Let A be in $\mathbb{K}^{D\times D}$. Let $s_1, \dots, s_r$ be the invariant factors of $tI - A$, ordered in such a way that s_2 divides s_1, s_3 divides s_2, etc, and let $d_i = \deg(s_i)$ for all i; for $i > r$, we let $s_i = 1$, with $d_i = 0$.

We fix an integer M and we define $\nu = d_1 + \cdots + d_M \leq D$ and $\delta = \lceil \nu/M \rceil \leq \lceil D/M \rceil$. We choose matrices $X, Y \in \mathbb{K}^{D\times M}$, and we let $F_X^{A,Y}$ be the minimal generator in Popov form of the sequence $(X^\perp A^i Y)_{i\geq 0}$. We also denote by $\sigma_1, \dots, \sigma_t$ the invariant factors of $F_X^{A,Y}$, for some $t \leq M$. As above, for $i > t$, we let $\sigma_i = 1$.

Theorem 1. *For a generic choice of X and Y, we have:*

- *$F_X^{A,Y}$ has degree δ;*
- *$s_i = \sigma_i$ for $1 \leq i \leq M$.*

Proof. Let $\mathscr{Y}$ be a matrix of indeterminates of size $N \times M$. Then, by [**?**, Proposition 6.1], the minimal generating polynomial $F_{A,\mathscr{Y}}$ for the generic sequence $(A^i \mathscr{Y})_{i\geq 0}$ has determinantal degree ν and degree $\delta = \lceil \nu/n \rceil$.

- $\dim(\langle \mathscr{Y} \rangle) = \nu$.

  This is proved in the proof of [**?**, Proposition 6.1].

- For a generic Y in $\mathbb{K}^{N\times n}$, $N_Y = N_{\mathscr{Y}}$, where $N_Y = \dim(\langle Y \rangle)$.

  This is because $\langle \mathscr{Y} \rangle$ is the span of $K_{\mathscr{Y}} = [\mathscr{Y}|A\mathscr{Y}|\cdots|A^{N-1}\mathscr{Y}]$, whereas $\langle Y \rangle$ is the span of $K_Y = [Y|AY|\cdots|A^{N-1}Y]$. Take a maximal non-zero minor μ of $K_{\mathscr{Y}}$; as soon as $\mu(Y) \neq 0$, we have equality of the dimensions.

- For any Y (including $\mathscr{Y}$), the degree δ_Y of $F_{A,Y}$ is equal to the first index d such that $\dim(\mathrm{span}([Y|AY|\cdots|A^{d-1}Y])) = N_Y$.

  This is [**?**, Lemma 4.3].

- For a generic Y, $\delta_Y = \lceil \nu/n \rceil$.

  By [**?**, Proposition 6.1], the minimal generating polynomial $F_{A,\mathscr{Y}}$ for the generic sequence $(A^i \mathscr{Y})_{i\geq 0}$ has degree $\delta_{\mathscr{Y}} = \lceil \nu/n \rceil$. The first restriction on Y is that $N_Y = N_{\mathscr{Y}}$. Then the claim follows from the previous item.

- For any Y in $\mathbb{K}^{N\times n}$, for a generic X in $\mathbb{K}^{n\times N}$, $F_{A,Y,X} = F_{A,Y}$.

  By [**?**, Lemma 4.2], there exists matrices P_Y in $\mathbb{K}^{N\times N_Y}$ and $A_Y \in \mathbb{K}^{N_Y\times N_Y}$, with P_Y of full rank N_Y, such that $F_{A,Y,X} = F_{A,Y}$ if and only if the dimension of the span of the rows of $XP_Y, XP_Y A_Y, XP_Y A_Y^2, \cdots$ is equal to N_Y.

  Let B_Y be the transpose of A_Y. Then, the dimension above is the dimension of the span of $(XP_Y)^t, B_Y(XP_Y)^t, B_Y^2(XP_Y)^t, \dots$ The number of invariant factors of A_Y (and thus of B_Y) is at most n. As a result, for a generic Z in $\mathbb{K}^{N_Y\times n}$, by the previous items, $\mathrm{span}([Z|B_Y Z|B_Y^2 Z|\cdots])$ has dimension N_Y (since the number of columns in Z is at least equal to the number of invariant factors of B_Y).

- $\mathrm{rank}(\mathrm{Hk}(X, Y)) = \deg(\det(F_{A,Y,X}))$.

  By [?, Eq. (2.6)], $\mathrm{rank}(\mathrm{Hk}_{e,d}) = \deg(\det(F_{A,Y,X}))$ for $d \geq \deg(F_{A,Y,X})$ and $e \geq n$. We take $e = n$, so that $\mathrm{rank}(\mathrm{Hk}_{n,d}) = \deg(\det(F_{A,Y,X}))$ for $d \geq \deg(F_{A,Y,X})$. On the other hand, for fixed n, the sequence $\mathrm{rank}(\mathrm{Hk}_{n,d})$ is constant for $d \geq n$. As a result, $\mathrm{rank}(\mathrm{Hk}_{n,n}) = \deg(\det(F_{A,Y,X}))$.

- For generic X, Y, the invariant factors of $F_{A,Y,X}$ are $s_1, \ldots, s_n$.

  By [?, Theorem 2.12], for any X and Y in $\mathbb{K}^{n\times N} \times \mathbb{K}^{N\times n}$, for $i = 1, \ldots, n$, the i^{th} invariant factor of $F_{A,Y,X}$ divides s_i, so that $\deg(\det(F_{A,Y,X})) \leq \nu$.

  Furthermore, there exist matrices X_0, Y_0 such that for $i = 1, \ldots, n$, the i^{th} invariant factor of F_{A,Y_0,X_0} is equal to s_i. In this case, $\deg(\det(F_{A,Y_0,X_0})) = \nu$.

  Thus, $\mathrm{rank}(\mathrm{Hk}(X, Y))$ has rank at most ν, and exactly ν for at least one pair (X_0, Y_0). So we have equality for a generic (X, Y). When equality holds, the i^{th} invariant factor of $F_{A,Y,X}$ equals s_i for all i.

$\square$