

Limits of foliations in quasi-Fuchsian manifolds

Keaton Quinn

University of Illinois at Chicago

June 10, 2020

Introduction

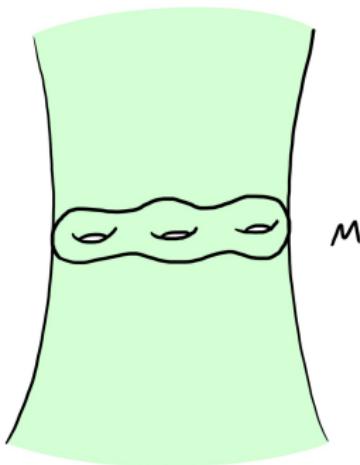
Let S be an oriented closed surface of genus at least 2.

Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

Introduction

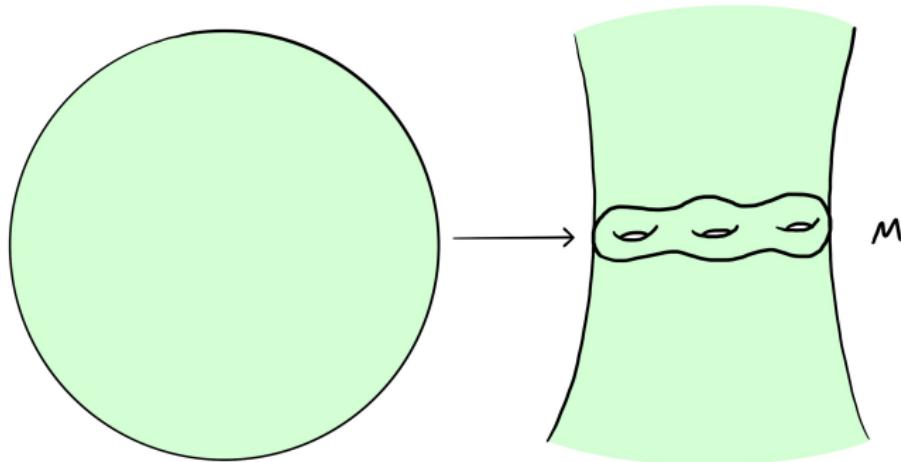
Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.



Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

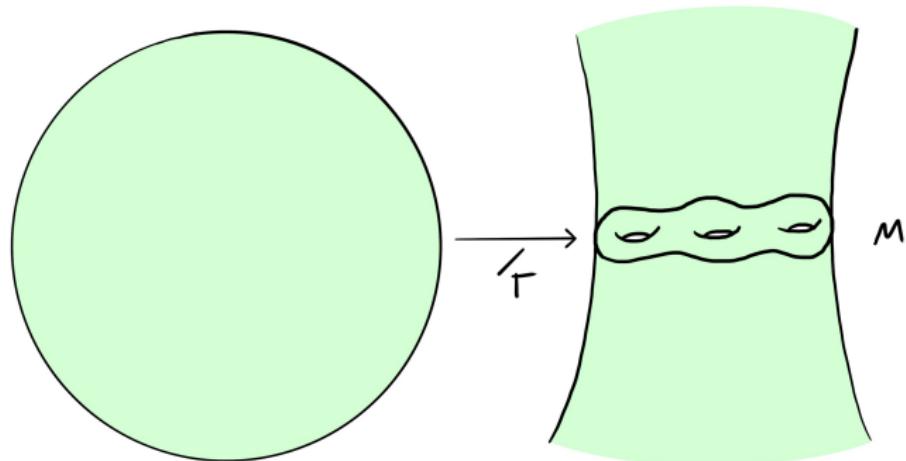
$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$



Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$

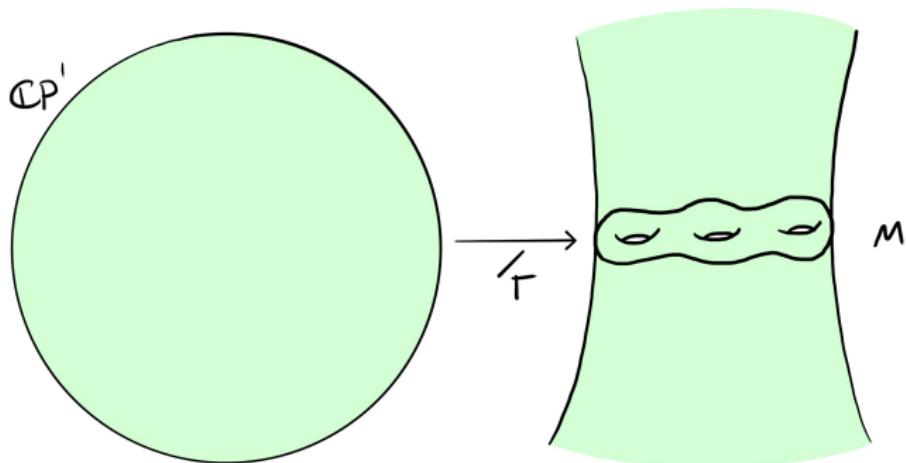


$$\pi_1(M) \simeq \pi_1(S) \xrightarrow{\sim} \Gamma \subset \mathrm{SL}_2 \mathbb{C}$$

Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$

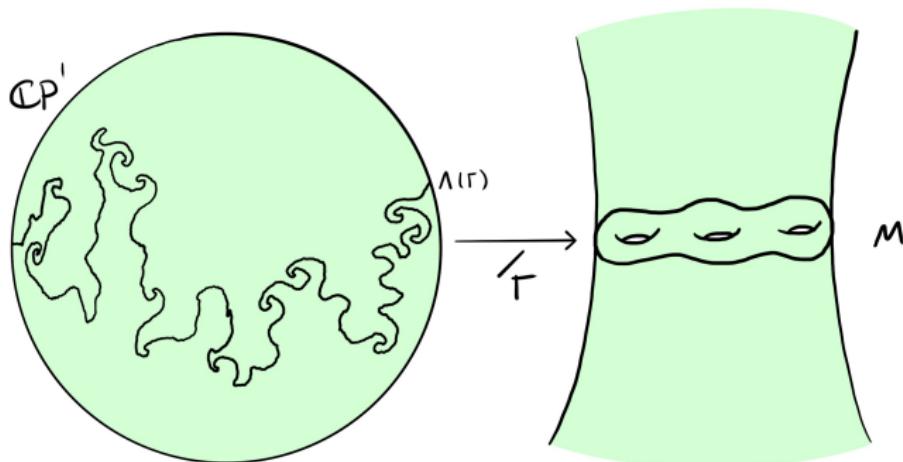


$$\pi_1(M) \simeq \pi_1(S) \xrightarrow{\sim} \Gamma \subset \mathrm{SL}_2 \mathbb{C}$$

Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$

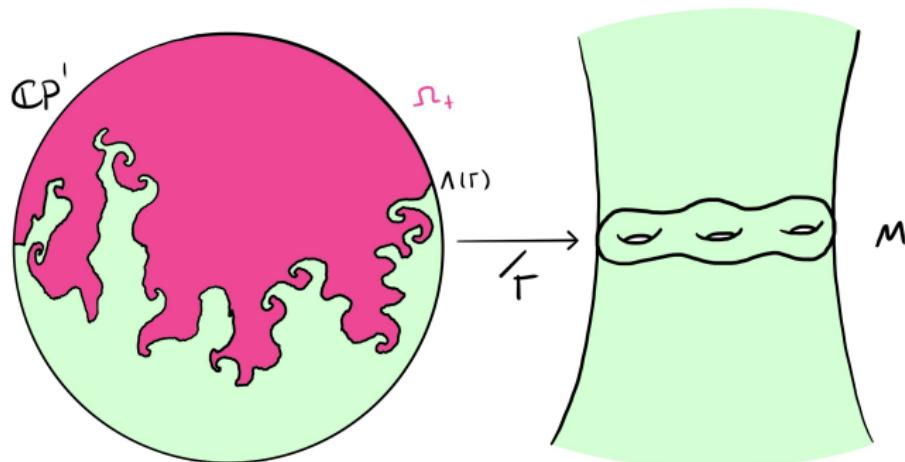


$$\pi_1(M) \cong \pi_1(S) \xrightarrow{\sim} \Gamma \subset \text{SL}_2 \mathbb{C}$$

Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$

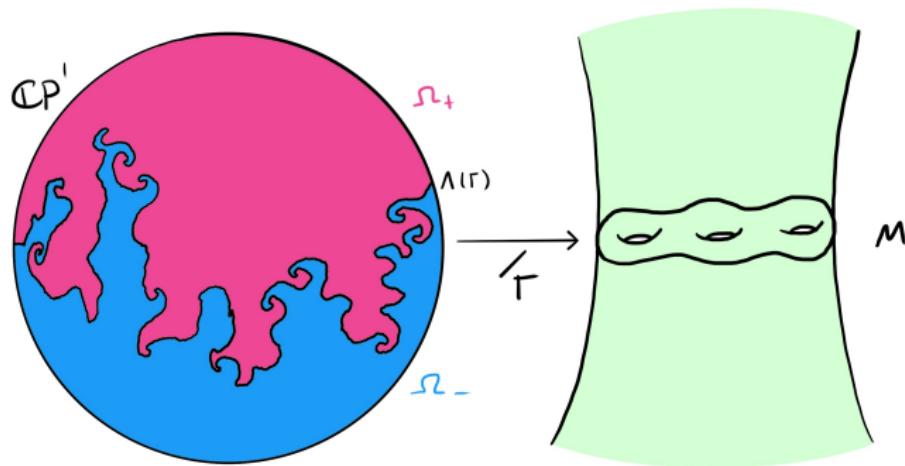


$$\pi_1(M) \simeq \pi_1(S) \xrightarrow{\sim} \Gamma \subset \text{SL}_2 \mathbb{C}$$

Introduction

Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.

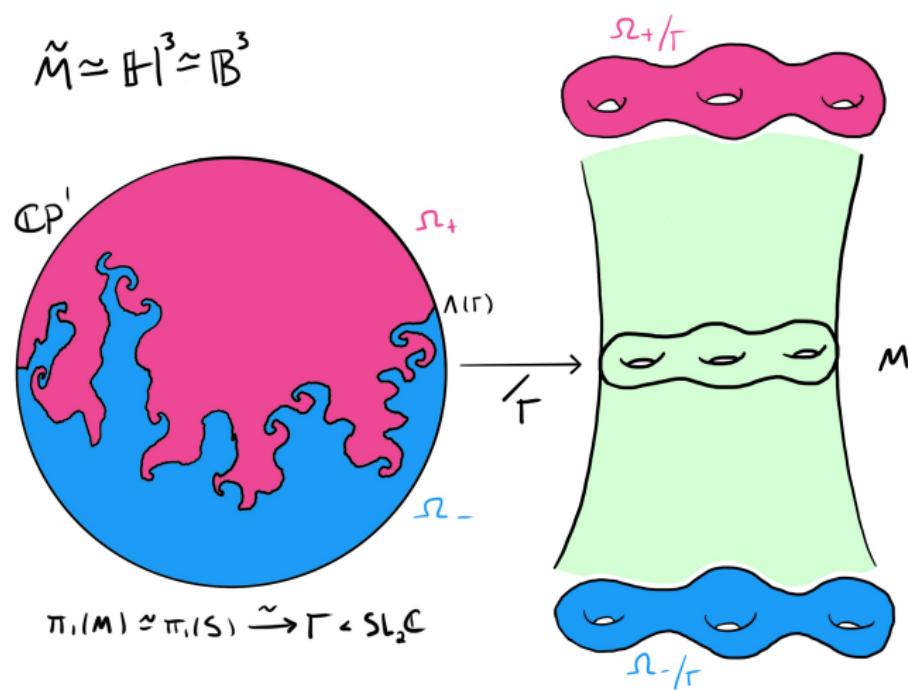
$$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$$



$$\pi_1(M) \cong \pi_1(S) \xrightarrow{\sim} \Gamma \subset \mathrm{SL}_2 \mathbb{C}$$

Introduction

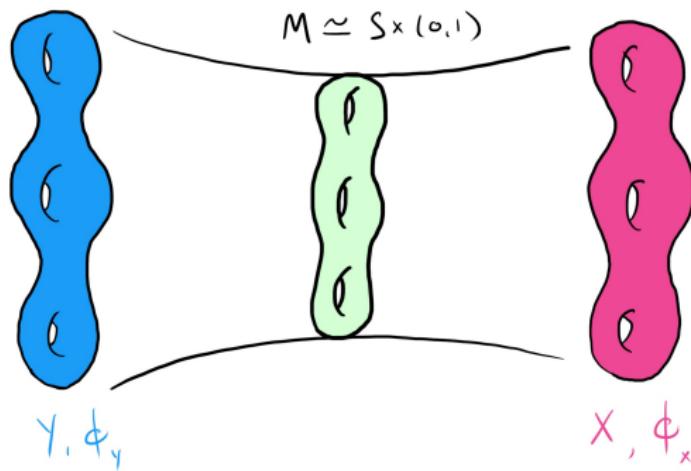
Let S be an oriented closed surface of genus at least 2. A quasi-Fuchsian structure M on $S \times (0, 1)$ is a complete hyperbolic metric such that there exists a nonempty, compact, geodesically convex subset.



Introduction

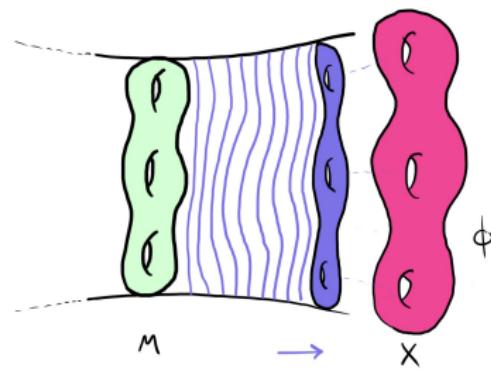
$X = \Omega_+/\Gamma$ and $Y = \Omega_-/\Gamma$ are called the surfaces at infinity and they inherit both conformal structures and complex projective structures.

