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Monotone Comparative Statics[1]

1 Overview

Given an optimization problem indexed by some parameter θ, comparative statics seeks a qualitative understanding of how the solution changes with θ. If, for example, wages decrease, does a firm hire more labor?

One way to obtain results of this type is to assume (or use the Implicit Function theorem to establish the existence of) a differentiable solution function, and then, having substituted the solution function into the first order condition, apply the Chain Rule to the first order condition to try to determine the sign of the derivatives of the solution function.

These notes briefly survey an alternative approach. Relative to the calculus-based approach, the approach described here has certain advantages. It makes assumptions that are weaker, either necessary or close to necessary. In particular, the results below can be applied in settings where differentiability, or even continuity, of the solution function cannot be assumed. The approach below yields arguments that are often remarkably concise. And those arguments are often also relatively transparent.

Work in this area was pioneered by Topkis in the 1970s. Over the subsequent two decades, economists gradually became persuaded that Topkis's approach was the right one for handling certain types of problems in economics. Key papers on comparative statics include Topkis (1978), Milgrom and Roberts (1990*a*), Milgrom and Shannon (1994), Athey (2002), and Quah and Strulovici (2009). Topkis (1998) is the standard general reference. There is a related literature analyzing equilibria, especially equilibria in games. Key papers there include Topkis (1979), Vives (1990), Milgrom and Roberts (1990*b*), and Milgrom and Roberts (1994). Finally, the lattice machinery that is characteristic of these literatures has proved important in some applications in competitive general equilibrium theory; see Aliprantis and Brown (1982) and Mas-Colell (1986).

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Thanks to Asaf Plan for spotting an error, now corrected, in an earlier version.

2 The Basic Montonicity Result

The material in this section and the next owes to Topkis. Topkis (1998) provides a comprehensive overview of the state of the art at the time of its publication.

To simplify notation, I take function domains to be all of some $\mathbb{R}^K$. But the definitions and theorems below extend immediately to functions whose domains are "nice" subsets of $\mathbb{R}^K$. $\mathbb{R}^K_+$ is an example of a nice subset.

To begin with, let $K = 2$ and express $z \in \mathbb{R}^2$ in the form $z = (x, \theta)$, where $x, \theta \in \mathbb{R}$. Think of x as a decision variable and θ as a parameter. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, say that f is *supermodular* iff, whenever $x^\circ \leq x^*$ and $\theta^\circ \leq \theta^*$,

$$f(x^*, \theta^*) - f(x^\circ, \theta^*) \geq f(x^*, \theta^\circ) - f(x^\circ, \theta^\circ).$$

That is, fixing $x^\circ \leq x^*$, the function

$$g(\theta) = f(x^*, \theta) - f(x^\circ, \theta),$$

which measures the benefit (which could be negative) of switching from x° to x^*, is weakly increasing in θ. Informally, supermodularity expresses complementarity between x and θ: an increase in θ increases the benefit of increasing x. f is *submodular* iff $-f$ is supermodular. In these notes, I focus primarily on supermodular functions.

In practice, it is frequently the case in applications that f is differentiable. In this case, there is an easy characterization, due to Topkis, of supermodularity in terms of derivatives.

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^2$. Then the following are equivalent.*

1. *f is supermodular.*
2. *$D_x f(x^*, \theta^*)$ is weakly increasing in θ, for every x^*.*
3. *$D_\theta f(x^*, \theta^*)$ is weakly increasing in x, for every θ^*.*
4. *$D^2_{\theta,x} f(x^*, \theta^*) \geq 0$ for every (x^*, θ^*).*

Proof. It is routine that (4) implies (2) and (3). To see that (2) implies (1), note that if $\theta^* \geq \theta^\circ$ and $x^* \geq x^\circ$ then (2) implies

$$\begin{aligned} f(x^*, \theta^*) - f(x^\circ, \theta^*) &= \int_{x^\circ}^{x^*} D_x f(y, \theta^*)\, dy \\ &\geq \int_{x^\circ}^{x^*} D_x f(y, \theta^\circ)\, dy \\ &= f(x^*, \theta^\circ) - f(x^\circ, \theta^\circ) \end{aligned}$$

Finally, to see that (1) implies (4), set $x^* = x^\circ + \delta$, $\delta > 0$. Then the definition of supermodularity implies, dividing both sides by δ and taking the limit as $\delta \downarrow 0$, that $D_x f(x^\circ, \theta^*) \geq D_x f(x^\circ, \theta^\circ)$. Thus $D_x f$ is weakly increasing in θ, for all $\theta \in \mathbb{R}$, which implies (4). ■

Example 1. Suppose $f : \mathbb{R}^2_{++} \to \mathbb{R}$ is defined by

$$f(x, \theta) = (x^\rho + \theta^\rho)^{1/\rho}$$

where $\rho \in \mathbb{R}$, $\rho \neq 0$. Then you can check that f is supermodular if $\rho \leq 1$ and submodular if $\rho \geq 1$. □

Theorem 2 below states that if f is supermodular then the solution $\phi(\theta)$ to the problem

$$\max_{x \in C} f(x, \theta)$$

is weakly increasing in θ. To make sense of this in the case where the solution $\phi(\theta)$ is a set, I need a way to order sets.

