# CONFORMALLY FLAT STRUCTURES VIA HYPERBOLIC GEOMETRY

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Abstract.

## 1. Introduction

## 2. Preliminaries

2.1. **Tensors.** Let $M$ be a smooth manifold carrying a Riemannian metric $g$. The non-degeneracy of $g$ at each point allows us to identify $TM$ with $T^*M$ via the map that sends $v \in T_pM$ to $g_p(v, \cdot) \in T_p^*M$ and this extends to an identification of vector fields and 1-forms. Abusing notation, we write $g : TM \to T^*M$ for this map and $g^{-1} : T^*M \to TM$ for its inverse. If $B : TM \to TM$ is a $(1, 1)$-tensor, self-adjoint with respect to $g$, then $gB$ is a symmetric 2-tensor defined by

$$gB(X, Y) = g(BX, Y)$$

for all vector fields $X$ and $Y$. Similarly, if $T$ is a symmetric 2-tensor, then the endomorphism field $g^{-1}T$ is defined by

$$T(X, Y) = g((g^{-1}T)X, Y)$$

is self-adjoint. The trace of such a 2-tensor is the trace of the corresponding $(1, 1)$-tensor

$$\mathrm{tr}_g(T) = \mathrm{tr}(g^{-1}T).$$

The Levi-Civita connection $\nabla$ for $g$ is the unique torsion free connection on $TM$ compatible with $g$. The exterior covariant derivative $d^\nabla$ is the alternization of the connection. For $B$ is an endomorphism, $d^\nabla B$ is defined by

$$\begin{aligned} d^\nabla B(X, Y) &= (\nabla_X B)Y - (\nabla_Y B)X \\ &= \nabla_X(BY) - \nabla_Y(BX) - B([X, Y]). \end{aligned}$$

For a symmetric 2-tensor, it is

$$\begin{aligned} d^\nabla T(X, Y, Z) &= (\nabla_Y T)(X, Z) - (\nabla_Z T)(X, Y) \\ &= YT(X, Z) - T(\nabla_Y X, Z) - T(X, \nabla_Y Z) \\ &\quad - YT(X, Z) + T(\nabla_Z X, Y) + T(X, \nabla_Z Y). \end{aligned}$$

The two are related via $d^\nabla T(X, Y, Z) = g(X, d^\nabla(g^{-1}T)(Y, Z))$ so that one vanishes if and only if the other does. The curvature of $\nabla$ is

$$R^\nabla(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the full curvature tensor

$$Rm(X, Y, Z, W) = g(R^\nabla(X, Y)Z, W),$$

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for vector fields $X$,$Y$,$Z$ and $W$. The curvature tensor has several important symmetries. Given two symmetric 2-tensors $T$ and $S$, there is an operation that produces a 4-tensor with these same symmetries called the Kulkarni-Nomizu product of $T$ and $S$. It is defined by

$$\begin{aligned} T \owedge S(X,Y,Z,W) &= T(X,W)S(Y,Z) + T(Y,Z)S(X,W) \\ &\quad - T(X,Z)S(Y,W) - T(Y,W)S(X,Z) \end{aligned}$$

and is a symmetric and bilinear product on symmetric 2-tensors. Now suppose $g$ is a Riemannian metric and $T$ a symmetric 2-tensor. The trace of their Kulkarni-Nomizu product is computed in terms of $T$ and its trace:

$$tr_g(g \owedge T) = (n-2)T + \mathrm{tr}_g(T)g.$$

The metric $g$ will have constant sectional curvature $K$ if and only if its curvature tensor is of the form

$$Rm = \frac{1}{2} K g \owedge g.$$

2.2. **Tangent Bundles and Hyperbolic Space.** The tangent bundle $\pi : TM \to M$ of a smooth manifold records differential information and so second order differential information naturally lives in the double tangent bundle $T(TM) \to TM$. A connection $\nabla$ on $TM$ lets us split the double tangent bundle into a horizontal bundle a vertical bundle $T(TM) \simeq H \oplus V$. Here the vertical bundle $V$ is canonically defined as $V = \ker d\pi$ while the horizontal bundle $H$ depends on the connection. It consists of all tangent vectors to curves $(\gamma(t), X(t))$ in $TM$ for which $D_t X = 0$, where $D_t = \gamma^* \nabla$.

In fact, with respect to the splitting, if $\alpha(t) = (\gamma(t), X(t))$ is a path in $TM$ for a vector field $V$ along $\gamma$ then $\alpha'(t) = (\gamma'(t), D_t X) \in H \oplus V$. Now suppose $f : \Sigma \to M$ is a smooth map and that $F : \Sigma \to TM$ is given by $F(x) = (f(x), V(x))$ for some vector field $V \in \Gamma(f^* TM)$ along $f$. Then the derivative of $F$ is $dF_x(u) = (df_x(u), (f^*\nabla)_u V)$. If $M$ carries a Riemannian metric with compatible connection, when $V = N$ is a unit normal vector field along $f$ we call $F = (f, N)$ a unit normal lift of $f$, and the Weingarten equation says $(f^*\nabla)_u N = -df(Bu)$ so that

$$dF_x(u) = (df_x(u), -df_x(Bu)).$$

**Lemma 2.1.** *Let $\mathbb{H}^n \to \mathbb{R}^{n,1}$ be the hyperboloid model of hyperbolic space as an embedded Riemannian submanifold of Mikowski space. Call $\bar{\nabla}$ the Levi-Civita connection of Minkowski space and $\nabla^{\mathbb{H}}$ that of hyperbolic space. Then the Gauss formula of this embedding is*

$$\bar{\nabla}_X Y = \nabla^{\mathbb{H}}_X Y + \langle X, Y \rangle N$$

*for $N(p) = p$ the unit normal vector field of the hyperboloid.*

The geodesic flow $\mathcal{G}^t : U\mathbb{H}^n \to \mathbb{H}^n$ (projected down) has a nice expression in this model. For $p$ a point in hyperbolic space and $v$ a unit vector tangent at $p$ the geodesic flow is

$$\mathcal{G}^t(p,v) = \cosh(t)p + \sinh(t)v.$$

The derivative of the geodesic flow is a map $d(\mathcal{G}^t) : T(U\mathbb{H}^n) \to T\mathbb{H}^n$. We can compute it as follows

