loop changes from $u \cdot (v \cdot w)$ to $(u \cdot v) \cdot w$. A graphic illustration is given below:



More explicitly, the homotopy $H : I \times I \to X$ is given by:

$$H(t,s) = \begin{cases} u\left(4t/\left(2-s\right)\right), & 0 \le t \le 1/2 - 1/4s \\ v\left(4t-2+s\right), & 1/2 - 1/4s \le t \le 3/4 - 1/4s \\ w\left(4t-3+s/\left(1+s\right)\right), & 3/4 - 1/4s \le t \le 1 \end{cases}$$

Therefore,

$$[u] * ([v] * [w]) = ([u] * [v]) * [w]$$

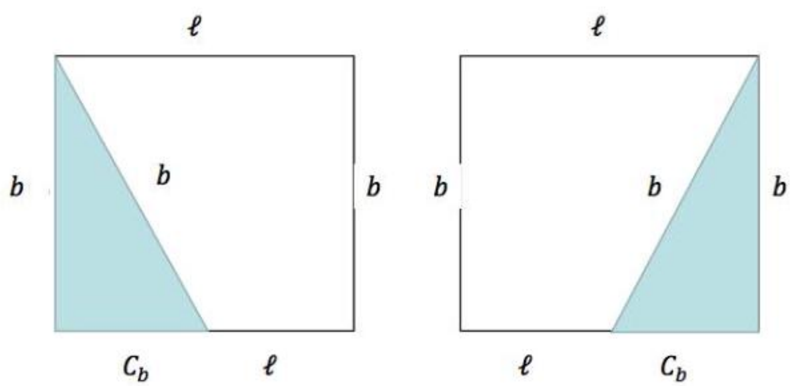
3. Identity: The identity element is the constant map, i.e., let $c_b : I \to X$ by $c_b(t) = b, \forall t$, and let $e = [c_b]$. Then it suffices to show

$$e * [\ell] = [\ell] * e = [\ell] \Leftrightarrow [c_b \cdot \ell] = [\ell \cdot c_b] = [\ell]$$

Or equivalently,

$$c_b \cdot \ell \simeq \ell, \ \ell \cdot c_b \simeq \ell$$

The graphic homotopy is shown below:



4. Inverse: the inverse of $[u]$, where u is a loop, should be $[u']$, where u' is the reverse of the traveling of u. More explicitly, for all loop $u : I \to X$ based at b, define $u^{-1} : I \to X$ by $u^{-1}(t) = u(1-t)$. Note that

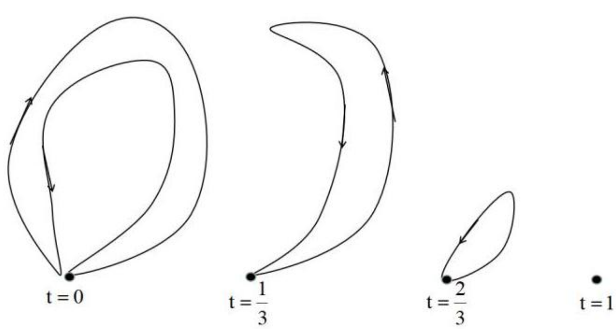
$$[u] * \left[u^{-1}\right] = \left[u \cdot u^{-1}\right], \ e = [c_b]$$

So it suffices to show $u \cdot u^{-1} \simeq c_b$ and $u^{-1} \cdot u \simeq c_b$:

The homotopy below gives $u \cdot u^{-1} \simeq c_b$.

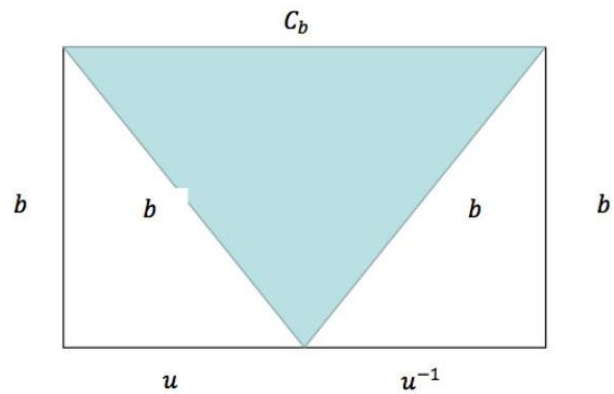
$$H(t,s) = \begin{cases} u(2t(1-s)), & 0 \le t \le 1/2 \\ u((2-2t)(1-s)), & 1/2 \le t \le 1 \end{cases}$$

The graphic illustration is given below:



So the result follows. □

Remark 7.5. Note that the figure below does not define a homotopy from $u \cdot u^{-1}$ to c_b:



The reason is that when we move from bottom to above in the figure, i.e. $s \to 1$, if we look at the white triangles, the time to travel along u and u^{-1} reduces from $1/2$ seconds to 0 second. In other words, when $s \to 1$ one has to go along u and u^{-1} at infinite speed, and hence it is not well-defined.

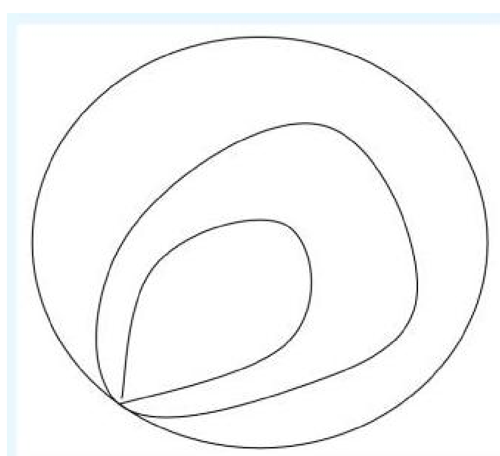
As an exercise, try to write down the function $H : [0,1] \times [0,1] \to X$ explicitly for the above diagram, and check that $H(t,s) \to \infty$ as $s \to 1$.

Example 7.6. We now check $\pi_1(\mathbb{R}^2, b) = \{e\}$ is trivial: For any $u : I \to \mathbb{R}^2$ with $u(0) = u(1) = b$, consider the homotopy

$$H(t,s) = (1-s)\,u(t) + sb.$$

Therefore, $u \simeq c_b$ for any loop u based at b.

Check the diagram below for graphic illustration of this homotopy.



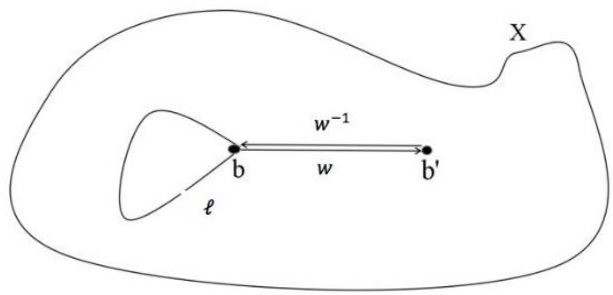
More generally, if $X \simeq \{x\}$ is contractible, then $\pi_1(X,b) = \{e\}$. The same argument cannot work for $(\mathbb{R}^2 \backslash \{0\}, \mathbf{b})$, since the mapping $H : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\}$ with $H(\mathbf{t}, s) = (1-s)\,u(\mathbf{t}) + s\mathbf{b}$ is not well-defined. In particular, the value $H(s,t)$ may hit the origin 0.

Proposition 7.7. *If b, b' are path-connected in X, then $\pi_1(X,b) \cong \pi_1(X,b')$.*

Proof. Let w be a path from b to b', and define

$$w_{\#}: \pi_1(X,b) \to \pi_1(X,b') \quad \text{with } [\ell] \mapsto \left[w^{-1}\ell w\right].$$

1. $w_{\#}$ is well-defined: Check that $\ell \simeq \ell'$ implies $w^{-1}\ell w \simeq w^{-1}\ell' w$. See the figure below for graphic illustration.



2. $w_{\#}$ is a homomorphism:

$$w_{\#}([\ell_1]) \cdot w_{\#}([\ell_2]) = \left[w^{-1} \cdot \ell_1 w\right] \cdot \left[w^{-1} \cdot \ell_2 w\right] = \left[w^{-1} \cdot \ell_1 \ell_2 w\right] = w_{\#}([\ell_1 \ell_2])$$

where the second equality is due to the fact that $w \cdot w^{-1} \simeq c_b$.