Given two sets $S^*, S^\circ \subseteq \mathbb{R}$, write $S^* \geq S^\circ$ iff for any $x^\circ \in S^\circ$ and any $x^* \in S^*$, if $x^\circ \geq x^*$ then $x^\circ \in S^*$ and $x^* \in S^\circ$. This order on subsets of $\mathbb{R}$ is the *strong set order*. In the strong set order, if $S^* \geq S^\circ$ then the union of S° and S^* consists of three sets (some of which could be empty): (a) points in S^* that are larger than any point in S°, (b) points in both sets, and (c) points in S° that are smaller than any point in S^*. In the special case that S° and S^* are singletons, $S^\circ = \{s^\circ\}$ and $S^* = \{s^*\}$, then $S^* \geq S^\circ$ iff $s^* \geq s^\circ$.

Theorem 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, let $C \subseteq \mathbb{R}$, and for each $\theta \in \mathbb{R}$, let $\phi(\theta)$ be the set of solutions, assumed non-empty, to the problem*

$$\max_{x \in C} f(x, \theta).$$

If f is supermodular then ϕ is weakly increasing: for any θ^*, θ°, if $\theta^* \geq \theta^\circ$ then $\phi(\theta^*) \geq \phi(\theta^\circ)$ in the strong set order.

Proof. The following proof is not the simplest possible but has the advantage that it generalizes easily; it is the template for the proofs to follow. Take any $\theta^\circ < \theta^*$. Let $x^\circ \in \phi(\theta^\circ)$, $x^* \in \phi(\theta^*)$. If $x^* \geq x^\circ$ then there is nothing to show. Suppose that $x^\circ > x^*$. I must show that $x^\circ \in \phi(\theta^*)$ and $x^* \in \phi(\theta^\circ)$.

To see that $x^\circ \in \phi(\theta^*)$, note that, by definition of ϕ, $f(x^\circ, \theta^\circ) \geq f(x^*, \theta^\circ)$, hence,

$$f(x^\circ, \theta^\circ) - f(x^*, \theta^\circ) \geq 0.$$

Since $x^* > x^\circ$, $\theta^* > \theta^\circ$, and f is supermodular,

$$f(x^\circ, \theta^*) - f(x^*, \theta^*) \geq f(x^\circ, \theta^\circ) - f(x^*, \theta^\circ).$$

Combining, $f(x^\circ, \theta^*) - f(x^*, \theta^*) \geq 0$, or

$$f(x^\circ, \theta^*) \geq f(x^*, \theta^*),$$

hence $x^\circ \in \phi(\theta^*)$ as was to be shown.

The proof that $x^* \in \phi(\theta^\circ)$ is similar. By definition of ϕ, $f(x^*, \theta^*) \geq f(x^\circ, \theta^*)$, hence

$$0 \geq f(x^*, \theta^*) - f(x^\circ, \theta^*).$$

By supermodularity,

$$f(x^\circ, \theta^*) - f(x^*, \theta^*) \geq f(x^\circ, \theta^\circ) - f(x^*, \theta^\circ).$$

Combining, $0 \geq f(x^\circ, \theta^\circ) - f(x^*, \theta^\circ)$, or

$$f(x^*, \theta^\circ) \geq f(x^\circ, \theta^\circ),$$

which implies $x^* \in \phi(\theta^\circ)$, as was to be shown. ■

If C is compact and f is continuous then one can show that $\phi(\theta)$ is a compact subset of C, so that, in particular, $\phi(\theta)$ has a largest and smallest element. In this case, Theorem 2 implies that the largest and smallest elements of $\phi(\theta)$ are both weakly increasing in θ.

Example 2. Consider a monopoly facing an inverse demand function $P : \mathbb{R}_+ \to \mathbb{R}$ giving price as a function of quantity. If the firm has constant marginal cost $c > 0$ then its profit is

$$\pi(x, c) = P(x)x - cx.$$

As a function of c, this is not supermodular, since $D_{xc}f(x, c) = -1 < 0$. But defining $\theta = -c$,

$$f(x, \theta) = P(x)x + \theta x$$

is supermodular, since $D_{x\theta}f(x, \theta) = 1 > 0$.

By Theorem 2, the set of profit maximizing x is weakly increasing in θ, hence weakly decreasing in c: a decrease in marginal cost implies an increase in output.

