Text as Data

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A General Framework for the Class

A focus on tasks

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1) Decide on some objective to accomplish

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   - Describe

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1) Decide on some objective to accomplish
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   - Cluster
   - Predict
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   - Measure covariance, discover latent structure, find nearest neighbor, ...

A General Framework for the Class

2) Use an objective function $\rightsquigarrow$ measure performance at task

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- Data generating process

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   - Analytic methods (Calculus)
   - Computational methods

Today: Using (Bayesian) Statistics to Obtain Objective Functions

- Encode assumptions in data generating process $\rightsquigarrow$ hierarchical model
- Assume parameters and data are random variables
- Conditional probability statement $\rightsquigarrow$ objective function
- Use computational tools to optimize objective function

Today: Using (Bayesian) Statistics to Obtain Objective Functions

Plan of Attack:

1) Write out any joint density function as conditional relationship
2) Show how this can be an objective function even if you've never taken likelihood/Bayesian/...
3) Discuss how to computationally optimize

Joint Distributions of Random Variables

Definition

Suppose that we have random variables $\boldsymbol{X} = (X_1, X_2, \ldots, X_K)$*. We will say that* $\boldsymbol{X}$ *is a jointly continuous random variable if for all* $\boldsymbol{X} \in \Re^K$ *there exists a function* $f : \Re^K \rightarrow \Re$ *such that for all* $C \subset \Re^K$*,*

$$P(\boldsymbol{X} \in C) = \iint \ldots \iint_{(\boldsymbol{X}) \in C} f(\boldsymbol{x}) d\boldsymbol{X}$$

- A joint density $f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_K)$ encodes information about the behavior of the random variable $\boldsymbol{X}$

Marginal Distribution

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_K)$ is a jointly continuous random variable. Define $f_{X_j}(x)$ as the marginal *probability density function for X_j,*

$$
\begin{aligned}
f_{X_j}(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x})dx_1 dx_2 \ldots dx_{j-1}dx_{j+1}\ldots dx_{K-1}dx_K \\
&= \int_{\Re^{K-1}} f(\boldsymbol{x})d\boldsymbol{X}_{-j}
\end{aligned}
$$

- To obtain the marginal distribution, $f_{X_j}(x)$ we integrate over all dimensions but j

Conditional Distributions and Independence of Random Variables

Definition

Suppose $\boldsymbol{X}$ *is a jointly continuous random variable. Define* $f_{\boldsymbol{X}_{-j}|X_j}(\boldsymbol{x})$ *as the conditional density function,*

$$f_{\boldsymbol{X}_{-j}|X_j}(\boldsymbol{x}_{-j}|x_j) = \frac{f(x_1, x_2, \ldots, x_K)}{f_{X_j}(x_j)}$$

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Two random variables X_1 *and* X_2 *are independent if*

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

Conditional Independence of Random Variables

Definition

Two random variables X_1 and X_2 are conditionally independent given X_3 if

$$f_{(X_1,X_2)|X_3}(x_1, x_2|x_3) = f_{X_1|X_3}(x_1|x_3)f_{X_2|X_3}(x_2|x_3)$$

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Equivalently,

$$f_{X_1|X_2,X_3}(x_1|x_2, x_3) \quad = \quad f_{X_1|X_3}(x_1|x_3)$$

Joint Density as Conditional Relationship

Theorem

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_K)$ *is a jointly continuous random variable. Then*

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\begin{aligned}
f(\boldsymbol{x}) &= f(x_1, x_2, \ldots, x_K) \\
&= f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) f_{X_3|X_2 X_1}(x_3|x_2, x_1) \ldots f_{X_K|X_1 \ldots X_{K-1}}(x_K|x_1, \ldots, x_{K-1})
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- We can write joint distributions as a product of conditional distributions
- If there are conditional independences in density we can simplify $\rightsquigarrow$ simplify some conditional expressions

Beta-Binomial Model

Definition

Suppose Y is a continuous random variable with $Y \in [0,1]$ and pdf of Y given by

$$f(y) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y^{\alpha_1 - 1}(1-y)^{\alpha_2 - 1}$$

Then we will say Y is a Beta distribution with parameters α_1 and α_2. Equivalently,

$$Y \sim Beta(\alpha_1, \alpha_2)$$

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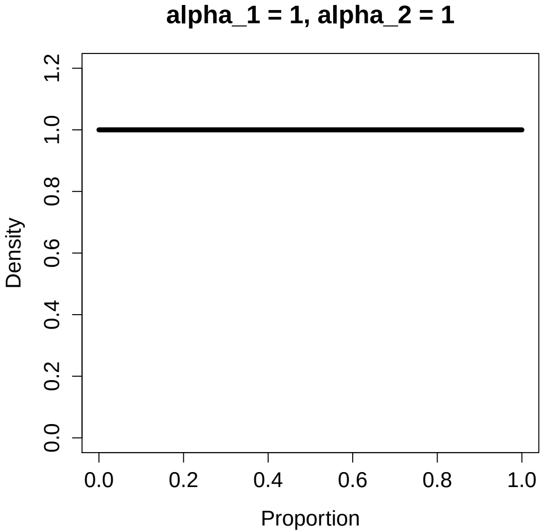
- Beta is a special case of the Dirichlet distribution
- $E[Y] = \frac{\alpha_1}{\alpha_1 + \alpha_2}$

Beta Distribution



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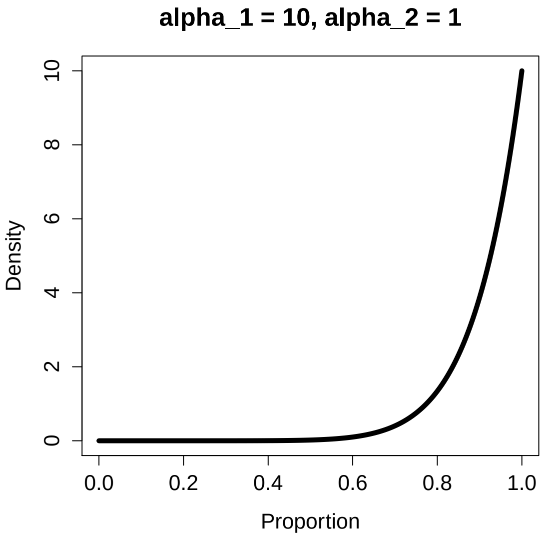

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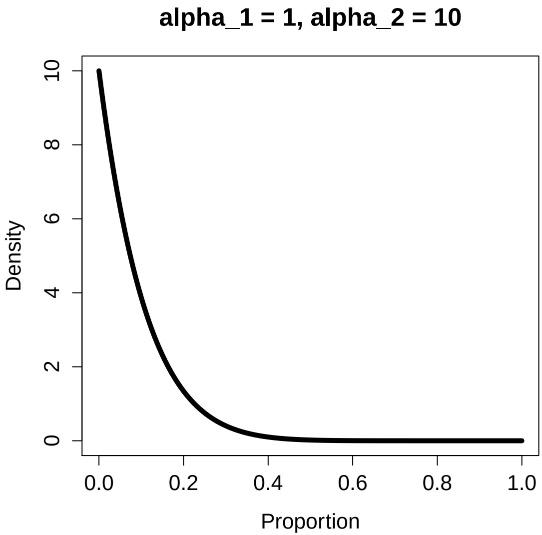

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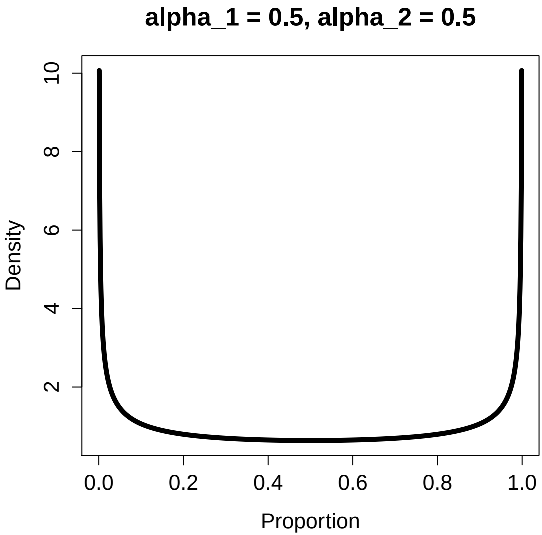

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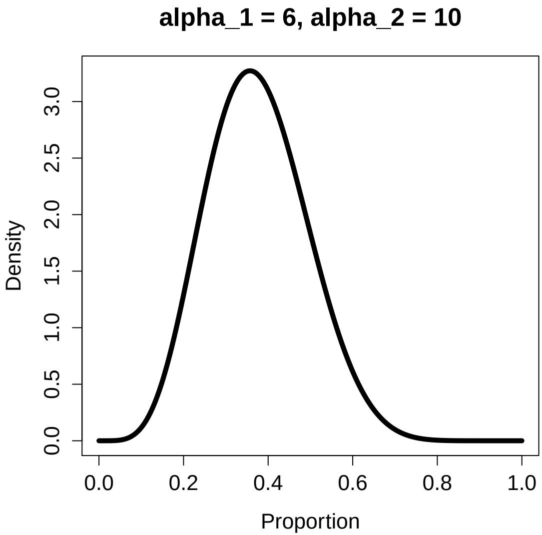

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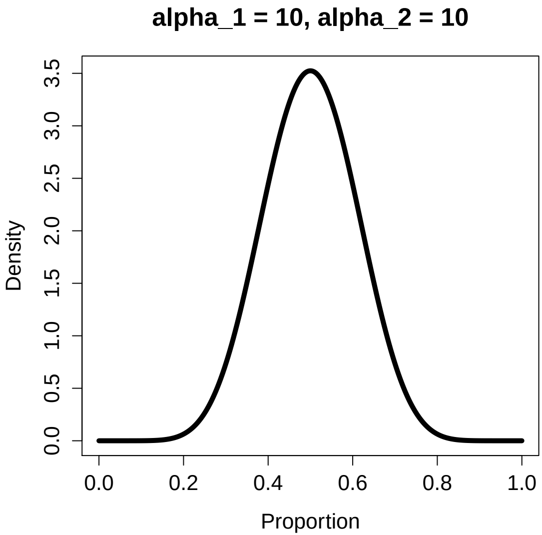

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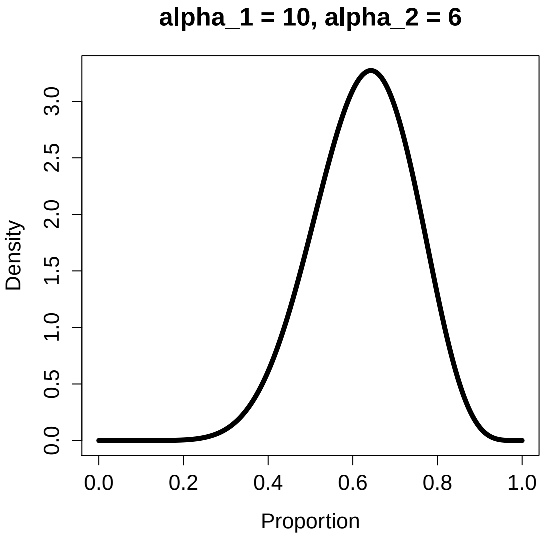