3. $w_{\#}$ is injective: If the loops ℓ_1, ℓ_2 are such that $w_{\#}(\ell_1) = w_{\#}(\ell_2)$, i.e.

$$\left[w^{-1}\ell_1 w\right] = \left[w^{-1}\ell_2 w\right]$$

Extend the definition of $[\ell]$ to allow ℓ to be a path, and the equivalence class is defined by the relation " $\sim$ ": $\ell_1 \sim \ell_2$ iff they are homotopic relative to $\{0,1\}$, then

$$[\ell_1] = [w]\left[w^{-1}\ell_1 w\right]\left[w^{-1}\right] = [w]\left[w^{-1}\ell_2 w\right]\left[w^{-1}\right] = [\ell_2] \tag{11.5}$$

4. $w_{\#}$ is surjective: we give the inverse of $w_{\#}$ explicitly:

$$w_{\#}^{-1}: \pi_1(X,b') \to \pi_1(X,b) \quad \text{with } [m] \mapsto \left[w \cdot m \cdot w^{-1}\right]$$

□

From now on, for any path connected space X, we will just write $\pi_1(X)$ instead of $\pi_1(X,x)$ if it causes no confusion.

Proposition 7.8. *Let (X,x) and (Y,y) be topological spaces with basepoints x and y, and $f: X \to Y$ be a continuous map with $f(x) = y$. Then every loop $\ell: I \to X$ based at x gives a loop $f \circ \ell: I \to Y$ based at y, i.e., the continuous map f induces a homomorphism of groups*

$$f_*: \pi_1(X,x) \to \pi_1(Y,y), \qquad [\ell] \mapsto [f \circ \ell] := f_*([\ell])$$

Moreover,

1. $(\mathrm{id}_{X\to X})_* = \mathrm{id}_{\pi_1(X,x)\to\pi_1(X,x)}$
2. $(g\circ f)_* = g_*\circ f_*$
3. *If* $f\simeq f'$ *relative to* $x\in X$, *then* $f_* = (f')_*$

Proof. Well-definedness: Suppose that $\ell\simeq\ell'$, then $f\circ\ell\simeq f\circ\ell'$ by Proposition 5.5. Therefore, $[f\circ\ell]=[f\circ\ell']$.

Homomorphism: It is clear that

$$f\circ(\ell\circ\ell') = (f\circ\ell)\circ(f\circ\ell')$$

Therefore, $f_*[\ell\ell'] = (f_*[\ell]) * (f_*[\ell'])$ The other three statements are obvious. □

Proposition 7.9. *Let* X, Y *be path-connected such that* $X\simeq Y$ *are homotopy equivalent (i.e., there exists* $f: X\to Y$ *and* $g: Y\to X$ *such that* $g\circ f\simeq\mathrm{id}_X, f\circ g\simeq\mathrm{id}_Y$ *). Then*

$$\pi_1(X)\cong\pi_1(Y).$$

In particular, if X, Y *are path-connected such that* $X\cong Y$ *are homeomorphic, then* $\pi_1(X)\cong\pi_1(Y)$.

Proof. Consider the mapping

$$\pi_1(X,x_0)\xrightarrow{f_*}\pi_1(Y,y_0)\xrightarrow{g_*}\pi_1(X,x_1)$$

By Proposition 7.8, f_* and g_* are homomorphisms. So it suffices to show that f_* and g_* are bijective.

Wrong proof: $g\circ f\simeq\mathrm{id}_X$ implies $(g\circ f)_* = (\mathrm{id}_X)_*$ implies $g_*\circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$.

Reason: $(g\circ f)\simeq\mathrm{id}_X$ is not relative to x_0.

So we have to work harder – Consider the homotopy $H : g\circ f\simeq\mathrm{id}_X$, where $H(x_0,s)$ is not necessarily a constant for $s\in I$. It follows that $H(x_0,0)=x_1$ and $H(x_0,1)=x_0$, i.e., $w(s) := H(x_0,s)$ defines a path from x_1 to x_0.

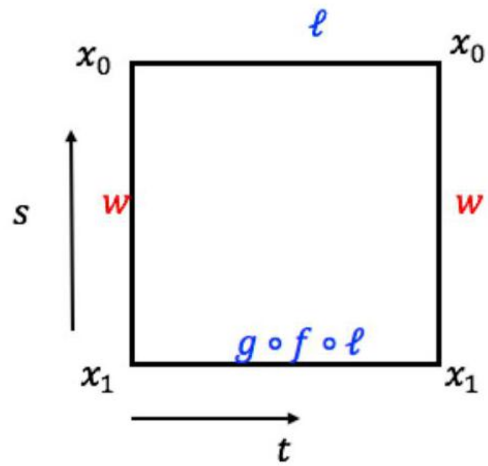
For any loop $\ell: I\to X$ based at x_0, consider the homotopy

$$K = H\circ(\ell\times\mathrm{id}_I):\ I\times I\to X$$

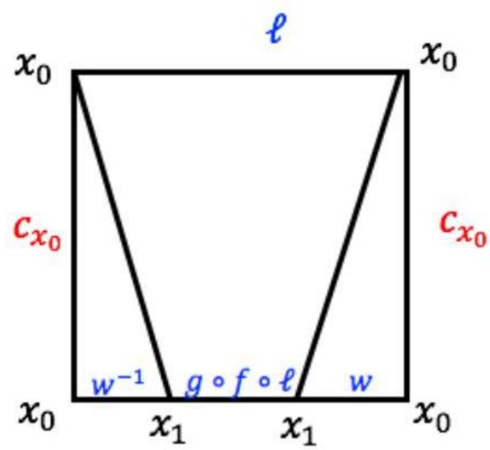
given by

$$\begin{aligned}
K(t,s) &= H((\ell(t),s))\\
K(t,0) &= H(\ell(t),0) = g\circ f(\ell(t))\\
K(t,1) &= H(\ell(t),1) = \ell(t)\\
K(0,s) &= w(s) = K(1,s)
\end{aligned}$$

The graphic plot of K is given in the figure below:



The homotopy K between ℓ and $g \circ f \circ \ell$ is not relative to $\{0, 1\}$. But we can modify it to get a homotopy between ℓ and $w^{-1} \circ g \circ f \circ \ell \circ w$ relative to $\{0, 1\}$:



Therefore,

$$[\ell] = \left[w^{-1} g f \ell w\right] = w_{\#}\left([g f \ell]\right) = \left(w_{\#} \circ g_{*} \circ f_{*}\right)[\ell]$$

which follows that $w_{\#} \circ g_{*} \circ f_{*} = \mathrm{id}_{\pi_1(X, x_0)}$. By Proposition 7.7, $w_{\#} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a group isomorphism. Therefore, f_* is injective, g_* is surjective.

Similarly, one can modify the above argument and get

$$w_{\#} \circ f_{*} \circ g_{*} = \mathrm{id}_{\pi_1(Y, y_0)}$$

Therefore, f_* is surjective, g_* is injective, and the result follows. □

Definition 7.10. [Simply-Connected] A space X is *simply-connected* if X is path connected, and X has trivial fundamental group, i.e., $\pi_1(X) = \{e\}$ for some point $e \in X$.

Example 7.11. If X is contractible, then $X \simeq \{x\}$ is simply-connected. Indeed, Proposition 7.9,

$$\pi_1(X) \cong \pi_1(\{x\}) = \{e\}.$$

Therefore, all contractible spaces (e.g. $\mathbb{R}^n$) are simply-connected.

However, not all simply-connected spaces are contractible, e.g., $\pi_1\left(S^2\right) \cong \{e\}$, but S^2 is not homotopy equivalent to a point.

Chapter 8

Calculations of $\pi_1(X)$

In this chapter, we will calculate the fundamental group of various topological spaces.

More explicitly, we begin with a simplicial complex K and its topological realization $X = |K|$. For $b \in X$, we will study $\pi_1(X, b)$ by the combinatorial structure of K.

Definition 8.1 (Edge Loop)**.** Let $K = (V, \Sigma)$ be a simplicial complex.

1. An edge path $(v_0, \dots, v_n)$ is such that
   (a) $a_i \in V(K)$
   (b) For each i, $\{a_{i-1}, a_i\}$ spans a simplex of K
2. An edge loop is an edge path with $a_n = a_0$.
3. Let $\alpha = (v_0, \dots, v_n), \beta = (w_0, \dots, w_m)$ be two edge paths with $v_n = w_0$, then we define

$$\alpha \circ \beta = (v_0, \dots, v_n, w_1, \dots, w_m)$$

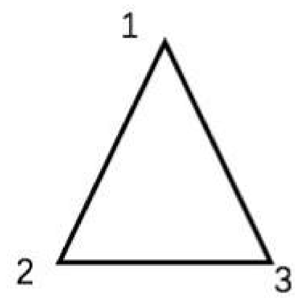
Definition 8.2 (Elementary Contraction/Expansion)**.** Let α, β be two edge paths.