These induce $[X]$ and $[Y]$ in $\mathcal{T}(S)$, the Teichmüller space of S and we get holomorphic quadratic differentials ϕ_X and ϕ_Y that parametrize the projective structures.



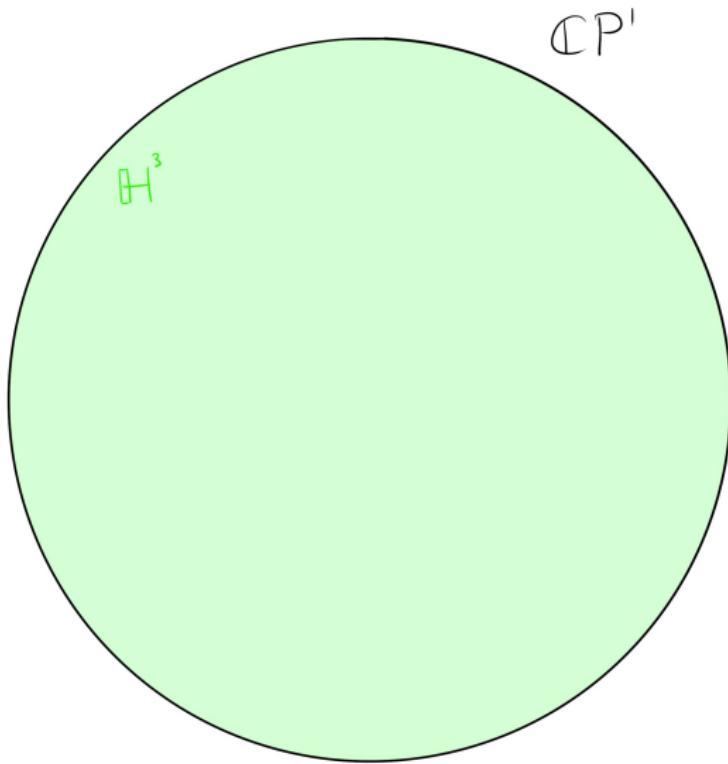
Introduction

We will consider certain foliations of the ends of M and investigate their limits as the foliations leave the ends.

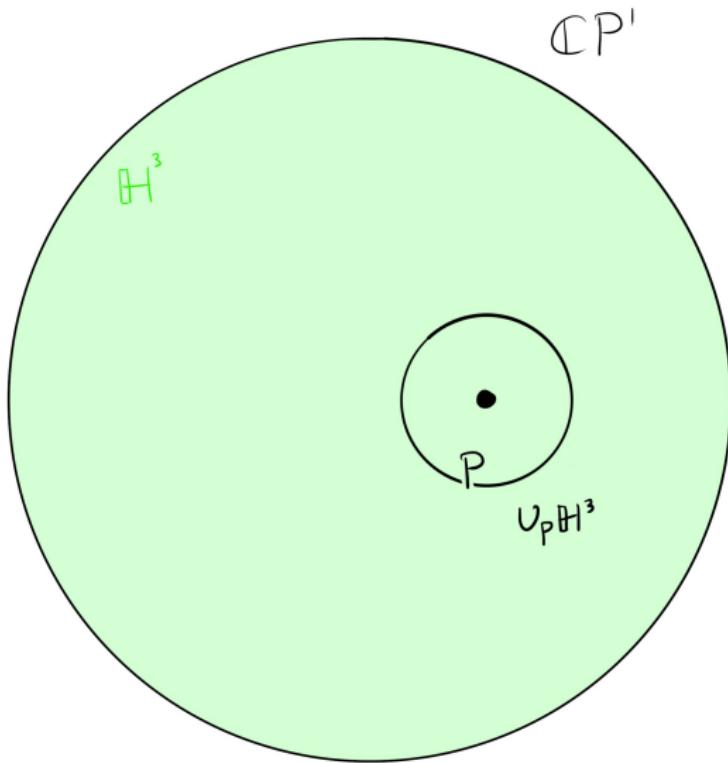


We will focus on just one end of M . From now on let X denote its surface at infinity and ϕ its holomorphic quadratic differential.

