Chapter 18

Implicitization using Moving Lines

This chapter presents a new way of looking at implicitization that has some geometric meaning. It also shows how to express the determinant in more compact form, how to locate the double point of a Béziercurve, and discusses the concept of duality.

18.1 Definition

A pencil of lines can be described by the equation

$$(a_0x + b_0y + c_0)(1 - t) + (a_1x + b_1y + c_1)t = 0 \tag{18.1}$$

where the equations $a_0x + b_0y + c_0 = 0$ and $a_1x + b_1y + c_1 = 0$ define any two distinct lines.

It is known that any conic section can be "generated by the intersection of corresponding lines of two related pencils in a plane" [Som51], p.388. In other words, given two distinct pencils, $(a_{00}x+b_{00}y+c_{00})(1-t)+(a_{10}x+b_{10}y+c_{10})t = 0$ and $(a_{01}x+b_{01}y+c_{01})(1-t)+(a_{11}x+b_{11}y+c_{11})t = 0$, to each value of t corresponds exactly one line from each pencil, and those two lines intersect in a point. The locus of points thus created for $-\infty \leq t \leq \infty$ is a conic section, as illustrated in Figure 18.1. This is reviewed in section 18.2.

This chapter examines the extension of that idea to higher degrees. A degree n family of lines intersects a degree m family of lines in a curve of degree $m + n$, which is discussed in section 18.3. Section 18.4 shows that *any* rational curve can be described as the intersection of two families of lines, from which the multiple points and the implicit equation of the curve can be easily obtained. For example, any cubic rational curve can be described as the intersection of a pencil of lines and a quadratic family of lines. The pencil axis lies at the double point of the cubic curve. Section 18.5 discusses the family of lines which is tangent to a given rational curve. Such families of lines are useful for analyzing the singularities of the curve, such as double points, cusps, and inflection points, and also for calculating derivative directions.

18.1.1 Homogeneous Points and Lines

In projective geometry, the point whose homogeneous coordinates are (X, Y, W) has Cartesian coordinates $(x, y) = (X/W, Y/W)$. Of course, X, Y, and W cannot all be zero. The equation of a line in homogenous form is

$$aX + bY + cW = 0 \tag{18.2}$$

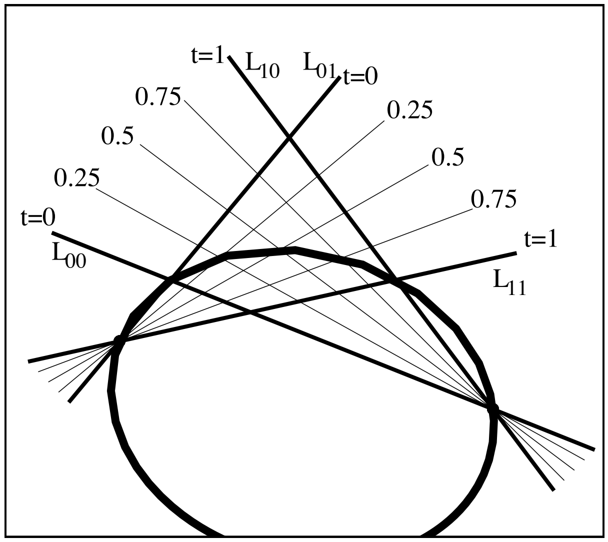

Figure 18.1: Intersection of Two Pencils of Lines

(a, b, and c not all zero).

Given two triples, (a, b, c) and (d, e, f), the operation of cross product is defined

$$(a, b, c) \times (d, e, f) = (bf - ec, dc - af, ae - db) \tag{18.3}$$

and the dot product is defined

$$(a, b, c) \cdot (d, e, f) = ad + be + cf. \tag{18.4}$$

In this chapter, all single characters in bold typeface signify a triple. In particular, the symbol $\mathbf{P}$ denotes the triple (X, Y, W), and $\mathbf{L}$ symbolizes the triple (a, b, c). When we refer to "the line $\mathbf{L}$", we mean

$$\{(X, Y, W) | \mathbf{L} \cdot \mathbf{P} = (a, b, c) \cdot (X, Y, W) = aX + bY + cW = 0\}. \tag{18.5}$$

Thus, we can say that a point $\mathbf{P}$ lies on a line $\mathbf{L} = (a, b, c)$ if and only if $\mathbf{P} \cdot \mathbf{L} = 0$.

The cross product has the following two applications: The line $\mathbf{L}$ containing two points $\mathbf{P}_1 = (X_1, Y_1, W_1)$ and $\mathbf{P}_2 = (X_2, Y_2, W_2)$ is given by

$$\mathbf{L} = \mathbf{P}_1 \times \mathbf{P}_2. \tag{18.6}$$

The point $\mathbf{P}$ at which two lines $\mathbf{L}_1 = (a_1, b_1, c_1)$ and $\mathbf{L}_2 = (a_2, b_2, c_2)$ intersect is given by

$$\mathbf{P} = \mathbf{L}_1 \times \mathbf{L}_2. \tag{18.7}$$

This illustrates the principle of *duality*. Loosely speaking, general statements involving points and lines can be expressed in a reciprocal way. For example, the statement "a unique line passes through two distinct points" has a dual expression, "a unique point lies at the intersection of two distinct lines".

In general, any triple (α, β, γ) can be interpreted as a line $(\{(X, Y, W)| = \alpha X + \beta Y + \gamma W = 0\})$ or as a point (whose Cartesian coordinates are $(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma})$). To remove the ambiguity, triples which are

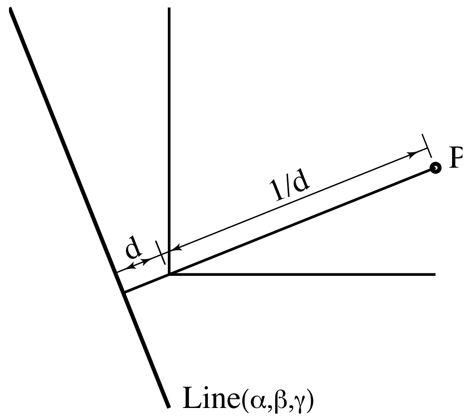

Figure 18.2: Dual Point and Line

to be interpreted as points are preceded by the prefix 'Point' (*i.e.* Point(α, β, γ)), and triples which signify lines are preceded by 'Line'. The *dual* of Line(α, β, γ) is Point(α, β, γ), and vice versa. Dual points and lines are related geometrically as follows:

- If $d = \frac{\gamma}{\sqrt{\alpha^2+\beta^2}}$ is the distance from Line(α, β, γ) to the origin, then $1/d$ is the distance from Point(α, β, γ) to the origin.
- The line from the origin to Point(α, β, γ) is perpendicular to Line(α, β, γ).

This relationship is illustrated in Figure 18.2.

Discussion of the duals of Bézier and B-spline curves can be found in [Hos83, Sab87].

18.1.2 Curves and Moving Lines

A homogeneous point whose coordinates are functions of a variable is denoted by appending the variable, enclosed in square brackets:

$$\mathbf{P}[t] = (X[t], Y[t], W[t]) \tag{18.8}$$

which amounts to the rational curve

$$x = \frac{X[t]}{W[t]}; \qquad y = \frac{Y[t]}{W[t]}. \tag{18.9}$$

If these functions are polynomials in Bernstein form

$$X[t] = \sum_{i=0}^{n} X_i B_i^n[t]; \quad Y[t] = \sum_{i=0}^{n} Y_i B_i^n[t]; \quad W[t] = \sum_{i=0}^{n} W_i B_i^n[t] \tag{18.10}$$

with $B_i^n[t] = \binom{n}{i}(1-t)^{n-i}t^i$, then equation (18.8) defines a rational Bézier curve

$$\mathbf{P}[t] = \sum_{i=0}^{n} \mathbf{P}_i B_i^n[t] \tag{18.11}$$

with homogeneous control points $\mathbf{P}_i = (X_i, Y_i, W_i)$. Customarily, these control points are thought of as having Cartesian coordinates $(\frac{X_i}{W_i}, \frac{Y_i}{W_i})$ with weights W_i. If $W[t] \equiv \gamma$, that is, if $W[t]$ is a constant, then the curve is referred to as a *polynomial curve.* This happens if $W_0 = W_1 = \ldots = W_n = \gamma$. In this chapter, the word *curve* generally means *rational curve* unless it is referred to as polynomial curve.

Likewise,

$$\mathbf{L}[t] = (a[t], b[t], c[t]) \tag{18.12}$$

denotes the family of lines

$$a[t]x + b[t]y + c[t] = 0. \tag{18.13}$$

Such a family of lines is traditionally known as a *pencil* of lines if $a[t]$, $b[t]$, and $c[t]$ are linear functions of t. In the general case, [Win23] refers to families of lines as *line equations*, and families of points (*i.e.*, curves) as *point equations.* In this chapter, it is more comfortable to refer to a parametric family of lines as in equation (18.12) as a *moving line.* Also, a curve $\mathbf{P}[t]$ will be referred to as a *moving point* when the varying position of point $\mathbf{P}[t]$ is of interest.

A moving point $\mathbf{P}[t]$ *follows* a moving line $\mathbf{L}[t]$ (or, equivalently, the moving line *follows* the moving point) if

$$\mathbf{P}[t] \cdot \mathbf{L}[t] \equiv 0, \tag{18.14}$$

that is, if point $\mathbf{P}[t]$ lies on line $\mathbf{L}[t]$ for all values of t. Two moving points $\mathbf{P}_1[t]$ and $\mathbf{P}_2[t]$ follow a moving line $\mathbf{L}[t]$ if

$$\mathbf{P}_1[t] \times \mathbf{P}_2[t] \equiv \mathbf{L}[t]k[t] \tag{18.15}$$

where $k[t]$ is a scalar rational function of degree zero or greater. Two moving lines $\mathbf{L}_1[t]$ and $\mathbf{L}_2[t]$ *intersect* at a moving point $\mathbf{P}[t]$ if

$$\mathbf{L}_1[t] \times \mathbf{L}_2[t] \equiv \mathbf{P}[t]k[t]. \tag{18.16}$$

18.1.3 Weights and Equivalency

Two homogeneous points (X_1, Y_1, W_1) and (X_2, Y_2, W_2) represent the same point in Cartesian coordinates $(X_1/W_1, Y_1/W_1) = (X_2/W_2, Y_2/W_2)$ if and only if $(X_1, Y_1, W_1) = k(X_2, Y_2, W_2)$ where k is a non-zero constant. Likewise, lines $\mathbf{L}_1$ and $\mathbf{L}_2$ are the same if and only if $\mathbf{L}_1 = k\mathbf{L}_2$. This fact can be extended to curves and moving lines. For example, two curves $\mathbf{P}_1[t]$ and $\mathbf{P}_2[t]$ are equivalent if $\mathbf{P}_1[t] \equiv k[t]\mathbf{P}_2[t]$ where $k[t]$ is a scalar rational function of t. Two curves with identical shape are not equivalent if the parametrizations are different each other.

