

# Limits of foliations in quasi-Fuchsian manifolds

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June 10, 2020

# Introduction

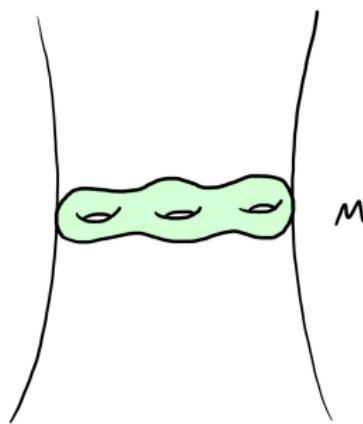
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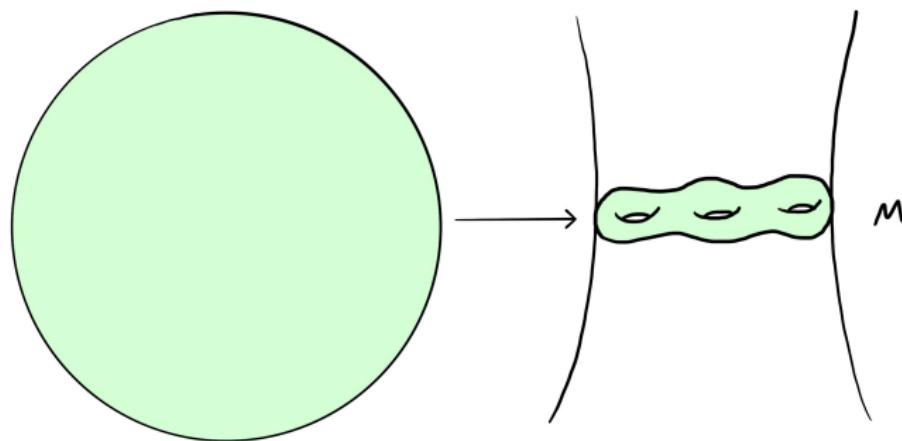
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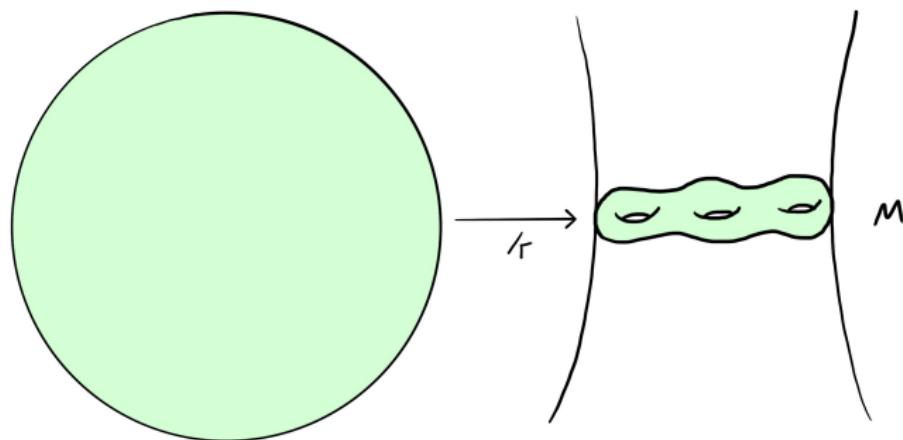
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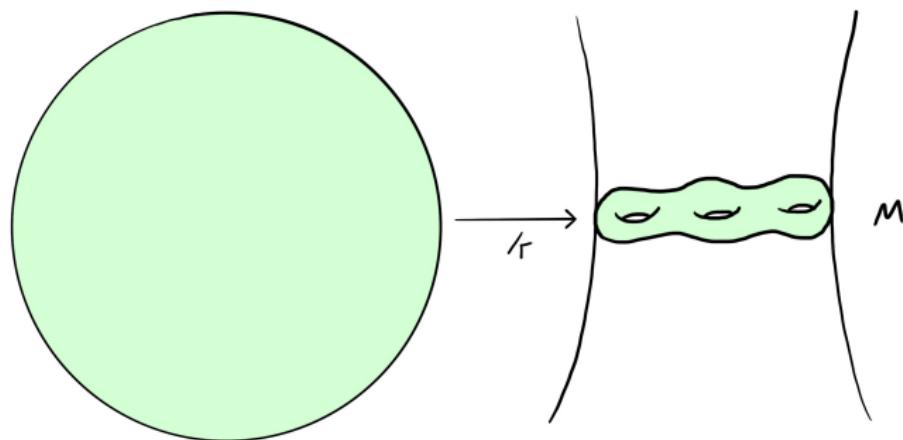


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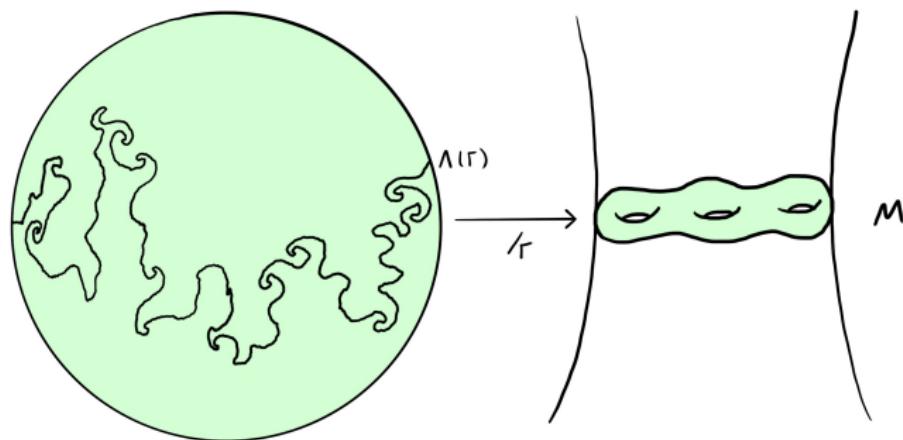


