MAT 4002 Homework 4

1. In this question, we show that there are infinitely many primes using topology. The idea is given by Furstenburg in 1955 as an undergraduate student.

   Let $X = \mathbf{Z}$ be the set of all integers. Consider the collection $\mathcal{A}$ of all subsets of X formed by arithmetic progressions. More precisely, by writing $P(a, b) := \{na + b \mid n \in \mathbf{Z}\}$, $\mathcal{A}$ is defined as

   $$\mathcal{A} := \{P(a, b) \mid a = 1, 2, \ldots; b = 0, 1, 2, \ldots\}$$

   e.g. $P(5, 3) = \{\ldots, -7, -2, 3, 8, \ldots\} \in \mathcal{A}$.

   (a) Show that $\mathcal{A}$ forms a basis of a topology in X .

   (b) Let (X, τ) be the topology constructed in (a). Show that every $P(a, b) \in \mathcal{A}$ is closed in (X, τ) .

   (c) Write down explicitly all the elements appearing in the set

   $$X \setminus \bigcup_{p \text{ prime number}} P(p, 0) .$$

   (d) Using (b) and (c), show that there are infinitely many prime numbers.

2. Prove that if $f : X \to Y$ is a continuous map of a space X to a Hausdorff space Y , then its graph

   $$\Gamma_f := \{(x, f(x)) \in X \times Y \mid x \in X\}$$

   is a closed subset of $X \times Y$.

3. Show that the subset of $\mathbf{R}^n$ with at least one rational coordinate is connected.

4. Let X be a topological space. For $x, y \in X$ we define the relation $\sim$ by:

$$x \sim y \Leftrightarrow \text{ there exists a connected subset } C \subset X \text{ s.t. } x, y \in C$$

Show that $\sim$ is an equivalence relation. (we say $x, y \in X$ are in the same connected component if $x \sim y$)