SUPER COOL THESIS

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1. Introduction

2. Preliminaries

2.1. **The Geometry of Surfaces.** Let S be a closed, oriented, smooth surface. A Riemannian metric g on S is a smooth section of the symmetric tensor product of the cotangent bundle of S, i.e., $g \in \Gamma(\Sigma^2 T^*S)$, such that at each point, the associated symmetric bilinear map $g(p) : T_pS \times T_pS \to \mathbb{R}$ is an inner product. The non-degeneracy of g at each point allows us to identity TS with T^*S via the map that sends $v \in T_pS$ to $g_p(\cdot, v) \in T_p^*S$ and this extends to an identification of vector fields and 1-forms. We will abuse notation and write $g : TS \to T^*S$ for this map and $g^{-1} : T^*S \to TS$ for its inverse. A Riemannian metric induces a (pointwise) norm on vector field in the familiar way by $|X| = g(X, X)^{1/2}$. It induces metrics and norms on all tensor products of TS and T^*S. It also distinguishes a volume form $dVol(g)$ on S that evaluates to $+1$ on all oriented orthonormal frames. Therefore the integral of a smooth function f is well defined by $\int_S f \; dVol(g)$.

The Levi-Civita connection ∇ of g on S is the unique torsion-free, metric connection on TS and it induces connections on T^*S and tensor products of the two. We will denote all induced connections by ∇., i.e.,

$$T^\nabla = 0 \text{ and } \nabla g = 0.$$

Given a torsion-free connection ∇ on S and a q-form ω with values in TS, the exterior covariant derivative of ω is

$$d^\nabla \omega = \mathrm{Alt}(\nabla \omega).$$

By considering vector fields as 0-forms with values in TS we define the Riemann curvature endomorphism of ∇ as the 2-form with values in $\mathrm{End}(TS)$ given by

$$R^\nabla(X, Y)Z = (d^\nabla)^2 Z\,(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemann curvature tensor is then

$$Rm(X, Y, Z, W) = g(R^\nabla(X, Y)Z, W).$$

and the Gaussian curvature of S at a point p is the function $K(g)$ defined by

$$K(g)(p) = \frac{Rm(v, w, w, v)}{\|v\|^2\|w\|^2 - g(v, w)}$$

where v, w is any basis for T_pS. In the 3 dimensional setting of M, the sectional curvature of $\tilde{g}$ is a function on 2-planes in the tangent bundle, i.e., a map $sec : \mathrm{Gr}_2(TM) \to \mathbb{R}$ whose value on a plane $\Pi \leq T_pM$ is the Gaussian curvature at p of the image of Π in M under the exponential map. It may be computed with the above formula with v and w now a basis for Π.

Suppose $(M, \tilde{g})$ is a 3-dimensional Riemannian manifold. When $f : S \to M$ is an immersion, the pullback tensor $I = f^*\tilde{g}$ is a Riemannian metric on S called the First Fundamental Form. If we regard f as an identification of S with its image in M, then we may also identify T_pS with its image in $T_{f(p)}M$. We therefore have a $\tilde{g}$-orthogonal splitting

$$T_pM = T_pS \oplus N_pS.$$

Here, N_pS is the normal space to the surface in M given by all vectors orthogonal to S at p. The disjoint union of the normal spaces forms a vector bundle (in this case a line bundle) called the Normal Bundle of S. By our assumptions, there is a unit normal vector field n, that is $n \in \Gamma(NS)$, so that S is co-oriented in M.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$ on M. By extending vector fields X and Y on S to a neighborhood of S in M, we may take the covariant derivative $\widetilde{\nabla}_X Y$. This resulting vector field is not necessarily also tangent to S and so we may decompose it into its tangential part $(\widetilde{\nabla}_X Y)^\top$ and its normal part $(\widetilde{\nabla}_X Y)^\perp$. The Gauss Formula tells us $(\widetilde{\nabla}_X Y)^\top = \nabla_X Y$ and since the normal bundle is spanned by n we have $(\widetilde{\nabla}_X Y)^\perp = I\!I(X, Y)n$ for a symmetric 2-tensor field $I\!I \in \Gamma(\Sigma^2 T^*S)$. This $I\!I$ is called the Second Fundamental Form.

The Gauss Equation relates the Gaussian curvature K of S to the sectional curvature of the ambient manifold M via the first and second fundamental forms. Since $I\!I$ is symmetric and since I is non-degenerate, we may form the shape operator $I^{-1}I\!I : TM \to TM$. The Gauss Equation states

$$K(I) = sec(TS) + \det(I^{-1}I\!I).$$

If M has constant sectional curvature, then the second fundamental form satisfies the Codazzi Equation $d^\nabla(I^{-1}I\!I) = 0$. Together the Gauss and Codazzi Equations form the integrability conditions for simply connected surfaces in constant curvature 3-manifolds:

Theorem 2.1 (Bonnet, Fundamental Theorem of Surface Theory)**.** *Suppose S is a simply connected surface, I a Riemannian metric on S, and $I\!I$ a 2-tensor on S such that I and $I\!I$ satisfy the Gauss-Codazzi equations*

$$\begin{aligned} K(I) &= \kappa + \det(I^{-1}I\!I) \\ d^\nabla(I^{-1}I\!I) &= 0. \end{aligned}$$

Then there is an isometric immersion $S \to \mathbb{M}^3_\kappa$ into the simply connected 3-manifold $\mathbb{M}^3_\kappa$ of constant sectional curvature κ, such that the second fundamental form of the immersion is $I\!I$. And this immersion is unique up to composition with an isometry of $\mathbb{M}^3_\kappa$.

Since our main interest is the hyperbolic setting $\mathbb{M}^3_\kappa = \mathbb{H}^3$, we note that the Gauss Equation reads

$$K(I) = -1 + \det(I^{-1}I\!I).$$

We also record here the definition of mean curvature. As opposed to the Gaussian curvature, this depends on the immersion $f : S \to M$ and is defined by

$$H(f) = \frac{1}{2}\mathrm{tr}(I^{-1}I\!I).$$

The Gaussian curvature is related to the global topology of S in the following sense.

Theorem 2.2 (Gauss-Bonnet)**.** *Let S be a closed, oriented, smooth surface, and g a Riemannian metric on S with Gaussian curvature $K(g)$. Then*

$$\int_S K(g)\ dVol(g) = 2\pi\chi(S).$$

The Gauss-Bonnet Theorem implies that if S has genus at least 2 (so that $\chi(S) < 0$), then there is no metric with curvature identically 0 or $+1$ on S.

2.2. **The Teichmüller Space of a Surface.** A complex structure on the topological surface S is a maximal atlas of charts to $\mathbb{C}$ whose transition functions are holomorphic. The surface S with a complex structure is called a Riemann surface and is a 1-dimensional complex manifold. Two Riemann surfaces X and Y with the same underlying smooth manifold S will be called equivalent if there is an $f \in \mathrm{Diff}_0(S)$ that is a biholomorphim $f : X \to Y$.

