Chapter 5

Properties of Blending Functions

We have just studied how the Bernstein polynomials serve very nicely as blending functions. We have noted that a degree n Bézier curve always begins at $\mathbf{P}_0$ and ends at $\mathbf{P}_n$. Also, the curve is always tangent to the control polygon at $\mathbf{P}_0$ and $\mathbf{P}_n$.

Other popular blending functions exist for defining curves. In fact, you can easily make up your own set of blending functions. And by following a few simple rules, you can actually create a new type of free-form curve which has desirable properties.

Consider a set of control points $\mathbf{P}_i$, $i = 0, ..., n$ and blending functions $f_i(t)$ which define the curve

$$\mathbf{P}(t) = \sum_{i=0}^{n} f_i(t)\mathbf{P}_i.$$

We can select our blending functions such that the curve has any or all of the following properties:

1. **Coordinate system independence**. This means that the curve will not change if the coordinate system is changed. In other words, imagine that the control points are drawn on a piece of paper and we move that piece of paper around so that the (x, y) coordinates of the control points change. It would be nice if the curve did not change relative to the control points. Actually, if we were to pick an arbitrary set of blending functions, the curve *would* change. In order to provide coordinate system independence, the blending functions must form a *partition of unity*, which is math jargon that means that the blending functions sum identically sum to one:

$$\sum_{i=0}^{n} f_i(t) \equiv 1.$$

   The property of coordinate system independence is also called *affine invariance.*

2. **Convex hull property**. The convex hull property was introduced in Section 2.5. This property exists in curves which are coordinate system independent and for which the blending functions are all non- negative:

$$\sum_{i=0}^{n} f_i(t) \equiv 1; \quad f_i(t) \geq 0, \quad 0 \leq t \leq 1, \quad i = 0, ..., n$$

3. **Symmetry**. Curves which are symmetric do not change if the control points are ordered in reverse sequence. For a curve whose domain is $[0,1]$, symmetry is assured if and only if

$$\sum_{i=0}^{n} f_i(t)\mathbf{P}_i \equiv \sum_{i=0}^{n} f_i(1-t)\mathbf{P}_{n-i}.$$

This holds if

$$f_i(t) = f_{n-i}(1-t).$$

For a curve whose domain is $[t_0, t_1]$, symmetry requires

$$\sum_{i=0}^{n} f_i(t)\mathbf{P}_i \equiv \sum_{i=0}^{n} f_i(t_0+t_1-t)\mathbf{P}_{n-i}.$$

4. **Variation Diminishing Property**. This is a property which is obeyed by Bézier curves and B-spline curves. It states that if a given straight line intersects the curve in c number of points and the control polygon in p number of points, then it will always hold that

$$c = p - 2j$$

where j is zero or a positive integer. This has the practical interpretation that a curve which

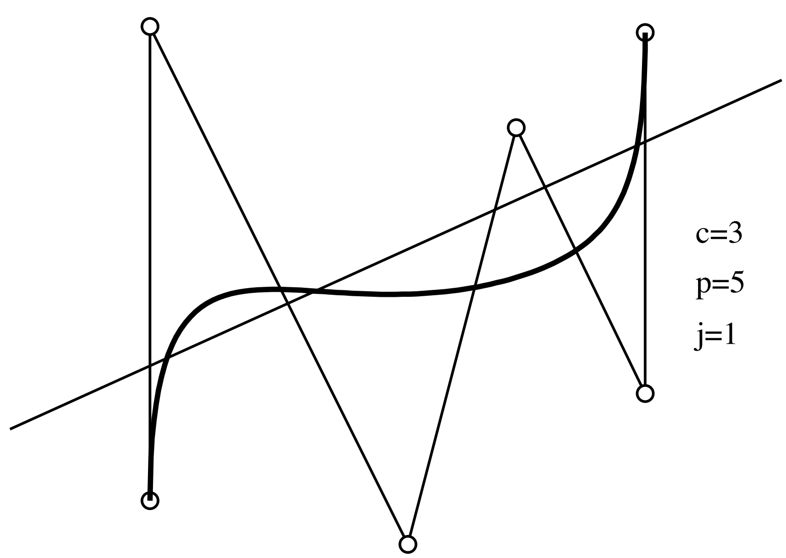

Figure 5.1: Variation Diminishing Property

obeys the variation diminishing property will “wiggle” no more than the control polygon.

