MAT 5210 Homework 6

1. Let $K/\mathbf{Q}$ be Galois, with $\mathrm{Gal}(K/\mathbf{Q}) = \{\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n\}$. Fix a prime integer $p \in \mathbf{Z}$.

   (a) Let P be a nonzero prime ideal of $\mathcal{O}_K$ lying above p, i.e., $P \cap \mathbf{Z} = p\mathbf{Z}$. Show that $\sigma_i(P)$ is also a prime ideal of $\mathcal{O}_K$ lying above p.

   We now study an alternative proof that all prime ideals lying above p are of the form $\sigma_i(P)$ for some i - Suppose on the contrary that there exists a prime ideal J above p such that $J \neq \sigma_i(P)$ for all i. Consider

   $$A := \prod_{i=1}^{n} \sigma_i(P), \quad B := \prod_{i=1}^{n} \sigma_i(J)$$

   (b) Show that there exists $x \in A$ such that $x - 1 \in B$.

   (c) Let $m := \mathrm{Norm}_{K/\mathbf{Q}}(x) \in \mathbf{Z}$. Show that $m \in A$.

   (d) Show that $m \in p\mathbf{Z} = J \cap \mathbf{Z}$.

   (e) Using (d), show that $x \in \sigma_j(J)$ for some j.

   (f) Prove that for the j obtained in (e), $1 \in \sigma_j(J)$, which contradicts the fact that $\sigma_j(J)$ is a (proper) prime ideal.

2. Let $K : \mathbf{Q}$ be a number field with $[K : \mathbf{Q}] = n$.

   (a) Using the Minkowski bound, show that $\sqrt{|\Delta^2(K)|} \geq b_n$, where $b_n := \left(\frac{\pi}{4}\right)^n \frac{n^n}{n!}$.

   (b) Prove that for all $n > 1$,

   $$\frac{b_{n+1}}{b_n} \geq \frac{\pi}{2}$$

   (c) Consequently, show that for $K \neq \mathbf{Q}$, $|\Delta^2(K)| > 1$.

3. Suppose a prime integer $p \in \mathbf{Z}$ does not divide the class number of a number field K. Show that if I is a non-zero ideal of $\mathcal{O}_K$, and I^p is a principal ideal, then I is also a principal ideal.

4. Calculate the class number of $K = \mathbf{Q}(\sqrt{-23})$.

5. Prove that the class number of $K = \mathbf{Q}(\sqrt{-47})$ is 5. Show that if $x, y \in \mathbf{Z}$ with $y^3 = 4x^2 + 47$, then $x = \pm 250$.