Chapter 19

Genus and Parametrization of Planar Algebraic Curves

19.1 Genus and Parametrization

We have seen that every parametric curve can be expressed in implicit form. The reverse is not generally true. The condition under which an implicit curve can be parametrized using rational polynomials is that its *genus* must be zero. Basically, the genus of a curve is given by the formula $g = \frac{(n-1)(n-2)}{2} - d$ where g is the genus, n is the degree, and d is the number of double points. There are some subtleties involved in this equation, but we will not concern ourselves with them. They deal with more complicated multiple points.

We see immediately that all curves of degree one and two have genus zero and thus can be parametrized using rational polynomials. Curves of degree three must have one double point in order to qualify.

It is also the case that an irreducible curve of degree n can have at most $(n-1)(n-2)/2$ double points. Thus, a rational degree n curve has the largest possible number of double points for a curve of its degree. An irreducible curve is one whose implicit equation $f(x, y) = 0$ cannot be factored. For example, the degree two curve $xy + x + y + 1$ can be factored into $(x + 1)(y + 1)$ and is thus actually two straight lines. Note that the point at which those two lines intersect is a double point of the curve, even though an irreducible conic cannot have any double points.

Another example of a reducible curve is given by the quartic which is factored into

$$(x^2 + y^2 - 1)(x^2 + y^2 - 4)$$

which is two concentric circles. This leads to another characteristic of rational curves - you can sketch an entire rational curve without removing your pencil from the paper (with the possible exception of a finite number of acnodes). For this reason, the classical algebraic geometry literature sometimes refers to rational curves as “unicursal” curves.

It should also be noted that an algebraic curve which consists of more than one component is not necessarily reducible. For example, many non-rational cubic curves consist of an oval and a branch which does not touch the oval, and yet the equation does not factor.

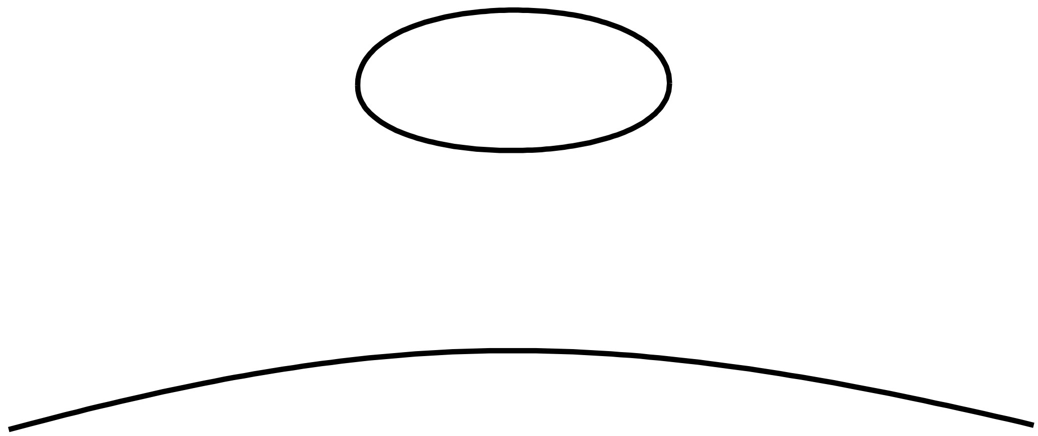

Figure 19.1: Irreducible Cubic Curve

19.2 Detecting Double Points

How does one verify the existence of a double point? A computational method is to verify that any straight line through the alleged double point hits the curve at least twice at that point. For example, any curve whose equation has no constant or linear terms has a double point at the origin. Consider the cubic algebraic curve:

$$x^3 - 2xy^2 + xy + 3y^2 = 0.$$

To determine how many times a general line through the origin hits this curve, we define such a line parametrically so that the point $t = 0$ on the line corresponds to the origin:

$$x = at, \qquad y = bt.$$

The intersection of the line and the curve yields

$$(a^3 - 2ab^2)t^3 + (ab + 3b^2)t^2 = 0.$$

Since this equation has at least a double root at $t = 0$ for any values of a and b (which control the slope of the line), we conclude that the curve has a double point at the origin.