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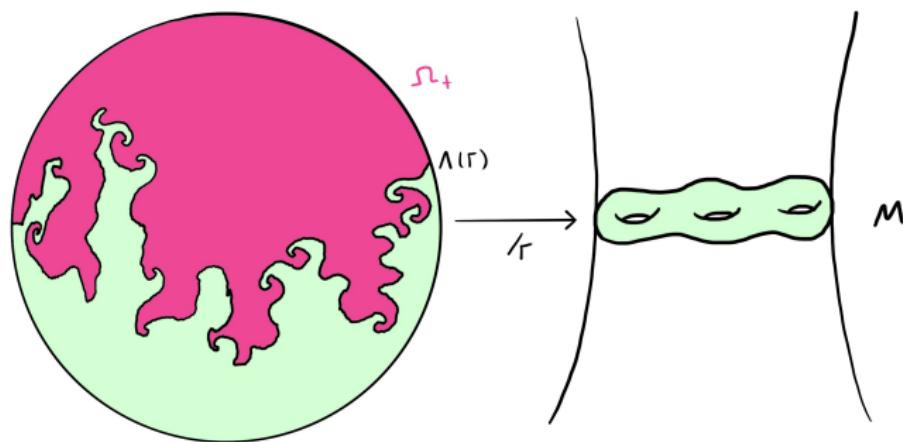


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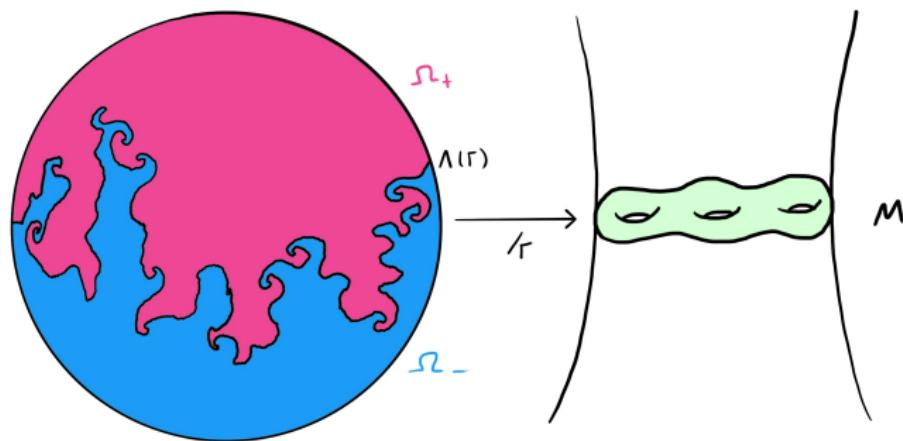


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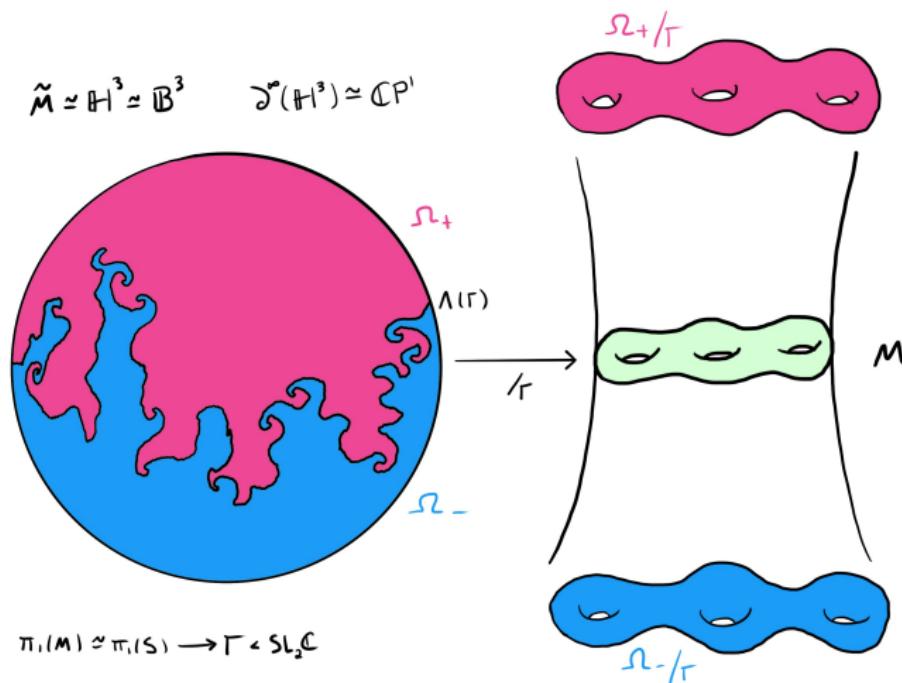
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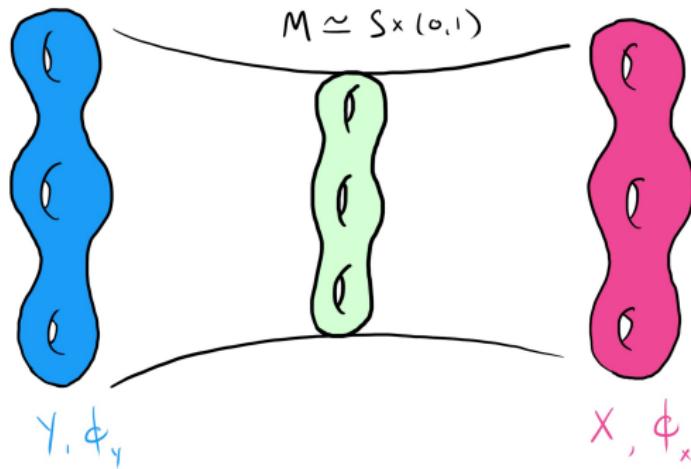
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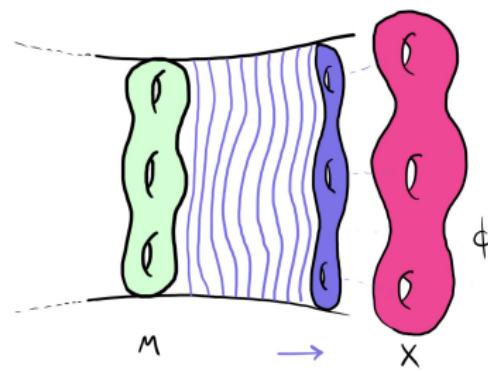
$X = \Omega_+/\Gamma$  and  $Y = \Omega_-/\Gamma$  are called the surfaces at infinity and they inherit both conformal structures and complex projective structures.

These induce  $[X]$  and  $[Y]$  in  $\mathcal{T}(S)$ , the Teichmüller space of  $S$  and we get holomorphic quadratic differentials  $\phi_X$  and  $\phi_Y$  that parametrize the projective structures.