The space of equivalence classes of Riemann surface structures on S is called the Teichmüller space of S and is denoted by $\mathcal{T}(S)$. This space is itself a complex manifold with complex dimension $3 \cdot \mathrm{genus}(S) - 3$. The cotangent space $T^*_{[X]}\mathcal{T}(S)$ is naturally identified with the space of all symmetric 2-tensors ϕ on S which, in a holomorphic chart z on X, may be written as $\phi = qdz^2$ for q a holomorphic function (on the domain of z). Such tensors are called holomorphic quadratic differentials and we denote the space of them by $Q(X)$.

Given a Riemann surface X, there are distinguished Riemannian metrics on S—called conformal metrics—defined by the property that in complex chart z they may be written as $\sigma = e^{2\eta}|dz|^2$, for η a smooth function. We will denote the space of conformal metrics on X by $\mathrm{Conf}(X)$. A Riemann surface that is simply connected and biholomoprhic to a proper subset of $\mathbb{C}$ has a unique hyperbolic metric, called the Poincaré metric of Ω, that is conformal. The universal cover $\tilde{X}$ of a Riemann surface X with genus at least 2 is such a Riemann surface. Its Poincaré metric is invariant under the action of $\pi_1(X)$ (by uniqueness) and so induces a hyperbolic metric on X. Hence, for a fixed X of genus at least 2, there exists a unique hyperbolic conformal metric which we will usually denote by h. We therefore assume from now on that the genus of S is greater than or equal to 2.

A definition of Teichmüller space may be given solely in terms of conformal metrics. Indeed, any other conformal metric can be written as $\sigma = e^{2u}h$ for u a smooth function on X. This is to say, conformal metrics are those that are conformally equivalent to h. This leads us to consider the action of smooth positive functions on the set of Riemannian metrics on S, i.e., $P(S)$ acting on $\mathrm{Met}(S)$. The action of $\mathrm{Diff}_0(S)$ on $\mathrm{Met}(S)$ may be packaged together with that of $P(S)$ as the action of the semi-direct product $\mathrm{Diff}_0(S) \times P(S)$. The quotient space is another model of Teichmüller space

$$\mathrm{Met}(S)/(\mathrm{Diff}_0(S) \times P(S)) = \mathcal{T}(S).$$

We have so far proceeded formally and we will continue doing so for the rest of this section. But a word of warning: the space $\mathrm{Met}(S)$ of smooth Riemannian metrics on S is an infinite dimensional Fréchet manifold and we are dealing with an action on it by an infinite dimensional Lie group $\mathrm{Diff}_0(S)$. There are many subtleties here that are best dealt with by using Sobolev spaces and universal properties. We will do this in later sections when they become relevant.

The set $\mathrm{Met}(S)$ is an open cone in the vector space of symmetric 2-tensors on S and hence is a manifold (of infinite dimension). The tangent space to $\mathrm{Met}(S)$ at a metric g is then naturally identified with this vector space. Given two tensors σ_1 and σ_2 in $T_g\mathrm{Met}(S)$, we may turn them into endomorphisms $g^{-1}\sigma_1$ and $g^{-1}\sigma_2$ and take their Frobenius inner product $\mathrm{tr}(g^{-1}\sigma_1 \circ g^{-1}\sigma_2)$, which is a function on S. Using integration we can define a Riemannian metric on $\mathrm{Met}(S)$ by

$$\langle \sigma_1, \sigma_2 \rangle_g = \int_S \mathrm{tr}(g^{-1}\sigma_1 \circ g^{-1}\sigma_2)\ dVol(g)$$

(this metric induces a multiple of the Weil-Petersson metric on the quotient, but we will not use this (see [**?**])).

Using this metric we may decompose the tangent space to $\mathrm{Met}(S)$ as the direct sum of the tangent space to the $\mathrm{Diff}_0(S) \times P(S)$-orbit and its orthogonal complement. We have

$$T_g\mathrm{Met}(S) = \{\dot{g} \mid \mathrm{tr}_g(\dot{g}) = 0 = \mathrm{div}_g(\dot{g})\} \oplus \{\mathcal{L}_X g + fg \mid f \in C^\infty(S) \text{ and } X \in \Gamma(TS)\}.$$

Here $\mathrm{div}_g(\dot{g})$ is the divergence of $\dot{g}$ given by $\mathrm{tr}_g(\nabla \dot{g})$. The right summand is tangent to the group orbit at g. The left summand is its orthogonal complement and consists of trace-free and divergence-free symmetric 2-tensors, which are referred to as transverse-traceless tensors. Since they are orthogonal to the group orbit, they may be identified with the tangent space to the quotient

$$T_{[g]}\mathcal{T}(S) = T_g\mathrm{Met}(S)/T_g(\mathrm{Diff}_0(S) \times P(S) \cdot g) \simeq \{\dot{g} \mid \mathrm{tr}_g(\dot{g}) = 0 = \mathrm{div}_g(\dot{g})\},$$

and the derivative of the projection $\pi : \mathrm{Met}(S) \to \mathcal{T}(S)$ at g is given by the orthogonal projection onto these transverse-traceless tensors.

There is a more useful description of transverse-traceless tensors than their definition. It turns out (see [**?**]), the trace-free condition implies that they are the real part of quadratic differentials and the divergence-free condition says the quadratic differential is holomorphic (with respect to the complex structure for which g is conformal). We have

Lemma 2.3 ([**?**]). *Let X the a Riemann surface and g a conformal metric on X. If ϕ is a holomorphic quadratic differential, then* $\mathrm{Re}(\phi)$ *is a g-transverse-traceless tensor. That is*

$$T_{[g]}\mathcal{T}(S) = \{\mathrm{Re}(\phi) \mid \phi \in Q(X)\}.$$

2.3. **Complex Projective Structures.** A complex projective structure on a surface S is a geometric structure with group $\mathrm{PSL}_2\mathbb{C}$ and topological space $\mathbb{C}\mathrm{P}^1$. That is, it is a maximal atlas of charts to $\mathbb{C}\mathrm{P}^1$ whose transition functions are the restrictions of Möbius transformations. The surface S together with a complex projective structure we will call a complex projective surface, or just a projective surface for short. In particular, a projective surface does not refer to a submanifold of $\mathbb{C}\mathrm{P}^n$. Since Möbius transformations are holomorphic, a projective structure also induces a complex structure on the surface S. We now describe a parametrization of the set of all projective structures with the same underlying complex structure using the space of quadratic differentials on S that are holomorphic with respect to the fixed underlying complex structure. See [**?**] and [**?**] for more details.

Suppose U is an open subset of the complex plane $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ is a locally injective holomorphic function. For each $z \in U$, there is a unique Möbius transformation $M_f(z) \in \mathrm{PSL}_2\mathbb{C}$ that agrees with f at z to second order:

$$f(w) = M_f(z) \cdot w + o((w-z)^2).$$

This defines a map $M_f : U \to \mathrm{PSL}_2\mathbb{C}$ called the osculating Möbius transformation of f. Intuitively, the derivative of M_f is a measure of how far f is from being Möbius.

