Surreal # ($\mathbb{S}$:= {all the surreal #s})

Part 1 Definitions.

Def 1: A surreal number x is a pair of previously created surreal numbers, known as the left set X_L and the ~~ortho~~ right set X_R with no member of the right set may be less than or equal to any member of X_L.

Def 2. $x, y \in \mathbb{S}$ $x \leq y$ iff the followings:

- $\neg \exists x_L \in X_L : y \leq x_L$ (*1)
- $\neg \exists y_R \in Y_R : y_R \leq x$ (* 2)

Def (zero). We define zero as: $0 :\equiv \{\,|\,\} := \{\emptyset | \emptyset\}$

Thm: $0 \leq 0$

Proof: Since 0_L and 0_R are empty, then the conditions in def 2 holds. $\Rightarrow 0 \leq 0$ ☑.

Def $-1 :\equiv \{\,|0\}$, $1 :\equiv \{0|\,\}$.

Rmk: Check the well-definedness of -1 and 1. Since $(-1)_L$ and $(1)_R$ and empty, then the condition for well-definedness in def 1 holds naturally.

Thm $0 \leq 1$ and $-1 \leq 0$

Proof: For "$0 \leq 1$", 0_L & 1_R are empty. Then "$0 \leq 1$" holds. For "$-1 \leq 0$", $(-1)_L$ and 0_R are empty. Then "$-1 \leq 0$" holds. Hence, we're done. ☑

Notational Conventions

$x \not\leq y$ means $\neg(x \leq y)$

$x < y$ means $x \leq y$ and $y \not\leq x$

$x \not< y$ means $\neg(x < y) \Leftrightarrow x \not\leq y \vee y \leq x$

$x = y$ means $x \leq y$ and $y \leq x$

$x \neq y$ means ~~$\neg(x \leq y)$~~ $\neg(x = y)$

Theorem. $0=0$

Proof: $0 \leq 0$ & $0 \leq 0 \Rightarrow 0 = 0$

Thm: ① $1=1$, ② $-1=-1$, ③ $0<1$, ④ $-1<0$, ⑤ $-1<1$

Proof:

① W.T.S.: $1 \leq 1$. $\Leftrightarrow \underbrace{\neg \exists x \in 1_L : 1 \leq x}_{(i)} \wedge \underbrace{\neg \exists y \in 1_R : y \leq 1}_{(ii)}$

Since $1_R = \emptyset$, then (ii) holds. For (i), $1_L = \{0\}$, so (i) holds because $0 \leq 1$ by the previous thm.

② $\underbrace{\neg \exists x \in (-1)_L : -1 \leq x}_{(iii)} \wedge \underbrace{\neg \exists y \in (-1)_R : y \leq -1}_{(iv)}$

Since $(-1)_L = \emptyset$, then (iii) holds. For (iv), $(-1)_R = \{0\}$, so (iv) holds because $-1 \leq 0$ by the previous thm.

③ W.T.S.: $1 \nleq 0 \Leftrightarrow \neg(1 \leq 0) \Leftrightarrow \underline{\exists x \in 1_L : 0 \leq x} \vee \underline{\exists y \in 0_R : y \leq 1}$

Since $1_L = \{0\}$ and $0 \leq 0$, then "$\exists x \in 1_L : 0 \leq x$" holds.

④ W.T.S.: $0 \nleq -1 \Leftrightarrow \underline{\exists x \in 0_L : -1 \leq 0_L} \vee \underline{\exists y \in (-1)_R : y \leq 0}$

Since $(-1)_R = \{0\}$ and $0 \leq 0$, then "$\exists y \in (-1)_R : y \leq 0$" holds.

⑤ W.T.S.: $-1 \leq 1 \wedge 1 \nleq -1$

(a) $\neg \exists x \in (-1)_L = \emptyset \wedge \neg \exists y \in 1_R = \emptyset \Rightarrow \neg \exists x \in (-1)_L : x \leq 1 \wedge \neg \exists y \in 1_R : y \leq -1$

$\Rightarrow -1 \leq 1$

(b) $-1 \leq 0 \Rightarrow \exists x \in 1_L = \{0\}\ (x=0) : -1 \leq x \Rightarrow 1 \nleq -1$ ☐

Def $2 :\equiv \{1|\}$, $-2 :\equiv \{|-1\}$, $\frac{1}{2} :\equiv \{0|1\}$, $-\frac{1}{2} :\equiv \{-1|0\}$

Rmk: The well-definedness of $2, -2, \frac{1}{2}, -\frac{1}{2}$ can be easily check by the thms on the left side of this page.