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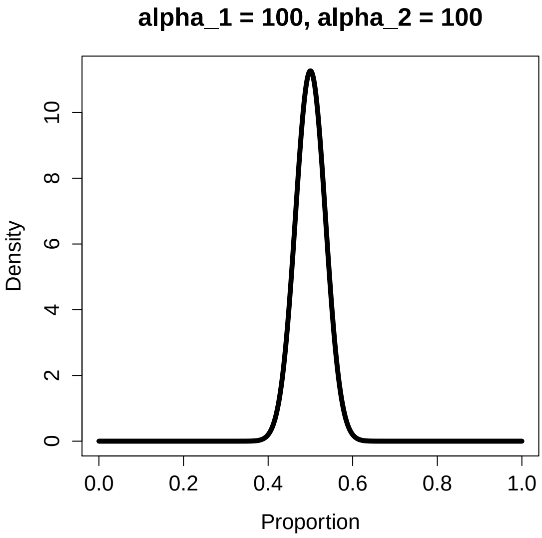

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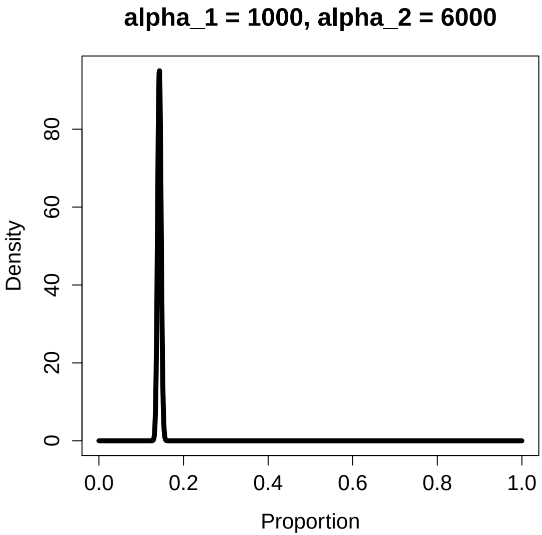

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\end{aligned}$$

Beta-Binomial Model

Suppose now we observe n statements

$$\begin{aligned} \pi &\sim \text{Beta}(\alpha_1, \alpha_2) \\ Y_i|\pi &\sim \text{Bernoulli}(\pi) \text{ for } i = 1, \ldots, n \end{aligned}$$

$\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$

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&= \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \pi^{\sum_{i=1}^{n} y_i + \alpha_1 - 1}(1-\pi)^{n + \alpha_2 - \sum_{i=1}^{n} y_i - 1}
\end{aligned}
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Bayes' Theorem for Continuous Random Variables

Theorem

Suppose we have jointly continuous random variables X_1 and X_2. Then,

$$f_{X_1|X_2}(x_1|x_2) \quad = \quad \frac{f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1)}{f_{X_2}(x_2)}$$

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We observe $\boldsymbol{y}$ and we want to learn about π

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Condition on data, describe function of π.

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$$p(\pi|\boldsymbol{y}, \alpha_1, \alpha_2) = \frac{p(\pi|\alpha_1, \alpha_2)p(\boldsymbol{y}|\pi, \alpha_1, \alpha_2)}{p(\boldsymbol{y})}$$

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\end{aligned}
$$

Defines a function of π, which we can use to describe the data.

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We observe $\boldsymbol{y}$ and we want to learn about π

Condition on data, describe function of π.

$$\begin{aligned} p(\pi|\boldsymbol{y}, \alpha_1, \alpha_2) &= \frac{p(\pi|\alpha_1, \alpha_2)p(\boldsymbol{y}|\pi, \alpha_1, \alpha_2)}{p(\boldsymbol{y})} \\ &\propto p(\pi|\alpha_1, \alpha_2)p(\boldsymbol{y}|\pi) \\ &\propto \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \pi^{\sum_{i=1}^n y_i + \alpha_1 - 1}(1-\pi)^{n + \alpha_2 - \sum_{i=1}^n y_i - 1} \end{aligned}$$

Defines a function of π, which we can use to describe the data.

Optimize $\rightsquigarrow$ analytically or computationally.

Theorem

Suppose $f : \Re^K \rightarrow \Re_+$. *If* $\boldsymbol{x}^*$ *maximizes* $\log(f(x))$, *then* $\boldsymbol{x}^*$ *maximizes* $f(x)$.

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\log\left(p(\pi|\boldsymbol{y}, \alpha_1, \alpha_2)\right) &= \left(\sum_{i=1}^{n} y_i + \alpha_1 - 1\right)\log \pi \\
&\quad + (n + \alpha_2 - \sum_{i=1}^{n} y_i - 1)(1 - \log \pi) + c
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Second derivative test $\rightsquigarrow$ maximum

The Probit Model

Suppose if we're interested in regression

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Suppose if we're interested in regression $\rightsquigarrow$ prediction, classification, description

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Implicit (improper) prior

The Probit Model $\rightsquigarrow$ Objective Function

$$
\begin{aligned}
p(\boldsymbol{\beta}|\boldsymbol{y}, \boldsymbol{X}) & \propto p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{X}) \\
& \propto \prod_{i=1}^{N} \Phi\left(\boldsymbol{X}_i \boldsymbol{\beta}\right)^{y_i} \left(1 - \Phi(\boldsymbol{X}_i \boldsymbol{\beta})\right)^{1-y_i}
\end{aligned}
$$

The Probit Model$\rightsquigarrow$ Optimization

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$$\log\left(p(\beta|\boldsymbol{y}, \boldsymbol{X})\right) = \sum_{i=1}^{N} \left[y_i \log(\Phi(\boldsymbol{X}_i\beta)) + (1 - y_i)\log(1 - \Phi(\boldsymbol{X}_i\beta))\right] + c$$

Find $\boldsymbol{\beta}^*$ to maximize $\log\left(p(\beta|\boldsymbol{y}, \boldsymbol{X})\right) \rightsquigarrow$ computational method

Computational Optimization Approaches

Analytic (Closed form)$\rightsquigarrow$ Often difficult, impractical, or unavailable

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  - Branch and Bound ...

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Iterative procedure to find a root

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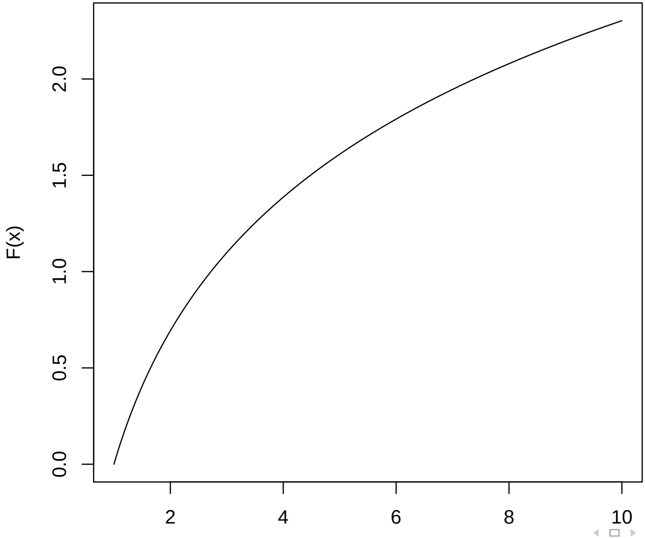
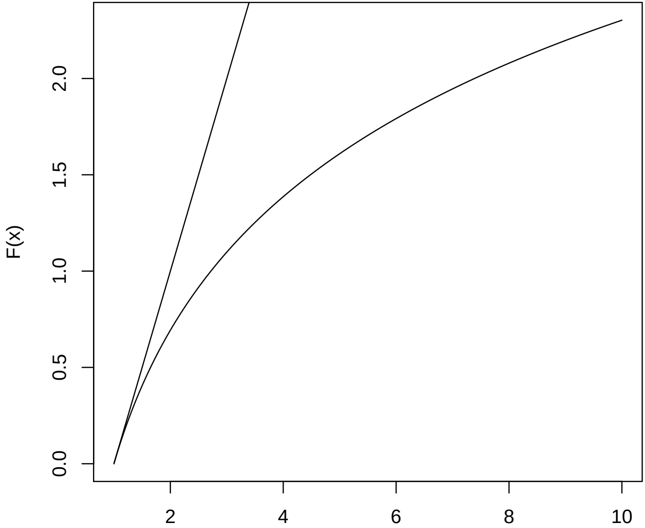
Approximate with tangent line, iteratively update

Tangent Line



Tangent Line



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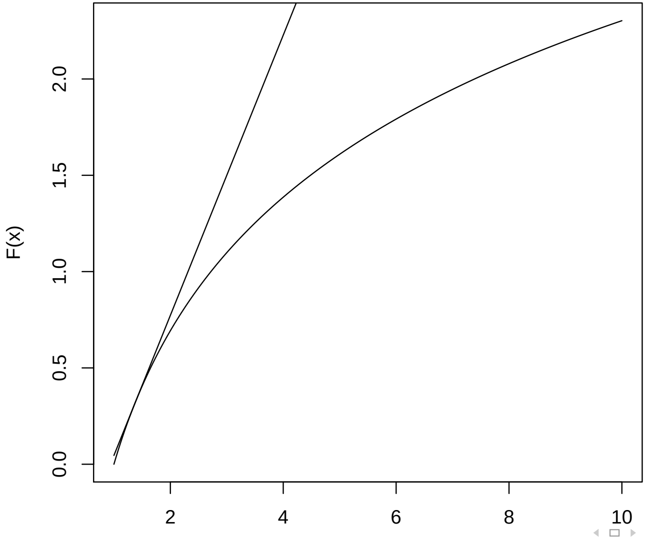

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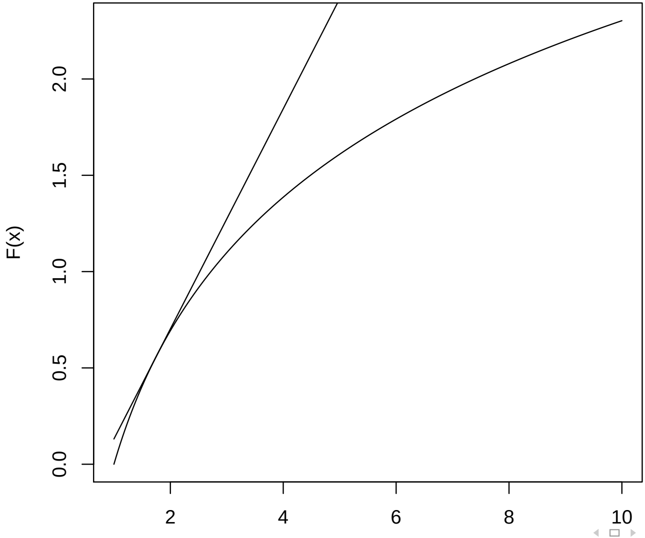

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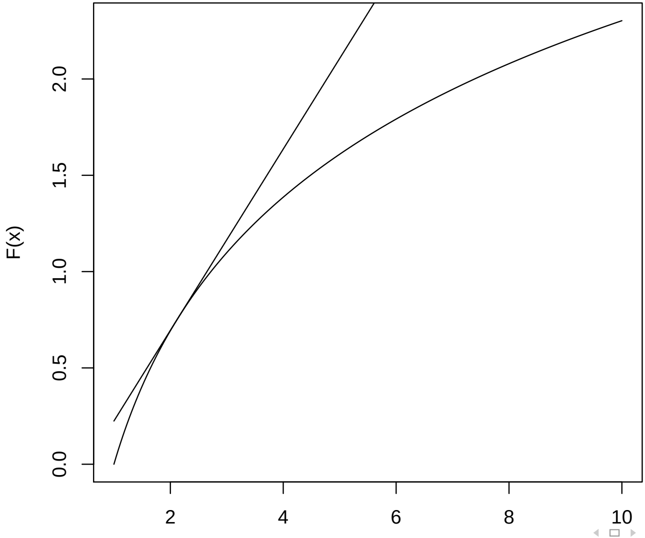

