Lemma 1: Let $U, V \in \mathbb{K}^{D\times M}$ be matrices with random entries and $s = (U^{tr}T_1^iV)_{i\geq 0}$. If $Z = \sum_{i=0}^{\infty} s^{(i)}/t^{i+1}$, then each entry of Z is in the form n_*/P, where P is the minimum scalar generator for s.

Proof: Rewrite U, V as $U = [u_1, u_2, \cdots, u_M], V = [v_1, v_2, \cdots, v_M]$, then

$$s^{(i)} = \begin{bmatrix} u_1^{tr}T_1^iv_1 & u_1^{tr}T_1^iv_2 & \cdots & u_1^{tr}T_1^iv_M \\ u_2^{tr}T_1^iv_1 & \cdots & \cdots & u_2^{tr}T_1^iv_M \\ \vdots & \ddots & \ddots & \vdots \\ u_M^{tr}T_1^iv_1 & \cdots & \cdots & u_M^{tr}T_1^iv_M \end{bmatrix}$$

Thus,

$$Z = \begin{bmatrix} \sum u_1^{tr}T_1^iv_1/t^{i+1} & \cdots & \cdots & \sum u_1^{tr}T_1^iv_M/t^{i+1} \\ \vdots & \ddots & \ddots & \vdots \\ \sum u_M^{tr}T_1^iv_1/t^{i+1} & \cdots & \cdots & \sum u_M^{tr}T_1^iv_M/t^{i+1} \end{bmatrix}$$

So each entry separately is what would be computed in the scalar case. Therefore, we can rewrite each entry as n_*/P for some numerator n_*.

Lemma 2: Let S be the minimum generating polynomial matrix for s and $D = ASB$ be the Smith normal form of S. Furthermore, let $s_1, \cdots s_D$ be invariant factors of S such that $s_1|s_2|\cdots|s_D$. Then, there exists a vector $\tilde{u}$ such that $\tilde{u}S = [0, \cdots, 0, s_D]$

Proof: Let $[b_1, \cdots, b_D]$ be the last row of B and $w = [\frac{s_Db_1}{s1}, \frac{s_Db_2}{s2}, \cdots, \frac{s_Db_{D-1}}{s_{D-1}}, b_D]$ (since $s_i|s_D$), then

$$\begin{aligned} (wA)A^{-1}D &= [\frac{s_Db_1}{s1}, \frac{s_Db_2}{s2}, \cdots, \frac{s_Db_{D-1}}{s_{D-1}}, b_D] \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_D \end{bmatrix} \\ &= [s_Db_1, s_Db_2, \cdots, s_Db_D] \\ &= [0, \cdots, 0, s_D]B \end{aligned}$$

Therefore, if we choose $\tilde{u} = wA$, we get $\tilde{u}S = (wA)A^{-1}DB^{-1} = [0, \cdots, 0, s_D]$ as needed.

Lemma 3: If $\sum_{i=0}^{\infty} s^{(i)}t^i = S^{-1}N$, then $deg(N)$ is less than or equal to $deg(S)$. TODO

Theorem 1: If $S^{-1}N = \sum_{i=0}^{\infty} s^{(i)}/t^{i+1}$, the first entry of the last row of $\tilde{u}N$ is the numerator of the generating function for $(u_M^{tr}T_1^iv_1)_{i\geq 0}$.

Proof: Let $S^{-1}N = \sum_{i=0}^{\infty} s^{(i)}/t^{i+1}$, then by lemma 1

$$N = S\sum_{i=0}^{\infty} s^{(i)}/t^{i+1} = S \begin{bmatrix} n_{1,1}/P & \cdots & n_{1,M}/P \\ \vdots & \ddots & \vdots \\ n_{M,1}/P & \cdots & n_{M,M}/P \end{bmatrix}$$

From theorem 1 of (randomXY-proof), we know that the i^{th} invariant factor of $XI - A$ is equal to the i^{th} invariant factor of S for generic choice of U, V. Thus, $s_D = P$ and by lemma 2

$$\begin{aligned}
\tilde{u}N &= \tilde{u}S \begin{bmatrix} n_{1,1}/P & \cdots & n_{1,M}/P \\ \vdots & \ddots & \vdots \\ n_{M,1}/P & \cdots & n_{M,M}/P \end{bmatrix} \\
&= [0, \cdots, 0, P] \begin{bmatrix} n_{1,1}/P & \cdots & n_{1,M}/P \\ \vdots & \ddots & \vdots \\ n_{M,1}/P & \cdots & n_{M,M}/P \end{bmatrix} \\
&= [n_{M,1}, \cdots, n_{M,M}]
\end{aligned}$$