The traditional calculus approach goes as follows. The first order condition for the maximization problem is

$$0 = D_x\pi(x, c).$$

Let $\phi(c)$ denote the solution for c. Assume this is single valued and differentiable. Then for any c

$$0 = D_x\pi(\phi(c), c).$$

By the Chain Rule, $0 = D^2_{xx}\pi(x, c)D\phi(c) + D^2_{xc}\pi(x, c)$, hence

$$D\phi(c) = -\frac{D^2_{x,c}\pi(x, c)}{D^2_{xx}\pi(x, c)} = \frac{1}{D^2_{xx}\pi(x, c)}.$$

This will be negative (a decrease in marginal cost implies an increase in output), provided $D^2_{xx}\pi(x, \theta) < 0$, that is, provided π is differentially strictly concave in x. $\square$

In Example 2, the concavity assumption on π is obnoxious because it is equivalent to a concavity assumption on P, and there is no general reason to to think that inverse demand will be concave. In contrast, Theorem 2 makes no mention of concavity. The two approaches to comparative statics are not, however, quite as far apart as they might seem in this respect. If $x^* = \phi(c^*)$, then $\pi(x, c^*)$ cannot be locally strictly convex in x near x^*, since then x^* would be a local *minimum*. Thus, there is some sense in which π cannot be too far from being concave near (x^*, c^*), and in this sense Theorem 2 implicitly builds in a local concavity-like condition at x^* simply by assuming that x^* is a solution. That said, the assumption that $D^2_{xx}\pi(x, c) < 0$ is clearly strong relative to what is used by Theorem 2.

3 A Generalized Monotonicity Result

The goal is to extend Theorem 2 to functions $f : \mathbb{R}^{N+M} \to \mathbb{R}$, with the decision variable $x \in \mathbb{R}^N$ and the parameter $\theta \in \mathbb{R}^M$.

If $N \geq 2$ then we face the following issue: if $x^* \in \phi(\theta^*)$ and $x^\circ \in \phi(\theta^\circ)$ then there is no general reason to expect *either* $x^* \geq x^\circ$ or $x^* \leq x^\circ$, even in extremely well behaved problems. Here, as usual, $x^* \geq x^\circ$ means that $x^*_n \geq x^\circ_n$ for each coordinate n. The two vectors could be like $(1, 0)$ and $(0, 1)$, which are non-comparable in the standard partial order on $\mathbb{R}^N$.

We deal with this problem as follows. For any x^* and x° in $\mathbb{R}^N$, write

$$x^* \vee x^\circ = (\max\{x^*_1, x^\circ_1\}, \ldots, \max\{x^*_N, x^\circ_N\})$$

and

$$x^* \wedge x^\circ = (\min\{x^*_1, x^\circ_1\}, \ldots, \min\{x^*_N, x^\circ_N\})$$

In particular, $x^* \vee x^\circ \geq x^*$ and x° and $x^* \wedge x^\circ \leq x^*$ and x°. Say that a set S is a *sublattice* of $\mathbb{R}^N$ iff for any $x^*, x^\circ \in S$, $x^* \vee x^\circ \in S$ and $x^* \wedge x^\circ \in S$.

Given a set $B \subseteq \mathbb{R}^N$, let $\overline{b} = \sup B$, if this point exists, and $\underline{b} = \inf B$, if this point exists.

For $K \geq 2$, a function $f : \mathbb{R}^K \to \mathbb{R}$ is *supermodular* iff for any $z^*, z^\circ \in \mathbb{R}^K$,

$$f(z^* \vee z^\circ) - f(z^*) \geq f(z^\circ) - f(z^* \wedge z^\circ).$$

f is *submodular* if $-f$ is supermodular.

If $K = 2$ then this version of supermodularity is equivalent to the previous one. Explicitly, let $x = z_1$ and let $\theta = z_2$. Suppose that $x^* \geq x^\circ$ and $\theta^* \geq \theta^\circ$. If $z^* = (x^\circ, \theta^*)$, $z^\circ = (x^*, \theta^\circ)$ then $z^* \vee z^\circ = (x^*, \theta^*)$ and $z^* \wedge z^\circ = (x^\circ, \theta^\circ)$ and supermodularity requires,

$$f(x^*, \theta^*) - f(x^\circ, \theta^*) \geq f(x^*, \theta^\circ) - f(x^\circ, \theta^\circ),$$

which is exactly the condition from before. (If $z^* = (x^\circ, \theta^*)$, $z^\circ = (x^*, \theta^\circ)$, then one gets the equivalent inequality, $f(x^*, \theta^*) - f(x^*, \theta^\circ) \geq f(x^\circ, \theta^*) - f(x^\circ, \theta^\circ)$. If $z^* = (x^*, \theta^*)$, $z^\circ = (x^\circ, \theta^\circ)$ then supermodularity just says $0 \geq 0$.)