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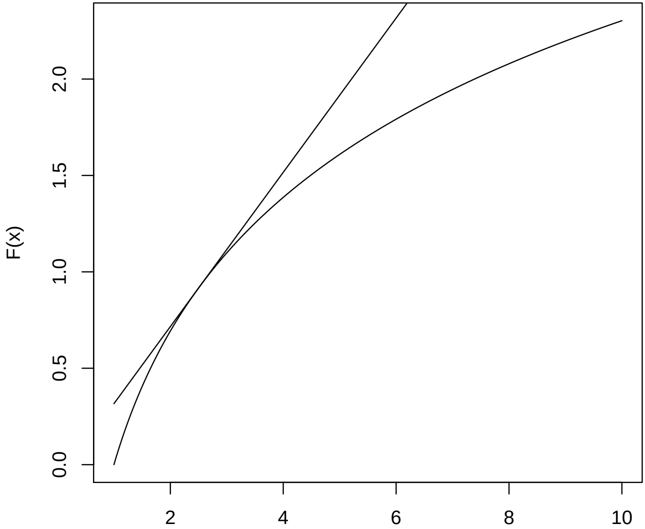

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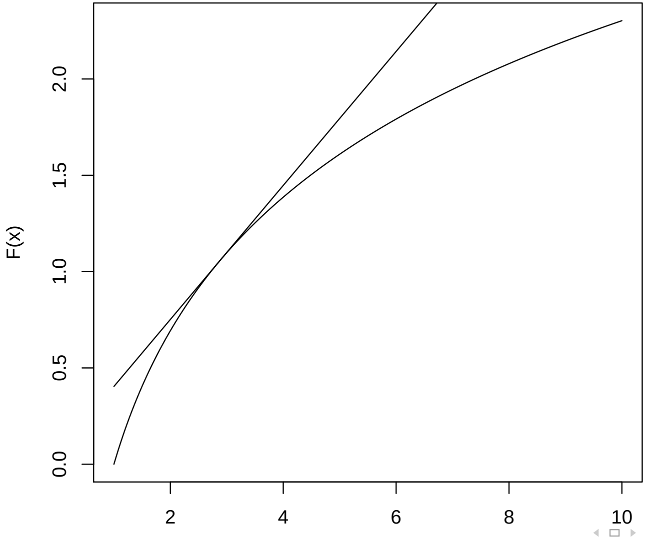

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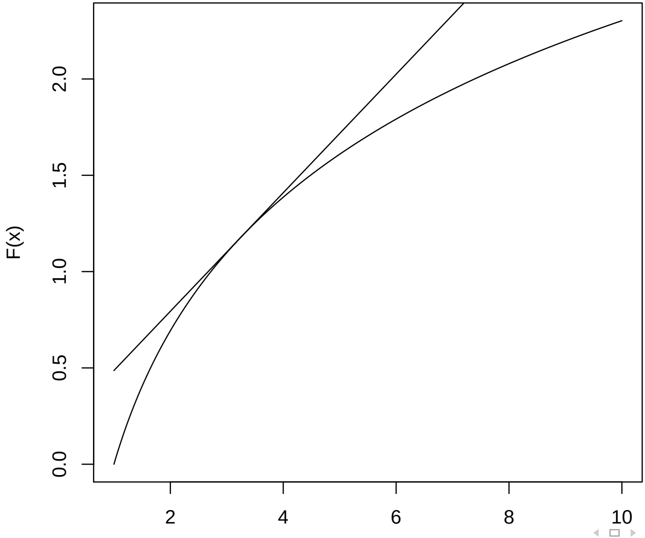

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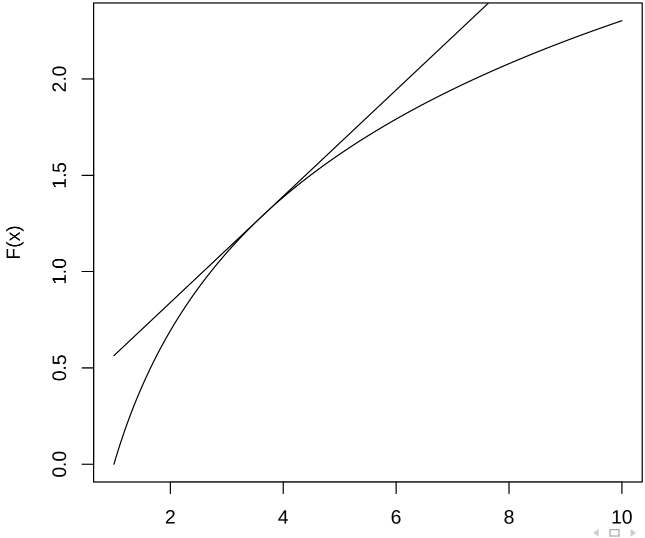

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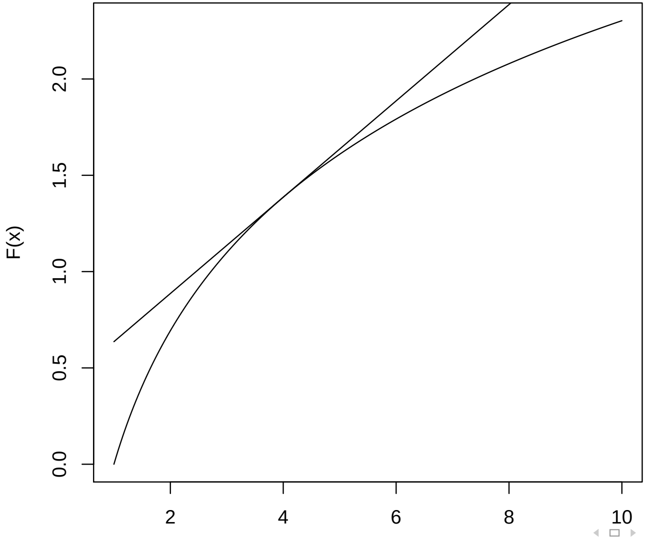

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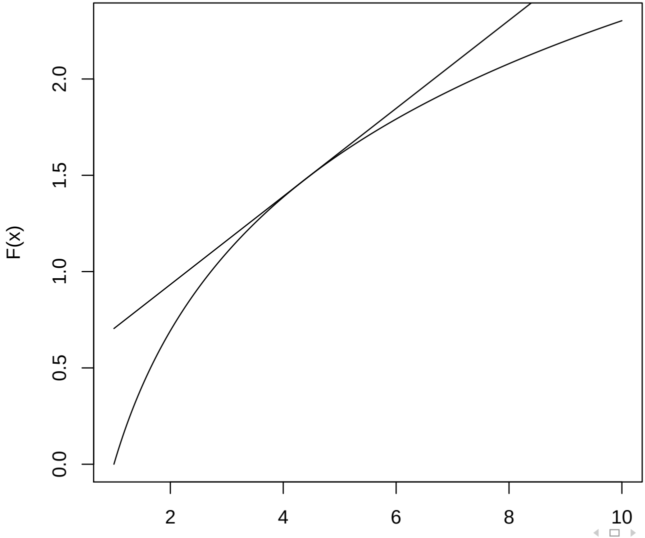

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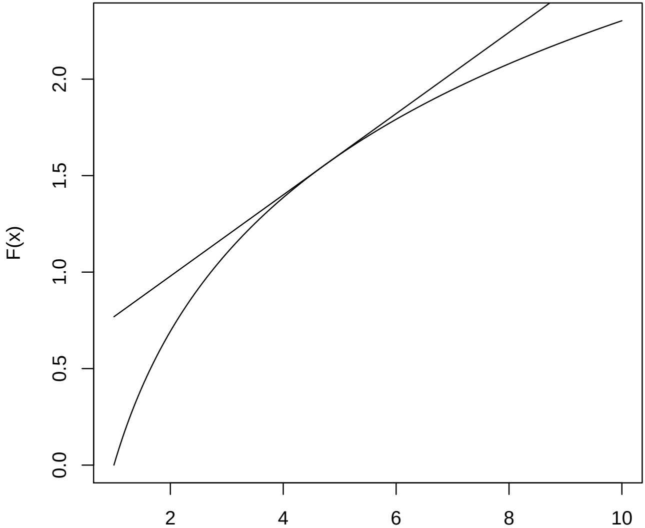

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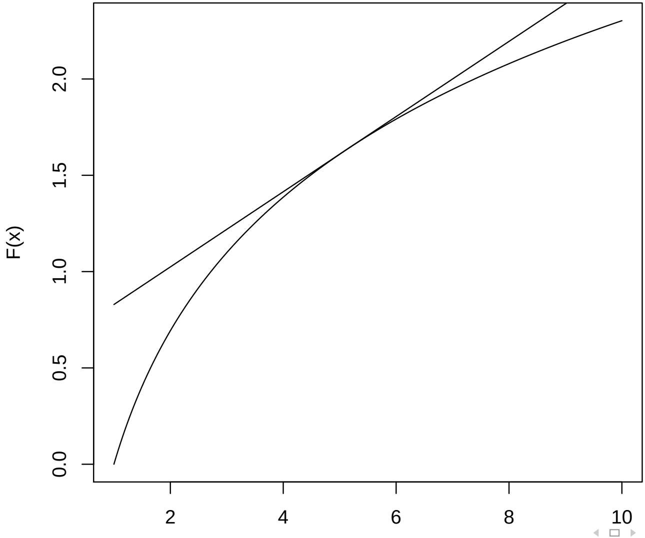

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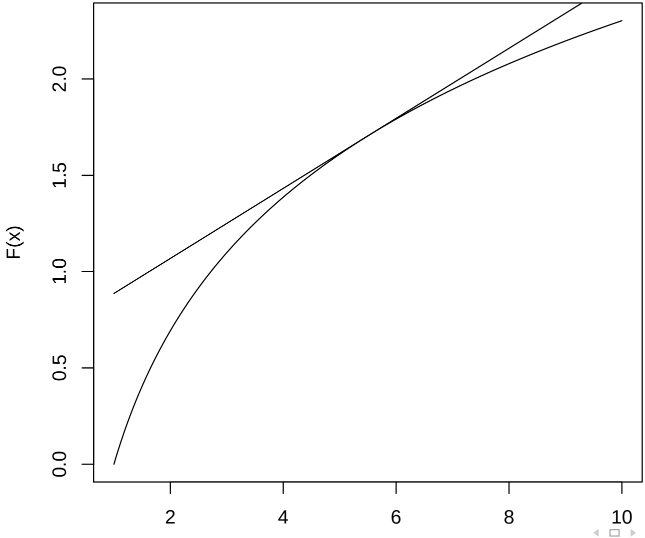

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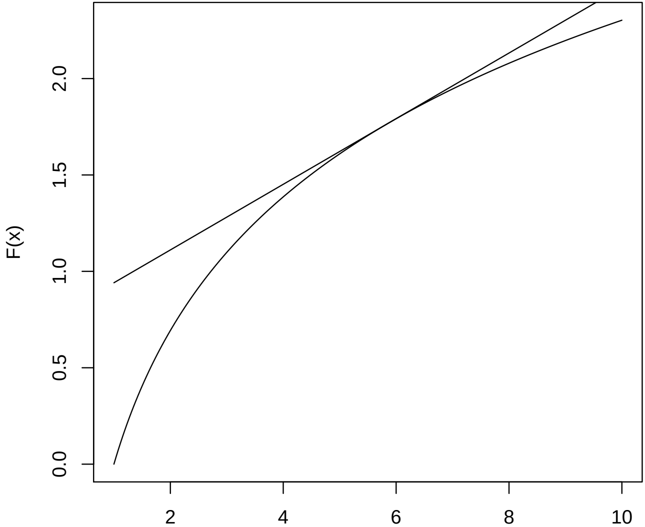