Even though two homogeneous points (or lines) may map to the same Cartesian point (or line), they do not always create identical results. For example, a control point $\mathbf{P}_i$ of a rational curve as in equation (18.11) cannot be replaced with a scale of itself without altering the curve, for scaling would change its weight. Of course, if all the control points were scaled by the same value, the curve would be unchanged.

18.2 Pencils and Quadratic Curves

18.2.1 Pencils of lines

Given any two lines $\mathbf{L}_0 = (a_0, b_0, c_0)$ and $\mathbf{L}_1 = (a_1, b_1, c_1)$, a pencil of lines $\mathbf{L}[t]$ can be expressed

$$\mathbf{L}[t] = \mathbf{L}_0(1 - t) + \mathbf{L}_1 t. \tag{18.17}$$

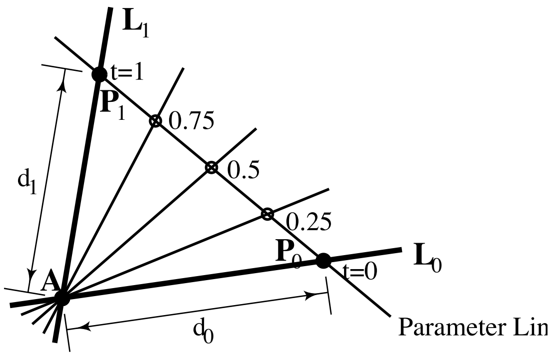

Figure 18.3: Pencil of Lines

All lines in the pencil contain the point at which $\mathbf{L}_0$ and $\mathbf{L}_1$ intersect. We will refer to this point as the pencil *axis*. There is a one-to-one correspondence between lines in the pencil and parameter values t.

The rate at which the moving line $\mathbf{L}[t]$ rotates about its axis as t varies is most easily visualized by introducing a *parameter line* as shown in Figure 18.3. The parameter line is a degree 1 polynomial Bézier curve

$$\mathbf{P}[t] = (1-t)\mathbf{P}_0 + t\mathbf{P} \tag{18.18}$$

which follows $\mathbf{L}[t]$. $\mathbf{P}_0$ lies on $\mathbf{L}_0$ a distance d_0 from the axis, and $\mathbf{P}_1$ lies on $\mathbf{L}_1$ a distance d_1 from the axis where

$$\frac{d_0}{d_1} = \frac{\sqrt{a_0^2 + b_0^2}}{\sqrt{a_1^2 + b_1^2}}. \tag{18.19}$$

If $\mathbf{L}[t]$ is to rotate counter clockwise as t increases, then

$$a_0 b_1 > a_1 b_0. \tag{18.20}$$

If parameter values are marked off evenly along the parameter line as shown, the line $\mathbf{L}[t]$ passes through the parameter line at the point corresponding to t.

Notice that d_0 and d_1 control the rate at which $\mathbf{L}[t]$ rotates. Figure 18.4 shows an example in which d_0 is larger than d_1, with the effect that lines defined by evenly spaced increments are concentrated near $\mathbf{L}_0$.

An alternate way to specify a pencil of lines is with an axis $\mathbf{P}_A = (X_A, Y_A, W_A)$ and a rational linear Bézier curve

$$\mathbf{P}[t] = \mathbf{P}_0(1-t) + \mathbf{P}_1 t = (X_0, Y_0, W_0)(1-t) + (X_1, Y_1, W_1)t. \tag{18.21}$$

A pencil $\mathbf{L}[t]$ can then be defined as the set of lines connecting $\mathbf{P}_A$ with points on $\mathbf{P}[t]$:

$$\mathbf{L}[t] = \mathbf{P}_A \times \mathbf{P}[t] = \mathbf{P}_A \times (\mathbf{P}_0(1-t) + \mathbf{P}_1 t). \tag{18.22}$$

This representation of a pencil is related to the representation in equation (18.17). The two lines $\mathbf{L}_0$ and $\mathbf{L}_1$ can be expressed

$$\begin{aligned} \mathbf{L}_0 &= \mathbf{P}_A \times \mathbf{P}_0 \\ \mathbf{L}_1 &= \mathbf{P}_A \times \mathbf{P}_1 \end{aligned} \tag{18.23}$$

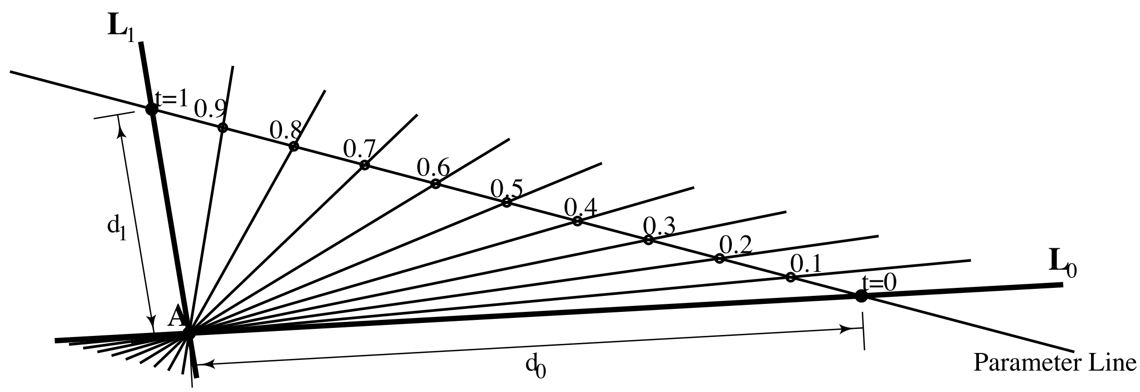

Figure 18.4: Pencil of Lines

and the parameter line is positioned with distances d_0 and d_1 which satisfy

$$\frac{d_0}{d_1} = \frac{\overline{\mathbf{P}_A\mathbf{P}_0}W_0}{\overline{\mathbf{P}_A\mathbf{P}_1}W_1}, \tag{18.24}$$

as shown in Figure 18.5. Thus, we see that every pencil of lines follows a degree one polynomial curve.

18.2.2 Intersection of Two Pencils

Consider what happens when two pencils of lines intersect, line by line. Figure 18.6 shows two pencils

$$\begin{aligned} \mathbf{L}_0[t] &= \mathbf{L}_{00}(1-t) + \mathbf{L}_{01}t \\ \mathbf{L}_1[t] &= \mathbf{L}_{10}(1-t) + \mathbf{L}_{11}t, \end{aligned} \tag{18.25}$$

and the points at which they intersect for parameter values $t = 0, .25, .5, .75, 1$. It is clear that those five sample points are not collinear, and in fact as t varies continuously from 0 to 1, the two pencils intersect in a smooth curve as shown. This curve turns out to be a conic section, which can be expressed as a rational Bézier curve $\mathbf{P}[t]$ as follows.

$$\begin{aligned} \mathbf{P}[t] &= \mathbf{L}_0[t] \times \mathbf{L}_1[t] \\ &= (\mathbf{L}_{00}(1-t) + \mathbf{L}_{01}t) \times (\mathbf{L}_{10}(1-t) + \mathbf{L}_{11}t) \\ &= (\mathbf{L}_{00} \times \mathbf{L}_{10})(1-t)^2 + \frac{1}{2}(\mathbf{L}_{00} \times \mathbf{L}_{11} + \mathbf{L}_{01} \times \mathbf{L}_{10})2(1-t)t + (\mathbf{L}_{01} \times \mathbf{L}_{11})t^2, \end{aligned} \tag{18.26}$$

which expresses $\mathbf{P}$[t] as a quadratic rational Bézier curve whose control points are:

$$\begin{aligned} \mathbf{P}_0 &= \mathbf{L}_{00} \times \mathbf{L}_{10} \\ \mathbf{P}_1 &= \frac{1}{2}(\mathbf{L}_{00} \times \mathbf{L}_{11} + \mathbf{L}_{01} \times \mathbf{L}_{10}) \\ \mathbf{P}_2 &= \mathbf{L}_{01} \times \mathbf{L}_{11}. \end{aligned} \tag{18.27}$$

