Day 2 Notes

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1 Introduction

Tensor analysis is designed to make it so that you don't have to worry about the coordinate system until you actually do a calculation.

Example:

$$\vec{F} = m\vec{a}$$

which is independent of coordinates.

A moment to explain Covariant/contravariant tensors in practice:

Example: Let us define a function $f(\vec{x})$:

$$f(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$
$$f(\vec{x}) = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Where a_i is a constant, and x_i is a variable representing position. $\vec{a}$ is covariant, while $\vec{x}$ is contravariant.

Another notation is:

$$a[\vec{x}] = f(\vec{x})$$

¡++¿

$\vec{F}$ and $\vec{a}$ are <u>vectors</u>, which are Rank 1 contravariant tensors. In (Einstein) index notation,

$$F^i = ma^i$$
$$\mathrm{d}x^i = v^i \mathrm{d}t$$
$$\mathrm{d}v^i = a^i \mathrm{d}t$$

Note: Upper index is for contravariant tensors

Rank must be conserved, just like dimensional analysis units must be conserved.

Let $\phi(x)$ be a scalar function, which is a Rank 0 contravariant tensor. Then

$$-\frac{\partial\phi(\underline{x})}{\partial x^i} = F_i$$

is a covariant tensor of rank 1.

2 Covariant and Contravariant Tensors

Transformations:

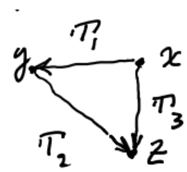

Figure 1: Transformation T_3 is the composition of transformations T_1 and T_2

$$T_1 := \Big\{ y^i = y^i(x^1, x^2, x^3)$$
$$T_2 := \Big\{ z^i = z^i(y^1, y^2, y^3)$$

Now think of a composite function $\circ$, (like multiplication, addition, etc of $\mathbb{R}$)

> For example, if we were dealing with scalars, $A \circ B$ could be $A \times B$, $A+B$, or $A \div B$.
>
> $\circ$ needs to be Associative $(T_4 \circ (T_2 \circ T_1) = (T_4 \circ T_2) \circ T_1$
>
> The point you need to get to is that T_1, T_2, etc. form a group

$$T_2 \circ T_1 := \Big\{ z^i = z^i(y^i(x))$$
$$T_2 \circ T_1 = T_3$$

$$T_4 \circ (T_2 \circ T_1) = (T_4 \circ T_2) \circ T_1$$
$$1 + (2 + 3) = (1 + 2) + 3$$

There are coordinate systems so that I can

$$T_i \circ T_j = T_3 \text{for all well posed coordinate systems}$$

Continuity condition $y^i = y^i(x) - C^1$ 1st derivatives also continuous

$$\mathrm{d}T = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{pmatrix}$$

The determinant of $\mathrm{d}T$ is called the jacobian $J(T) \neq 0$, this also implies that the inverso of T, T^{-1} also exists.

$$T \circ T^{-1} = T^{-1} \circ T = I$$

$I := y^i = x^i$

Exsitence of T, Definition of $\circ$, I have $J \neq 0$, identity, inverse - This system is a group GROUP.

2.1 Example of the Jacobian being 0

PICTURE Example: The convesion between cartesian and spherical coordinates. At a finite number of points, the Jacobian will be equal to zero. (R=0) The GROUP here includes the transformation between cartesian and spherical coordinates, as well as every other transformation you could make.

3 Consider

$$y^i = y^i(\underline{x})$$
$$\mathrm{d}y^i = \frac{\partial Y^i}{\partial x^j}\partial x^j$$

Using the Einstein sum convention A^i_{kl} suppose $A^i_{il} \equiv \sum_{i=1}^3 A^i_{il}$

Note: $\mathrm{d}y^i$ is a contravariant Rank 1 tensor transformed from $\mathrm{d}x^j$ through the matrix $\frac{\partial y^i}{\partial x^j}$

$$\text{The } y^i \text{ coordinate system} \leftarrow \text{the } x^i \text{ coordinate system}$$
$$A^i \leftarrow \frac{\partial y^i}{\partial x^i} A^j$$

$$\frac{\partial \phi}{\partial x^i}\frac{\partial x^i}{\partial y^j} = \frac{\partial \phi}{\partial y^j} \text{ a } \underline{\text{covariant}} \text{ vector}$$

4 Matrices

Let us define variables

$$\vec{e}_1, \vec{e}_2, \vec{e}_3$$
$$\vec{e}_1 \to \frac{\partial y^i}{\partial x^1}, \text{etc.}$$

and

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

Which is to say that $\boldsymbol{g}$ is a metric tensor.

Now, let us look at derivatives:

$$\mathrm{d}S^i = \begin{pmatrix} \mathrm{d}x^1 & \mathrm{d}x^2 & \mathrm{d}x^3 \end{pmatrix} \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} = (\mathrm{d}x)^t \mathrm{d}x$$

in cartesian coordinates.

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial y^j} \partial y^j$$

Gone from

$$\begin{aligned} dS^2 &= (\mathrm{d}x^1)^2 + (dx^2)^2 + (dx^3)^2 \\ \delta_{ij} &= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \\ \mathrm{d}S^2 &= \delta_{ij} \mathrm{d}x^i dx^j \\ \mathrm{d}S^2 &= g_{ij} \mathrm{d}x^i \mathrm{d}x^j \\ J &= \sqrt{g} \\ g &= \det(\boldsymbol{g}) \end{aligned}$$

Now that we have g_{ij} defined, does there exist g^{kl} such that

$$g_{ij} g^{jl} = \delta_i^l = \begin{cases} 1 \text{ if } i = l \\ 0 \text{ if } i \neq l \end{cases} \quad ?$$

It turns out that g^{kl} is the matrix inverse of g_{ij}

$$\begin{aligned} A^i &= \text{contravariant} \\ A^i g_{ij} &= A_j \\ A_j g^{ij} &= A^i \end{aligned}$$

$\frac{\partial\phi}{\partial x^i} g^{ij} = \left(\frac{\partial\phi}{\partial x^i}\right)^j_{\text{contravariant}} | B_i g^{ij} = B^j$

Note:

$$\vec{F} = m\vec{a} + -\nabla\phi$$

$\vec{a}$ is contravariant, $-\nabla\phi$ is covariant, so we need to rewrite this.

$$F^i = ma^i - \frac{\partial\phi}{\partial x^j} g^{ji}$$

Now this is tensor consistent, both sides of the "=" are contravariant.

Now, the sort of tensors we use in this course, we don't have to worry about the coordinate system that we're in. A place where you have to be very careful is when the surface is curved. For formulating mass/momentum transport to deal with interfaces, we can deal with just cartesian tensors.

5 Experimental Methods

Define surface tension as a function of T, P, c, etc.

1. Geometric Methods
   - Capillary Rise (since the 1700's!) Take a glass tube with a small inside diameter, put it into your solution, and you will get liquid that rises up to a head height. That is related to the surface tension.
2. Force Methods