**Lemma 2.2.** *Let $\mathcal{G}^t : U\mathbb{H}^n \to \mathbb{H}^n$ be the geodesic flow projected down to hyperbolic space. With respect to the splitting defined above, for $(x, y)$ tangent to $U\mathbb{H}^n$ at $(p, v)$, the derivative of the geodesic flow is*

$$d(\mathcal{G}^t)_{(p,v)}(x, y) = \cosh(t)x + \sinh(t)y + \sinh(t) \langle x, v\rangle p.$$

*Proof.* Let $\alpha(s) = (\gamma(s), V(s))$ be a curve in $U\mathbb{H}^n$ with $\alpha(0) = (p, v)$ and $\alpha'(0) = (x, y)$. Recall this means that $\gamma'(0) = x$ and $D_sV(0) = y$. We compute

$$\begin{aligned} d(\mathcal{G}^t)_{(p,v)}(x, y) &= \frac{d}{ds}\mathcal{G}^t(\gamma(s), V(s))\bigg|_{s=0} \\ &= \frac{d}{ds} \cosh(t)\gamma(s) + \sinh(t)V(s)\bigg|_{s=0} \\ &= \cosh(t)\gamma'(0) + \sinh(t)V'(0) \end{aligned}$$

where $V'$ is the derivative of $V : (-1, 1) \to \mathbb{R}^{n,1}$, which is equal to $\bar{D}_sV(0)$. By the Gauss equation for hyperbolic space Lemma 2.1, this derivative is

$$\begin{aligned} \bar{D}_sV(0) &= D_sV(0) + \langle\gamma'(0), V(0)\rangle \gamma(0) \\ &= y + \langle x, v\rangle p, \end{aligned}$$

and this gives the result. ☐

The hyperbolic Gauss map $\mathcal{G} : U\mathbb{H}^n \to S^{n-1}$ is the map that sends $(p, v)$ to the ideal endpoint of the geodesic staring at $p$ traveling in the direction $v$. One of multiple equivalent ways to interpret this is to to take the geodesics in the hyperboloid model, send them to the Poincaré model with $\pi : \mathbb{H}^n \to \mathbb{B}^n$ and take a limit as $t \to \infty$ in the Euclidean topology. Doing this, one gets

$$\begin{aligned} \mathcal{G}(p, v) &= \lim_{t\to\infty} \pi(\mathcal{G}^t(p, v)) \\ &= \lim_{t\to\infty} \frac{1}{1 + \langle\cosh(t)p + \sinh(t)v, e_{n+1}\rangle}(\cosh(t)p + \sinh(t)v) \\ &= \frac{1}{\langle p + v, e_{n+1}\rangle}(p + v) \end{aligned}$$

The derivative of the hyperbolic Gauss map may be computed similarly to the derivative of the geodesic flow.

**Lemma 2.3.** *Let $\mathcal{G} : U\mathbb{H}^n \to S^{n-1}$ be the hyperbolic Gauss map on the hyperboloid model given by*

$$\mathcal{G}(p, v) = \frac{1}{\langle p + v, e_{n+1}\rangle}(p + v),$$

*then the derivative is*

$$d\mathcal{G}_{(p,v)}(x, y) = -\frac{\langle x + y + \langle x, v\rangle p, e_{n+1}\rangle}{\langle p + v, e_{n+1}\rangle^2}(p + v) + \frac{1}{\langle p + v, e_{n+1}\rangle}(x + y + \langle x, v\rangle p)$$

*with respect to the splitting $T(U\mathbb{H}^n) \simeq H \oplus V$.*

2.3. **Parallel Surfaces.** Let $M$ be a smooth $n$-dimensional manifold and let $f : M \to \mathbb{H}^{n+1}$ be an immersion for which $M$ has a smooth unit normal vector field $N$. A parallel family of hypersurfaces is obtained by flowing $M$ in its normal direction.

**Convention:** Following the literature (refs), we define the shape operator $B : TM \to TM$ as usual by $g^{-1}I\!I$ relative to the normal vector field $N$, but we flow in the *opposite* direction $-N$.

Given $f$ and $N$ one can form $F = (f, -N)$ the unit normal lift $M \to U\mathbb{H}^{n+1}$. If one then follows this with the geodesic flow (projected down to $\mathbb{H}^{n+1}$) then we get the desired parallel surfaces.

To actually compute things, we work in the hyperboloid model of $\mathbb{H}^{n+1}$, in which the geodesic flow has a simple expression given above.

**Lemma 2.4.** *Let $f^t : M \to \mathbb{H}^{n+1}$ be defined by $f^t = \mathcal{G}^t \circ F$, then $f^t$ is an immersion provided none of the eigenvalues of $B$ are equal to $-\coth(t)$ and the induced metric is given by*

$$g_t(X, Y) = g((\cosh(t)Id + \sinh(t)B)X, (\cosh(t)Id + \sinh(t)B)Y)$$

*Proof.* The derivative of $f^t$ may be computed via the chain rule $df^t_x = d\mathcal{G}^t_{F(x)} \circ dF_x$. Since $F : M \to U\mathbb{H}^{n+1}$, the derivative takes values in $T_{F(x)}U\mathbb{H}^{n+1}$. The connection splits this into the horizontal and vertical space, and via this decomposition, the derivative is

$$dF_x(v) = (df_x(v), \nabla_x(-N)) = (df_x(v), df_x(Bv))$$

(recall our convention to still use $B$ relative to $+N$). The derivative of the geodesic flow is given by Lemma 2.2 as

$$d\mathcal{G}^t_{(p,v)}(x, y) = \cosh(t)x + \sinh(t)y + \sinh(t)\langle x, v\rangle p.$$

Since $df_x(v)$ and $N(x)$ are orthogonal, we get

$$\begin{aligned} df^t_x(v) &= d\mathcal{G}^t_{F(x)}(dF_x(v)) \\ &= \cosh(t)df_x(v) + \sinh(t)df_x(Bv) \\ &= df_x(\cosh(t)Id + \sinh(t)B)v. \end{aligned}$$

This identifies the pullback tensor as $g_t = (f^t)^* g_{\mathbb{H}^{n+1}} = A_t^* g$, for $A_t = \cosh(t)Id + \sinh(t)B$. This is a metric (equivalently $f^t$ is an immersion) whenever it is positive definite. That is, whenever the eigenvalues of $A_t^2$ are all positive, which reduces to the condition that the eigenvalues of $A_t$ be nonzero. In terms of the eigenvalues $\lambda_i$ for $B$, the requirement is that $\cosh(t) + \sinh(t)\lambda_i \neq 0$, or that $\lambda_i \neq -\coth(t)$ for all $i$. This gives the result. □