1. An elementary contraction of α is a new edge path obtained by performing one of the following on α:
   - Replacing $\cdots a_{i-1} a_i \cdots$ by $\cdots a_{i-1} \cdots$ provided that $a_{i-1} = a_i$
   - Replacing $\cdots a_{i-1} a_i a_{i+1} \cdots$ by $\cdots a_{i-1} \cdots$ provided that $a_{i-1} = a_{i+1}$
   - Replacing $\cdots a_{i-1} a_i a_{i+1} \cdots$ by $\cdots a_{i-1} a_{i+1} \cdots$ provided that $\{a_{i-1}, a_i, a_{i+1}\}$ spans a 2-simplex of K.
2. An elementary expansion is the reverse of the elementary contraction.
3. Two edge paths α, β are equivalent if α and β differs by a finite sequence of elementary contractions or expansions.

4. The equivalence class of edge loops is given by:

$$[\alpha] = \{\alpha' \mid \alpha' \sim \alpha, \alpha' \text{ is the edge loop based at } b\}$$

Note that $\alpha' \sim \alpha$ if they differ from finitely many elementary contractions or expansions. For instance, let K in the figure below denote a triangle:



Then the canonical form of any equivalence form $[\alpha]$ can be expressed as:

$$[\alpha] = [bcabc \cdots ab] \, ,$$

where $a, b, c \in \{1, 2, 3\}$ are distinct.

Proposition 8.3. *Let*

$$E\,(K, b) := \{[\alpha] \mid \alpha \textit{ is edge loop based at } b\}$$

Then $E\,(K, b)$ is a group with multiplication

$$[\alpha] * [\beta] = [\alpha \cdot \beta]$$

We call $E(K, b)$ the edge loop group based at *b.*

Proof. 1. Well-definedness: Note that by definition,

$$\alpha \sim \alpha', \beta \sim \beta' \Rightarrow \alpha \cdot \beta \sim \alpha' \cdot \beta'$$

2. Associativity is clear.
3. The identity is $e = [b]$: for any edge loop $[\alpha] = [bv_1 \cdots b]$,

$$[\alpha] * e = [bv_1 \cdots v_n b] * [b] = [bv_1 \cdots v_n bb] = [bv_1 \cdots v_n b] = [\alpha] \, .$$

Also, $e * [\alpha] = [\alpha]$.

4. The inverse of any edge loop $[bv_1 \cdots v_n b]$ is $[bv_n \cdots v_1 b]$:

$$\begin{aligned}
[bv_1 \cdots v_n b]^{-1} * [bv_1 \cdots v_n b] &= [bv_n \cdots v_1 bbv_1 \cdots v_n b] \\
&= [bv_n \cdots v_1 bv_1 \cdots v_n b] \\
&= [bv_n \cdots v_2 v_1 v_2 \cdots v_n b] \\
&= \cdots \\
&= [b]
\end{aligned}$$

Similarly, $[bv_1 \cdots v_n b] * [bv_1 \cdots v_n b]^{-1} = [b]$. □

Indeed, the edge loop group of a simplicial complex gives us a combinatorial way of studying the fundamental group of its topological realization:

Theorem 8.4. $E(K, b) \cong \pi_1(|K|, b)$.

This is the most difficult theorem that we have faced so far. To do so, we recall (a slight generalization of) the simplicial approximation theorem (Theorem 5.25): Suppose that $f\colon |K| \to |L|$ be such that for all $v \in V(K)$, there exists $g(v) \in V(L)$ satisfying

$$f(\mathrm{st}_K(v)) \subseteq \mathrm{st}_L(g(v)).$$

Then there is a simplicial map $g: K \to L$ with $v \mapsto g(v)$ and $|g| \simeq f$.

A generalization of the simplicial approximation theorem needed for the proof of Theorem 8.4 is as follows: if $A \subseteq K$ and $B \subseteq L$ are simplicial subcomplexes, and

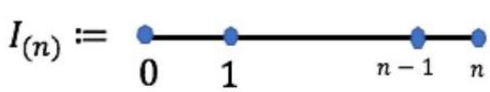
$$f(|A|) \subseteq |B|,$$

then the above g can be chosen such that

$$g|_A : A \to B$$

and the homotopy $|g| \simeq f$ sends $|A|$ to $|B|$.

Proof. For each edge loop $\alpha = (v_0, \ldots, v_n)$ based at b, consider the simplicial complex



together with the simplicial map

$$g_\alpha : I_{(n)} \to K \quad \text{with } g_\alpha(i) = v_i.$$

Note that it is well-defined, since $\{i, i+1\} \in \Sigma_{I_{(n)}}$, and $\{v_i, v_{i+1}\} \in \Sigma_K$.

Now construct the mapping

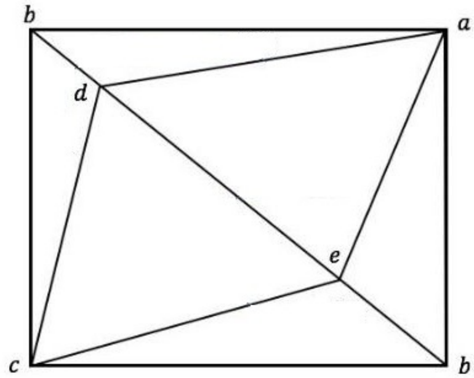
$$\theta : \{\alpha \mid \alpha \text{ edge loop based at } b\} \to \pi_1(K, b)$$

by

$$\alpha \mapsto [|g_\alpha|],$$

where $|g_\alpha| : \left|I_{(n)}\right| (\cong [0,1]) \to |K|$ satisfies $|g_\alpha|(i/n) = v_i$.

For example, let K be given by



and $\alpha = (\text{ bdeabcb })$, then the simplicial map $g_\alpha : I_{(6)} \to K$ has realization given by:

$$|g_\alpha|(0) = b, |g_\alpha|(1/6) = d, |g_\alpha|(2/6) = e, \cdots, |g_\alpha|(1) = b,$$

So $|g_\alpha|$ is a loop based at b, and $[|g_\alpha|] \in \pi_1(|K|, b)$.

Now, suppose $\alpha \sim \alpha'$ be two edge loops differ by an elementary contraction, e.g.,

$$\alpha' = (\text{ bdebcb }) \sim \alpha = (\text{ bdeabcb }).$$

Then one can easily find a homotopy $|g_{\alpha'}| \simeq |g_\alpha|$ relative to $\{0,1\}$. As a result, $[|g_\alpha|] = [|g_{\alpha'}|]$, and we have a well-defined map:

$$\widetilde{\theta} : \{\text{ edge loops based at } b\}/\sim := E(K, b) \to \pi_1(|K|, b)$$

with $[\alpha] \mapsto [|g_\alpha|]$.

Now we show $\widetilde{\theta}$ **is a homomorphism**, i.e.

$$\widetilde{\theta}([\alpha] * [\beta]) = \widetilde{\theta}([\alpha])\,\widetilde{\theta}([\beta]),$$

which suffices to show $[|g_{\alpha\cdot\beta}|] = [|g_\alpha|\,|g_\beta|]$, i.e., $|g_{\alpha\cdot\beta}| \simeq |g_\alpha|\,|g_\beta|$. Note that $|g_{\alpha\cdot\beta}|$ and $|g_\alpha|\,|g_\beta|$ are the same path with different "speeds", therefore, one can easily construct a homotopy between the maps.

Next, we show $\widetilde{\theta}$ **is surjective**: Let $\ell : [0,1] \to |K|$ be a loop based at b. It suffices to find an edge loop α such that $[|g_\alpha|] = [\ell]$, i.e., $|g_\alpha| \simeq \ell$.

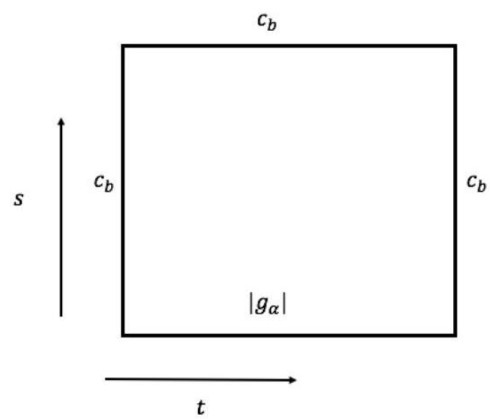
To do so, apply the simplicial approximation theorem such that there exist a large n and $g : I_{(n)} \to K$ such that $|g| \simeq \ell$. Here we can choose $g : I_{(n)} \to K$ to be such that $g(0) = b = g(n)$, and $|g| \simeq \ell$ relative to $\{0,1\}$. Then one can take

$$\alpha = (g(0), g(1), \ldots, g(n)),$$

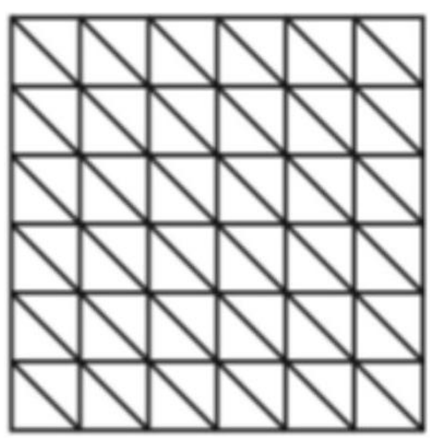
so that $g(0) = b = g(n)$, with $g_\alpha = g$. Therefore, $[|g_\alpha|] = [\ell]$, and hence $\widetilde{\theta}$ is surjective.