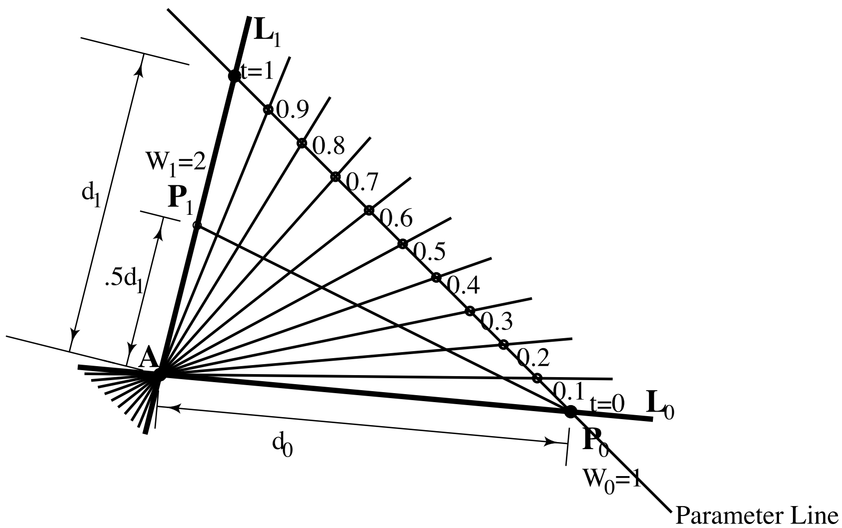

Figure 18.5: Pencil of Lines

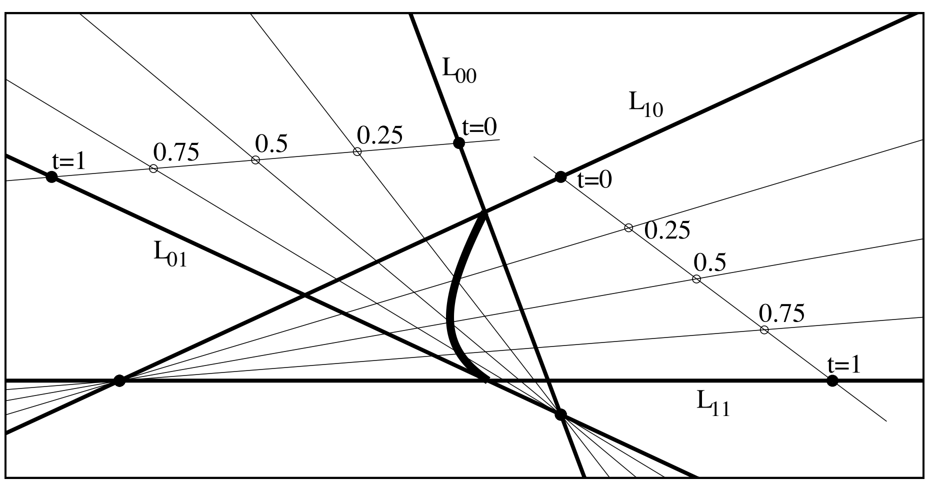

Figure 18.6: Intersection of Two Pencils of Lines

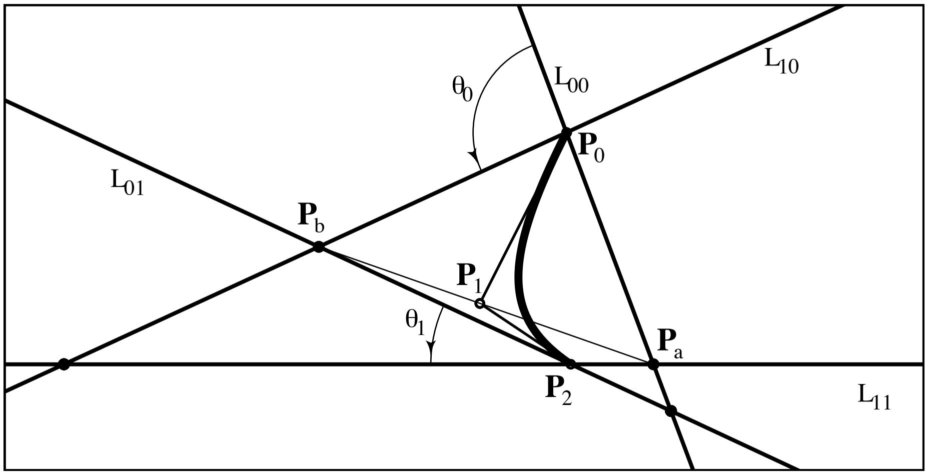

Figure 18.7: Rational Quadratic Curve

The geometric relation between the control points and weights are shown in Figure 18.7. Control point $\mathbf{P}_0$ lies at the intersection of $\mathbf{L}_{00}$ and $\mathbf{L}_{10}$ and has a weight of $d_{00}d_{10}\sin(\theta_0)$, where $d_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2}$. Control point $\mathbf{P}_2$ lies at the intersection of $\mathbf{L}_{01}$ and $\mathbf{L}_{11}$ and has a weight of $d_{01}d_{11}\sin(\theta_1)$. Control point $\mathbf{P}_1$ and its weight is computed using the auxiliary points $\mathbf{P}_a$ and $\mathbf{P}_b$ which are the intersection of $\mathbf{L}_{00}$ and $\mathbf{L}_{11}$, and $\mathbf{L}_{01}$ and $\mathbf{L}_{10}$, respectively.

The two pencils of lines provide a useful intermediate representation of a conic section, from which the implicit equation and parametric equations can be derived with equal ease. The implicit equation can be found by eliminating the pencil parameter t from the two pencil equations

$$\begin{aligned} L_{00}(1-t) + L_{10}t &= 0 \\ L_{10}(1-t) + L_{11}t &= 0 \end{aligned} \tag{18.28}$$

where

$$L_{ij} = \mathbf{L}_{ij} \cdot \mathbf{P} = a_{ij}X + b_{ij}Y + c_{ij}W. \tag{18.29}$$

In matrix form,

$$\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} \begin{Bmatrix} 1-t \\ t \end{Bmatrix} = 0 \tag{18.30}$$

from which the implicit equation of the intersection locus is

$$L_{00}L_{11} - L_{01}L_{10} = 0. \tag{18.31}$$

18.2.3 Pencils on Quadratic Curves

An arbitrary rational quadratic Bézier curve

$$\mathbf{P}[t] = \mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2(1-t)t + \mathbf{P}_2 t^2 \tag{18.32}$$

can be represented as the intersection of two pencils. Consider a moving line $\mathbf{L}[t]$ which goes through a certain fixed point $\mathbf{P}[k]$ on the curve and follows the moving point $\mathbf{P}[t]$. This moving line

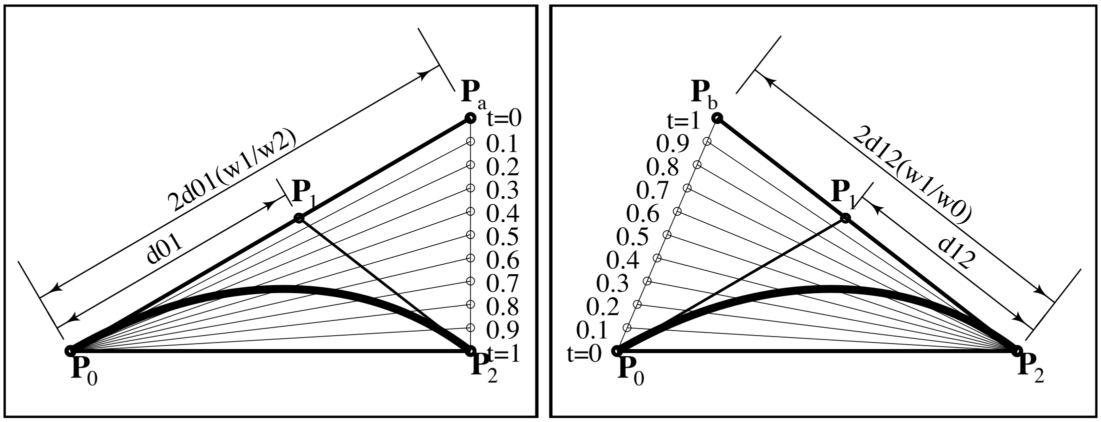

Figure 18.8: Quadratic Bézier Curve

is expressed as follows.

$$\begin{aligned}
\mathbf{L}[t] &= \mathbf{P}[k] \times \mathbf{P}[t] \\
&= (\mathbf{P}_0(1-k)^2 + \mathbf{P}_1 2(1-k)k + \mathbf{P}_2 k^2) \times (\mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2(1-t)t + \mathbf{P}_2 t^2) \\
&= (t-k) \left\{ \begin{array}{cc} 1-k & k \end{array} \right\} \left[\begin{array}{cc} 2\mathbf{P}_0 \times \mathbf{P}_1 & \mathbf{P}_0 \times \mathbf{P}_2 \\ \mathbf{P}_0 \times \mathbf{P}_2 & 2\mathbf{P}_1 \times \mathbf{P}_2 \end{array} \right] \left\{ \begin{array}{c} 1-t \\ t \end{array} \right\},
\end{aligned} \tag{18.33}$$

from which this $\mathbf{L}[t]$ is always equivalent to a linear moving line (*i.e.* a pencil). Thus, two pencils can be found by choosing arbitrary two parameter values k_0 and k_1 and calculating $\mathbf{P}[k_i] \times \mathbf{P}[t]$. The simplest example is $k_0 = 0$, $k_1 = 1$. In this case,

$$\begin{aligned}
\mathbf{P}[0] \times \mathbf{P}[t] &= t\,[\mathbf{P}_0 \times (2\mathbf{P}_1(1-t) + \mathbf{P}_2 t)] \\
\mathbf{P}[1] \times \mathbf{P}[t] &= (1-t)\,[\mathbf{P}_2 \times (\mathbf{P}_0(1-t) + 2\mathbf{P}_1 t)]\,,
\end{aligned} \tag{18.34}$$

from which the two pencils intersecting at the moving point $\mathbf{P}[t]$ are

$$\begin{aligned}
\mathbf{L}_0 &= \mathbf{P}_0 \times (2\mathbf{P}_1(1-t) + \mathbf{P}_2 t) \\
\mathbf{L}_1 &= \mathbf{P}_2 \times (\mathbf{P}_0(1-t) + 2\mathbf{P}_1 t).
\end{aligned} \tag{18.35}$$

Figure 18.8 shows the geometric relationship. The parameter lines of pencils $\mathbf{L}_0[t]$ and $\mathbf{L}_1[t]$ are $\mathbf{P}_a\mathbf{P}_2$ and $\mathbf{P}_0\mathbf{P}_b$, respectively, where

$$\begin{aligned}
\overline{\mathbf{P}_0\mathbf{P}_a} &= \frac{2w_1}{w_2}\overline{\mathbf{P}_0\mathbf{P}_1}, \\
\overline{\mathbf{P}_2\mathbf{P}_b} &= \frac{2w_1}{w_0}\overline{\mathbf{P}_2\mathbf{P}_1}.
\end{aligned} \tag{18.36}$$

These parameter lines have the property of a *nomogram*. Consider a point $\mathbf{P}_p$ on the curve. If the line $\mathbf{P}_0\mathbf{P}_p$ intersects the parameter line $\mathbf{P}_a\mathbf{P}_2$ at the point $\mathbf{P}_q$, then the parameter value t at $\mathbf{P}_p$ can be found as

$$t = \frac{\overline{\mathbf{P}_a\mathbf{P}_q}}{\overline{\mathbf{P}_a\mathbf{P}_2}}. \tag{18.37}$$

18.3 Moving Lines

18.3.1 Bernstein Form

In equation (18.12), if the polynomials $a[t]$, $b[t]$, and $c[t]$ are of degree n, one way to define the moving line $\mathbf{L}[t]$ is to use $n+1$ *control lines* $\mathbf{L}_i$ where

$$\mathbf{L}[t] = \sum_{i=0}^{n} \mathbf{L}_i B_i^n[t]. \tag{18.38}$$

Since this is the dual of a Bézier curve, it has many nice properties. For example, $\mathbf{L}[t]$ moves from $\mathbf{L}_0$ to $\mathbf{L}_n$ when t is changing from 0 to 1. The line $\mathbf{L}[\tau]$ for any parameter value τ can be calculated using the de Casteljau algorithm. Denote

$$\mathbf{L}_i^{<k>}[\tau] = (1-\tau)\mathbf{L}_i^{<k-1>}[\tau] + \tau\mathbf{L}_{i+1}^{<k-1>}[\tau] \tag{18.39}$$

where

$$\mathbf{L}_i^{<0>}[\tau] = \mathbf{L}_i. \tag{18.40}$$

Then, $\mathbf{L}_0^{<n>}[\tau]$ is the required line $\mathbf{L}[\tau]$. This algorithm also *subdivides* the moving line in the same way as for a Bézier curve.