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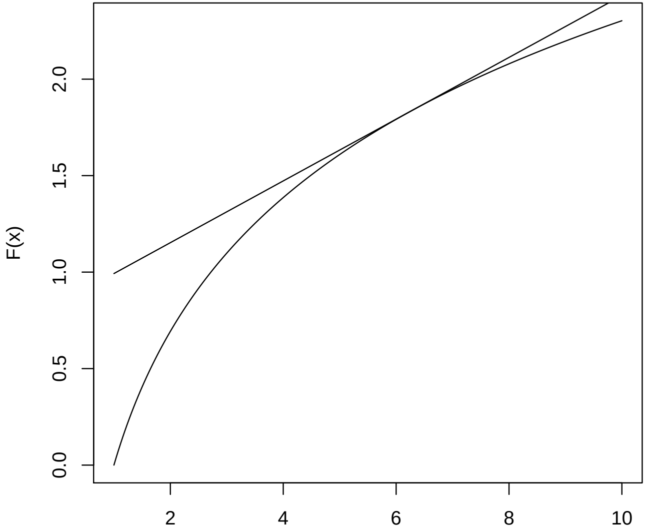

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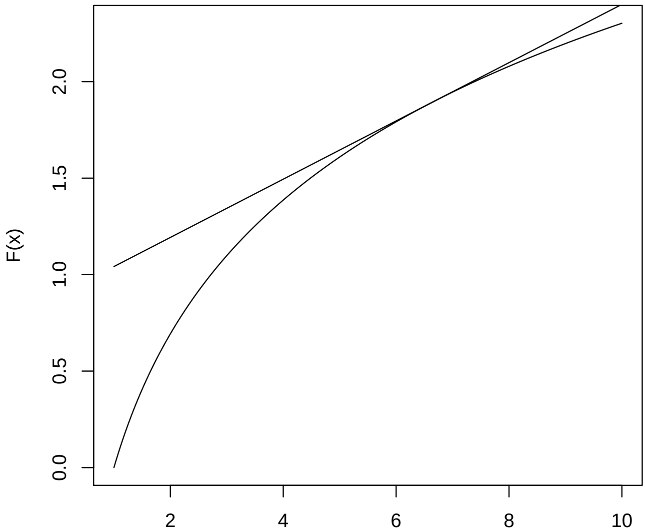

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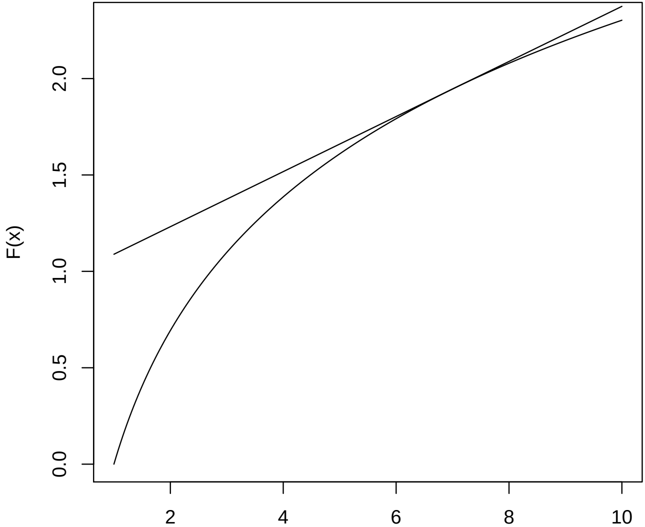

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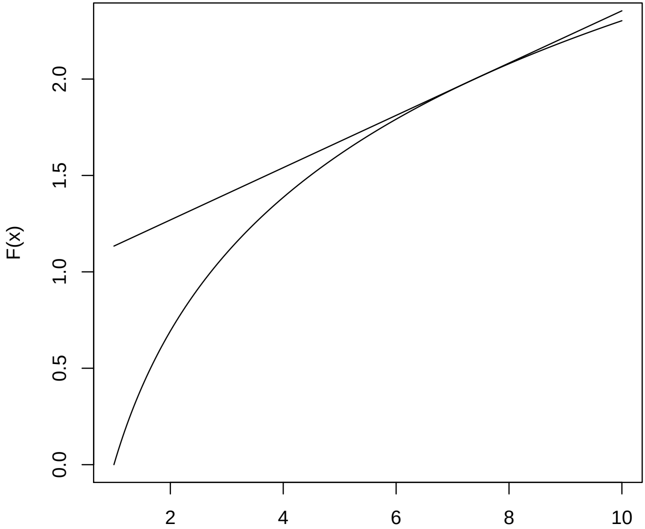

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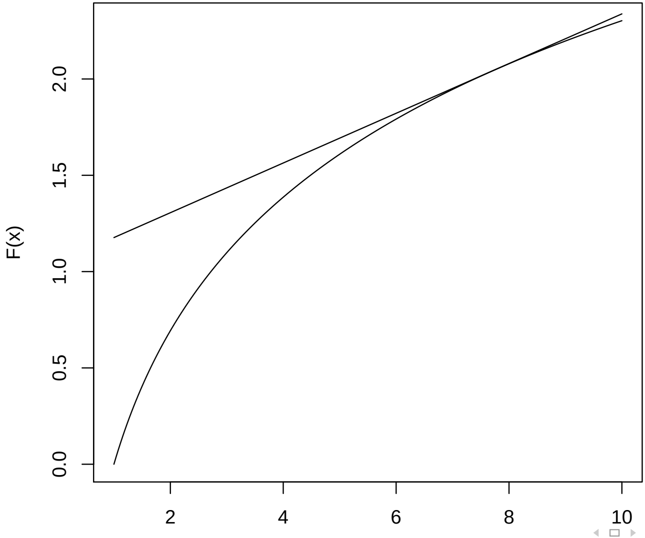

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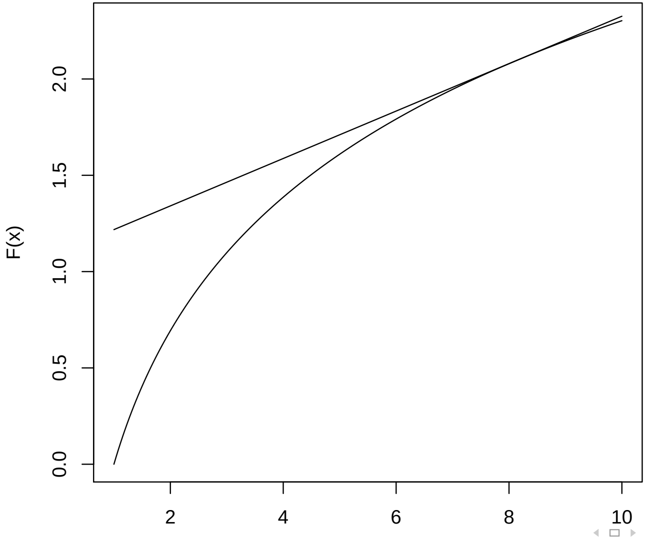

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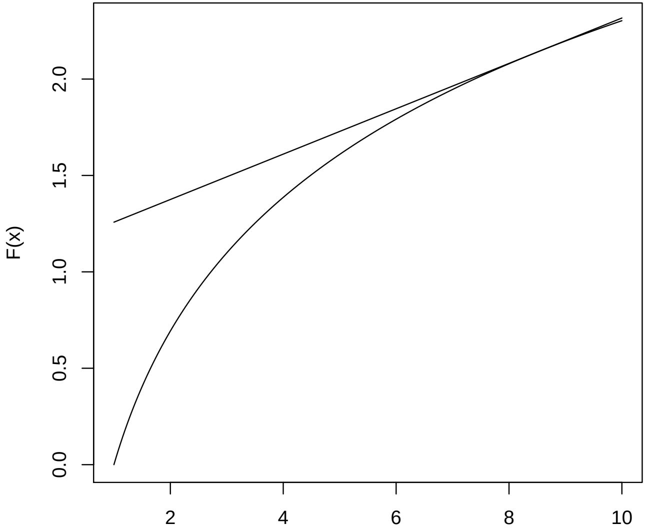

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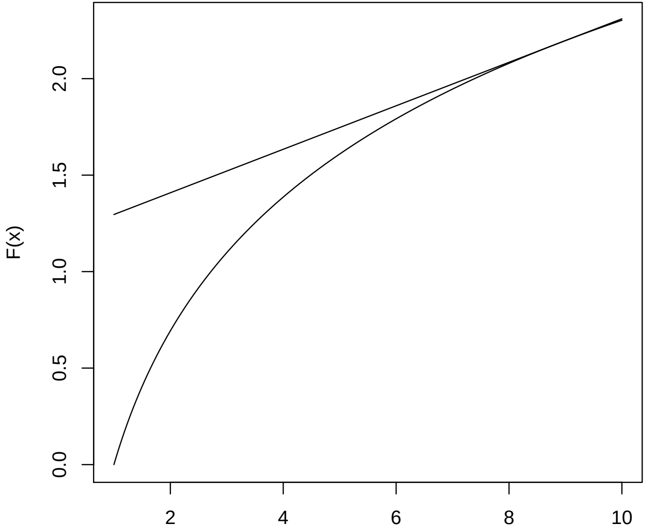

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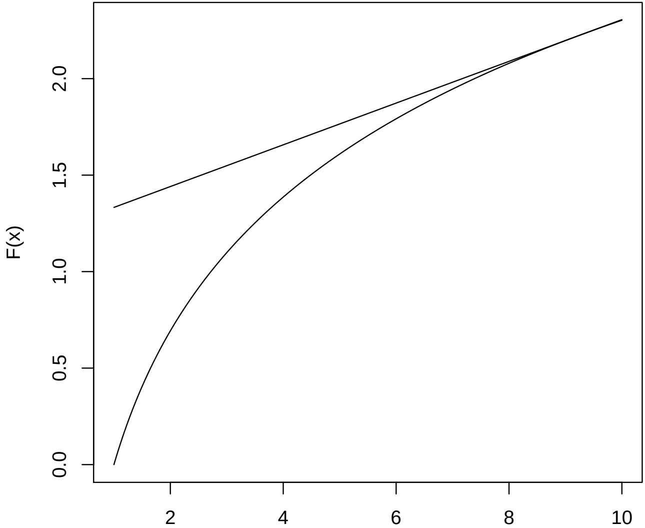

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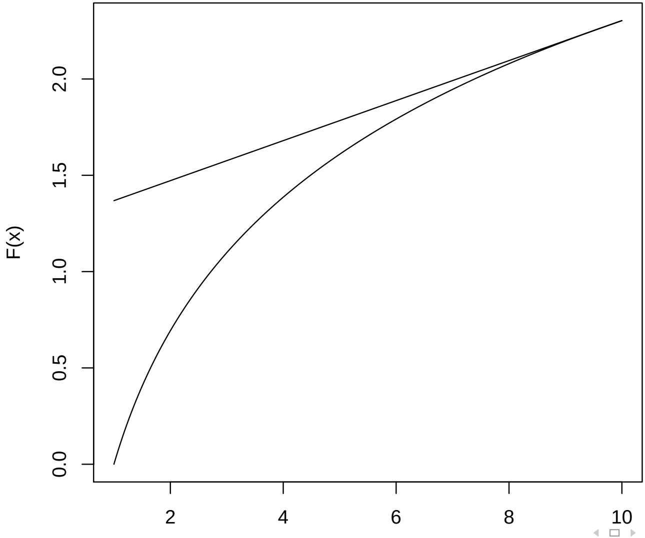