Now we show that $\overline{\theta}$ **is injective**: Let $\alpha = (v_0, \ldots, v_n)$ be an edge loop based at b such that $\theta([\alpha]) = e$, i.e., $|g_\alpha| \simeq c_b$. It suffices to show that $[\alpha]$ is the identity element of $E(K, b)$.

Choose a homotopy $H : I \times I \to |K|$ between $|g_\alpha| \simeq c_b$:



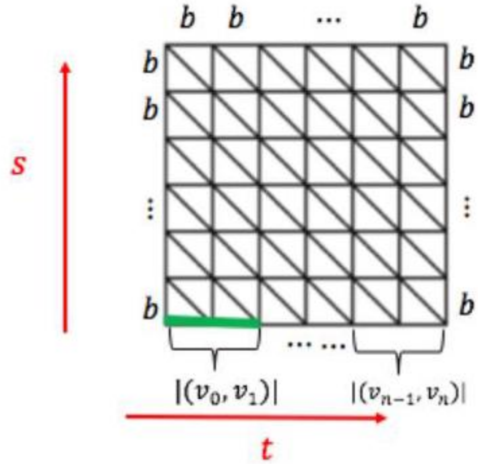
Apply the simplicial approximation theorem on H, so that there exists a subdivision $(I \times I)_{(r)}$ of $I \times I$ with $r \times r$ squares:



such that $\left|(I \times I)_{(r)}\right| \cong I \times I$ are isomorphic, and there exists a simplicial map

$$G : (I \times I)_{(r)} \to K \quad \text{such that} \quad |G| \simeq H.$$

Without loss of generality, assume that r is a sufficiently large multiple of n. Then the graphic illustration of $|G|$ is:



In particular, $|G|$ maps $\{0,1\} \times I$ into $\{b\}; I \times \{1\}$ into $\{b\}; (i/n, 0)$ into $\{v_i\}, i = 0, \ldots, n$, and $[i/n, (i+1)/n]$ into $|(v_i, v_{i+1})|, i = 0, \ldots, n-1$.

Now look at the green line (with $\eta := r/n$ vertices) at the bottom of $|G|$: it is the path from $v_0 = b$ to v_1, i.e.

$$G(0,0) = v_0 = b, \quad \ldots, \quad G(0, \gamma/r) = v_1$$

and hence G defines an edge path

$$\Big(G(0,0),\ G(0, \frac{1}{n}), \ldots, G(0, \frac{\gamma - 1}{n}),\ G(0, \frac{\gamma}{n})\Big) = (b = v_0, v_0, \ldots, v_0, v_1, \ldots, v_1).$$

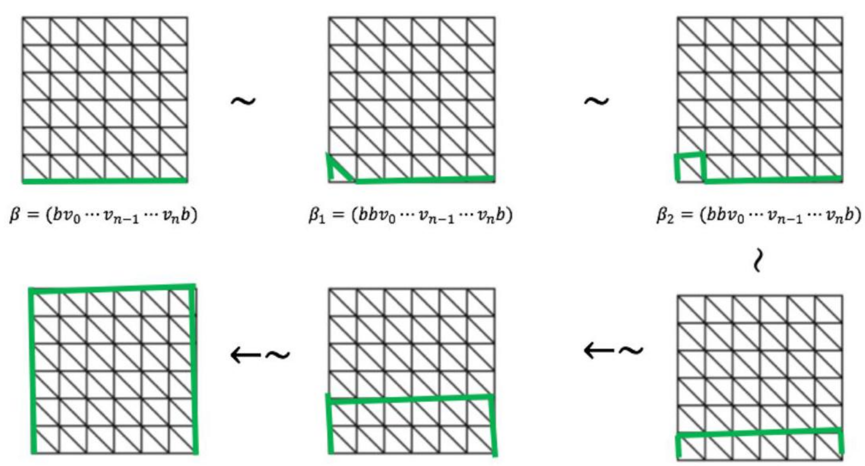
As a result, the edge loop corresponding to the bottom edge of the square reads

$$\Big(G(0,0), G(0, \frac{1}{n}), \ldots, G(0, \frac{n-1}{n}), G(0,1)\Big) = (bv_0 \cdots v_0 v_1 \cdots v_1 \cdots v_n \cdots v_n b)$$

and clearly

$$\beta \sim (bv_0 v_1 v_2 \cdots v_{n-1} v_n b) \sim (bv_1 v_2 \cdots v_{n-1} b) = \alpha,$$

i.e. $[\beta] = [\alpha] \in E(K, b)$. So it suffices to show $\beta \sim e$ as edge loops, which can be seen by the sequence of elementary contractions and expansions:



By tracing the above elementary contractions, one has $\beta \sim (b \cdots b) = (b)$ on the bottom left hand picture, and hence $[\beta] = e$. $\square$

Note that the definition of $E(K, b)$ only involves n -simplicials for $n \leq 2$, so one has:

Proposition 8.5. *For any simplicial complex K, consider the simplicial subcomplex* $\mathrm{Skel}^n(K) = (V_k, \Sigma_K^n)$*, where Σ_K^n consists of $\sigma \in \Sigma_K$ with $|\sigma| \leq n+1$ (this is the n-skeleton of K). Then*

$$\pi_1(|K|, b) \cong \pi_1\left(\left|\mathrm{Skel}^2(K)\right|, b\right)$$

Proof. Since $E(K, b)$ only involves n -simplicials for $n \leq 2$, we imply $E(K, b) \cong E\left(\mathrm{Skel}^2(K), b\right)$.

Moreover, $\pi_1(|K|, b) \cong E(K, b)$ and $\pi_1\left(\left|\mathrm{Skel}^2(K)\right|, b\right) \cong E\left(\mathrm{Skel}^2(K), b\right)$. So the proof is complete. $\square$

Corollary 8.6. *For $n \geq 2, \pi_1(S^n)$ is simply connected.*

Proof. Consider the simplicial complex K with

$$V = \{1, 2, \ldots, n+2\},\ \Sigma = \{\text{ all proper subsets of } V\}$$

It is clear that $|K| \cong \Delta^n \cong S^n$, and $\mathrm{Skel}^2(K)$ has

- $V : \{1, \ldots, n+2\}$
- Σ^2: all subsets of V with less or equal to 3 elements.

For any edge loop a in $\pi_1\left(\left|\mathrm{skel}^2(K)\right|\right)$, we have

$$a = (bv_0v_1v_2 \cdots v_n) \sim (bv_1v_2 \cdots v_{n-2}v_{n-1}b) \sim \cdots \sim (b)$$

Therefore, all edge loops α in $\pi_1\left(\left|\mathrm{skel}^2(K)\right|\right)$ satisfies $[\alpha] = [(b)] = e$., i.e.,

$$\pi_1\left(\left|\mathrm{skel}^2(K)\right|\right) \cong \{e\}$$

which implies $\pi_1(|K|) \cong \pi_1\left(\left|\mathrm{skel}^2(K)\right|\right) \cong \{e\}$. Since $|K| \cong S^n$, we have

$$\pi_1(S^n) \cong \pi_1(|K|) \cong \{e\}.$$

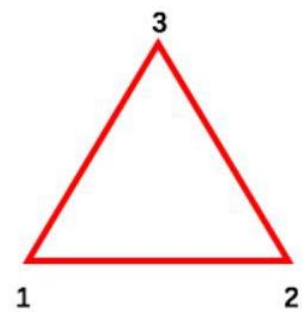
$\square$

8.1 Fundamental Group of S^1

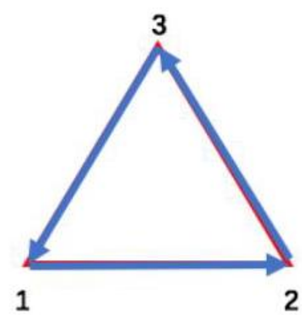
Corollary 8.6 does not hold for S^1, since the constructed Σ^2 for S^1 does not contain $\{1, 2, 3\}$. Instead, we have

Theorem 8.7. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Consider the simplicial complex K below with $|K| \cong S^1$:



It suffices to show $E(K,1) \cong \mathbb{Z}$. Define the orientation of $|K|$ as shown below.