**Corollary 2.5.** *If the eigenvalues of $B$ are in $[-1, 1]$ then $f_t$ is an immersion for all $t$ and $M$ may be flown in the forwards and backwards direction for all time.*

2.4. **Tensors at Infinity.** Expanding the hyperbolic trig functions in terms of exponentials gives the induced metric on the parallel surface $M_t$ as

$$g_t = \frac{1}{4}e^{2t}(g + 2I\!I + I\!I\!I) + \frac{1}{2}(g - I\!I\!I) + \frac{1}{4}e^{-2t}(g - 2I\!I + I\!I\!I).$$

Define $\hat{g} = g + 2I\!I + I\!I\!I$. The induced metrics are asymptotic to $\hat{g}$ in the sense that their conformal classes converge to that of $\hat{g}$ as $t \to \infty$ since $4e^{-2t}g_t \to$

$g + 2I\!I + I\!I\!I$. Following (refs) we refer to $\hat{g}$ as the metric at infinity. This name is mostly accurate in the following sense. If $\mathcal{G} : U\mathbb{H}^{n+1} \to S^n$ is the hyperbolic Gauss map $\mathcal{G} = \lim_{t\to\infty} \mathcal{G}^t$, then the composition $f_\infty = \mathcal{G} \circ F$ gives a surface in $S^n = \partial_\infty(\mathbb{H}^{n+1})$, which the surfaces $M_t$ can be shown to limit to (in a Euclidean topology). If we pull back the round metric $\mathring{g}$ on the sphere by $f_\infty$ then we obtain a metric on $M$ that is conformal to $\hat{g}$.

**Lemma 2.6.** *Let $f : M \to \mathbb{H}^{n+1}$ be an immersion with a smooth unit normal vector field $N$. If $\hat{g}$ and $f_\infty$ are defined as above then*

$$f_\infty^* \mathring{g} = \frac{1}{\langle f + N, e_{n+2}\rangle^2}\ \hat{g}$$

*for $\mathring{g}$ the round metric on the sphere. In particular, the metrics $\hat{g}$ and $f_\infty^*\mathring{g}$ are conformal.*

*Proof.* The hyperbolic Gauss map in the hyperboloid model is given by

$$\mathcal{G}(p, v) = \frac{1}{\langle p + v, e_{n+2}\rangle}(p + v)$$

which has derivative, using the splitting of $U\mathbb{H}^{n+1}$

$$d\mathcal{G}_{(p,v)}(x, y) = -\frac{\langle x + y + \langle x, v\rangle\, p, e_{n+2}\rangle}{\langle p + v, e_{n+2}\rangle^2}(p + v) + \frac{1}{\langle p + v, e_{n+2}\rangle}(x + y + \langle x, v\rangle\, p).$$

This can be used to compute the pullback tensor $\mathcal{G}^*\mathring{g}$ as

$$\mathcal{G}^*\mathring{g}_{(p,v)}((x, y), (u, w)) = \frac{1}{\langle p + v, e_{n+2}\rangle^2}\left(\langle x + y, u + w\rangle - \langle x, v\rangle\langle u, v\rangle\right)$$

Recalling that the derivative of $F$ is $dF_x(v) = (df_x(v), df_x(Bv))$ and that $df_x$ is orthogonal to $N$ gives

$$\begin{aligned}(f_\infty)^*\mathring{g} = F^*(\mathcal{G}^*)\mathring{g} &= \frac{1}{\langle f + N, e_{n+2}\rangle^2}\langle df(Id + B), df(Id + B)\rangle\\ &= \frac{1}{\langle f + N, e_{n+2}\rangle^2}\hat{g},\end{aligned}$$

as claimed. □

Looking again at the expansion of $g_t$ one makes the following definitions:

$$\hat{I\!I} = g - I\!I\!I, \quad \hat{B} = \hat{I\!I}^{-1}\hat{g}, \quad \hat{I\!I\!I} = g - 2I\!I + I\!I\!I.$$

In particular,

$$\begin{aligned}\hat{g} &= g(Id + B, Id + B)\\ \hat{B} &= (Id + B)^{-1}(Id - B)\end{aligned} \tag{1}$$

and these algebraic equalities can be inverted. Namely,

$$\begin{aligned}g &= \frac{1}{4}\hat{g}(Id + \hat{B}, Id + \hat{B})\\ B &= (Id + \hat{B})^{-1}(Id - \hat{B}).\end{aligned} \tag{2}$$

If tensors $(g, B)$ obtained via (2) from $(\hat{g}, \hat{B})$ happen to be the induced metric and shape operator of some immersion in to $\mathbb{H}^{n+1}$ then $(\hat{g}, \hat{B})$ will be the metric and shape operator at infinty.

## 3. The Gauss-Codazzi Equations

We now discuss a system of equations that serve as the integrability conditions for a pair of tensors $(g, B)$ to be the induced metric and shape operator of an immersion $M \to \mathbb{H}^{n+1}$ as well as an equivalent dual set of equations for $(\hat{g}, \hat{B})$

Given an immersion $f : M \to \mathbb{H}^{n+1}$ with smooth unit normal vector field $N$ along $f$ one has the induced metric $g = f^* g_{\mathbb{H}^{n+1}}$ and second fundamental form $\mathit{I\!I}$ defined by the Gauss formula

$$(f^*\nabla^{\mathbb{H}})_X df(Y) = df(\nabla^g_X Y) + \mathit{I\!I}(X,Y)N,$$

where $\nabla^{\mathbb{H}}$ and $\nabla^g$ are the corresponding Levi-Civita connections of $g_{\mathbb{H}^{n+1}}$ and $g$, and where $f^*\nabla^{\mathbb{H}}$ is the pullback connection on $f^*(T\mathbb{H}^{n+1}) \to M$.