More precisely, the differential $dM_f : TU \to T\mathrm{PSL}_2\mathbb{C}$ takes values in the tangent bundle of $\mathrm{PSL}_2\mathbb{C}$, which is trivialized by left multiplication. By composing with left multiplication we can consider the Darboux derivative of M_f, a 1-form on U with values in $\mathrm{Lie}(\mathrm{PSL}_2\mathbb{C}) = \mathfrak{sl}_2\mathbb{C}$. See [**?**] for details. An explicit computation gives

$$M_f(z)^{-1} d(M_f)_z = \frac{1}{2}\left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right)\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix} dz.$$

The coefficient of the matrix in the Darboux derivative transforms as a quadratic differential. Hence, the Schwarzian Derivative of a locally injective holomorphic function is defined as

$$S(f)(z) = \left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) dz^2.$$

We see from its appearance in the Darboux derivative that f is locally a Möbius transformation if and only if $S(f) = 0$. Furthermore, a computation shows the Schwarzian chain rule is given by

$$S(f \circ g) = g^* S(f) + S(g).$$

A complex projective structure on a surface allows us to extend the Schwarzian derivative to holomorphic functions defined on the surface. Indeed, suppose $z : U \to \mathbb{C}$ is a projective chart on S, we can define the Schwarzian of $f : S \to \mathbb{C}$ on U by $z^* S(f \circ z^{-1})$. When w is another projective chart overlapping with z, we have

$$\begin{aligned} z^* S(f \circ z^{-1}) &= z^* S(f \circ w^{-1} \circ (w \circ z^{-1})) \\ &= w^* S(f \circ w^{-1}) + z^* S(w \circ z^{-1}). \end{aligned}$$

And since z and w are two compatible projective charts, the second term on the right vanishes, and the local Schwarzian derivative of f patches together to a global holomorphic quadratic differential. For a map $f : Z \to W$ from one projective surface to another, we may define its Schwarzian derivative in charts and show in the same manner that they patch together to a global object on Z.

So, the Schwarzian derivative of a map between projective surfaces is a holomorphic quadratic differential that we think of as measuring how far the map is from being a projective transformation between the surfaces. If we restrict ourselves to projective structures Z and W with the same underlying complex structure X, we can consider the identity map $Id : Z \to W$ and take its Schwarzian as a measure of how compatible the two atlases are. We define the difference of the two projective structures as this measure of compatability

$$Z - W := S(Id).$$

Indeed, if Z and W are the same projective structure, then the identity map is a projective transformation and its Schwarzian derivative is zero. By fixing a projective structure Z_0 on X, we see the set of projective structures with the same underlying complex structure X is an affine space modeled on $Q(X)$, the map given by $Z \mapsto Z - Z_0 \in Q(X)$.

There is a convenient choice for Z_0. Since X is a compact Riemann surface of genus at least 2, there is a unique hyperbolic metric in its conformal class and hence we can present X as $\mathbb{H}/\Gamma_F$, a quotient of the upper half plane by a Fuchsian group Γ_F. This hyperbolic structure is also a complex projective structure Z_F that is called the standard Fuchsian projective structure on X. Under this identification the identity map between Z and Z_F lifts to a Riemann map between $\tilde{X}$ and $\mathbb{H}$ and we may take its Schwarzian derivative to obtain a holomorphic quadratic differential $\tilde{\phi}$ on $\tilde{X}$. This quadratic differential is Γ invariant since it is the Schwarzian of a lift of a map $Z \to Z_F$. The quadratic differential induced by this $\tilde{\phi}$ is the desired $S(Id) = Z - Z_F$.

2.4. **Schwarzian Derivatives of Conformal Metrics.** Osgood and Stowe in [**?**] define a tensor generalizing the Schwarzian derivative in the setting of conformal changes to Riemannian metrics. For two metrics related by $\sigma_2 = e^{2u}\sigma_1$ on a manifold of dimension n, they define the Schwarzian tensor as the trace-free part of $\mathrm{Hess}(u) - du \otimes du$, i.e., as

$$\mathrm{Hess}(u) - du \otimes du - \frac{1}{n}\left(\Delta u - \|\nabla u\|^2\right)\sigma_1,$$

with all relevant quantities taken with respect to σ_1. In the Riemann surface setting ($n = 2$), we define the Schwarzian derivative of the conformal metric σ_2 with respect to the conformal metric σ_1, $B(\sigma_1, \sigma_2)$, as the $(2, 0)$ part of Osgood and Stowe's Schwarzian tensor: $B(\sigma_1, \sigma_2) = (\mathrm{Hess}(u) - du \otimes du)_{(2,0)}$. In a coordinate chart z write $\sigma_i = e^{2\eta_i}|dz|^2$, then the Schwarzian derivative is the quadratic differential

$$B(\sigma_1, \sigma_2) = \left((\eta_2)_{zz} - (\eta_2)_z^2 - (\eta_1)_{zz} + (\eta_1)_z^2\right) dz^2.$$

As opposed to the Schwarzian derivative of a function, this need not be holomorphic. However, when f is locally injective and holomorphic we have

$$\mathcal{S}(f) = 2B(|dz|^2, f^*|dz|^2).$$

We also have a cocycle property $B(\sigma_1, \sigma_3) = B(\sigma_1, \sigma_2) + B(\sigma_2, \sigma_3)$ and naturality $f^*B(\sigma_1, \sigma_2) = B(f^*\sigma_1, f^*\sigma_2)$ (again, for f holomorphic). When σ is a conformal metric on a domain in $\mathbb{C}$ and $B(|dz|^2, \sigma) = 0$, there exists a constant $a > 0$ and $A \in \mathrm{SL}(2, \mathbb{C})$ such that $aA^*\sigma$ is the restriction of either the hyperbolic metric on the unit disk, the Euclidean metric on $\mathbb{C}$, or the spherical metric on $\mathbb{CP}^1$. Any such metric σ is called a Möbius flat metric and we will denote it by $g_{\mathbb{CP}^1}$. Working in an affine coordinate chart z for $\mathbb{CP}^1$, the cocycle property gives us that the Schwarzian derivative of $\sigma = e^{2\eta}|dz|^2$ relative to any Möbius flat metric may be computed by

$$B(g_{\mathbb{CP}^1}, \sigma) = (\eta_{zz} - \eta_z^2)dz^2.$$

Suppose we have a complex projective surface Z whose underlying complex structure is X. Then we may compute the holomorphic quadratic differential $\phi = Z - Z_F$ corresponding to Z using the Schwarzian derivative of conformal metrics. Indeed, we know ϕ is induced by $\tilde{\phi} = S(f)$, the quadratic differential on $\tilde{X}$ that is the Schwarzian of the Riemann map $\tilde{X} \to \mathbb{H}$. Then, noting that the Poincaré metric h

of $\tilde{X}$ is the pullback of the standard Poincaré metric ρ on $\mathbb{H}$ by f, we can compute

$$\begin{aligned}\tilde{\phi} = S(f) = 2B(|dz|^2, f^*|dz|^2) &= 2B(|dz|^2, f^*\rho) + 2B(f^*\rho, f^*|dz|^2) \\ &= 2B(|dz|^2, h) + 2f^*B(\rho, |dz|^2) = 2B(g_{\mathbb{C}\mathrm{P}^1}, h).\end{aligned}$$

That is, ϕ is induced by the Schwarzian derivative of the Poincaré metric of $\tilde{X}$ relative to any Möbius flat metric $g_{\mathbb{C}\mathrm{P}^1}$.