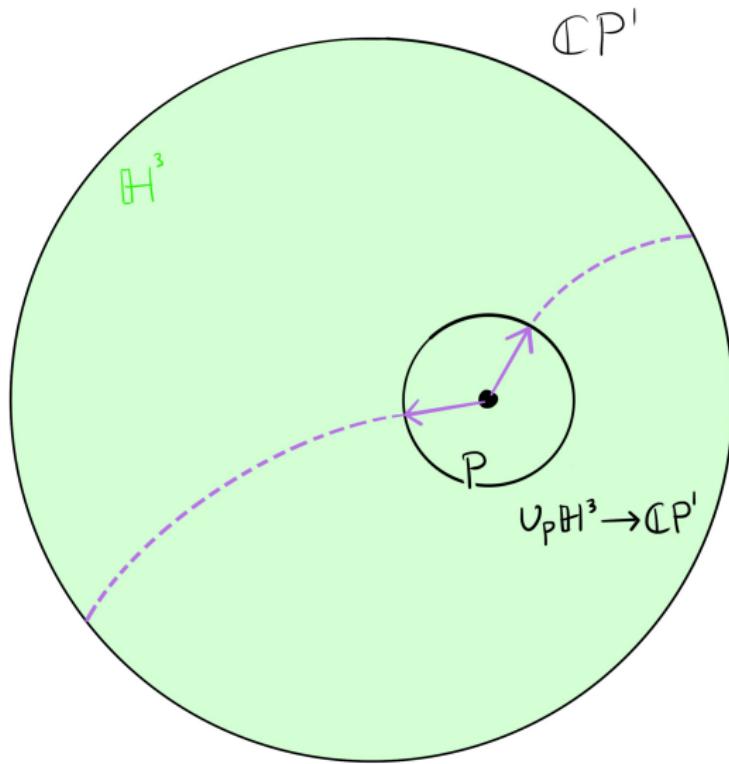
Visual Metrics



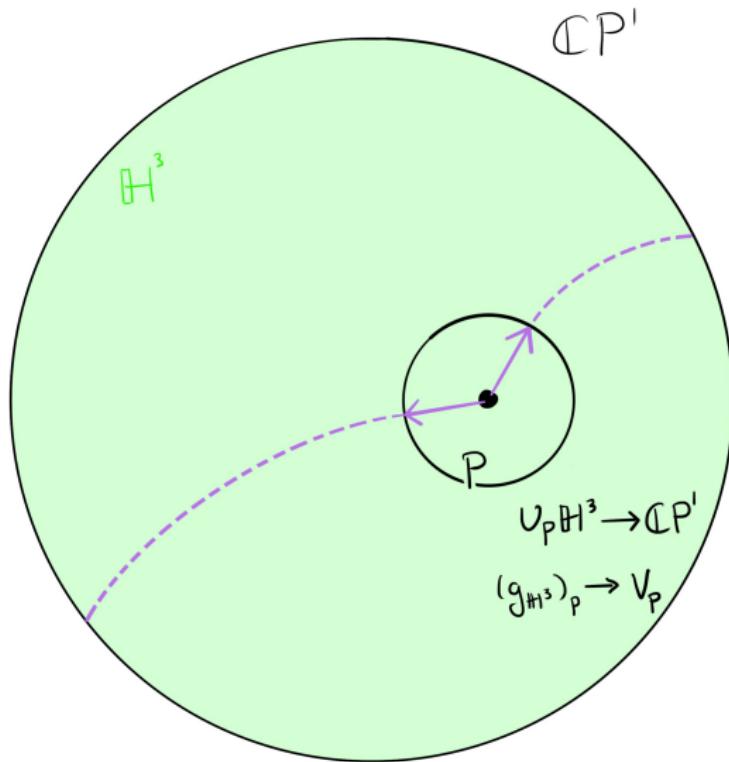
Visual Metrics



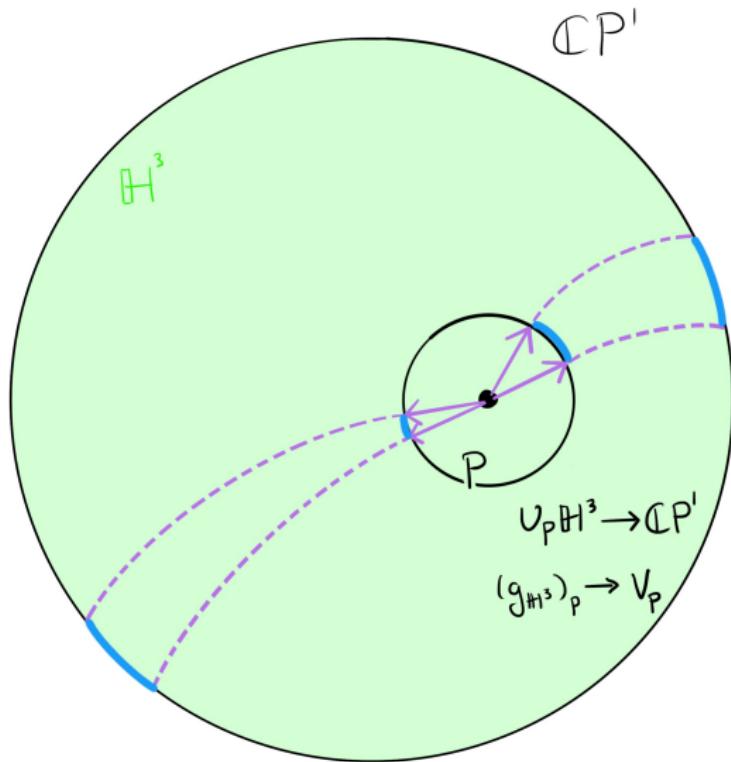
Visual Metrics



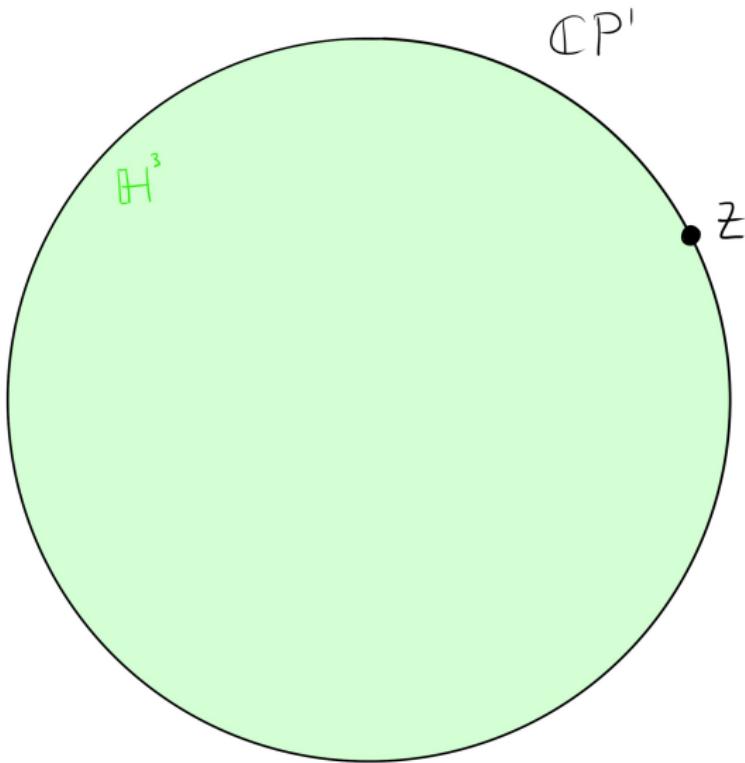
Visual Metrics



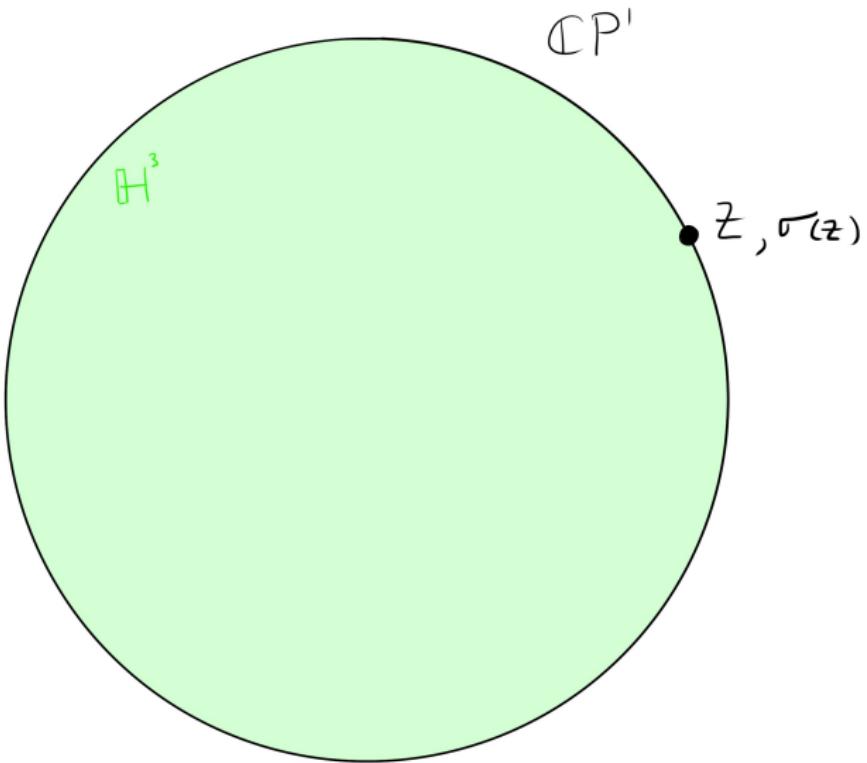
Visual Metrics



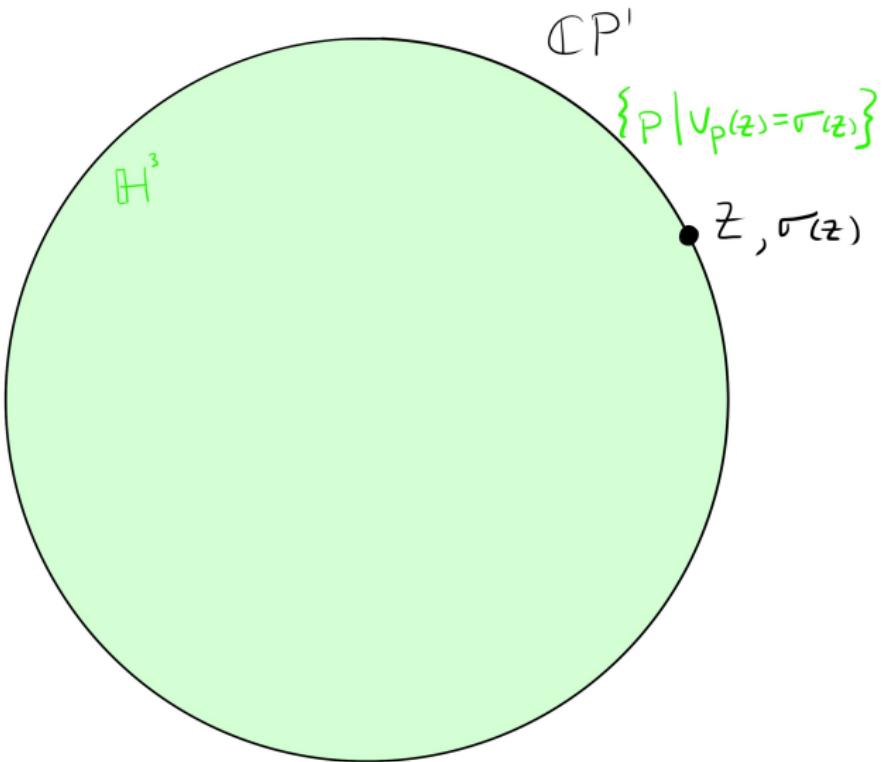
Epstein Surfaces



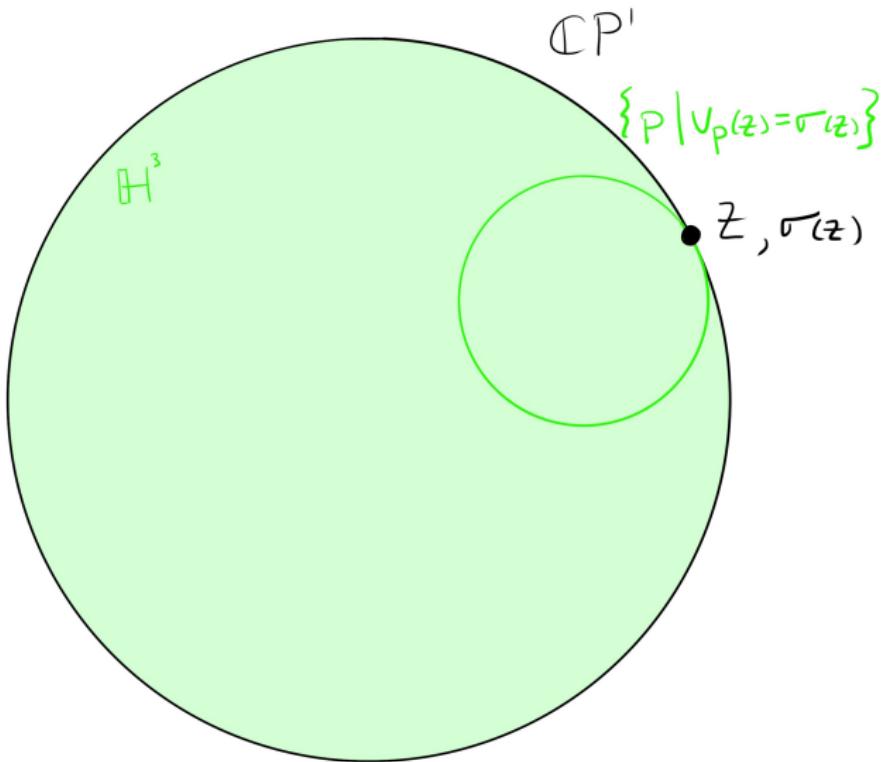
Epstein Surfaces



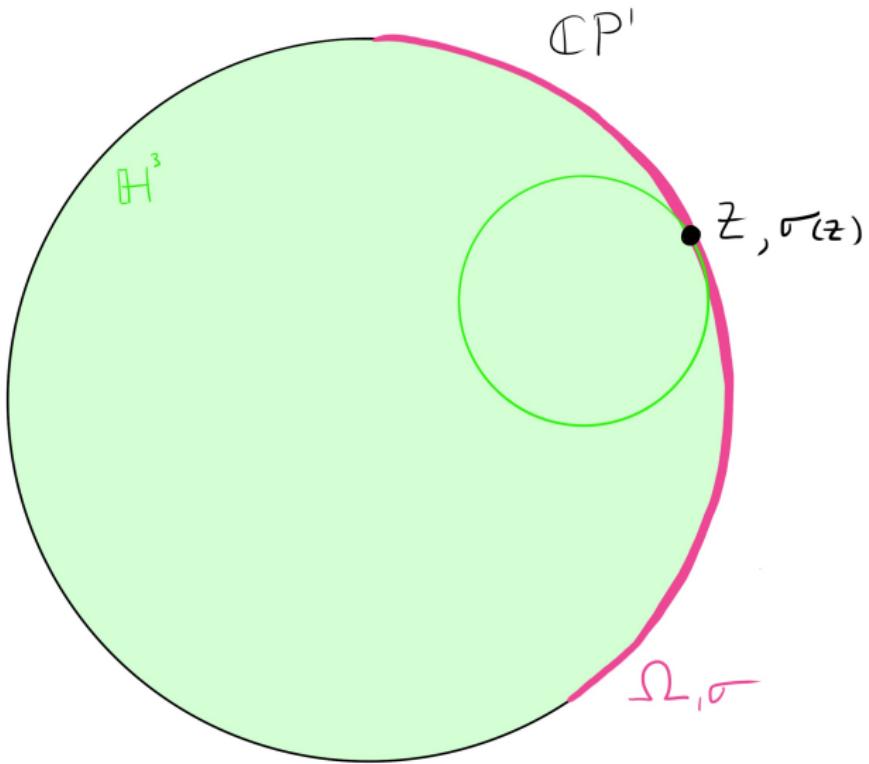
Epstein Surfaces



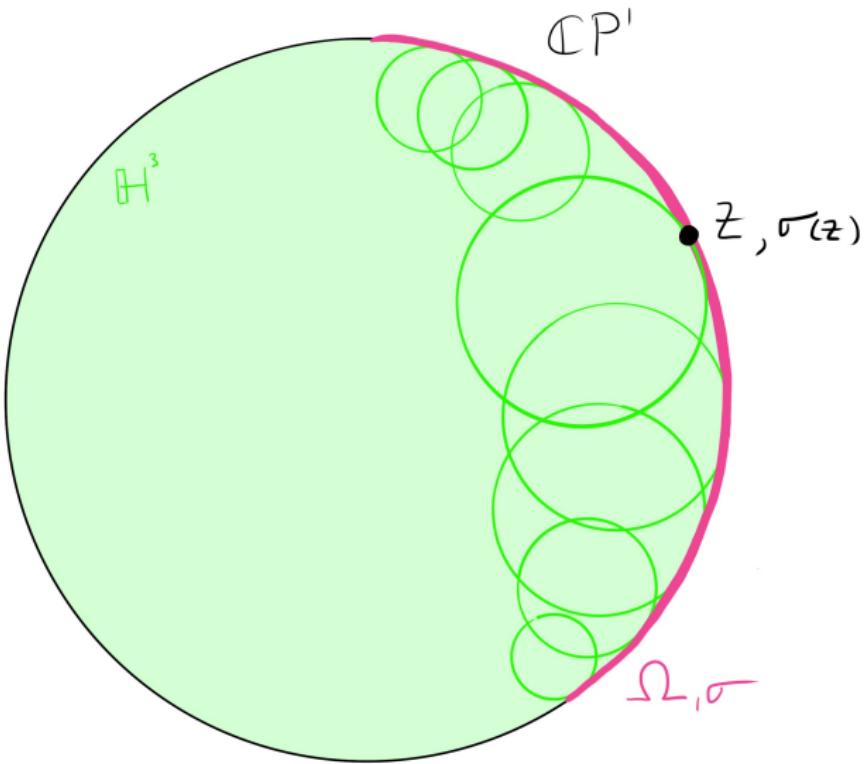
Epstein Surfaces



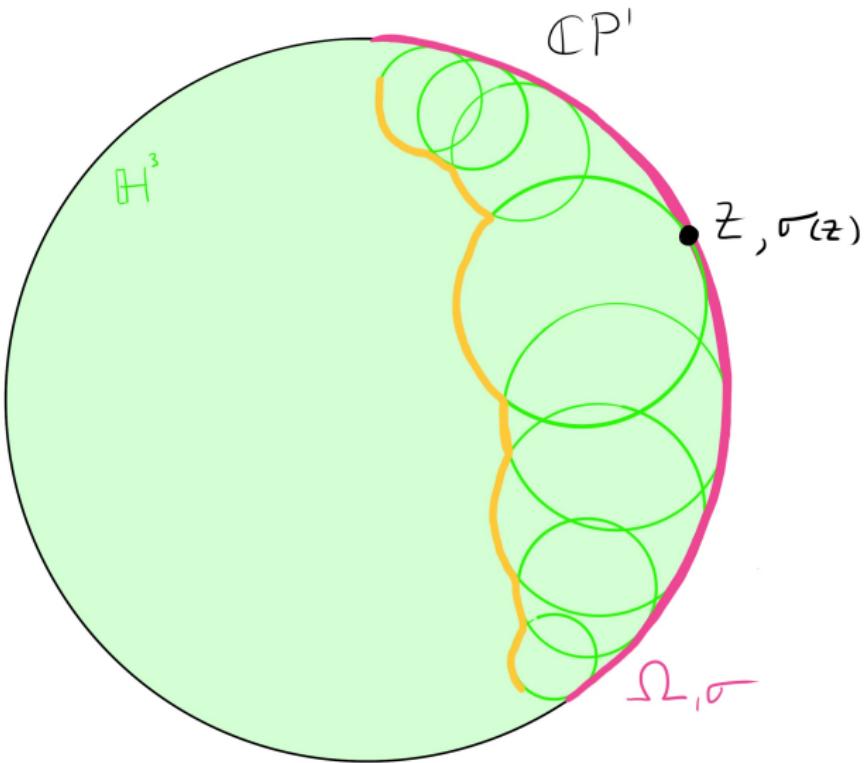
Epstein Surfaces



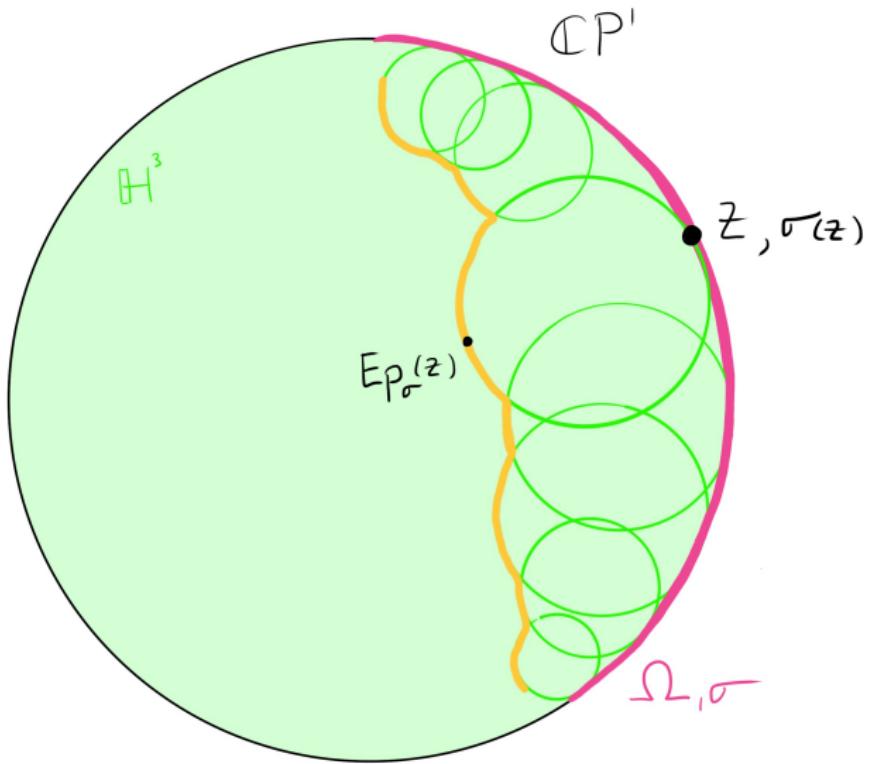
Epstein Surfaces



Epstein Surfaces



Epstein Surfaces



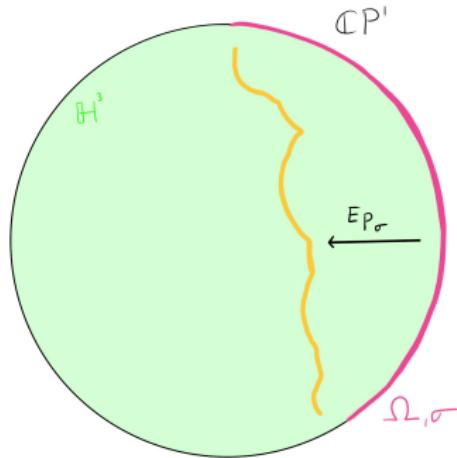
Epstein Surfaces

Theorem (Epstein 1984, Thurston)

Let Ω be a domain in $\mathbb{C}\mathbb{P}^1$ and σ a C^k conformal metric on Ω , then there exists a unique C^{k-1} map $\text{Ep}_\sigma : \Omega \rightarrow \mathbb{H}^3$, called the Epstein map of Ω for the metric σ , such that for all $z \in \Omega$,

$$V_{\mathrm{Ep}_\sigma(z)}(z) = \sigma(z).$$

Moreover, the image of a point z depends only on the 1-jet of σ at z .



Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

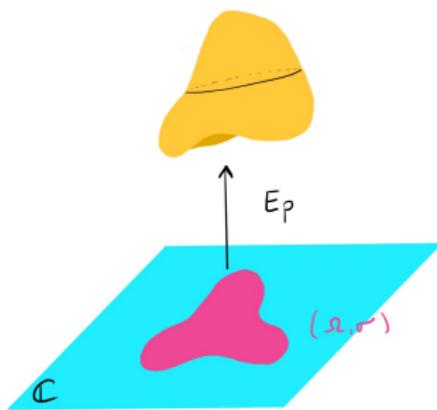
$$\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$$



Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

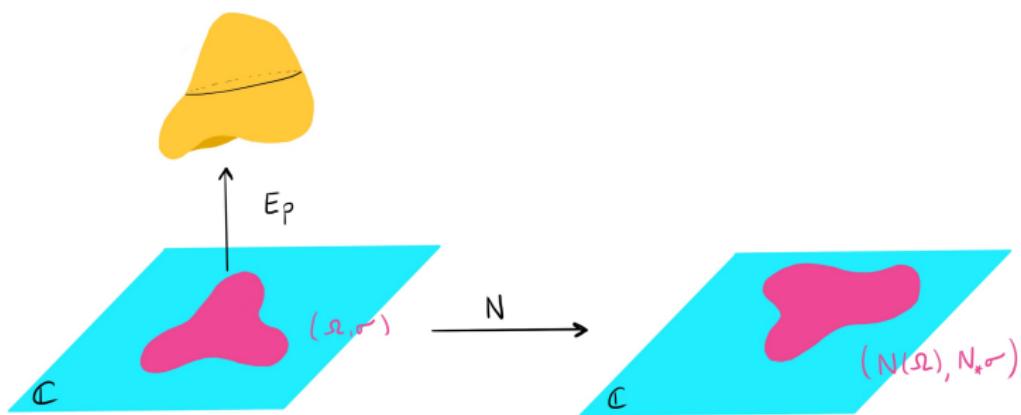
$$\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$$



Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

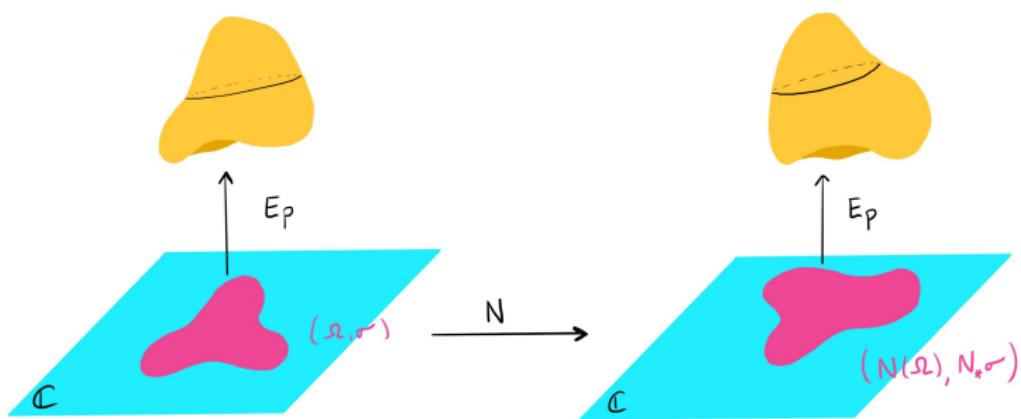
$$\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$$



Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

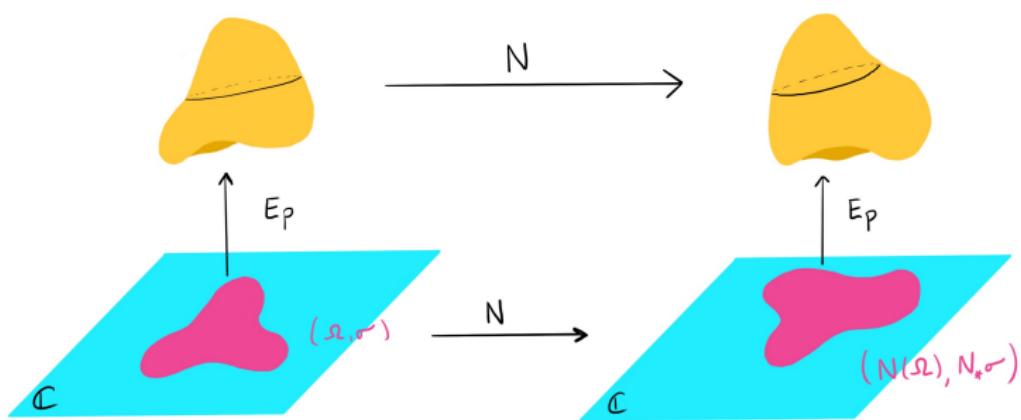
$$\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$$



Properties

This construction is equivariant with respect to the actions of $\mathrm{SL}_2\mathbb{C}$ on \mathbb{CP}^1 and \mathbb{H}^3 .

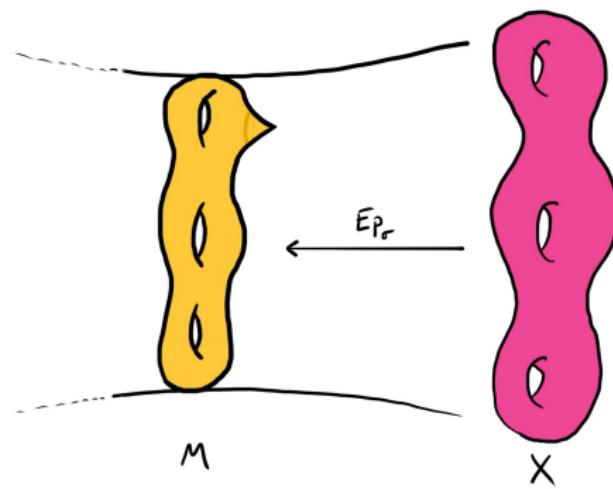
$$\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$$



Properties

From $\text{Ep}_{N_*\sigma}(N \cdot z) = N \cdot \text{Ep}_\sigma(z)$ we get a variant of this Epstein construction for quotients.

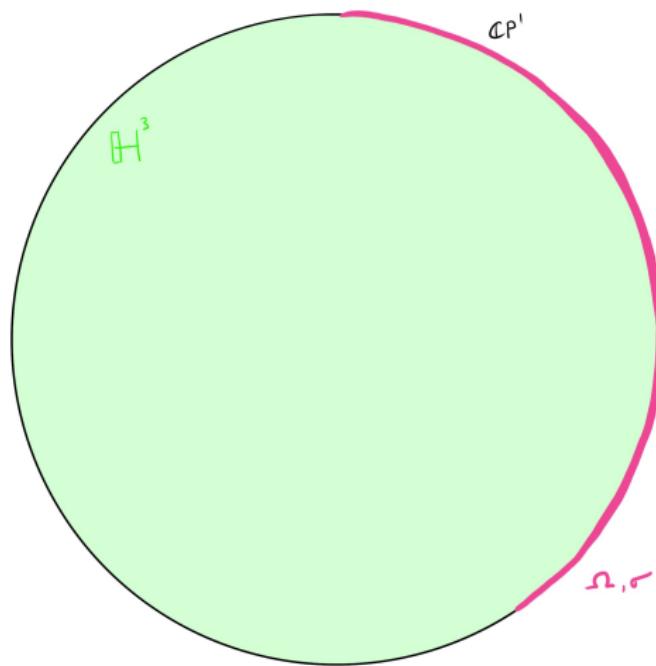
In the quasi-Fuchsian case this means if we take a conformal metric σ on the surface at infinity X , we get an Epstein surface $\text{Ep}_\sigma : X \rightarrow M$.
 (Caution: we say surface even though it may fail to be immersed)



Properties

The Epstein construction behaves well under a normal flow.

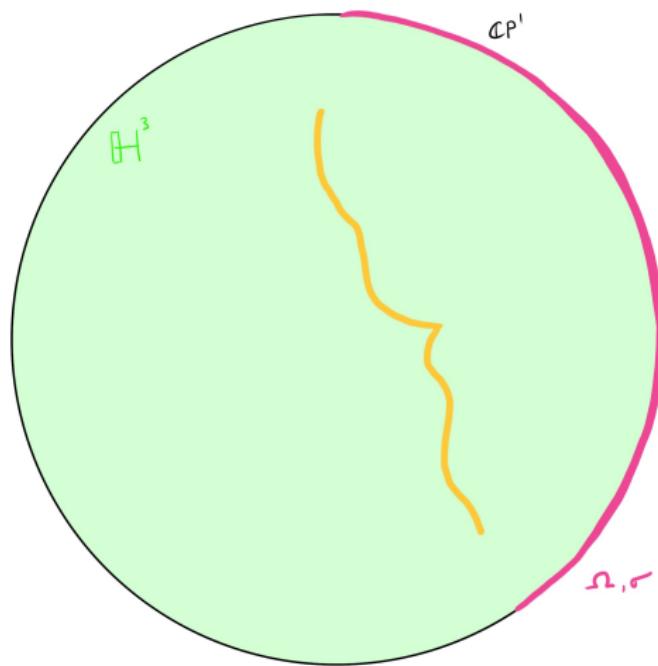
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

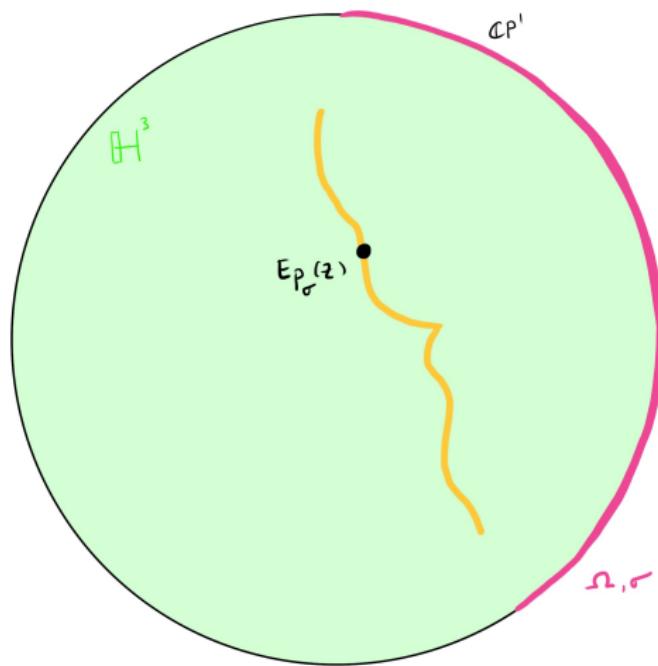
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

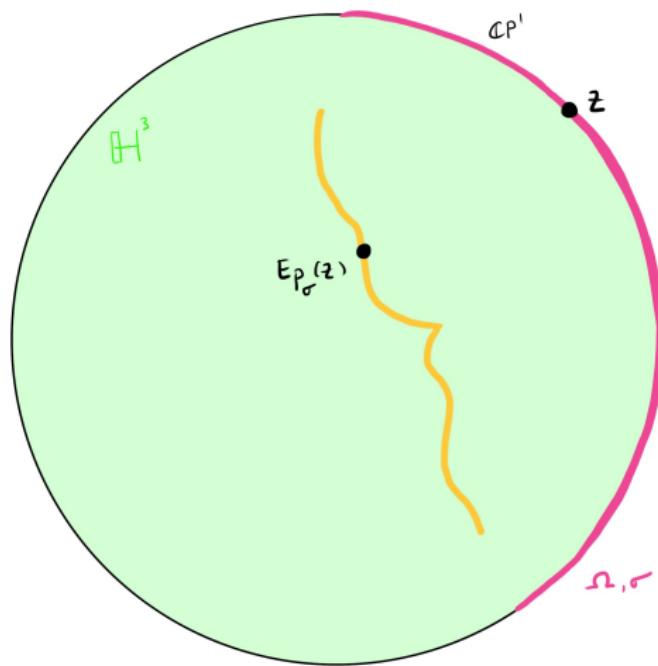
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

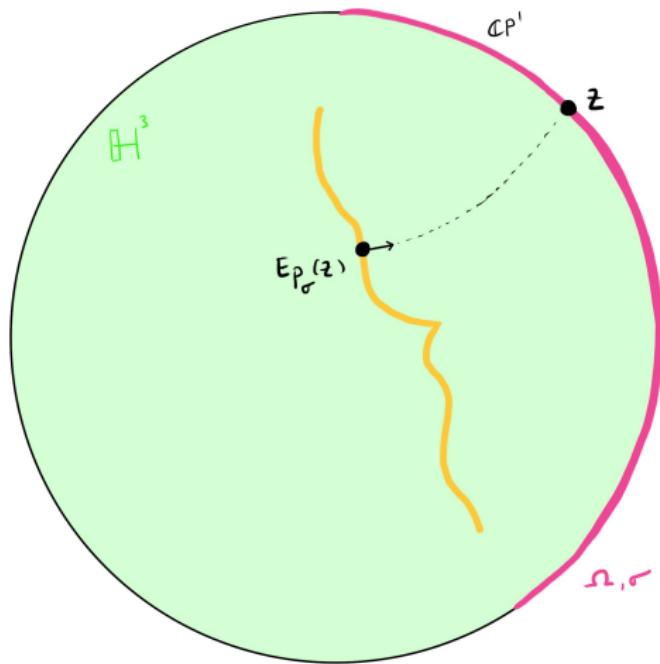
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