The conditions under which a curve will obey the variation diminishing property are rather complicated. Suffice it to say that Bézier curves and B-spline curves obey this property, and most other curves do not.

5. **Linear Independence**. A set of blending functions is linearly independent if there do no exist a set of constants $c_0, \ldots, c_n$, not all zero, for which

$$\sum_{i=0}^{n} c_i f_i(t) \equiv 0.$$

Linearly independent blending functions are desirable for many reasons. For example,if they are not linearly independent, it is possible to express one blending function in terms of the other ones. This has the practical disadvantage that any given curve can be represented by infinitely many different control point positions. It also means that, for certain control point arrangements, the curve collapses to a single point, even though the control points are not all at that point.

If a set of blending functions are linearly independent, they can be called *basis functions*. Thus, Bernstein polynomials are basis functions for Bézier curves, and B-spline blending functions are basis functions.

6. **Endpoint Interpolation** If a curve over the domain $[t_0, t_1]$ is to pass through the first and last control points, as in the case of Bézier curves, the following conditions must be met:

$$f_0(t_0) = 1, \quad f_i(t_0) = 0, \; i = 1, ..., n$$

$$f_n(t_1) = 1, \quad f_i(t_1) = 0, \; i = 0, ..., n-1.$$

Any set of blending functions can be analyzed in terms of the above properties.

Example

Check which of the above-mentioned properties are obeyed by the sample curve

$$\mathbf{P}(t) = f_0(t)\mathbf{P}_0 + f_1(t)\mathbf{P}_1 + f_2(t)\mathbf{P}_2 = \frac{t^2 - 2t + 1}{t^2 + 1}\mathbf{P}_0 + \frac{2t - 2t^2}{t^2 + 1}\mathbf{P}_1 + \frac{2t^2}{t^2 + 1}\mathbf{P}_2$$

Answer:

This curve is **coordinate system independent** because

$$\frac{t^2 - 2t + 1}{t^2 + 1} + \frac{2t - 2t^2}{t^2 + 1} + \frac{2t^2}{t^2 + 1} = \frac{t^2 + 1}{t^2 + 1} \equiv 1.$$

The curve also obeys the **convex hull property** because its blending functions are non-negative for $t \in [0, 1]$. This is most easily seen by factoring the blending functions:

$$\frac{t^2 - 2t + 1}{t^2 + 1} = \frac{(t-1)^2}{t^2 + 1}$$

is non-negative for any real number because the square of a real number is always non-negative.

$$\frac{2t - 2t^2}{t^2 + 1} = \frac{2t(1-t)}{t^2 + 1}$$

is clearly non-negative for $t \in [0, 1]$.

The curve is **not symmetric** because

$$\frac{(1-t)^2 - 2(1-t) + 1}{(1-t)^2 + 1} = \frac{t^2}{t^2 - 2t + 2} \neq \frac{2t^2}{t^2 + 1}$$

Variation diminishing is very difficult to prove, and will not be attempted here.
Linear independence is assessed by checking if there exist constants c_0, c_1, c_2, not all zero, for which

$$c_0 \frac{t^2 - 2t + 1}{t^2 + 1} + c_1 \frac{2t - 2t^2}{t^2 + 1} + c_2 \frac{2t^2}{t^2 + 1} \equiv 0.$$