We construct the isomorphism between $E(K,b)$ and $\mathbb{Z}$ directly:

$$\phi: \ E(K,b) \to \mathbb{Z} \text{ with } [\alpha] \mapsto \text{winding number of } \alpha$$

where the winding number of α is

$$\text{number of times it traverses } (2,3) - \text{number of times it traverses } (3,2).$$

Note that:

1. The winding number for $(123\cdots1231) = m$, where 23 shows up for m times
2. The winding number for $(132\cdots1321) = -n$, where 32 shows up for n times
3. The winding number is invariant under elementary contractions and elementary expansions, since (for instance) $(1231321) \sim (123321) \sim (1221) \sim (1)$, that is, the 32 and 23 'cancel out' each other.

Therefore, any edge loop α based at 1 is equivalent uniquely to one of the canonical form:

$$\alpha \sim (1bc1bc\cdots1bc1), \quad \text{where } bc = 32 \text{ or } 23.$$

and $\phi: E(K,1) \to \mathbb{Z}$ is well-defined.

To see ϕ is a homomorphism, consider any two edge loops α, β based at 1. suppose that $[\alpha] = [(1bc1bc \cdots 1bc1)]$ and $[\beta] = [(1pq1pq \cdots 1pq1)]$ are at their canonical forms, then

$$\phi\left([\alpha] \cdot [\beta]\right) = \phi\left([\alpha \cdot \beta]\right) = [(1bc1bc \cdots 1bc11pq1pq \cdots 1pq1)]$$

Then one can easily check that $\phi\left([\alpha] \cdot [\beta]\right) = \phi\left([\alpha]\right) + \phi\left([\beta]\right)$. Moreover, it is easy to see that ϕ is bijective. Therefore, ϕ is an isomorphism. □

Actually, we can show that the loop based at 1 given by:

$$\ell\ I \to S^1 \text{ with } t \mapsto e^{2\pi i t}$$

is a generator for $\pi_1\left(S^1, 1\right)$.

Corollary 8.8 (Fundamental Theorem of Algebra)**.** *All non-constant polynomials in $\mathbb{C}$ has at least one root in $\mathbb{C}$.*

Proof. Suppose on the contrary that

$$p\left(x\right) = a_n x^n + \cdots + a_1 x + a_0 \neq 0$$

has no complex roots, then p is a continuous mapping from $\mathbb{C}$ to $\mathbb{C} \smallsetminus \{0\}$.

It is clear that $\mathbb{C} \smallsetminus \{0\} \simeq \{z \in \mathbb{C} \mid |z| = 1\}$, and therefore $\pi_1\left(\mathbb{C} \smallsetminus \{0\}\right) = \pi_1\left(S^1\right) \cong \mathbb{Z}.$, and the induced homomorphism p_* of p is given by:

$$p_* : \ \pi_1\left(\mathbb{C}\right) \cong \{e\} \to \pi_1\left(\mathbb{C} \smallsetminus \{0\}\right) \cong \mathbb{Z}$$

Since p_* is a group homomorphism, $p_*\left(e\right) = 0$.

Consider the inclusion $i : \ C_r \hookrightarrow \mathbb{C}$, where $C_r = \{z \in \mathbb{C} \,||\, z \,|= r\}$ is the circle of radius r, and consider the diagram given below: Or equivalently,

$$\begin{tikzcd}
C_r \arrow[r, "i"] \arrow[dr, "p|_{C_r}"'] & \mathbb{C} \arrow[d, "p"] \\
& \mathbb{C} \setminus \{0\}
\end{tikzcd}$$

As a result, the induced homomorphism i_* of i satisfies the diagram

$$\begin{tikzcd}
\pi_1(C_r) \cong \mathbb{Z} \arrow[r, "i_*"] \arrow[dr, "(p|_{C_r})_*"'] & \pi_1(\mathbb{C}) \cong \{e\} \arrow[d, "p_*"] \\
& \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}
\end{tikzcd}$$

Therefore, $p_* \circ i_*$ is a zero map since $p_*(e) = 0$, i.e., $\left(p|_{C_r}\right)_*$ is a zero homomorphism.

Now study $p|_{C_r} : C_r \to \mathbb{C} \setminus \{0\}$. Construct

$$q(z) = k \cdot z^n, \quad \text{where } k := \frac{p(r)}{r^n}$$

Then $p|_{C_r}, q|_{C_r} : C_r \to \mathbb{C} \setminus \{0\}$ with $p(r) = q(r)$.

We **claim** that $p|_{C_r} \simeq q|_{C_r}$ for large r: First construct the mapping

$$H: \ C_r \times [0,1] \to \mathbb{C} \ \text{ with } H(z,t) = tp(z) + (1-t)q(z)$$

and $H(z,0) = q(z), H(z,1) = p(z)$. If

$$H : C_r \times [0,1] \to \mathbb{C} \setminus \{0\}$$

i.e. $H(z,t) \neq 0$ for all $z \in C_r$ and $t \in [0,1]$, H defines a homotopy between $p|_{C_r}$ and $q|_{C_r}$.

Suppose on the contrary that there exists (z,t) such that

$$(1-t)p(z) + tq(z) = 0, \ |z| = r, t \in [0,1]$$

Or equivalently,

$$(1-t)(a_n z^n + \cdots + a_1 z + a_0) + t \cdot kz^n = 0.$$

Substituting k with $p(r)/r^n$ gives

$$a_n z^n + \cdots + a_1 z + a_0 = t\left(a_{n-1}z^{n-1} + \cdots + a_0 - a_{n-1}\frac{z^n}{r} - \cdots - a_1\frac{z^n}{r^{n-1}} - a_0\frac{z^n}{r^n}\right)$$

The LHS has leading order n, while the RHS has leading order less or equal to $n-1$. As $r = |z| \to \infty, t \to \infty$. Therefore, the equality does not hold in the range $t \in [0,1]$ when r is sufficiently large, and the claim is proved for r sufficiently large.

Therefore, $p|_{C_r} \simeq q|_{C_r}$ and $\left(p|_{C_r}\right)_* = \left(q|_{C_r}\right)_*$.

Consider the induced mapping $\left(q|_{C_r}\right)_* : \mathbb{Z} \to \mathbb{Z}$. In particular, we check the value of $\left(q|_{C_r}\right)_*(1)$, where 1 is the generator in $\pi_1(C_r)$.

Recall that the loop $\ell : I \to C_r$ with $\ell(t) = re^{2\pi i t}$ is a generator of $\pi_1(C_r) \cong \mathbb{Z}$, i.e. $[\ell] = 1$. It follows that

$$\left(q|_{C_r}\right)_*(1) = \left(q|_{C_r}\right)_*([\ell]) = \left[q|_{C_r}(\ell)\right] = q\left(re^{2\pi i t}\right) = k \cdot r^n \cdot e^{2\pi i n t} \neq 0.$$

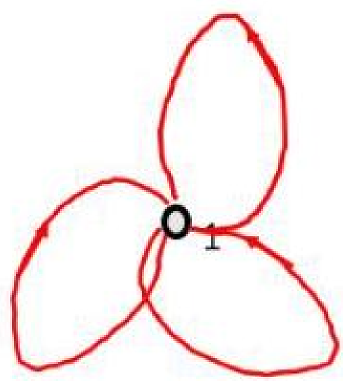
Therefore, $\left(q|_{C_r}\right)_* : \mathbb{Z} \cong \pi_1(C_r) \to \pi_1(\mathbb{C}\backslash\{0\}) \cong \mathbb{Z}$ is the nonzero map $1 \mapsto n$, contradicting the fact that $\left(q|_{C_r}\right)_* = \left(p|_{C_r}\right)_*$ is the zero homomorphism. □

8.2 Fundamental Group of a Graph

Definition 8.9 (Graph)**.** A graph $T = (V, E)$ is defined by the following components:

- V is a finite or countable set, called vertex set;
- E is a finite or countable set, called edge set;
- A function $\delta : E \to V \times V$ with $\delta(e) = (\ell(e), \tau(e))$, where $\ell(e), \tau(e)$ is known as the endpoints of e.

