Q1)

Q1.a)

First, we argue that f^* is well-defined. Let $f : X \to Y$ be a holomorphic map, and let $\phi \in \mathcal{O}(Y)$, such that $\phi : Y \underset{\text{hol}}{\to} \mathbb{C}$. Hence,

$$f^*(\phi) = \phi \circ f$$
$$\Rightarrow f^*(\phi) : X \underset{\text{hol}}{\to} Y \underset{\text{hol}}{\to} \mathbb{C}$$
$$\Rightarrow f^*(\phi) \in \mathcal{O}(X)$$

Since f, ϕ are well-defined holomorphic maps, so is their composition. It remains to show f^* is a ring homomorphism. Let $\phi, \psi \in \mathcal{O}(Y)$, so we get distribution over addition:

$$\begin{aligned} f^*(\phi + \psi) &= (\phi + \psi) \circ f \\ &= \phi \circ f + \psi \circ f \\ &= f^*(\phi) + f^*(\psi) \end{aligned}$$

and multiplication:

$$\begin{aligned} f^*(\phi\psi) &= (\phi\psi) \circ f \\ &= (\phi \circ f)(\psi \circ f) \\ &= f^*(\phi) f^*(\psi) \end{aligned}$$

Also, f^* preserves the identity:

$$\begin{aligned} f^*(1_Y) &= 1_Y \circ f \\ &= 1_X \end{aligned}$$

Hence, f^* is a ring homomorphism. Suppose f is non-constant (so that $f(X)$ is open), and that $f^*(\phi) = \phi \circ f = f^*(\psi) = \psi \circ f$. Hence, $f^*(\phi - \psi) = f^*(\phi) - f^*(\psi) = 0$, but since f is non-constant and holomorphic, if $\phi - \psi$ is nonconstant then $f^*(\phi - \psi)$ is an open mapping. However, since this is not true, $\phi - \psi = 0 \Rightarrow \phi = \psi$, thus f is an injective homomorphism. We have shown that f^* is a monomorphism when f is non-constant. ■

Q1.b)

Let X, Y be connected Riemann surfaces, with $f : X \to Y$ a holomorphic map. Suppose f is non-constant and proper, and A the set of all branch points of f. We argue that $B := f(A)$ is closed and discrete. Let S be a closed and discrete set.

Q1.b.i) Firstly, f is closed

Since X, Y are Riemann surfaces, they are locally compact and Hausdorff.

Let $C \subseteq X$ be closed, and consider $y \in Y \setminus f(C)$. Then, y admits an open neighborhood U with compact closure, so $f^{-1}\left(\overline{U}\right)$ is compact. Then, $C \cap f^{-1}\left(\overline{U}\right)$ is compact, hence $f\left(C \cap f^{-1}\left(\overline{U}\right)\right)$ is also compact (therefore closed). Then, we have an open neighborhood $U \setminus f\left(C \cap f^{-1}\left(\overline{U}\right)\right)$ of y disjoint from $f(C)$, so $Y \setminus f(C)$ is open. Therefore, $f(C)$ is closed. Hence f is closed.

Q1.b.ii) Also, f preserves discreteness

Let K be a compact set in Y. If we can show $f(S) \cap K$ is always finite, then we have that $f(S)$ is discrete. Note that:

$$f(S) \cap K = f\left(S \cap f^{-1}(K)\right)$$

Since f is proper, $f^{-1}(K)$ is compact. Since S is discrete, we have that $S \cap f^{-1}(K)$ is finite. Hence, the image $f(S) \cap K$ is finite. Therefore, $f(S)$ is discrete.

Q1.b.iii) Hence, B is a closed, discrete subset of X

Since the set of branch points is closed and discrete, B is discrete. ■

Q2)

Q2.a)

Recall from PMATH 352 that:

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = \frac{e^{2iz} - 1}{i\left(e^{2iz} + 1\right)}$$

Now, observe that $\tan(z) = \mu \circ \psi \circ \phi(z)$, where:

$$\phi(z) = 2iz \qquad \psi(z) = e^z \qquad \mu(z) = \frac{z-1}{i(z+1)}$$

Firstly, ϕ is a linear function, hence trivially a local homeomorphism. Next, $\psi(z) = e^z \Rightarrow \psi'(z) = e^z$, so by the inverse function theorem, ψ is a local homeomorphism. Finally, because μ is a Möbius map, $\det\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = 2i$ implies μ is bijective, hence a local homeomorphism.

