RESEARCH STATEMENT

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1. Overview

My research lies in differential geometry and global analysis. Specifically, my research projects often seek to answer geometric questions using analytic tools, typically from the theory of PDEs. These questions usually involve Riemannian manifolds, which are topological spaces modeled on Euclidean space and equipped with a notion of distance based on a quadratic form at each point. These quadratic forms bundled together are called a Riemannian metric, and this metric lets us investigate the geometry of these spaces. Indeed, the metric is used to define several familiar geometric concepts. For example, the notion of a straight line in this setting is captured by geodesics, and these are locally solutions to a system of nonlinear ODEs with coefficient defined from the metric. The curvature of a Riemannian manifold may be thought of as the obstruction to the manifold being locally isometric to flat Euclidean space and it can be expressed solely in terms of the metric and its derivatives. My work deals with similar topics in the setting of 3-dimensional hyperbolic manifolds, that is, 3-manifolds whose curvature is everywhere equal to -1.

One of the main strategies in my work is to study surfaces in hyperbolic manifolds, as it is a general technique to study codimension-one objects to gain understanding of an ambient space. These surfaces may be in the context of embedded minimal surfaces (surfaces which locally minimize area), or in foliations of a manifold or its ends, or as surfaces which naturally compactify a 3-manifold. Since surfaces are so vital to the study of hyperbolic 3-manifolds, Teichmüller theory is frequently employed. We denote by $\mathcal{T}(S)$ the Teichmüller space of S, which is the space of all of its complex structures (modeling S on $\mathbb{C}$), up to an equivalence. This space may also be described as a space of conformal classes of metrics (up to another equivalence), where two metric are conformal if they assign the same angles.

I have mainly focused on quasi-Fuchsian manifolds, which are complete hyperbolic manifolds M diffeomorphic to $S \times (-1, 1)$ that contain a non-empty compact geodesically-convex subset. Here S is a smooth closed surface of genus greater than 1. These manifolds are naturally compactified by two copies of S, called the surfaces at infinity, and these copies inherit both complex structures and complex projective structures (modeled on $\mathbb{CP}^1$) induced from M. My main results relate the limiting behavior of foliations of ends of quasi-Fuchsian manifolds to the complex projective geometry on the surfaces at infinity.

More specifically, I characterize the infinitesimal behavior of certain foliations that are asymptotic to a canonical foliation defined using the complex structures at infinity. This class of foliations I define contains the constant Gaussian curvature foliations of Labourie, and my main result resolves a conjecture he states in [Lab92] concerning their behavior as they leave the end of the manifold. Moreover, the constant mean curvature foliations of Mazzeo and Pacard [MP11] form another example of my class of foliations and so my results dictate their asymptotics as well.

My main research techniques deal with taking infinitesimal data on the boundary at infinity of hyperbolic space and constructing surfaces embedded inside hyperbolic space. The main example of this is the construction due to Epstein in [Eps84], which is a manifestation of an equivariant isomorphism between a bundle over hyperbolic space and a jet-bundle of Riemannian metrics on the boundary. I indent to use the same construction to investigate surfaces in 3-dimensional deSitter space, which is in a certain sense dual to hyperbolic space. I will investigate whether Epstein's construction in higher dimensions is still a consequence of some similar isomorphism, and whether it can be used to determine the geometric behavior of hypersurfaces in higher dimensional hyperbolic ends. I will also look in general for other examples of equivariant isomorphisms between bundles over symmetric spaces and bundles of infinitesimal data over the boundaries of the symmetric spaces, with one specific example of interest being complex hyperbolic space.

2. Past Research

Foliations have been a successful tool in studying quasi-Fuchsian manifolds. Indeed, the Renormalized Volume of such a manifold M, a concept from physics, can be defined more geometrically (see for example [KS08]) in terms of foliations of the ends of M. Many have studied the renormalized volume including [Sch13],[CM16], and [BBB19], where it has been used to obtain bounds on the geometry of the quasi-Fuchsian manifold and the surfaces at infinity. In the case where M contains a unique minimal surface (with "small" principal curvatures) Uhlenbeck showed in [Uhl83] that parallel copies of the minimal surface foliate the entire manifold, and when considered as a path in Teichmüller space, this foliation starts and ends at the complex structures on the surfaces at infinity.

As another example, in [Lab91] Labourie proved that hyperbolic ends of 3-manifolds admit foliations by surfaces of constant curvature. These surfaces he called k-surfaces and for each k in $(-1, 0)$ there is a surface of constant Gaussian curvature k belonging to the foliation. Each metric with curvature k induces a complex structure and Labourie in [Lab92] describes the foliation as a path in Teichmüller space. He shows how the path converges to the conformal class of the hyperbolic metric on the surface at infinity of the end of the manifold (as $k \to 0$). He asks about the infinitesimal behavior of this path and suggests that it should be related to the projective structure on the surface at infinity.