One can show, although I will not do so explicitly, that Theorem 1 extends to this case. Explicitly, if f is $\mathcal{C}^2$ then f is supermodular iff for every z, all cross partials are non-negative: for $k \neq k'$,

$$D^2_{z_k z_{k'}} f(z) \geq 0$$

It turns out that the full strength of supermodularity is not needed for Theorem 3, which is the generalization of Theorem 2. Writing $z = (x, \theta)$, with $x \in \mathbb{R}^N$, $\theta \in \mathbb{R}^M$, and $K = N + M$, Theorem 3 assumes that f is supermodular in x (i.e., supermodular holding the θ coordinates fixed) and exhibits *increasing differences* in (x, θ), where the latter means that for any $x^*, x^\circ \in \mathbb{R}^N$ and any $\theta^*, \theta^\circ \in \mathbb{R}^M$, if $x^* \geq x^\circ$ and $\theta^* \geq \theta^\circ$, then

$$f(x^*, \theta^*) - f(x^\circ, \theta^*) \geq f(x^*, \theta^\circ) - f(x^\circ, \theta^\circ).$$

If $N = 1$ then any f is supermodular in x.

Example 3. Suppose $f : \mathbb{R}^3_{++} \to \mathbb{R}$ is defined by

$$f(x, \theta_1, \theta_2) = x(\theta_1^\rho + \theta_2^\rho)^{1/\rho},$$

where $\rho \in \mathbb{R}$, $\rho \neq 0$. Then f is supermodular in x (x is one dimensional) and satisfies increasing differences in (x, θ) ($D^2_{x\theta_m} > 0$), but if $\rho > 1$ then it is not supermodular in θ. $\square$

Finally, given two sets $S^*, S^\circ \subseteq \mathbb{R}^N$, write $S^* \geq S^\circ$ iff for any $x^\circ \in S^\circ$ and any $x^* \in S^*$, $x^* \vee x^\circ \in S^*$ and $x^* \wedge x^\circ \in S^\circ$. This order on subsets of $\mathbb{R}^N$ is the *strong set order*. When $N = 1$, this definition of the strong set order is equivalent to the one given earlier.

Theorem 3. *Let $f : \mathbb{R}^{N+M} \to \mathbb{R}$, let C be a sublattice of $\mathbb{R}^N$, and for each $\theta \in \mathbb{R}^M$, let $\phi(\theta)$ be the set of solutions, assumed non-empty, to the problem*

$$\max_{x \in C} f(x, \theta).$$

If f is supermodular in x and exhibits increasing differences in (x, θ) then ϕ is weakly increasing: for any θ^*, θ°, if $\theta^* \geq \theta^\circ$ then $\phi(\theta^*) \geq \phi(\theta^\circ)$ in the strong set order.

Proof. Consider any $\theta^*, \theta^\circ \in \mathbb{R}^M$, with $\theta^* \geq \theta^\circ$. Consider any $x^* \in \phi(\theta^*)$ and $x^\circ \in \phi(\theta^\circ)$. Note that, since C is a sublattice of $\mathbb{R}^N$, $x^* \vee x^\circ, x^* \wedge x^\circ \in C$. I must show that $x^* \vee x^\circ \in \phi(\theta^*)$ and $x^* \wedge x^\circ \in \phi(\theta^\circ)$.

To see that $x^* \vee x^\circ \in \phi(\theta^*)$, note that by definition of ϕ,

$$f(x^\circ, \theta^\circ) \geq f(x^\circ \wedge x^*, \theta^\circ),$$

hence

$$f(x^\circ, \theta^\circ) - f(x^\circ \wedge x^*, \theta^\circ) \geq 0.$$

By increasing differences, since $x^\circ \geq x^\circ \wedge x^*$ and $\theta^* \geq \theta^\circ$,

$$f(x^\circ, \theta^*) - f(x^\circ \wedge x^*, \theta^*) \geq f(x^\circ, \theta^\circ) - f(x^\circ \wedge x^*, \theta^\circ).$$

Since f is supermodular in x,

$$f(x^* \vee x^\circ, \theta^*) - f(x^*, \theta^*) \geq f(x^\circ, \theta^*) - f(x^\circ \wedge x^*, \theta^*).$$

Combining all this,

$$f(x^* \vee x^\circ, \theta^*) - f(x^*, \theta^*) \geq 0,$$

or

$$f(x^* \vee x^\circ, \theta^*) \geq f(x^*, \theta^*).$$

Therefore, since $x^* \in \phi(\theta^*)$, $x^* \vee x^\circ \in \phi(\theta^*)$, as was to be shown.