18.3.2 Moving Line which Follows Two Moving Points

Consider two moving points $\mathbf{P}[t]$ and $\mathbf{Q}[t]$ of degree m and n respectively. The moving line $\mathbf{L}[t]$ that follows the two moving points is

$$\mathbf{L}[t] = \mathbf{P}[t] \times \mathbf{Q}[t], \tag{18.41}$$

and its degree is generally $m+n$. If $\mathbf{P}[t]$ and $\mathbf{Q}[t]$ are Bézier curves

$$\begin{aligned} \mathbf{P}[t] &= \sum_{i=0}^{m} B_i^m[t]\mathbf{P}_i \\ \mathbf{Q}[t] &= \sum_{j=0}^{n} B_j^n[t]\mathbf{Q}_j, \end{aligned} \tag{18.42}$$

then the control lines $\mathbf{L}_k$ of the moving line are expressed as follows:

$$\mathbf{L}[t] = \sum_{k=0}^{m+n} B_k^{m+n}[t]\mathbf{L}_k \tag{18.43}$$

$$\mathbf{L}_k = \frac{1}{\begin{pmatrix} m+n \\ k \end{pmatrix}} \sum_{i+j=k} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \mathbf{P}_i \times \mathbf{Q}_j. \tag{18.44}$$

For example, if a moving line $\mathbf{L}[t]$ follows a degree one moving point

$$\mathbf{P}[t] = \mathbf{P}_0(1-t) + \mathbf{P}_1 t \tag{18.45}$$

and a degree two moving point

$$\mathbf{Q}[t] = \mathbf{Q}_0(1-t)^2 + \mathbf{Q}_1 2(1-t)t + \mathbf{Q}_2 t^2, \tag{18.46}$$

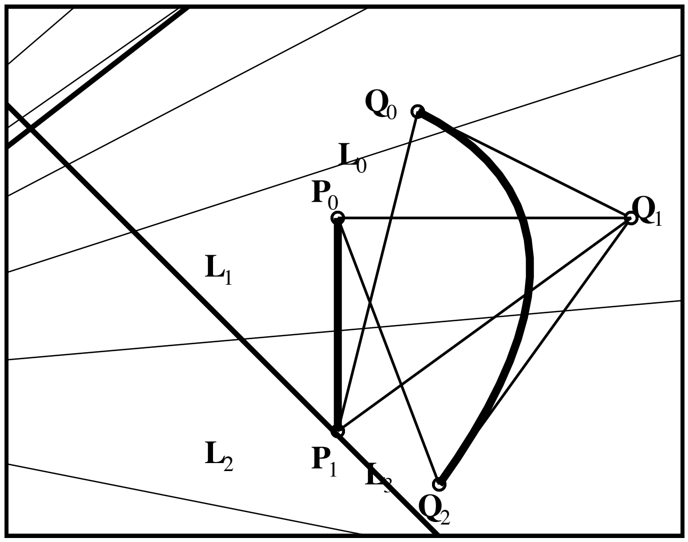

Figure 18.9: Cubic Moving Line which Follow Linear and Quadratic Moving Points

then $\mathbf{L}[t]$ is a cubic moving line with control lines

$$\begin{aligned}
\mathbf{L}_0 &= \mathbf{P}_0 \times \mathbf{Q}_0 \\
\mathbf{L}_1 &= \frac{1}{3}(2\mathbf{P}_0 \times \mathbf{Q}_1 + \mathbf{P}_1 \times \mathbf{Q}_0) \\
\mathbf{L}_2 &= \frac{1}{3}(\mathbf{P}_0 \times \mathbf{Q}_2 + 2\mathbf{P}_1 \times \mathbf{Q}_1) \\
\mathbf{L}_3 &= \mathbf{P}_1 \times \mathbf{Q}_2.
\end{aligned} \tag{18.47}$$

The relationship between the control lines and the control points is shown in Figure 18.9.

18.3.3 Intersection of Two Moving Lines

Two moving lines $\mathbf{L}_0[t]$ and $\mathbf{L}_1[t]$ of degree m and n respectively, intersect at a moving point

$$\mathbf{P}[t] = \mathbf{L}_0[t] \times \mathbf{L}_1[t], \tag{18.48}$$

whose degree is generally $m+n$. This is the dual of what we discussed in section 18.3.2. Therefore, the relationship between the control lines of $\mathbf{L}_0[t]$ and $\mathbf{L}_1[t]$ and the control points of $\mathbf{P}[t]$ is

$$\mathbf{P}_k = \frac{1}{\binom{m+n}{k}} \sum_{i+j=k} \binom{m}{i}\binom{n}{j} \mathbf{L}_{0i} \times \mathbf{L}_{1j}. \tag{18.49}$$

Figure 18.10 shows an example of cubic moving point $\mathbf{P}[t]$ which is the intersection of a pencil $\mathbf{L}_0[t]$ and a quadratic moving line $\mathbf{L}_1[t]$. Notice that each control point of $\mathbf{P}[t]$ can be calculated

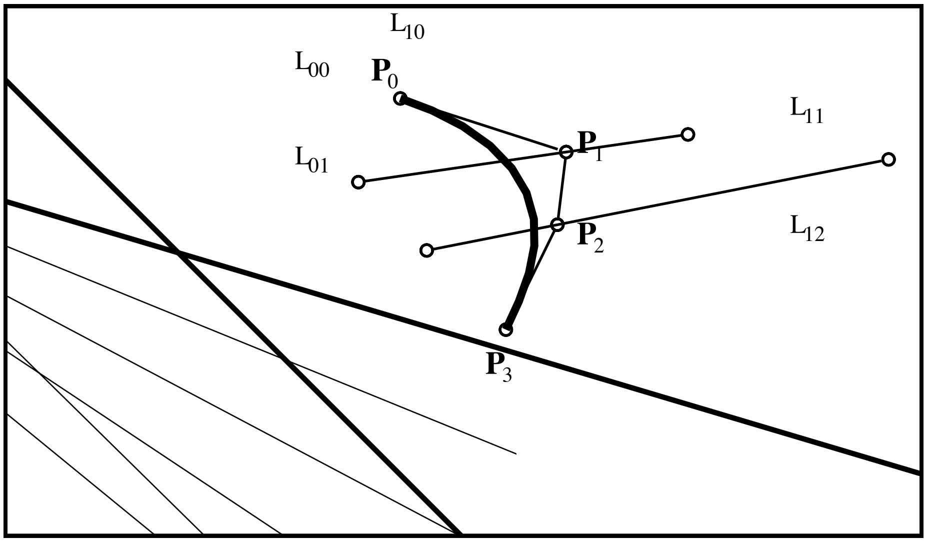

Figure 18.10: Intersection of Linear and Quadratic Moving Lines

from intersection(s) of the control lines as follows:

$$
\begin{aligned}
\mathbf{P}_0 &= \mathbf{L}_{00} \times \mathbf{L}_{10} \\
\mathbf{P}_1 &= \frac{1}{3}(2\mathbf{L}_{00} \times \mathbf{L}_{11} + \mathbf{L}_{01} \times \mathbf{L}_{10}) \\
\mathbf{P}_2 &= \frac{1}{3}(\mathbf{L}_{00} \times \mathbf{L}_{12} + 2\mathbf{L}_{01} \times \mathbf{L}_{11}) \\
\mathbf{P}_3 &= \mathbf{L}_{01} \times \mathbf{L}_{12}.
\end{aligned}
\tag{18.50}
$$

18.3.4 Base Points

Any parameter value t for which $\mathbf{P}[t] = (0,0,0)$ or $\mathbf{L}[t] = (0,0,0)$ is called a *base point.* Any curve or moving line which has a base point can be replaced by an equivalent curve or moving line of degree one less. For example, if for some parameter value $t = \tau$, $\mathbf{P}[\tau] = (0,0,0)$, then $t - \tau$ would be a factor of $X[t]$, $Y[t]$, and $W[t]$ and we could divide out that factor to get

$$
\mathbf{P}[t] = (t-\tau)(\frac{X[t]}{t-\tau}, \frac{Y[t]}{t-\tau}, \frac{W[t]}{t-\tau}). \tag{18.51}
$$

In equation (18.41), assume $\mathbf{P}$ and $\mathbf{Q}$ have no base points themselves. Then, the degree of $\mathbf{L}[t]$ is generally $m+n$. The degree will be less than $m+n$ only if $\mathbf{L}$ has a base point. This can only happen if there exists a value τ such that $\mathbf{P}[\tau] \times \mathbf{Q}[\tau] = (0,0,0)$, which can only happen if at one of the $m \times n$ points at which $\mathbf{P}$ and $\mathbf{Q}$ intersect, both curves have the same parameter value τ. Thus, the degree of $\mathbf{L}$ is $m+n-r$ where r is the number of base points. The same thing happens in equation (18.48).

18.3.5 Axial Moving Lines

Moving lines of degree $n > 1$ generally have no axis, that is, no single point about which the line rotates. The special case in which a moving line does have an axis will be referred to as an *axial moving line*. A degree n axial moving line $\mathbf{L}[t]$ can be expressed

$$\mathbf{L}[t] = \mathbf{P}_A \times \mathbf{P}[t] \tag{18.52}$$

where $\mathbf{P}_A$ is the axis and a $\mathbf{P}[t]$ is a degree n curve. An axial moving line can be thought of as following a degree 0 moving point (*i.e.* a fixed point) and a degree n moving point (*i.e.* curve). The relationship between the control lines of the axial moving line and the control points of the curve is

$$\mathbf{L}_i = \mathbf{P}_A \times \mathbf{P}_i. \tag{18.53}$$

Notice that all the control lines $\mathbf{L}_i$ of the axial pencil go through the axis.

18.4 Curve Representation with Two Moving Lines

18.4.1 Axial Moving Line on a Curve

Consider an axial moving line $\mathbf{L}[t]$ which follows a degree n Bézier curve $\mathbf{P}[t]$:

$$\mathbf{L}[t] = \mathbf{P}_A \times \mathbf{P}[t]. \tag{18.54}$$

Though the degree of this moving line is generally n, it can always be reduced to degree $n-1$ if the axis lies on the curve

$$\mathbf{P}_A = \mathbf{P}[\tau], \tag{18.55}$$

because then the moving line has a base point.

