Coppie di variabili aleatorie / 01-04

Variabili aleatorie discrete X e Y CONGIUNTE (già fornite in coppia)

$X \{x_1, \dots, x_N\}$ $\quad Y \{y_1, \dots, y_M\}$ e si vogliono descrivere assieme

pmf $p(i,j)$ a ogni coppia di valori assegno un numero

$$p(i,j) = p(x_i, y_j) = p(X = x_i, Y = y_j) \qquad i = 1, \dots, N \quad j = 1, \dots, M$$

- $0 \leq p(i,j) \leq 1$
- $\sum_{i=1}^{N} \sum_{j=1}^{M} p(i,j) = 1$

cdf $F(a,b) = \sum_{x_i \leq a} \sum_{y_j \leq b} p(x_i, y_j)$

si introduce il concetto di marginale: a partire da $p(x_i, y_j)$ determino $p(x_i)$ o $p(y_j)$

$$p_X(x_i) = \sum_{j=1}^{M} p(x_i, y_j) \quad \text{"itero le righe"} \qquad p_Y(y_j) = \sum_{i=1}^{N} p(x_i, y_j) \quad \text{"itero le colonne"}$$

$x_i \backslash y_j$	0	1	2	3	somma 220 ($tot = 1$)
0	$10/220$	$40/220$	$30/220$	$4/220$	$84/220$
1	$30/220$	$60/220$	$18/220$		$108/220$
2	$15/220$	$12/220$			$27/220$
3	$1/220$				$1/220$

marginali per la x $= p(x_0)$, $= p(x_1)$, $\vdots$

$$\boxed{\frac{56}{220} \quad \frac{112}{220} \quad \frac{48}{220} \quad \frac{4}{220}}$$ } somma 220 (tot = 1)

marginali per le y

$p(y_0)$ $p(y_1)$ ...

Es 3 palline estratte: 3R, 4W, 5B. $X = \# R$ $Y = \# W$

Calcola pmf congiunta e marginali

casi possibili: $\binom{12}{3} = 220$ casi

$(0,0)$; $\binom{5}{3}$ — $(0,0)$: R, W; $\binom{5}{3}$: tutte B

$(0,1)$; $\binom{4}{1}\binom{5}{2}$ — R, W; $\binom{4}{1}$: 1 fra 4 W; $\binom{5}{2}$: B

$(0,2)$; $\binom{4}{2}\binom{5}{1}$ — 2 bianche, 1 blu

$(0,3)$; $\binom{4}{3}$

$(1,0)$; $\binom{3}{1}\binom{5}{2}$ ... — 1 R

⋮

Dividendo per i casi possibili ottengo le singole probabilità (come per i marginali)

$p(x_i, y_j)$

$$p(x_i) = \sum_{j=1}^{M} p(x_i, y_j)$$

$$F(a,b) = \sum_{x_i \le a} \sum_{y_j \le b} p(x_i, y_j)$$

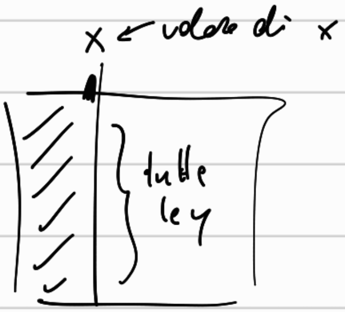
cdf marginale

$$F_X(a) = \sum_{x \le a} \sum_{y_j}^{M} p(x_i, y_j) = P(X \le a, \underbrace{Y \le +\infty}_{\text{sempre vera}})$$

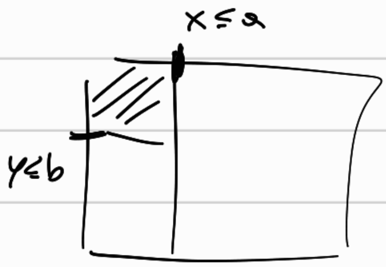
M: le prendo tutte

Y (~~no~~ $p(y)$, può essere qualsiasi numero)

$$F_Y(b) = \sum_{x_i}^{N} \sum_{y_j \le b} p(x_i, y_j) = P(X \le +\infty, Y \le b)$$



marginale



congiunta

Es $P(X > a, Y > b) = 1 + F(a,b) - F_X(a) - F_Y(b)$ Dimostro.