To see that $x^* \wedge x^\circ \in \phi(\theta^\circ)$, note that by definition of ϕ,

$$f(x^*, \theta^*) \geq f(x^* \vee x^\circ, \theta^*),$$

hence

$$0 \geq f(x^* \vee x^\circ, \theta^*) - f(x^*, \theta^*).$$

By increasing differences, since $x^* \vee x^\circ \geq x^*$ and $\theta^* \geq \theta^\circ$,

$$f(x^* \vee x^\circ, \theta^*) - f(x^*, \theta^*) \geq f(x^* \vee x^\circ, \theta^\circ) - f(x^*, \theta^\circ).$$

Since f is supermodular in x,

$$f(x^* \vee x^\circ, \theta^\circ) - f(x^*, \theta^\circ) \geq f(x^\circ, \theta^\circ) - f(x^* \wedge x^\circ, \theta^\circ).$$

Combining all this,

$$0 \geq f(x^\circ, \theta^\circ) - f(x^* \wedge x^\circ, \theta^\circ),$$

or

$$f(x^* \wedge x^\circ, \theta^\circ) \geq f(x^\circ, \theta^\circ).$$

Since $x^\circ \in \phi(\theta^\circ)$, this implies $x^* \wedge x^\circ \in \phi(\theta^\circ)$ as was to be shown. ■

A lattice $S \subseteq \mathbb{R}^N$ is a *complete sublattice* of $\mathbb{R}^N$ iff for every $B \subseteq S$, $\sup B$ and $\inf B$ exist in $\mathbb{R}^N$ and are contained in S. One can prove that if $S \subseteq \mathbb{R}^N$ is a complete sublattice of $\mathbb{R}^N$ then it is compact. One can then prove that if the constraint set C is a complete sublattice of $\mathbb{R}^N$ and f is continuous then $\phi(\theta)$ is a complete sublattice of C (and hence of $\mathbb{R}^N$) and in particular has a largest and smallest element. Theorem 3 implies that the largest and smallest elements of $\phi(\theta)$ are both weakly increasing in θ.

4 An Application: the Le Chatelier Principle

This section follows Milgrom and Roberts (1996). The Le Chatelier Principle compares the effects of a parameter change on a decision variable in two settings, one where other decision variables are fixed and another where those variable are free to adjust. The name "Le Chatelier Principle," was coined in Samuelson (1947) and alludes to Le Chatelier's Principle in chemistry.

Consider the effect of the change in the wage on a competitive firm. In this setting, the claim is that a decrease in the wage causes the firm to increase employment, and the effect is larger in the long-run, when the firm can also increase capital, than in the short-run, when capital is fixed. The goal in this section is to prove this claim.

Let $f : \mathbb{R}^3 \to \mathbb{R}$, with the decision variable $x \in \mathbb{R}^2$ and the parameter $\theta \in \mathbb{R}$. Assume that f is supermodular. The interpretation will be that x_1 is labor, x_2 is capital, θ indexes wages, and f measures profit. I'll be more explicit about θ and f in a moment. Let

$$\phi_S(x_2, \theta) = \text{argmax}_{x_1} f(x_1, x_2, \theta)$$

and let

$$\phi_L(\theta) = \text{argmax}_{x_1, x_2} f(x_1, x_2, \theta).$$

I assume that these solutions exist; if there are multiple solutions, set ϕ_1 and ϕ_2 equal to the largest solutions (which I assume exist; see also the comment following the proof of Theorem 3). ϕ_S is the short-run response to θ: x_1 adjusts but x_2 is held fixed. ϕ_L is the long-run response to θ: both x_1 and x_2 adjust.

Take any $\theta^* \geq \theta^\circ$. Let $(x_1^\circ, x_2^\circ) = \phi_L(\theta^\circ)$, $(x_1^*, x_2^*) = \phi_L(\theta^*)$. Since f is supermodular, Theorem 3 (and the discussion following it) implies that

$$(x_1^*, x_2^*) \geq (x_1^\circ, x_2^\circ),$$

and in particular that $x_2^* \geq x_2^\circ$.

Define

$$\tilde{x} = \phi_S(x_2^\circ, \theta^*).$$

and notice that $x_1^\circ = \phi_S(x_2^\circ, \theta^\circ)$ and $x_1^* = \phi_S(x_2^*, \theta^*)$. Then, since f is supermodular, Theorem 3 implies that

$$\phi_S(x_2^\circ, \theta^\circ) \leq \phi_S(x_2^\circ, \theta^*) \leq \phi_S(x_2^*, \theta^*),$$

or

$$x_1^\circ \leq \tilde{x}_1 \leq x_1^*.$$

In words, following an increase in θ, x_1 initially increases, and then x_1 increases by even more once x_2 adjusts as well.

As an application, consider a competitive firm with profit

$$\pi(x_1, x_2, w) = pg(x_1, x_2) - wx_1 - rx_2,$$

where p is output price, g is a production function, w is the wage, x_1 is labor, r is a capital rental price, and x_2 is capital.

Suppose that g is supermodular. As in Example 2, π is *not* supermodular because $D_{x_1 w}\pi = -1 < 0$. Therefore, set $\theta = -w$ and let

$$f(x_1, x_2, \theta) = pg(x_1, x_2) + \theta x_1 - rx_2.$$

Then f is supermodular in (x_1, x_2, θ). By the above inequalities, an increase in θ, and hence a decrease in w, causes the firm to hire more labor in the short run (with capital fixed at x_2°) and even more so in the long run when capital increases to x_2^*. Intuitively, supermodularity of g captures complementarity between labor and capital. The decrease in w induces the firm to hire more labor initially, which raises the productivity of capital (since labor and capital are complements) which leads the firm to hire more capital, which raises the productivity of labor (again since labor and capital are complements) which leads the firm to hire even more labor.