There are basically three types of double points: the crunode (or simply *node*, or self intersection), the cusp, and the acnode (or isolated node). An example of a cubic curve with a crunode at the origin is the curve in Figure 19.2 whose implicit equation is

$$x^3 + 9x^2 - 12y^2 = 0.$$

You can verify by implicitization that this curve can be expressed as a Bézier curve with control points $(3, 3)$, $(-13, -15)$, $(-13, 15)$, $(3, -3)$.

An example of a cubic curve with a cusp at the origin is given by

$$x^3 - 3y^2 = 0$$

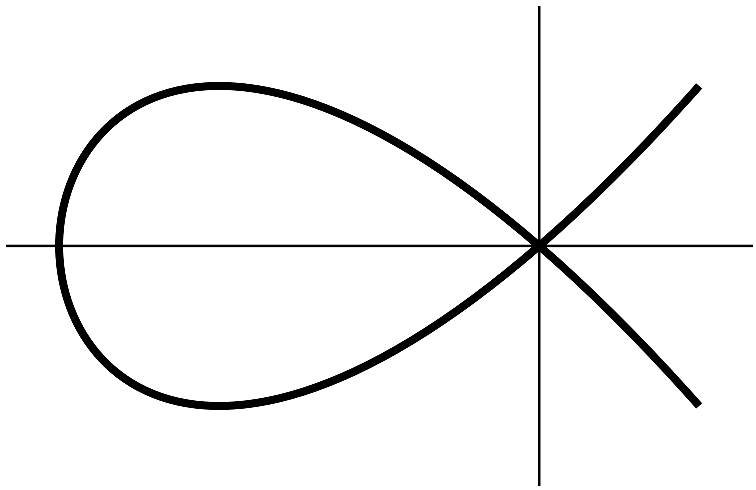

Figure 19.2: Crunode: $x^3 + 9x^2 - 12y^2 = 0$

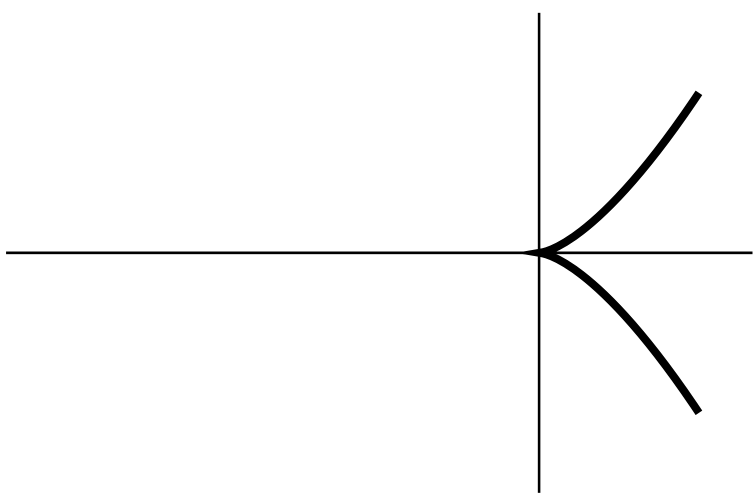

Figure 19.3: Cusp: $x^3 - 3y^2 = 0$

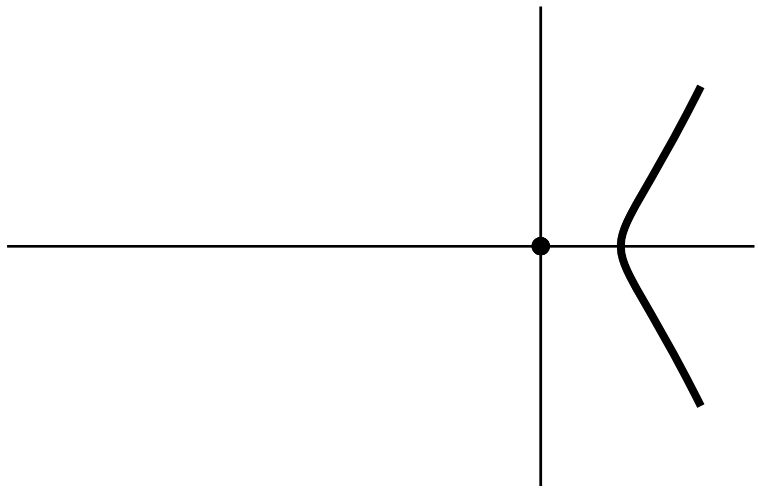

Figure 19.4: Acnode: $x^3 - 3x^2 - 3y^2 = 0$

which is illustrated in Figure 19.2.