We have already confirmed this fact for quadratic Bézier curves in section 18.2.2. For cubic Béziers,

$$\begin{aligned}
\mathbf{L}[t] &= \mathbf{P}[\tau] \times \mathbf{P}[t] \\
&= (t-\tau)\left\{\; (1-\tau)^2 \quad (1-\tau)\tau \quad \tau^2 \;\right\} \\
&\quad \begin{bmatrix} 3\mathbf{P}_0 \times \mathbf{P}_1 & 3\mathbf{P}_0 \times \mathbf{P}_2 & \mathbf{P}_0 \times \mathbf{P}_3 \\ 3\mathbf{P}_0 \times \mathbf{P}_2 & \mathbf{P}_0 \times \mathbf{P}_3 + 9\mathbf{P}_1 \times \mathbf{P}_2 & 3\mathbf{P}_1 \times \mathbf{P}_3 \\ \mathbf{P}_0 \times \mathbf{P}_3 & 3\mathbf{P}_1 \times \mathbf{P}_3 & 3\mathbf{P}_2 \times \mathbf{P}_3 \end{bmatrix} \left\{ \begin{array}{c} (1-t)^2 \\ (1-t)t \\ t^2 \end{array} \right\}
\end{aligned} \tag{18.56}$$

where $\mathbf{P}_i$ $(i = 0, 1, 2, 3)$ are the control points of the Bézier curve. In general,

$$\begin{aligned}
\mathbf{L}[t] &= (t-\tau)\left\{\; (1-\tau)^{n-1} \quad (1-\tau)^{n-2}\tau \quad \ldots \quad \tau^{n-1} \;\right\} \\
&\quad \begin{bmatrix} \mathbf{L}_{0,0} & \mathbf{L}_{0,1} & \ldots & \mathbf{L}_{0,n-1} \\ \mathbf{L}_{1,0} & \mathbf{L}_{1,1} & \ldots & \mathbf{L}_{1,n-1} \\ \vdots & \vdots & ::: & \vdots \\ \mathbf{L}_{n-1,0} & \mathbf{L}_{n-1,1} & \ldots & \mathbf{L}_{n-1,n-1} \end{bmatrix} \left\{ \begin{array}{c} (1-t)^{n-1} \\ (1-t)^{n-2}t \\ \vdots \\ t^{n-1} \end{array} \right\}
\end{aligned} \tag{18.57}$$

where

$$\mathbf{L}_{i,j} = \sum_{l+m=i+j+1} \binom{n}{l} \binom{n}{m} \mathbf{P}_l \times \mathbf{P}_m \tag{18.58}$$

Notice that the determinant of the $n \times n$ matrix is equivalent to the Bezout's resultant which gives the implicit equation of the curve.

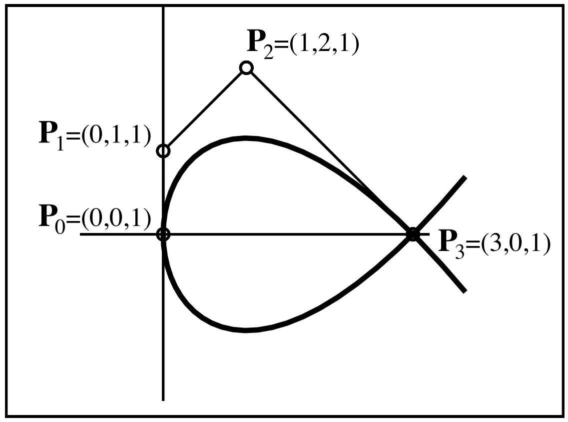

Figure 18.11: Cubic Bézier Curve

18.4.2 Axial Moving Line on a Double Point

Consider what is happening when the axis is on a double point of a degree n curve. Since the curve $\mathbf{P}[t]$ goes through the axis twice at certain parameter values $t = \tau_0, \tau_1$ (which may be imaginary), the moving line has two base points

$$\mathbf{L}[\tau_0] = \mathbf{L}[\tau_1] = 0. \tag{18.59}$$

Therefore, the degree of the axial moving line is reducible to $n - 2$. In general, if the axis is on a point of multiplicity m, then the degree of the axial moving line which follows the degree n curve is $n - m$.

Figure 18.11 shows an example of a cubic curve. This cubic Bézier curve has a double point at $\mathbf{P}_3$. Thus the axial moving line with the axis $\mathbf{P}_3$ is a pencil:

$$\begin{aligned}
\mathbf{L}[t] &= \mathbf{P}_3 \times \mathbf{P}[t] \\
&= (3,0,1) \times \left\{(0,0,1)(1-t)^3 + (0,1,1)3(1-t)^2 t + (1,2,1)3(1-t)t^2 + (3,0,1)t^3\right\} \\
&= (t-1)(3t^2 + 3t, 3t + 3, -9t^2 - 9t) \\
&= (t-1)(t+1)(3t, 3, -9t) \\
&= (t-1)(t+1)\left\{(0,3,0)(1-t) + (3,3,-9)t\right\}.
\end{aligned} \tag{18.60}$$

Two roots of $\mathbf{L}[t] = 0$ shows the type of the double point. In this case, it has two distinct real roots $t = \pm 1$, so the double point is a *crunode*. If the two roots are identical or imaginary, then the double point is a *cusp* or an *acnode*, respectively.

18.4.3 Cubic Curves

For any rational cubic Bézier curve, we can find a pencil and a quadratic moving line which intersects at the moving point. Let us calculate them for the example of Figure 18.11. First of all, make axial moving lines which follow the moving point by using equation (18.56)

$$\begin{aligned}
\mathbf{P}[\tau] \times \mathbf{P}[t] = (t-\tau)&\left\{ \begin{matrix} (1-\tau)^2 & (1-\tau)\tau & \tau^2 \end{matrix} \right\} \\
&\begin{bmatrix} (-3,0,0) & (-6,3,0) & (0,3,0) \\ (-6,3,0) & (-9,12,-9) & (3,9,-9) \\ (0,3,0) & (3,9,-9) & (6,6,-18) \end{bmatrix} \left\{ \begin{matrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{matrix} \right\}.
\end{aligned} \tag{18.61}$$

Since this moving line follows the moving point $\mathbf{P}[t]$ for any τ, the following equation is obtained

$$\begin{bmatrix} (-3,0,0) & (-6,3,0) & (0,3,0) \\ (-6,3,0) & (-9,12,-9) & (3,9,-9) \\ (0,3,0) & (3,9,-9) & (6,6,-18) \end{bmatrix} \left\{ \begin{array}{c} (1-t)^2 \\ (1-t)t \\ t^2 \end{array} \right\} \cdot \mathbf{P}[t] = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}. \tag{18.62}$$

In this equation, each row of the matrix is a quadratic moving line that follows the moving point, and any linear combination of these three moving line also follow the moving point. Thus we can make linear operations between the three rows. Since the left most three elements of the matrix shows three lines which intersect at the same point $\mathbf{P}_0$, we can zero out the bottom left corner of the matrix

$$\begin{bmatrix} (-3,0,0) & (-6,3,0) & (0,3,0) \\ (-6,3,0) & (-9,12,-9) & (3,9,-9) \\ (0,0,0) & (0,3,0) & (3,3,-9) \end{bmatrix} \left\{ \begin{array}{c} (1-t)^2 \\ (1-t)t \\ t^2 \end{array} \right\} \cdot \mathbf{P}[t] = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}. \tag{18.63}$$

Now the bottom moving line has a base point at $t = 0$ and it is reducible to a linear moving line

$$\begin{bmatrix} (0,3,0) & (3,3,-9) \end{bmatrix} \left\{ \begin{array}{c} 1-t \\ t \end{array} \right\} \cdot \mathbf{P}[t] = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}. \tag{18.64}$$

As a result, this pencil and one of the three quadratic moving lines are the required two moving lines

$$\begin{aligned} \mathbf{L}_0[t] &= (0,3,0)(1-t) + (3,3,-9)t \\ \mathbf{L}_1[t] &= (-6,3,0)(1-t)^2 + (-9,12,-9)(1-t)t + (3,9,-9)t^2. \end{aligned} \tag{18.65}$$

Notice that the axis of the pencil always lies at the double point of the cubic curve. It can be calculated from the two control lines. In this example,

$$\begin{aligned} (0,3,0) \times (3,3,-9) &= (-27,0,-9) \\ &= -9(3,0,1), \end{aligned} \tag{18.66}$$

from which the double point is $(3, 0)$.

18.4.4 Quartic Curves

Any rational quartic Bézier curve can be expressed either with two quadratic moving lines, or with one linear and one cubic moving line. We can apply the same approach as in section 18.4.3.

Figure 18.12 shows a sample quartic Bézier curve, with the curve plotted beyond the traditional $[0, 1]$ parameter interval. First of all, using equation (18.57), calculate four cubic moving lines which follow the curve by

$$\begin{bmatrix} (-4,0,0) & (-12,6,0) & (-8,12,0) & (0,3,0) \\ (-12,6,0) & (-32,36,-24) & (-16,50,-48) & (3,9,-9) \\ (-8,12,0) & (-16,50,-48) & (4,56,-104) & (12,6,-24) \\ (0,2,0) & (4,8,-8) & (12,6,-24) & (8,-4,-16) \end{bmatrix} \left\{ \begin{array}{c} (1-t)^3 \\ (1-t)^2 t \\ (1-t)t^2 \\ t^3 \end{array} \right\} \cdot \mathbf{P}[t] = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}. \tag{18.67}$$

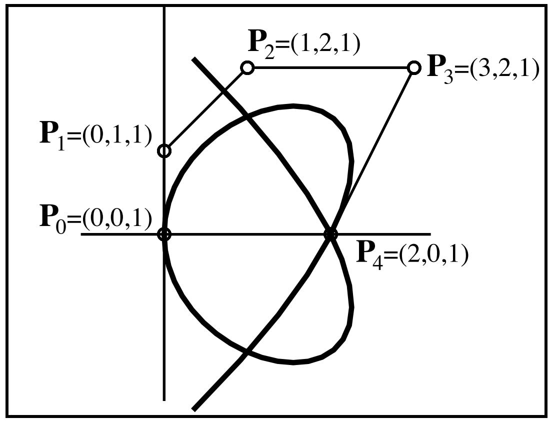

Figure 18.12: Quartic Bézier Curve

In this 4×4 matrix, the left most elements of the bottom two rows can be eliminated by applying row operations

$$\begin{bmatrix} (-4,0,0) & (-12,6,0) & (-8,12,0) & (0,3,0) \\ (-12,6,0) & (-32,36,-24) & (-16,50,-48) & (3,9,-9) \\ (0,0,0) & (0,2,0) & (4,4,-8) & (4,-2,-8) \\ (0,0,0) & (8,6,0) & (28,4,-24) & (20,-14,-40) \end{bmatrix} \begin{Bmatrix} (1-t)^3 \\ (1-t)^2 t \\ (1-t)t^2 \\ t^3 \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{18.68}$$

from which we can obtain two quadratic moving lines

$$\begin{bmatrix} (0,2,0) & (4,4,-8) & (4,-2,-8) \\ (8,6,0) & (28,4,-24) & (20,-14,-40) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18.69}$$

that follow the curve, and hence which intersect at the curve.

