CUR Decomposition and Subspace Clustering

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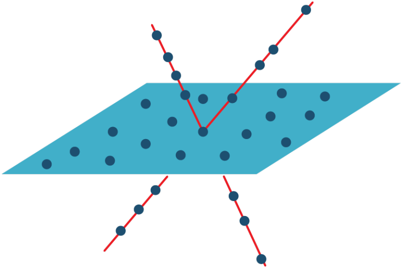
ICCHA7

Joint with A. Aldroubi, A. Sekmen, and A. B. Koku

The Subspace Clustering Problem

$$\mathscr{U} := \bigcup_{i=1}^{L} S_i, \quad S_i \subset \mathbb{R}^m$$



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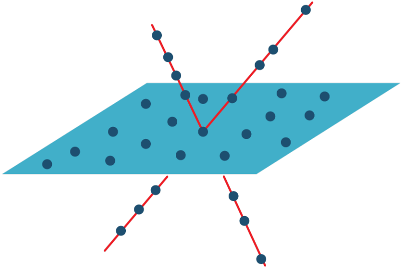
$$W = [w_1 \dots w_n] \in \mathbb{R}^{m \times n}, \quad w_i \in \mathscr{U}$$



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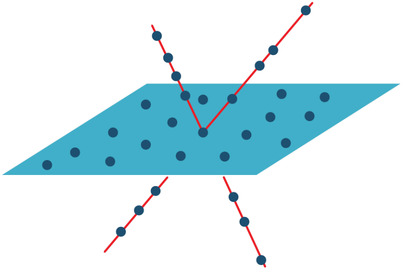

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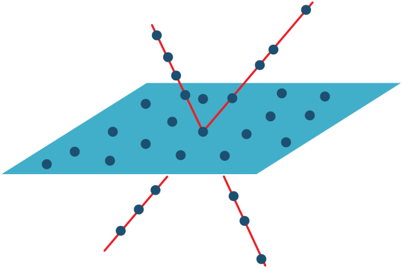

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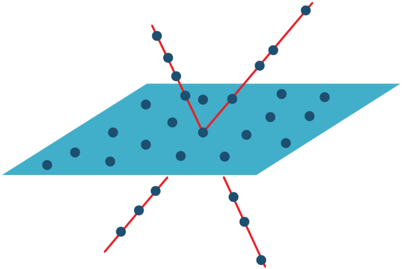

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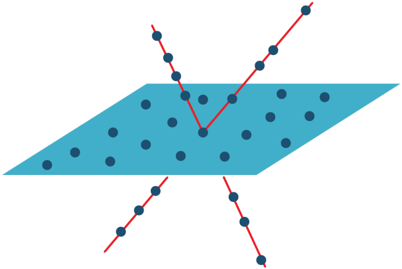

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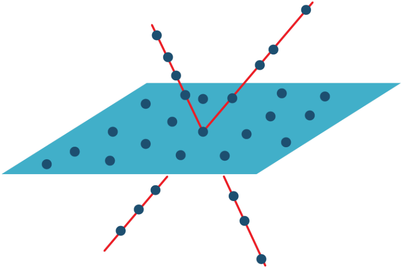