$= 1 - P(((X > a, Y > b)^c)$

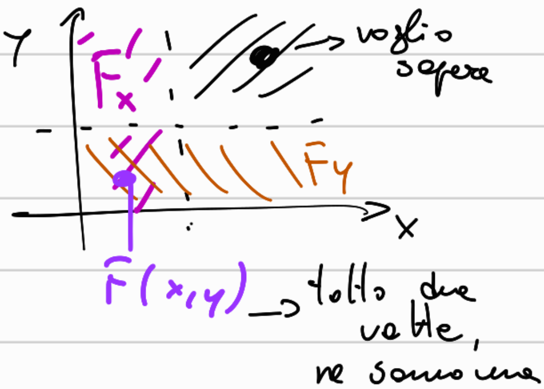
$= 1 - P((X > a)^c \cup (Y > b)^c)$

$= 1 - P((X \leq a) \cup (Y \leq b))$

$= 1 - (P(X \leq a) + P(Y \leq b) - P(X \leq a, Y \leq b))$

↳ perché avevo sommato due volte l'intersezione

$= 1 - F(a,b) - F_X(a) - F_Y(b)$ ✓



Variabili casuali indipendenti

$P(X \in A, Y \in B) = P_{(X)}(X \in A) P_{(Y)}(Y \in B)$ una variabile non dipende dall'altra

(A ↓ insieme di valori (evento))

$P(X \leq a, Y \leq b) = P_X(X \leq a) P_Y(Y \leq b)$

$\quad \parallel$

$F(a,b) = F_X(a) F_Y(b)$

Valore atteso

Con una var.: $E[X] = \sum_{i=1}^{N} x_i \, p(x_i)$

$$E[g(x)] = \sum_{i=1}^{N} g(x_i) \, p(x_i) \qquad g : \{x_1, \dots, x_n\} \to \mathbb{R}$$

$$E[g(X,Y)] = \sum_{x_i}^{N} \sum_{y_j}^{M} g(x_i, y_j) \, p(x_i, y_j) \qquad g : \{x_1, \dots x_N\} \times \{y_1 \dots y_n\} \to \mathbb{R}$$

es. Somma di variabili aleatorie

$$g(X,Y) = X + Y$$

$$E[X+Y] = \sum_{x_i}^{N} \sum_{y_j}^{M} (x_i + y_j) \, p(x_i, y_j)$$

$$= \sum_{x_i}^{N} \sum_{y_j}^{M} x_i \, p(x_i, y_j) + \sum_{x_i}^{N} \sum_{y_j}^{M} y_j \, p(x_i, y_j)$$

$$= \sum_{i=1}^{N} x_i \underbrace{\sum_{j=1}^{M} p(x_i, y_j)}_{} + \sum_{j}^{M} y_j \underbrace{\sum_{i}^{N} p(x_i, y_j)}_{}$$

non dipende da j

$= $ marginale di x_i $P_x(x_i)$

marginale di y_j $P_y(y_j)$

media delle x rispetto alla p marginale

$$= E[X] + E[Y]$$

es. Prodotto fra var aleatorie

$$g(X,Y) = X \cdot Y$$

$$E[X \cdot Y] = \sum_{x_i}^{N} \sum_{y_j}^{M} (x_i \cdot y_j)\, p(x_i, y_j)$$

solo se X e Y indipendenti posso separare in $p(x_i)\,p(y_j)$

come prima raggruppo $= \left(\sum_{i}^{N} x_i \cdot P_X(x_i) \right) \cdot \left(\sum_{j}^{M} y_j\, P_Y(y_j) \right)$

Quindi $\qquad E[X] \qquad\qquad E[Y]$

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

solo se indipendenti