We have shown that $\tan(z)$ is the composition of local homeomorphisms, so $\tan(z)$ must be a local homeomorphism. ■

Q2.b)

Again, consider $\tan(z) = \mu \circ \psi \circ \phi(z)$. First note that ϕ is a linear map, and hence is a covering map of $\mathbb{C}$.

Next, we wish to show that ψ is a covering map of $\mathbb{P}^1 \setminus \{0, \infty\} \cong \mathbb{C} \setminus \{0\}$. Let $z \in \mathbb{C}$ such that $z \neq 0$. Thus, we have that $\psi^{-1}(z) = \{\log(|z|) + \arg(z) + 2\pi ik \mid k \in \mathbb{Z}\}$. Thus, we can easily take disjoint open neighborhoods around each point in $\psi^{-1}(z)$ (e.g. balls of radius π). Furthermore, we have already shown that ψ is a local homeomorphism. Hence, ψ is a covering map of $\mathbb{C} \setminus \{0\} \cong \mathbb{P}^1 \setminus \{0, \infty\}$.

Finally, we know that μ is an automorphism of $\mathbb{C}$. Using PMATH 352, we have that $\mu(\{0, \infty\}) = \{\pm i\}$.

Therefore,

$$\tan(z) : \mathbb{C} \underset{\phi}{\to} \mathbb{C} \underset{\psi}{\to} \mathbb{P}^1 \setminus \{0, \infty\} \underset{\mu}{\to} \mathbb{P}^1 \setminus \{\pm i\}$$

is a composition of covering maps. ■

Q2.c)

In the previous parts of this question, we expressed $\tan(z)$ as a composition of three functions, μ, ψ, and ϕ. We argued that ϕ is a linear map, hence bijective, and ϕ is a Möbius map induced from an invertible matrix, hence bijective as well. Furthermore, these functions are holomorphic.

However, to deal with $\psi(z) = e^z$, we remark that this function is unique on the strip $S_k := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in (-k\pi, k\pi), k \in \mathbb{Z}\}$, which is the image of the set $S'_k := \left\{z \in \mathbb{C} \mid \mathrm{Re}(z) \in \left(-\frac{k\pi}{2}, \frac{k\pi}{2}\right)\right\}$ under ϕ. However, note that $\tan(S'_k) = X$, so ψ is effectively bijective.

Therefore, tan has a unique holomorphic inverse $\arctan_k : X \to \mathbb{C}$ such that $\tan \circ \arctan_k = \mathrm{id}_X$.

Note that $\tan^{-1}(0) = k\pi, k \in \mathbb{Z}$. Thus, since $\tan(z)$ is a covering map of $\mathbb{P}^1 \setminus \{\pm i\}$, we have that $\tan(z)$ is a local homeomorphism sending $k\pi$ to 0 with the curve lifting property. Therefore, for $\tan \circ \arctan_k = \mathrm{id}_X$, we have that $\arctan(0) = k\pi$. ■

Q3)

Q3.a)

We will show that $\mathscr{B}$, defined as follows, is a basis for a topology on R:

$$\begin{aligned}
\mathscr{S} &:= \{D((z_0,k),r) \mid z_0 \in \mathbb{C}^* \setminus \mathbb{R}^+, r < d(z,\mathbb{R}^+)\} \\
\mathscr{T} &:= \{A((z_0,k),r) \mid z_0 \in \mathbb{R}^+\} \\
\mathscr{B} &:= \mathscr{S} \cup \mathscr{T}
\end{aligned}$$

Let $x := (z,k) \in R$. Suppose $z \notin \mathbb{R}^+$. Then, since $\mathbb{C}^*$ is Hausdorff, we have that $r := d(x,\mathbb{R}^+) > 0$, so there is a disc $D\left(x, \frac{r}{2}\right)$ containing x. Next, if $z \in \mathbb{R}^+$, then we have that $A\left(x, \frac{z}{2}\right)$ contains x. Therefore, each point in R is contained in an element of $\mathscr{B}$.

Next, we will show that for $B, B' \in \mathscr{B}$ where $B \cap B' \neq \emptyset$, there exists a $B'' \subseteq B \cap B'$.