2.5. **Quasi-Fuchsian Manifolds.** A quasi-Fuchsian structure on $S \times (0,1)$ is a choice of complete hyperbolic metric g such that there exists a non-empty, compact, geodesically convex subset. We will call $S \times (0,1)$ with a quasi-Fuchsian metric g a quasi-Fuchsian manifold and say two quasi-Fuchsian manifolds $M = (S \times (0,1), g_1)$ and $N = (S \times (0,1), g_2)$ are equivalent if there exists an $f \in \mathrm{Diff}_0(S \times (0,1))$ that is an isometry $f : M \to N$. The space of equivalence classes of quasi-Fuchsian structures on $S \times (0,1)$ will be denoted by $\mathcal{QF}(S)$

A quasi-Fuchsian manifold M is naturally compactified by two copies of S with one having the opposite orientation. These are called the surfaces at infinity and are in bijection with the ends of M. The smallest non-empty, compact, geodesically convex subset of M is called the convex core. The fundamental group $\pi_1(M) \simeq \pi_1(S)$ considered as the group of deck transformations is a discrete subgroup $\Gamma < \mathrm{PSL}_2\mathbb{C}$ acting properly discontinuously on $\mathbb{H}^3$. A group Γ obtained in this way is called a quasi-Fuchsian group.

The action of $\mathrm{PSL}_2\mathbb{C}$ on $\mathbb{H}^3$ extends to an action on $\mathbb{C}\mathrm{P}^1$ by Möbius transformation. The limit set Λ of a quasi-Fuchsian group Γ is the smallest non-empty, closed, Γ-invariant subset of $\mathbb{C}\mathrm{P}^1$ and, in this setting, is a Jordan Curve. The convex hull of Λ in $\mathbb{H}^3$ is also Γ-invariant and its quotient in M is the convex core. The complement of Λ in $\mathbb{C}\mathrm{P}^1$ is called the domain of discontinuity and consists of two domains $\Omega_\pm$, each Γ-invariant. The quotients $\Omega_\pm/\Gamma$ are the surfaces at infinity of M.

The following description applies to both ends of M and so we focus on one, calling the corresponding component of the domain of discontinuity Ω. Since Ω is an open subset of $\mathbb{C}\mathrm{P}^1$, it is trivially a complex manifold. Since Γ acts on Ω by Möbius transformations, which are holomorphic, the quotient Ω/Γ (i.e., the surface at infinity) inherits a Riemann surface structure X. This surface X is then a compact Riemann surface of genus genus(S) and thus defines a point in $\mathcal{T}(S)$. Hence, given a quasi-Fuchsian manifold, we obtain two Riemann surfaces as its surfaces at infinity. Bers showed that this procedure is invertible.

Theorem 2.4 (Bers' Simultaneous Uniformization [**?**])**.** *Given two Riemann surfaces X and Y diffeomorphic to S, there exists an isomorphism $\rho : \pi_1(S) \to \Gamma < \mathrm{PSL}_2\mathbb{C}$ such that $\mathbb{H}^3/\Gamma$ is a quasi-Fuchsian manifold with surfaces at infinity $\Omega_+/\Gamma = X$ and $\Omega_-/\Gamma = \bar{Y}$, where $\bar{Y}$ is the complex conjugate of Y, which is diffeomorphic to $\bar{S}$, the surface with the opposite orientation.*

Moreover, the space of equivalence classes of quasi-Fuchsian structures on $S \times \mathbb{R}$ is isomorphic to the product of two copies of Teichmüller space

$$\mathcal{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(\bar{S}).$$

(The presence of the complex conjugate is due to the surfaces at infinity having opposite orientation.)

The universal cover of X is identified with Ω, which again is an open subset of $\mathbb{C}\mathrm{P}^1$. Hence Ω also has a trivial complex projective structure. Since Γ is acting on Ω by Möbius transformations this projective structure descends to a projective structure Z_M on the surface at infinity Ω/Γ and has underlying complex structure X. We therefore have a holomorphic quadratic differential ϕ on X parametrizing this structure $\phi = Z_M - Z_F$. Recall that ϕ is induced by the Schwarzian derivative of the Riemann map $\Omega \to \mathbb{H}$. It may also be computed as $2B(g_{\mathbb{CP}^1}, h)$, for h the Poincaré metric of Ω, as described above. This ϕ is called the holomorphic quadratic differential at infinity (for the given end of M).

3. Epstein Surfaces

3.1. **The Visual Metric Construction.** A natural trivialization of the unit tangent bundle of hyperbolic space $U\mathbb{H}^3$ is given as

$$U\mathbb{H}^3 \to \mathbb{H}^3 \times \mathbb{CP}^1 \quad \text{by} \quad (p, v) \mapsto (p, \lim_{t\to\infty} \exp_p(tv)).$$

That is we map (p, v) to the ideal endpoint of the geodesic through p in the direction v. Restricting to a point p and using the diffeomorphism $U_p\mathbb{H}^3 \to \mathbb{CP}^1$, we can push forward to $\mathbb{CP}^1$ the induced metric on $U_p\mathbb{H}^3$ considered as a submanifold of $T_p\mathbb{H}^3$ with metric given by the inner product $g_{\mathbb{H}^3}(p)$. The resulting metric V_p on $\mathbb{CP}^1$ is called the visual metric from p. As an example, the visual metric from the origin in the ball model V_0 is just the spherical metric $\overset{\circ}{\sigma}$ on S^2 (which is identified with $\mathbb{CP}^1$ in this model). In general, if $M \in \mathrm{PSL}_2\mathbb{C}$ is an isometry taking 0 to the point p, then $V_p = M_* V_0$.