# Introduction

We will consider certain foliations of the ends of  $M$  and investigate their limits as the foliations leave the ends.



Without loss of generality, we will focus on the end of  $M$  with surface at infinity  $X$  and holomorphic quadratic differential at infinity  $\phi$ .

# Epstein Surfaces

C. Epstein describes a way of taking geometric data on the boundary of  $\mathbb{H}^3$  and obtaining a surface in  $\mathbb{H}^3$ .

## Theorem

*Let  $\Omega$  be a domain in  $\mathbb{CP}^1$  and  $\sigma$  a  $C^k$  conformal metric on  $\Omega$ , then there exists a unique  $C^{k-1}$  map  $\text{Ep}_\sigma : \Omega \rightarrow \mathbb{H}^3$ , called the Epstein map of  $\Omega$  for the metric  $\sigma$ , such that for all  $z \in \Omega$ ,*

$$V_{\text{Ep}_\sigma(z)}(z) = \sigma(z).$$

*Moreover, the image of a point  $z$  depends only on the 1-jet of  $\sigma$  at  $z$ .*

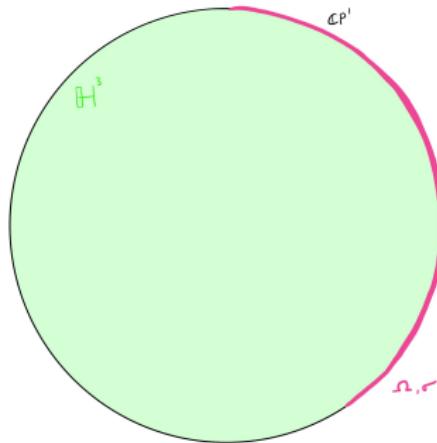
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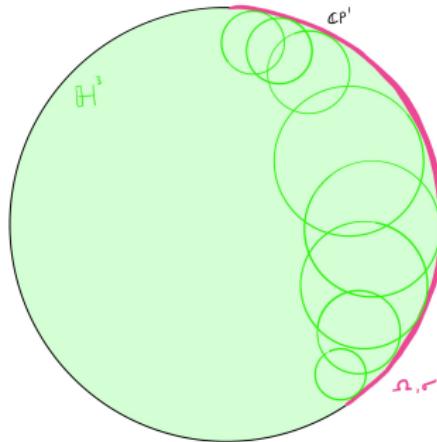


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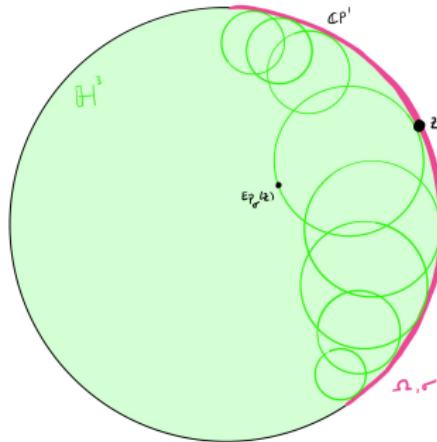


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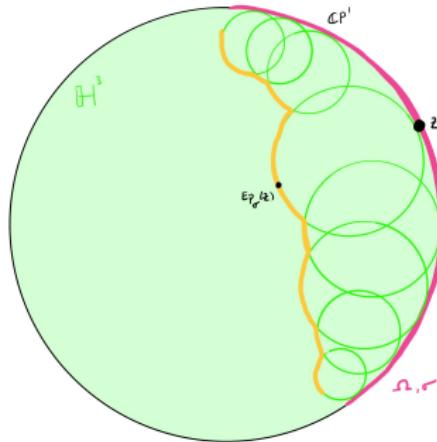


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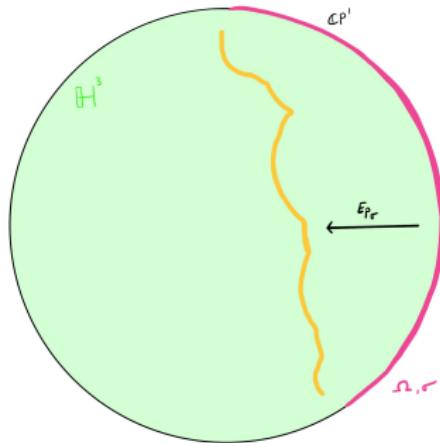