Case 1: Suppose $B, B' \in \mathscr{S}$. Since $B \cap B' \neq \emptyset$, they must be at the same "k", so $B = D((z_0,k),r)$ and $B' = D((z_0',k),r')$. Let $p \in B \cap B'$ and let $r'' < \min(r - |p - z_0|, r' - |p - z_0'|)$, so that $D(p,r'') \subseteq B \cap B'$.

Case 2: Suppose $B, B' \in \mathscr{T}$. Again, since $B \cap B' \neq \emptyset$, they must be at the same "k", so $B = A((z_0,k),r)$ and $B' = A((z_0',k),r')$. We use the same trick as before to obtain r'' such that $D(p,r'') \subseteq B \cap B'$.

Case 3: Suppose $B \in \mathscr{S}, B' \in \mathscr{T}$. This splits into two more cases, where if $B = A((z_0,k),r)$, then B' can either be of the form $D((z_0',k),r')$ or $D((z_0',k-1),r')$. In either case, let $p \in B \cap B'$, and let $r'' = \min(r - |p - z_0|, r' - |p - z_0'|)$ such that $D(p,r'') \subseteq B \cap B'$.

Therefore, we have shown that $\mathscr{B}$ is a basis for the topology τ. $\blacksquare$

Q3.b)

We proceed by showing R is Hausdorff and locally homeomorphic to $\mathbb{C}$ with a holomorphic atlas.

(*Hausdorff*) Suppose $x, y \in \mathbb{C}$ such that $x \neq y$. In our first case, we have $x, y \in \mathscr{S}$, such that $x, y \neq \mathbb{R}^+$. Therefore, if we let $r < |x-y|$, then we have open sets $D\big((x,k_1), \frac{r}{2}\big)$ and $D\big((y,k_2), \frac{r}{2}\big)$ such that $D\big((x,k_1), \frac{r}{2}\big) \cap D((y,k_2), \frac{r}{s}) = \emptyset$. Alternatively, if $x, y \in \mathbb{R}^+$, then if we let $r < |x-y|$, we again have open neighbourhoods $A\big((x,k_1), \frac{r}{2}\big)$ and $A\big((y,k_2), \frac{r}{2}\big)$ of x and y, respectively, such that $A\big((x,k_1), \frac{r}{2}\big) \cap A((y,k_2), \frac{r}{s}) = \emptyset$. Finally, if $x \in \mathbb{C} \setminus R^+$ and $y \in \mathbb{R}^+$, then again we may have $r < \min(|x-y|, y)$ such that $D\big((x,k_1), \frac{r}{2}\big) \cap A\big((y,k_2), \frac{r}{2}\big) = \emptyset$. Hence, in every case, we may find disjoint open neighbourhoods of x, y, so R is Hausdorff with respect to τ.

(*Locally Homeomorphic to* $\mathbb{C}$) Now, we claim that R is locally homeomorphic to $\mathbb{C}$ by finding homeomorphisms from $\mathscr{B}$ to $\mathbb{C}$. Let $B \in \mathscr{S}$, so B is of the form $D((z_0,k),r)$. Then, the map $\phi : D((z_0,k),r) \to D(z_0,r), (z,k) \mapsto z$ is a continuous projection map. Likewise, for $B \in \mathscr{T}$, where B is of the form $A((z_0,k),r)$, we have the map $\psi : A((z_0,k),r) \to D(z_0,r)$ where if "$\Im$"$(z) \geq 0$, then $\psi((z,k)) = z$ and if "$\Im$"$(z) < 0$, then $\psi((z,k-1)) = z$. This is clearly continuous. Hence, we have homeomorphisms sending $\mathscr{B}$ to the topology of $\mathbb{C}$, so R is locally homeomorphic to $\mathbb{C}$ since $\mathscr{B}$ is a basis for R.

It remains to check that the atlas is holomorphic. Consider the atlas $(\mathscr{B}, \{\phi, \psi\})$, defined previously. Then, for $B, B' \in \mathscr{B}$, such that $B \cap B' \neq \emptyset$, we can split it up into cases.