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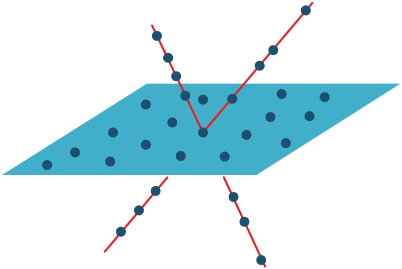
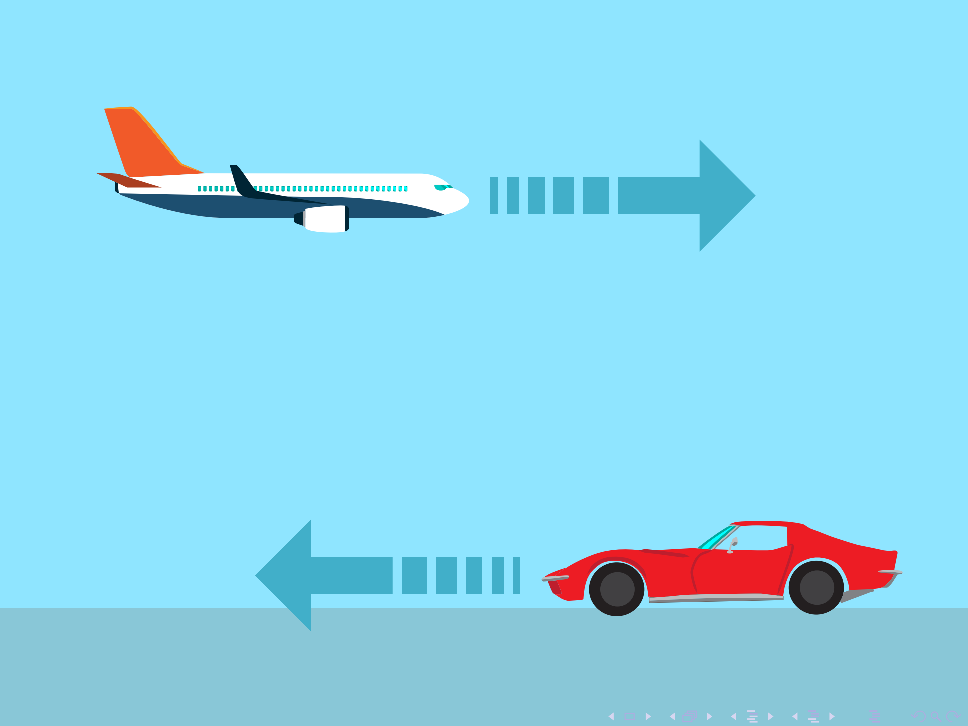
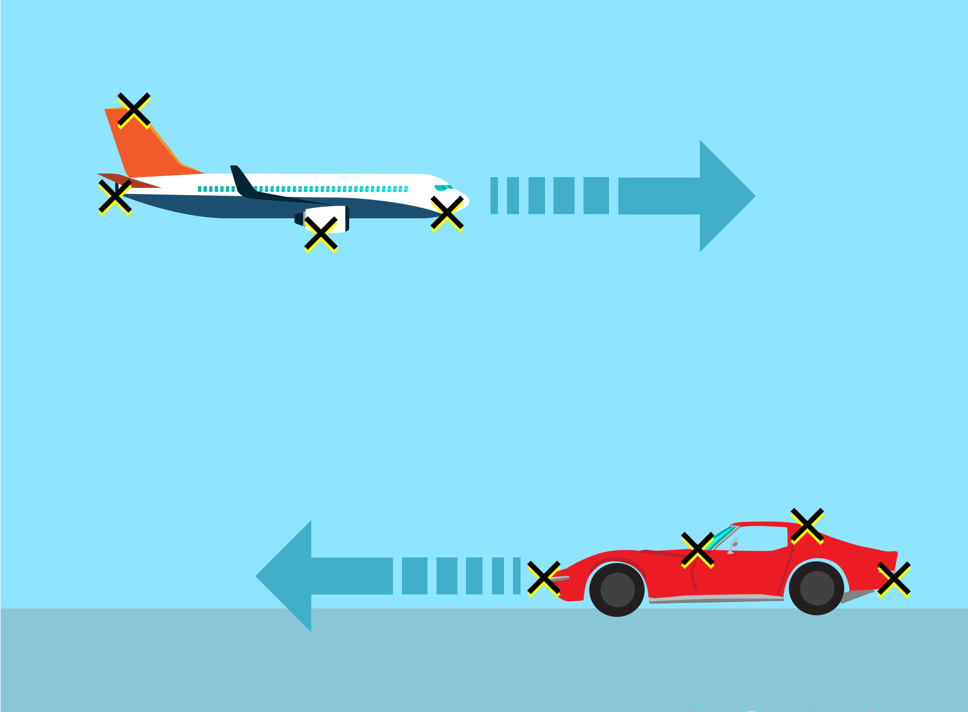
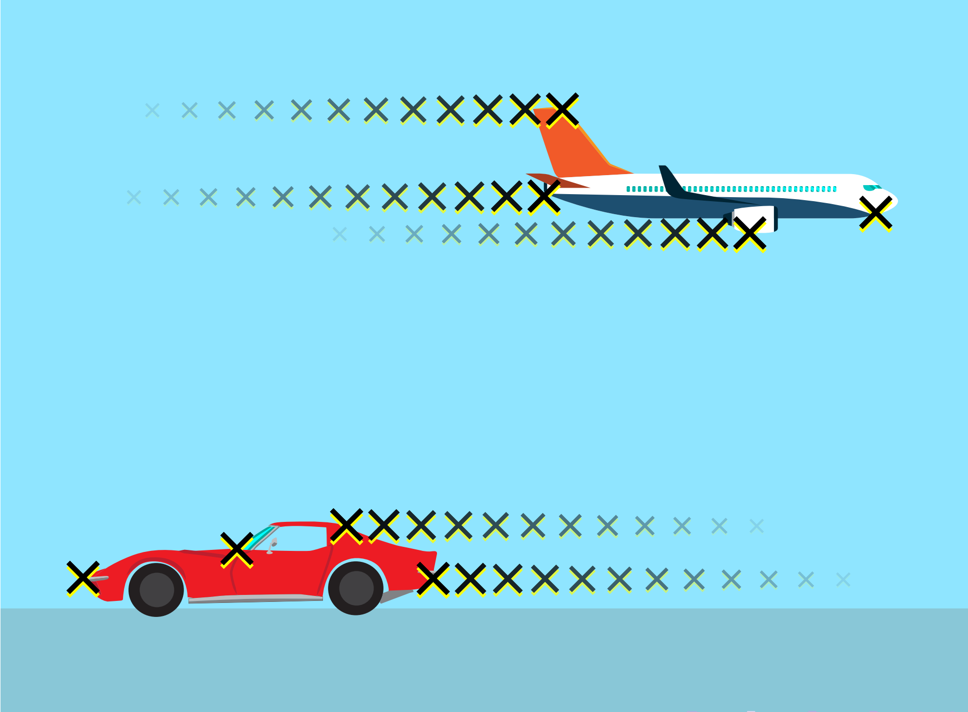
- Data from each subspace is *generic*