As the spherical metric belongs to the conformal class of $\mathbb{CP}^1$, we have that V_0 is a conformal metric. Since Möbius transformations are biholomorphisms of $\mathbb{CP}^1$, each V_p is also a conformal metric. If we work in the ball model of hyperbolic space $\mathbb{H}^3 \cong \mathbb{B}^3$ we can actually be explicit regarding the conformal factor between V_p and $\overset{\circ}{\sigma}$ using the affine parameter of a horosphere. If H is a horosphere, then its affine parameter is the signed hyperbolic distance from $0 \in \mathbb{H}^3$ to H, positive if 0 is outside H and negative if inside. Then for $p \in \mathbb{H}^3$ there is a unique horosphere based at $z \in \mathbb{CP}^1$ that contains p. Denote by $[p, z]$ the affine parameter of this horosphere. Then

$$V_p(z) = e^{2[p,z]} \overset{\circ}{\sigma}(z).$$

We now describe the Visual Metric Construction. This is a process that, given a strictly convex surface S in $\mathbb{H}^3$, gives a domain Ω in $\mathbb{CP}^1$ and a conformal metric σ on Ω. The idea is this: Given S, we have its image under the Gauss map on $\mathbb{CP}^1$. That is, given a unit normal vector field n on S for which S is strictly convex, define

$$\mathcal{G} : S \to \mathbb{CP}^1 \text{ by } \mathcal{G}(p) = \lim_{t\to\infty} \exp_p(tn(p)).$$

The strict convexity of S guarantees the map $\mathcal{G}$ be a homeomorphism. The image surface also comes equipped with a metric σ by defining $\sigma(\mathcal{G}(p)) = V_p(\mathcal{G}(p))$. Since for each p, the visual metric from p is a conformal metric on $\mathbb{CP}^1$ we have σ itself is a conformal metric.

Charles Epstein in [**?**] describes an inverse process to the visual metric construction, which we describe below.

3.2. **The Epstein Map.** An inverse to the visual metric construction would take a domain Ω in $\mathbb{CP}^1$ and a conformal metric σ on Ω and return a strictly convex surface in $\mathbb{H}^3$. That is we would get a map $f : \Omega \to \mathbb{H}^3$ such that the image surface S has a unit normal vector field n with respect to which the second fundamental form of S is positive definite, and so that for all $z \in \Omega$ we have $V_{f(z)}(z) = \sigma(z)$. Expanding upon this last condition and using the affine parameter discussed above, we see that

$$\sigma(z) = V_{f(z)}(z) = e^{2[f(z),z]}\overset{\circ}{\sigma}(z),$$

or that $f(z)$ lies on the horosphere based at z with affine parameter

$$[f(z), z] = \frac{1}{2}\log\left(\frac{\sigma(z)}{\overset{\circ}{\sigma}(z)}\right) =: \rho(z)$$

Hence, we know which horosphere based at z that $f(z)$ must lie on.

There is a convenient choice of normal vector field. At $f(z)$, the geodesic in the direction $n(f(z))$ must end at z in order for f to be an inverse to the visual metric construction. The normal vectors to a horosphere pointing to its base have this property, so we define $n(f(z))$ to be the normal vector to the horosphere based at z with affine parameter $\rho(z)$. Since n must be normal to the image surface $S = f(\Omega)$, this identifies the tangent spaces to the sought-after S with the tangent spaces to the horospheres. And this identifies the surface S with the envelope of the family of horospheres

$$\mathcal{H}(\Omega, \sigma) = \{\text{Horosphere based at } z \text{ with parameter } \rho(z) \mid z \in \Omega\}.$$

In [**?**], Epstein derives an equation for such an envelope. Working in the ball model and taking z both as a point in $\mathbb{C}\mathrm{P}^1$ as well as a unit vector in $\mathbb{R}^3$, he shows the desired map is

$$\mathrm{Ep}_\sigma(z) = \frac{|\overset{\circ}{\nabla}\rho(z)|^2 + e^{2\rho(z)} - 1}{|\overset{\circ}{\nabla}\rho(z)|^2 + (e^{\rho(z)} + 1)^2} z + \frac{2}{|\overset{\circ}{\nabla}\rho(z)|^2 + (e^{\rho(z)} + 1)^2}\overset{\circ}{\nabla}\rho(z),$$

where $\overset{\circ}{\nabla}\rho$ is the sphereical gradient of ρ. This construction leads to the following theorem.

Theorem 3.1 (Epstein [**?**])**.** *Let Ω be a domain in $\mathbb{C}\mathrm{P}^1$ and σ a C^k conformal metric on Ω, then there exists a unique C^{k-1} map $\mathrm{Ep}_\sigma : \Omega \to \mathbb{H}^3$, called the Epstein map of Ω for the metric σ, such that for all $z \in \Omega$,*

$$V_{\mathrm{Ep}_\sigma(z)}(z) = \sigma(z).$$

Moreover, the image of a point z depends only on the 1-jet of σ at z.

Epstein's original construction uses the ball model of hyperbolic space to define the Epstein map. In [**?**], Dumas gives a model independent definition of the map using an $\mathrm{SL}_2\mathbb{C}$-frame field. It proceeds as follows. Choose an affine chart z on $\mathbb{C}\mathrm{P}^1$ that distinguishes a point $0 \in \Omega$ and $\infty \notin \Omega$. Then, on the geodesic in $\mathbb{H}^3$ with ideal endpoints 0 and ∞, there exists a unique point p such that the visual metric from p at 0 is the Euclidean metric of this affine chart, $V_p(0) = |dz|^2$. The Epstein map is an $\mathrm{SL}_2\mathbb{C}$-frame orbit of this point.

Proposition 3.2 ([**?**])**.** *On a domain Ω in $\mathbb{C}\mathrm{P}^1$ write $\sigma = e^{2\eta}|dz|^2$. Define the $\mathrm{SL}_2, \mathbb{C}$-frame field $\widetilde{\mathrm{Ep}}_\sigma : \Omega \to \mathrm{SL}_2\mathbb{C}$ by*

$$\widetilde{\mathrm{Ep}}_\sigma(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \eta_z & 1 \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix},$$

then the Epstein map is given by

$$\mathrm{Ep}_\sigma(z) = \widetilde{\mathrm{Ep}}_\sigma(z) \cdot p.$$

Even though we call the image an Epstein surface, the Epstein map need not be an immersion. Indeed, if σ is itself a visual metric then the Epstein map for σ is constant. However, the lift of Ep_σ from Ω to the unit tangent bundle of hyperbolic space given by

$$\widehat{\mathrm{Ep}}_\sigma(z) = (\mathrm{Ep}_\sigma(z), z)$$

is an immersion (here we are using the trivialization $U\mathbb{H}^3 \simeq \mathbb{H}^3 \times \mathbb{C}\mathrm{P}^1$ defined above). This lift can be thought of as providing a unit "normal" vector field for the Epstein surface even when the Epstein map is not an immersion. Indeed, this lift agrees with a unit normal vector field when the surface is immersed and so we will simply refer to it as the normal field from now on.

Because the Epstein map is unique, it is natural with respect to the action of $\mathrm{SL}_2\mathbb{C}$ in the following sense. Suppose $M \in \mathrm{SL}_2\mathbb{C}$, then the following diagram commutes:

$$\begin{array}{ccc} (\Omega, \sigma) & \xrightarrow{M} & (M(\Omega), M_*\sigma) \\ \mathrm{Ep}\downarrow & & \downarrow\mathrm{Ep} \\ \mathbb{H}^3 & \xrightarrow{M} & \mathbb{H}^3 \end{array}$$

That is, $\mathrm{Ep}_{M_*\sigma}(M(z)) = M(\mathrm{Ep}_\sigma(z))$. This is because both $M \circ \mathrm{Ep}_\sigma$ and $\mathrm{Ep}_{M_*\sigma} \circ M$ as maps on Ω satisfy the visual metric condition from Theorem 3.1 (see [?] for more details).