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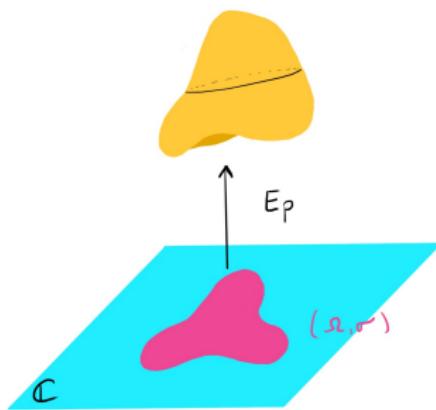
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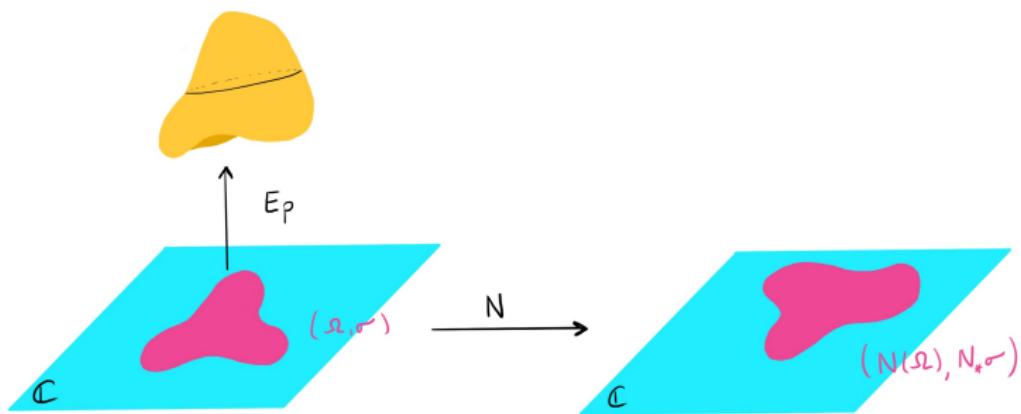
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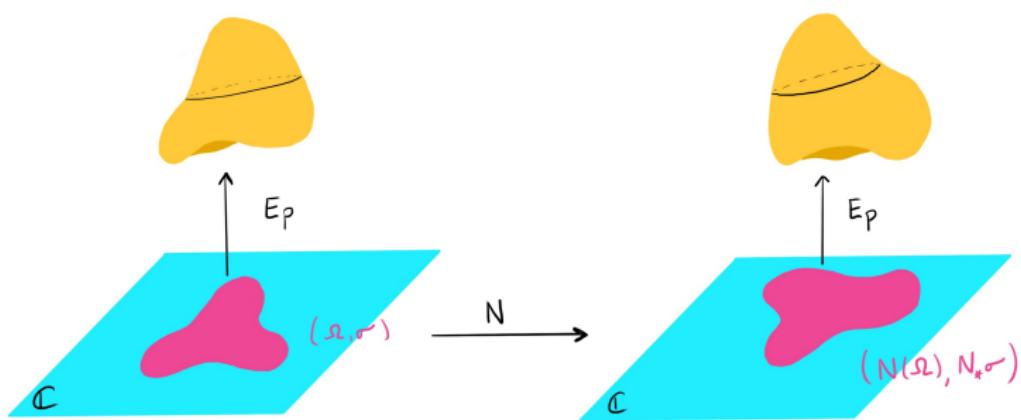
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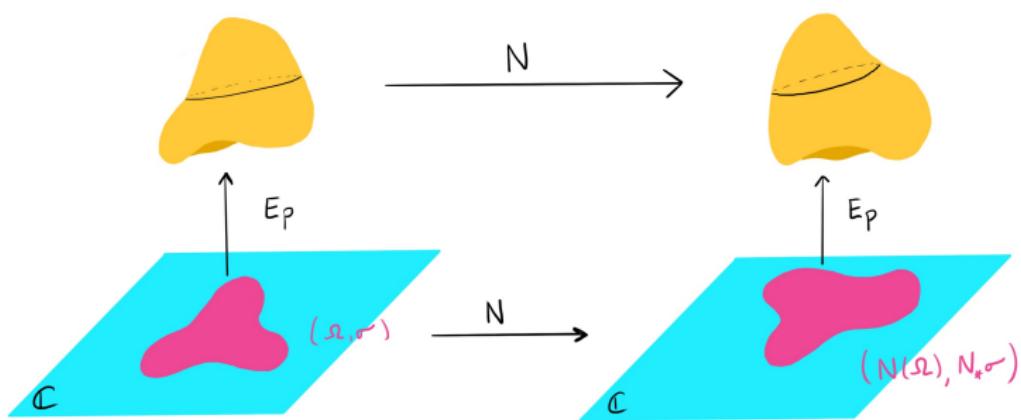
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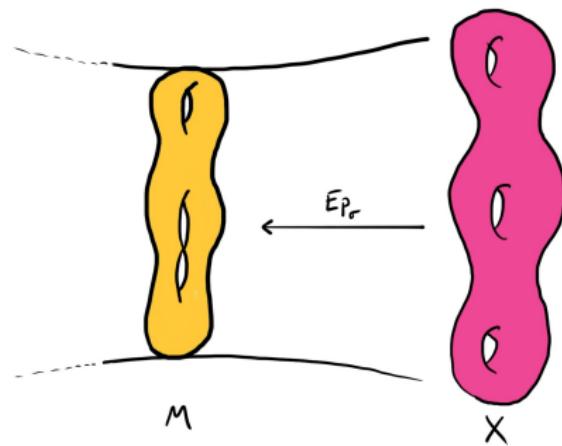
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From  $\text{Ep}_{N_*\sigma}(N \cdot z) = M \cdot \text{Ep}_\sigma(z)$  we get a variant of this Epstein construction for quotients.

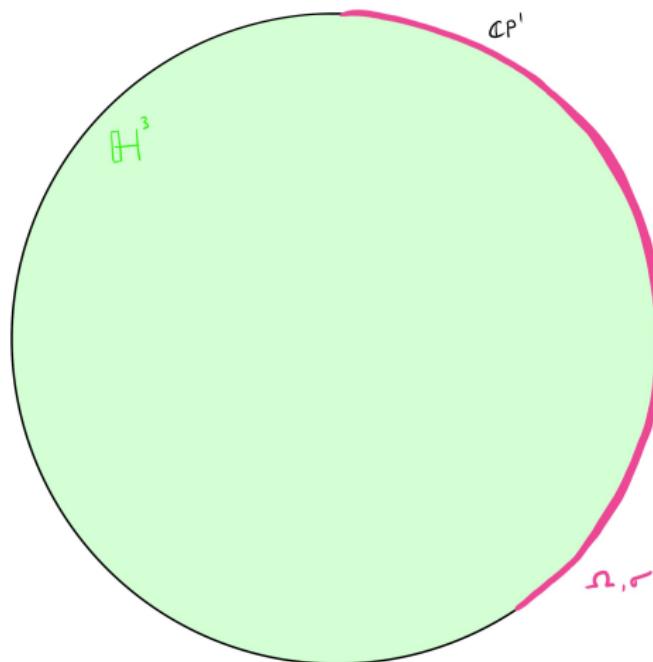
In the quasi-Fuchsian case this means if we take a conformal metric  $\sigma$  on the surface at infinity  $X$ , we get an Epstein surface  $\text{Ep}_\sigma : X \rightarrow M$ .  
(Caution: we say surface even though it may fail to be immersed)



