MAT 4002 Homework 5

1. Last week we studied the equivalence relation $x \sim y$ iff $x, y \in C$ for a connected set $C \subset X$.

   (a) Let $C_a := \{y \in X \mid a \sim y\}$ be the equivalence class of $a \in X$. Show that C_a is the largest connected subset containing a . This is called a connected component of X (containing a).

   (b) Prove that connected components of X are either disjoint or they coincide.

   (c) Let A be a subset of $\mathbf{R}$ such that $A^0 = \phi$. Prove that the connected components of A are the singletons. What are the connected components of $\mathbf{Q}$ with the topology induced from $\mathbf{R}$?

2. Prove that if $A \subset X$ is connected and $A \subset B \subset \bar{A}$, then B is also connected.

3. Let I be an open interval in $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be a differentiable function.

   (a) Prove that the set $T = \{(x, y) \in I \times I : x < y\}$ is a connected subset of $\mathbf{R}^2$ with the standard topology.

   (b) Let $g : T \to \mathbf{R}$ be the function defined by $g(x, y) = \frac{f(x) - f(y)}{x - y}$. Prove that
   $$g(T) \subseteq f'(I) \subseteq \overline{g(T)}$$

   (c) Show that $f'(I)$ is an interval.

   Thus the derivative f' of any differentiable function $f : I \to \mathbf{R}$ always has the intermediate value property (without necessarily being continuous). This is Darboux's theorem.

4. Let X be a Hausdorff space and let A, B be disjoint compact subsets of X . Show that there exist disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$.