This allows us to define Epstein maps on certain quotients. Suppose in general that Γ is a subgroup of $\mathrm{SL}_2\mathbb{C}$ acting freely and properly discontinuously on $\mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ leaving a domain Ω invariant. Then Ω/Γ inherits a Riemann surface structure. Call this structure X and let σ be a conformal metric on X. Lift this to $\tilde{\sigma}$ on Ω, which is Γ-invariant. Then $\mathrm{Ep}_{\tilde{\sigma}} : \Omega \to \mathbb{H}^3$ is Γ-equivariant and therefore descends to a map $\mathrm{Ep}_\sigma : X \to \mathbb{H}^3/\Gamma$. In particular, when Γ is a quasi-Fuchsian group and Ω a component of the domain of discontinuity, each conformal metric σ on the surface at infinity X gives rise to a map from X into the quasi-Fuchsian manifold M.

Uniqueness also shows us that the surfaces parallel to an Epstein surface are themselves Epstein surfaces. More specifically, let $g^t : U\mathbb{H}^3 \to \mathbb{H}^3$ denote the time-t geodesic flow projected down to $\mathbb{H}^3$. Thus for a unit tangent vector v on $\mathbb{H}^3$ we have $g^t(v) = \exp_p(tv)$. Using the lift of an Epstein surface to $U\mathbb{H}^3$ described above, each Epstein surface gives rise to a family of surfaces by applying the geodesic flow (and projecting to $\mathbb{H}^3$). That is, we have the flowed surfaces $g^t \circ \widehat{\mathrm{Ep}}_\sigma(\Omega)$. These surfaces are themselves Epstein surfaces corresponding to scalar multiples of σ. Indeed, since the parallel flow of a horosphere is a horosphere, we know $[g^t(\widehat{\mathrm{Ep}}_\sigma(z)), z] = [\mathrm{Ep}_\sigma(z), z] + t$. This shows us

$$V_{g^t(\widehat{\mathrm{Ep}}_\sigma(z))}(z) = e^{2[g^t(\widehat{\mathrm{Ep}}_\sigma(z)),z]}\mathring{\sigma}(z) = e^{2t}e^{2[\mathrm{Ep}_\sigma(z),z]}\mathring{\sigma}(z) = e^{2t}\sigma(z).$$

But the unique map that satisfies this equality is $\mathrm{Ep}_{e^{2t}\sigma}$. In summary, we have the following lemma, attributed to Thurston (unpublished work) by Epstein in [?].

Lemma 3.3 (Thurston, see [**?**])**.** *Let Ω be a domain in $\mathbb{CP}^1$ and σ a conformal metric on Ω. Then*

$$g^t \circ \widehat{\mathrm{Ep}}_\sigma = \mathrm{Ep}_{e^{2t}\sigma}.$$

That is, flowing the Epstein surface for σ for time t in the normal direction corresponds to taking the Epstein surface for the metric $e^{2t}\sigma$.

3.3. **Geometry of Epstein Surfaces.** The first fundamental form of the Epstein surface for the metric σ is given by $I(\sigma) = \mathrm{Ep}_\sigma^*(g_{\mathbb{H}^3})$ for $g_{\mathbb{H}^3}$ the metric of $\mathbb{H}^3$. It is given by

$$I(\sigma) = \frac{4}{\sigma}|B(g_{\mathbb{CP}^1}, \sigma)|^2 + \frac{1}{4}(1 - K(\sigma))^2\sigma + 2(1 - K(\sigma))\mathrm{Re}(B(g_{\mathbb{CP}^1}, \sigma)).$$

The second fundamental form (relative to the normal lift $\widehat{\mathrm{Ep}}_\sigma$) is

$$I\!I(\sigma) = \frac{4}{\sigma}|B(g_{\mathbb{CP}^1}, \sigma)|^2 - \frac{1}{4}(1 - K(\sigma)^2)\sigma - 2K(\sigma)\mathrm{Re}(B(g_{\mathbb{CP}^1}, \sigma))$$

(see [**?**, Eqns. 3.2-3.3]). Here $K(\sigma)$ is the Gaussian curvature of σ and $B(g_{\mathbb{CP}^1}, \sigma)$ the Schwarzian derivative of σ with respect to a Möbius flat metric. A computation gives the ratio between the determinant of $I(\sigma)$ and the determinant of σ:

$$\det(I(\sigma)) = \left((1 - K(\sigma))^2 - 16\frac{|B(g_{\mathbb{CP}^1}, \sigma)|^2}{\sigma^2}\right)^2 \det(\sigma).$$

Also we can compute the Gaussian curvature by $K(I(\sigma)) = -1 + \det(I(\sigma)^{-1} I\!I(\sigma))$ and the mean curvature by $H(\mathrm{Ep}_\sigma) = \frac{1}{2}\mathrm{tr}(I(\sigma)^{-1} I\!I(\sigma))$. We obtain

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16\frac{|B(g_{\mathbb{CP}^1}, \sigma)|^2}{\sigma^2}}$$

and

$$H(\mathrm{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16\frac{|B(g_{\mathbb{CP}^1}, \sigma)|^2}{\sigma^2}}{(K(\sigma) - 1)^2 - 16\frac{|B(g_{\mathbb{CP}^1}, \sigma)|^2}{\sigma^2}}.$$

In the quasi-Fuchsian setting, if σ is a Γ-invariant conformal metric on Ω then each term in the above equations is also Γ-invariant. This is maybe less clear for the quadratic differential $B(g_{\mathbb{CP}^1}, \sigma)$ since the Möbius flat metric $g_{\mathbb{CP}^1}$ is not itself Γ-invariant. However, we see that for $\gamma \in \Gamma$ we have $\gamma^* B(g_{\mathbb{CP}^1}, \sigma) = B(\gamma^* g_{\mathbb{CP}^1}, \gamma^*\sigma) = B(\gamma^* g_{\mathbb{CP}^1}, \sigma)$, by naturality of Schwarzian derivatives of conformal metrics. The metric $\gamma^* g_{\mathbb{CP}^1}$ is still a Möbius flat metric, and so $B(\gamma^* g_{\mathbb{CP}^1}, \sigma) = B(g_{\mathbb{CP}^1}, \sigma)$, implying $B(g_{\mathbb{CP}^1}, \sigma)$ is Γ-invariant. Therefore, $B(g_{\mathbb{CP}^1}, \sigma)$ induces a quadratic differential on X, which we will denote by $B(\sigma)$.

In summary of the above, we have the following Gaussian and mean curvatures of the Epstein surfaces in M.

Lemma 3.4. *The Gaussian curvature for the Epstein surface $\mathrm{Ep}_\sigma : X \to M$ is given by*

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16\frac{|B(\sigma)|^2}{\sigma^2}},$$

and the mean curvature by

$$H(\mathrm{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16\frac{|B(\sigma)|^2}{\sigma^2}}{(1 - K(\sigma))^2 - 16\frac{|B(\sigma)|^2}{\sigma^2}}. \qquad \square$$

These are now equations on the compact Riemann surface X.