If $B, B' \in \mathscr{S}$, then we have that $B = D((z_0,k),r), B' = D((z'_0,k),r')$, and so $\phi \circ \phi^{-1} = \text{id}$. Likewise, if $B, B' \in \mathscr{T}$, then $B = A((z_0,k),r), B' = A((z'_0,k),r')$, and $\psi \circ \psi^{-1} = \text{id}$. There are two cases for if $B \in \mathscr{S}, B' \in \mathscr{T}$: B can either be of the form $D((z_0,k),r)$ or $D((z_0,k-1),r)$ if B' is of the form $A((z'_0,k),r')$. In the first case, $\psi \circ \phi^{-1}(z) = \psi(z,k) = z$, and in the second, $\psi \circ \phi^{-1}(z) = \psi(z, k -$

$1) = z$, so $\psi \circ \phi^{-1} = \text{id}$. Therefore, we have verified that the atlas is holomorphic. ■

Q3.c)

To show that $\hat{f}$ is single-valued, we first note that $\hat{f}$ is clearly single-valued for some fixed k, since it's just one branch of the complex logarithm function. Then, since varying k shifts the image by $2\pi i$, there is no overlap between $\hat{f}(R, k_1)$ and $\hat{f}(R, k_2)$ if $k_1 \neq k_2$. Hence, $\hat{f}$ is single-valued.

Next, we claim that $\hat{f}$ is holomorphic. We verify that each of the charts of the form $\hat{f} \circ \phi^{-1} : \phi(B) \to \mathbb{C}$ and $\hat{f} \circ \psi^{-1} : \psi(B') \to \mathbb{C}$ are holomorphic, for $B \in \mathscr{S}$ and $B' \in \mathscr{T}$. (Recall that $\mathscr{B} = \mathscr{S} \cup \mathscr{T}$ is the basis for the topology τ.)

If $z \notin \mathbb{R}^+$, then $z \in D((z,k), r) \in \mathscr{S}$, so $\hat{f} \circ \phi^{-1}(z) = \hat{f}(z,k) = \ln|z| + i(\arg_0 z + 2\pi k)$, which is holomorphic as it coincides with the complex logarithm function shifted by $2\pi i k$. Furthermore, notice that $\arg_0(z) \in [0, 2\pi)$, so it follows that $i(\arg_0 z + 2\pi k) \in [2\pi i k, 2\pi i(k+1))$, hence:

$$\begin{aligned}
\hat{f} \circ \phi^{-1}(z) &= \ln|z| + i(\arg_0 z + 2\pi k) \\
&= \ln|z| + i(\arg_{2\pi k} z) \\
&= L_{2\pi k}(z)
\end{aligned}$$

In the other case, $z \in A((z,k), r)$, so

$$\begin{aligned}
\hat{f} \circ \psi^{-1}(z) &= \begin{cases} \hat{f}(z,k) & \text{if } \operatorname{Im}(z) \geq 0 \\ \hat{f}(z,k-1) & \text{if } \operatorname{Im}(z) < 0 \end{cases} \\
&= \begin{cases} L_{2\pi k}(z) & \text{if } \operatorname{Im}(z) \geq 0 \\ L_{2\pi(k-1)}(z) & \text{if } \operatorname{Im}(z) < 0 \end{cases} \\
&= \begin{cases} \ln|z| + i(\arg_{2\pi k} z) & \text{if } \operatorname{Im}(z) \geq 0 \\ \ln|z| + i\left(\arg_{2\pi(k-1)} z\right) & \text{if } \operatorname{Im}(z) < 0 \end{cases}
\end{aligned}$$

We remark that this function is continuous along $\mathbb{R}^+$. (As soon as you cross the positive real line, $\arg_0 z$ gets sent to 2π, but out function now adds $2\pi(k-1)$ instead of $2\pi k$, so it balances out.) Hence, $\hat{f} \circ \psi^{-1}(z)$ is holomorphic, and we have shown that $\hat{f}$ is holomorphic with respect to an atlas of R and equal to $L_{2\pi k}(z)$ for $z \in [2\pi k, 2\pi(k+1))$. $\blacksquare$

Q4)

Let $\Gamma = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$ and let f be a non-constant, elliptic function relative to Γ.

First, we will show that f must have a pole. Suppose for contradiction f has no poles in $\mathbb{C}/\Gamma$, hence f is holomorphic over $\mathbb{C}/\Gamma$. This implies that over $\mathbb{C}/\Gamma$, $|f|$ is bounded by a constant B. However, by periodicity, this implies $|f|$ is bounded over $\mathbb{C}$, and by Liouville's theorem, f is therefore constant. Contradiction — f must have a pole!