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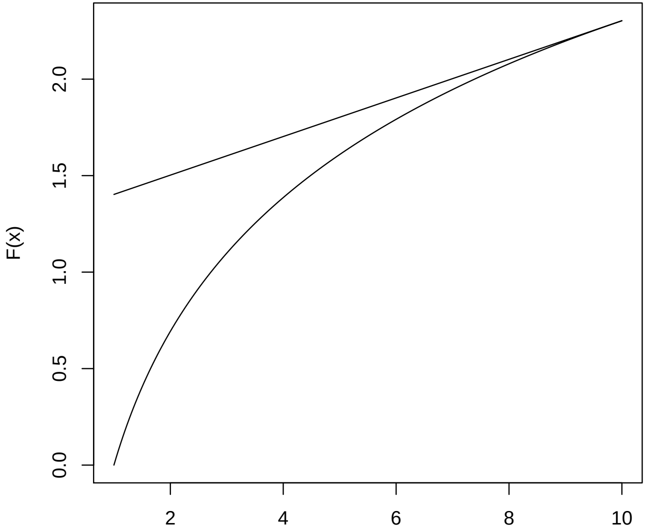

Tangent Line

Formula for Tangent line at x_0:

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$$g(x) = f^{'}(x_0)(x - x_0) + f(x_0)$$

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$$g(x) \quad = \quad f^{''}(x_0)(x - x_0) + f^{'}(x_0)$$

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$$\begin{aligned} g(x) &= f^{''}(x_0)(x - x_0) + f^{'}(x_0) \\ 0 &= f^{''}(x_0)(x_1 - x_0) + f^{'}(x_0) \end{aligned}$$

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Perform iteratively until change in $|x_{t+1} - x_t| <$ threshold

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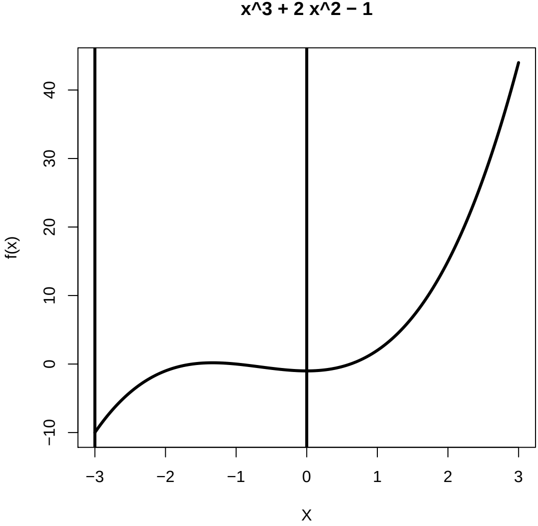
Example Function

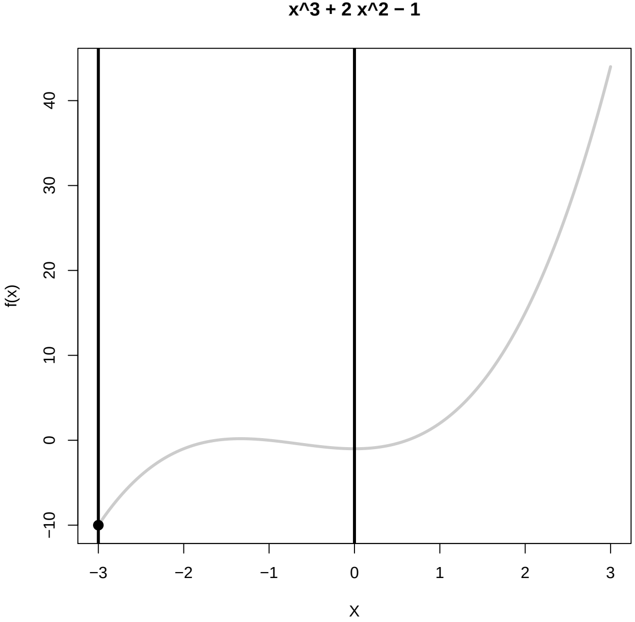
$f(x) = x^3 + 2x^2 - 1$ find x that maximizes $f(x)$ with $x \in [-3, 0]$



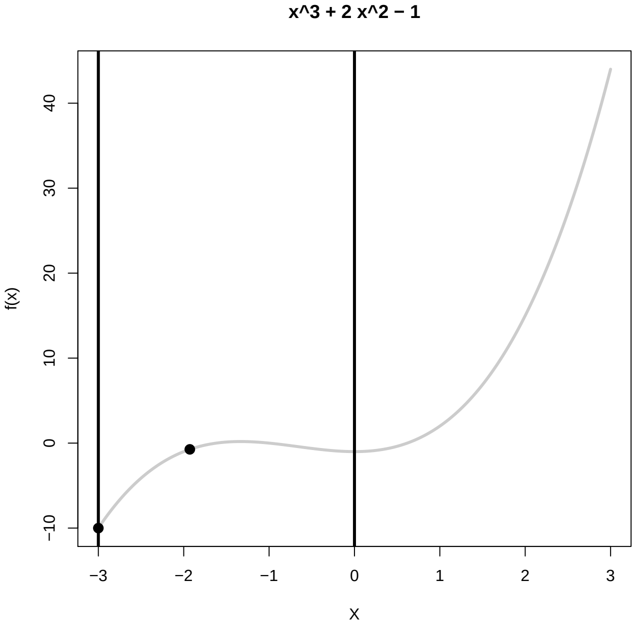
$$
\begin{aligned}
f^{'}(x) &= 3x^2 + 4x \\
f^{''}(x) &= 6x + 4
\end{aligned}
$$

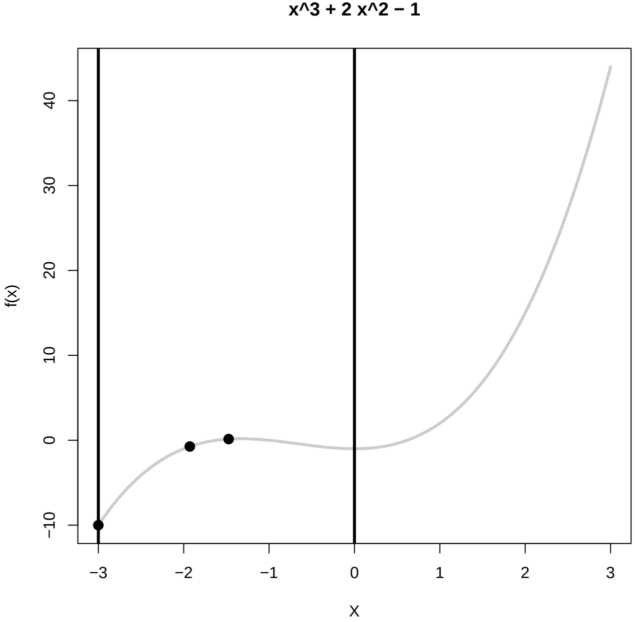
Suppose we have guess x_t then the next step is:

$$
x_{t+1} = x_t - \frac{3x_t^2 + 4x_t}{6x_t + 4}
$$



x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
f(x)
−10
0
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20
30
40
−3
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2
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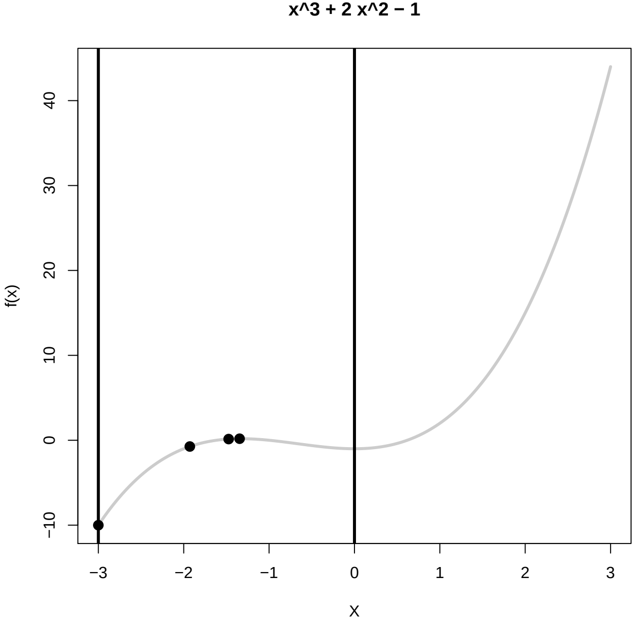
X


x^3 + 2 x^2 − 1
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40
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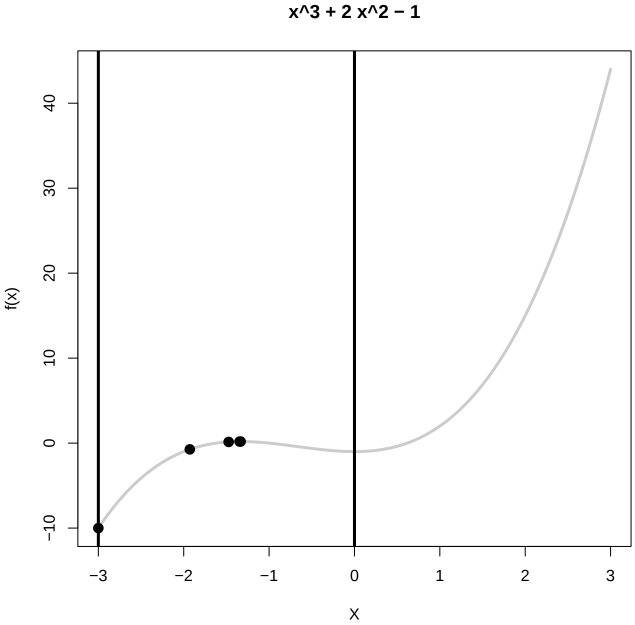
x^3 + 2 x^2 − 1
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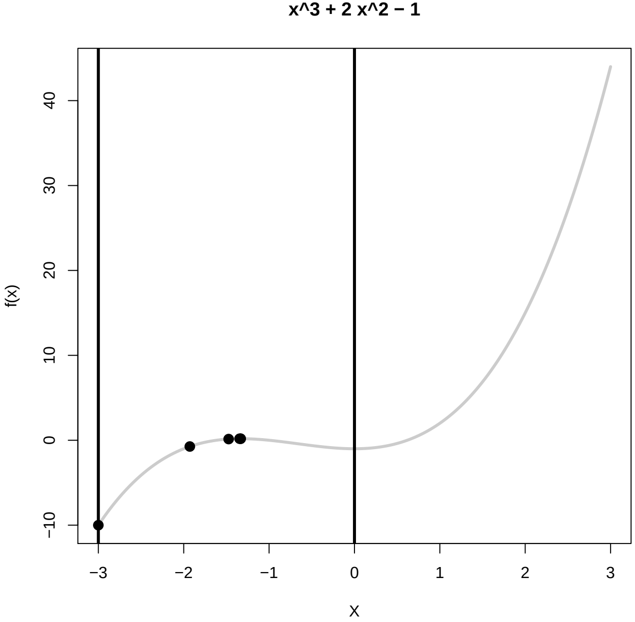
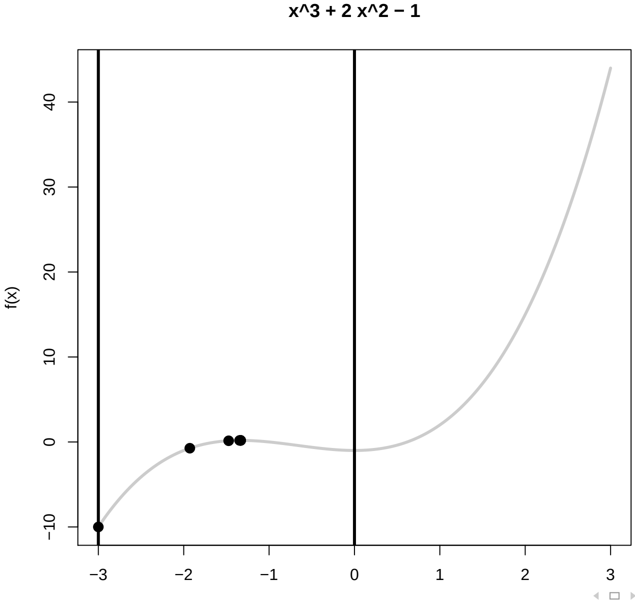
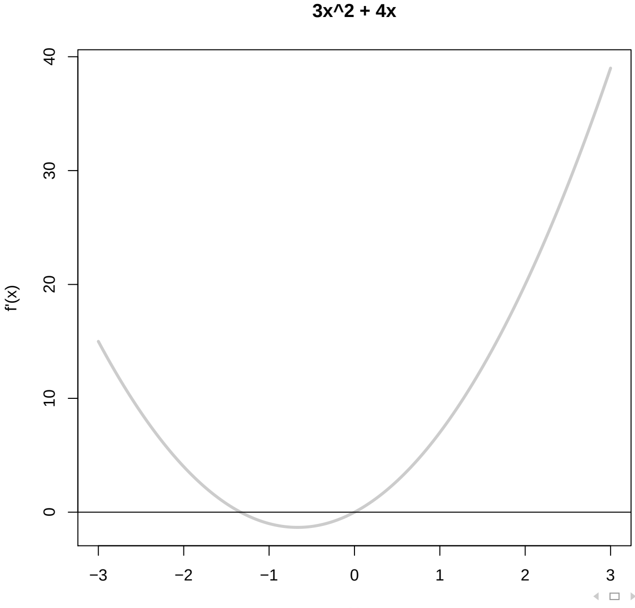