From $\mathit{I\!I}$, one has the shape operator $B = g^{-1}\mathit{I\!I}$ and sees that $(g, B)$ obey the Gauss-Codazzi equations

$$\begin{aligned} Rm &= -\frac{1}{2} g \owedge g + \frac{1}{2}\mathit{I\!I} \owedge \mathit{I\!I} \\ d^{\nabla} B &= 0. \end{aligned} \tag{GC}$$

Because hyperbolic space has constant sectional curvature $-1$, these equations may be written as

$$\begin{aligned} sec(X,Y) &= -1 + \mathit{I\!I}(X,X)\mathit{I\!I}(Y,Y) - \mathit{I\!I}(X,Y)^2 \\ d^{\nabla} B &= 0. \end{aligned}$$

When $n = 2$ we recover the familiar formulas for surfaces in hyperbolic space.

$$\begin{aligned} K(g) &= -1 + \det(B) \\ d^{\nabla} B &= 0. \end{aligned}$$

The Gauss-Codazzi equations are the integrability equations for a pair of tensors to be induced by an immersion

**Theorem 3.1.** *Let $g$ be a Riemannian metric on a simply connected manifold $M$ and let $B$ be a self adjoint endomorphism of $TM$. Suppose $(g, B)$ solve the Gauss-Codazzi equations. Then there exists an immersion $f : M \to \mathbb{H}^{n+1}$ such that $g$ is the induced metric of the immersion and $B$ is the shape operator. In addition, the immersion $f$ is unique up to post composition with an isometry of hyperbolic space.*

If $(g, B)$ solve the Gauss-Codazzi equations then we can isometrically immerse $M$ into hyperbolic space and via parallel flowing $M$ we get the tensors at infinity $(\hat{g}, \hat{B})$. So, given two tensors $(g, B)$ solving the Gauss-Codazzi equations we obtain another pair of tensors $(\hat{g}, \hat{B})$ and these two solve their own set of equations, which, following (refs), we call the Gauss-Codazzi equations at infinity. In the general $n$-dimensional setting they are

$$\begin{aligned} \hat{Rm} &= -\frac{1}{2}\hat{g} \owedge \hat{\mathit{I\!I}} \\ d^{\hat{\nabla}} \hat{B} &= 0. \end{aligned} \tag{$\widehat{\mathrm{GC}}$}$$

When $n = 2$ these equations reduce to those of (names) in (refs). Indeed, at a point $p \in M$, take $v_1$ and $v_2$ to be an orthonormal basis of $T_pM$ which are eigenvectors of $\hat{B}$ with corresponding eigenvalues $\lambda_i$. Then, at $p$,

$$\begin{aligned} K(\hat{g}) = sec(v_1, v_2) &= -\frac{1}{2}\hat{g} \owedge \hat{I\!I}(v_1, v_2, v_2, v_1) \\ &= -\frac{1}{2}(\lambda_1 + \lambda_2) = -\frac{1}{2}\mathrm{tr}(\hat{B}) \end{aligned}$$

This dual set of equations is equivalent to the Guass-Codazzi equations in the following sense.

**Theorem 3.2.** *Let the tensors $(g, B)$ and $(\hat{g}, \hat{B})$ on $M$ be related by the algebraic identities (1) and (2). Then $(g, B)$ solves the Gauss-Codazzi equations if and only if $(\hat{g}, \hat{B})$ solves the Gauss-Codazzi equations at infinity.*

*Proof.* That $(\hat{g}, \hat{B})$ solves the Codazzi equation at infinity if and only if $(g, B)$ solves the Codazzi equation follows from the relationship between the connections of $g$ and $\hat{g}$. For vector fields $X$ and $Y$,

$$\hat{\nabla}_X Y = (Id + B)^{-1}\nabla_X (Id + B)Y.$$

A computation then shows that, because the connections are torsion-free,

$$d^{\hat{\nabla}}\hat{B} = -(Id + B)^{-1} d^{\nabla} B.$$

So, $d^{\hat{\nabla}}\hat{B} = 0$ if and only if $d^{\nabla}B = 0$.

Now suppose $(g, B)$ solves the Gauss equation. We will show $(\hat{g}, \hat{B})$ solves the Gauss equation at infinity. Recall the identities $g = \frac{1}{4}(\hat{g}+2\hat{I\!I}+\hat{I\!I\!I})$ and $I\!I = \frac{1}{4}(\hat{g}-\hat{I\!I\!I})$. The Gauss equation then becomes

$$Rm = -\frac{1}{32}(\hat{g} + 2\hat{I\!I} + \hat{I\!I\!I}) \owedge (\hat{g} + 2\hat{I\!I} + \hat{I\!I\!I}) + \frac{1}{32}(\hat{g} - \hat{I\!I\!I}) \owedge (\hat{g} - \hat{I\!I\!I}),$$

and some expansion and cancellation reduces this to

$$-8Rm = (\hat{g} + \hat{I\!I}) \owedge (\hat{I\!I} + \hat{I\!I\!I}).$$

Since, for example, $\hat{g}(X, Y) + \hat{I\!I}(X, Y) = \hat{g}((Id + \hat{B})X, Y)$, we abuse notation and write that

$$-8Rm = \hat{g}(Id, Id + \hat{B}) \owedge \hat{g}(\hat{B}, Id + \hat{B}).$$

Now, a computation shows that because $\hat{g} = g(Id + B, Id + B)$, the curvature tensors are related by $\hat{Rm}(X, Y, Z, W) = Rm(X, Y, (Id+B)Z, (Id+B)W)$, meaning we are ultimately interested in computing

$$\hat{g}(Id, Id + \hat{B}) \owedge \hat{g}(\hat{B}, Id + \hat{B})(X, Y, (Id + B)Z, (Id + B)W).$$

By virtue of $(Id + \hat{B})(Id + B) = 2Id$, the first term of this is

$$\begin{aligned} &\hat{g}(X, (Id + \hat{B})(Id + B)W)\hat{g}(\hat{B}Y, (Id + \hat{B})(Id + B)Z) \\ &\qquad = 4\hat{g}(X, W)\hat{g}(\hat{B}Y, Z) \\ &\qquad = 4\hat{g}(X, W)\hat{I\!I}(Y, Z). \end{aligned}$$

The other terms simplify similarly and all together yield

$$-2Rm(X, Y, (Id + B)Z, (Id + B)W) = \hat{g} \owedge \hat{I\!I}(X, Y, Z, W),$$

which is equivalent to the Gauss equation at infinity. That the pair $(\hat{g}, \hat{B})$ solving $\widehat{\mathrm{GC}}$ implies $(g, B)$ solves GC is a similar computation. $\square$

3.1. **Conformally Flat and Möbius Structures.** A locally conformally flat structure on a manifold $M$ is an atlas of charts to $\mathbb{R}^n$ whose transition functions are conformal maps of the Euclidean metric. There is an equivalent definition in terms of Riemannian metrics. We say a metric $g$ is locally conformally flat if each point in $M$ has a chart to $\mathbb{R}^n$ on which $g$ is conformal to the pullback of the Euclidean metric. A locally conformally flat structure on $M$ is also the conformal class $[g]$ of a locally conformally flat metric $g$.