## Properties

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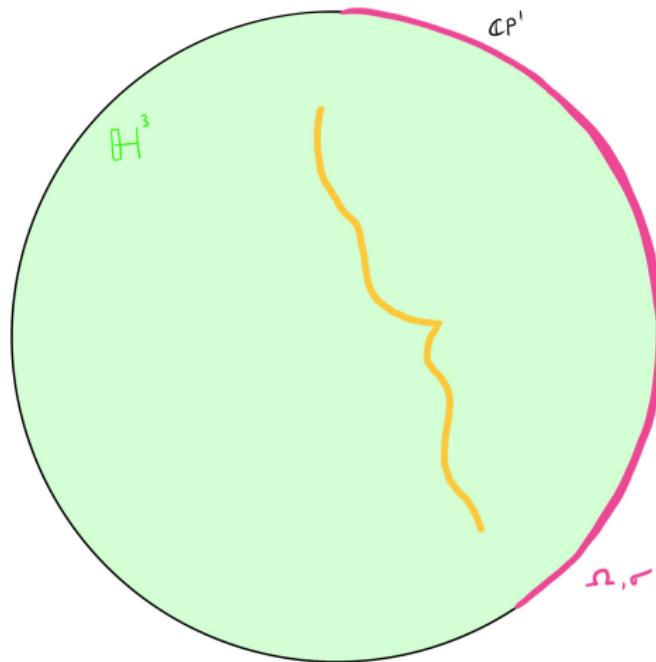
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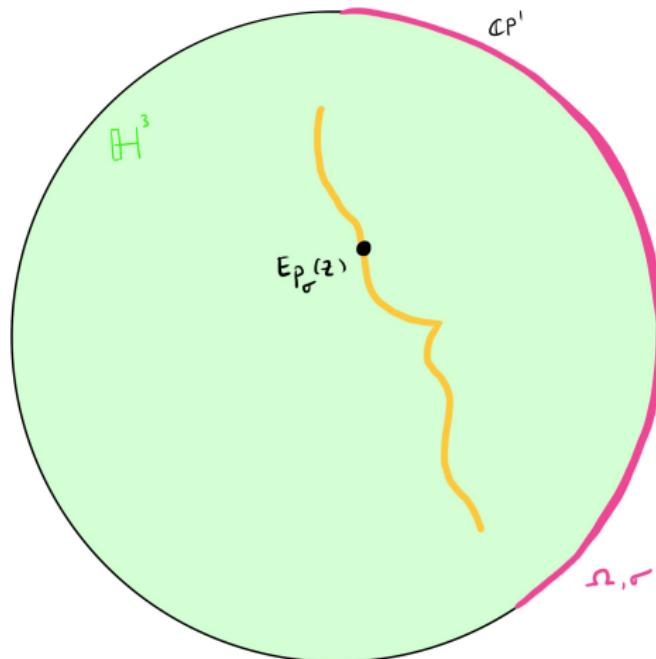
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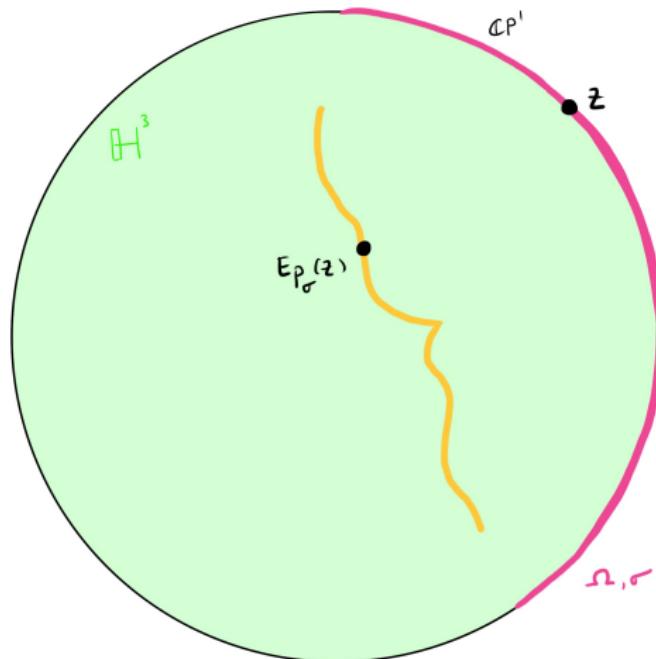
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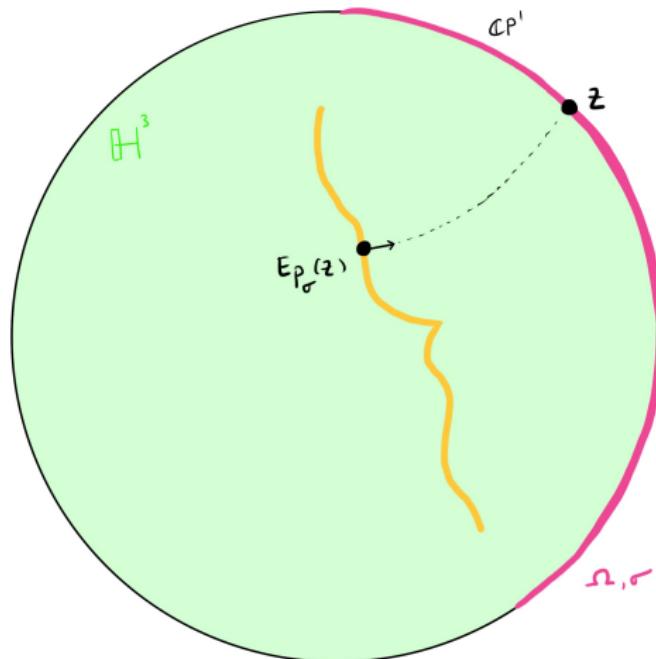
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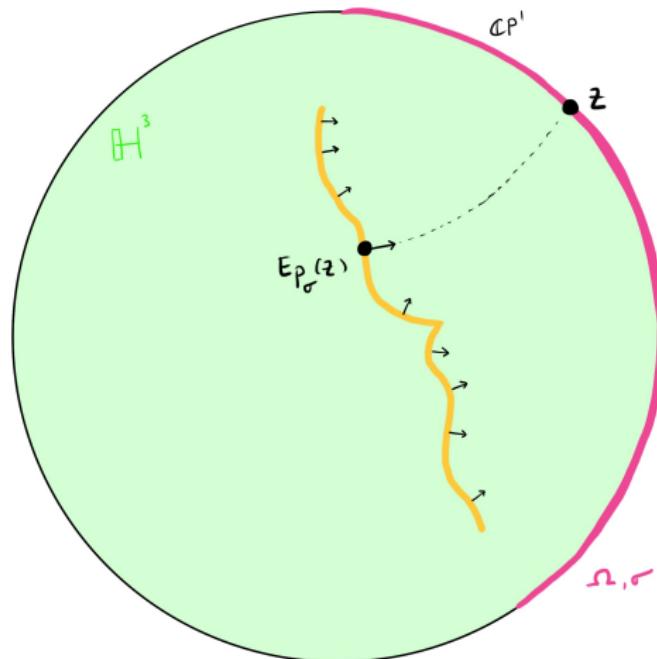
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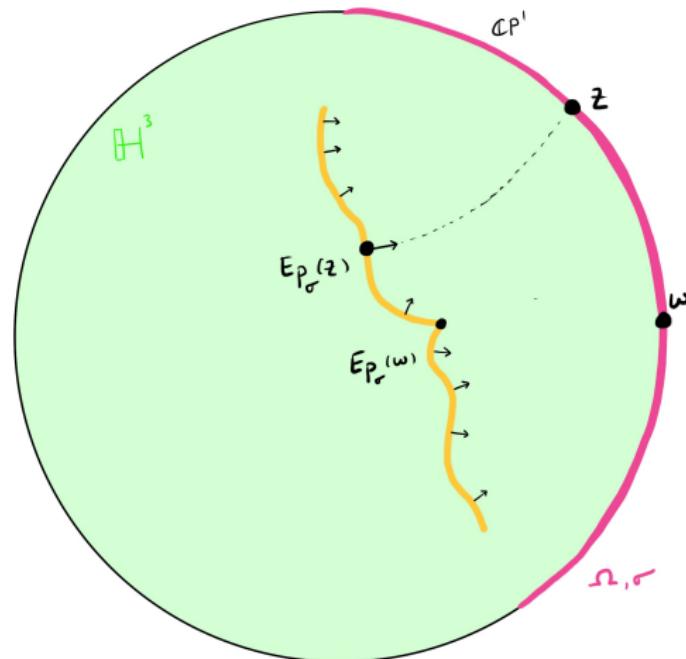