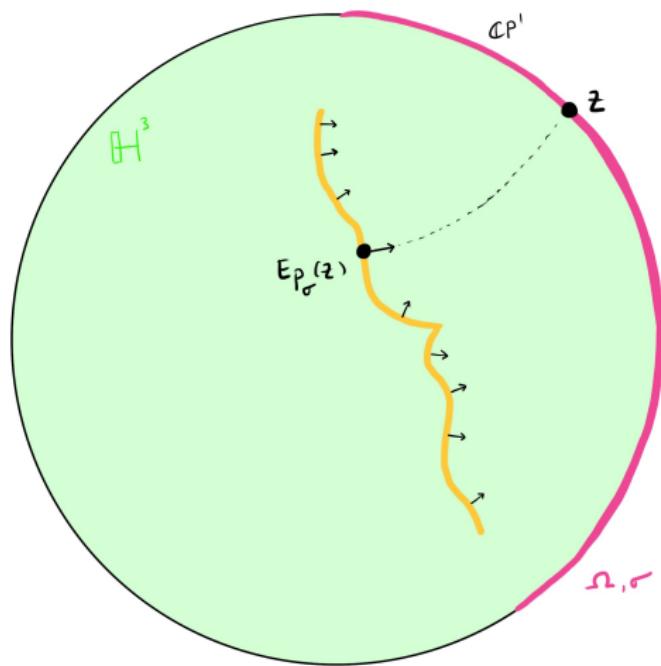
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

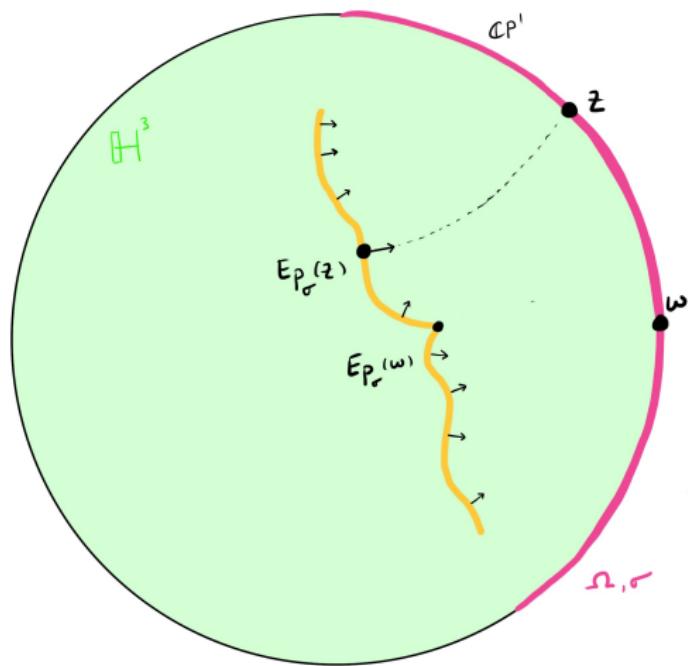
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

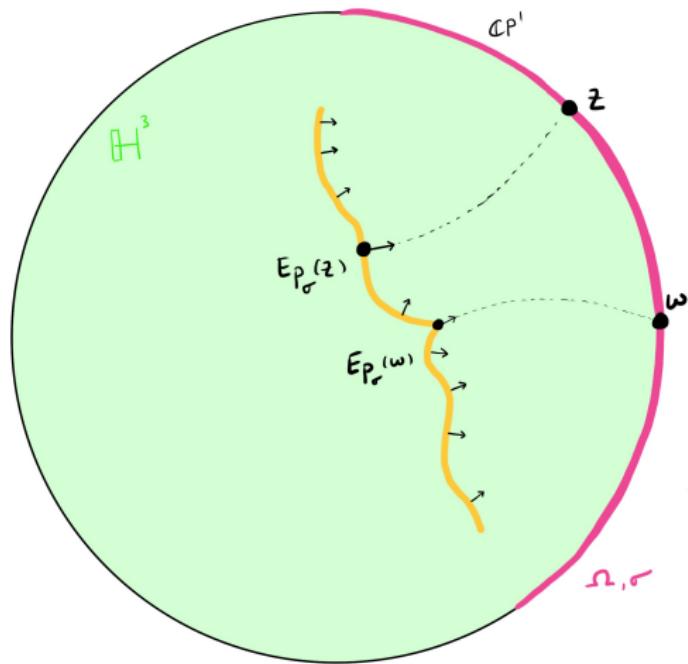
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

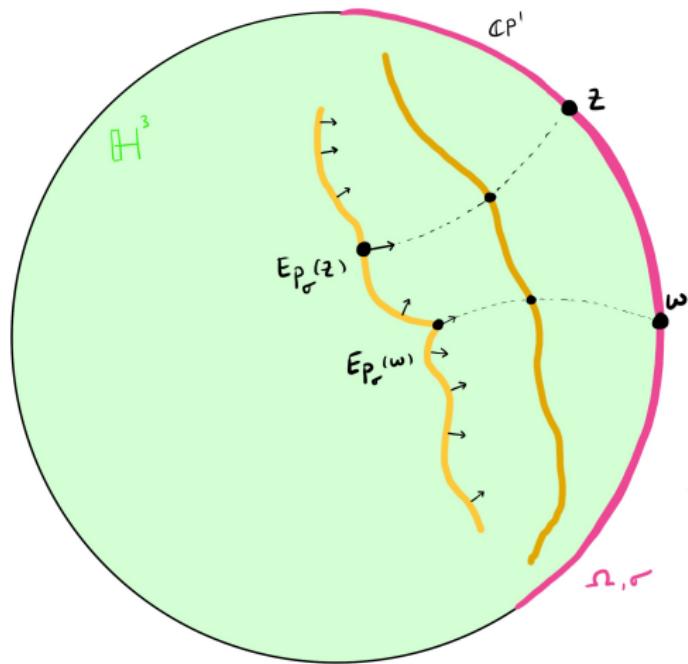
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

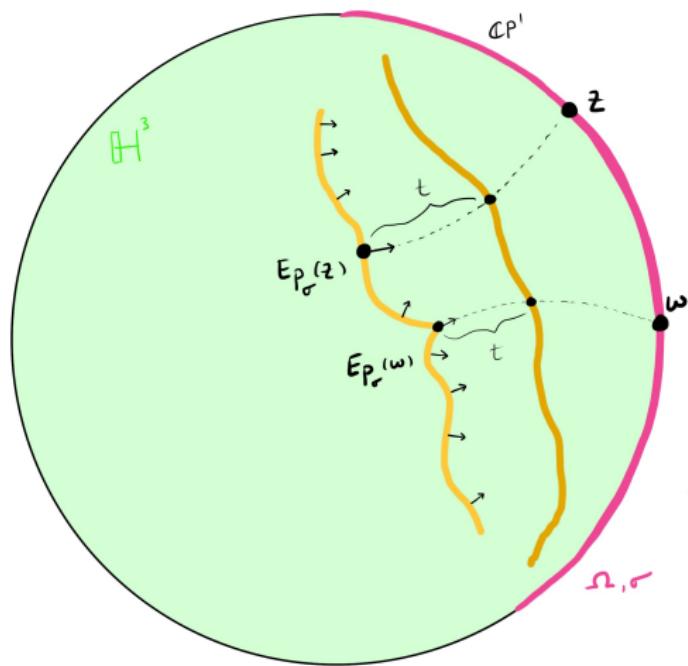
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

The Epstein construction behaves well under a normal flow.

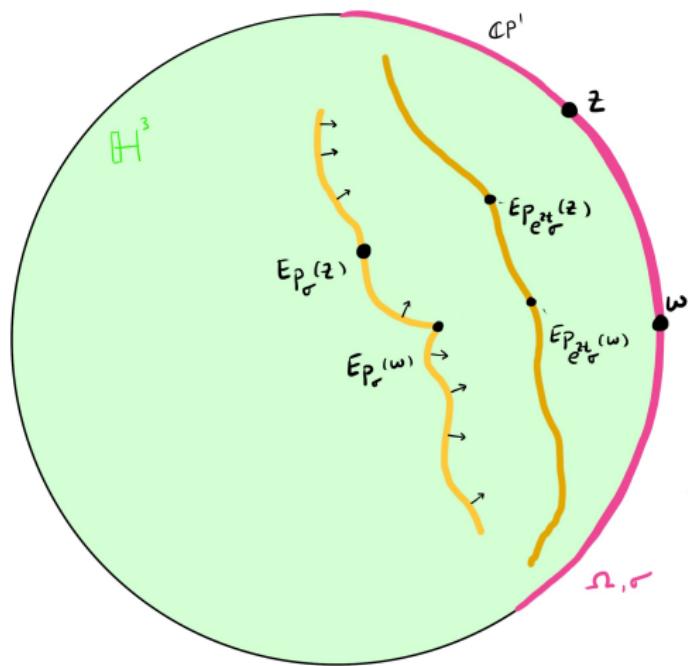
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



Properties

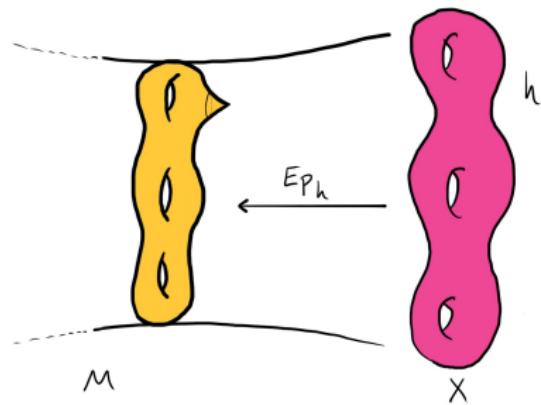
The Epstein construction behaves well under a normal flow.

Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\text{Ep}_{e^{2t}\sigma}$.



The Poincaré Family

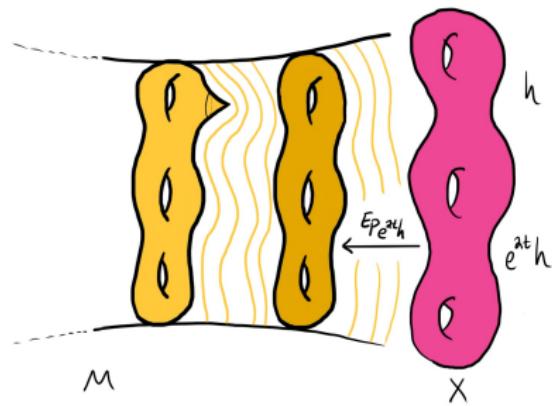
We call the unique hyperbolic conformal metric h on X the Poincaré metric and its Epstein surface $\text{Ep}_h : X \rightarrow M$ the Poincaré surface of X .



The Poincaré Family

We call the unique hyperbolic conformal metric h on X the Poincaré metric and its Epstein surface $\text{Ep}_h : X \rightarrow M$ the Poincaré surface of X .

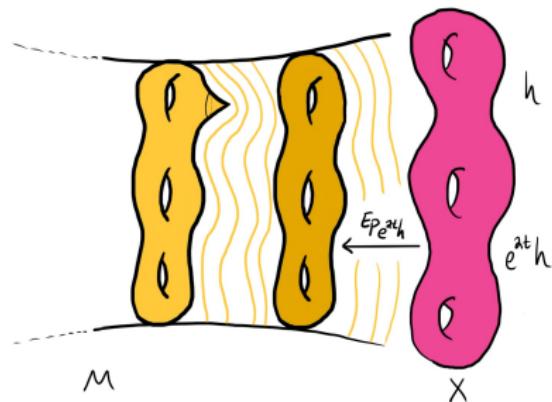
The family obtained by parallel flowing Ep_h we call the Poincaré Family, and this family is given as $(\text{Ep}_{e^{2t}h})_{t \geq 0}$.



The Poincaré Family

We call the unique hyperbolic conformal metric h on X the Poincaré metric and its Epstein surface $\text{Ep}_h : X \rightarrow M$ the Poincaré surface of X .

The family obtained by parallel flowing Ep_h we call the Poincaré Family, and this family is given as $(\text{Ep}_{e^{2t}h})_{t \geq 0}$.



For $t > c = c(\phi, h)$, the surfaces $\text{Ep}_{e^{2t}h}$ are embedded, and so for t sufficiently large, this family forms a foliation of the end of M .

The Poincaré Family

Notice that the family of conformal metrics $e^{2t} h$ for the Poincaré family has the property that it can be rescaled to be equal to h for all t .

The Poincaré Family

Notice that the family of conformal metrics $e^{2t}h$ for the Poincaré family has the property that it can be rescaled to be equal to h for all t .

Specifically, define $\epsilon = e^{-2t}$ then the conformal metrics ρ_ϵ of the Poincaré family satisfy $\epsilon\rho_\epsilon = h$ for all ϵ .

The Poincaré Family

Notice that the family of conformal metrics $e^{2t}h$ for the Poincaré family has the property that it can be rescaled to be equal to h for all t .

Specifically, define $\epsilon = e^{-2t}$ then the conformal metrics ρ_ϵ of the Poincaré family satisfy $\epsilon\rho_\epsilon = h$ for all ϵ .

We can consider weakening this condition to only ask that $\epsilon\rho_\epsilon \rightarrow h$ as $\epsilon \rightarrow 0$.

The Poincaré Family

Notice that the family of conformal metrics $e^{2t}h$ for the Poincaré family has the property that it can be rescaled to be equal to h for all t .

Specifically, define $\epsilon = e^{-2t}$ then the conformal metrics ρ_ϵ of the Poincaré family satisfy $\epsilon\rho_\epsilon = h$ for all ϵ .

We can consider weakening this condition to only ask that $\epsilon\rho_\epsilon \rightarrow h$ as $\epsilon \rightarrow 0$.

