A Finite Element Domain Decomposition Combined With Algebraic Multigrid Method for Large-Scale Electromagnetic Field Computation

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A nesting combination of the finite element domain decomposition method and the algebraic multigrid method is presented in this paper, which does not use the algebraic multigrid method as a pure iterative solver, but nests the iteration steps of the domain decomposition method into the iteration steps of the algebraic multigrid method. Mainly the step of updating inner boundary values in the domain decomposition iteration is nested into the transition steps of the coarse and the fine grids in algebraic multigrid method. The number of iteration between the sub-domains of the domain decomposition method can be decreased greatly by the nesting combination approach. The approach is suitable for parallel computation of large-scale problems.

***Index Terms*—Algebraic multigrid, domain decomposition method, electromagnetic field, finite element method.**

I. Introduction

THE domain decomposition method (DDM) is the dominant paradigm in contemporary large-scale partial differential equation solution. It implements the computation of electromagnetic or other kinds of fields by splitting the original large domain into several small sub-domains with overlapping common regions, and the sub-domains are calculated by an iteration process of updating the inner boundary conditions gradually [1]–[3]. The main advantage of the DDM is saving computer memory, so that more complicated problems can be solved by current computer facilities. The DDM is suitable for parallel computation. Another advantage of the DDM is that when a modification is made in a sub-domain, the previous computation results in other sub-domains can be used directly.

The algebraic multigrid (AMG) approach is a pure solver for large-scale system of equations. The approach solves the system of equations iteratively between a few sets of grids with different nodes or degrees of freedom. The AMG has been used for solving electromagnetic problems and widely applied [4], [5]. It can reduce the computational time significantly.

However, the direct combination of the AMG and the DDM, i.e., the AMG is used only as a solver in the solution of each sub-domain, is only able to reduce the time consumed by solving the equation system, but not able to reduce the number of iteration times of the DDM. To overcome this shortcoming, a combination approach by nesting the process of the FEM-DDM and the AMG is presented in this paper, which is named nesting FEM-DDM-AMG approach. It is able to reduce the number of DDM iteration times, and increase the convergence speed greatly. The mathematic background of the new method can refer to the works of Gundolf Haase [6].

II. Domain Decomposition and Multigrid Method

To describe the nesting combination of FEM-DDM and AMG presented in this paper, the conventional DDM and AMG method must be introduced first.

A. The Domain Decomposition Method

The DDM is a technique for arriving at the solution of problem defined over a domain from the solution of related problems posed on sub-domains. It is a practical approach to translate a large-scale problem into smaller and manageable ones. The conventional DDM is the overlapping Schwarz's method [7].

In the DDM, the whole solution domain Ω is divided into a few overlapping sub-domains, such as $\Omega_1, \Omega_2 \ldots \Omega_{\mathrm{n}}$. For the sake of simplify, we take two sub-domains for example as shown in Fig. 1, and $\Omega = \Omega_1 \cup \Omega_2, \Omega_i = \Omega_1 \cap \Omega_2$. $\Gamma_{1\mathrm{a}}$ and $\Gamma_{2\mathrm{a}}$ are the outer boundaries; $\Gamma_{1\mathrm{b}}$ and $\Gamma_{2\mathrm{b}}$ are the inner boundaries. By using the FEM (or other numerical methods) in domain Ω_1, the system of equations can be formed for the static fields as follows:

$$\begin{bmatrix} K_{11} & K_{1i} \\ K_{i1} & K_{ii} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_i \end{bmatrix} = \begin{bmatrix} b_1 \\ b_i \end{bmatrix} \tag{1}$$

where K is the stiffness matrix, ϕ is the freedom of nodes, and b is the right-hand term. The subscript i is related to the overlapping domain Ω_i, subscript 1 to Ω_1. b_1 is a set of given values associated with the outer boundary conditions, and b_{i} remains as unknown value associated with the inner boundaries. The DDM performs as follows:

- Step 1: Assign a set of initial values (may be arbitrary values) to the inner boundary $\Gamma_{1\mathrm{b}}$, as its boundary condition is unknown.
- Step 2: Solve the equation of sub-domain Ω_1 with the boundary conditions on $\Gamma_{1\mathrm{b}}$ and $\Gamma_{1\mathrm{a}}$, so that the value or the boundary condition on $\Gamma_{2\mathrm{b}}$ can be got from the obtained solution of Ω_1.
- Step 3: Solve the equation of sub-domain Ω_2 with boundary conditions on $\Gamma_{2\mathrm{b}}$ and $\Gamma_{2\mathrm{a}}$, so that a new boundary condition on $\Gamma_{1\mathrm{b}}$ can be obtained.

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Repeat step 2 and 3 till the desired accuracy is reached, which can be estimated by the maximum error between the values on inner boundaries of two neighboring iterations.

The above procedure is a serial version or process. It can be slightly changed to a parallel process, in which the initial values on all inner boundaries $(\Gamma_{1b}, \Gamma_{2b})$ are assigned at first, then every sub-domain is solved at the same time or in parallel, then the values on all inner boundaries are updated by the currently obtained sub-domain solutions.

The conventional DDM has a main shortcoming: the convergence speed of the inner boundary iteration is relatively low after the first few times of iteration, i.e., the convergence speed at the beginning of iteration is fast, but the later will be slow, which is determined by the property of the iteration algorithm.

B. The Multigrid Method

The motive of the multigrid method comes from the convergence analysis of error distribution function in terms of space in frequency domain [8]. A low frequency function in a fine grid or short space scale becomes a high frequency function in a coarse grid, i.e., the low varying rate of function relative to a short space scale looks like a high varying rate to a long space scale. In the solution of an equation system based on a grid, the ordinary iteration solvers tend to eliminate the error of high frequency component more rapidly than that of low frequency. Therefore, to eliminate the error of low frequency to the fine grid, a coarse grid with longer element sides can be employed to solve the same problem. Consequently, a small number of iteration times in the fine grid can eliminate high frequency residual rapidly, and a small number of iteration times in the coarse grid can eliminate the low frequency residual.

The setup of the coarse grid relative to a fine grid must submit certain criteria [8]. Therefore, the element sides of the coarse grid can not be too large and the number of elements can not be too few. Suppose a fine and a coarse grid are generated for a field domain, whose corresponding equation system in the fine grid is $K\varphi = b$, and the current solution in the fine grid is φ_f^{old}, where subscript f denotes the variable in fine grid. The procedure of obtaining a new value denoted by φ_f^{new} in the fine grid by a V-cycle, i.e., a process from fine to coarse then return to fine grid, is as follows [8].

Step 1: *Pre-smoothing Operation*. Solve the equation system in the fine grid by an iteration algorithm with the initial value φ_f^{old}, but only carrying out a few iteration times (e.g., n that may be set to around 3), whose result is denoted by $\overline{\varphi}_f = \text{Relax}^n(\varphi_f^{\text{old}}, b_f)$.

Step 2: Error Estimation. Calculate the residual of the equation system with the solution $\overline{\varphi}_f$, which is $r_f = b_f - K_f\overline{\varphi}_f$.

Step 3: *Coarse Grid Setup*. Form an equation system in the coarse grid as follows:

$$K_c w_c = r_c \tag{2}$$

where subscript c denotes the variables in coarse grid, $K_c = I_f^c K_f I_c^f$. The prolongation operator I_c^f from the coarse to the fine grid and the restriction operator I_f^c from fine to coarse are associated with the distribution of the fine and the coarse grids (see [8] in details), $r_c = I_f^c r_f$.