In my thesis [Qui20] I showed this is the case by establishing a precise formula relating the projective structure and limiting k-surface behavior. If I_k is the family of first fundamental forms, i.e., the metrics of the k-surfaces induced from being submanifolds of M, then Labourie showed that the underlying conformal structures $[I_k]$ form a path in Teichmüller space that converges to the class of the hyperbolic metric $[h]$ representing the complex structure at infinity. These same considerations apply to II_k, the second fundamental forms of the k-surfaces which also induce points in $\mathcal{T}(S)$. We proved the following regarding the infinitesimal behavior of the paths at $[h]$.

Theorem 2.1. *Let $[I_k]$ and $[II_k]$ be the paths of first and second fundamental forms of the k-surfaces in Teichmüller space. Let the complex projective structure at infinity have corresponding holomorphic quadratic differential ϕ. Then the tangent vectors to $[I_k]$ and $[II_k]$ at $k = 0$ are given by*

$$[\dot{I_k}] = -\mathrm{Re}(\phi) \text{ and } [\dot{II_k}] = 0.$$

My approach to this theorem gives an alternate construction of the k-surface foliation in terms of data at infinity. Specifically it uses the construction of Epstein in [Eps84], which describes a way to construct surfaces in hyperbolic space given a domain in $\partial^\infty(\mathbb{H}^3) \simeq \mathbb{CP}^1$ and geometric data on that domain. These data take the form of a conformal metric, which can be characterized as a metric conformally equivalent to h. Epstein surfaces have been used by several other others to study the geometry of projective structures and hyperbolic 3-manifolds. See, for example, [And98], [Bro04], and [KS08]. One of the benefits of these Epstein surfaces is they have a very concrete description in terms of the defining conformal metric and its derivatives. Indeed there are explicit formulas for the first and second fundamental forms (seen for example in [Dum17]) and for the Gaussian and mean curvatures (seen for example in [Qui20]).

My main technique in the proof of Theorem 2.1 is to describe k-surfaces as Epstein surfaces, at least for k near zero. This prescribed curvature problem is solved in this case by turning the constant Gaussian curvature k condition into a family, indexed by k, of fully nonlinear PDEs in terms of the defining conformal metric. This leads to a single PDE with a deformation parameter k describing the entire family which, critically, reduces to the condition of having constant curvature -1 at $k = 0$. An Implicit Function Theorem and elliptic regularity argument shows the PDE has a solution σ_k for each k near zero. Incidentally, this method is what gives another proof of the existence of Labourie's k-surface foliation. Finally, using the explicit formulas mentioned above, it is then easy to understand I_k and II_k in terms of these solutions.

There is a more general class of surface foliations to which the same argument naturally applies. If one drops the k-surfaces setting and assumes there is a family of conformal metrics σ_ϵ and a family of constants $f(\epsilon)$ such that $f\sigma$ converges to the hyperbolic metric h (as $\epsilon \to 0$, in the C^∞ topology), then we get a more general result of a similar kind for the family of Epstein surfaces for the metrics (σ_ϵ).

Theorem 2.2. *Let $[I_\epsilon]$ and $[II_\epsilon]$ be the the family of first and second fundamental forms representing the Epstein surface in Teichmuller space. Then*

$$[I_\epsilon] \to [h] \text{ and } [II_\epsilon] \to [h] \text{ as } \epsilon \to 0,$$

and the tangent vectors at $\epsilon = 0$ are give by

$$[\dot{I_\epsilon}] = -4f'(0)\mathrm{Re}(\phi) \text{ and } [\dot{II_\epsilon}] = 0.$$

We call such families *Asymptotically Poincaré Families* of surfaces since the family corresponding to multiples of the Poincaré metric h is a canonical example (where $f(\epsilon) = \epsilon$). This Poincaré family consists of parallel surfaces, i.e., copies of a surface flowed in its normal direction. We showed an asymptotically Poincaré family gives a foliation of the end by approximately parallel surfaces.

Theorem 2.3. *Let (S_ϵ) be an asymptotically Poincaré families of surfaces, then the distance between S_ϵ flowed for time t in the normal direction and the surface $S_{e^{-2t}\epsilon}$ tends towards zero as ϵ does. Moreover, there exists an $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, the surfaces S_ϵ form a foliation of the end of M.*

These results apply to a wide collection of surfaces, a notable example being Labourie's k-surfaces. They form an asymptotically Poincaré family and so Theorem 2.2 has Theorem 2.1 as a corollary.

Another example is given by the constant mean curvature foliation produced by the work of Mazzeo and Pacard (see [MP11]). They show that an end of a quasi-Fuchsian manifold admits a foliation by constant mean curvature surfaces. I showed this foliation may be presented as a family of Epstein surfaces, which give another proof of the existence of such a foliation in the end of a quasi-Fuchsian manifold. Moreover, this family satisfies the conditions to be an asymptotically Poincaré family, and so our results characterize their limiting behavior as well.