5 Extensions and Generalizations

5.1 Overview.

In Theorem 3, supermodularity is sufficient but not necessary and in fact cannot be necessary. It is a basic feature of optimization problems that the solution remains the same if the objective function is transformed by an increasing function. One can easily use such transformations to destroy supermodularity. This raises the question of whether one can find conditions that are in the same spirit as supermodularity but that are weaker and in particular ordinal, hence immune to the problem just noted. The question is of interest both as a purely mathematical exercise and also because there are circumstances in which weaker conditions might apply but supermodularity does not.

5.2 Single Crossing.

The best known generalization of Theorem 3 appears in Milgrom and Shannon (1994), which replaces supermodularity with an ordinal analog called quasisupermodularity and replaces increasing differences with an ordinal analog called the single crossing property.

Explicitly, $f : \mathbb{R}^{N+M} \to \mathbb{R}$ is *quasisupermodular* in $x \in \mathbb{R}^N$ iff for any $x^*, x^\circ \in \mathbb{R}^N$ and any $\theta \in \mathbb{R}^M$, then

$$f(x^\circ, \theta) - f(x^* \wedge x^\circ, \theta) \geq 0,$$

implies

$$f(x^* \vee x^\circ, \theta) - f(x^*, \theta) \geq 0,$$

with a strict inequality in the first implying a strict inequality in the second. It is immediate that supermodularity in x implies quasisupermodularity in x. If $N = 1$ then any f is quasisupermodular in x.

The function $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ satisfies *the single crossing property* in (x, θ) iff for any $x^*, x^\circ \in \mathbb{R}^N$ and any $\theta^*, \theta^\circ \in \mathbb{R}^M$, if $x^* \geq x^\circ$ and $\theta^* \geq \theta^\circ$, then

$$f(x^*, \theta^\circ) - f(x^\circ, \theta^\circ) \geq 0,$$

implies

$$f(x^*, \theta^*) - f(x^\circ, \theta^*) \geq 0,$$

with a strict inequality in the first implying a strict inequality in the second. Quasisupermodularity in (x, θ) (not just in x) implies single crossing.

It is immediate that increasing differences implies the single crossing property. Single crossing can be strictly weaker than increasing differences, as illustrated by the following example.

Example 4. Suppose that $M = N = 1$, that $\theta^* > \theta^\circ$, and that $f(x, \theta^\circ) = -x^2$ and $f(x, \theta^*) = -(x - 1/10)^4$. (For example, $f(x, \theta) = -(x - (\theta - 1)/10)^{2\theta}$, with $\theta^\circ = 1$ and $\theta^* = 2$.) One can verify that f satisfies single crossing for these θ, and that $\phi(\theta^*) = 1/10 > \phi(\theta^\circ) = 0$, so that the comparative statics do indeed go in the expected direction. But f fails increasing differences. The problem is that, near its maximum, $f(\cdot, \theta^*)$ is flatter than $f(\cdot, \theta^\circ)$, and increasing differences requires that $f(\cdot, \theta^*)$ be steeper whenever both functions are increasing. □

The name "single crossing property" refers to the fact that (when $M = 1$), for any fixed x^*, x°, the function $g(\theta)$ defined by

$$g(\theta) = f(x^*, \theta) - f(x^\circ, \theta),$$

which measures the benefit of switching from x° to x^*, crosses 0 at most once, and from below. The single crossing condition here is related to the Spence-Mirlees single crossing condition, which is important in signaling and optimal taxation, among other economic applications. The link between the two versions of single crossing is discussed in Milgrom and Shannon (1994).

Theorem 4. *Let $f : \mathbb{R}^{N+M} \to \mathbb{R}$, let C be a sublattice of $\mathbb{R}^N$, and for each $\theta \in \mathbb{R}^M$, let $\phi(\theta)$ be the set of solutions, assumed non-empty, to the problem*

$$\max_{x \in C} f(x, \theta).$$

If f is quasisupermodular in x and satisfies the single crossing property in (x, θ) then ϕ is weakly increasing: for any θ^*, θ°, if $\theta^* \geq \theta^\circ$ then $\phi(\theta^*) \geq \phi(\theta^\circ)$ in the strong set order.

Proof. Consider any $\theta^*, \theta^\circ \in \mathbb{R}^M$, with $\theta^* \geq \theta^\circ$. Consider any $x^* \in \phi(\theta^*)$ and $x^\circ \in \phi(\theta^\circ)$. I must show that $x^* \vee x^\circ \in \phi(\theta^*)$ and $x^* \wedge x^\circ \in \phi(\theta^\circ)$.