Any linear combination of these two quadratic moving lines is also a quadratic moving line that follows the curve. If the moving line is axial, then the axis is a double point. Any rational quartic curve has three double points, which may possibly coalesce to form one triple point. In the case of three distinct double points, their locations can be obtained by following method. Let $\mathbf{L}_Q[t]$ be a linear combination of the two moving lines:

$$\mathbf{L}_Q[t] = \begin{Bmatrix} 1-\tau & \tau \end{Bmatrix} \begin{bmatrix} (0,2,0) & (4,4,-8) & (4,-2,-8) \\ (8,6,0) & (28,4,-24) & (20,-14,-40) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{Bmatrix} \tag{18.70}$$

$$= (8\tau, 4\tau+2, 0)(1-t)^2 + (24\tau+4, 4, -16\tau-8)(1-t)t + (16\tau+4, -12\tau-2, -32\tau-8)t^2 \tag{18.71}$$

If it is axial, three control lines must intersect at one point. This condition is expressed with the following equation

$$(8\tau, 4\tau+2, 0) \times (24\tau+4, 4, -16\tau-8) \cdot (16\tau+4, -12\tau-2, -32\tau-8) = 0 \tag{18.72}$$

$$32\tau^3 - 32\tau^2 - 8\tau = 0$$

$$\tau = 0, \frac{1 \pm \sqrt{2}}{2}.$$

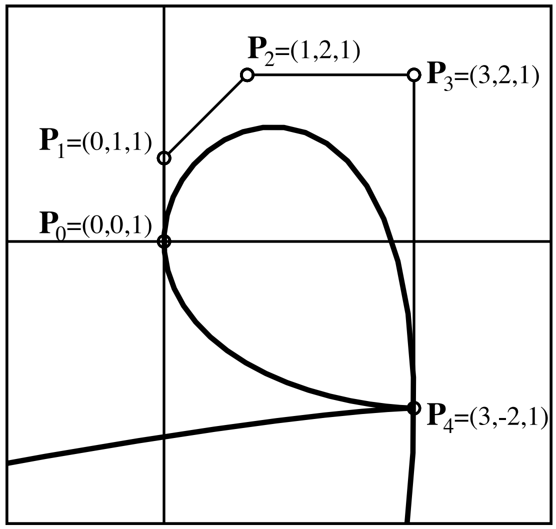

Figure 18.13: Quartic Bézier Curve with a Triple Point

Three double points can be obtained by calculating the axis of $\mathbf{L}_Q[t]$

$$(8\tau, 4\tau+2, 0) \times (24\tau+4, 4, -16\tau-8) = \begin{cases} -1(2,0,1) & (\tau = 0) \\ (-12 \mp 8\sqrt{2})(1, \mp\sqrt{2}, 1) & (\tau = \frac{1\pm\sqrt{2}}{2}), \end{cases} \tag{18.73}$$

from which the double points are $(2,0)$ and $(1, \pm\sqrt{2})$.

If there is a triple point on the curve, however, there is no pair of quadratic moving lines that can represent the curve. Figure 18.13 is an example quartic Bézier curve with a triple point at $\mathbf{P}_4$. We can obtain four cubic moving lines that follow the curve

$$\begin{bmatrix} (-4,0,0) & (-12,6,0) & (-8,12,0) & (2,3,0) \\ (-12,6,0) & (-32,36,-24) & (-14,51,-48) & (12,12,-12) \\ (-8,12,0) & (-14,51,-48) & (12,60,-108) & (24,12,-48) \\ (2,3,0) & (12,12,-12) & (24,12,-48) & (16,0,-48) \end{bmatrix} \begin{Bmatrix} (1-t)^3 \\ (1-t)^2 t \\ (1-t)t^2 \\ t^3 \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{18.74}$$

form which we can get two quadratic moving lines

$$\begin{bmatrix} (2,3,0) & (8,6,-12) & (8,0,-24) \\ (8,12,0) & (30,21,-48) & (28,0,-84) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{18.75}$$

These two quadratic moving lines (call them $\mathbf{L}_1$ and $\mathbf{L}_2$) are linearly independent, but their intersection does not express the curve because all control points obtained from $\mathbf{L}_1 \times \mathbf{L}_2$ are $(0,0,0)$. This means that both moving lines have exactly the same rotation, but their weights are different. In fact, each of them has a base point and both are identical to the same pencil. The pencil can be obtained by eliminating the bottom left element

$$\begin{bmatrix} (0,0,0) & (2,3,0) & (-4,0,12) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{18.76}$$

As a result, the curve can be expressed as the intersection of this pencil and one of the cubic moving lines. The axis of the pencil is

$$\begin{aligned}(2,3,0)\times(-4,0,12) &= (36,-24,12)\\ &= 12(3,-2,1)\end{aligned} \tag{18.77}$$

and thus, the triple point is $(3,-2)$.

Notice that a quartic curve with a triple point cannot be expressed as the intersection of two quadratic moving lines. If it were possible, there would be three quadratic moving lines that are linearly independent. In that case, they could represent a cubic curve.

18.4.5 General Case

The methods for cubic and quartic curves can be extended to higher degree curves. For a degree n curve, there exists an n parameter family of moving lines whose degree is $n-1$. A basis for that family of moving lines is given by

$$\begin{bmatrix} \mathbf{L}_{0,0} & \mathbf{L}_{0,1} & \dots & \mathbf{L}_{0,n-1} \\ \mathbf{L}_{1,0} & \mathbf{L}_{1,1} & \dots & \mathbf{L}_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{L}_{n-1,0} & \mathbf{L}_{n-1,1} & \dots & \mathbf{L}_{n-1,n-1} \end{bmatrix} \begin{Bmatrix} (1-t)^{n-1} \\ (1-t)^{n-2}t \\ \vdots \\ t^{n-1} \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{18.78}$$

where

$$\mathbf{L}_{i,j} = \sum_{l+m=i+j+1} \binom{n}{l}\binom{n}{m} \mathbf{P}_l \times \mathbf{P}_m. \tag{18.79}$$

By calculating linear combinations of these rows of the $n \times n$ matrix, we can always zero out all but two elements in the left-most column, since all such elements are lines which pass through the point $\mathbf{P}_0$:

$$\begin{bmatrix} \mathbf{L}_{0,0} & \mathbf{L}_{0,1} & \dots & \mathbf{L}_{0,n-1} \\ \mathbf{L}_{1,0} & \mathbf{L}_{1,1} & \dots & \mathbf{L}_{1,n-1} \\ 0 & \mathbf{L}^{<1>}_{2,1} & \dots & \mathbf{L}^{<1>}_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \mathbf{L}^{<1>}_{n-1,1} & \dots & \mathbf{L}^{<1>}_{n-1,n-1} \end{bmatrix} \begin{Bmatrix} (1-t)^{n-1} \\ (1-t)^{n-2}t \\ \vdots \\ t^{n-1} \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{18.80}$$

$\mathbf{L}^{<k>}_{i,j}$ denotes the element after k-th calculation. Now, we have $n-2$ moving lines whose degree is $n-1$:

$$\begin{bmatrix} \mathbf{L}^{<1>}_{2,1} & \mathbf{L}^{<1>}_{2,2} & \dots & \mathbf{L}^{<1>}_{2,n-1} \\ \mathbf{L}^{<1>}_{3,1} & \mathbf{L}^{<1>}_{3,2} & \dots & \mathbf{L}^{<1>}_{3,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{L}^{<1>}_{n-1,1} & \mathbf{L}^{<1>}_{n-1,2} & \dots & \mathbf{L}^{<1>}_{n-1,n-1} \end{bmatrix} \begin{Bmatrix} (1-t)^{n-2} \\ (1-t)^{n-3}t \\ \vdots \\ t^{n-2} \end{Bmatrix} \cdot \mathbf{P}[t] = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{18.81}$$

In this $(n-2)\times(n-1)$ matrix, it turns out that we can *again* zero out all but two elements of the left-most column, since magically, all lines $\mathbf{L}^{<1>}_{i,1}$ contain the point $\mathbf{P}_0$. This can be seen by evaluating the set of equations (18.81) at $t=0$:

$$\begin{Bmatrix} \mathbf{L}^{<1>}_{2,1} \\ \mathbf{L}^{<1>}_{3,1} \\ \vdots \\ \mathbf{L}^{<1>}_{n-1,1} \end{Bmatrix} \cdot \mathbf{P}[0] = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{18.82}$$

This zeroing out process can be repeated until one or two rows remain.

If the degree is $n = 2m$, we can repeat the element zeroing process $m-1$ times and obtain two degree m moving lines

$$\left[\begin{array}{cccc} \mathbf{L}_{n-2,m-1}^{<m-1>} & \mathbf{L}_{n-2,m}^{<m-1>} & \dots & \mathbf{L}_{n-2,n-1}^{<m-1>} \\ \mathbf{L}_{n-1,m-1}^{<m-1>} & \mathbf{L}_{n-1,m}^{<m-1>} & \dots & \mathbf{L}_{n-1,n-1}^{<m-1>} \end{array}\right] \left\{\begin{array}{c} (1-t)^m \\ (1-t)^{m-1}t \\ \vdots \\ t^m \end{array}\right\} \cdot \mathbf{P}[t] = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}. \tag{18.83}$$

If the degree is $n = 2m+1$, we can repeat the zero out process m times. In this case, only one element can be eliminated in the last step, and thus, the bottom two rows express degree $m+1$ and degree m moving lines

$$\left[\begin{array}{cccc} \mathbf{L}_{n-2,m-1}^{<m-1>} & \mathbf{L}_{n-2,m}^{<m-1>} & \dots & \mathbf{L}_{n-2,n-1}^{<m-1>} \\ 0 & \mathbf{L}_{n-1,m}^{<m>} & \dots & \mathbf{L}_{n-1,n-1}^{<m>} \end{array}\right] \left\{\begin{array}{c} (1-t)^{m+1} \\ (1-t)^{m}t \\ \vdots \\ t^{m+1} \end{array}\right\} \cdot \mathbf{P}[t] = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}. \tag{18.84}$$

If there is a triple or higher order multiple point, however, the bottom two moving lines may be reducible. In such case, a lower degree moving line can be obtained as discussed in section 18.4.4.