An example of a curve with an acnode at the origin is shown in Figure 19.2, given by the equation

$$x^3 - 3x^2 - 3y^2 = 0.$$

We can classify a double point as being a crunode, acnode or cusp by examining the tangent lines to the curve at the double point. A crunode has two distinct real tangent lines, a cusp has two identical tangent lines, and an acnode has two complex tangent lines. A tangent line through a double point hits the curve three times at the double point. We can find the slope of the tangent lines by computing values of a and b for which the intersection of the line and the curve has a triple root at $t = 0$. In the previous example, this occurs if $ab + 3b^2 = 0$ from which $a = 1$, $b = 0$ and $a = -3$, $b = 1$ are the two independent solutions. These represent two real, distinct lines, so the double point in a crunode.

The exercise that we just performed only lets us verify that a given point is singular (a singular point is a double point, triple point, or multiple point in general). To determine the location of singular points, we can use a tool known as the *discriminant* (discussed later).

19.3 Implicit Curve Intersections

We have seen how the intersections of two Bezier curves can be computed, and also how the intersection points of a parametric and an implicit curve can be found. What about two implicit curves?

One direct method for computing the intersection points of two implicit curves is to take the resultant of the curves with respect to x or y. The X-resultant is computed by treating the implicit equations as polynomials in x whose coefficients are polynomials in y. The X-resultant eliminates x from the two equations and produces a polynomial in y whose roots are the y coordinates of the intersection points.

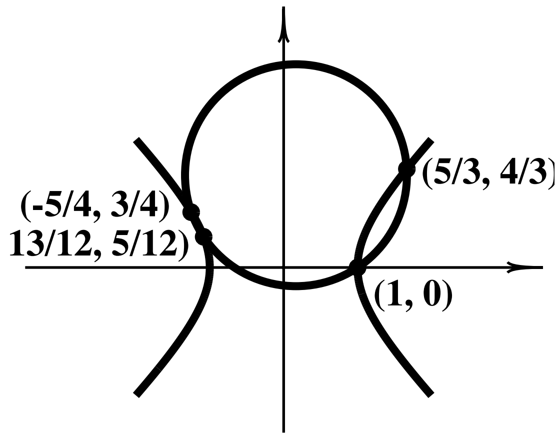

Figure 19.5: Circle and Hyperbola

We illustrate with a circle $6x^2 + 6y^2 - 2x - 15y - 4 = 0$ and a hyperbola $x^2 - y^2 - 1 = 0$.

The x-resultant of these two implicit equations is $144y^4 - 360y^3 + 269y^2 - 60y$ whose roots are $y = 0$, $y = 4/3$, $y = 3/4$, and $y = 5/12$. These are the y-coordinates of the points of intersection of the two curves.

We can use the y-resultant to find the x-coordinates of the points of intersection. The y-resultant is $144x^4 - 48x^3 - 461x^2 + 40x + 325$ which has roots $x = 1$, $x = 5/3$, $x = -5/4$, and $x = -13/12$.

We now find ourselves in the interesting situation of knowing the x and y components of the points of intersection, but we don't know which x goes with which y! One way to determine that is simply to evaluate each curve equation with every x and every y to see which (x, y) pairs satisfy both curve equations simultaneously. A more clever way is to use Euclid's algorithm. In fact, Euclid's algorithm spares us the trouble of computing both the x-resultant and the y-resultant.

Suppose we had only computed the y-resultant and we wanted to find the y-coordinate of the point of intersection whose x-coordinate is 5/3. That is to say, we want to find a point $(\frac{5}{3}, y)$ which satisfies both curve equations. We substitute $x = 5/3$ into the circle equation to get $16/6 - y^2 = 0$ and into the hyperbola equation to get $6y^2 - 15y + 28/3 = 0$. We now simply want to find a value of y which satisfies both of these equations. Euclid's algorithm tells us that the GCD of these two $3y - 4 = 0$, and thus one point of intersection is $(\frac{5}{3}, \frac{4}{3})$.

19.4 Discriminants

The discriminant of a univariate polynomial is the resultant of the polynomial and its first derivative. If the discriminant is zero, the polynomial has a double root. (Why?) We all encountered discriminants as early as 7th grade in connection with the quadratic equation. For a degree two polynomial $at^2 + bt + c$, the value $b^2 - 4ac$ is referred to as the discriminant, although if we actually compute the resultant of $at^2 + bt + c$ and its derivative $2at + b$, we find that the discriminant is actually $a^2(b^2 - 4ac)$. However, since it is understood that $a \neq 0$, the discriminant can only vanish if $b^2 - 4ac = 0$.