Thm $0 = \{-1|1\}$

Proof:

① $\underline{0 \leq \{-1|1\}}$: $\neg \exists x \in 0_L = \emptyset$ ~~$\wedge \neg \exists y$~~ $\Rightarrow \neg \exists x \in 0_L : \{-1|1\} \leq x$ (#1)

$0 < 1 \Rightarrow 1 \nleq 0 \Rightarrow \neg \exists y \in \{1\} : y \leq 0$

$\Rightarrow \neg \exists y \in \{-1|1\}_R : y \leq 0$ — (#2)

Then by (#1) (#2), $\underline{0 \leq \{-1|1\}}$

② $\underline{\{-1|1\} \leq 0}$:

$\neg \exists y \in 0_R = \emptyset \Rightarrow \neg \exists y \in 0_R : y \leq \{-1|1\}$ (*1)

$-1 < 0 \Rightarrow 0 \nleq -1 \Rightarrow \neg \exists x \in \{-1\} : 0 \leq x$

$\Rightarrow \neg \exists x \in \{-1|1\}_L : 0 \leq x$ — (*2)

By (*1)(*2), $\underline{\{-1|1\} \leq 0}$

Then by ①②, we get that $0 = \{-1|1\}$. ☐

Def (Representative).

The different forms that represent the same surreal number are called representatives of that surreal number

Example: $\{|\}$ and $\{-1|1\}$ are two representatives of the surreal number 0

Def (Birthday)

- $0 :\equiv \{|\}$ is born on day 0
- $-1 :\equiv \{|0\}$, $1 :\equiv \{0|\}$ are born on day 1.
- $\vdots$

Part 2 Basic Properties of Surreal Numbers

Def (Set comparisons). Let A be a set of surreal numbers, c be a surreal number. Then we define:

$$A \le c \iff \forall a \in A : a \le c$$

$$c \le A \iff \forall a \in A : c \le a$$

$$A \le B \iff \forall a \in A, \forall b \in B : a \le b.$$

Similarly for $A < c$, $A \not\le c$.

Rmk: $\neg(A \le b)$ is <u>NOT</u> equivalent to $A \not\le b$

Example: "$\{3,5\} \le 4$" is <u>false</u> $\Rightarrow \neg(\{3,5\} \le 4)$ is <u>true</u>

However, $\{3,5\} \not\le 4$ is also <u>false</u>

$\Rightarrow (\{3,5\} \not\le 4)$ is <u>NOT</u> equivalent to $\neg(\{3,5\} \le 4)$

Def (Set Equality) $A, B \subseteq \mathbb{S}$.

$$A = B \iff (\forall a \in A\ \exists b \in B : a = b) \wedge (\forall b \in B\ \exists a \in A : b = a)$$

Rmk: A and B are <u>not</u> necessarily identical forms

Example: $\{\{|\}, \{1|\}\} = \{\{-1|1\}, \{|\}, \{-1,0,1|\}\}$,

since $\{|\} = \{-1|1\}$, $\{|\} = \{|\}$, $\{1|\} = \{-1,0,1|\}$.

Thm (Alternative Form of Comparison)

Let $x = \{X_L | X_R\}$, $y = \{Y_L | Y_R\}$ Then

$$x \le y \iff (Y_R \not\le x) \wedge (y \le X_L)$$

where

$$\begin{cases} Y_R \not\le x & \text{means } \neg\exists\, y_R \in Y_R : y_R \le x \\ y \not\le X_L & \text{means } \neg\exists\, x_L \in X_L : y \le x_L. \end{cases}$$

Rmk: It's just a notational rewriting of <u>Def 2</u>.

Def (Parents (or Options))

The members of the left set X_L and the right set X_R of a surreal number $x = \{X_L | X_R\}$ are called the parents (or options) of x.

Thm (Reflexivity of $\le$) If $x \in \mathbb{S}$, then $x \le x$.

Proof: By induction.

Basic Case: $0 \le 0$, which we've shown

Inductive Step: Suppose that for all parents of x, the thm holds, that is, $\forall x_L \in X_L$ & $\forall x_R \in X_R$ we have $x_L \le x_L$ & $x_R \le x_R$. W.T.S.: $x \le x$

$$\Leftrightarrow \underbrace{\neg \exists x_L \in X_L : x \le x_L}_{(i)} \wedge \underbrace{\neg \exists x_R \in X_R : x_R \le x}_{(ii)}$$

For (i), suppose not, i.e.: $\exists x_L \in X_L : x \le x_L$

$$\Rightarrow \neg \exists x_L' \in X_L : x_L \le x_L' \wedge \neg \exists x_{LR} \in X_{LR} : x_{LR} \le x$$

Since $x_L \le x_L$, then let $x_L' = x_L$. Then contradict!

Hence, (i) is true. Similarly, (ii) is true. $\square$

Coro (Reflexivity of $=$). If $x \in \mathbb{S}$, then $x = x$.

Proof: $x \le x \Rightarrow x = x$, by def of "$=$". $\square$

Thm Let A, A', B, B' be sets of surreal numbers, and let $a_1, a_2, \dots$ be the members of A, let $a_1', a_2', \dots$ be the members of A', let $b_1, b_2, \dots$ be the members of B, let $b_1', b_2', \dots$ be the members of B'. If $\forall a_i \in A$, $\exists a_j' \in A'$, s.t.: $a_i \le a_j'$ and $\forall b_k \in B$ $\exists b_p' \in B'$. s.t.: $b_k \le b_p'$, then $\{A|B\} \le \{A'|B'\}$

Proof: W.T.S.: $\underbrace{\neg \exists a \in A : \{A'|B'\} \le a}_{(i)} \wedge \underbrace{\neg \exists b' \in B' : b' \le \{A|B\}}_{(ii)}$

For (i), suppose that $\exists a \in A : \{A'|B'\} \le a$. Then $\neg \exists a' \in A' : a \le a'$, Contradict! $\Rightarrow$ (i) holds.