Theorem 2.4. *Let $[I_c]$ and $[II_c]$ be the paths of first and second fundamental forms of the constant mean curvature surfaces in Teichmüller space. Let the complex projective structure at infinity have corresponding holomorphic quadratic differential ϕ. Then the tangent vectors to $[I_c]$ and $[II_c]$ at $c = 0$ are given by*

$$[\dot{I_c}] = -\mathrm{Re}(\phi) \text{ and } [\dot{II_c}] = 0.$$

3. Future Research

3.1. **Generalizing Asymptotically Poincaré Families.** A natural extension of my thesis is to consider not just families (σ_ϵ) that converge to h in the asymptotically Poincaré sense, but to ask the same questions for families (σ_ϵ) that converge in the same sense to some fixed conformal metric σ_0. For such a family of surfaces S_ϵ we can ask similar questions:

- Do the surfaces S_ϵ still converge to the conformal class at infinity? That is, do $[I_\epsilon]$ and $[II_\epsilon]$ converge to $[\sigma_0] = [h]$ as $\epsilon \to 0$?

and

- What are the tangent vectors to $[I_\epsilon]$ and $[II_\epsilon]$ at $\epsilon = 0$. How are they related to the Schwarzian derivative of σ_0 and to the holomorphic quadratic differential ϕ?

The first of these should follow using the same ideas as in Theorem 2.2. Preliminary calculations suggest that yes, $[I(\sigma_\epsilon)]$ and $[I\!I(\sigma_\epsilon)]$ converge to $[\sigma_0] = [h]$. Moreover, we expect that it is still the case that $[I\!I(\dot{\sigma_\epsilon})] = 0$. It also appears that $[I(\dot{\sigma_\epsilon})]$ is indeed related to $\mathrm{Re}(B(\sigma_0))$. Here $B(\sigma_0)$ is the Schwarzian tensor of Osgood and Stowe [OS92]. It is a type of generalization of the Schwarzian derivative of a locally injective holomophic function to the setting of conformally equivalent metrics.

However, $B(\sigma_0)$ need not be holomorphic, i.e., if σ_0 does not have constant curvature, so $\mathrm{Re}(B(\sigma_0))$ is not a tangent vector to Teichmüller space (see [Tro92]). The actual tangent vector should be (the real part of) the holomorphic part of $B(\sigma_0)$. It would be nice to find a more explicit description of the holomorphic part of a quadratic differential, and to verify these preliminary computations.

3.2. **The Epstein-frame in higher dimensions.** Epstein's construction in [Eps84] works in $\mathbb{H}^n$ for $n > 3$ as well. It is possible to express this Epstein map as the orbit of a point in $\mathbb{H}^n$ by an $O(n,1)$-frame. Write $\sigma = e^{2\eta}\bar{g}$ where $\bar{g}$ is the Euclidean metric of a domain Ω in S^{n-1}. Then we have $\mathrm{Ep}_\sigma(x) = \widetilde{\mathrm{Ep}}_\sigma(x)p$ for the frame $\widetilde{\mathrm{Ep}}_\sigma : \Omega \to O(n,1)$ given by $\widetilde{\mathrm{Ep}}_\sigma(x) = A(x)B_\sigma(x)C_\sigma(x)$, where

$$A(x) = \begin{pmatrix} Id_{n-1} & -x & x \\ x^t & 1-\frac{1}{2}|x|^2 & \frac{1}{2}|x|^2 \\ x^t & -\frac{1}{2}|x|^2 & 1+\frac{1}{2}|x|^2 \end{pmatrix}, \quad B_\sigma(x) = \begin{pmatrix} Id_{n-1} & \frac{1}{2}\nabla\eta & \frac{1}{2}\nabla\eta \\ -\frac{1}{2}\nabla\eta^t & 1-\frac{1}{8}|\nabla\eta|^2 & -\frac{1}{8}|\nabla\eta|^2 \\ \frac{1}{2}\nabla\eta^t & \frac{1}{8}|\nabla\eta|^2 & 1+\frac{1}{8}|\nabla\eta|^2 \end{pmatrix},$$

$$C_\sigma(x) = \begin{pmatrix} Id_{n-1} & 0 & 0 \\ 0 & \cosh(\eta) & -\sinh(\eta) \\ 0 & -\sinh(\eta) & \cosh(\eta) \end{pmatrix},$$

and the point $p = (0,\dots,0,3/4,5/4)^t$.

