MAT 4002 Homework 10

1. Show that for a space X , the following three conditions are equivalent:

   (a) Every map $f : S^1 \to X$ is homotopic to a constant map.

   (b) Every map $f : S^1 \to X$ can be extended to a map $\widetilde{f} : D^2 \to X$.

   (c) $\pi_1 (X, x_0) = \{e\}$ for all $x_0 \in X$.

   Deduce that a space X is simply-connected iff all maps $f : S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

2. Show that $\pi_1 (X \times Y, (x_0, y_0)) \cong \pi_1 (X, x_0) \times \pi_1 (Y, y_0)$. (Hint: use the projections $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$) .

3. Read the supplementary notes on blackboard proving the fundamental group of the circle S^1 is given by

   $$\pi_1 \left(S^1\right) \cong \mathbf{Z}$$

   (we will give another proof using simplicial complexes this week). Hence show that the fundamental group of the torus is equal to

   $$\pi_1 \left(S^1 \times S^1\right) \cong \mathbf{Z} \times \mathbf{Z}$$

4. By finding a simplicial complex K with $|K| \cong S^n$ and using the isomorphism

   $$E (K, b) \cong \pi_1 (|K| , b) ,$$

   show that S^n is simply connected for $n \geq 2$.