Let A be in $\mathbb{K}^{D\times D}$. Let $s_1, \dots, s_r$ be the invariant factors of $tI - A$, ordered in such a way that s_2 divides s_1, s_3 divides s_2, etc, and let $d_i = \deg(s_i)$ for all i; for $i > r$, we let $s_i = 1$, with $d_i = 0$.

We fix an integer M and we define $\nu = d_1 + \cdots + d_M \le D$ and $\delta = \lceil \nu/M \rceil \le \lceil D/M \rceil$. We choose matrices $X, Y \in \mathbb{K}^{D\times M}$, and we let $F_X^{A,Y}$ be the minimal generator in Popov form of the sequence $(X^\perp A^i Y)_{i\ge 0}$. We also denote by $\sigma_1, \dots, \sigma_t$ the invariant factors of $F_X^{A,Y}$, for some $t \le M$. As above, for $i > t$, we let $\sigma_i = 1$.

Theorem 1. *For a generic choice of X and Y, we have:*

- *$F_X^{A,Y}$ has degree δ;*
- *$s_i = \sigma_i$ for $1 \le i \le M$.*

Proof. Let $F^{A,Y}$ be the minimal generating polynomial of the sequence $(A^i Y)_{i\ge 0}$. We denote by $\langle Y \rangle$ the vector space generated by the columns of $Y, AY, A^2Y, \dots$. We also write $N_Y = \dim(\langle Y \rangle)$.

First, we prove that for any Y in $\mathbb{K}^{N\times M}$, for a generic X in $\mathbb{K}^{N\times M}$, $F_X^{A,Y} = F^{A,Y}$. Indeed, by [**?**, Lemma 4.2], there exists matrices P_Y in $\mathbb{K}^{N\times N_Y}$ and $A_Y \in \mathbb{K}^{N_Y\times N_Y}$, with P_Y of full rank N_Y, and where A_Y is a matrix of the restriction of A to $\langle Y \rangle$, such that $F_X^{A,Y} = F^{A,Y}$ if and only if the dimension of the span of $[Z\ B_Y Z\ B_Y^2 Z\ \cdots]$ is equal to N_Y, with $B_Y = A_Y^\perp$ and $Z = P_Y^\perp X \in \mathbb{K}^{N_Y\times M}$.

We prove that this is the case for a generic X. By construction, one can find a basis of $\langle Y \rangle$ in which the matrix of A_Y is block-companion, with $M' \le M$ blocks (take the A_Y-span of the first column of Y, then of the second column, working modulo the previous vector space, etc.) Thus, B_Y is similar to a block-companion matrix with M' blocks as well; since Z has M columns, S has full dimension N_Y for a generic Z (and for a generic X, since P_Y has rank N_Y). Thus, for generic choices of X and Y, $F_X^{A,Y} = F^{A,Y}$.

Let us next introduce a matrix $\mathscr{Y}$ of indeterminates of size $N \times M$, and let $F^{A,\mathscr{Y}}$ be the minimal generating polynomial of the "generic" sequence $(A^i \mathscr{Y})_{i\ge 0}$. The notation $\langle \mathscr{Y} \rangle$ and $N_{\mathscr{Y}}$ are defined as above. In particular, by [**?**, Proposition 6.1], the minimal generating polynomial $F^{A,\mathscr{Y}}$ has degree δ and determinantal degree ν.

Now, for a generic Y in $\mathbb{K}^{N\times M}$, $N_Y = N_{\mathscr{Y}}$. Indeed, $\langle \mathscr{Y} \rangle$ is the span of $[\mathscr{Y}\ A\mathscr{Y}\ \cdots\ A^{N-1}\mathscr{Y}]$, whereas $\langle Y \rangle$ is the span of $[Y\ AY\ \cdots\ A^{N-1}Y]$. Take a maximal non-zero minor μ of $K_{\mathscr{Y}}$; as soon as $\mu(Y) \ne 0$, we have equality of the dimensions. On the other hand, by [**?**, Lemma 4.3], for any Y (including $\mathscr{Y}$), the degree of $F^{A,Y}$ is equal to the first index d such that $\dim(\mathrm{span}([Y\ AY\ \cdots\ A^{d-1}Y])) = N_Y$. As a result, for generic Y, $F^{A,Y}$ and $F^{A,\mathscr{Y}}$ have the same degree, that is, δ. The first item is proved.

We conclude by proving that for generic X, Y, the invariant factors $\sigma_1, \dots, \sigma_M$ of $F_X^{A,Y}$ are $s_1, \dots, s_M$. By [**?**, Theorem 2.12], for any X and Y in $\mathbb{K}^{N\times M}$, for $i = 1, \dots, M$, the i^{th} invariant factor σ_i of $F_X^{A,Y}$ divides s_i, so that $\deg(\det(F_X^{A,Y})) \le \nu$, with equality if and only if $\sigma_i = s_i$ for all $i \le M$.

For Y as above and any integers e, d, we let $\mathrm{Hk}_{e,d}(Y)$ be the block Hankel matrix

$$\mathrm{Hk}_{e,d}(Y) = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{e-1} \end{bmatrix} \begin{bmatrix} Y & AY & A^2Y & \cdots & A^{d-1}Y \end{bmatrix}$$

By [**?**, Eq. (2.6)], $\mathrm{rank}(\mathrm{Hk}_{e,d}(Y)) = \deg(\det(F^{A,Y}))$ for $d \geq \deg(F^{A,Y})$ and $e \geq N$. We take $e = N$, so that $\mathrm{rank}(\mathrm{Hk}_{N,d}(Y)) = \deg(\det(F^{A,Y}))$ for $d \geq \deg(F^{A,Y})$. On the other hand, the sequence $\mathrm{rank}(\mathrm{Hk}_{N,d}(Y))$ is constant for $d \geq N$; as a result, $\mathrm{rank}(\mathrm{Hk}_{N,N}(Y)) = \deg(\det(F^{A,Y}))$. For the same reason, we also have $\mathrm{rank}(\mathrm{Hk}_{N,N}(\mathscr{Y})) = \deg(\det(F^{A,\mathscr{Y}}))$, so that for a generic Y, $F^{A,Y}$ and $F^{A,\mathscr{Y}}$ have the same determinantal degree, that is, ν. As a result, for generic X and Y, we also have $\deg(\det(F_X^{A,Y})) = \nu$, and the conclusion follows. $\square$