A Möbius structure on a manifold $M$ is an atlas of charts to $S^n$ whose transition functions are (the restrictions of) Möbius transformations. This is a geometric structure with topological space $S^n$ and group $\text{Möb}(S^n)$. As such, any Möbius structure on a manifold $M$ can be given via a developing map $f : \tilde{M} \to S^n$ on its universal cover and a holonomy representation $\rho : \pi_1(M) \to \text{Möb}(S^n)$ satisfying $f(\gamma \cdot x) = \rho(\gamma) f(x)$ for all $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$.

Conformal transformations in dimensions $n \geq 3$ are incredibly rigid in the following sense.

**Theorem 3.3** (Liouville)**.** *Suppose $U$ is a domain in $\mathbb{R}^n$ for $n \geq 3$ and $\varphi : U \to \mathbb{R}^n$ is a conformal map for the Euclidean metric. Then $\varphi$ is a Möbius transformation.*

**Corollary 3.4.** *Any (locally) conformally flat structure is a Möbius structure.*

*Proof.* The transition functions of a conformally flat structure are conformal maps and hence Möbius transformations. □

There is a tensor that detects whether a map is a Möbius transformation. Given two conformal metrics $g_2 = e^{2u} g_1$, Osgood and Stowe define in (ref) the symmetric 2-tensor $\text{OS}(g_2, g_1)$ as the traceless part of $\text{Hess}(u) - du^2$. That is,

$$\text{OS}(g_2, g_1) = \text{Hess}(u) - du \otimes du - \frac{1}{n} \left( \Delta u - |\nabla u|^2 \right) g_1,$$

with all relevant objects defined with respect to $g_1$. We will refer to this as the Osgood-Stowe differential of $g_2$ with respect to $g_1$, or sometimes simply as the Osgood-Stowe tensor. They prove that OS has a cocycle property for three conformal metrics

$$\text{OS}(g_3, g_1) = \text{OS}(g_3, g_2) + \text{OS}(g_2, g_1).$$

There is also a naturality property that if $f : M \to M$ is a smooth map then for two conformal metrics

$$f^* \text{OS}(g_2, g_1) = \text{OS}(f^* g_2, f^* g_1).$$

It also detects Möbius transformations in the following sense. If $f : U \to \mathbb{R}^n$ is a smooth map of a domain in $\mathbb{R}^n$ then for $\bar{g}$ the Euclidean metric, $\text{OS}(f^* \bar{g}, \bar{g}) = 0$ if and only if $f$ is a Möbius transformation.

These facts have the following consequence. If $g$ is a locally conformally flat metric on $M$ and $\varphi : U \to \mathbb{R}^n$ is a chart for which $e^{2u} g = \varphi^* \bar{g}$, then $\text{OS}(g, \varphi^* \bar{g})$ patches together to form a global tensor on $M$. To see this, if $\psi : V \to \mathbb{R}^n$ is any other conformal chart overlapping with $\varphi$ then

$$\begin{aligned} \text{OS}(g, \psi^* \bar{g}) &= \text{OS}(g, \varphi^* \bar{g}) + \text{OS}(\varphi^* \bar{g}, \psi^* \bar{g}) \\ &= \text{OS}(g, \varphi^* \bar{g}) + \varphi^* \text{OS}(\bar{g}, (\psi \circ \varphi^{-1})^* \bar{g}) \end{aligned}$$

and $\mathrm{OS}(\bar{g}, (\psi \circ \varphi^{-1})^* \bar{g}) = 0$ since $\psi \circ \varphi^{-1}$ is a conformal map and hence a Möbius transformation. For a conformal metric $g$ on $M$ we will refer to this globally defined object as $\mathrm{OS}(g)$.

## 4. Results

**Lemma 4.1.** *If $(\hat{g}, \hat{B})$ are a pair of tensors that solve the Gauss-Codazzi equations at infinity, then $\hat{g}$ is locally conformally flat.*

*Proof.* Let $p$ be a point in $M$ and take a simply connected neighborhood $U$ around $p$. On $U$, the tensors $(\hat{g}, \hat{B})$ solving $\widehat{\mathrm{GC}}$ implies $(g, B)$, defined via (2), solve GC and so we have an isometric immersion $f : U \to \mathbb{H}^{n+1}$. From this we get $f_\infty : U \to S^n$ via composing with the hyperbolic Gauss map and $\hat{g}$ is conformal to $f_\infty^* \mathring{g}$ by Lemma 2.6. Since the round metric on the sphere is locally conformally flat, $f_\infty^* \mathring{g}$ is locally conformal to a flat metric and, after shrinking $U$ if necessary, $\hat{g}$ will be as well. □

When $\hat{g}$ is locally conformally flat, the Gauss-Codazzi equations at infinity are a system of equations for the tensor $\hat{B}$. We are able to fully identify the solutions.

**Theorem 4.2.** *Let $\hat{g}$ be a locally conformally flat metric on $M$. Then $(\hat{g}, \hat{B})$ solves the Gauss-Codazzi equations at infinity if and only if*

$$\hat{g}\hat{B} = 2\mathrm{OS}(\hat{g}) + \frac{1}{n - n^2} S(\hat{g})\hat{g}.$$

In preparation for the proof of this theorem, we list some useful identities.