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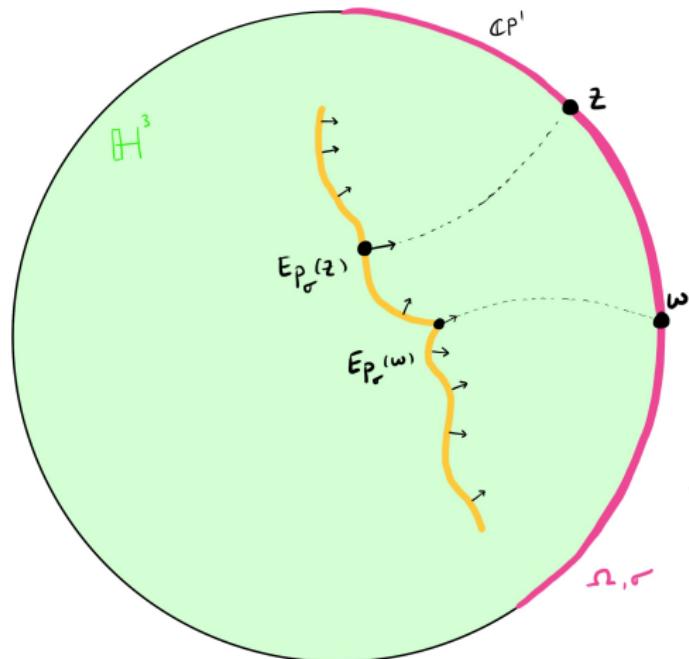
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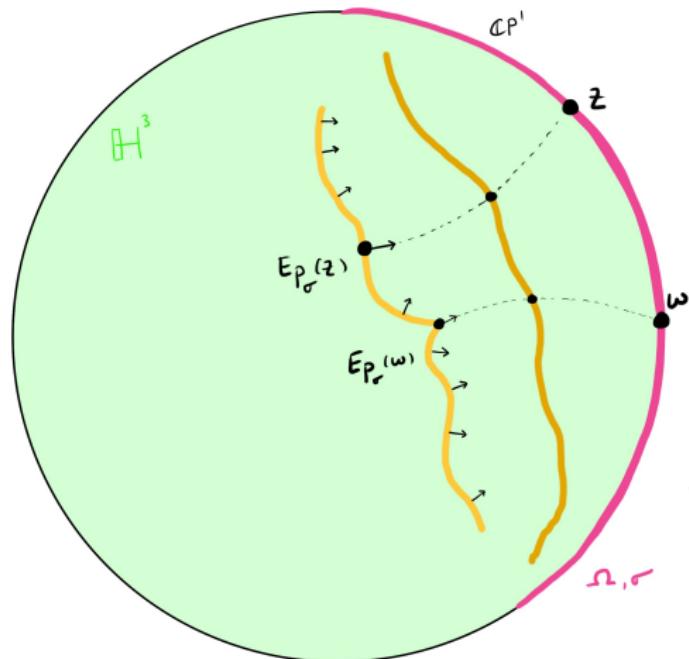
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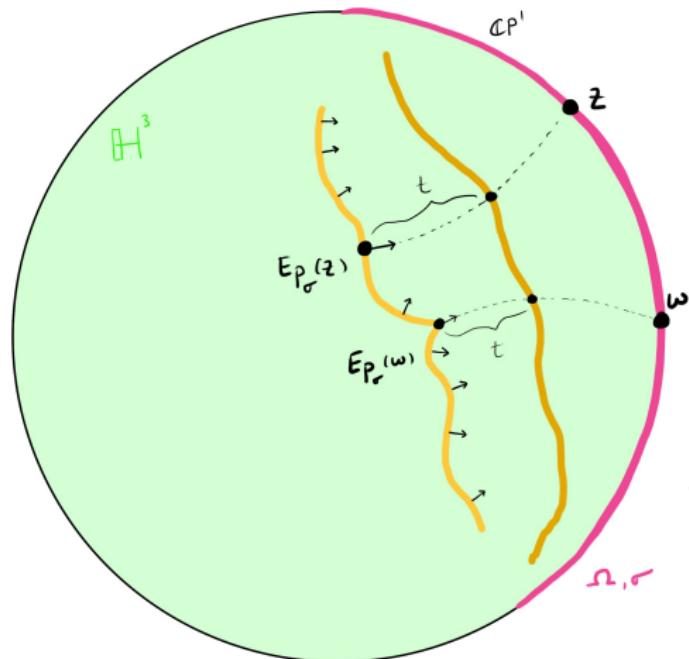
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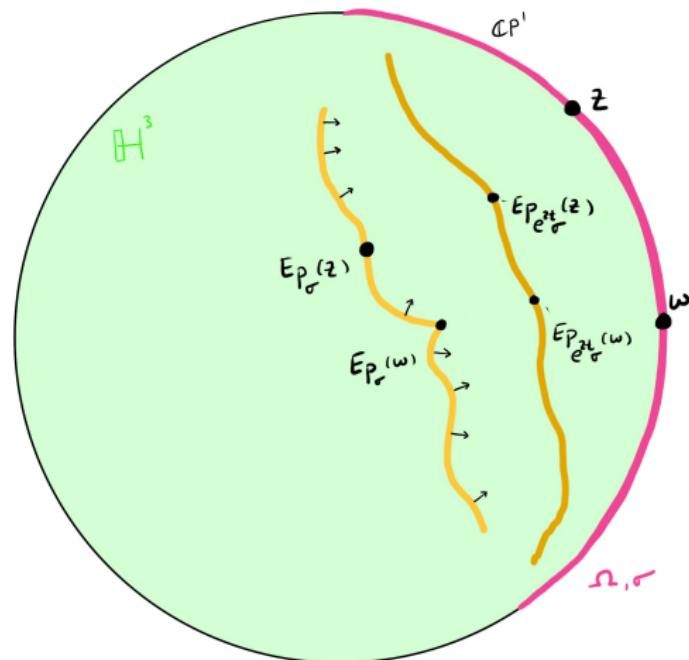
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Notice that after rescaling by  $\epsilon = e^{-2t}$ , the conformal metrics of Poincaré family satisfies  $\epsilon \rho_\epsilon \rightarrow h$  as  $\epsilon \rightarrow 0$ .