18.4.6 Implicitization

Conventionally, the implicit equation of a rational Bézier curve is calculated as the determinant of the $n \times n$ matrix in equation (18.78) [SAG84], which is called *Bezout's resultant*. It is also possible to implicitize using a pair of moving lines. An arbitrary point $\mathbf{P} = (x, y, 1)$ lies on the curve $\mathbf{P}[t]$ if and only if it lies on both moving lines in equation (18.83) (or (18.84)):

$$\left[\begin{array}{cccc} \mathbf{L}_{n-2,m-1}^{<m-1>}\cdot\mathbf{P} & \mathbf{L}_{n-2,m}^{<m-1>}\cdot\mathbf{P} & \dots & \mathbf{L}_{n-2,n-1}^{<m-1>}\cdot\mathbf{P} \\ \mathbf{L}_{n-1,m-1}^{<m-1>}\cdot\mathbf{P} & \mathbf{L}_{n-1,m}^{<m-1>}\cdot\mathbf{P} & \dots & \mathbf{L}_{n-1,n-1}^{<m-1>}\cdot\mathbf{P} \end{array}\right] \left\{\begin{array}{c} (1-t)^m \\ (1-t)^{m-1}t \\ \vdots \\ t^m \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}. \tag{18.85}$$

This is equivalent to saying that, if $\mathbf{P} = (x, y, 1)$ lies on the curve, then there exists a value of t which satisfies both equations (18.85), which means that the resultant of the two equations is zero.

Let us calculate the implicit equation of the quartic curve in Figure 18.12. By calculating the dot product with $\mathbf{P}$ for each element, equation (18.69) is expressed as follows

$$\left[\begin{array}{ccc} 2y & 4x+4y-8 & 4x-2y-8 \\ 8x+6y & 28x+4y-24 & 20x-14y-40 \end{array}\right] \left\{\begin{array}{c} (1-t)^2 \\ (1-t)t \\ t^2 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}. \tag{18.86}$$

The implicit equation is the resultant of these two quadratic equations of t,

$$\begin{aligned} & \left| \begin{array}{cc} \left|\begin{array}{cc} 2y & 4x+4y-8 \\ 8x+6y & 28x+4y-24 \end{array}\right| & \left|\begin{array}{cc} 2y & 4x-2y-8 \\ 8x+6y & 20x-14y-40 \end{array}\right| \\ \left|\begin{array}{cc} 2y & 4x-2y-8 \\ 8x+6y & 20x-14y-40 \end{array}\right| & \left|\begin{array}{cc} 4x+4y-8 & 4x-2y-8 \\ 28x+4y-24 & 20x-14y-40 \end{array}\right| \end{array} \right| \\ = & \left| \begin{array}{cc} -32x^2-16y^2+64x & -32x^2+32xy-16y^2+64x-32y \\ -32x^2+32xy-16y^2+64x-32y & -32x^2+64xy-48y^2-64y+128 \end{array} \right| \\ = & \; 512(y^4+4x^3+2xy^2-16x^2-6y^2+16x). \end{aligned} \tag{18.87}$$

This method is generally faster than the conventional method. In most cases, the resultant of the two moving lines ends up in a $(\frac{n}{2})\times(\frac{n}{2})$ determinant, while you must calculate the $n\times n$ determinant in conventional method.

18.5 Tangent Moving Lines

18.5.1 Tangent Moving Lines and Envelope Curves

Consider the moving line which is tangent to a curve $\mathbf{P}[t]$ at each parameter value t. We call this the *tangent moving line* of $\mathbf{P}[t]$. The tangent moving line is computed

$$\mathrm{Line}(\mathbf{P}[t]\times\mathbf{P}'[t]). \tag{18.88}$$

where $\mathbf{P}'[t]=(X'[t],Y'[t],W'[t])$. Thus, one would expect the tangent moving line to generally be degree $2m-1$ if $\mathbf{P}[t]$ is a degree m curve. However, it happens that $\mathbf{P}[t]\times\mathbf{P}'[t]$ always has a base point at $t=\infty$. This is most easily seen by using power basis polynomials

$$\mathbf{P}[t] \quad = \quad (\sum_{i=0}^{m} X_i t^i, \sum_{i=0}^{m} Y_i t^i, \sum_{i=0}^{m} W_i t^i); \tag{18.89}$$

$$\mathbf{P}'[t] \quad = \quad (\sum_{i=0}^{m-1}(i+1)X_{i+1}t^i, \sum_{i=0}^{m-1}(i+1)Y_{i+1}t^i, \sum_{i=0}^{m-1}(i+1)W_{i+1}t^i); \tag{18.90}$$

$$\mathbf{P}[\infty] \quad = \quad (X_m, Y_m, W_m); \tag{18.91}$$

$$\mathbf{P}'[\infty] \quad = \quad m(X_m, Y_m, W_m). \tag{18.92}$$

Thus, when equation (18.88) is expanded, the coefficient of t^{2m-1} is $(X_m,Y_m,W_m)\times m(X_m,Y_m,W_m)=(0,0,0)$. Therefore, the degree of the tangent moving line is at most $2m-2$.

The control lines of the tangent moving line can be obtained as follows. Both $\mathbf{P}[t]$ and $\mathbf{P}'[t]$ can be expressed with degree $m-1$ Bernstein polynomials:

$$\mathbf{P}[t] \quad = \quad \sum_{i=0}^{m} B_i^m[t]\mathbf{P}_i = (1-t)\sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_i + t\sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_{i+1} \tag{18.93}$$

$$\mathbf{P}'[t] \quad = \quad m\sum_{i=0}^{m-1} B_i^{m-1}[t](\mathbf{P}_{i+1}-\mathbf{P}_i) = m\left[-\sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_i + \sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_{i+1}\right]. \tag{18.94}$$

Thus,

$$\mathbf{P}[t]\times\mathbf{P}'[t] = m\left[\sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_i \times \sum_{i=0}^{m-1} B_i^{m-1}[t]\mathbf{P}_{i+1}\right]. \tag{18.95}$$

Any non-axial moving line $\mathbf{L}[t]$ can be represented as the set of lines which are tangent to a given parametric curve — its *envelope curve*. The envelope curve is given by the equation

$$\mathrm{Point}(\mathbf{L}[t]\times\mathbf{L}'[t]) \tag{18.96}$$

where $\mathbf{L}'[t]=(a'[t],b'[t],c'[t])$ [Win23], p. 244. Thus, $\mathbf{L}[\tau]$ contains $\mathrm{Point}(\mathbf{L}[\tau]\times\mathbf{L}'[\tau])$ and is tangent to its envelope curve at that point. Figure 18.14 shows an example of a cubic moving line and its envelope curve. Notice that equations (18.88) and (18.96) are the dual of each other. The degree of $\mathbf{L}[t]\times\mathbf{L}'[t]$ is at most $2m-2$ if $\mathbf{L}[t]$ is a degree m moving line. The control points can be calculated using the dual of equation (18.95).

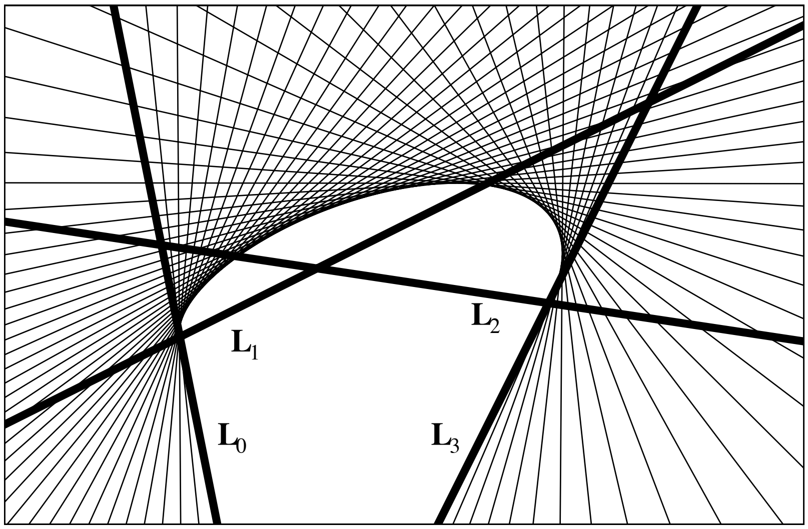

Figure 18.14: Envelope curve

18.5.2 Reciprocal Curves

We would expect that

$$\text{Point}\left((\mathbf{P}[t] \times \mathbf{P}'[t]) \times (\mathbf{P}[t] \times \mathbf{P}'[t])'\right) = \text{Point}(\mathbf{P}[t]). \tag{18.97}$$

That is, if we compute the envelope curve of the tangent moving line of $\mathbf{P}[t]$, we would expect to end up with $\mathbf{P}[t]$. However, the expression on the left is degree $4m-6$ and the on the right is degree m. Thus, we conclude that the expression on the left must have $3m-6$ base points. How do we account for those base points?

In equation (18.88), Line($\mathbf{P}[t]\times\mathbf{P}'[t]$) has a base point at $t=\tau$ if and only if $\mathbf{P}[\tau]=0$ or $\mathbf{P}'[\tau]=0$. The latter condition means that the curve $\mathbf{P}[t]$ has a *cusp* (or *stationary point* [Sal34], p.25) at $t=\tau$. Likewise, in equation (18.96), Point($\mathbf{L}[t]\times\mathbf{L}'[t]$) has a base point at $t=\tau$ if and only if $\mathbf{L}[\tau]=0$ or $\mathbf{L}'[\tau]=0$. Notice that the latter condition is equivalent to saying that the envelope curve $\mathbf{L}[t]\times\mathbf{L}'[t]$ has an *inflection point* (or *stationary tangent* [Sal34], p.33) at $t=\tau$. Now, let m be the degree of the curve $\mathbf{P}[t]$, and let κ and ι be its number of cusps and inflection points respectively. Assume that the curve has no base point. Then, the degree of its tangent moving line, Line($\mathbf{P}[t]\times\mathbf{P}'[t]$), is $2m-2-\kappa$, and the degree of the envelope curve of the moving line, Point($(\mathbf{P}[t]\times\mathbf{P}'[t])\times(\mathbf{P}[t]\times\mathbf{P}'[t])'$) (*i.e.* the original curve $\mathbf{P}[t]$), is $4m-6-2\kappa-\iota$. This fact concludes that any rational curve of degree m must have the following number of cusps and/or inflection points

$$2\kappa+\iota = 3m-6. \tag{18.98}$$

Equation (18.98) is confirmed by one of *Plücker's six equations* [Sal34], p.65 for implicit curves