4. Asymptotically Poincaré Families

Previously, we have discussed Epstein surfaces for domains and for quotients. Generalizing this, we say a closed surface $i : S \to M$ of genus greater than one is an Epstein surface of X if there exists a Γ-invariant conformal metric $\tilde{\sigma}$ on Ω and a diffeomorphism $\varphi : X \to S$ such that the diagram

$$\begin{tikzcd} \Omega \arrow[r, "\mathrm{Ep}_{\tilde{\sigma}}"] \arrow[d, "\varphi \circ \pi_X"'] & \mathbb{H}^3 \arrow[d, "\pi_M"] \\ S \arrow[r, "i"] & M \end{tikzcd}$$

commutes. Here the π_X and π_M are the respective quotient maps $\Omega \to X$ and $\mathbb{H}^3 \to M$. When S is an Epstein surface of X, the conformal metric $\tilde{\sigma}$ induces a conformal metric σ on X that we call the conformal metric at infinity. We have $\mathrm{Ep}_\sigma = \varphi \circ i$. The Epstein surface S will then refer to the embedded image of $\mathrm{Ep}_\sigma : X \to M$.

Since X is a closed surface of genus at least 2, it possesses a unique hyperbolic conformal metric h. Hence, there is a distinguished Epstein surface in M.

Definition 1. *The Poincaré surface of X is the Epstein induced by the Poincaré metric of Ω. Since the Poincaré metric induces the hyperbolic metric h on X, the Poincaré surface of X is the Epstein surface $\mathrm{Ep}_h : X \to M$.*

Again, we call this a surface but it need not be immersed. In fact, we can identify cases when it is not. When $\sigma = h$ we know $K(h) = -1$ and $B(h) = \frac{1}{2}\phi$. Then using the formula for determinant of $I(h)$ given in Section 3.3, we compute

$$\det(I(h)) = 16 \left(1 - \frac{|\phi|^2}{h^2}\right)^2 \det(h).$$

And so we see Ep_h is not an immersion at points in X where $\frac{|\phi|^2}{h^2} = 1$. However, by parallel flowing of the Poincaré surface we eventually obtain an immersed (embedded, actually) surface, which we now discuss.

The Poincaré surface together with its parallel copies forms a family of surfaces we call the Poincaré Family. Recall from Lemma 3.3 that the parallel copies are given as the Epstein surfaces $\mathrm{Ep}_{e^{2t}h} : X \to M$ for $t \geq 0$. The surfaces in the Poincaré family are eventually embedded as was shown by Anderson[**?**] and [**?**](BBB). We omit the proof here as we will show a more general result in Proposition 4.3. Although we will show a bound for t after which the Epstein surface for $e^{2t}\sigma$ is immersed. Denote by $\|B(\sigma)\|_\sigma$ and $\|K(\sigma)\|$ the supremums over X of the functions $|B(\sigma)|/\sigma$ and $K(\sigma)$, then we have the following.

Lemma 4.1. *Let σ be a conformal metric on X. If*

$$t > \ln \sqrt{4\|B(\sigma)\|_\sigma + \|K(\sigma)\|},$$

then $\mathrm{Ep}_\sigma : X \to M$ is an immersion.

Proof. If t satisfies this bound then we have

$$e^{2t} > 4\|B(\sigma)\|_\sigma + \|K(\sigma)\| > 4\frac{|B(\sigma)|}{\sigma} + K(\sigma)$$

Rearranging and simplifying, this implies that

$$(e^{2t} - K(\sigma))^2 - 16\frac{|B(\sigma)|^2}{\sigma^2} > 0.$$

It then follows that $\det(I(e^{2t}\sigma)) > 0$ so that $I(e^{2t}\sigma)$ is positive definite. □

In the Poincaré case the bound becomes $t > \ln\sqrt{2\|\phi\|_h + 1}$, which is the bound obtained by [?] [?], although we do not get embeddedness from this lemma.

So, the Poincaré family is the family of Epstein surfaces for the conformal metrics at infinity $\rho_t = e^{2t}h$ and for t sufficiently large these surfaces are embedded. We now discuss their asymptotic behavior. In order to consider derivatives more easily, let us reparametrize the Poincaré family using $\epsilon = e^{-2t}$, so that $\rho_\epsilon = h/\epsilon$. We are interested in the behavior as $\epsilon \to 0^+$.

Using the formula for the first fundamental form of an Epstein surface we can compute

$$I(\rho_\epsilon) = \frac{1}{4\epsilon}h + \frac{1}{2}h + \mathrm{Re}(\phi) + \epsilon\left(\frac{1}{4}h + \mathrm{Re}(\phi) + \frac{|\phi|^2}{h}\right).$$

We can therefore consider the Poincaré family as a path in the space of metrics $\mathrm{Met}(S)$. The following arguments are formal; they will be made precise in the next section. We see the family of first fundamental forms diverges as $\epsilon \to 0$. However, rescaling gives

$$4\epsilon I(\rho_\epsilon) = h + 4\epsilon\left(\frac{1}{2}h + \mathrm{Re}(\phi)\right) + O(\epsilon^2).$$

And so, if we project to the quotient, we can consider the Poincaré family as a path in Teichmüller space that converges: $[I(\rho_\epsilon)] \to [h]$ as $\epsilon \to 0$. Moreover, differentiating at 0 yields

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} 4\epsilon I(\rho_\epsilon) = 2h + 4\mathrm{Re}(\phi).$$

Since $2h$ is pure-trace (with respect to h) and since $\mathrm{Re}(\phi)$ is an h-transverse-traceless tensor by Lemma 2.3, we see the tangent vector to this path in Teichmüller space at $[h]$ is $[I(\dot{\rho_\epsilon})] = d\pi_h(2h+4\mathrm{Re}(\phi)) = 4\mathrm{Re}(\phi)$. Similar computations apply to $I\!I(\rho_\epsilon)$ and show that $[I\!I(\rho_\epsilon)] \to [h]$ as $\epsilon \to 0$ with tangent vector given by $[I\!I(\dot{\rho_\epsilon})] = 0$.

Summarizing: The Poincaré family has the property that the surfaces form a parallel family foliating the end of M such that the paths $[I(\rho_\epsilon)] \to [h]$ and $[I\!I(\rho_\epsilon)] \to [h]$ in Teichmüller space as $\epsilon \to 0$, and $[I(\dot{\rho_\epsilon})] = 4\mathrm{Re}(\phi)$ and $[I\!I(\dot{\rho_\epsilon})] = 0$. It turns out the only thing needed about the the Poincaré family to obtain these results is that there is a function f so that $f(\epsilon)\rho_\epsilon \to h$ as $\epsilon \to 0$. The function here is the identity $f(\epsilon) = \epsilon$. In the next section we introduce a generalization of the Poincaré family and discuss the corresponding results that are true for this generalization.