It is also possible to compute the discriminant of an implicit curve by taking the resultant of the implicit equation and its partial derivative with respect to one of the variables. To do this, we treat the implicit equation as a polynomial in y whose coefficients are polynomials in x (or, vice versa). The resultant will then be a polynomial in x with constant coefficients. The roots of that polynomial will correspond to the x coordinates of the vertical tangents and of any double points.

Graphically, the discriminant can be thought of as the silhouette of the curve.

19.5 Parametrizing Unicursal Curves

There are several ways in which a parametrization may be imposed on a curve of genus zero. For a conic, we may establish a one-one correspondence between points on the curve and a family of lines through a point on the curve. This is most easily illustrated by translating the curve so that it passes through the origin, such as does the curve

$$x^2 - 2x + y^2 = 0$$

which is a unit circle centered at $(0,1)$. We next make the substitution $y = tx$ and solve for x as a function of t:

$$x^2(1 + t^2) - 2x = 0$$

$$x = \frac{2}{1 + t^2}$$

$$y = tx = \frac{2t}{1 + t^2}$$

Notice that $y = tx$ is a family of lines through the origin. A one parameter family of lines (or, of any implicitly defined curves) is known as a *pencil* of lines or curves. The variable line $y = tx$ intersects the curve once at the origin, and at exactly one other point (because of Bezout's theorem). Thus, we have established a one-one correspondence between points on the curve and values t which correspond to lines containing that point and the origin.

An example of a circle parametrized in this manner is shown in Figure 19.6.

The same trick can be played with a genus zero cubic curve which has been translated so that its double point lies on the origin. if this happens, its implicit equation involves terms of degree two and three only. For example, consider the curve

$$(x^3 + 2x^2y + 3xy^2 + 4y^3) + (5x^2 + 6xy + 7y^2) = 0$$

which has a double point at the origin. Again make the substitution $y = tx$ to obtain

$$x^3(1 + 2t + 3t^2 + 4t^3) + x^2(5 + 6t + 7t^2) = 0.$$

From which

$$x = -\frac{5 + 6t + 7t^2}{1 + 2t + 3t^2 + 4t^3}, \quad y = tx = -\frac{5t + 6t^2 + 7t^3}{1 + 2t + 3t^2 + 4t^3}$$

An example of a cubic curve parametrized in this manner is shown in Figure 19.7.

In general, a cubic curve does not have a double point, and if it does, that double point is not generally at the origin. Thus, before one can parametrize a cubic curve, one must first compute the location of its double point (or, determine that rational parametrization is not possible if there is no double point).

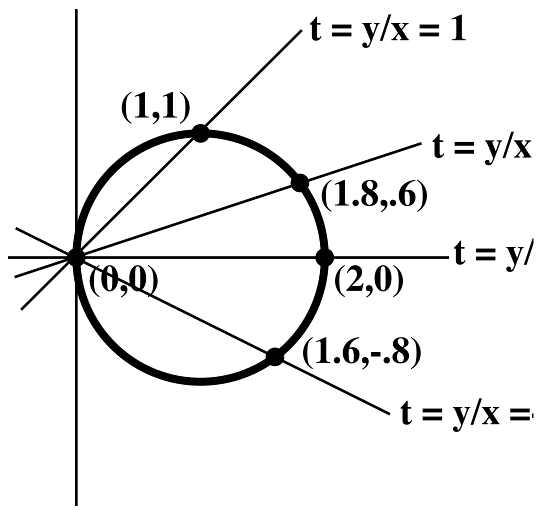

Figure 19.6: Parametrizing a Circle

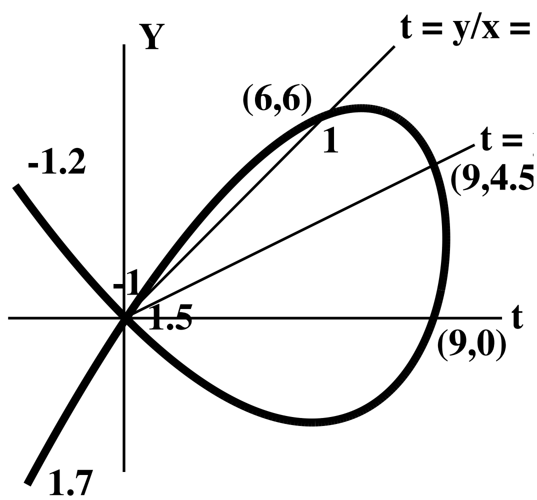

Figure 19.7: Parametrizing a Cubic Curve

Double points satisfy the equations $f(x,y) = f_x(x,y) = f_y(x,y) = 0$. Consider the cubic curve