The families of surfaces we consider are in this sense asymptotic to the Poincaré family.

Asymptotically Poincaré Families

Definition

Let S_ϵ for $\epsilon \in (0, 1)$ be a family of embedded Epstein surfaces for the conformal metrics $\sigma(\epsilon)$. We call this family asymptotically Poincaré if

Asymptotically Poincaré Families

Definition

Let S_ϵ for $\epsilon \in (0, 1)$ be a family of embedded Epstein surfaces for the conformal metrics $\sigma(\epsilon)$. We call this family asymptotically Poincaré if

1. there exists a scaling function $f : [0, 1) \rightarrow [0, \infty)$ so that the path

$$f\sigma : (0, 1) \rightarrow \text{Met}^\infty(X)$$

is differentiable and converges to h , as $\epsilon \rightarrow 0$, in the space $\text{Met}^\infty(X)$ of smooth metrics on X ,

Asymptotically Poincaré Families

Definition

Let S_ϵ for $\epsilon \in (0, 1)$ be a family of embedded Epstein surfaces for the conformal metrics $\sigma(\epsilon)$. We call this family asymptotically Poincaré if

1. there exists a scaling function $f : [0, 1) \rightarrow [0, \infty)$ so that the path

$$f\sigma : (0, 1) \rightarrow \text{Met}^\infty(X)$$

is differentiable and converges to h , as $\epsilon \rightarrow 0$, in the space $\text{Met}^\infty(X)$ of smooth metrics on X ,

2. the function f is smooth, $f(0) = 0$, and $f'(0) \neq 0$,

Asymptotically Poincaré Families

Definition

Let S_ϵ for $\epsilon \in (0, 1)$ be a family of embedded Epstein surfaces for the conformal metrics $\sigma(\epsilon)$. We call this family asymptotically Poincaré if

1. there exists a scaling function $f : [0, 1) \rightarrow [0, \infty)$ so that the path

$$f\sigma : (0, 1) \rightarrow \text{Met}^\infty(X)$$

is differentiable and converges to h , as $\epsilon \rightarrow 0$, in the space $\text{Met}^\infty(X)$ of smooth metrics on X ,

2. the function f is smooth, $f(0) = 0$, and $f'(0) \neq 0$,
3. the continuous extension $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$ of $f\sigma$ is differentiable.

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ be an asymptotically Poincaré family for the conformal metrics $\sigma(\epsilon)$. There exists a $\delta > 0$ (depending on the family) such that for $\epsilon \in (0, \delta)$ the surfaces S_ϵ foliate the end of M .

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ be an asymptotically Poincaré family for the conformal metrics $\sigma(\epsilon)$. There exists a $\delta > 0$ (depending on the family) such that for $\epsilon \in (0, \delta)$ the surfaces S_ϵ foliate the end of M .

Proof.

- ▶ The 1-parameter family of associated Epstein maps gives a map $\text{Ep}_\sigma : [0, 1) \times X \rightarrow M \cup X$ that restricts to the identity on the boundary $\{0\} \times X \rightarrow X$.

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ be an asymptotically Poincaré family for the conformal metrics $\sigma(\epsilon)$. There exists a $\delta > 0$ (depending on the family) such that for $\epsilon \in (0, \delta)$ the surfaces S_ϵ foliate the end of M .

Proof.

- ▶ The 1-parameter family of associated Epstein maps gives a map $\text{Ep}_\sigma : [0, 1) \times X \rightarrow M \cup X$ that restricts to the identity on the boundary $\{0\} \times X \rightarrow X$.
- ▶ The function given by $E(\epsilon, x) = \text{Ep}_\sigma(\epsilon^2, x)$ has invertible derivative on the boundary and so it is a local C^1 -diffeomorphism from a neighborhood of the boundary into $M \cup X$.

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ be an asymptotically Poincaré family for the conformal metrics $\sigma(\epsilon)$. There exists a $\delta > 0$ (depending on the family) such that for $\epsilon \in (0, \delta)$ the surfaces S_ϵ foliate the end of M .

Proof.

- ▶ The 1-parameter family of associated Epstein maps gives a map $\text{Ep}_\sigma : [0, 1) \times X \rightarrow M \cup X$ that restricts to the identity on the boundary $\{0\} \times X \rightarrow X$.
- ▶ The function given by $E(\epsilon, x) = \text{Ep}_\sigma(\epsilon^2, x)$ has invertible derivative on the boundary and so it is a local C^1 -diffeomorphism from a neighborhood of the boundary into $M \cup X$.
- ▶ Therefore, the restriction of E to $[0, \delta^2) \times X$, for some small enough δ , is a diffeomorphism onto a collar neighborhood of X in $M \cup X$.

Main Results

Theorem (Q.)

Let S_ϵ for $\epsilon \in (0, 1)$ be an asymptotically Poincaré family of surfaces for the conformal metrics $\sigma(\epsilon)$. If h is the Poincaré metric of X and ϕ the holomorphic quadratic differential at infinity, then in Teichmüller space we have

$$[I(\sigma(\epsilon))] \rightarrow [h] \quad \text{and} \quad [II(\sigma(\epsilon))] \rightarrow [h] \quad \text{as } \epsilon \rightarrow 0,$$

where $[g]$ denotes the point in $\mathcal{T}(X)$ represented by the Riemannian metric g .

Moreover, the tangent vectors at $\epsilon = 0$ in $T_{[h]}\mathcal{T}(X)$ are given by

$$[\dot{I}(\sigma(\epsilon))] = 4f'(0)\operatorname{Re}(\phi) \quad \text{and} \quad [\dot{II}(\sigma(\epsilon))] = 0.$$

Riemannian Model of $\mathcal{T}(X)$

We model $\mathcal{T}(X)$ as the quotient

$$\text{Met}^\infty(X)/(\text{Diff}_0^\infty(X) \ltimes P^\infty(X))$$

of smooth metrics on X by the semi-direct product of $\text{Diff}_0^\infty(X)$, the group of smooth diffeomorphisms isotopic to the identity, and $P^\infty(X)$, the group of smooth positive functions on X .

Riemannian Model of $\mathcal{T}(X)$

We model $\mathcal{T}(X)$ as the quotient

$$\text{Met}^\infty(X)/(\text{Diff}_0^\infty(X) \ltimes P^\infty(X))$$

of smooth metrics on X by the semi-direct product of $\text{Diff}_0^\infty(X)$, the group of smooth diffeomorphisms isotopic to the identity, and $P^\infty(X)$, the group of smooth positive functions on X .

However, we will first work with Sobolev tensors.

Riemannian Model of $\mathcal{T}(X)$

We model $\mathcal{T}(X)$ as the quotient

$$\text{Met}^\infty(X)/(\text{Diff}_0^\infty(X) \ltimes P^\infty(X))$$

of smooth metrics on X by the semi-direct product of $\text{Diff}_0^\infty(X)$, the group of smooth diffeomorphisms isotopic to the identity, and $P^\infty(X)$, the group of smooth positive functions on X .

However, we will first work with Sobolev tensors.

Denote by $\text{Met}^s(X)$ the set of Riemannian metrics of Sobolev class H^s for a fixed $s > 3$ and note that $\text{Met}^\infty(X) = \cap_{s>3} \text{Met}^s(X)$

With these regularity assumptions $T_h \text{Met}^s(X)$ is a Hilbert space.

Riemannian Model of $\mathcal{T}(X)$

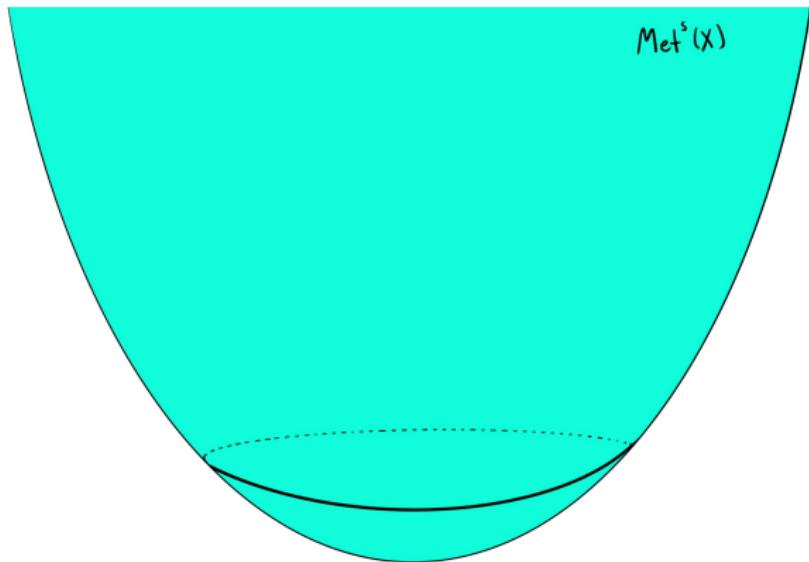
$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

Riemannian Model of $\mathcal{T}(X)$

$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

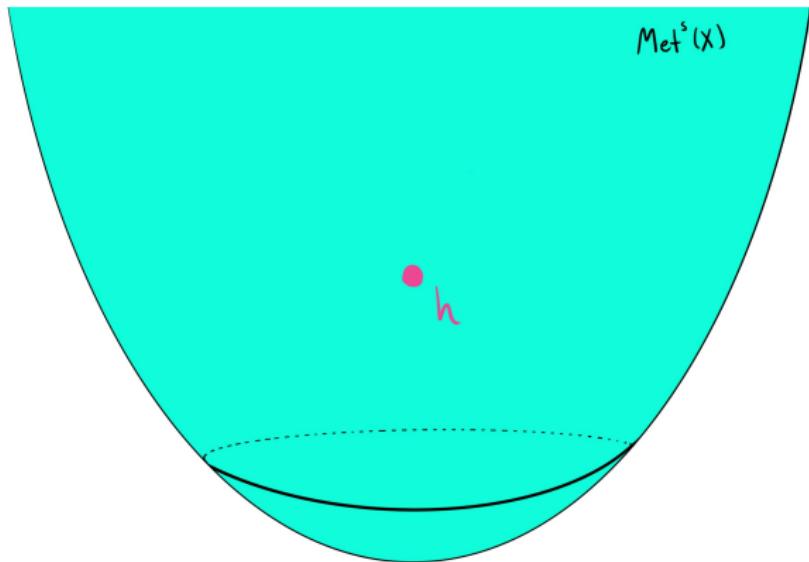


$$\Gamma^s(\Sigma^2)$$

Riemannian Model of $\mathcal{T}(X)$

$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

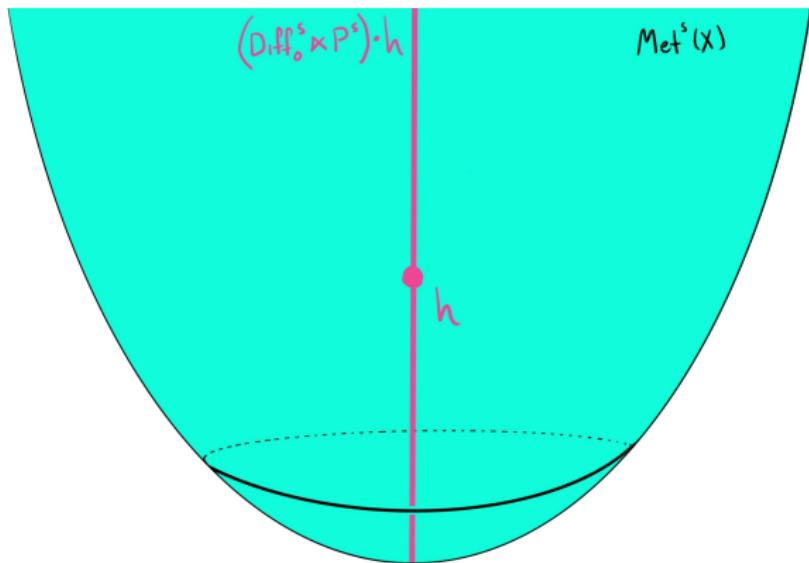


$$\Gamma^s(\Sigma^2)$$

Riemannian Model of $\mathcal{T}(X)$

$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

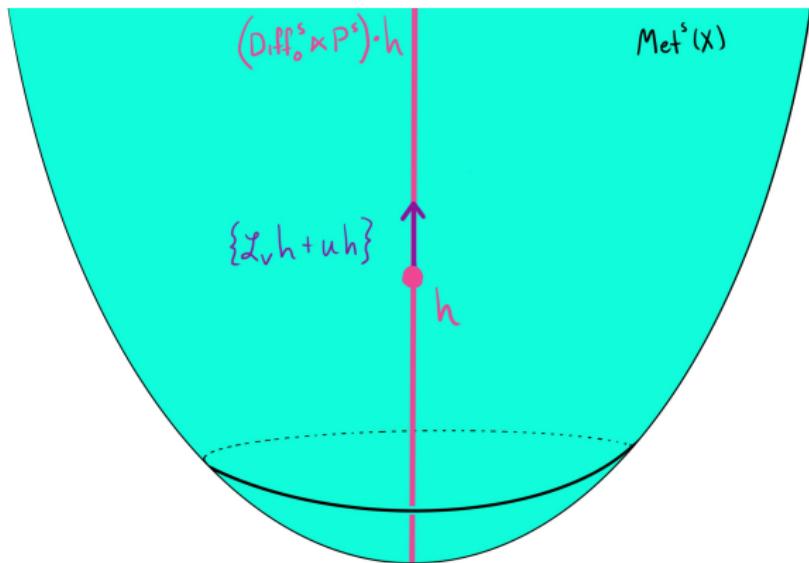


$$\Gamma^s(\Sigma^2)$$

Riemannian Model of $\mathcal{T}(X)$

$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

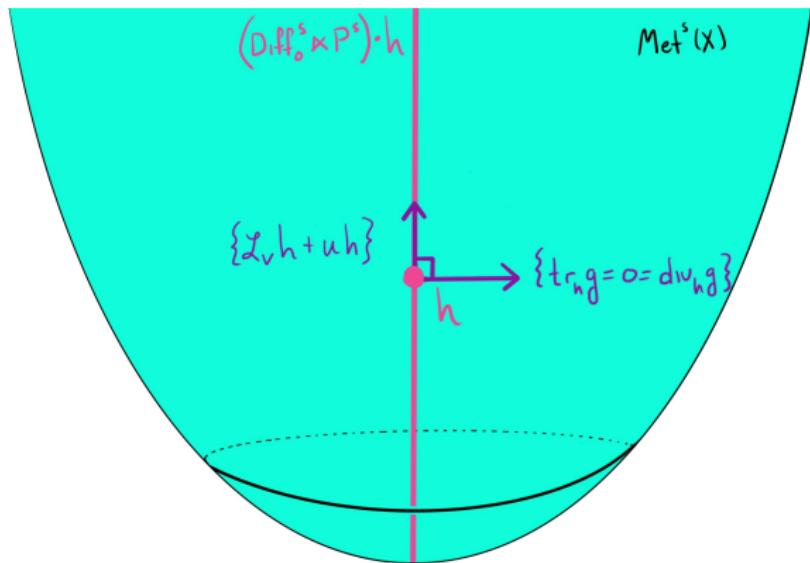


$$\Gamma^s(\Sigma^2)$$

Riemannian Model of $\mathcal{T}(X)$

$T_h \text{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$



$$\Gamma^s(\Sigma)$$

Riemannian Model of $\mathcal{T}(X)$

The set $\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\}$ is a model for the tangent space to the quotient $\text{Met}^s(X)/(\text{Diff}_0^s(X) \ltimes P^s(X))$ at $[h]$.