For (ii), suppose that $\exists b' \in B' : b' \le \{A|B\}$. Then $\neg \exists b \in B : b' \le b$, Contradict! $\Rightarrow$ (ii) holds. $\square$

Coro. If $A = A'$ and $B = B'$, then $\{A|B\} = \{A'|B'\}$

Proof: $A = A' \Rightarrow \forall a \in A$, $\exists a' \in A'$, s.t.: $a = a' \Rightarrow a \le a'$. Similarly for B and B' $\Rightarrow \{A|B\} \le \{A'|B'\}$ & $\{A'|B'\} \le \{A|B\}$ $\square$

Thm Let $x = \{A|B\}$ be a surreal number. Then $\forall a \in A : a < x$ and $\forall b \in B : x < b$, that is,

$A < x$ and $x < B$.

Proof: W.T.S.: ~~$A<$~~ (a) $\forall a \in A : a < x \Leftrightarrow \overset{(a1)}{a \le x}\ \&\ \overset{(a2)}{a \ne x}$

(b) $\forall b \in B : x < b \Leftrightarrow \overset{(b1)}{x \le b}\ \&\ \overset{(b2)}{b \ne x}$

(a1): $a \le x \Leftrightarrow \neg \exists a_L \in A_L : x \le a_L \wedge \underbrace{\neg \exists b \in B : b \le a}_{\text{holds by def 1}}$

Suppose $\exists a_L \in A_L : x \le a_L$. Then $\neg \exists a \in A : a_L \le a$, which contradicts to the inductive step. $\Rightarrow$ (a1) holds.

(a2): W.T.S.: $x \not\le a \Leftrightarrow \neg(x \le a) \Leftrightarrow \exists a' \in A : a \le a' \vee \exists b \in B : a \le b$

The right-hand-side holds by def 1. $\Rightarrow$ (a2) holds

(b1): $x \le b \Leftrightarrow \underbrace{\neg \exists a \in A : b \le a}_{\text{holds by def 1}} \wedge \neg \exists b_R \in B_R : b_R \le x$

Suppose $\exists b_R \in B_R : b_R \le x$ Then $\neg \exists b \in B : b \le b_R$, which contradicts to the inductive step $\Rightarrow$ (b1) holds.

(b2): W.T.S.: $b \not\le x \Leftrightarrow \neg(b \le x) \Leftrightarrow \exists b'_L \in B_L : b' \le b_L \vee \exists b' \in B : b' \le b$

By inductive step, left-hand-side holds

$\Rightarrow$ (b2) holds.

Then we're done.

Coro. If $x = \{a|\}$, then $a < x$

Thm (Transitive law for $\le$). If $x, y, z \in S$, then $x \le y \wedge y \le z$ can imply $x \le z$.

Proof: Suppose not, i.e.: $x \not\le z$ when $x \le y$ & $y \le z$.

Take $p(x,y,z) \Leftrightarrow x \le y \wedge y \le z \wedge x \not\le z$. Then

$\neg \exists x_L \in X_L : y \le x_L$ (1); $\neg \exists y_R \in Y_R : y_R \le x$ (2);

$\neg \exists y_L \in Y_L : z \le y_L$ (3); $\neg \exists z_R \in Z_R : z_R \le y$ (4);

$\exists x_L \in X_L : z \le x_L \vee \exists z_R \in Z_R : z_R \le x$.

Case 1: $\exists x_L \in X_L : z \le x$. By (1), we have

$p(y, z, x_L) \Leftrightarrow y \le z \wedge z \le x_L \wedge y \not\le x_L$

Case 2: $\exists z_R \in Z_R : z_R \le x$. By (4), we have

$p(z_R, x, y) \Leftrightarrow z_R \le x \wedge x \le y \wedge z_R \not\le y$

Hence, we have $p(x,y,z) \Leftrightarrow \exists x_L \in X_L : p(y, z, x_L) \vee \exists z_R \in Z_R : p(z_R, x, y)$

Note that x_L, z_R are the parents of x and z respectively. ~~Then by induction, we can prove it.~~

~~Basic case: $0 \le 0$ & $0 \le 0 \Rightarrow 0 = 0$.~~

~~Inductive step: Suppose that the transitive law holds for all parents of x, y, z Then~~

~~$p(x,y,z) \Leftrightarrow \exists x_L \in X_L : p(y, z, x_L) \vee \exists z_R \in Z_R : p(z_R, x, y)$~~

Then repeat that ~~proc~~ procedure and we ~~to~~ will reach 0. Then we have contradiction, since $p(0,0,0)$ is false. Hence, the assumption never holds. Then we're done. $\square$