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3. the continuous extension  $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$  of  $f\sigma$  is differentiable.

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## Proposition (Q.)

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- ▶ Therefore, the restriction of  $E$  to  $[0, \delta^2) \times X$ , for some small enough  $\delta$ , is a diffeomorphism onto a collar neighborhood of  $X$  in  $M \sqcup X$ .

□

# Main Results

## Theorem (Q.)

Let  $S_\epsilon$  for  $\epsilon \in (0, 1)$  be an asymptotically Poincaré family of surfaces for the conformal metrics  $\sigma(\epsilon)$ . If  $h$  is the Poincaré metric of  $X$  and  $\phi$  the holomorphic quadratic differential at infinity, then in Teichmüller space  $\mathcal{T}(X)$  we have

$$[I(\sigma(\epsilon))] \rightarrow [h] \quad \text{and} \quad [II(\sigma(\epsilon))] \rightarrow [h] \quad \text{as } \epsilon \rightarrow 0.$$

Moreover, the tangent vectors in  $T_{[h]}\mathcal{T}(S)$  are given by

$$[\dot{I}(\sigma(\epsilon))] = 4f'(0)\operatorname{Re}(\phi) \quad \text{and} \quad [\dot{II}(\sigma(\epsilon))] = 0.$$

## Proof

Denote by  $\text{Met}^s(X)$  the set of Riemannian metrics of Sobolev class  $H^s$  for a fixed  $s > 3$  and note that  $\text{Met}^\infty(X) = \cap_{s>3} \text{Met}^s(X)$ .

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We are given that  $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$  is continuous and differentiable at  $\epsilon = 0$ . Therefore, we also have that  $\gamma$  is continuous to  $\text{Met}^s(X)$  and differentiable at  $\epsilon = 0$ .

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By noting that  $\sigma(\epsilon) = f(\epsilon)^{-1}\gamma(\epsilon)$ , in  $\text{Met}^s(X)$ , we can write

$$I(\sigma(\epsilon)) = \frac{1}{4\epsilon f'(0)}h + \frac{1}{4f'(0)}\dot{\gamma} + \frac{1}{2}h + \text{Re}(\phi) + O(\epsilon),$$

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so that

$$I_\epsilon := 4\epsilon f'(0)I(\sigma(\epsilon)) = h + \epsilon(\dot{\gamma} + 2f'(0)h + 4f'(0)\text{Re}(\phi)) + O(\epsilon^2).$$

## Proof

From the expression

$$I_\epsilon = h + \epsilon(\dot{\gamma} + 2f'(0)h + 4f'(0)\operatorname{Re}(\phi)) + O(\epsilon^2)$$

we can see that  $I_\epsilon \rightarrow h$  in  $\operatorname{Met}^s(X)$  for all  $s$ , implying  $I_\epsilon \rightarrow h$  in  $\operatorname{Met}^\infty(X)$ .

Moreover, the derivative at  $\epsilon = 0$  is

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Similar considerations to the second fundamental form give

$$II_\epsilon := 4\epsilon f'(0)II(\sigma(\epsilon)) = h + \epsilon\dot{\gamma} + O(\epsilon^2)$$

$$\dot{II}_\epsilon = \dot{\gamma}.$$

## Proof

We model  $\mathcal{T}(X)$  as the quotient  $\text{Met}^\infty(X)/(\text{Diff}_0^\infty(X) \ltimes P^s(X))$ .  
But again, we work with Sobolev tensors and functions. With these regularity assumptions  $T_h \text{Met}^s(X)$  is a Hilbert space and it decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_X h + f h \mid f \in H^s, X \in \Gamma(TX)\}.$$

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The second summand is tangent to the  $\text{Diff}_0^s(X) \ltimes P^s(X)$ -orbit of  $h$  so the first summand is a model for the tangent space to the quotient at  $[h]$ . This is true for any  $s$ , so it is identified with the tangent space to Teichmüller space.

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$$\begin{aligned} T_{[h]} \mathcal{T}(X) &= \{\dot{g} \mid \text{tr}_h(\dot{g}) = 0 = \text{div}_h(\dot{g})\} \\ &= \{\text{Re}(\psi) \mid \psi \text{ a holomorphic quadratic differential on } X\}. \end{aligned}$$

## Proof

We have that the projection  $\text{Met}^\infty(X) \rightarrow \mathcal{T}(X)$  is continuous and its derivative at  $h$  is given by orthogonal projection onto  $\{\text{Re}(\psi)\}$ .

Therefore, as  $I_\epsilon \rightarrow h$  in  $\text{Met}^\infty(X)$  we have that  $[I(\sigma(\epsilon))] = [I_\epsilon] \rightarrow [h]$  in  $\mathcal{T}(X)$ .

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The same arguments work for  $II(\sigma(\epsilon))$ .