This is equivalent to

$$c_0(t^2 - 2t + 1) + c_1(2t - 2t^2) + c_2(2t^2) = (c_0 - 2c_1 + 2c_2)t^2 + (-2c_0 + 2c_1)t + c_0 \equiv 0$$

which can only happen if

$$c_0 - 2c_1 + 2c_2 = 0; \qquad -2c_0 + 2c_1 = 0; \qquad c_0 = 0.$$

There is no solution to this set of linear equations, other than $c_0 = c_1 = c_2 = 0$, so we conclude that these blending functions are linearly independent.
The curve does interpolate the endpoints, because

$$f_0(0) = 1, \quad f_1(0) = f_2(0) = 0; \qquad f_2(1) = 1, \quad f_0(1) = f_1(1) = 0.$$

The Bézier and B-spline curves are currently the most popular curve forms. Historically, other curve forms evolved independently at several different industrial sites, each faced with the common problem of making free-form curves accessible to designers. In this section, we will review three such curves: Timmer's Parametric Cubic, Ball's Cubic, and the Overhauser curve. Each of these curves is coordinate system independent and symmetric, but only Ball's cubic obeys the convex hull property.

5.1 Timmer's Parametric Cubic

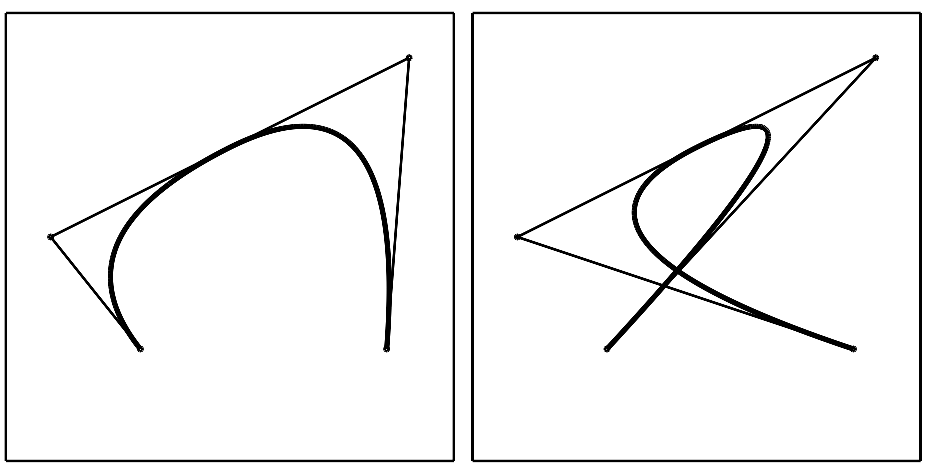

Figure 5.2: Timmer's PC

Timmer's Parametric Cubic (or PC) was created by Harry Timmer of McDonnell Douglas [Tim80]. It was modeled after the Bézier curve. Timmer felt that he could improve upon the Bézier curve if he could make it follow the control polygon more tightly. This he did by forcing the curve to interpolate the endpoints of the control polygon and to be tangent to the control polygon at those points (just like Bézier curves) and in addition, he forced the curve to go through the midpoint of the line segment $\mathbf{P}_1$ - $\mathbf{P}_2$. The resulting blending functions are:

$$f_0(t) = (1-2t)(1-t)^2 = -2t^3 + 5t^2 - 4t + 1$$
$$f_1(t) = 4t(1-t)^2 = 4t^3 - 8t^2 + 4t$$
$$f_2(t) = 4t^2(1-t) = -4t^3 + 4t^2$$
$$f_3(t) = (2t-1)t^2 = 2t^3 - t^2$$

Figure 5.2 may mislead one into thinking that Timmer's curve is tangent to $\mathbf{P}_1 - \mathbf{P}_2$. This is not generally so (and is not exactly so even in Figure 5.2).