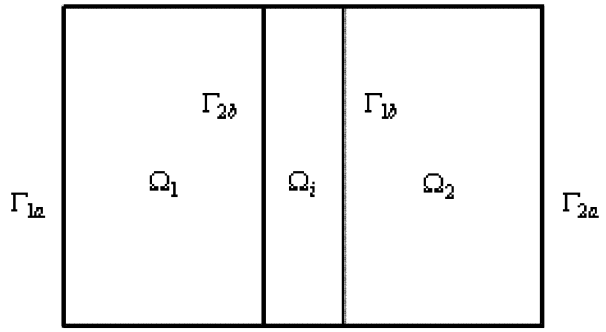

Fig. 1. Domain and sub-domains of DDM.

Step 4: *Solution of Coarse Grid*. Solve (2) to get w_c by any algorithm or solver.

Step 5: *Determine Revised Value in the Fine Grid*. Based on the solution in the coarse grid and the old value of the fine grid $\overline{\varphi}_f$, define a new value of freedoms in the fine grid by $\overline{\overline{\varphi}}_f = \overline{\varphi}_f + I_c^f w_c$.

Step 6: *Post-smoothing Operation*. Solve the equation system by an iteration algorithm with the initial value $\overline{\overline{\varphi}}_f$ to get the new value φ_f^{new}, i.e., $\varphi_f^{\text{new}} = \text{Relax}^{n_2}(\overline{\overline{\varphi}}_f, b_f)$ where n_2 is the number of iteration times (may be set to around 3), which is able to eliminate the errors caused by the coarse grid correction.

In step 4, it is indicated that any algorithm can be employed to solve (2). Obviously, if the fine grid is in a large scale, the scale of (2) concerning the first level of coarse grid can not be very small, because of the restriction of setting coarse grid by the criteria. Therefore, the multigrid approach can be employed again to solve (2). By analogy, coarser and coarser grids can be set up till the dimension of the equation system is small enough to be solved by a direct algorithm.

C. The Algebraic Multigrid Method

The multigrid can only be used on structured computation grids, since the coarse grid must be generated in the geometrical sense. However, the structured grid or FE mesh may not be generated for the domains with complex structures. The algebraic multigrid (AMG) method is able to implement the multigrid approach on un-structured grids, based on the strategy that the coarse grid is put into a black box, i.e., it does not require a geometry-visible coarse grid but forms the coefficient matrix of (2) directly by algebraic operations [5]. The AMG method preserves most advantages of the multigrid method. However, the approach for setting the virtual coarse grid is a complicated algebraic operation process.

The AMG solution process can be divided into two phases. The first is setup-phase, in which hierarchical virtual coarse grids are formed based on the fine grid. The second is the ordinary multigrid cycles. In the six steps of the multigrid procedure described above, the setup-phase includes the construction of the prolongation operator I_c^f and the restriction operator I_f^c, which is introduced in details in [8]. The defection of the AMG method is that the setup-phase needs a relatively long time. But this disadvantage can be minimized when combining AMG with DDM, since the sub-domains in DDM are iteratively solved for many times, and every time the stiffness matrix of FEM keeps

the same, and the setup-phase is only carried out one time in the whole iteration process of DDM.

III. Nesting Combination of FEM-DDM-AMG

In the ordinary FEM-DDM, several equation systems with sparse matrixes associated with the sub-domains are established by the FEM, and the Gauss-Seidel iteration algorithm is usually used to solve the system of equations. This iteration process is called interior iteration. The DDM iteration between the sub-domains is called exterior iteration. These two iterations always lead to a long time consumed for solving large-scale problems, though they use much smaller computer memory in comparison with the conventional FEM. Therefore, efficient interaction approaches should be investigated to increase the iteration efficiency.

The AMG approach is able to increase the efficiency of the interior iteration, but not able to increase the efficiency of the exterior iteration, i.e., not able to decrease the number of iteration times of DDM.

A nesting combination of FEM-DDM and AMG can decrease the number of iteration times of DDM, whose procedure is as follows (the associated method of each step is given in the end of step).