# Applications

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Theorem (Labourie, 1991 - 1992)

*Let  $E$  be an end of a quasi-Fuchsian manifold  $M$ , then for each  $k \in (-1, 0)$ , there exists a unique (incompressible) surface embedded in  $E$  with constant Gaussian curvature  $k$ . Moreover, this family of surfaces foliates the end  $E$ .*

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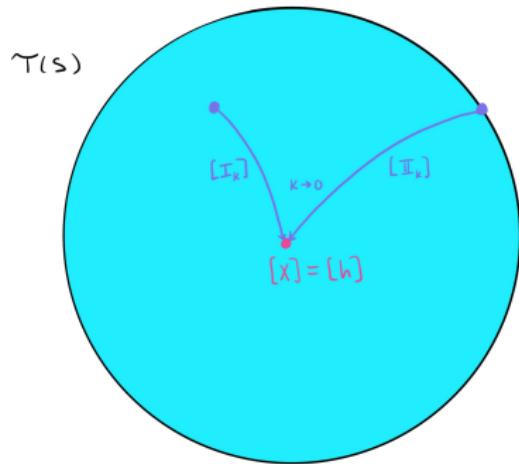
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## Theorem (Mazzeo-Pacard, 2011)

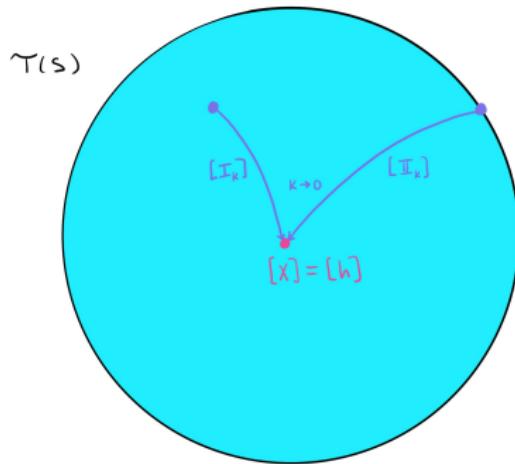
*The ends of a quasi-Fuchsian manifold admit unique foliations by constant mean curvature surfaces.*

# A Conjecture of Labourie

Labourie called these  $k$ -surfaces and he discusses how their first and second fundamental forms may be considered as paths in Teichmüller space.



# A Conjecture of Labourie



He shows that as  $k \rightarrow 0$ , the paths  $[I_k]$  and  $[II_k]$  converge to  $[X] = [h]$  and he asks after the tangent vectors to these paths at  $k = 0$ .

He conjectures  $[I_k]$  is related to the holomorphic quadratic differential  $\phi$ .

# A Conjecture of Labourie

## Theorem (Q.)

Let  $I_k$  and  $\Pi_k$  be the first and second fundamental forms of the  $k$ -surface. Let  $\phi$  be the holomorphic quadratic differential at infinity of  $M$ . Then, as  $k \rightarrow 0$ , the tangent vectors to  $[I_k]$  and  $[\Pi_k]$  in Teichmüller space are given by

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$$[\dot{I}_k] = -\operatorname{Re}(\phi) \quad \text{and} \quad [\dot{\Pi}_k] = 0.$$

We prove this by presenting the  $k$ -surfaces as Epstein surfaces (for  $k$  near 0) and showing they form an asymptotically Poincaré family.

This also gives another proof of the existence of  $k$ -surfaces, for  $k$  near 0.

## Proof

To present a  $k$ -surface as an Epstein surface we must find a conformal metric  $\sigma$  so that

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}} = k.$$

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as for small  $k$  this suffices.

Notice though, that when  $k = 0$  this reads  $K(\sigma) = 0$ , which has no solutions on  $X$ .

## Proof

Our technique is to use the Implicit Function Theorem to obtain solution for each  $k$  near 0. But since there are no solutions for  $k = 0$  we rescale the equation by considering the function  $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$ .

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If  $\sigma$  solves  $(*)$  then  $\tau = f(k)\sigma$  solves

$$(2 + k)(1 + K(\tau))^2$$

$$+ 2\sqrt{1+k} \left(1 - K(\tau)^2\right) + 16 \left(2\sqrt{1+k} - 2 - k\right) \frac{|B(\tau)|^2}{\tau^2} = 0$$

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Notice that  $F(0, \tau) = 0$  is equivalent to  $K(\tau) = -1$ , which has the solution  $\tau = h$ .

## Proof

Since we have a solution for  $k = 0$ , we can then use the Implicit Function Theorem to get solution for  $k$  near zero as well.

The function  $F$  extends from the smooth setting to one of  $(-1, 0) \times \text{Conf}^s(X) \rightarrow H^{s-2}(X)$ .

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We compute

$$DF_{(0,h)}(\dot{k}, \dot{\tau}) = 4DK_h(\dot{\tau}) = -2(\Delta_h - Id)\frac{\dot{\tau}}{h}$$

and see that  $D_2F_{(0,h)} = 4DK_h : H^s(X) \rightarrow H^{s-2}(X)$  is an isomorphism.

## Proof

Consequently, there exists a neighborhood  $V$  of 0 and a curve  $\gamma : V \rightarrow \text{Conf}^s(X)$  with  $\gamma(0) = h$  and  $F(k, \gamma(k)) = 0$ .

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To complete the proof we take  $\sigma(k) := f(k)^{-1}\gamma(k)$  for  $k \in (-\delta, 0)$ .

The family of Epstein surfaces for  $\sigma(k)$  then satisfy the definition of an asymptotically Poincaré family and our main theorem applies. □

# Constant Mean Curvature Surfaces

## Theorem (Mazzeo-Pacard, 2011)

*The ends of a quasi-Fuchsian manifold admit unique foliations by constant mean curvature surfaces.*

## Theorem (Q.)

*Let  $I_k$  and  $II_k$  be the first and second fundamental forms of the Epstein surface with constant mean curvature  $-\sqrt{1+k}$ . Let  $\phi$  be the holomorphic quadratic differential at infinity. Then, as  $k \rightarrow 0$ , the tangent vectors to  $[I_k]$  and  $[II_k]$  in Teichmüller space are given by*

$$[I_k] = -\operatorname{Re}(\phi) \quad \text{and} \quad [II_k] = 0.$$

## Proof

The proof proceeds the same as in the constant Gaussian curvature case. However, here we wish to solve the equation

$$H(\text{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16|B(g_{\mathbb{CP}^1}, \sigma)|^2\sigma^{-2}}{(K(\sigma) - 1)^2 - 16|B(g_{\mathbb{CP}^1}, \sigma)|^2\sigma^{-2}} = -\sqrt{1 + k}.$$

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Using the same scaling function  $f(k)$  we are led to consider solutions to a function  $G(k, \tau) = 0$ , which has partial derivative

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The Implicit Function Theorem then gives solutions for  $k$  near zero. □

Thank you!