

# RESEARCH STATEMENT

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## 1. BACKGROUND

My research lies in differential geometry and global analysis. These subjects deal with answering geometric questions using analytic tools, typically from the theory of PDEs. These geometric questions are usually with respect to Riemannian manifolds, which are topological spaces modeled on Euclidean space and equipped with a Riemannian metric. This metric (which is actually a tensor) induces a notion of distance and angle, letting us ask geometric questions about these spaces. Analytic tools are helpful for defining several intuitive concepts. For example, the notion of a straight line in this setting is captured by geodesics, and these are locally solutions to a system of nonlinear ODEs. The curvature of a Riemannian manifold may be thought of as the obstruction to the manifold being locally isometric to flat Euclidean space and it can be expressed solely in terms of the metric and its derivatives. A natural question that frequently arises is whether a given function is the curvature of a Riemannian metric. This prescribed curvature question may be answered by constructing solutions to a nonlinear PDE on the manifold. It is interesting that the existence of solutions to this PDE depends on the topology of the manifold, and so which curvatures a manifold can support is dictated by its topology. My work deals with similar topics, including some prescribed curvature questions, in the setting of hyperbolic manifolds, i.e., Riemannian manifolds whose (sectional) curvature is everywhere equal to  $-1$ .

More specifically, my research concerns 3-dimensional hyperbolic geometry. Thurston's Geometrization Conjecture, which has since been proved, states that every closed 3-manifold may be decomposed into pieces that have one of eight model geometries, one among these being hyperbolic geometry. In some sense, the hyperbolic 3-manifolds are the least understood while also being (in ways that can be made precise) the most prevalent of the eight. And so, they have received much attention. One of the main methods of investigation here is to study the surfaces related—in one way or another—to a hyperbolic 3-manifold. These surfaces may be in the context of embedded minimal surfaces (surfaces which locally minimize area), or in foliations of the manifold or its ends, or as surfaces which naturally compactify the 3-manifold.

Since surfaces are so vital to the study of hyperbolic 3-manifolds, Teichmüller theory is frequently employed. We denote by  $\mathcal{T}(S)$  the Teichmüller space of  $S$ , which is the space of all of its complex structures (modeling  $S$  on  $\mathbb{C}$ ), up to an equivalence. This space may also be described as the space of equivalence classes of metrics that assign the same angles to tangent vectors (up to homotopy). When two metrics assign the same angles we call them conformally equivalent, and  $[g]$  will refer to all metrics conformally equivalent  $g$  (up to homotopy).

My work so far has focused on quasi-Fuchsian manifolds, which are complete hyperbolic manifolds  $M$  diffeomorphic to  $S \times (-1, 1)$  that contain a non-empty compact geodesically-convex subset. Here  $S$  is a smooth closed surface of genus greater than 1. These manifolds are naturally compactified by two copies of  $S$ , called the surfaces at infinity, and these copies inherit both complex structures and complex projective structures (modeled on  $\mathbb{CP}^1$ ) induced from  $M$ . Moreover, such a hyperbolic manifold is uniquely determined (up to an equivalence) by the complex structures on these surfaces at infinity. And so, the space of all such hyperbolic 3-manifolds (up to an equivalence) is in one-to-one correspondence with the product of two copies of  $\mathcal{T}(S)$ .

For a quasi-Fuchsian manifold, when the complex structures on the surfaces at infinity are complex conjugates, the 3-manifold is called Fuchsian. These manifolds contain a unique totally-geodesic minimal surface. Almost-Fuchsian manifolds are those manifolds  $M$  where the complex structures at infinity are sufficiently close together such that  $M$  contains a unique minimal surface (satisfying a principal curvature condition). In general, when the complex structures at infinity take any value in  $\mathcal{T}(S)$ , the manifolds are simply known as quasi-Fuchsian manifolds.

Foliations have been a successful tool in studying quasi-Fuchsian manifolds. Indeed, the Renormalized Volume of  $M$ , a concept from physics, can be defined more geometrically (see for example [KS08]) in terms of foliations of the ends of  $M$ . Many have studied the renormalized volume including [Sch13], [CM16], and [BBB19], where it has been used to obtain bounds on the geometry of the quasi-Fuchsian manifold and the surfaces at infinity. In the almost-Fuchsian case, Uhlenbeck showed in [Uhl83] that parallel copies of the unique minimal surface foliate the entire manifold, and when considered as a path in Teichmüller space, this foliation starts and ends at the complex structures on the surfaces at infinity. Labourie in [Lab91] showed the ends of quasi-Fuchsian manifolds admit foliations by constant Gaussian curvature surfaces. These constant curvature foliations are where my research began.

## 2. PAST RESEARCH

In [Lab91] Labourie proved that hyperbolic ends of 3-manifolds admit foliations by surfaces of constant curvature. These surfaces he called  $k$ -surfaces and for each  $k$  in  $(-1, 0)$  there is a surface of constant Gaussian curvature  $k$  belonging to the foliation. Each metric with curvature  $k$  induces a complex structure and Labourie in [Lab92] describes the foliation as a path in Teichmüller space. He shows how the path converges to the conformal class of the hyperbolic metric on the surface at infinity of the end of the manifold (as  $k \rightarrow 0$ ). He asks about the infinitesimal behavior of this path and guesses that it should be related to the projective structure on the surface at infinity.