**Lemma 4.3.** *Suppose $g$ is a Riemannian metric and $\hat{g} = e^{2u} g$ is a metric conformal to $g$. Let $\hat{\nabla}$ and $\nabla$ be the corresponding Levi-Civita connections. If $T$ is a symmetric 2-tensor then the exterior covariant derivatives are related by*

$$d^{\hat{\nabla}} T(X, Y, Z) = d^{\nabla} T(X, Y, Z) + T \owedge g(\nabla u, X, Y, Z).$$

*Proof.* We have

$$\begin{aligned} d^{\hat{\nabla}} T(X, Y, Z) &= Y T(X, Z) - T(\hat{\nabla}_Y X, Z) - T(X, \hat{\nabla}_Y Z) \\ &\quad - Z T(X, Y) + T(\hat{\nabla}_Z X, Y) + T(X, \hat{\nabla}_Z Y). \end{aligned}$$

To simplify, we need how the connections for $\hat{g}$ and $g$ relate:

$$\hat{\nabla}_U V = \nabla_U V + du(U)V + du(V)U - g(U, V)\nabla u$$

for vectors fields $U$ and $V$. Using this on each relevant term in the derivative leads to a good amount of cancellation, eventually resulting in

$$\begin{aligned} d^{\hat{\nabla}} T(X, Y, Z) &= Y T(X, Z) - T(\nabla_Y X, Z) - T(X, \nabla_Y Z) \\ &\quad - Z T(X, Y) + T(\nabla_Z X, Y) + T(X, \nabla_Z Y) \\ &\quad + T(\nabla u, Z) g(X, Y) + T(X, Y) du(Z) \\ &\quad - T(\nabla u, Y) g(X, Z) - T(X, Z) du(Y). \end{aligned}$$

The first set of the terms form $d^{\nabla} T(X, Y, Z)$ and after writing $du = g(\nabla u, \cdot)$, we notice the second set of the terms to be $T \owedge g(\nabla u, X, Y, Z)$. □

**Lemma 4.4.** *Let $g$ be a Riemannian metric with Levi-Civita connection $\nabla$ and let $u$ be a smooth function, then*

$$d^{\nabla} \mathrm{Hess}_g(u)(X, Y, Z) = Rm(\nabla u, X, Y, Z)$$

*Proof.* To ease notation, we write Hess for $\mathrm{Hess}_g(u)$ when no confusion is likely. The derivative is

$$d^{\nabla}\mathrm{Hess}_g(u)(X,Y,Z) = (\nabla_Y \mathrm{Hess})(X,Z) - (\nabla_Z \mathrm{Hess})(X,Y)$$

and $\nabla_Y\mathrm{Hess}$ can be computed via

$$\begin{aligned}(\nabla_Y \mathrm{Hess})(X,Z) &= \nabla_Y(\mathrm{Hess}(X,Z)) - \mathrm{Hess}(\nabla_Y X, Z) - \mathrm{Hess}(X, \nabla_Y Z)\\ &= Yg(\nabla_Z(\nabla u), X) - g(\nabla_Z(\nabla u), \nabla_Y X) - g(\nabla_{\nabla_Y Z}(\nabla u), X)\\ &= g(\nabla_Y\nabla_Z(\nabla u), X) + g(\nabla_Z(\nabla u), \nabla_Y X)\\ &\qquad - g(\nabla_Z(\nabla u), \nabla_Y X) - g(\nabla_{\nabla_Y Z}(\nabla u), X)\\ &= g((\nabla_Y\nabla_Z - \nabla_{\nabla_Y Z})\nabla u, X).\end{aligned}$$

Similarly,

$$(\nabla_Z \mathrm{Hess})(X,Y) = g((\nabla_Z\nabla_Y - \nabla_{\nabla_Z Y})\nabla u, X).$$

The exterior covariant derivative is then

$$\begin{aligned}d^{\nabla}\mathrm{Hess}_g(u)(X,Y,Z) &= g(\nabla_Y\nabla_Z\nabla u - \nabla_Z\nabla_Y\nabla u - \nabla_{\nabla_Y Z - \nabla_Z Y}\nabla u, X)\\ &= g(R^{\nabla}(Y,Z)\nabla u, X),\end{aligned}$$

where we have used that the connection is torsion free so that $\nabla_Y Z - \nabla_Z Y = [Y,Z]$. The lemma then follows from symmetries of the curvature tensor. □

*Proof of main theorem.* We start by showing the given tensor satisfies the Codazzi equation at infinity. Because $d^{\hat\nabla}(g^{-1}T) = 0$ if and only if $d^{\hat\nabla}T = 0$ for any symmetric 2-tensor $T$, it suffices to compute the derivative of $2\mathrm{OS} + \frac{1}{n-n^2}S\hat{g}$ and show it vanishes. And, since terms in the Osgood-Stowe tensor are (locally) computed relative to a flat metric $\bar{g}$, we can use Lemma 4.3 to instead compute things in terms of $d^{\bar\nabla}$. In fact, the computation becomes more manageable once we simplify our second fundamental form at infinity so that every thing is in terms of $\bar{g}$. This is already the case for the Osgood-Stowe tensor, so we focus on how the pure-trace portion can be expressed in these terms. Write $\hat{g} = e^{2u}\bar{g}$. Then the scalar curvatures are related by

$$S(\hat{g}) = e^{-2u}(S(\bar{g}) - 2(n-1)\Delta u - (n-2)(n-1)|\bar\nabla u|^2).$$

Because $\bar{g}$ is a flat metric, $S(\bar{g}) = 0$ and we have the pure-trace part as

$$\frac{1}{n-n^2}S(\hat{g})\hat{g} = \left(\frac{2}{n}\Delta u + \left(\frac{n-2}{n}\right)|\bar\nabla u|^2\right)\bar{g}.$$

The full second fundamental form at infinity can now be simplified to

$$\begin{aligned}&2\mathrm{OS}(\hat{g},\bar{g}) + \frac{1}{n-n^2}S(\hat{g})\hat{g}\\ &\qquad = 2\mathrm{Hess}_{\bar{g}}(u) - 2du^2 - \frac{2}{n}\left(\Delta u - |\bar\nabla u|^2\right)\bar{g} + \left(\frac{2}{n}\Delta u + \left(\frac{n-2}{n}\right)|\bar\nabla u|^2\right)\bar{g}\\ &\qquad = 2\mathrm{Hess}_{\bar{g}}(u) - 2du^2 + |\bar\nabla u|^2\bar{g}. \qquad (3)\end{aligned}$$

We now compute the exterior covariant derivative of each term in (3) and show the sum vanishes.