1) Divide the domain to sub-domains (DDM).
2) Mesh sub-domains as the fine grids (FEM).
3) Set initial values on inner boundaries (step 1 of DDM).
4) Form the stiffness matrix of each sub-domain (FEM).
5) Set up the virtual coarse grids (setup-phase of AMG).
6) Pre-smooth the solution of sub-domains (step 1 of AMG).
7) Update the inner boundary values (DDM)
8) Renew the right-hand term concerning inner boundaries (FEM).
9) Renew the value of freedoms in the fine grid by the coarse grid revision (step 2–5 of AMG).
10) Update the inner boundary values (DDM).
11) Renew the right-hand term (FEM).
12) Post-smooth the solution of sub-domains (step 6 of AMG).
13) Update the inner boundary values (DDM).

Repeat step 6 to 13 till the maximum difference between the values on inner boundaries of two neighboring iterations is small enough. The nesting FEM-DDM-AMG method is very efficient, and the number of iteration times of FEM-DDM can be decreased greatly, which can be shown by the following example.

IV. Numerical Example

A simple two-dimensional electrostatic model is employed to illustrate the performance of the nesting FEM-DDM-AMG method. The model with the sub-domains is shown as Fig. 1. The four boundary sides of the model are assigned potential of 1, 2, 3, and 4 V, starting from the bottom side and others in anticlockwise order. The finite element mesh with 2659 nodes and 1280 elements is generated as shown in Fig. 2, which is used as the fine grid in the FEM-DDM.

In the first step of DDM or the third step of FEM-DDM-AMG, the initial values on inner boundaries should be assigned. Theoretically, they can be arbitrary values, but it is easy to understand that a proper definition of initial values is able to decrease the number of iteration times of DDM.

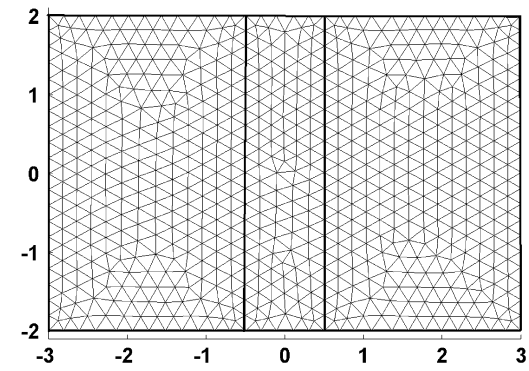

Fig. 2. Finite element mesh as the fine grid of DDM.

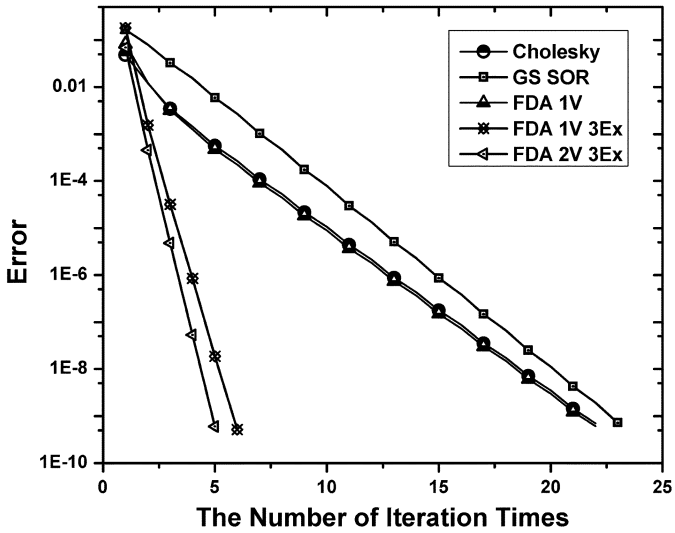

Fig. 3. Comparison of the convergence of the error on inner boundaries between several methods versus the number of iteration times of DDM.

If the domain can be solved as a whole, we can employ a very coarse FE mesh to calculate the problem first, then the initial inner boundary values in the fine mesh can be obtained by the interpolation from the coarse mesh to the fine mesh. Although these values on the fine mesh are approximate, they are indeed proper initial values to the inner boundaries. For the example shown in Fig. 2, we take a coarse mesh with 36 elements and 25 nodes to define the initial values.