Example Problem
Given a Timmer curve with control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, find the control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ of an equivalent cubic Bézier curve $\mathbf{P}(t)$.
Solution
Note that if $\mathbf{P}(t) \equiv \mathbf{Q}(t)$,then $\mathbf{P}(0) = \mathbf{Q}(0)$, $\mathbf{P}(1) = \mathbf{Q}(1)$, $\mathbf{P}'(0) = \mathbf{Q}'(0)$, and $\mathbf{P}'(1) = \mathbf{Q}'(1)$. This strategy provides a simple way to perform the conversion.

$$\mathbf{P}(0) = \mathbf{Q}(0) \rightarrow \mathbf{P}_0 = \mathbf{Q}_0.$$
$$\mathbf{P}(1) = \mathbf{Q}(1) \rightarrow \mathbf{P}_3 = \mathbf{Q}_3.$$

For derivatives, we differentiate the Timmer basis functions (it does not work to use the Bézier hodograph):

$$\mathbf{Q}'(t) = (-6t^2 + 10t - 4)\mathbf{Q}_0 + (12t^2 - 16t + 4)\mathbf{Q}_1 + (-12t^2 + 8t)\mathbf{Q}_2 + (6t^2 - 2t)\mathbf{Q}_3$$

so $\mathbf{Q}'(t) = 4(\mathbf{Q}_1 - \mathbf{Q}_0)$ and $\mathbf{Q}'(1) = 4(\mathbf{Q}_3 - \mathbf{Q}_2)$.

$$\mathbf{P}'(0) = \mathbf{Q}'(0) \rightarrow 3(\mathbf{P}_1 - \mathbf{P}_0) = 4(\mathbf{Q}_1 - \mathbf{Q}_0) \rightarrow \mathbf{Q}_1 = \frac{\mathbf{P}_0 + 3\mathbf{P}_1}{4}$$
$$\mathbf{P}'(1) = \mathbf{Q}'(1) \rightarrow 3(\mathbf{P}_3 - \mathbf{P}_2) = 4(\mathbf{Q}_3 - \mathbf{Q}_2) \rightarrow \mathbf{Q}_2 = \frac{\mathbf{P}_3 + 3\mathbf{P}_2}{4}$$

5.2 Ball's Rational Cubic

Alan Ball first published his cubic curve formulation in 1974 [Bal74]. Ball worked for the British Aircraft Corporation, and his cubic curve form was used in BAC's in-house CAD system.

Like Timmer's PC curve, Ball's cubic can be considered a variant of the cubic Bézier curve. It's distinguishing feature is that it handles conic sections as a special case in a natural way. The blending functions for the non-rational case are:

$$f_0(t) = (1-t)^2$$
$$f_1(t) = 2t(1-t)^2$$
$$f_2(t) = 2t^2(1-t)$$

$$f_3(t) = t^2$$

Notice that if $\mathbf{P}_1 = \mathbf{P}_2$, then the curve becomes a quadratic Bézier curve:

$$\begin{aligned} &\mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2t(1-t)^2 + \mathbf{P}_1 2t^2(1-t) + \mathbf{P}_3 t^2 \\ &= \mathbf{P}_0(1-t)^2 + \mathbf{P}_1[2t(1-t)^2 + 2t^2(1-t)] + \mathbf{P}_3 t^2 \\ &= \mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2t(1-t) + \mathbf{P}_3 t^2 \end{aligned}$$