We will start with the last term and work our way to the first. From Lemma 4.3, for vector fields $X$, $Y$ and $Z$ we have

$$d^{\hat\nabla}(|\bar\nabla u|^2\bar{g})(X,Y,Z) = d^{\bar\nabla}(|\bar\nabla u|^2\bar{g})(X,Y,Z) + |\bar\nabla u|^2\bar{g} \owedge \bar{g}(\bar\nabla u, X, Y, Z).$$

The product term is

$$\begin{aligned}|\bar\nabla u|^2\bar g \owedge \bar g(\bar\nabla u, X, Y, Z) &= 2|\bar\nabla u|^2(\bar g(\bar\nabla u, Z)\bar g(X,Y) - \bar g(\bar\nabla u, Y)\bar g(X,Z)) \\ &= 2|\bar\nabla u|^2(du(Z)\bar g(X,Y) - du(Y)\bar g(X,Z))\end{aligned} \tag{4}$$

and the derivative term would need a product rule, but the connection is compatible with the metric so $d^{\bar\nabla}\bar g = 0$. Therefore,

$$d^{\bar\nabla}(|\bar\nabla u|^2\bar g)(X,Y,Z) = Y(|\bar\nabla u|^2)\bar g(X,Z) - Z(|\bar\nabla u|^2)\bar g(X,Y)$$

and writing $|\bar\nabla u|^2 = \bar g(\bar\nabla u, \bar\nabla u)$ shows the derivative of $|\bar\nabla u|^2$ is twice the Hessian:

$$d^{\bar\nabla}(|\bar\nabla u|^2\bar g)(X,Y,Z) = 2\mathrm{Hess}(\bar\nabla u, Y)\bar g(X,Z) - 2\mathrm{Hess}(\bar\nabla u, Z)\bar g(X,Y). \tag{5}$$

For the second term in (3), we again have from Lemma 4.3 that $d^{\hat\nabla}(du^2) = d^{\bar\nabla}(du^2) + du^2 \owedge \bar g$. Using that $du^2(U,V) = du(U)du(V)$ for any two vector fields $U$ and $V$ we have that the first term here is

$$\begin{aligned}d^{\bar\nabla}(du^2)(X,Y,Z) &= Y(du(X)du(Z)) - du(\bar\nabla_Y X)du(Z) - du(X)du(\bar\nabla_Y Z) \\ &\quad - Z(du(X)du(Y)) - du(\bar\nabla_Z X)du(Y) - du(X)du(\bar\nabla_Z Y) \\ &= (Y du(X) - du(\bar\nabla_Y X))du(Z) + du(X)(Y du(Z) - du(\bar\nabla_Y Z)) \\ &\quad - (Z(du(X) - du(\bar\nabla_Z X))du(Y) - du(X)(Z du(Y) - du(\bar\nabla_Z Y)).\end{aligned}$$

Each expression in parentheses is a Hessian, and using that the Hessian is a symmetric tensor lets us cancel the two terms that are equal and finally get

$$d^{\bar\nabla}(du^2)(X,Y,Z) = \mathrm{Hess}(X,Y)du(Z) - \mathrm{Hess}(X,Z)du(Y). \tag{6}$$

As for the other term,

$$\begin{aligned}du^2 \owedge \bar g(\bar\nabla u, X, Y, Z) &= du(\bar\nabla u)du(Z)\bar g(X,Y) + du(X)du(Y)du(Z) \\ &\quad - du(\bar\nabla u)du(Y)\bar g(X,Z) - du(X)du(Z)du(Y) \\ &= |\bar\nabla u|^2(du(Z)\bar g(X,Y) - du(Y)\bar g(X,Z)),\end{aligned}$$

and by (4) this is

$$du^2 \owedge \bar g = \frac{1}{2}|\bar\nabla u|^2 \bar g \owedge \bar g. \tag{7}$$

For the first term in (3), by Lemmas 4.3 and 4.4,

$$\begin{aligned}d^{\hat\nabla}\mathrm{Hess} &= d^{\bar\nabla}\mathrm{Hess} + \mathrm{Hess}\owedge\bar g \\ &= \bar{Rm} + \mathrm{Hess}\owedge\bar g,\end{aligned}$$

and $\bar{Rm} = 0$ since $\bar g$ is flat. Consequently, we only need to determine $\mathrm{Hess}\owedge\bar g$. To this end,

$$\begin{aligned}\mathrm{Hess}\owedge\bar g(\bar\nabla u, X, Y, Z) &= \mathrm{Hess}(\bar\nabla u, Z)\bar g(X,Y) + \mathrm{Hess}(X,Y)du(Z) \\ &\quad - \mathrm{Hess}(\bar\nabla u, Y)\bar g(X,Z) - \mathrm{Hess}(X,Z)du(Y)\end{aligned}$$

The first and third terms sum to $-(1/2)d^{\bar\nabla}(|\bar\nabla u|^2\bar g)(X,Y,Z)$ by (5), and the second and last terms sum to $d^{\bar\nabla}(du^2)(X,Y,Z)$ by (6). Together we have

$$\mathrm{Hess}\owedge\bar g(\bar\nabla u, X, Y, Z) = -\frac{1}{2}d^{\bar\nabla}(|\bar\nabla u|^2\bar g)(X,Y,Z) + d^{\bar\nabla}(du^2)(X,Y,Z). \tag{8}$$

The derivatives of each term in (3) have been computed and substituting equations (4) and (8) lets us cancel terms

$$\begin{aligned} d^{\hat{\nabla}}\left(\mathrm{Hess}_{\bar{g}}(u) - 2du^2 + |\bar{\nabla}u|^2\bar{g}\right) &= -d^{\bar{\nabla}}(|\bar{\nabla}u|^2\bar{g}) + d^{\bar{\nabla}}(du^2) \\ &\quad - d^{\bar{\nabla}}(du^2) - |\bar{\nabla}u|^2\bar{g} \owedge \bar{g} \\ &\quad + d^{\bar{\nabla}}(|\bar{\nabla}u|^2\bar{g}) + |\bar{\nabla}u|^2\bar{g} \owedge \bar{g} \\ &= 0. \end{aligned}$$

We now show the given tensor satisfies the Gauss equation at infinity. This is a quick consequence of how the curvatures behave under a conformal change. Indeed, if we have two metrics related by $\hat{g} = e^{2u}g$ then

$$\hat{Rm} = e^{2u}Rm - e^{2u}g \owedge (\mathrm{Hess}_g(u) - du^2 - \frac{1}{2}|\nabla u|^2 g). \tag{9}$$

