MAT4002 Spring 2020 Final Examination

Problem 1

Let $(X, \tau), (Y, \sigma)$ be topological spaces with (Y, σ) being Hausdorff.

1. Suppose $f, g : (X, \tau) \to (Y, \sigma)$ are continuous. Show that the set
$$A = \{x \in X \mid f(x) = g(x)\}$$
is closed in X.

2. Using (a), show that $\Delta\{(y, y) \in Y \times Y \mid y \in Y\}$ is closed in $Y \times Y$ under the product topology.

3. Let $X = [0, 1]$ be equipped with the usual topology, and $A = [0, 1] \cap \mathbb{Q}$ equipped with the subspace topology. Consider the function $s : A \to \mathbb{R}$ given by
$$s(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$
Using (a), show that there is no continuous function $f : X \to \mathbb{R}$ satisfying $f|_A = s$.

Problem 2

Let X be any topological space. Let the *cone* of X be $CX := (X \times [0, 1])/\sim$, where $(x_1, t_1) \sim (x_2, t_2)$ iff $(x_1, t_1) = (x_2, t_2)$ or $t_1 = 1 = t_2$.

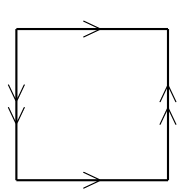
1. Show that CX is path-connected.

2. Write down the definition for two continuous maps $f, g : X \to Y$ to be homotopic.

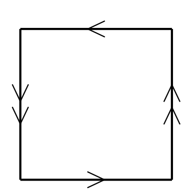
3. Show that a continuous map $f : X \to Y$ is homotopic to a constant map if and only if there exists $\tilde{f} : CX \to Y$ such that $\tilde{f}([(x, 0)]) = f(x)$ for all $x \in X$.

Problem 3

Let K be the Klein bottle and $\mathbb{RP}^2$ be the real projective plane given by the quotient space of $[0, 1] \times [0, 1]$ shown below:

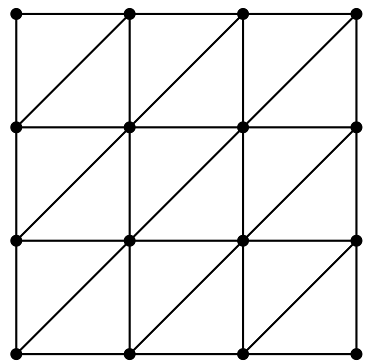


Klein Bottle K



Projective Plane $\mathbb{RP}^2$

1. It is known that the diagram



does NOT define a triangulation on either K or $\mathbb{RP}^2$. Which space, K or $\mathbb{RP}^2$, does not have the above diagram as its triangulation? Explain your reasoning.

2. For the space in (a) where the above diagram does not give a triangulation, find a triangulation of the space.

3. Using the triangulations you obtained above, what is the number $\chi :=$ (number of 0-simplices)$-$(number of 1-simplices) $+$ (number of 2-simplices) for K and $\mathbb{RP}^2$?

Problem 4

Let X be a topological space and $S \subset X$ be a subspace of X.

1. Write down the definition of S to be a homotopy retract of X.

2. Suppose S is a homotopy retract of X. Show that if every continuous function $f : X \to X$ has a fixed point, then every continuous function $g : S \to S$ also has a fixed point.

3. State the Brouwer's fixed point theorem for $f : D^2 \to D^2$.

4. Let A be a 3×3 matrix with positive entries. Using (c), show that A must have an eigenvector with positive eigenvalue. You can use the fact that D^2 is homeomorphic to the 2-simplex $\Delta^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x + y + z = 1\}$.
Hint: For all (x', y', z') with $x', y', z' \geq 0$ and $x' + y' + z' > 0$, the map $(x', y', z') \mapsto \left(\frac{x'}{x'+y'+z'}, \frac{y'}{x'+y'+z'}, \frac{z'}{x'+y'+z'}\right) \in \Delta^2$ is continuous.

Problem 5

Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\|_2 = 1\}$ be the n-sphere.

1. Write down the definition of a homotopy retract $r : X \to A$ for a subset $A \subset X$.

2. By drawing pictures, show that:

(i) The 2-sphere S^2 with the "equator" $\{(x_1, x_2, x_3) \in S^2 \mid x_1^2 + x_2^2 = 1\} \cong S^1$ removed can be retracted to the "north and south poles" $\{N = (0,0,1), S = (0,0,-1)\} \cong S^0$.

(ii) The 2-sphere S^2 with the poles removed can be retracted to the "equator".

3. By finding an inverse of the continuous map $g : S^2 \setminus \{N, S\} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ given by $g(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$, prove that $S^2 \setminus \{N, S\}$ is homotopy equivalent to S^1.

4. More generally, let $S^{m+n+1} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbb{R}^{m+1}, \mathbf{y} \in \mathbb{R}^{n+1}, \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = 1\}$. Consider S^m as a subspace by $\mathbf{y} = \mathbf{0}$. Show that $S^{m+n+1} \setminus S^m$ is homotopy equivalent to $S^n := \{(\mathbf{x}, \mathbf{y}) \in S^{m+n+1} \mid \mathbf{x} = \mathbf{0}\}$.

Problem 6

1. Show that the space $\mathbb{R}^3$ is contractible to a point.

2. Show that the space $Y := \mathbb{R}^3 \setminus \{(0,0,t) \mid t \in \mathbb{R}\}$ has a homotopy retract to the "cylinder" $C := \{(x, y, t) \mid x^2 + y^2 = 1, t \in \mathbb{R}\} \cong S^1 \times \mathbb{R}$. What is $\pi_1(Y)$?

3. State the Seifert-van Kampen Theorem.

4. Let $Z_m = \mathbb{R}^3 \setminus (\{(0,0,t) \mid t \in \mathbb{R}\} \cup \{(1,1,t) \mid t \in \mathbb{R}\} \cup \cdots \cup \{(m,m,t) \mid t \in \mathbb{R}\})$. Using (b) and (c), compute $\pi_1(Z_m)$.