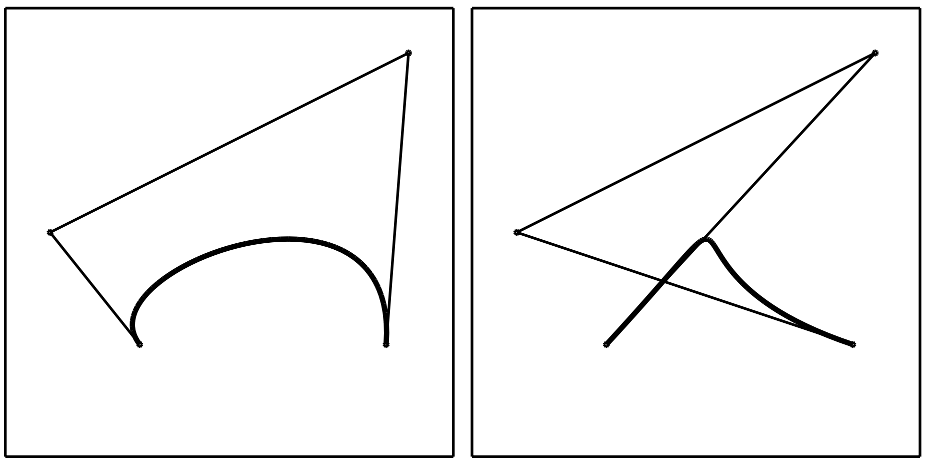

Figure 5.3: Ball's Cubic

5.3 Overhauser Curves

Overhauser curves were developed and used at Ford Motor Company [Ove68]. They are also known as cubic Catmull-Rom splines [CR74].

One complaint of Bézier curves is that the curve does not interpolate all of the control points. Overhauser curves do interpolate all control points in a piecewise string of curve segments. A single Overhauser curve is defined with the following blending functions:

$$f_0(t) = -\frac{1}{2}t + t^2 - \frac{1}{2}t^3$$

$$f_1(t) = 1 - \frac{5}{2}t^2 + \frac{3}{2}t^3$$

$$f_2(t) = \frac{1}{2}t + 2t^2 - \frac{3}{2}t^3$$

$$f_3(t) = -\frac{1}{2}t^2 + \frac{1}{2}t^3$$

A single Overhauser curve segment interpolates $\mathbf{P}_1$ and $\mathbf{P}_2$. Furthermore, the slope of the curve at $\mathbf{P}_1$ is only a function of $\mathbf{P}_0$ and $\mathbf{P}_2$ and the slope at $\mathbf{P}_2$ is only a function of $\mathbf{P}_1$ and $\mathbf{P}_3$:

$$\mathbf{P}'(0) = \frac{1}{2}(\mathbf{P}_2 - \mathbf{P}_0)$$

$$\mathbf{P}'(1) = \frac{1}{2}(\mathbf{P}_3 - \mathbf{P}_1)$$

This means that a second curve segment will be tangent to the first curve segment if its first three control points are identical to the last three control points of the first curve. This is illustrated in Fig. 5.4.

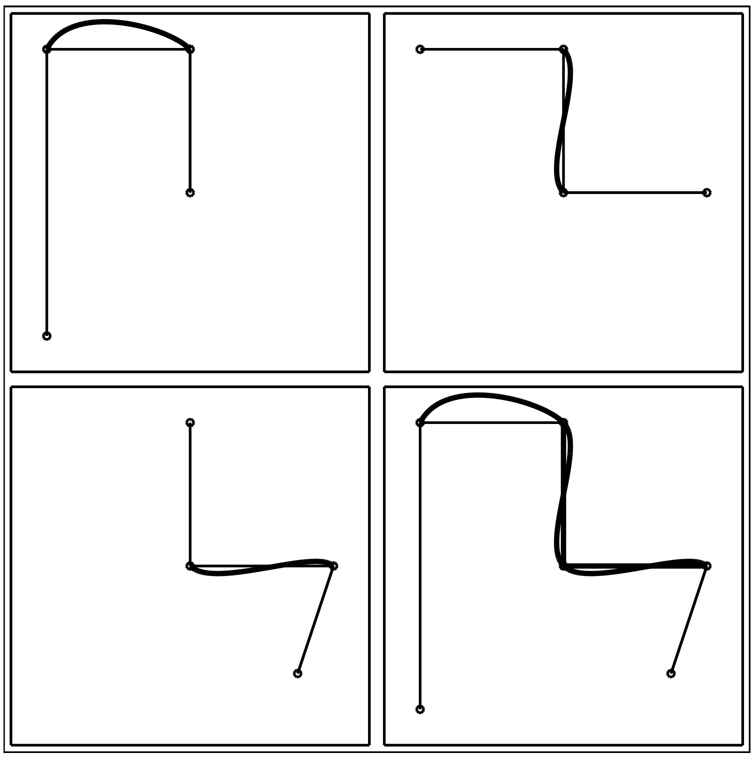

Figure 5.4: Overhauser curves