In the iteration process of DDM, the Cholesky decomposition (a direct algorithm), Gauss-Seidel successive over-relaxation iteration or the nesting FEM-DDM-AMG method presented in this paper can be employed. To illustrate the efficiency of the FEM-DDM-AMG, those different methods are compared by investigating the convergence of the inner boundary values.

Fig. 3 gives the maximum error or difference between the values on inner boundaries of two neighboring iterations versus the number of iteration times of DDM, in which FDA is the shortening of FEM-DDM-AMG, 1 V, and 2 V mean one and two V-cycles respectively as shown in Section II-B, and 3Ex means three exchanging or updating processes of inner boundary values, i.e., all the updating processes in the FDA procedure are employed.

From the results we can see that the nested FDA method presented in this paper is able to increase the convergence speed greatly. The accuracy obtained by the FDA of one V-cycle with

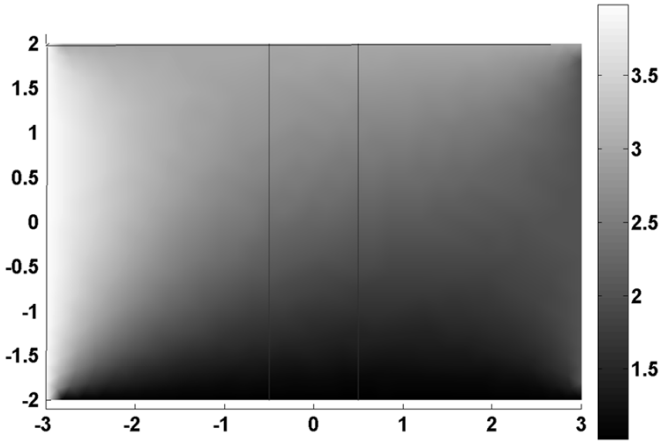

Fig. 4. Distribution of the potential.

only one exchanging process of inner boundary values, which is step 7 of FDA procedure (denoted by FDA 1 V in Fig. 3), is higher than that obtained by the Gauss-Seidel iteration. To obtain certain accuracy, the number of iteration times of FDA with three times of exchanging inner boundary values in the process of FDA is only one third of the times used by the Cholesky decomposition algorithm. The solution of the electric field is show in Fig. 4. The continuity around the inner boundaries can demonstrate the effect of DDM.

Table I gives a few relationships between the number of FE mesh nodes and the maximum error of the inner boundary values after six times of iteration with one V-cycle, and five times of iteration with two V-cycles for the solution of the model. From the results in the table we can see that the error decreases as the number of nodes increase, which is only an individual but not a general conclusion. However it really illustrates another advantage of the AMG-FE-DDM, which is that the number of its iteration times keeps almost a constant as the scale of the problem or the dimension of the equation system increases.

Although the above example is simple, it is able to demonstrate the advantage and the efficiency of the FDA. This approach can be extended to three-dimensional models directly.

V. Conclusion

The algebraic multigrid (AMG) method is really a superexcellent method in the solution of equation systems, which can increase the solution efficiency greatly. The domain decomposition method (DDM) is a practical approach to translate a large-scale problem into smaller. The direct combination of AMG and DDM is only able to reduce the time consumed by solving the equation system, but not able to reduce the number of the iteration times of the DDM. The nesting combination of FEM-DDM and AMG presented in this paper is able to reduce the number of iteration times of DDM greatly. The approach is very suitable for parallel computations.

TABLE I
Number of Nodes Versus the Error of Inner Boundary Values

The number of nodes	FDA with one V-cycle		FDA with two V-cycles	
	Iteration times	error	Iteration times	error
2659	6	5.21e-10	5	6.02e-10
8743	6	2.89e-10	5	3.87e-10
15323	6	2.01e-10	5	2.34e-10
23727	6	1.75e-10	5	2.05e-10

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