Chapter 8

Offset Curves

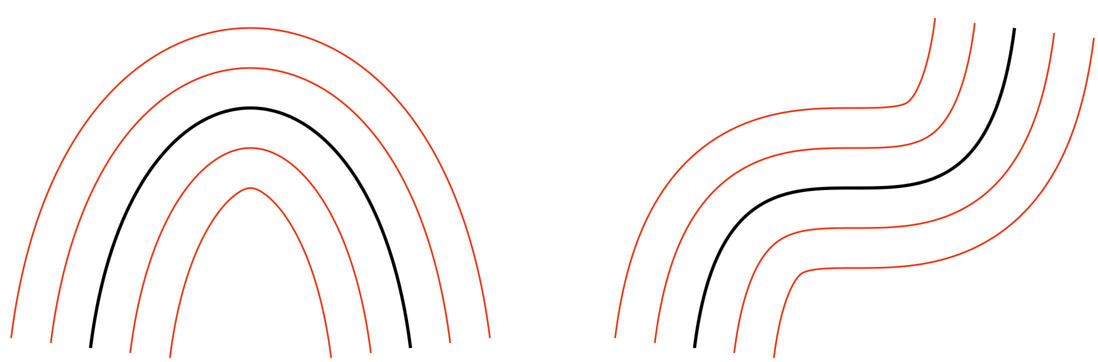
An offset curve is the set of all points that lie a perpendicular distance ρ from a given curve in R^2. The scalar ρ is called the *offset radius*. If the parametric equation of the given curve is

$$\mathbf{P}(t) = (x(t), y(t)) \tag{8.1}$$

then the offset curve with offset radius ρ is given by

$$\Omega(\rho, \mathbf{P}(t)) = \mathbf{P}(t) + \rho \frac{(y'(t), -x'(t))}{\sqrt{x'^2(t) + y'^2(t)}} \tag{8.2}$$

Note that in this definition, if ρ is positive, the offset is on our right as we walk along the base curve in the direction of increasing parameter value.



(a) Offsets of a convex curve.

(b) Offset of a curve that has an inflection point.

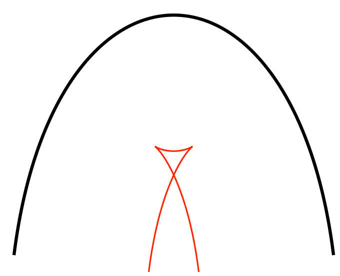
Figure 8.1: Offset Curves.

In Figure 8.1, the red curves are offsets of the black curves.

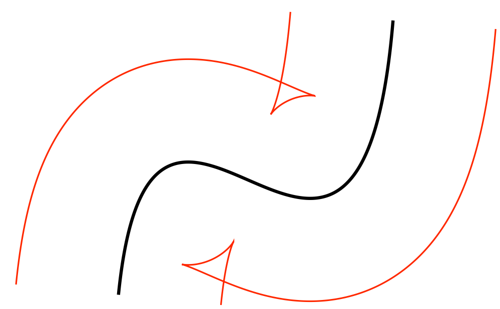
Offset curves play an important roll in computer aided design and manufacturing (CAD/CAM). If a numerically controlled machine is used to cut out a shape, the cutting tool has a finite radius.

Therefore, the path that the tool traverses is an offset curve and the offset radius is the radius of the cutting tool.

It is important that the tool radius is less than the minimum radius of curvature of the curve, otherwise the tool will perform unintended gouging. Figure 8.2 illustrates an offset curve whose radius exceeds the radius of curvature along part of the base curve. Every point at which the radius of curvature in the base curve matches the offset radius creates a cusp in the offset curve. At the cusp, the offset curve changes direction. Between cusps, the first derivative vectors of the base curve and offset curve point in opposite directions.



(a) Offsets of a convex curve.



(b) Offset of a curve that has an inflection point.

Figure 8.2: Offset Curves in which the Offset Radius Exceeds the Radius of Curvature for a Portion of the Base Curve.

If $\mathbf{P}$ is an offset of $\mathbf{Q}$, the reverse is generally true, as long as the offset radius is everywhere less than the radius of curvature of both curve segments:

$$\Omega(-\rho, \Omega(\rho, \mathbf{P}(t))) = \mathbf{P}(t). \tag{8.3}$$

Ini general, offset curves cannot be represented in Bézier form because (8.2) contains a square root of a polynomial. The obvious exceptions are circles and straight lines. A non-obvious exception is that the offset of any parabola can be represented as a degree eight rational Bézier curve.

In addition, a family of curves has been identified by Rida Farouki which can be represented in rational Bézier form. There curves are called Pythagorean Hodograph curves, and they defined by the property that the sum $x'^2(t) + y'^2(t)$ is a perfect square of a polynomial. Farouki has written many papers on this subject that detail some elegant geometric properties that can be imposed on the control polygon of a Bézier curve to assure that it satisfies the Pythagorean Hodograph requirement.