x^3 + 2 x^2 − 1
f(x)
40
30
20
10
0
−10
−3
−2
−1
0
1
2
3
X

$x^* = -1.3333$



What is Happening with the Roots



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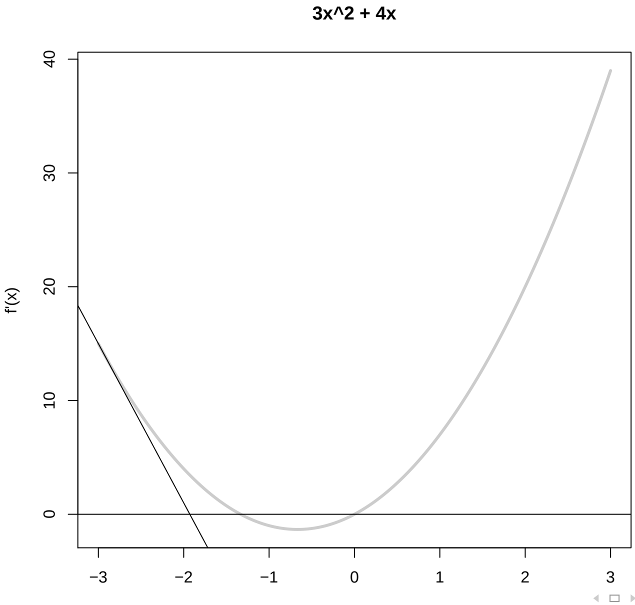

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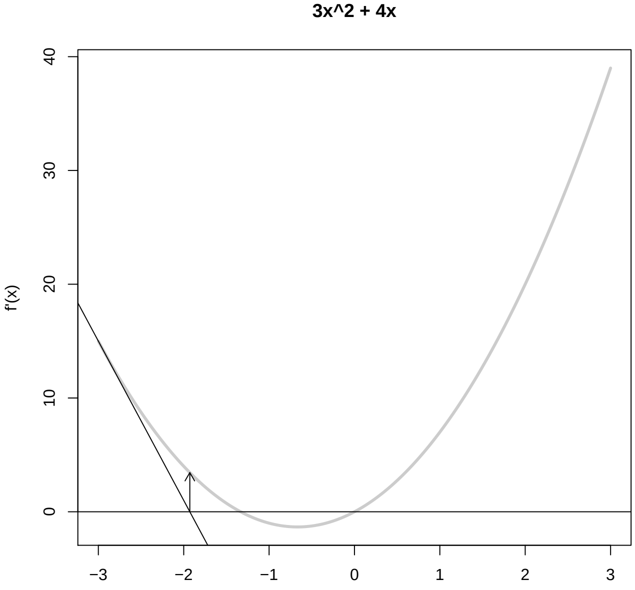

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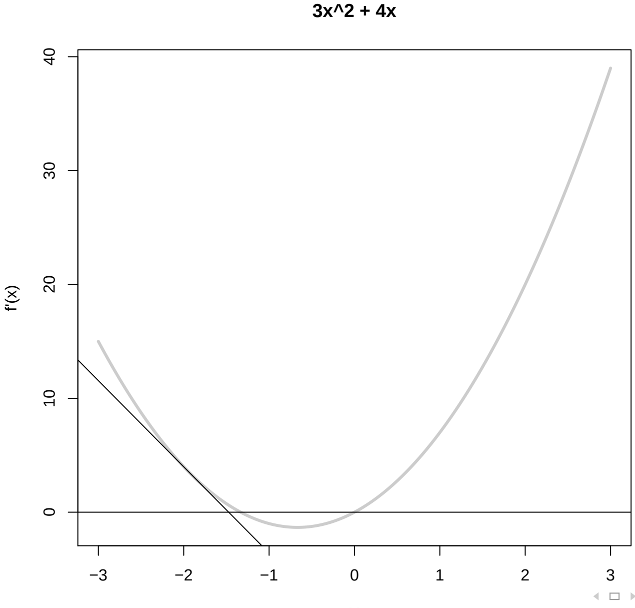

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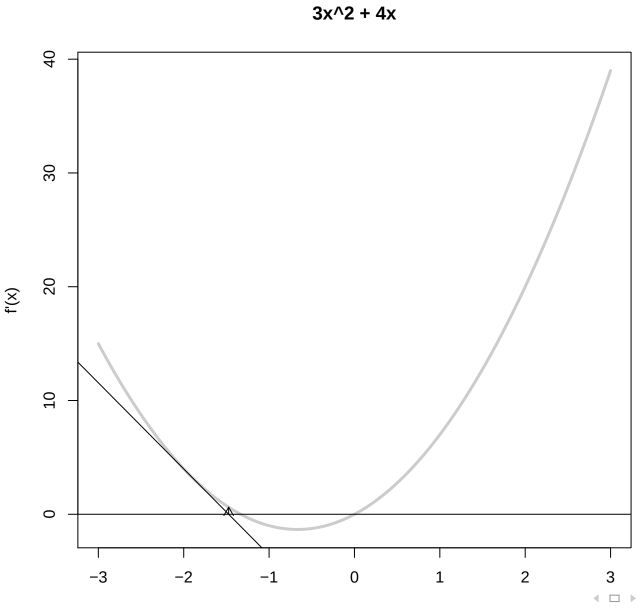

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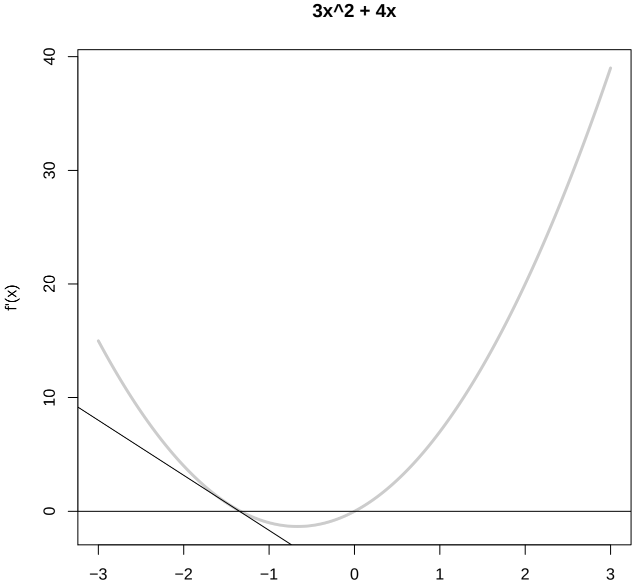

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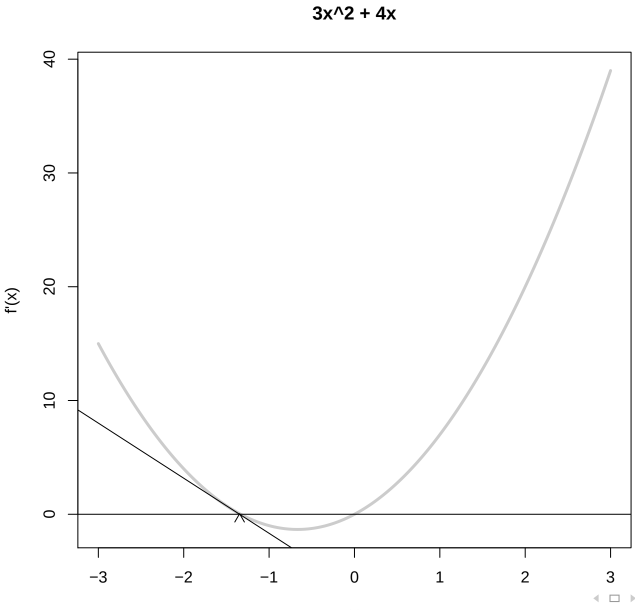

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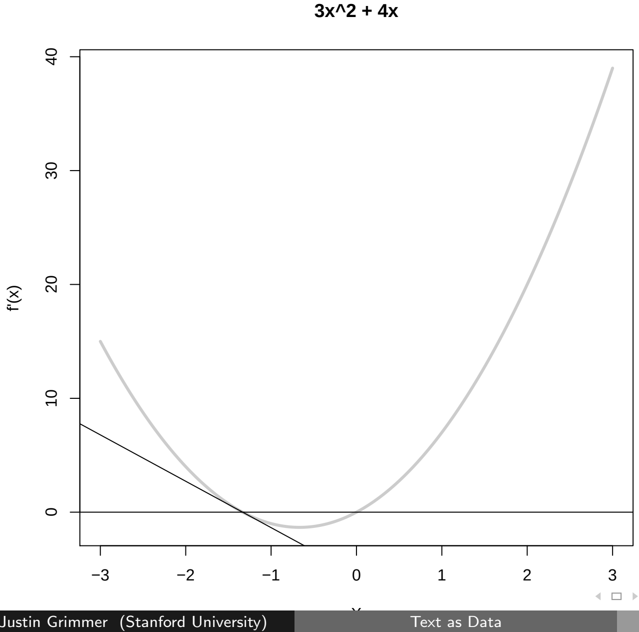

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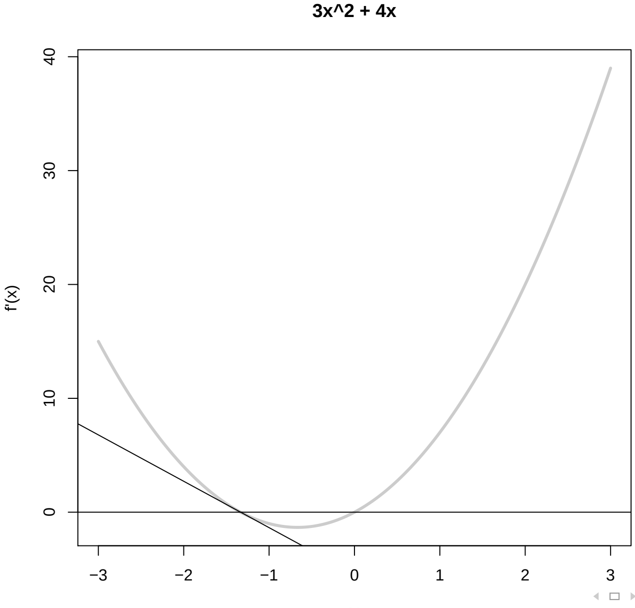

