MAT 5210 Homework 5

1. Let $K = \mathbb{Q}(\sqrt[3]{3})$. Show that $\omega = \{1, \sqrt[3]{3}, \sqrt[3]{9}\}$ is an integral basis of $\mathcal{O}_K$, i.e. $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{3}]$. (Hint: Use Question 2 and Question 7 of Homework 4).

2. Let $[I], [J]$ be ideal classes in C_K. Show that the multiplication
$$[I] * [J] := [IJ]$$
in C_K is well-defined.

3. Find the prime ideal factorizations of $(3), (5), (7)$ in $\mathbf{Z}[\sqrt{-5}]$. Show that the prime ideal factors of (7) are not principal.

4. Let $K \subseteq L$ be fields and let I be an ideal of $\mathcal{O}_K$. Define $I \cdot \mathcal{O}_L$ to be the ideal of $\mathcal{O}_L$ generated by the products $i\ell$ with $i \in I$ and $\ell \in \mathcal{O}_L$. Show that, for any ideals I, J of $\mathcal{O}_K$, any $n \in \mathbf{N}$ and any principal ideal $(a)_K := a\mathcal{O}_K$ of $\mathcal{O}_K$,

   (a) $(IJ) \cdot \mathcal{O}_L = (I \cdot \mathcal{O}_L)(J \cdot \mathcal{O}_L)$;
   (b) $I^n \cdot \mathcal{O}_L = (I \cdot \mathcal{O}_L)^n$; and
   (c) $(a)_K \cdot \mathcal{O}_L = (a)_L = a\mathcal{O}_L$, i.e., the principal ideal of $\mathcal{O}_L$ generated by the same element.

   Let $K = \mathbf{Q}(\sqrt{-13})$ and let $I = (2, \sqrt{-13} + 1)$.

   (d) Show that $I^2 = (2)$ and that I is not principal.

   Let $L = \mathbf{Q}(\sqrt{-13}, \sqrt{2})$.

   (e) Show that $I \cdot \mathcal{O}_L = (\sqrt{2})_L$ is the principal ideal of $\mathcal{O}_L$ generated by $\sqrt{2}$.

5. Let P, Q be distinct prime ideals in $\mathcal{O}_K$. Show that $P + Q = \mathcal{O}_K$ and $P \cap Q = PQ$.

6. Let d be a square-free integer not $\equiv 1(\mathrm{mod} 4)$, so that $K = \mathbf{Q}(\sqrt{d})$ and $\mathcal{O}_K = \mathbf{Z}[\sqrt{d}]$. Let $p \in \mathbf{N}$ be a prime integer such that $d \equiv a^2(\mathrm{mod} p)$ for some $a \in \mathbf{Z}$. Consider
$$P := (p, a + \sqrt{d}), \quad P' := (p, a - \sqrt{d})$$

   (a) Show that P, P' are both prime ideals with $N(P) = N(P') = p$.
   (b) Show that $(p) = PP'$.
   (c) Show that $P = P'$ if and only if $p \mid 2d$.