$$\iota = 3m^2-6m-6\delta-8\kappa \tag{18.99}$$

where δ is the number of double points except cusps. For any rational curve of degree m, there are fixed number of double points (include cusps)

$$\delta+\kappa = \frac{1}{2}(m-1)(m-2). \tag{18.100}$$

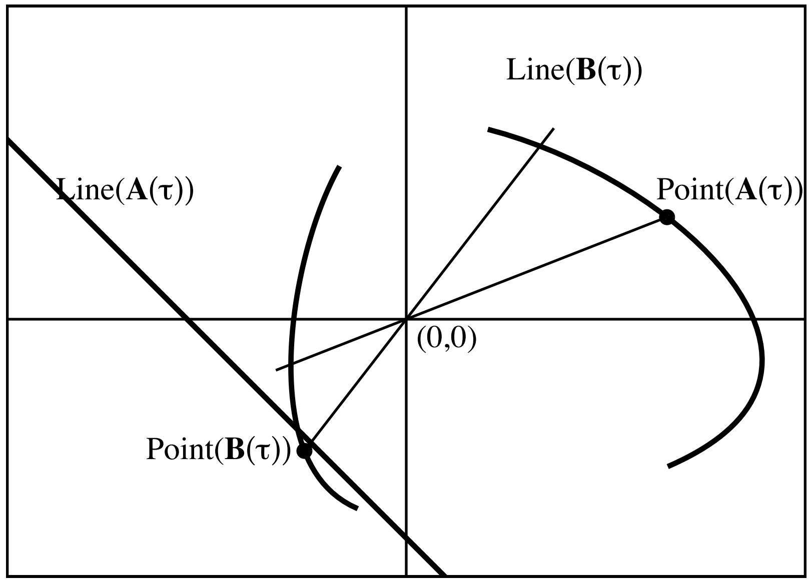

Figure 18.15: Dual and Reciprocal

By eliminating δ from equations (18.99) and (18.100), we can obtain equation (18.98).

Given $\mathbf{A}[t]$, Point($\mathbf{A}[\tau]$) is the dual of Line($\mathbf{A}[\tau]$) for every value of τ, as defined in Figure 18.2. Thus, we say that the moving point Point($\mathbf{A}[t]$) is dual with the moving line Line($\mathbf{A}[t]$). Consider a second moving point Point($\mathbf{B}[t]$) and its dual moving line Line($\mathbf{B}[t]$). If

$$\mathbf{B}[t] = k_1[t](\mathbf{A}[t] \times \mathbf{A}'[t]) \tag{18.101}$$

where $k_1[t]$ is a rational function, then there also exists a rational function $k_2[t]$ such that

$$\mathbf{A}[t] = k_2[t](\mathbf{B}[t] \times \mathbf{B}'[t]). \tag{18.102}$$

When this relationship exists, Point($\mathbf{A}[t]$) and Point($\mathbf{B}[t]$) are referred to as *reciprocal curves* [Sal34], p.54. The rational functions are related

$$\begin{aligned} k_2[t] &= \frac{1}{k_1^2[t]\mathbf{A}[t] \times \mathbf{A}'[t] \cdot \mathbf{A}''[t]} \\ k_1[t] &= \frac{1}{k_2^2[t]\mathbf{B}[t] \times \mathbf{B}'[t] \cdot \mathbf{B}''[t]} \end{aligned} \tag{18.103}$$

The geometric relationship between two reciprocal curves is shown in Figure 18.15. For any value of $t = \tau$, Point($\mathbf{A}[\tau]$) has a dual Line($\mathbf{A}[\tau]$) which is tangent to the curve Point($\mathbf{B}[t]$) at Point($\mathbf{B}[\tau]$). Likewise, Point($\mathbf{B}[\tau]$) has a dual Line($\mathbf{B}[\tau]$) which is tangent to the curve Point($\mathbf{A}[t]$) at Point($\mathbf{A}[\tau]$). If Point($\mathbf{A}[\tau]$) happens to be a cusp, then Point($\mathbf{B}[\tau]$) is an inflection point, and vice versa (see Figure 18.16). If Point($\mathbf{A}[\tau]$) = Point($\mathbf{A}[\beta]$) (that is, if there exists a self intersection or *crunode*), then Point($\mathbf{B}[\tau]$) and Point($\mathbf{B}[\beta]$) have the same tangent line (known as a *double tangent*). This is illustrated in Figure 18.17.

Table 18.1 summarizes the correspondences between a pair of reciprocal curves. The degree of a curve can be defined as the number of times (properly counting real, complex, infinite, and tangent

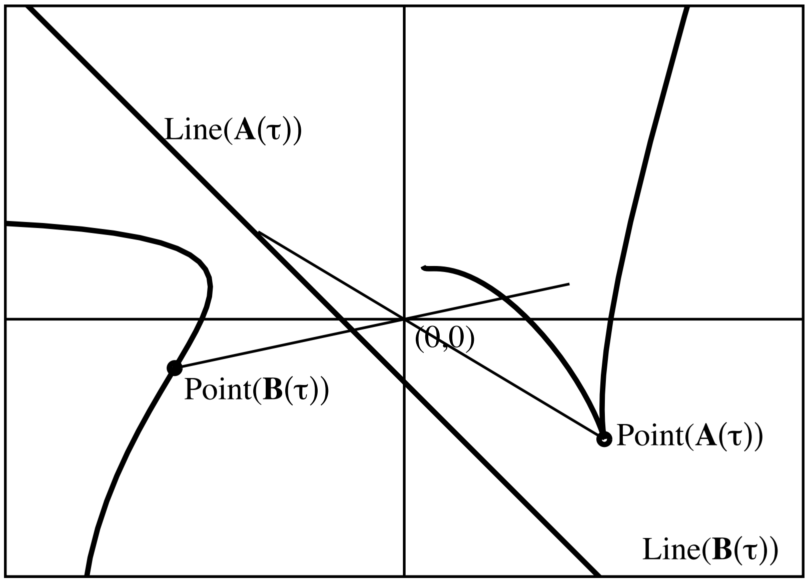

Figure 18.16: Cusp ⇔ Inflection Point

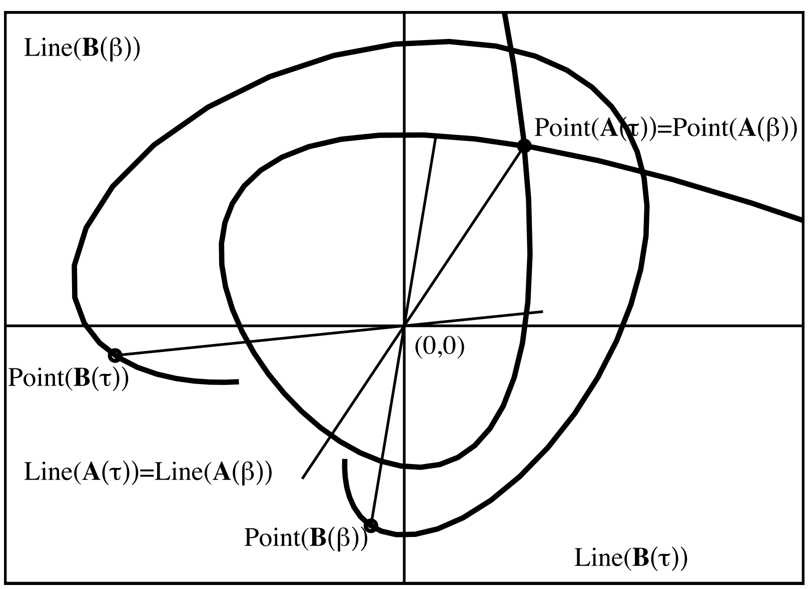

Figure 18.17: Crunode ⇔ Double Tangent

Table 18.1: Reciprocal Curve

curve 1	curve 2
degree	class
class	degree
double point	double tangent
double tangent	double point
cusp	inflection point
inflection point	cusp

intersections) any line intersects the curve. The *class* of a curve is the number of lines passing through a given point which are tangent to the curve.

Tangent moving lines and reciprocal curves provide a powerful tool for analyzing the singularities of rational curves. For example, you can obtain the inflection points of a curve by finding the cusps of its reciprocal curve. The lines tangent to a curve at two distinct real points are found as the crunodes of the reciprocal curve. Equation (18.95) provides an easy way to compute reciprocal curves in Bernstein form.

18.5.3 Tangent Directions

The *hodograph* $\mathbf{P}'[t]$ of a curve $\mathbf{P}[t]$ is its parametric first derivative. The vector with tail at the origin and tip at $\mathbf{P}'[t]$ indicates the magnitude and direction of the derivative of $\mathbf{P}[t]$. It is well-known that the hodograph of a degree n polynomial Bézier curve can be expressed as a degree $n-1$ polynomial Bézier curve. However the hodograph of a degree n rational curve is much more complicated, being a rational function of degree $2n$. [SW87] proposed a *scaled hodograph*, which shows only the derivative direction, and is a degree $2n-2$ polynomial Bźier curve. We can derive an equivalent result using tangent moving lines.

Let $\mathbf{L}_T[t]$ be the tangent moving line of a degree n curve $\mathbf{P}[t]$, and let

$$\mathbf{L}_{Ti} = \text{Line}(a_i, b_i, c_i) \quad (i = 0, 1, \ldots, 2n-2) \tag{18.104}$$

be its control lines. If we force all c_i to be 0

$$\mathbf{L}_H[t] = \sum_{i=0}^{2n-2} \mathbf{L}_{Hi} \tag{18.105}$$

$$\mathbf{L}_{Hi} = \text{Line}(a_i, b_i, 0), \tag{18.106}$$

then this moving line $\mathbf{L}_H[t]$ has an axis at the origin. Notice that $\mathbf{L}_H[t]$ follows the degree $2n-2$ polynomial curve

$$\mathbf{P}_H[t] = \sum_{i=0}^{2n-2} \mathbf{P}_{Hi} \tag{18.107}$$

$$\mathbf{P}_{Hi} = \text{Point}(b_i, -a_i, 1), \tag{18.108}$$

because

$$\text{Point}(0, 0, 1) \times \text{Point}(b_i, -a_i, 1) = \text{Line}(a_i, b_i, 0). \tag{18.109}$$

Since two moving lines $\mathbf{L}_T[t]$ and $\mathbf{L}_H[t]$ has the same direction at any t, $\mathbf{L}_H[t]$ gives the tangent direction of the curve $\mathbf{P}[t]$. Thus, the curve $\mathbf{P}_H[t]$ is a scaled hodograph of $\mathbf{P}[t]$.

This method is available for any rational curve $\mathbf{P}[t]$, however, the magnitude of the derivative is not correct except for polynomial curves.