# Constrained Convex Optimisation

## Introduction / motivation

Constrained convex optimisation appears in multiple areas of control engineering. From predictive and optimal controllers (MPC), to robust controllers (LPV, $H_\infty$) to control barrier and control Lyapunov functions. Constrained convex optimisation is a useful tool to understand and to be able to use.

## 1 Solving the dual problem

The first method proposed is solving the dual problem which involves solving a minimisation nested inside of a maximisation.

The **primal problem** is as shown in eq. (1). Where $f(x)$ is the equation you are minimising or the objective function, $g(x)$ is the inequality constraints and $h(x)$ is the equality constraints. The variable $x$ then is the value you can change otherwise known as you decision variable.

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, i = 1, ..., m \\ & h_i(x) = 0, j = 1, ..., n \end{aligned} \tag{1}$$

The **Lagrangian** is how we can express the objective and the constraints in one equation. This is shown in eq. (2). The variables $\lambda$ and $\nu$ are called Lagrange multipliers and they will aid in the constrained optimisation.

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{n} \nu_j h_j(x) \tag{2}$$

The **dual problem** then can be expressed as shown in eq. (3). There is an inner minimisation of $x$ and an outer maximisation of $\lambda$ and $\nu$. A complete inner minimisation is ran and an optimal $x^*$ is found, $\lambda$ and $\nu$ are constant. If the constraints have been violated then $g(x^*)$ will be positive or $h(x^*)$ will be positive or negative. Then one step is taken in the maximisation with $x^*$ as a constant. The corresponding multiplier is then updated (more positive $\lambda$ or more positive/more negative $\nu$) to increase the outer maximisation problem if constraints were violated. The inner optimisation is ran again with the new multipliers and $x$ is encouraged more to adhere to the constraints as it is more costly to violate the constraints with the new multiplier. This is repeated until an $x$ which minimises the inner problem and the multipliers can no longer increase the outer maximisation is found.

$$\max_{\lambda \geq 0, \nu} \min_{x} L(x, \lambda, \nu) \tag{3}$$

## 2 Interior point method

Before I had said something to the effect of when an $x$ and Lagrange multipliers are found that mean neither minimisation nor maximisation can further improve the optimisation should stop. But how should we check whether or not this solution is the optimal solution?

The Karush–Kuhn–Tucker conditions enable us to do this. The first condition eq. (4) is referred to as stationarity. That the gradient of the Lagrangian at the proposed solution is zero, meaning it is a critical point. Conditions eq. (5) and eq. (6) ensure primal feasibility. Condition eq. (7) ensures dual feasibility, the Lagrange multiplier for inequality constraints must be greater than or equal to zero to ensure constraints are adhered to. Condition eq. (8) is complimentary slackness. Complementary slackness means $\lambda g(x) = 0$ for inequality constraints, and $\nu h(x) = 0$ for equality constraints. This means either a constraint is strictly satisfied ($g(x) < 0$, leading to $\lambda = 0$) or it holds with equality ($g(x) = 0$, allowing $\lambda$ to be positive).

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{n} \nu_j^* \nabla h_j(x^*) = 0, \tag{4}$$

$$h_i(x^*) = 0, \quad i = 1, \dots, n, \tag{5}$$

$$g_i(x^*) \leq 0, \quad i = 1, \dots, m, \tag{6}$$

$$\lambda_i^* \geq 0, \quad i = 1, \dots, m, \tag{7}$$

$$\lambda_i^* g_i(x^*) = 0, \quad i = 1, \dots, m. \tag{8}$$

The clever part about the interior point method is that instead of doing nested minimisation and maximisation the primal-dual interior point method solves a relaxed version of the KKT conditions like a system of nonlinear equations using Newtons method.

$$\underbrace{\begin{bmatrix} H(x,\lambda,\nu) & A(x)^T & G(x)^T & 0 \\ A(x) & 0 & 0 & 0 \\ G(x) & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix}}_{J(z)} \underbrace{\begin{bmatrix} \Delta x \\ \Delta \nu \\ \Delta \lambda \\ \Delta s \end{bmatrix}}_{\Delta z} = - \underbrace{\begin{bmatrix} \nabla f(x) + G(x)^T \lambda + A(x)^T \nu \\ h(x) \\ g(x) + s \\ S\lambda - v \end{bmatrix}}_{f(z)} \tag{9}$$

Where,

$$H(x,\lambda,\nu) = \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x) + \sum_{i=1}^{n} \nu_i \nabla^2 h_i(x) \tag{10}$$

We can reorganise eq. (9) to get eq. (11)

$$\Delta z = -J(z)^{-1} f(z) \tag{11}$$

and finally eq. (12) where $\alpha$ is the step length.

$$z_{k+1} = z_k + \alpha \Delta z \tag{12}$$

# 3 Worked example

The primal of the problem which I have done is shown in eq. (13)

$$\begin{aligned} \min_{x} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 \geq 1, \\ & x_2 = 1.25 \end{aligned} \tag{13}$$

The solving dual problem method is shown in fig. 1, it can be seen the true minimum is found in the first step, but the solution is then refined back towards the feasible area.