To see that $x^* \vee x^\circ \in \phi(\theta^*)$, note that by definition of ϕ,

$$f(x^\circ, \theta^\circ) \geq f(x^\circ \wedge x^*, \theta^\circ).$$

By single crossing, since $x^\circ \geq x^\circ \wedge x^*$ and $\theta^* \geq \theta^\circ$, this implies

$$f(x^\circ, \theta^*) \geq f(x^\circ \wedge x^*, \theta^*).$$

By quasisupermodularity,

$$f(x^* \vee x^\circ, \theta^*) \geq f(x^*, \theta^*).$$

Therefore, since $x^* \in \phi(\theta^*)$, $x^* \vee x^\circ \in \phi(\theta^*)$, as was to be shown.

To see that $x^* \wedge x^\circ \in \phi(\theta^\circ)$, note that by definition of ϕ,

$$f(x^*, \theta^*) \geq f(x^* \vee x^\circ, \theta^*).$$

Since $x^* \vee x^\circ \geq x^*$ and $\theta^* \geq \theta^\circ$, the contrapositive of the single crossing property, in its strict inequality form, says that since it is *not* true that

$$f(x^* \vee x^\circ, \theta^*) > f(x^*, \theta^*),$$

then it is *not* true that

$$f(x^* \vee x^\circ, \theta^\circ) > f(x^*, \theta^\circ).$$

The contrapositive of quasisupermodularity, again in its strict inequality form, then implies that it is *not* true that

$$f(x^\circ, \theta^\circ) > f(x^* \wedge x^\circ, \theta^\circ).$$

That is,

$$f(x^* \wedge x^\circ, \theta^\circ) \geq f(x^\circ, \theta^\circ).$$

Since $x^\circ \in \phi(\theta^\circ)$, this implies $x^* \wedge x^\circ \in \phi(\theta^\circ)$ as was to be shown. ■

Again, one can prove that if the constraint set C is a complete sublattice of $\mathbb{R}^N$ and f is continuous then $\phi(\theta)$ is a complete sublattice of C and in particular has a largest and smallest element. Theorem 4 implies that the largest and smallest elements of $\phi(\theta)$ are both weakly increasing in θ.

The version of Theorem 4 stated in Milgrom and Shannon (1994) is stronger in two respects. First, in Milgrom and Shannon (1994), the solution ϕ depends explicitly on the constraint set C. The statement is then that if f is quasisupermodular in x and satisfies the single crossing property in (x, θ) then ϕ is weakly increasing in both θ and C, where $C^* \geq C^\circ$ refers to the strong set order. The proof of the strengthened version of Theorem 4 is essentially identical to the one just given. See Quah (2007) for an extension that uses a weaker set order on C.

Second, the result in Milgrom and Shannon (1994) is stated in "if and only if" form: f is quasisupermodular in x and satisfies the single crossing property in (x, θ) iff ϕ is weakly increasing in (x, C).

Consider single crossing. Take any $x^*, x^\circ \in \mathbb{R}^N$ with $x^* \geq x^\circ$ and any $\theta^*, \theta^\circ \in \mathbb{R}^M$ with $\theta^* \geq \theta^\circ$. Let $C = \{x^*, x^\circ\}$. Then it is immediate from the definitions that if ϕ is weakly increasing in θ then f must satisfy single crossing for these particular (x, θ). On the other hand, consider quasisupermodularity. Take any $x^*, x^\circ \in \mathbb{R}^N$ and any $\theta \in \mathbb{R}^M$. Let $C^* = \{x^* \vee x^\circ, x^*\}$, and $C^\circ = \{x^* \wedge x^*, x^\circ\}$. Note that $C^* \geq C^\circ$. Then it is immediate from the definitions that if ϕ is weakly increasing in C, for these particular C, then f must be quasisupermodular in x, for these particular x.

5.3 Interval Order Dominance.

As just discussed, Milgrom and Shannon (1994) show that quasisupermodularity in x and single crossing in (x, θ) is jointly necessary as well as sufficient for the conclusion that ϕ is weakly increasing in (θ, C). Nevertheless, these conditions are stronger than necessary in many applications. The reason is that the necessity argument exploits the fact that the set of possible constraint sets is unrestricted. If, as is often the case in applications, C must take a particular form, then weaker conditions on f may suffice to ensure that ϕ is weakly increasing.

To keep things simple, and to focus on the new idea, let $N = M = 1$. My discussion follows Quah and Strulovici (2009). The condition below is somewhat stronger than, but roughly in the same spirit as, checking single crossing only over intervals where the function $f(\cdot, \theta^\circ)$ is increasing.

Formally, f satisfies *interval order dominance* iff for any $\theta^*, \theta^\circ \in \mathbb{R}$, $\theta^* \geq \theta^\circ$, and any $x^*, x^\circ \in \mathbb{R}$, $x^* > x^\circ$, if $f(x^*, \theta^\circ) \geq f(x, \theta^\circ)$ for every $x \in [x^\circ, x^*]$ then

$$f(x^*, \theta^*) - f(x^\circ, \theta^*) \geq 0,$$

with a strict inequality if $f(x^*, \theta^\circ) > f(x, \theta^\circ)$. Quah and Strulovici (2009) provides examples in which interval order dominance is strictly weaker than single crossing.