His guess is correct. In [Qui20] we make his statement precise and prove it. If  $I_k$  is the family of first fundamental forms, i.e., the metrics of the  $k$ -surfaces induced from being submanifolds of  $M$ , then Labourie showed that the underlying conformal structures  $[I_k]$  form a path in Teichmüller space that converges to the class of the hyperbolic metric  $[h]$  representing the complex structure at infinity. These same considerations apply to  $II_k$ , the second fundamental forms of the  $k$ -surfaces which also induce points in  $\mathcal{T}(S)$ . We proved the following regarding the infinitesimal behavior of the paths at  $[h]$ .

**Theorem 2.1.** *Let  $[I_k]$  and  $[II_k]$  be the paths of first and second fundamental forms of the  $k$ -surfaces in Teichmüller space. Let the complex projective structure at infinity be parametrized by the holomorphic quadratic differential  $\phi$ . Then*

$$[\dot{I}_k] = -\text{Re}(\phi) \text{ and } [\dot{II}_k] = 0.$$

The proof uses the construction of Epstein in [Eps84], which describes a way to construct surfaces in hyperbolic space given a domain in  $\mathbb{CP}^1$  and geometric data on that domain. These data take the form of a conformal metric, which can be characterized as a metric conformally equivalent to  $h$ . A variant of this construction gives a surface in  $M$  from a conformal metric on the surface at infinity. One of the benefits of these Epstein surfaces is they have a very concrete description in terms of the defining conformal metric and its derivatives. Indeed there are explicit formulas for the first and second fundamental forms (seen for example in [Dum17]) and for the Gaussian and mean curvatures (seen for example in [Qui20]).

Our main technique in the proof of Theorem 2.1 is to describe  $k$ -surfaces as Epstein surfaces, at least for  $k$  near zero. This prescribed curvature problem was done by turning the constant Gaussian curvature  $k$  condition into a family, indexed by  $k$ , of fully nonlinear PDEs in terms of the defining

conformal metric. This family can be combined to a single PDE in terms of the conformal metric and  $k$  and may be rescaled to become a small deformation of the Poincaré Theorem (regarding the existence of a hyperbolic metric in a given conformal class, see [Tro92]). And so, an Implicit Function Theorem and elliptic regularity argument shows the PDE has a solution  $\sigma_k$  for each  $k$  near zero. Incidentally, this method also gives another proof of the existence of Labourie's  $k$ -surface foliation. Finally, using the explicit formulas mentioned above, it is then easy to understand  $I_k$  and  $II_k$  in terms of these solutions.

Analyzing the asymptotic behavior of  $I_k$  and  $II_k$  really depended on the fact that we found a family of constants  $f(k)$  such that  $f(k)\sigma_k$  converges to the hyperbolic metric. If one drops the  $k$ -surfaces setting and just assumes there is a family of conformal metrics  $\sigma_\epsilon$  and a family of constants  $f(\epsilon)$  such that  $f\sigma$  converges to the hyperbolic metric (as  $\epsilon \rightarrow 0$ , in the  $C^\infty$  topology), then we get a more general result of a similar kind for the family of Epstein surfaces for the metrics  $(\sigma_\epsilon)$ .

**Theorem 2.2.** *Let  $[I_\epsilon]$  and  $[II_\epsilon]$  be the the family of first and second fundamental forms representing the Epstein surface in Teichmuller space. Then*

$$[I_\epsilon] \rightarrow [h] \text{ and } [II_\epsilon] \rightarrow [h] \text{ as } \epsilon \rightarrow 0,$$

and

$$[\dot{I}_\epsilon] = -4f'(0)\text{Re}(\phi) \text{ and } [\dot{II}_\epsilon] = 0.$$

Such families we called *Asymptotically Poincaré Families of Surfaces* since the family corresponding to multiples of the Poincaré metric  $h$  is a canonical example (where  $f(\epsilon) = \epsilon$ ). This Poincaré family consists of parallel surfaces, i.e., copies of a surface flowed in its normal direction. We showed an asymptotically Poincaré family gives a foliation of the end by approximately parallel surfaces.

**Theorem 2.3.** *Let  $(S_\epsilon)$  be an asymptotically Poincaré families of surfaces, then the distance between  $S_\epsilon$  flowed for time  $t$  in the normal direction and the surface  $S_{e^{-2t\epsilon}}$  tends towards zero as  $\epsilon$  does. Moreover, there exists an  $\epsilon_0 > 0$  such that for  $\epsilon < \epsilon_0$ , the surfaces  $S_\epsilon$  form a foliation of the end of  $M$ .*

These results apply to a wide collection of surfaces, a notable example being Labourie's  $k$ -surfaces. They form an asymptotically Poincaré family and so Theorem 2.2 has Theorem 2.1 as a corollary. Another example is given by the constant mean curvature foliation produced by the work of Mazzeo and Pacard (see [MP11]), and so our results characterize the asymptotics of these surfaces as well. Our work on the constant mean curvature case also furnishes a new proof of their existence, similar to the  $k$ -surface case.