In our case, locally write $\hat{g} = e^{2u}\bar{g}$ as above. Then $\bar{Rm} = 0$ and, again as above, the scalar curvature obeys

$$\frac{1}{n-n^2}S(\hat{g})\hat{g} = \frac{2}{n}\left(\Delta u - |\bar{\nabla}u|^2\right)\bar{g} + |\bar{\nabla}u|^2\bar{g}.$$

Using this we can simplify the second fundamental form at infinity as

$$2\mathrm{OS}(g,\bar{g}) + \frac{1}{n-n^2}S(\hat{g})\hat{g} = 2\mathrm{Hess}(u) - 2du^2 + |\bar{\nabla}u|^2\bar{g},$$

and then substituting into (9) shows that $\hat{Rm} = -\frac{1}{2}\hat{g}\owedge\hat{\mathit{I\!I}}$. Consequently, $\hat{B}$ defined in the theorem satisfies the Gauss equation at infinity and we get $(\hat{g}, \hat{B})$ solves $\widehat{\mathrm{GC}}$, as claimed.

To show that $\hat{B}$ is the unique solution, we show that $\widehat{\mathrm{GC}}$ has a unique solution in the standard way. Assume $\hat{\mathit{I\!I}} = h$ and $\hat{\mathit{I\!I}} = k$ are two solutions to $\hat{Rm} = (-1/2)\hat{g}\owedge\hat{\mathit{I\!I}}$. Then we can write $\hat{g}\owedge(h-k) = 0$ and using the trace identity for the Kulkarni-Nomizu product we have $(n-2)(h-k) + \mathrm{tr}_{\hat{g}}(h-k)\hat{g} = 0$. Solve for $h-k$ to obtain $h - k = -\frac{1}{n-2}\mathrm{tr}_{\hat{g}}(h-k)\hat{g}$ and take another trace of sides. We have

$$\mathrm{tr}_{\hat{g}}(h-k) = -\frac{n}{n-2}\mathrm{tr}_{\hat{g}}(h-k) \qquad \Longrightarrow \qquad \mathrm{tr}_{\hat{g}}(h-k) = 0.$$

Substituting this into the trace identity gives $(n-2)(h-k) = 0$, which says $h = k$. Thus the solution is unique. Since our $\hat{B}$ gives a solution, it must be the only solution to $\widehat{\mathrm{GC}}$. □

## 5. The Weyl-Schouten Theorem

Given a Riemannian metric $g$ on a smooth manifold $M$, one forms the Ricci tensor by taking the trace of the full curvature tensor

$$Ric(g) = \mathrm{tr}_g(Rm(g)).$$

Here the trace may be thought of as a linear operator from $\mathcal{R}(T^*M)$, the sub-vector bundle of 4-tensors with the same symmetries as the curvature tensor, to $\Sigma^2(T^*M)$, the set of symmetric 2-tensors. For $n \geq 3$, a right inverse of the trace $\mathrm{tr}_g : R(T^*M) \to \Sigma^2(T^*M)$ is given by

$$G(h) = \frac{1}{n-2}\left(h - \frac{\mathrm{tr}_g(h)}{2(n-1)}g\right)\owedge g,$$

and the image of $G$ is the orthogonal complement to $\ker(\mathrm{tr}_g)$.

The Schouten tensor of $g$ is the symmetric 2-tensor $P(g)$ satisfying $G(Ric) = P(g) \owedge g$, i.e.,

$$P(g) = \frac{1}{n-2}\left(Ric(g) - \frac{S(g)}{2(n-1)}g\right),$$

and decomposing the curvature tensor via $\mathcal{R}(T^*M) = \ker(\mathrm{tr}_g) \oplus \ker(\mathrm{tr}_g)^\perp$ defines the Weyl tensor $W(g)$ by

$$Rm(g) = W(g) + P(g) \owedge g.$$

One sees that

$$W(g) = Rm(g) - \frac{1}{n-2}Ric(g) \owedge g + \frac{S(g)}{2(n-1)(n-2)} g \owedge g.$$

**Lemma 5.1.** *Suppose $g$ is locally conformally flat. Then*

$$2\mathrm{OS}(g) + \frac{1}{n-n^2}S(g)g = -2P(g).$$

*In particular, $(g, -2P)$ solves the Gauss-Codazzi equations at infinity if and only if $\hat{g}$ is locally conformally flat.*

*Proof.* The traceless Ricci tensor for a metric $g$ locally conformal to a flat metric $\bar{g}$ satisfies $Ric_0(g) = -(n-2)\mathrm{OS}(g, \bar{g})$. Substituting this into

$$P(g) = \frac{1}{n-2}\left(Ric_0(g) + \frac{1}{n}S(g)g - \frac{S(g)}{2(n-1)}g\right)$$

and simplifying gives the equality. That $(g, -2P)$ solves $\widehat{\mathrm{GC}}$ then follows from Theorem 4.2 and conversely if $(g, -2P)$ solves $\widehat{\mathrm{GC}}$ then Lemma 4.1 shows $\hat{g}$ is locally conformally flat. □

**Theorem 5.2** (Weyl-Schouten)**.** *Let $g$ be a Riemannian metric on an $n$ dimensional manifold.*

*(1) If $n = 3$ then $g$ is locally conformally flat if and only if $d^\nabla P = 0$.*
*(2) If $n \geq 4$ then $g$ is locally conformally flat if and only if $W(g) = 0$.*

*Proof.* Let $n \geq 3$. By Lemma 5.1, the metric $g$ is locally conformally flat if and only if $(g, -2P)$ solves the Gauss-Codazzi equations at infinity. This happens if and only if $d^\nabla P = 0$ and

$$Rm = -\frac{1}{2} g \owedge (-2P) = 0 + g \owedge P,$$

which happens if and only if $d^\nabla P = 0$ and $W = 0$.

When $n = 3$, for dimension reasons, the Weyl tensor of any metric will always vanish (see, for example, ref). So in dimension 3, the condition $d^\nabla P = 0$ is enough for the equivalence. When $n \geq 4$, the identity

$$d^\nabla P = -\frac{1}{n-3}\mathrm{tr}_g(\nabla W)$$

shows that $d^\nabla P$ will vanish if $W = 0$, so this condition is enough for the equivalence when $n = 4$. □