Theorem 5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, let $C \subseteq \mathbb{R}$ be an interval, and for each $\theta \in \mathbb{R}$, let $\phi(\theta)$ be the set of solutions, assumed non-empty, to the problem*

$$\max_{x \in C} f(x, \theta).$$

If f satisfies interval order dominace then ϕ is weakly increasing: for any θ^*, θ°, if $\theta^* \geq \theta^\circ$ then $\phi(\theta^*) \geq \phi(\theta^\circ)$ in the strong set order.

Proof. Suppose $\theta^* \geq \theta^\circ$ and let $x^* \in \phi(\theta^*)$, $x^\circ \in \phi(\theta^\circ)$. I need to show that if $x^\circ \geq x^*$ then $x^\circ \in \phi(\theta^*)$ and $x^* \in \phi(\theta^\circ)$. If $x^* = x^\circ$ then the result is immediate. So suppose $x^\circ > x^*$.

To see that $x^\circ \in \phi(\theta^*)$, note that since $x^\circ \in \phi(\theta^\circ)$ and $x^\circ > x^*$, $f(x^\circ, \theta^\circ) \geq f(x, \theta^\circ)$ for every $x \in [x^*, x^\circ]$. Interval order dominance then implies that $f(x^\circ, \theta^*) \geq f(x^*, \theta^*)$. Since $x^* \in \phi(\theta^*)$, this implies $x^\circ \in \phi(\theta^*)$, as was to be shown.

To see that $x^* \in \phi(\theta^\circ)$, note that, as above, $f(x^\circ, \theta^\circ) \geq f(x, \theta^\circ)$ for all $x \in [x^*, x^\circ]$. But, since $x^* \in \phi(\theta^*)$, $f(x^*, \theta^*) \geq f(x^\circ, \theta^*)$, hence it is *not* true that $f(x^\circ, \theta^*) > f(x^*, \theta^*)$. Therefore, the contrapositive of interval order dominance, in its strict inequality form, implies that it is *not* true that $f(x^\circ, \theta^\circ) > f(x^*, \theta^\circ)$, which implies that $f(x^*, \theta^\circ) \geq f(x^\circ, \theta^\circ)$, hence $x^* \in \phi(\theta^\circ)$ as was to be shown. ■

The theorem in Quah and Strulovici (2009) differs from the one just provided in three respects. First, Quah and Strulovici (2009) use a generalized notion of interval and thus their result applies to a richer set of possible constraint sets. Second, Quah and Strulovici (2009) allows C to vary as well as θ. The proof of the result is essentially unchanged. Finally, Quah and Strulovici (2009) establishes necessity, as well as sufficiency for monotone comparative statics when the constraint sets are (generalized) intervals.

References

Aliprantis, Charalambos and Donald Brown. 1982. "Equilibria in Markets with a Riesz Space of Commodities." *Journal of Mathematical Economics* 11:180–207.

Athey, Susan. 2002. "Monotone Comparative Statics and Uncertainty." *Quarterly Journal of Economics* 117(1):187–223.

Mas-Colell, Andreu. 1986. "The Price Equilibrium Existence Problem in Topological Vector Lattices." *Econometrica* 54(5):1039–1053.

Milgrom, Paul and Christina Shannon. 1994. "Monotone Comparative Statics." *Econometrica* 62(1):157–180.

Milgrom, Paul and John Roberts. 1990*a*. "The Economics of Modern Manufacturing: Technology, Strategy, and Organization." *American Economic Review* 80(3):511–528.

Milgrom, Paul and John Roberts. 1990*b*. "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities." *Econometrica* 58(6):1255–1277.

Milgrom, Paul and John Roberts. 1994. "Comparing Equilibria." *American Economic Review* 84(3):441–459.

Milgrom, Paul and John Roberts. 1996. "The LeChatelier Principle." *American Economic Review* 86(1):173–179.

Quah, John. 2007. "The Comparative Statics of Constrained Optimization Problems." *Econometrica* 75(2):401–431.

Quah, John and Bruno Strulovici. 2009. "Comparative Statics, Informativeness, and the Interval Dominance Order." *Econometrica* 77(6):1949–1992.

Samuelson, Paul. 1947. *Foundations of Economic Analysis.* Cambridge, MA: Harvard University Press.

Topkis, Donald. 1978. "Minimizing a Submodular Function on a Lattice." *Operations Research* 26(2):305–21.

Topkis, Donald. 1979. "Nash Equilibrium in Non-zero Sum n-Person Supermodular Games." *SIAM Journal of Control and Optimization* 17:773–787.

Topkis, Donald. 1998. *Supermodularity and Complementarity.* Princeton University Press.

Vives, Xavier. 1990. "Nash Equilibrium with Strategic Complementarities." *Journal of Mathematical Economics* 19:305–321.