

Matrix Berlekamp-Massey

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1 Definition of recurrence relations

First issue: which definition to use for matrix generator? There is one in [?], and a different one in [?]; both are related as follows.

Definition 1 ([?]). *Let $\mathcal{S} = (S_k)_{k \in \mathbb{Z}_{\geq 0}} \subset \mathbb{K}^{m \times n}$ be a sequence of $m \times n$ matrices over \mathbb{K} . We define the power series matrix $\mathbf{S} = \sum_{k \geq 0} S_k X^k \in \mathbb{K}[[X]]^{m \times n}$. Then, a vector $\mathbf{p} \in \mathbb{K}[X]^{n \times 1}$ is said to be a (linear recurrence) relation for \mathcal{S} if the product $\mathbf{S}\mathbf{p}$ has polynomial entries, that is, $\mathbf{S}\mathbf{p} \in \mathbb{K}[X]^{n \times 1}$.*

For some reason (which is unclear to me), in [?] the word “generator” is used for such relations. Here we will reserve this word for sets of vectors that indeed generate the set of all relations.

Assume there exists a nonzero relation for \mathcal{S} , and let \mathbf{p} be such a relation. Writing $\mathbf{p} = \sum_k p_k X^k$ for matrices $p_k \in \mathbb{K}^{n \times 1}$, then we have

$$\sum_{k=0}^d S_{\delta-k} p_k = 0 \quad \text{for all } d \geq \deg(\mathbf{p}) \text{ and } \delta \geq \max(d, \deg(\mathbf{S}\mathbf{p}) + 1). \quad (1)$$

One may have in mind that $\deg(\mathbf{S}\mathbf{p}) < \deg(\mathbf{p})$ since this typically holds for the output of existing algorithms; in general, this does not necessary hold. For example, if \mathcal{S} has only finitely many nonzero terms, and thus \mathbf{S} already has polynomial entries, any coordinate vector is a relation \mathbf{p} such that $\mathbf{S}\mathbf{p}$ has degree larger than $\deg(\mathbf{p})$.

Definition 2 ([?]). *Let $\mathcal{S} = (S_k)_{k \in \mathbb{Z}_{\geq 0}} \subset \mathbb{K}^{m \times n}$ be a sequence of $m \times n$ matrices over \mathbb{K} . We define the power series matrix $\mathbf{S} = \sum_{k \geq 0} S_k X^k \in \mathbb{K}[[X]]^{m \times n}$. Then, a vector $\mathbf{p} \in \mathbb{K}[X]^{n \times 1}$ of degree at most d is said to be a (linear recurrence) relation for \mathcal{S} if*

$$\sum_{k=0}^d S_{\delta+k} p_k = 0 \quad \text{for all } \delta \geq 0.$$

where $(p_k)_k$ are the matrices in $\mathbb{K}^{n \times 1}$ such that $\mathbf{p} = \sum_{0 \leq k \leq d} p_k X^k$.

Lemma 3. *For a given sequence $\mathcal{S} \subset \mathbb{K}^{m \times n}$, a nonzero vector $\mathbf{p} \in \mathbb{K}[X]^{n \times 1}$ is a relation for ?? if and only if there exists $d \geq \deg(\mathbf{p})$ such that the reverse $X^d \mathbf{p}(X^{-1})$ is a relation for ??.*

Proof. First, we assume that $X^d \mathbf{p}(X^{-1}) = \sum_{k=0}^d p_{d-k} X^k$ is a relation for ??, for some integer $d \geq \deg(\mathbf{p})$. This means that, for all $\delta \geq 0$, we have $0 = \sum_{k=0}^d S_{\delta+k} p_{d-k} = \sum_{k=0}^d S_{\delta+d-k} p_k$. This implies that \mathbf{Sp} has polynomial entries (and $\deg(\mathbf{Sp}) \leq d$).

Now, we assume that \mathbf{p} is a relation for ??. Taking $d = \max(\deg(\mathbf{p}), \deg(\mathbf{Sp}) + 1)$ in ??, we obtain $\sum_{k=0}^d S_{\delta-k} p_k = 0$ for all $\delta \geq d$. This implies $\sum_{k=0}^d S_{\delta-d+k} p_{d-k} = 0$ for all $\delta \geq d$, or equivalently, $\sum_{k=0}^d S_{\delta+k} p_{d-k} = 0$ for all $\delta \geq 0$. Therefore the reverse $X^d \mathbf{p}(X^{-1})$ is a relation for ??. \square