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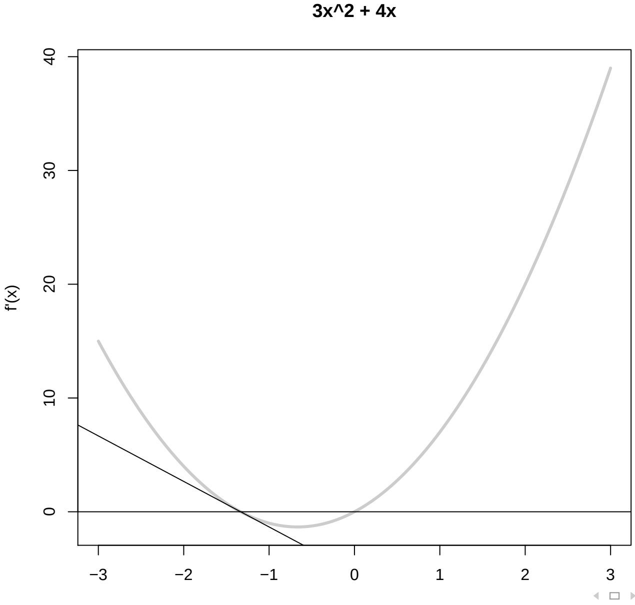

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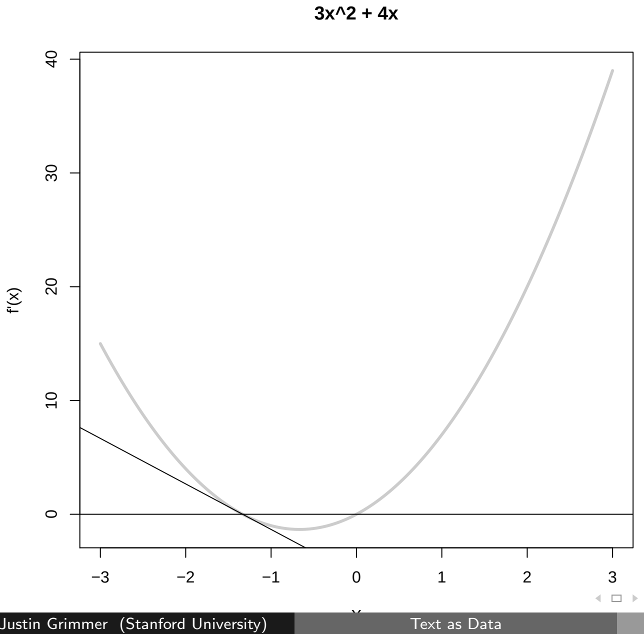

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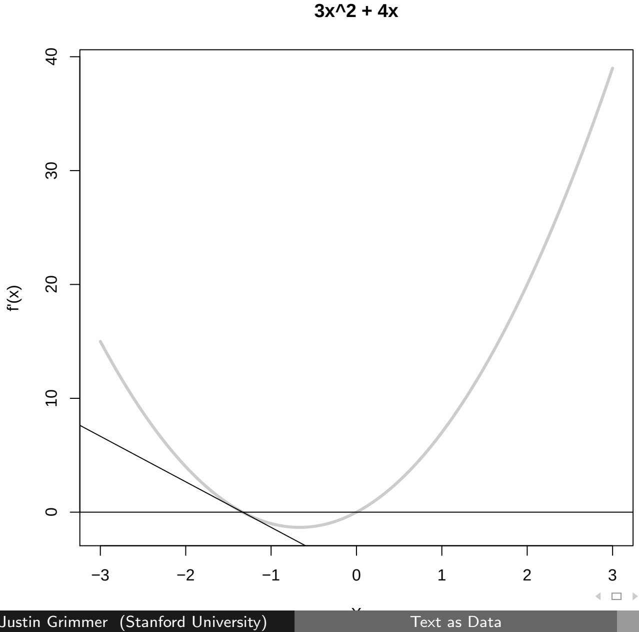

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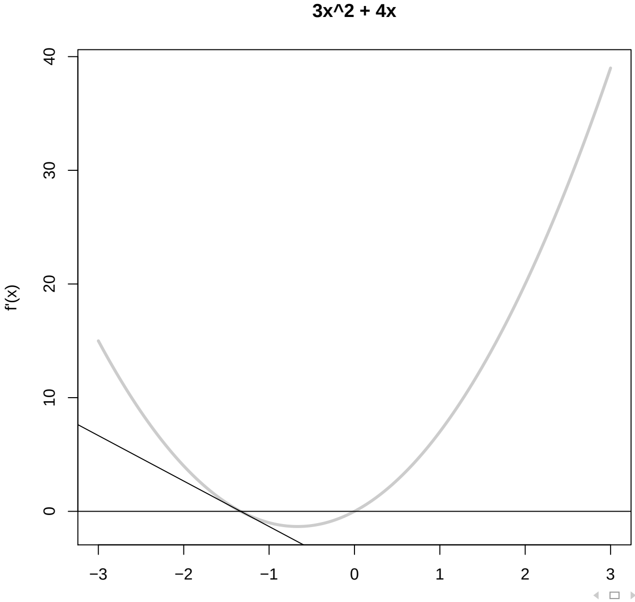