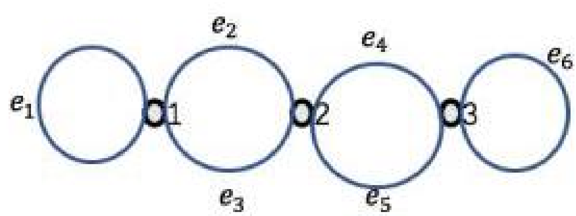
Example 8.10. 1. Let $V = \{1\}, E = \{e_1, e_2, e_3\}$, and define $\delta(e_i) = (1, 1), i = 1, 2, 3$. The graph (V, E) is represented below:



2. Let $V = \{1, 2, 3\}$ and $E = \{e_1, \ldots, e_6\}$, and define

$$\delta(e_1) = (1, 1),\ \delta(e_2) = (1, 2),\ \delta(e_3) = (1, 2), \delta(e_4) = (2, 3),\ \delta(e_5) = (2, 3),\ \delta(e_6) = (3, 3).$$

The graph (V, E) is represented below (We do not care the direction of edges for this graph):

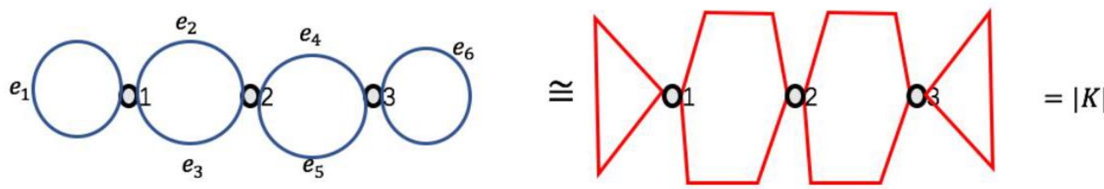


Definition 8.11 (Realization of a Graph)**.** For a given graph $\Gamma = (V, E)$, construct a realization by

$$\{|V| \times \{ \text{ zero simplices } \} \coprod |E| \times \{1 \text{ -simplices } \}/ \sim$$

where the equivalence class is induced from the function δ. We still call this realization of the graph as Γ.

In general, graphs are not simplicial complexes. But we can "sub-divide" each edge of Γ into three parts such that there exists simplicial complex K with $|K| \cong \Gamma$. For instance,



Definition 8.12. Let $\Gamma = (V, E)$ be a graph:

- A *subgraph* $\Gamma' \subseteq \Gamma$ is $\Gamma' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, and

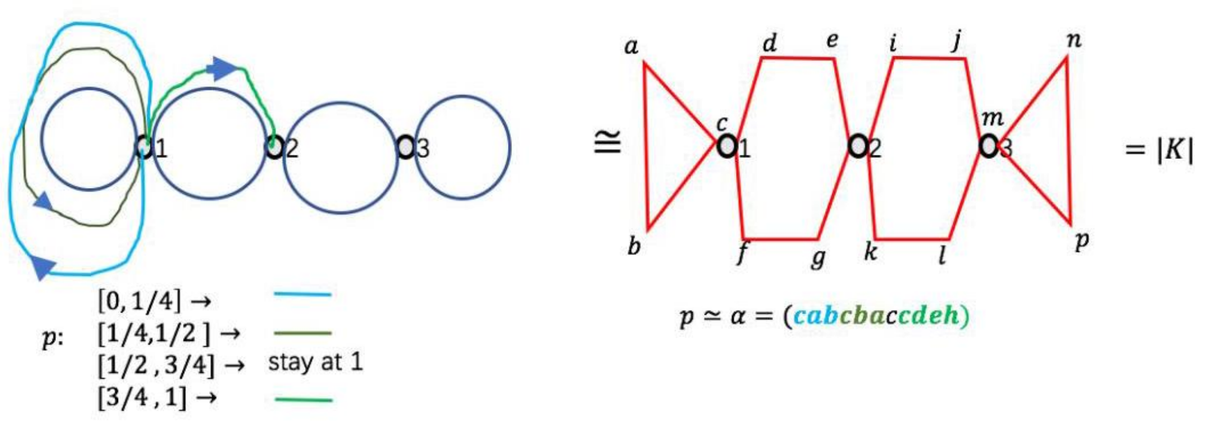
$$\delta|_{V'} : E' \to V' \times V'$$

- An *edge path* is a continuous function $p : [0, 1] \to \Gamma$ such that there exists $n \in \mathbb{N}$ satisfying

$$p|_{[i/n, i+1/n]} : \left[\frac{i}{n}, \frac{i+1}{n}\right] \to T$$

  is a path along an edge of Γ, or a constant function on a vertex of Γ, for $0 \leq i \leq n - 1$.

Under the homeomorphism $\Gamma \cong |K|$, each edge path is homotopic to $|g_\alpha|$ for some edge path α in the simplicial complex K. For instance:

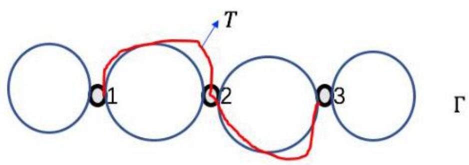


Here are more definitions:

Definition 8.13. Let $\Gamma = (V, E)$ be a graph:

- An edge loop is an edge path p such that $p(0) = p(1) = b \in V$.
- An embedded edge loop is an injective edge loop, i.e., $p : [0,1] \to \Gamma$ such that for $x \notin V$, $p^{-1}(x) = \varnothing$ or a single point.
- A tree is a connected graph T that contains no embedded edge loop $p : [0,1] \to T$. For instance, as shown in the figure, T_1 contains no edge loop, in particular, the edge loop (a, b, a) is not embedded; T_2 contains embedded edge loop (a, b, c, d, a).
- Maximal Tree of a connected graph Γ is a subgraph T of Γ such that
  - T is a tree; and
  - By adding an edge $e \in E(\Gamma) \setminus E(T)$ into T, the new graph is no longer a tree.

For instance, $T \subseteq \Gamma$ shown in the figure below is a maximal tree.



With all the definitions given above, we can finally compute the fundamental group of a (connected) graph:

Theorem 8.14. *Let Γ be a connected graph, and T is a subgraph of Γ such that T is a tree. Then T is a maximal tree if and only if $V(T) = V(\Gamma)$.*

Moreover, there always exists a maximal tree for all Γ.

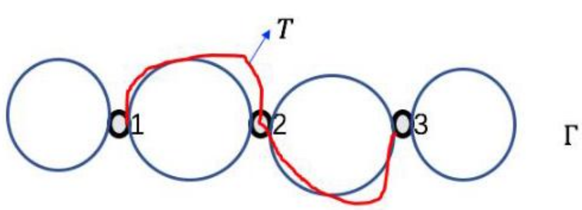
Proof. Outline for second part: Construct an ordering of $\{v_1, \ldots, v_i\} \subseteq V(\Gamma)$, such that for each integer $i \geq 2$, there is an edge connecting v_{i+1} with some vertex in $\{v_1, \ldots, v_i\}$. Then construct

$$T_1 \subseteq T_2 \subseteq \cdots,$$

where T_i is a tree containing vertices $\{v_1, \ldots, v_i\}$. As a result, $T = \cup_{i \in \mathbb{N}} T_i$ is a maximal tree. □

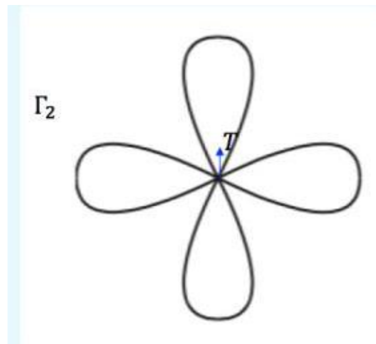
Theorem 8.15. *Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma) \setminus E(T)\}$ elements, for any maximal tree of Γ.*

For instance, a maximal tree of the the graph $T \subseteq \Gamma_1$ is:



Therefore, $\pi_1(\Gamma_1) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_1) \setminus E(T)\} = 4$.

On the other hand, a maximal tree of the graph with '4-petals' $T \subseteq \Gamma_2$ is just the middle vertexc:

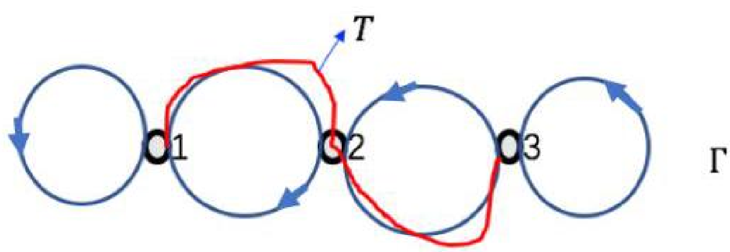


Therefore, $\pi_1(\Gamma_2) \cong \langle a, b, c, d \rangle$ since $\#\{E(\Gamma_2) \setminus E(T)\} = 4$.