$$f(x,y) = -21 + 46\,x - 13\,x^2 + x^3 + 25\,y - 23\,x\,y + 3\,x^2\,y - 9\,y^2 + 3\,x\,y^2 + y^3$$

for which

$$f_x(x,y) = 46 - 26\,x + 3\,x^2 - 23\,y + 6\,x\,y + 3\,y^2$$

and

$$f_y(x,y) = 25 - 23\,x + 3\,x^2 - 18\,y + 6\,x\,y + 3\,y^2$$

From section 19.3, we compute the x coordinates of the intersections of $f_x = 0$ and $f_y = 0$ by taking the resultant of f_x and f_y with respect to y:

$$\mathrm{Resultant(f_x, f_y, y) = 174 - 159\,x + 36\,x^2}$$

whose roots are $x = 2$ and $x = \frac{29}{12}$. Likewise the y coordinates of the intersections of $f_x = 0$ and $f_y = 0$ are found by taking the resultant of f_x and f_y with respect to x:

$$\mathrm{Resultant(f_x, f_y, x) = 297 - 207\,y + 36\,y^2}$$

whose roots are $y = 3$ and $y = \frac{11}{4}$. From these clues, we find that the only values of (x,y) which satisfy $f(x,y) = f_x(x,y) = f_y(x,y) = 0$ are $(x,y) = (2,3)$, which is therefore the double point.

This curve can be parametrized by translating the implicit curve so that the double point lies at the origin. This is done by making the substitution $x = x + 2$, $y = y + 3$, yielding

$$2\,x^2 + x^3 + 7\,x\,y + 3\,x^2\,y + 6\,y^2 + 3\,x\,y^2 + y^3$$

Parametrization is then performed using the method discussed earlier in this section,

$$x = -\frac{6t^2 + 7t + 2}{t^3 + 3t^2 + 3t + 1}; \quad y = -\frac{6t^3 + 7t^2 + 2t}{t^3 + 3t^2 + 3t + 1}$$

and the parametrized curve is translated back so that the doubled point is again at $(2,3)$:

$$x = -\frac{6t^2 + 7t + 2}{t^3 + 3t^2 + 3t + 1} + 2 = \frac{2t^3 - t}{t^3 + 3t^2 + 3t + 1};$$

$$y = -\frac{6t^3 + 7t^2 + 2t}{t^3 + 3t^2 + 3t + 1} + 3 = \frac{-3t^3 + 2t^2 + 7t + 3}{t^3 + 3t^2 + 3t + 1}$$

19.6 Undetermined Coefficients

An implicit curve equation has $(n+1)(n+2)/2$ terms. Since the equation $f(x,y) = 0$ can be scaled without changing the curve, we can freely assign any of the non-zero coefficients to be 1 (or, any other value that we choose) and we are left with $(n^2 + 3n)/2$ coefficients. This means that we can generally specify $(n^2 + 3n)/2$ points through which an implicit curve will pass. One way this is done is by using the method of undetermined coefficients.

Consider the general conic given by

$$c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y + c_{00} = 0$$

and five points (x_i, y_i) which we wish the curve to interpolate. The coefficients c_{ij} can be solved from the system of linear equations:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \left\{ \begin{matrix} {}_{20} \\ c_{11} \\ c_{02} \\ c_{10} \\ c_{01} \\ c_{00} \end{matrix} \right\} = 0.$$

Of course, this method can be applied to any special equation. For example, the general equation of a circle is $(x-a)^2 + (y-b)^2 - r^2 = 0$. A circle through three points can be found by applying the method of undetermined coefficients to the equation $Ax + By + C = x^2 + y^2$ to compute A, B, C. Then, we find $a = A/2$, $b = B/2$, and $r = \sqrt{C + a^2 + b^2}$.