### 3. FUTURE RESEARCH

**3.1. Generalizing Asymptotically Poincaré Families.** A natural extension of my thesis is to consider not just families  $(\sigma_\epsilon)$  that converge to  $h$  in the asymptotically Poincaré sense, but to ask the same questions for families  $(\sigma_\epsilon)$  that converge in the same sense to some fixed conformal metric  $\sigma_0$ . We can ask similar questions:

- Do  $[I_\epsilon]$  and  $[II_\epsilon]$  converge to  $[\sigma_0] = [h]$  as  $\epsilon \rightarrow 0$ ?

and

- What are the tangent vectors to  $[I_\epsilon]$  and  $[II_\epsilon]$  at  $\epsilon = 0$ . How are they related to the Schwarzian derivative of  $\sigma_0$  and to the holomorphic quadratic differential  $\phi$ ?

The first of these should follow easily using the same ideas as in Theorem 2.2. The second will need tools from the deformation theory of geometric structures to resolve.

**3.2. Extension to de Sitter Space.** Via the map  $U\mathbb{H}^3 \hookrightarrow TdS^3$  from the unit tangent bundle of hyperbolic space to the tangent bundle of de Sitter space given by  $(p, v) \mapsto (v, p)$ , a surface  $S$  with a normal vector field in hyperbolic space has a dual surface  $S^*$  in de Sitter space. And so, each  $k$ -surface in hyperbolic space has a dual surface in de Sitter space. It turns out the dual surface to a  $k$ -surface also has constant Gaussian curvature and this family of dual  $k$ -surfaces foliates an end of de Sitter space. Moreover, this family may similarly be considered as a path in Teichmüller space. One can show that the dual family also converges to  $[h]$  and so these two paths meet at  $[h]$  when  $k = 0$ . If one forms the concatenated path  $\gamma : (-1, 1) \rightarrow \mathcal{T}$  with  $\gamma(0) = [h]$ , then  $\gamma$  is continuous. A preliminary calculation using my results appears to show that this path is differentiable at  $[h]$ . Based on this, I conjecture the following, which I intend to prove.

**Conjecture 1.** *Let  $\gamma : (-1, 1) \rightarrow \mathcal{T}(S)$  be the concatenated path induced by the  $k$ -surface foliation and the dual foliation. Then  $\gamma$  is a smooth curve.*

**3.3. Constant Mean Curvature Foliations.** Foliations by constant mean curvature surfaces have been widely used to study the geometry of 3-manifolds. For example, in certain spacetimes, they may be used to define time functions. A natural question, attributed to Thurston in [HW13], is whether quasi-Fuchsian manifolds admit foliations by constant mean curvature surfaces. In [HW13] Huang and Wang constructed a quasi-Fuchsian manifold which does not admit such a foliation. However, this manifold contains two minimal surfaces and so the question may still be asked for the class of almost-Fuchsian manifolds.

**Conjecture 2** (Thurston). *An almost-Fuchsian manifold admits a foliation by constant mean curvature surfaces.*

Filippo Mazzoli is using my work (in [Maz19]) to show that as  $k \rightarrow 0$ , the parametrization by  $k$ -surfaces of hyperbolic ends (see [Lab92]) limits to the Schwarzian parametrization. Combining my work and his, he has shown that a foliation by  $k$ -surfaces can be seen as an integral curve of a time-dependent Hamiltonian vector field on  $T^*\mathcal{T}(S)$ . Similar arguments give the same results for constant mean curvature foliations. Therefore, a foliation by constant mean curvature surfaces is an integral curve of this vector field. By showing this vector field is complete, Mazzoli and I hope to prove Thurston’s conjecture.

**3.4. Complex Hyperbolic Space.** Finally, I am interested in how many of these questions can be answered in the complex hyperbolic setting. For example:

- Do complex hyperbolic ends admit foliations by constant curvature (hyper)surfaces?
- To this end, do any of the same tools transfer to the complex setting?

For my work, Epstein surfaces have been quite integral. So I ask:

- Is there an Epstein-like construction for complex hyperbolic space, i.e., an analogous way of taking geometric data on the ideal boundary  $\partial^\infty \mathbb{H}_{\mathbb{C}}^n$  and constructing a (hyper)surface in  $\mathbb{H}_{\mathbb{C}}^n$  such that this construction is equivariant with respect to the isometry group?
- For  $n = 2$ , in what way does the fact that the boundary at infinity  $\partial^\infty \mathbb{H}_{\mathbb{C}}^2$  is the 1-point compactification of the Heisenberg group affect this construction?

I intend to find a  $\text{Isom}(\mathbb{H}_{\mathbb{C}}^2)$  frame-field description of these complex Epstein surfaces, similar to that given in [Dum17] for the real hyperbolic case, and use this to answer these questions.

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