Now, we will show that f must have at least two poles, counting multiplicity. Let C be the contour along the perimeter of a translated fundamental parallelogram, along the path $[k, k+\omega_1, k+\omega_1+\omega_2, k+\omega_2]$. (If the poles of f lie in Γ, we may choose k without loss of generality such that this is no longer the case.) Hence, we compute the integral along C:

$$\begin{aligned}
\int_C f(z)dz &= \left(\int_k^{k+\omega_1} + \int_{k+\omega_1}^{k+\omega_1+\omega_2} + \int_{k+\omega_1+\omega_2}^{k+\omega_2} + \int_{k+\omega_2}^{k}\right) f(z)dz \\
&= \int_k^{k+\omega_1} f(z) - f(z+\omega_2)dz - \int_k^{k+\omega_2} f(z) - f(z+\omega_1)dz \\
&= 0, \text{ because } f(z) = f(z+\omega_1) = f(z+\omega_2)
\end{aligned}$$

Thus, by the residue theorem, the sum of the residues of f equals 0. If f has a simple pole at z_0, then $f = \frac{c}{z-z_0} + \sum_{i=0}^{\infty} c_i (z-z_0)^i$. However, notice that if $\text{Res}(f, z_0) = 0$, then $c = 0$, so the simple pole isn't a pole at all. Hence, the order of f must be at least two. ∎

Q5)

Q5.a)

Let X be a constant Riemann surface, and let f be a non-constant meromorphic function on X. Hence, $f : X \to \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$. We argue that f is proper.

Let $z_1, z_2, ..., z_n$ be poles of f (there are finitely many since $\mathbb{P}^1$ is compact). Then, since f is holomorphic on $X \setminus \{z_1, z_2, ..., z_n\}$, we have that $f^{-1}(K)$ is compact, for compact $K \subset \mathbb{C}$. Then, for any compact $K \subseteq \mathbb{C} \cup \{\infty\}$ containing ∞, we have that $f^{-1}(K) = \{\text{some compact set}\} \cup \{z_1, z_2, ..., z_n\}$, which is compact. Since f is meromorphic, it can be thought of as a holomorphic mapping onto $\mathbb{P}^1$. Therefore, f is proper.

Since f is a proper non-constant holomorphic map from X to $\mathbb{P}^1$, we know that $\exists n \in \mathbb{N}$ such that for $p \in \mathbb{P}^1$, $\left|f^{-1}(p)\right| = n$, counting multiplicity. Hence, $\left|f^{-1}(\infty)\right| = \left|f^{-1}(0)\right|$, as desired. ■

Q5.b)

($\Rightarrow$) Let f be a meromorphic function on X with a pole of multiplicity one.

First, we will show f is injective. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = c$. However, if we let $g(z) = f(z) - c$, g would be a meromorphic function on X with one pole and two zeroes, contradicting (a). Hence, f must be injective.

Next, we show that f is surjective. Using (a), we get that f has a single zero (of multiplicity one) at some $x_0 \in X$. Hence, we see that $\left|f^{-1}(0)\right| = 1$, so f is an unbranched map. In (a), we established that f is a proper map from X to $\mathbb{P}^1$, so f is an unbranched proper non-constant holomorphic mapping. Hence, via a theorem proved in class, f is a covering map of $\mathbb{P}^1$, implying f is surjective.

Therefore, $f : X \to \mathbb{P}^1$ is a bijective holomorphism, so the two surfaces are isomorphic.

($\Leftarrow$) Now, suppose X is isomorphic to $\mathbb{P}^1$. Hence, there exists a biholomorphism $f : X \to \mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$. Since f is injective, $\left|f^{-1}(\infty)\right| = 1$, i.e. there is one point in X which maps to ∞. Also, f is unbranched. Therefore, f has a biholomorphism from X to $\mathbb{C} \cup \{\infty\}$, where there is only one $x \in X$ such that $f(x) = \infty$. It follows that f is a meromorphic function with a single simple pole. ■

Q5.c)

Let f be a meromorphic function on $\mathbb{C}/\Gamma$. Hence, f is elliptic with respect to Γ. In a previous question, we showed that f must either be constant or admit at least two poles (counting multiplicity). Hence, $\mathbb{C}/\Gamma$ cannot admit a meromorphic function with a simple pole, so $\mathbb{C}/\Gamma \ncong \mathbb{P}^1$. ■