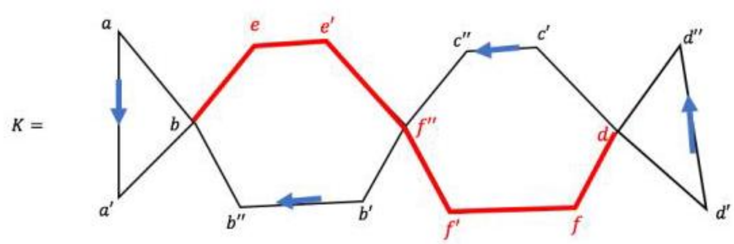
Indeed, one has $\Gamma_1 \simeq \Gamma_2$. The reason for such homotopy equivalence is in the link

https://www.math3ma.com/blog/clever-homotopy-equivalences

We will not prove the theorem. Instead, we present the idea of the proof in the special case of Γ with the following specified maximal tree T and orientation:



and the simplicial complex K with $|K| \cong \Gamma$:



Now we construct the group homomorphism

$$\phi : \langle \alpha, \beta, \gamma, \delta \rangle \to E(K, b) \cong \pi_1(\Gamma)$$

with

$$\begin{aligned}
\phi(\alpha) &= [ba'a''b] \\
\phi(\beta) &= [bee'f''b'b''b] \\
\phi(\gamma) &= [bee'f''f'fdc'c''f''e'eb] \\
\phi(\delta) &= [\text{ bee }'f''f'fdd''d'dff'f''e'eb]
\end{aligned}$$

We can show the group homomorphism ϕ is bijective. In particular, the inverse of ϕ is given by:

$$\Psi : \ E(K,b) \to \langle \alpha, \beta, \gamma, \delta \rangle$$

where for any $[\ell] := \ [bv_1 \cdots v_n] \in E(K,b)$, the mapping $\Psi[\ell]$ is constructed by

(a) Remove all other letters appearing in ℓ except $b, a', a'', b', b'', c', c'', d', d''$

(b) Assign

$$\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}, \delta, \delta^{-1}$$

for each appearance of

$$a'a'', a''a', b'b'', b''b', c'c'', c''c', d'd'', d''d'$$

respectively.

Chapter 9

The Seifert-Van Kampen Theorem

Here is the statement of Seifert-Van Kampen Theorem:

Theorem 9.1. *Let $K = K_1 \cup K_2$ be the union of two path-connected open sets, where $K_1 \cap K_2$ is also path-connected. Take $b \in K_1 \cap K_2$, and suppose the group presentations for $\pi_1(K_1, b), \pi_1(K_2, b)$ are*

$$\pi_1(K_1, b) \cong \langle X_1 \mid R_1 \rangle, \ \pi_1(K_2, b) \cong \langle X_2 \mid R_2 \rangle.$$

Let the inclusions be

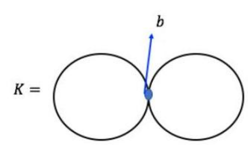
$$i_1 : K_1 \cap K_2 \hookrightarrow K_1, \ i_2 : K_1 \cap K_2 \hookrightarrow K_2,$$

then a presentation of $\pi_1(K, b)$ is given by:

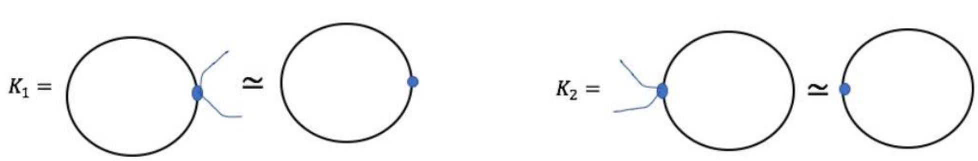
$$\pi_1(K, b) \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{(i_1)_*(g) = (i_2)_*(g) : \forall g \in \pi_1(K_1 \cap K_2, b)\}\rangle.$$

(Here $(i_1)_* : \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_1, b)$ and $(i_2)_* : \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_2, b)$.)

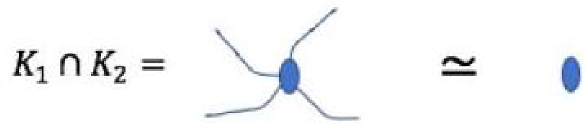
Example 9.2. Let $K = S^1 \vee S^1$ given by



Let b be the intersection between two circles, and construct K_1, K_2 as shown below:



We can see that $K_1 \cap K_2$ is contractible:



As we proved before, $\pi_1(S^1) \cong \mathbb{Z}$, which follows that

$$\pi_1(K_1, b) \cong \langle\alpha\rangle,\ \pi_1(K_2, b) \cong \langle\beta\rangle$$

Also, $\pi_1(K_1 \cap K_2, b) \cong \pi_1(\{b\}, b) \cong \{e\}$.

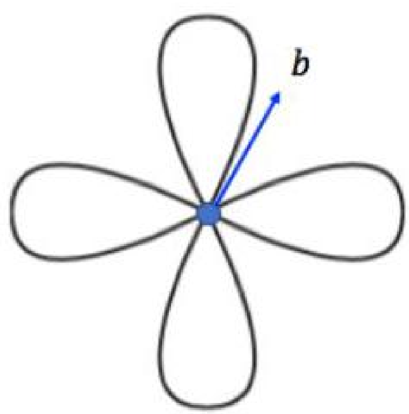
Also, it is easy to compute $(i_1)_*$ and $(i_2)_*$: For instance, $(i_2)_* : \ \pi_1(K_1 \cap K_2) \to \pi_1(K_2)$ is given by $e \mapsto e$. Therefore, by Seifert-Van Kampen Theorem,

$$\pi_1(K, b) \cong \langle\alpha, \beta \mid e = e\rangle \cong \langle\alpha, \beta\rangle$$

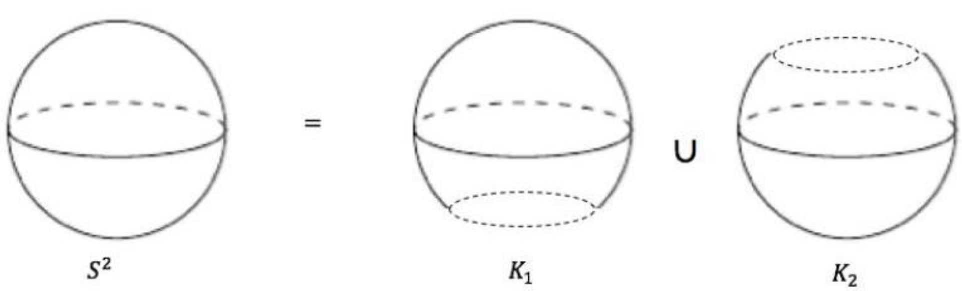
More generally, by induction

$$\pi_1\left(\vee^n S^1, b\right) \cong \langle a_1, \dots, a_n\rangle$$

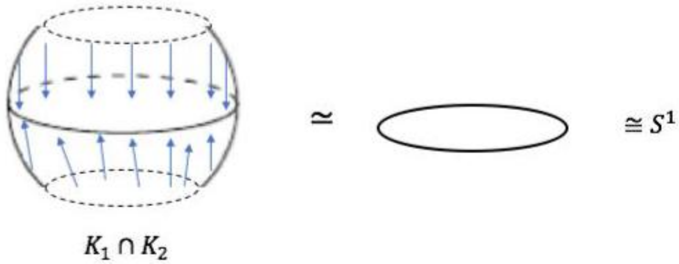
For instance, the figure illustration for $\vee^4 S^1$ and the basepoint b is given below:



Example 9.3. Consider$S^2 = K_1 \cup K_2$, which is shown below:



Therefore, we see that $K_1 \cap K_2 \simeq S^1$:



while K_1 and K_2 are contractible. Therefore

$$\pi_1(K_1) \cong \langle \beta \mid \beta \rangle, \ \pi_1(K_2) \cong \langle \gamma \mid \gamma \rangle$$

and $\pi_1(K_1 \cap K_2) \cong \pi_1(S^1) \cong \langle \alpha \rangle$.

Now we compute $(i_1)_*$ and $(i_2)_*$: For any loop γ in $K_1 \cap K_2$ based at b, Since K_1 is contractible, we imply γ in K_1 is homotopic to c_b, i.e.,

$$(i_1)_*([\gamma]) = [i_1(\gamma)] = e, \forall \gamma \in \pi_1(K_1 \cap K_2).$$

Similarly, $(i_2)_*([\gamma]) = e$.