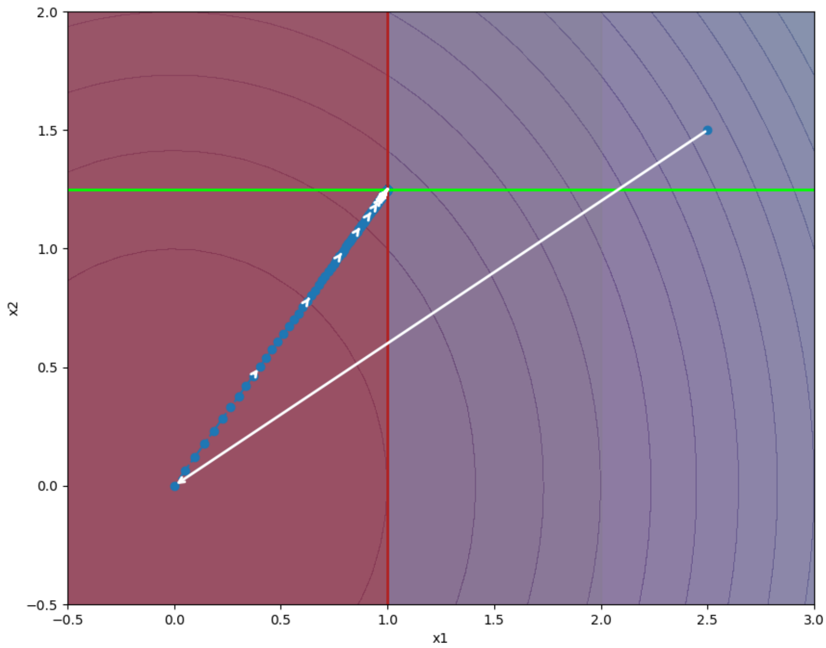


Figure 1: Result of optimisation by solving dual problem

The solution using the interior point method always stays inside the feasible area, fig. 2, hence the name. It also arrives at the optimal solution.

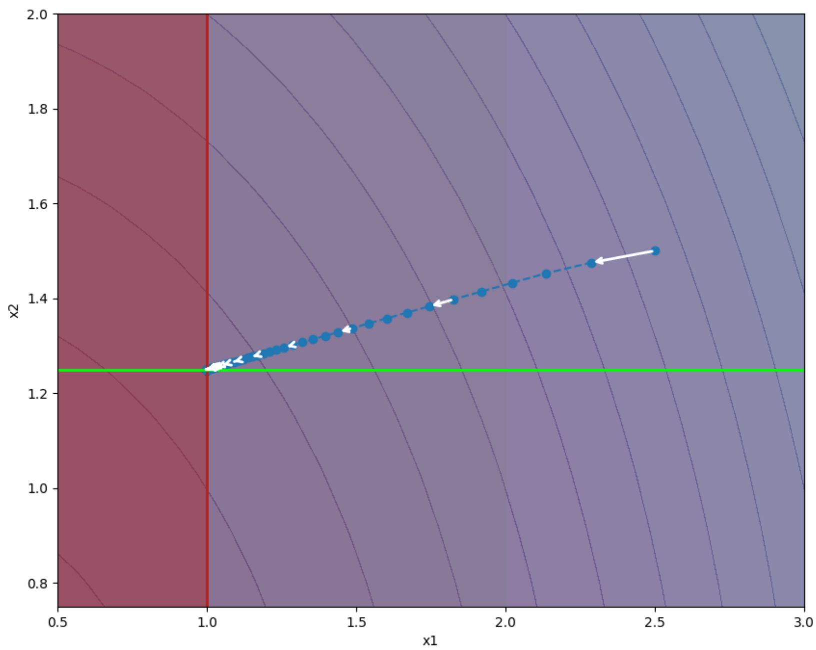


Figure 2: Result of optimisation using interior point method