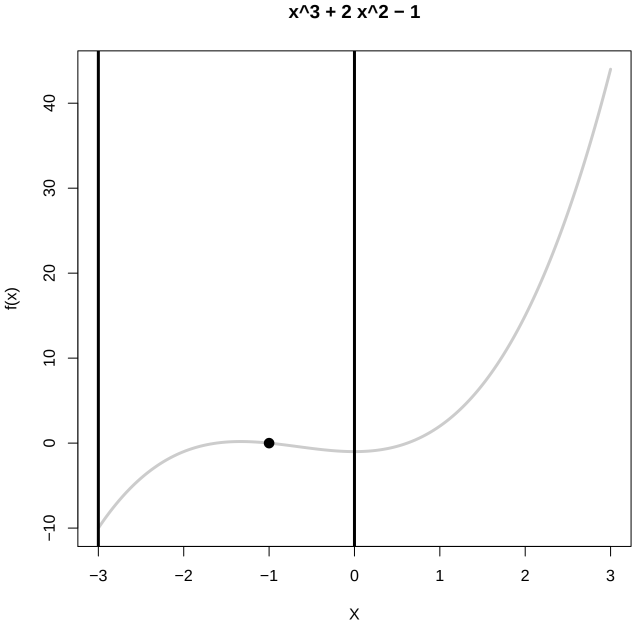
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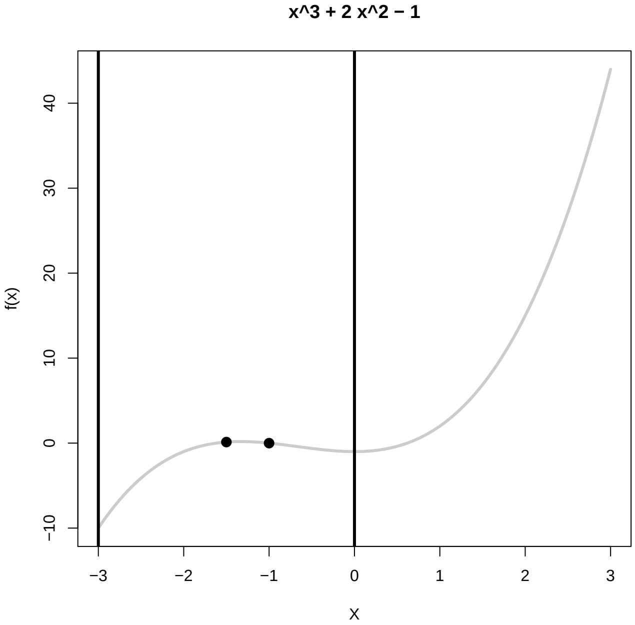
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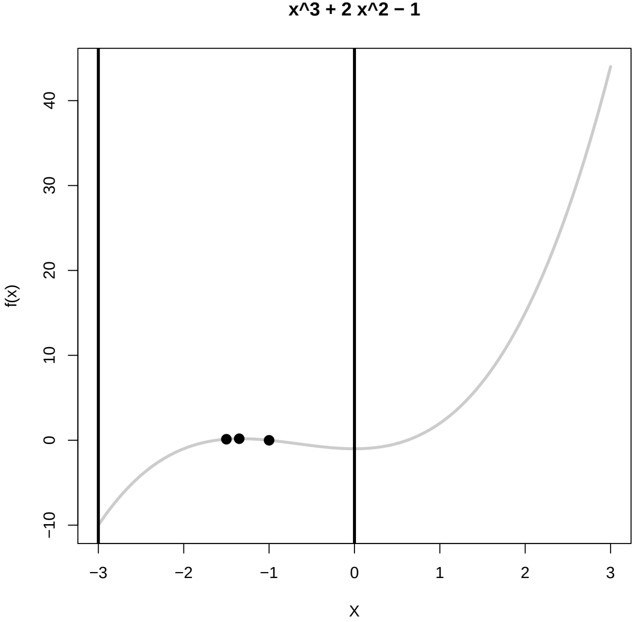
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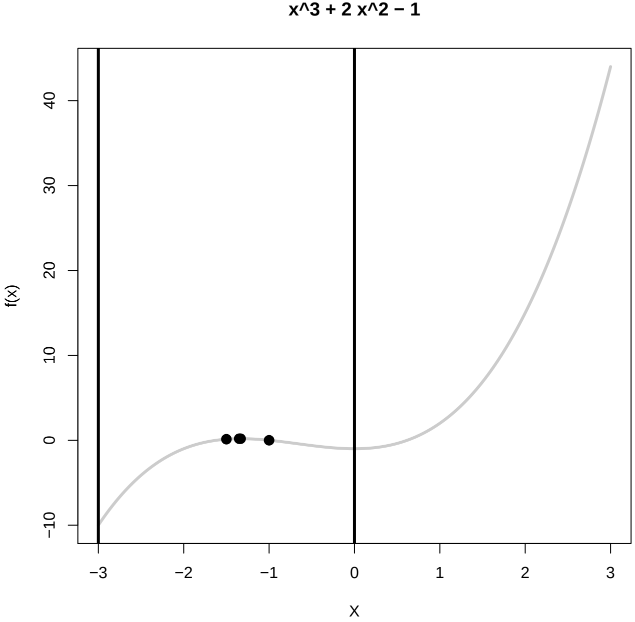
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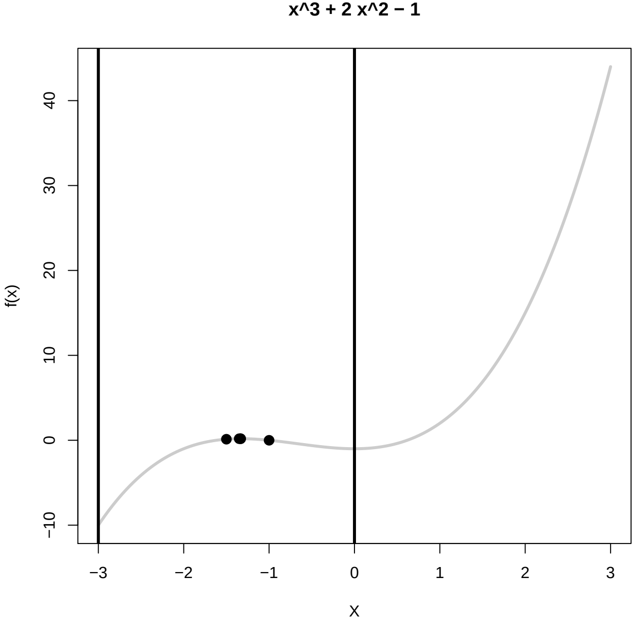
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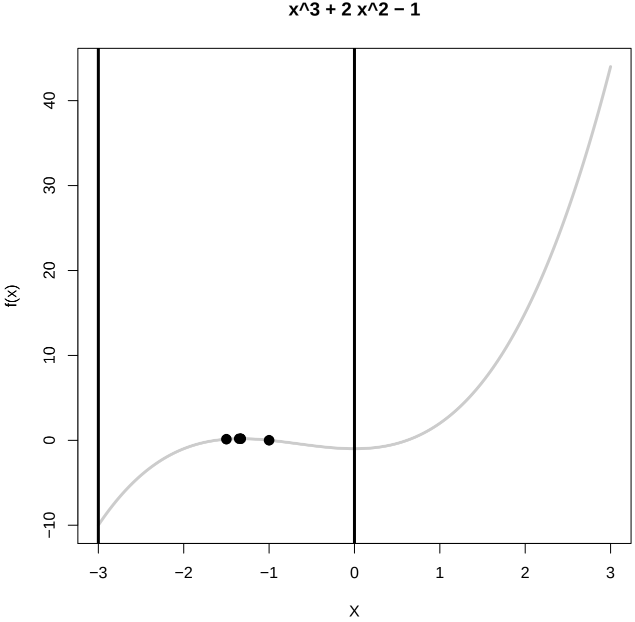
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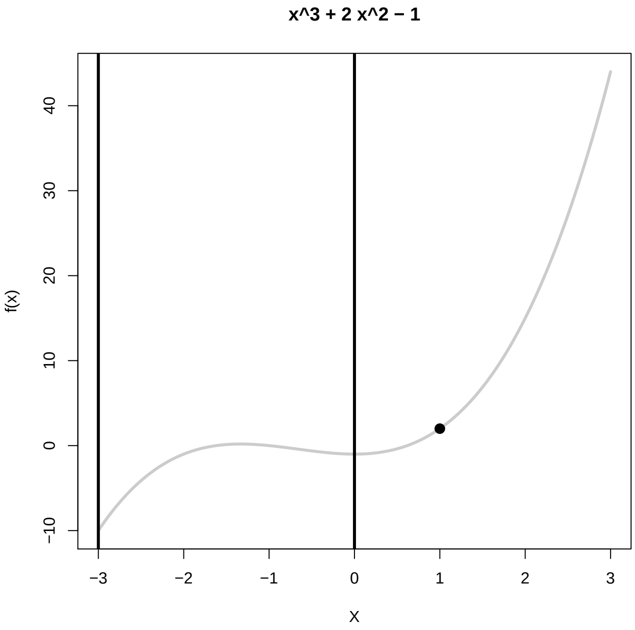
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X
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0
10
20
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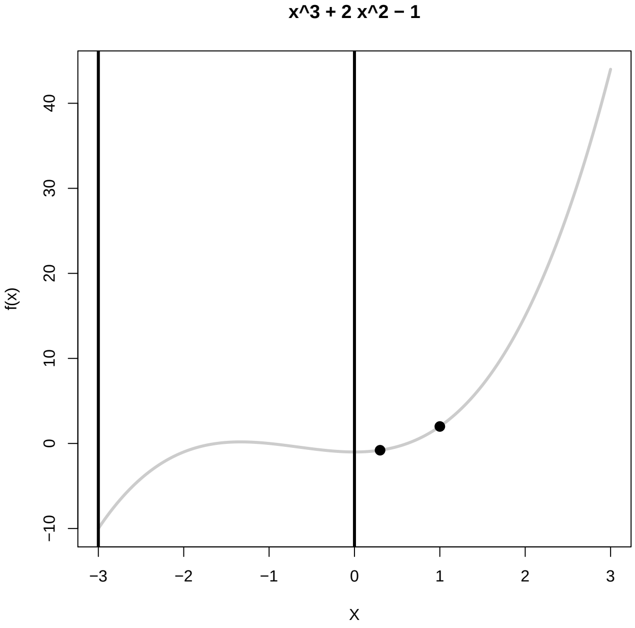
40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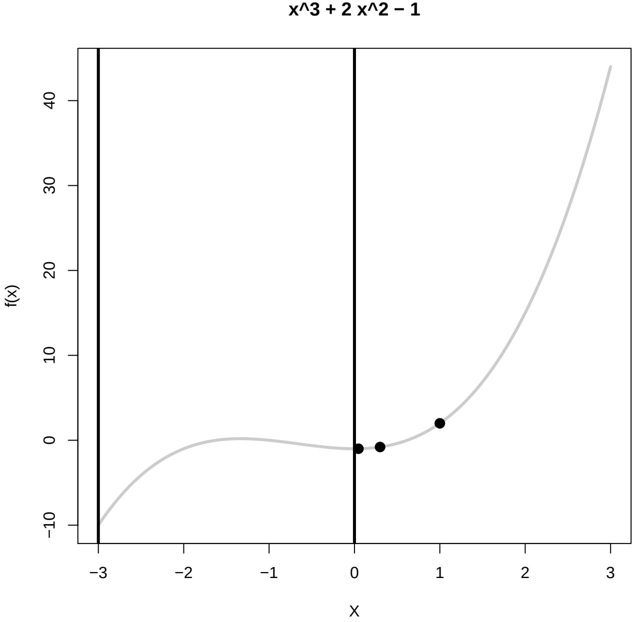
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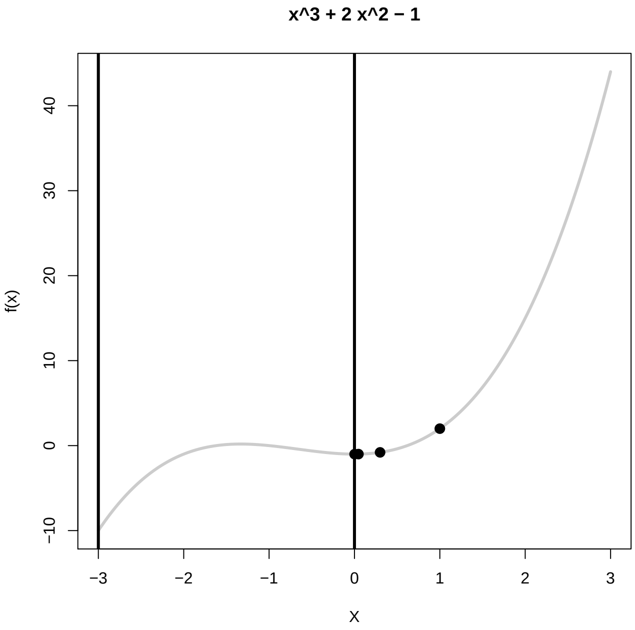
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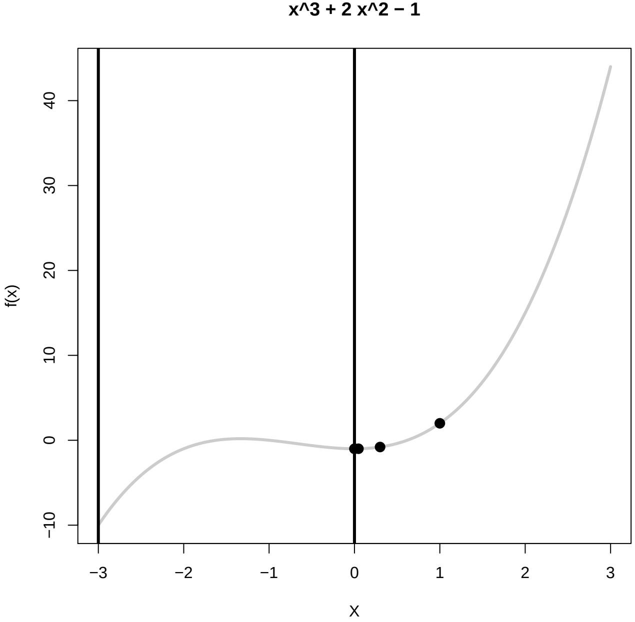
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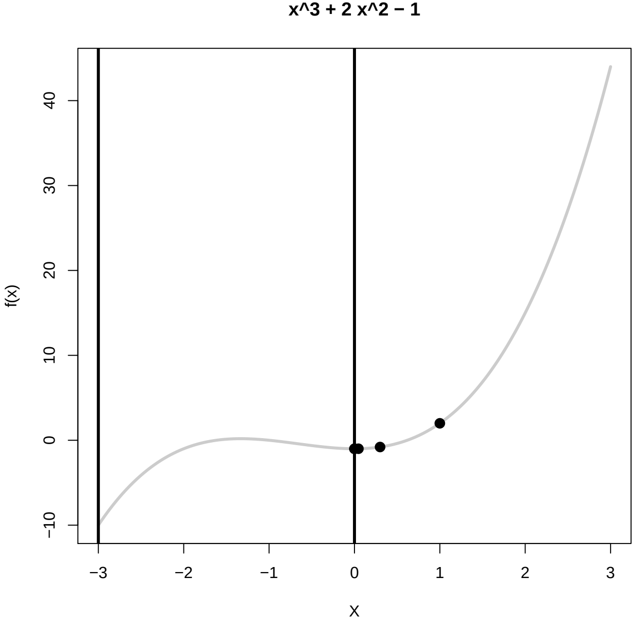
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0
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X

Multivariate Optimization

$$\log\left(p(\beta|\boldsymbol{y},\boldsymbol{X})\right) = \sum_{i=1}^{N}\left[y_i \log(\Phi(\boldsymbol{X}_i\beta)) + (1-y_i)\log(1-\Phi(\boldsymbol{X}_i\beta))\right] + c$$

To do so:
Apply BFGS (quasi-Newton) in `R`, in `optim`
`R code`

Multivariate Optimization

$$\log\left(p(\beta|\boldsymbol{y}, \boldsymbol{X})\right) \quad = \quad \sum_{i=1}^{N} \left[y_i \log(\Phi(\boldsymbol{X}_i\beta)) + (1-y_i)\log(1-\Phi(\boldsymbol{X}_i\beta))\right] + c$$

To do so:

Apply BFGS (quasi-Newton) in `R`, in `optim`

`R code`

Estimates: predict, classify, describe, ...

Probit Regression, with Prior

Consider the following data generation process

$$
\begin{aligned}
Y_i &\sim \text{Bernoulli}(\pi_i) \\
\pi_i &= \Phi(\boldsymbol{X}_i \boldsymbol{\beta}) \\
\beta_j &\sim \text{Normal}(\mu, \sigma^2)
\end{aligned}
$$

$$
\begin{aligned}
p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}, \mu, \sigma^2) &\propto p(\boldsymbol{\beta}|\mu, \sigma^2) \times p(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{\beta}) \\
&\propto \prod_{j=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\beta_j - \mu)^2}{2\sigma^2}\right) \times \prod_{i=1}^{N} \Phi(\boldsymbol{X}_i \boldsymbol{\beta})^{y_i}(1 - \Phi(\boldsymbol{X}_i \boldsymbol{\beta}))
\end{aligned}
$$

Homework $\rightsquigarrow$ explore behavior of $\widehat{\beta}$ as μ, σ^2 vary.

Where We're Going

1) Task
2) Objective Function
3) Optimization procedure

All supposes we have data.
Next week $\rightsquigarrow$ converting text data