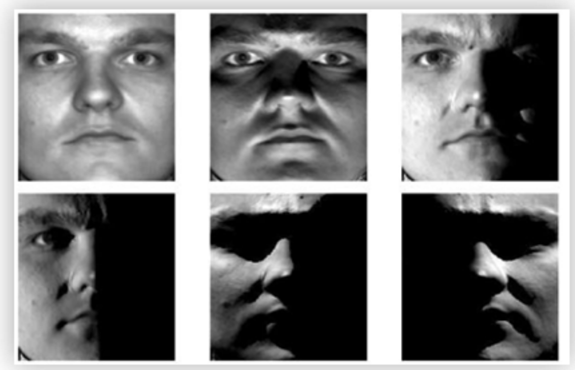








Example Application



Yale Face Database B: Georghiades, A.S. and Belhumeur, P.N. and Kriegman, D.J. From Few to Many: Illumination Cone Models for Face Recognition under Variable Lighting and Pose. IEEE Trans. Pattern Anal. Mach. Intelligence 23(6):643-660 (2001).

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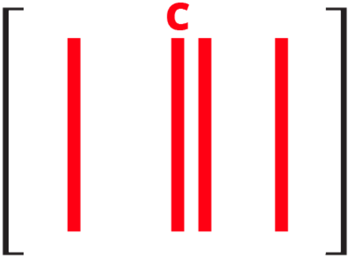

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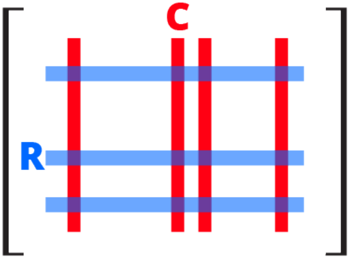

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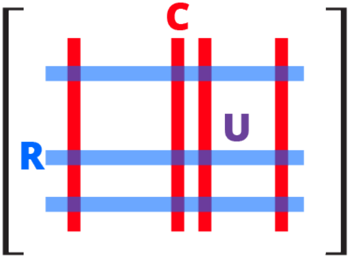

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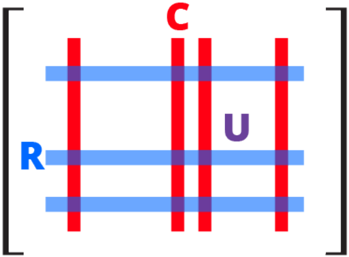

Theorem

If $rank(U) = rank(A)$, then

$$A = CU^{\dagger}R.$$

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Theorem (Aldroubi, H, Koku, Sekmen, '17)

Suppose W is drawn from $\mathscr{U}$. If $W = CU^\dagger R$, let $Y = U^\dagger R$, and $Q = |Y^T Y|$ or $Q = \mathrm{bin}(Y^T Y)$. Then $Q^{d_{\max}}$ is a similarity matrix for W.

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Proof of Corollary: Choose $C = R = W$, then $Q = |W^\dagger W| = |V_r V_r^T|$

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- Relates similarity matrices with Compressed Sensing-inspired minimization techniques

Relation to Minimization Problems

Low-Rank Representation[3]

$$\min_Z \operatorname{rank} Z \quad \text{s.t.} \quad W = WZ$$

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Theorem (Liu, et. al.)

$V_r V_r^T$ (hence $W^\dagger W$) is the unique minimizer of the ℓ_1 problem above.

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Hopkins155 Motion Dataset[4]

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Experiments

Hopkins155 Motion Dataset[4]

Method	GPCA	RANSAC	MSL	ALC	SSC	RSIM	NLS	SIM	CUR
Average	10.34%	9.76%	5.03%	3.56%	1.45%	0.89%	0.76%	7.08%	$\sim$ 1.97%

Table: % Classification errors for all 155 sequences in Hopkins155 database.

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- Cluster columns of S_W

Future Directions

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- Non-independent subspaces?

Thanks!

A. Aldroubi, K. Hamm, A. B. Koku, and A. Sekmen, CUR Decompositions, Similarity Matrices, and Subspace Clustering, arXiv: 1711.04178