This is true for any s , and each such g is a smooth tensor. So this set is identified with the tangent space to Teichmüller space.

Riemannian Model of $\mathcal{T}(X)$

The set $\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\}$ is a model for the tangent space to the quotient $\text{Met}^s(X)/(\text{Diff}_0^s(X) \ltimes P^s(X))$ at $[h]$.

This is true for any s , and each such g is a smooth tensor. So this set is identified with the tangent space to Teichmüller space.

$$\begin{aligned} T_{[h]}\mathcal{T}(X) &= \{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \\ &= \{\text{Re}(\psi) \mid \psi \text{ a holomorphic quadratic differential on } X\}. \end{aligned}$$

Riemannian Model of $\mathcal{T}(X)$

The set $\{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\}$ is a model for the tangent space to the quotient $\text{Met}^s(X)/(\text{Diff}_0^s(X) \ltimes P^s(X))$ at $[h]$.

This is true for any s , and each such g is a smooth tensor. So this set is identified with the tangent space to Teichmüller space.

$$\begin{aligned} T_{[h]}\mathcal{T}(X) &= \{g \mid \text{tr}_h(g) = 0 = \text{div}_h(g)\} \\ &= \{\text{Re}(\psi) \mid \psi \text{ a holomorphic quadratic differential on } X\}. \end{aligned}$$

The projection $\text{Met}^\infty(X) \rightarrow \mathcal{T}(X)$ is continuous and its derivative at h is given by orthogonal projection onto $\{\text{Re}(\psi)\}$

Proof

Returning to our theorem, we are given that $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$ is continuous and differentiable at $\epsilon = 0$. Therefore, we also have that γ is continuous to $\text{Met}^s(X)$, for each s , and is differentiable at $\epsilon = 0$.

Proof

Returning to our theorem, we are given that $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$ is continuous and differentiable at $\epsilon = 0$. Therefore, we also have that γ is continuous to $\text{Met}^s(X)$, for each s , and is differentiable at $\epsilon = 0$.

In $\text{Met}^s(X)$ we can write

$$I(\sigma(\epsilon)) = \frac{1}{4\epsilon f'(0)}h + \frac{1}{4f'(0)}\dot{y} + \frac{1}{2}h + \text{Re}(\phi) + O(\epsilon),$$

Proof

Returning to our theorem, we are given that $\gamma : [0, 1) \rightarrow \text{Met}^\infty(X)$ is continuous and differentiable at $\epsilon = 0$. Therefore, we also have that γ is continuous to $\text{Met}^s(X)$, for each s , and is differentiable at $\epsilon = 0$.

In $\text{Met}^s(X)$ we can write

$$I(\sigma(\epsilon)) = \frac{1}{4\epsilon f'(0)}h + \frac{1}{4f'(0)}\dot{\gamma} + \frac{1}{2}h + \text{Re}(\phi) + O(\epsilon),$$

so that

$$I_\epsilon := 4\epsilon f'(0)I(\sigma(\epsilon)) = h + \epsilon(\dot{\gamma} + 2f'(0)h + 4f'(0)\text{Re}(\phi)) + O(\epsilon^2).$$

Proof

From the expression

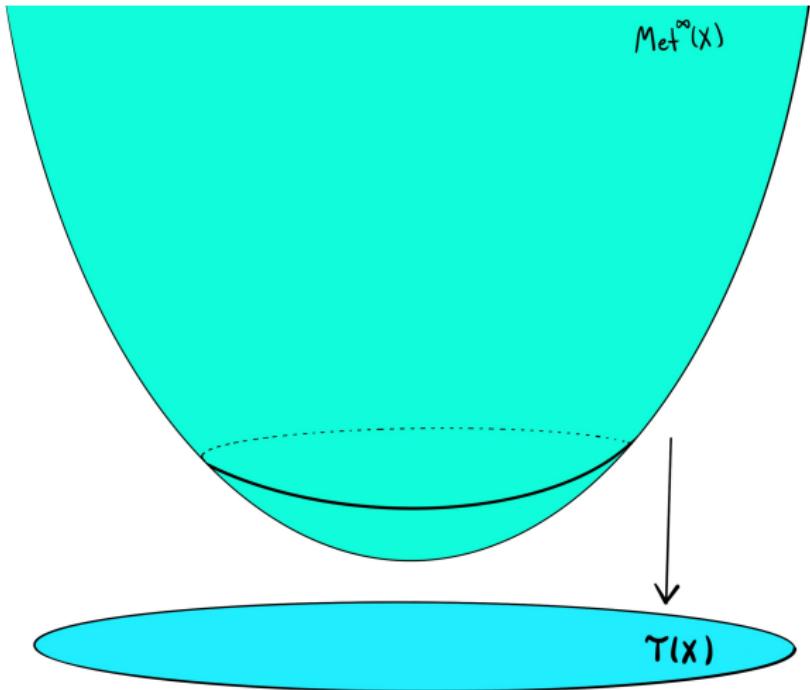
$$I_\epsilon = h + \epsilon(\dot{\gamma} + 2f'(0)h + 4f'(0)\operatorname{Re}(\phi)) + O(\epsilon^2)$$

we can see that $I_\epsilon \rightarrow h$ in $\operatorname{Met}^s(X)$ for all s , implying $I_\epsilon \rightarrow h$ in $\operatorname{Met}^\infty(X)$.

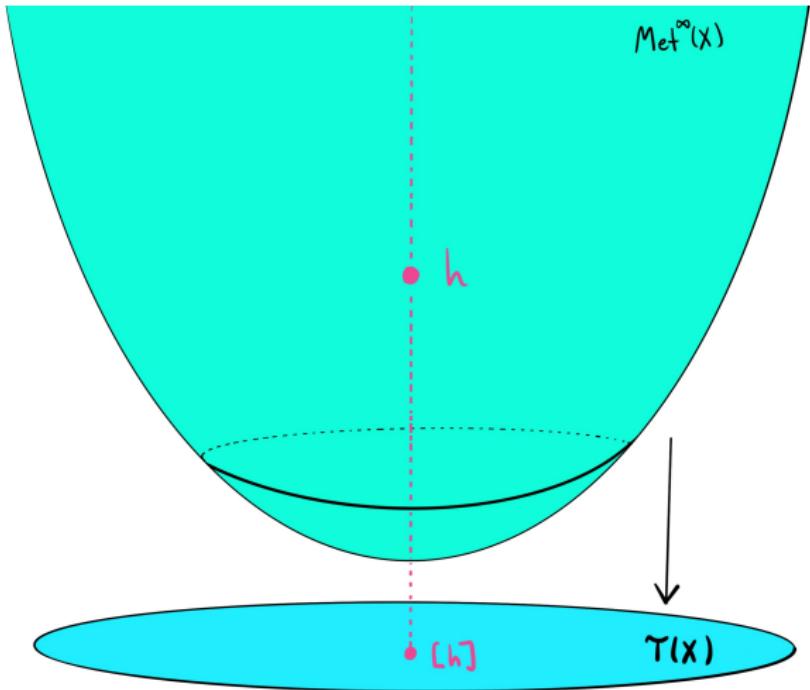
Moreover, the derivative at $\epsilon = 0$ is

$$\dot{I}_\epsilon = \dot{\gamma} + 2f'(0)h + 4f'(0)\operatorname{Re}(\phi).$$

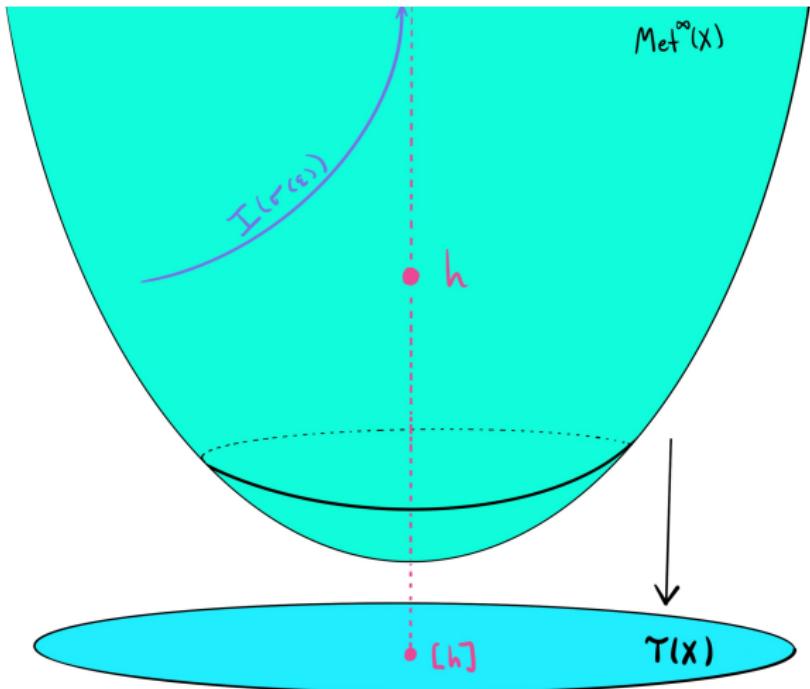
Proof



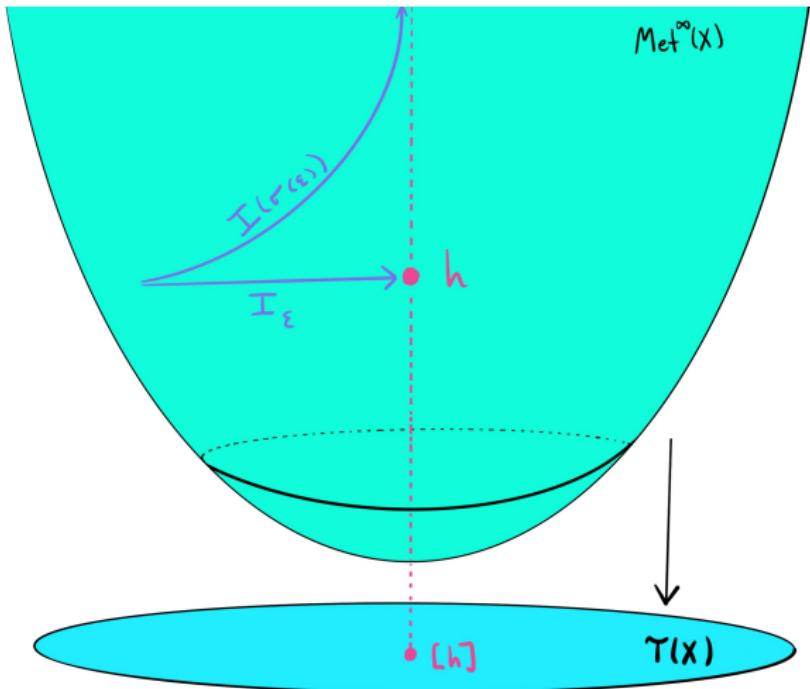
Proof



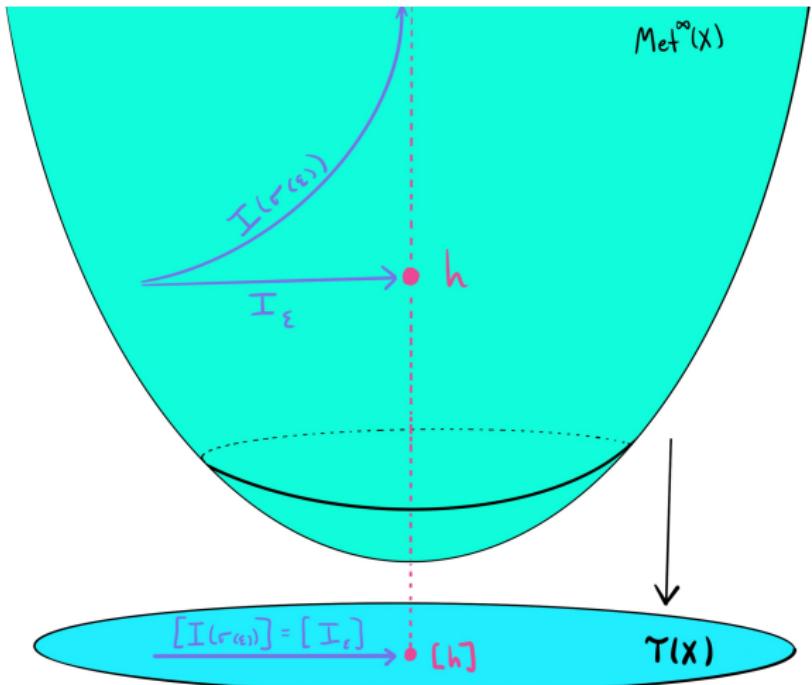
Proof



Proof



Proof



Therefore, as $I_\epsilon \rightarrow h$ in $\text{Met}^\infty(X)$ we have that $[I(\sigma(\epsilon))] = [I_\epsilon] \rightarrow [h]$ in $\mathcal{T}(X)$.