Therefore, by Seifert-Van Kampen Theorem, we conclude that

$$\pi_1\left(S^2\right) \cong \langle \beta, \gamma \mid \beta, \gamma, e = e \rangle \cong \{e\}$$

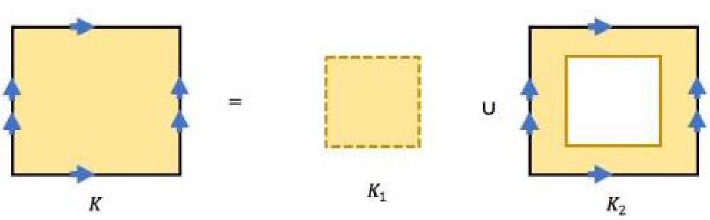
Homework: Use the same trick to check that $\pi_1(S^n) = \{e\}$ for all $n \geq 2$ (Hint: for S^3, construct

$$K_1 = \left\{(x_1, \ldots, x_4) \in S^3 \mid x_4 > -1/2\right\}$$

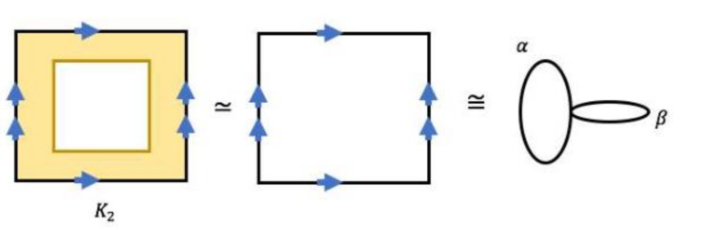
and

$$K_1 = \left\{(x_1, \ldots, x_4) \in S^3 \mid x_4 < 1/2\right\})$$

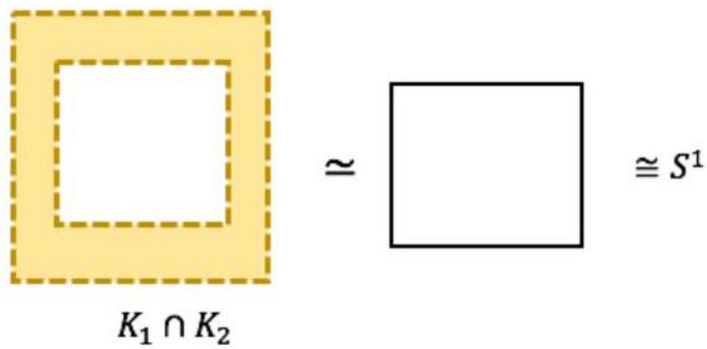
Example 9.4. Consider $K \cong \mathbb{T}^2$, and $K = K_1 \cup K_2$ given by:



Therefore, we can see that K_1 is contractible, and K_2 is homotopy equivalent to $S^1 \vee S^1$:



and $K_1 \cap K_2$ is homotopic equivalent to the circle:



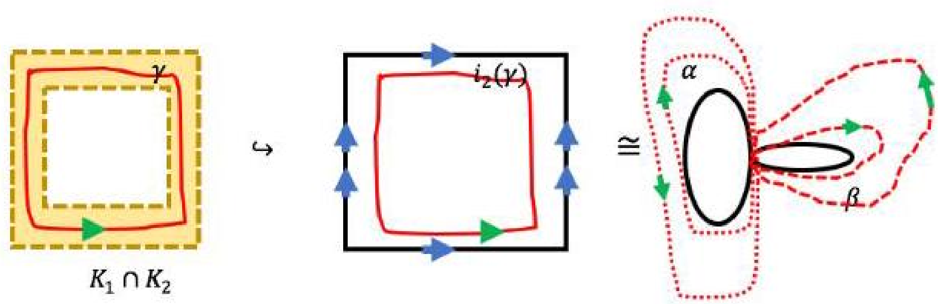
It follows that

$$\pi_1(K_1) \cong \{e\},\ \pi_1(K_2) \cong \langle \alpha, \beta \rangle,\ \pi_1(K_1 \cap K_2) \cong \langle \gamma \rangle.$$

Then we compute $(i_1)_*$ and $(i_2)_*$. Firstly, $(i_1)_*$ is trivial as in the previous example:

$$(i_1)_* : \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \quad \text{with } [\alpha] \mapsto e$$

As for $(i_2)_*$, let γ be any loop in $K_1 \cap K_2$. We draw the graph for $i_2(\gamma)$:



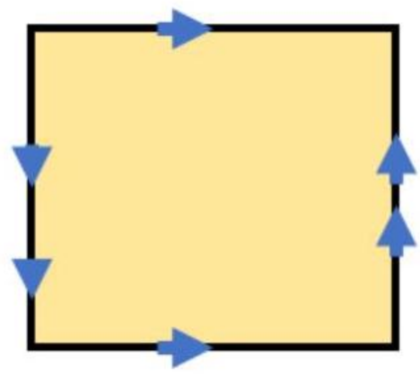
Therefore,

$$(i_2)_*[\gamma] = [i_2(\gamma)] = \left[\alpha\beta\alpha^{-1}\beta^{-1}\right]$$

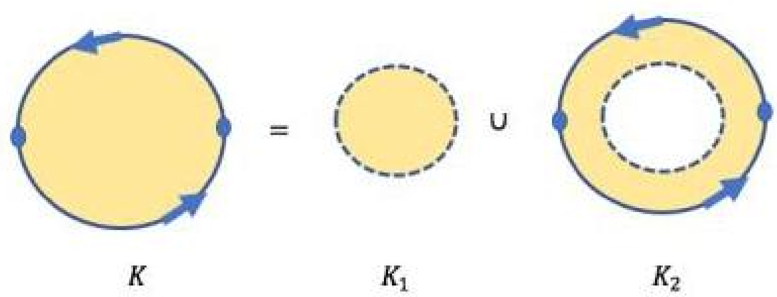
And by Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \left\langle \alpha, \beta \mid \beta, \alpha\beta\alpha^{-1}\beta^{-1} = e \right\rangle \cong \langle \alpha, \beta \mid, \alpha\beta = \beta\alpha \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

Exercise: The Klein bottle K shown in graph below satisfies $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.



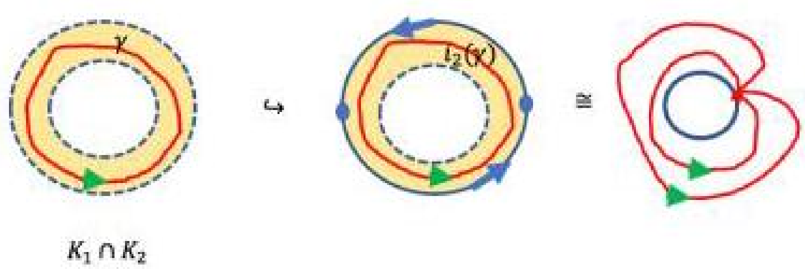
Example 9.5. Consider the space $K = \mathbb{R}P^2$ with $K = K_2 \cup K_2$ given by:



It is clear that K_1 is contractible. And in Homework 3, we can see that $K_2 \simeq S^1$. Moreover, $K_1 \cap K_2 \simeq S^1$ as before. Therefore,

$$\pi_1(K_1) = \{e\},\ \pi_1(K_2) = \langle\alpha\rangle,\ \pi_1(K_1 \cap K_2) = \langle\gamma\rangle.$$

It's easy to see that $(i_1)_*([\gamma]) = e$ for any loop γ. As for $(i_2)_*([\gamma])$:



Therefore, $(i_2)_*([\gamma]) = [i_2(\gamma)] = [\alpha^2]$.

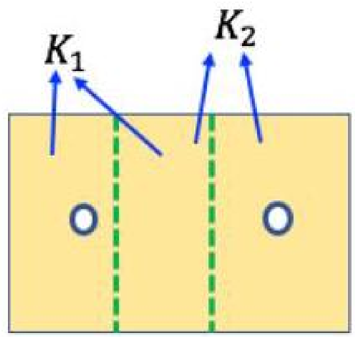
By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \left\langle \alpha \mid \alpha^2 = e \right\rangle \cong \mathbb{Z}/2\mathbb{Z}$$

Example 9.6. Let $K = \mathbb{R}^2 \setminus \{2 \text{ points } \alpha, \beta\}$. As we have shown in Homework 3, $K \simeq S^1 \vee S^1$, which implies

$$\pi_1(K) \cong \pi_1\left(S^1 \wedge S^1\right) \cong \langle \alpha, \beta \rangle.$$

We can compute the fundamental group for K using Seifert-Van Kampen Theorem directly: Construct $K = K_1 \cup K_2$ as follows:



It is clear that $K_1 \cong \mathbb{R}^2 \setminus \{\text{ one point }\} \simeq S^1$ and similarly $K_2 \simeq S^1$. Moreover, $K_1 \cap K_2$ is contractible. Therefore,

$$\pi_1(K_1) \cong \langle \alpha \rangle,\ \pi_1(K_2) \cong \langle \beta \rangle,\ \pi_1(K_1 \cap K_2) \cong \{e\}$$

Also, $(i_1)_*$ and $(i_2)_*$ is trivial, since $\pi_1(K_1 \cap K_2) \cong \{e\}$.

By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$