This gives a geometrically natural factorization of the Epstein map similar to that in [Dum17] as follows. The visual metric from a point q in hyperbolic space is a Riemannian metric on the boundary sphere at infinity that assigns lengths to tangent vector based on their apparent size as viewed from q. The Epstein map sends a point x to the point in hyperbolic space whose visual metric has the same 1-jet as σ at x (see [And98]). This proposed factorization of the frame gives this explicitly. The matrix C moves p along a geodesic to a point whose visual metric agrees with σ to 0th order. The matrix B moves the point along a horosphere until the visual metric agrees with σ to 1st order. And the matrix A moves the horosphere to be based at x.

So far, this has only been verified for $n = 3$ and $n = 4$. A natural next step would be to use this description of the Epstein map to get closed formulas for the induced metric on the Epstein surface and its curvature, perhaps in terms of the Schwarzian tensor of Osgood and Stowe [OS92]. We also hope to investigate constant curvature hypersurfaces similarly to our work done in the previous sections using these explicit formulas.

3.3. **Extension to de Sitter Space.** Via the map $U\mathbb{H}^3 \hookrightarrow TdS^3$ from the unit tangent bundle of hyperbolic space to the tangent bundle of de Sitter space given by $(p,v) \mapsto (v,p)$, a strictly convex surface S with a normal vector field in hyperbolic space has a dual surface S^* in de Sitter space, see [HR93] or [Sch02].

This correspondence extends to a duality between hyperbolic ends of 3-manifolds and certain de Sitter space-time 3-manifolds (see [Mes07]). Therefore, fix a quasi-Fuchsian manifold and an end E. Let E^* be the corresponding de Sitter space-time. The k-surface S_k in E has a dual surface S_k^* in E^* which also has constant Gaussian curvature, as shown by a computation. One can show, see [Lab92], that the induced metric I_k^* on S_k^* also satisfies $[I_k^*] \to [h]$ in $\mathcal{T}(S)$ as $k \to 0$. Hence the paths $[I_k]$ and $[I_k^*]$ meet at $[h]$ when $k = 0$.

A preliminary calculation shows that $[\dot{I}_k^*] = +\mathrm{Re}(\phi)$, and so if one forms the concatenated path $\gamma : (-1, 1) \to \mathcal{T}(S)$ by

$$\gamma(t) = \begin{cases} [I_t^*] & -1 < t \leq 0 \\ [I_{-t}] & 0 \leq t < 1 \end{cases}$$

then $\gamma'(0) = \mathrm{Re}(\phi)$, so that this path is differentiable. Based on this, I conjecture the following, which I intend to prove.

Conjecture. *Let $\gamma : (-1, 1) \to \mathcal{T}(S)$ be the concatenated path induced by the k-surface foliation and the dual foliation. Then γ is a smooth curve.*

3.4. **Other Equivariant Constructions.** Epstein surfaces are consequences of the fact that $U\mathbb{H}^3$ is isomorphic to the bundle of 1-jets of conformal metrics on $\partial^\infty(\mathbb{H}^3) \simeq \mathbb{C}\mathrm{P}^1$, and that this isomorphism is equivariant with respect to the isometry group of $\mathbb{H}^3$. Naturally, I wonder

- Are there other examples of equivariant isomorphisms between a symmetric space or bundles associated to the Frame bundle of the symmetric space and jet spaces of infinitesimal data on the (or a) boundary of the symmetric space?

Moreover,

- Even if there is not an isomorphism, are there embeddings of associated bundles on the symmetric space into, say, k-jets of Riemannian metrics on the boundary?
- Which k is optimal or sharp?
- If so, which metrics are in the image of the embedding?

One example I will investigate is complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. Here the Gauss map which sends a unit tangent vector to the ideal endpoint of the geodesic ray in the direction of the vector is still a diffeomorphism. So, we still have a visual metric construction sending the induced metric on a unit tangent sphere to a Riemannian metric on the visual boundary $\partial^\infty(\mathbb{H}^n_{\mathbb{C}}) \simeq S^{2n-1}$.

- Are the visual metrics from different points still conformally equivalent as they were in the real hyperbolic setting?
- What surfaces in $\mathbb{H}^n_{\mathbb{C}}$ can we get out of this construction? How explicitly can we determine the geometry of the surfaces? For example, can we get formulas for the induced metrics? For the curvatures?
- Can we find an $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$ frame-field description of these surfaces similar to that given in [Dum17] for the real hyperbolic case in dimension 3, or similar to the proposed frame given here in Section 3.2?
- For $n = 2$, in what way does the fact that the boundary at infinity $\partial^\infty\mathbb{H}^2_{\mathbb{C}}$ is the 1-point compactification of the Heisenberg group affect this construction?

Once I determine the answers to these questions I can ask how many of the results on foliations of hyperbolic ends transfer to the complex setting. Indeed,

- Do complex hyperbolic ends admit foliations by constant curvature (hyper)surfaces?
- Is the asymptotics of such a foliation determined by the geometry of the hyperbolic manifold and the boundary at infinity?

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