Let $A$ be in $\mathbb{K}^{D\times D}$. Let $s_1, \ldots, s_r$ be the invariant factors of $tI - A$, ordered in such a way that $s_2$ divides $s_1$, $s_3$ divides $s_2$, etc, and let $d_i = \deg(s_i)$ for all $i$; for $i > r$, we let $s_i = 1$, with $d_i = 0$.

We fix an integer $M$ and we define $\nu = d_1 + \cdots + d_M \le D$ and $\delta = \lceil \nu/M \rceil \le \lceil D/M \rceil$. We choose matrices $X, Y \in \mathbb{K}^{D\times M}$, and we let $F_X^{A,Y}$ be the minimal generator in Popov form of the sequence $(X^\perp A^i Y)_{i\ge 0}$. We also denote by $\sigma_1, \ldots, \sigma_t$ the invariant factors of $F_X^{A,Y}$, for some $t \le M$. As above, for $i > t$, we let $\sigma_i = 1$.

**Theorem 1.** *For a generic choice of $X$ and $Y$, we have:*

- *$F_X^{A,Y}$ has degree $\delta$;*
- *$s_i = \sigma_i$ for $1 \le i \le M$.*

*Proof.* Let $\mathscr{Y}$ be a matrix of indeterminates of size $N \times M$. Then, by [**?**, Proposition 6.1], the minimal generating polynomial $F_{A,\mathscr{Y}}$ for the generic sequence $(A^i\mathscr{Y})_{i\ge 0}$ has determinantal degree $\nu$ and degree $\delta = \lceil \nu/n \rceil$.

- $\dim(\langle \mathscr{Y} \rangle) = \nu$.

  This is proved in the proof of [**?**, Proposition 6.1].

- For a generic $Y$ in $\mathbb{K}^{N\times n}$, $N_Y = N_{\mathscr{Y}}$, where $N_Y = \dim(\langle Y \rangle)$.

  This is because $\langle \mathscr{Y} \rangle$ is the span of $K_{\mathscr{Y}} = [\mathscr{Y}|A\mathscr{Y}|\cdots|A^{N-1}\mathscr{Y}]$, whereas $\langle Y \rangle$ is the span of $K_Y = [Y|AY|\cdots|A^{N-1}Y]$. Take a maximal non-zero minor $\mu$ of $K_{\mathscr{Y}}$; as soon as $\mu(Y) \ne 0$, we have equality of the dimensions.

- For any $Y$ (including $\mathscr{Y}$), the degree $\delta_Y$ of $F_{A,Y}$ is equal to the first index $d$ such that $\dim(\mathrm{span}([Y|AY|\cdots|A^{d-1}Y])) = N_Y$.

  This is [**?**, Lemma 4.3].

- For a generic $Y$, $\delta_Y = \lceil \nu/n \rceil$.

  By [**?**, Proposition 6.1], the minimal generating polynomial $F_{A,\mathscr{Y}}$ for the generic sequence $(A^i\mathscr{Y})_{i\ge 0}$ has degree $\delta_{\mathscr{Y}} = \lceil \nu/n \rceil$. The first restriction on $Y$ is that $N_Y = N_{\mathscr{Y}}$. Then the claim follows from the previous item.

- For any $Y$ in $\mathbb{K}^{N\times n}$, for a generic $X$ in $\mathbb{K}^{n\times N}$, $F_{A,Y,X} = F_{A,Y}$.

  By [**?**, Lemma 4.2], there exists matrices $P_Y$ in $\mathbb{K}^{N\times N_Y}$ and $A_Y \in \mathbb{K}^{N_Y\times N_Y}$, with $P_Y$ of full rank $N_Y$, such that $F_{A,Y,X} = F_{A,Y}$ if and only if the dimension of the span of the rows of $XP_Y, XP_YA_Y, XP_YA_Y^2, \cdots$ is equal to $N_Y$.

  Let $B_Y$ be the transpose of $A_Y$. Then, the dimension above is the dimension of the span of $(XP_Y)^t, B_Y(XP_Y)^t, B_Y^2(XP_Y)^t, \ldots$ The number of invariant factors of $A_Y$ (and thus of $B_Y$) is at most $n$. As a result, for a generic $Z$ in $\mathbb{K}^{N_Y\times n}$, by the previous items, $\mathrm{span}([Z|B_YZ|B_Y^2Z|\cdots])$ has dimension $N_Y$ (since the number of columns in $Z$ is at least equal to the number of invariant factors of $B_Y$).

- $\mathrm{rank}(\mathrm{Hk}(X, Y)) = \deg(\det(F_{A,Y,X}))$.

  By [?, Eq. (2.6)], $\mathrm{rank}(\mathrm{Hk}_{e,d}) = \deg(\det(F_{A,Y,X}))$ for $d \geq \deg(F_{A,Y,X})$ and $e \geq n$. We take $e = n$, so that $\mathrm{rank}(\mathrm{Hk}_{n,d}) = \deg(\det(F_{A,Y,X}))$ for $d \geq \deg(F_{A,Y,X})$. On the other hand, for fixed $n$, the sequence $\mathrm{rank}(\mathrm{Hk}_{n,d})$ is constant for $d \geq n$. As a result, $\mathrm{rank}(\mathrm{Hk}_{n,n}) = \deg(\det(F_{A,Y,X}))$.

- For generic $X, Y$, the invariant factors of $F_{A,Y,X}$ are $s_1, \ldots, s_n$.

  By [?, Theorem 2.12], for any $X$ and $Y$ in $\mathbb{K}^{n\times N} \times \mathbb{K}^{N\times n}$, for $i = 1, \ldots, n$, the $i^{th}$ invariant factor of $F_{A,Y,X}$ divides $s_i$, so that $\deg(\det(F_{A,Y,X})) \leq \nu$.

  Furthermore, there exist matrices $X_0, Y_0$ such that for $i = 1, \ldots, n$, the $i^{th}$ invariant factor of $F_{A,Y_0,X_0}$ is equal to $s_i$. In this case, $\deg(\det(F_{A,Y_0,X_0})) = \nu$.

  Thus, $\mathrm{rank}(\mathrm{Hk}(X, Y))$ has rank at most $\nu$, and exactly $\nu$ for at least one pair $(X_0, Y_0)$. So we have equality for a generic $(X, Y)$. When equality holds, the $i^{th}$ invariant factor of $F_{A,Y,X}$ equals $s_i$ for all $i$.

□

# References