Therefore, as $I_\epsilon \rightarrow h$ in $\text{Met}^\infty(X)$ we have that $[I(\sigma(\epsilon))] = [I_\epsilon] \rightarrow [h]$ in $\mathcal{T}(X)$.

And since

$$\dot{I}_\epsilon = \dot{\gamma} + 2f'(0)h + 4f'(0)\text{Re}(\phi),$$

we see

$$[\dot{I}(\sigma(\epsilon))] = 4f'(0)\text{Re}(\phi).$$

Therefore, as $I_\epsilon \rightarrow h$ in $\text{Met}^\infty(X)$ we have that $[I(\sigma(\epsilon))] = [I_\epsilon] \rightarrow [h]$ in $\mathcal{T}(X)$.

And since

$$\dot{I_\epsilon} = \dot{\gamma} + 2f'(0)h + 4f'(0)\text{Re}(\phi),$$

we see

$$[\dot{I(\sigma(\epsilon))}] = 4f'(0)\text{Re}(\phi).$$

If we apply the same arguments to the Taylor series for $\dot{II}(\sigma(\epsilon))$ then we see $[\dot{II}(\sigma(\epsilon))] \rightarrow [h]$ and $[\dot{II}(\sigma(\epsilon))] = 0$.



Applications

We apply this result to two foliations by constant curvature surfaces.

Applications

We apply this result to two foliations by constant curvature surfaces.

Theorem (Labourie, 1992)

Let E be an end of a quasi-Fuchsian manifold M , then for each $k \in (-1, 0)$, there exists a unique (incompressible) surface embedded in E with constant Gaussian curvature k . Moreover, this family of surfaces foliates the end E .

Applications

We apply this result to two foliations by constant curvature surfaces.

Theorem (Labourie, 1992)

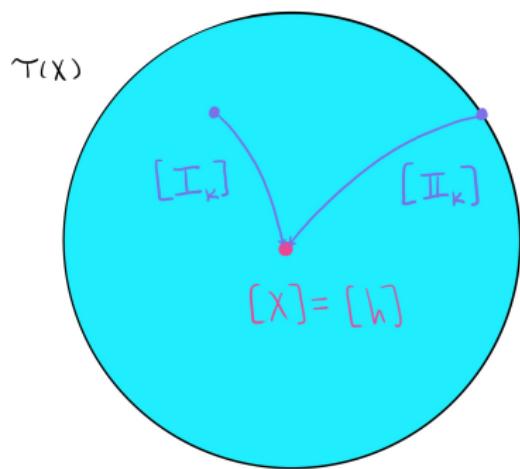
Let E be an end of a quasi-Fuchsian manifold M , then for each $k \in (-1, 0)$, there exists a unique (incompressible) surface embedded in E with constant Gaussian curvature k . Moreover, this family of surfaces foliates the end E .

Theorem (Mazzeo-Pacard, 2011)

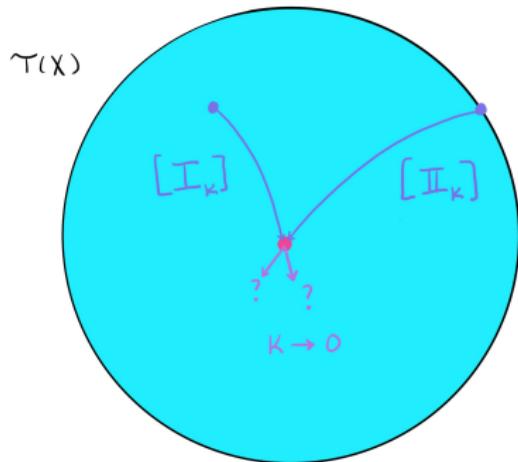
Each end of a quasi-Fuchsian manifold admits a unique foliation by constant mean curvature surfaces.

A Conjecture of Labourie

Labourie called the constant Gaussian curvature surfaces k -surfaces and he discusses how their first and second fundamental forms may be considered as paths in Teichmüller space.



A Conjecture of Labourie



He shows that as $k \rightarrow 0$, the paths $[I_k]$ and $[II_k]$ converge to $[X] = [h]$ and he asks after the tangent vectors to these paths at $k = 0$.

He conjectures $[I_k]$ is related to the holomorphic quadratic differential ϕ .

A Conjecture of Labourie

Theorem (Q.)

Let I_k and Π_k be the first and second fundamental forms of the family of k -surfaces in an end of a quasi-Fuchsian manifold. Let ϕ be the holomorphic quadratic differential at infinity of M . Then, as $k \rightarrow 0$, the tangent vectors to $[I_k]$ and $[\Pi_k]$ in Teichmüller space are given by

$$[\dot{I}_k] = -\operatorname{Re}(\phi) \quad \text{and} \quad [\dot{\Pi}_k] = 0.$$

A Conjecture of Labourie

Theorem (Q.)

Let I_k and II_k be the first and second fundamental forms of the family of k -surfaces in an end of a quasi-Fuchsian manifold. Let ϕ be the holomorphic quadratic differential at infinity of M . Then, as $k \rightarrow 0$, the tangent vectors to $[I_k]$ and $[II_k]$ in Teichmüller space are given by

$$[\dot{I}_k] = -\operatorname{Re}(\phi) \quad \text{and} \quad [\dot{II}_k] = 0.$$

This will follow from our main theorem if we can show the k -surfaces form an asymptotically Poincaré family. To do this we must present the k -surfaces as Epstein surfaces.

Proof

The Gaussian curvature of an Epstein surface for a conformal metric σ can be calculated as

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}}.$$

Proof

The Gaussian curvature of an Epstein surface for a conformal metric σ can be calculated as

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}}.$$

Here

- ▶ $K(\sigma)$ is the Gaussian curvature of σ itself.

Proof

The Gaussian curvature of an Epstein surface for a conformal metric σ can be calculated as

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}}.$$

Here

- ▶ $K(\sigma)$ is the Gaussian curvature of σ itself.
- ▶ $B(\sigma)$ is the Schwarzian tensor of σ . This is a quadratic differential (not necessarily holomorphic) depending on σ and its first two derivatives.

Proof

The Gaussian curvature of an Epstein surface for a conformal metric σ can be calculated as

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}}.$$

Here

- ▶ $K(\sigma)$ is the Gaussian curvature of σ itself.
- ▶ $B(\sigma)$ is the Schwarzian tensor of σ . This is a quadratic differential (not necessarily holomorphic) depending on σ and its first two derivatives.

To find a k -surface, we therefore need to solve $K(I(\sigma)) = k$ for σ .

Proof

We focus on solving the equivalent equation,

$$4K(\sigma) = k \left((1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2} \right) \quad (*)$$

Proof

We focus on solving the equivalent equation,

$$4K(\sigma) = k \left((1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2} \right) \quad (*)$$

Our technique is to use the Implicit Function Theorem to obtain solutions for each k near 0.

Notice though, that when $k = 0$ this reads $K(\sigma) = 0$, which has no solutions on X .

Proof

Since there are no solutions for $k = 0$ we rescale the equation by considering the function $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$.

Proof

Since there are no solutions for $k = 0$ we rescale the equation by considering the function $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$.

If σ solves $(*)$ then $\tau = f(k)\sigma$ solves

$$(2 + k)(1 + K(\tau))^2 + 2\sqrt{1 + k} \left(1 - K(\tau)^2\right) \\ + 16 \left(2\sqrt{1 + k} - 2 - k\right) \frac{|B(\tau)|^2}{\tau^2} = 0$$

or

$$F(k, \tau) = 0.$$

Proof

Since there are no solutions for $k = 0$ we rescale the equation by considering the function $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$.

If σ solves $(*)$ then $\tau = f(k)\sigma$ solves

$$(2 + k)(1 + K(\tau))^2 + 2\sqrt{1 + k} \left(1 - K(\tau)^2\right) \\ + 16 \left(2\sqrt{1 + k} - 2 - k\right) \frac{|B(\tau)|^2}{\tau^2} = 0$$

or

$$F(k, \tau) = 0.$$

Note that $F(0, \tau) = 0$ is equivalent to $K(\tau) = -1$, which has the solution $\tau = h$.

Proof

Since we have a solution for $k = 0$, we can then use the Implicit Function Theorem to get solution for k near zero as well.

The function F , while defined on smooth conformal metrics $\text{Conf}^\infty(X)$, extends to one of Sobolev conformal metrics $\text{Conf}^s(X)$. We have

$$F : (-1, 0) \times \text{Conf}^s(X) \rightarrow H^{s-2}(X).$$

Proof

Since we have a solution for $k = 0$, we can then use the Implicit Function Theorem to get solution for k near zero as well.

The function F , while defined on smooth conformal metrics $\text{Conf}^\infty(X)$, extends to one of Sobolev conformal metrics $\text{Conf}^s(X)$. We have

$$F : (-1, 0) \times \text{Conf}^s(X) \rightarrow H^{s-2}(X).$$

We compute

$$DF_{(0,h)}(\dot{k}, \dot{\tau}) = 4DK_h(\dot{\tau}) = -2(\Delta_h - Id)\frac{\dot{\tau}}{h}$$

and see that $D_2F_{(0,h)} = 4DK_h : H^s(X) \rightarrow H^{s-2}(X)$ is an isomorphism.

Proof

Consequently, there exists a neighborhood V of 0 and a curve $\gamma : V \rightarrow \text{Conf}^s(X)$ with $\gamma(0) = h$ and $F(k, \gamma(k)) = 0$.

Proof

Consequently, there exists a neighborhood V of 0 and a curve $\gamma : V \rightarrow \text{Conf}^s(X)$ with $\gamma(0) = h$ and $F(k, \gamma(k)) = 0$.

Elliptic regularity theory applied to DK_h tells us that there is a $\delta > 0$ such that

- ▶ $\gamma(k)$ is a smooth conformal metric for each $k \in (-\delta, \delta)$,

Proof

Consequently, there exists a neighborhood V of 0 and a curve $\gamma : V \rightarrow \text{Conf}^s(X)$ with $\gamma(0) = h$ and $F(k, \gamma(k)) = 0$.

Elliptic regularity theory applied to DK_h tells us that there is a $\delta > 0$ such that

- ▶ $\gamma(k)$ is a smooth conformal metric for each $k \in (-\delta, \delta)$,
- ▶ the path $\gamma : (-\delta, \delta) \rightarrow \text{Conf}^\infty(X)$ is a smooth path.

Proof

Consequently, there exists a neighborhood V of 0 and a curve $\gamma : V \rightarrow \text{Conf}^s(X)$ with $\gamma(0) = h$ and $F(k, \gamma(k)) = 0$.

Elliptic regularity theory applied to DK_h tells us that there is a $\delta > 0$ such that

- ▶ $\gamma(k)$ is a smooth conformal metric for each $k \in (-\delta, \delta)$,
- ▶ the path $\gamma : (-\delta, \delta) \rightarrow \text{Conf}^\infty(X)$ is a smooth path.

To complete the proof we take $\sigma(k) := f(k)^{-1}\gamma(k)$ for $k \in (-\delta, 0)$.

The family of Epstein surfaces for $\sigma(k)$ then satisfy the definition of an asymptotically Poincaré family and our main theorem applies. □

Constant Mean Curvature Surfaces

Theorem (Mazzeo-Pacard, 2011)

The ends of a quasi-Fuchsian manifold admit unique foliations by constant mean curvature surfaces.

Theorem (Q.)

Let I_k and II_k be the first and second fundamental forms of the Epstein surface with constant mean curvature $-\sqrt{1+k}$. Let ϕ be the holomorphic quadratic differential at infinity. Then, as $k \rightarrow 0$, the tangent vectors to $[I_k]$ and $[II_k]$ in Teichmüller space are given by

$$[I_k] = -\operatorname{Re}(\phi) \quad \text{and} \quad [II_k] = 0.$$

Proof

The proof proceeds the same as in the constant Gaussian curvature case. However, here we wish to solve the equation

$$H(\text{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16|B(\sigma)|^2\sigma^{-2}}{(K(\sigma) - 1)^2 - 16|B(\sigma)|^2\sigma^{-2}} = -\sqrt{1 + k}.$$

Proof

The proof proceeds the same as in the constant Gaussian curvature case. However, here we wish to solve the equation

$$H(\text{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16|B(\sigma)|^2\sigma^{-2}}{(K(\sigma) - 1)^2 - 16|B(\sigma)|^2\sigma^{-2}} = -\sqrt{1 + k}.$$

Using the same scaling function $f(k)$ we are led to consider solutions to a function $G(k, \tau) = 0$, which has partial derivative

$$D_2G_{(0,h)} = 2DK_h(\dot{\tau}).$$

Proof

The proof proceeds the same as in the constant Gaussian curvature case. However, here we wish to solve the equation

$$H(\text{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16|B(\sigma)|^2\sigma^{-2}}{(K(\sigma) - 1)^2 - 16|B(\sigma)|^2\sigma^{-2}} = -\sqrt{1 + k}.$$

Using the same scaling function $f(k)$ we are led to consider solutions to a function $G(k, \tau) = 0$, which has partial derivative

$$D_2G_{(0,h)} = 2DK_h(\dot{\tau}).$$

The Implicit Function Theorem then gives solutions for k near zero. □

Thank you!

