MAT 5210 Homework 2

1. Let $L := \mathbb{Q}(\sqrt[3]{2})$ and $K = \mathbb{Q}$. Show that $\mathrm{Aut}_K(L) = \{\mathrm{id}\}$. (Hence $L^{\mathrm{Aut}_K(L)} = L^{\mathrm{id}} = L \neq K$ and $L : K$ is not Galois).

2. Let $M : K$ be an algebraic extension (not necessarily Galois), and $G \leq \mathrm{Aut}_K(M)$. Show that $M : M^G$ is Galois. (Hint: Use the definition of Galois extension $M^{\mathrm{Aut}_{M^G}(M)} = M^G$).

3. Show that if f is a polynomial of degree n over K, then its splitting field has degree less than or equal to $n!$ over K.

4. Find the degrees of the splitting fields of the following polynomials:
   (a) $x^3 - 1$ over $\mathbb{Q}$;
   (b) $x^3 - 2$ over $\mathbb{Q}$.

5. Let $L = \mathbb{Q}\left(2^{1/3}, 3^{1/4}\right)$. Compute the degree of L over $\mathbb{Q}$.

6. Which of the following fields are normal extensions of $\mathbb{Q}$?
   (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;
   (b) $\mathbb{Q}\left(2^{1/4}\right)$;
   (c) $\mathbb{Q}(\alpha)$, where $\alpha^4 - 10\alpha^2 + 1 = 0$.

7. Find the Galois groups of the following polynomials and for each subgroup identify the corresponding subfield of the splitting field:
   (a) $x^2 + 1$ over $\mathbb{R}$;
   (b) $x^3 - 1$ over $\mathbb{Q}$.

8. Let G be a group (not necessarily finite), and let M be a field. A **character** of G is a group homomorphism $\chi : G \to M^*$.
   (a) Let $\chi_1, \cdots, \chi_k$ be distinct characters of M, and let $a_1, \cdots, a_k \in M$ such that
   $$a_1\chi_1(g) + \cdots + a_k\chi_k(g) = 0$$
   for all $g \in G$. Show that $a_1 = a_2 = \cdots = a_k = 0$. (Hint: Use induction on k. When $k = 2$, consider $a_1\chi_1(g) + a_2\chi_2(g) = 0 \Rightarrow \chi_1(h)(a_1\chi_1(g) + a_2\chi_2(g)) = 0$ for all $h, g \in G$)

(b) Let L be a field and let n be a positive integer with $(n, \mathrm{char}(L)) = 1$ and $x^n - 1$ splits in L with primitive root of unity $\omega \in L$.

Suppose that $M : L$ is a Galois extension, and $\mathrm{Gal}(M/L) \cong \langle \sigma \rangle$ is a cyclic group of order n. Show that there exists some $\alpha \in M$ such that

$$\beta := \alpha + \omega\sigma(\alpha) + \omega^2\sigma^2(\alpha) + \cdots + \omega^{n-1}\sigma^{n-1}(\alpha) \in M$$

is non-zero. (Hint: Take $G = M^*$ in (a))