4.1. **Asymptotically Poincaré Families.** The key property of the Poincaré family we wish to generalize is that $f(\epsilon)\rho_\epsilon \to h$ for $f(\epsilon) = \epsilon$ as $\epsilon \to 0$. We consider, then, families of Epstein surfaces whose conformal metrics at infinity $\sigma(\epsilon)$ satisfy $f(\epsilon)\sigma(\epsilon) \to h$ as $\epsilon \to 0$ where now f is any positive smooth function with the same behavior to first order at 0, i.e., that $f(0) = 0$, and $f'(0) \neq 0$. Up until now we have proceeded formally with respect to the topology on the space of smooth tensors. We will now be precise by using Sobolev spaces of tensors and the fact that C^∞ is the intersection over all Sobolev exponents of the Sobolev spaces. We will also add regularity designations, e.g, the space of smooth metrics on X will be denoted by $\mathrm{Met}^\infty(X)$, while the Sobolev class of metrics on H^s we will be $\mathrm{Met}^s(X)$.

Definition 2. *Let S_ϵ for ϵ in $(0,1)$ be a family of embedded Epstein surfaces with conformal metrics at infinity $\sigma(\epsilon)$. We call this family asymptotically Poincaré if*

(1) there exists a scaling function $f : [0,1) \to [0,\infty)$ so that the path

$$f\sigma : (0,1) \to \mathrm{Met}^\infty(X)$$

is differentiable and converges to the hyperbolic metric on X as $\epsilon \to 0$, that is, $f(\epsilon)\sigma(\epsilon) \to h$ as $\epsilon \to 0$,

(2) the function f is smooth and has simple zero at 0, and

(3) the continuous extension $\gamma : [0,1) \to \mathrm{Met}^\infty(X)$ of $f\sigma$ is differentiable.

The surfaces in the Poincaré family are parallel to one another. The surfaces in an asymptotically Poincaré family are asymptotically parallel in the sense of the following lemma.

Lemma 4.2. *Suppose S_ϵ is an asymptotically Poincaré family of surfaces. Let $t > 0$. If g^t is the geodesic flow operator defined above, then*

$$d_M\left(g^t(\widehat{\mathrm{Ep}}_{\sigma(\epsilon)}(z)), \mathrm{Ep}_{\sigma(e^{-2t}\epsilon)}(z)\right) \to 0 \qquad \text{as } \epsilon \to 0$$

uniformly in z. That is, the distance between the surface S_ϵ flowed for time t and the surface $S_{e^{-2t}\epsilon}$ tends towards zero as ϵ does.

Proof. We work with the universal covers. In general, a straight forward computation using the $\mathrm{SL}_2\mathbb{C}$-frame definition of the Epstein map $\Omega \to \mathbb{H}^3$ gives the distance between the image of z under the Epstein map for the metrics $\sigma = e^{2\eta}|dz|^2$ and $\tau = e^{2\lambda}|dz|^2$ as

$$d_{\mathbb{H}^3}\left(\mathrm{Ep}_\sigma(z), \mathrm{Ep}_\tau(z)\right) = 2\mathrm{arctanh}\left(\sqrt{\frac{(e^\eta - e^\lambda)^2 + 4|\eta_z - \lambda_z|^2}{(e^\eta + e^\lambda)^2 + 4|\eta_z - \lambda_z|^2}}\right)$$

where all functions are evaluated at z, which we have suppressed for brevity.

In our case, lift $\sigma(\epsilon)$ to $\tilde{\sigma}(\epsilon)$ and $\gamma(\epsilon) = f(\epsilon)\sigma(\epsilon)$ to $\tilde{\gamma}(\epsilon)$ on Ω. Write $\tilde{\sigma}(\epsilon) = e^{2\eta(\epsilon)}|dz|^2$ and $\tilde{\gamma}(\epsilon) = e^{2\lambda(\epsilon)}|dz|^2$, then $\eta(\epsilon) = \lambda(\epsilon) - (1/2)\ln(f(\epsilon))$. Recall from Lemma 3.3 that $g^t \circ \widehat{\mathrm{Ep}}_{\tilde{\sigma}(\epsilon)} = \mathrm{Ep}_{e^{2t}\tilde{\sigma}(\epsilon)}$. For ease of notation, let $c = e^{-2t}$. Then we have the distance between $\mathrm{Ep}_{c^{-1}\tilde{\sigma}(\epsilon)}$ and $\mathrm{Ep}_{\tilde{\sigma}(c\epsilon)}$ can be simplified to

$$2\mathrm{arctanh}\left(\sqrt{\frac{(1 - \sqrt{\frac{cf(\epsilon)}{f(c\epsilon)}}e^{\lambda(c\epsilon)-\lambda(\epsilon)})^2 + 4cf(\epsilon)e^{-2\lambda(\epsilon)}|\lambda_z(c\epsilon) - \lambda_z(\epsilon)|^2}{(1 + \sqrt{\frac{cf(\epsilon)}{f(c\epsilon)}}e^{\lambda(c\epsilon)-\lambda(\epsilon)})^2 + 4cf(\epsilon)e^{-2\lambda(\epsilon)}|\lambda_z(c\epsilon) - \lambda_z(\epsilon)|^2}}\right).$$

Since $\tilde{\gamma}(\epsilon)$ converges in $\mathrm{Met}^\infty(X)$, the function λ has a C^2 limit as $\epsilon \to 0$. Therefore, the argument of arctanh converges to zero uniformly in z.

Since the distance between the surfaces in the universal cover is converging to zero, and since the quotient map $\mathbb{H}^3 \to M$ is a local isometry, we get the Lemma. $\square$

We have required that an asymptotically Poincaré family consist of embedded surfaces. The next proposition gives a useful condition for a family of conformal metrics to give rise to an asymptotically Poincaré family of surfaces.

Proposition 4.3. *Let $\sigma : (0,1) \to \mathrm{Conf}^\infty(X)$ be a family of conformal metrics on X. Suppose there exists a smooth function $f : [0,1) \to [0,\infty)$ with simple zero at 0, such that $f\sigma \to h$ as $\epsilon \to 0$ and such that the extension $\gamma : [0,1) \to \mathrm{Conf}^\infty(X)$ is differentiable. Then there exists an $\epsilon_0 > 0$ so that for $\epsilon < \epsilon_0$, the Epstein map $\mathrm{Ep}_{\sigma(\epsilon)}$ is an embedding. Hence, the Epstein surfaces $\mathrm{Ep}_{\sigma(\epsilon)} : X \to M$, for $\epsilon < \epsilon_0$, form an asymptotically Poincaré family.*

Proof. Let $\tilde\sigma : (0,1) \to \mathrm{Conf}^\infty(\Omega)$ be the lift of the family σ. Define the Epstein family map $\mathrm{Ep}_{\tilde\sigma} : \Omega \times (0,1) \to \mathbb{H}^3$ by $\mathrm{Ep}_{\tilde\sigma}(z,\epsilon) = \mathrm{Ep}_{\tilde\sigma(\epsilon)}(z)$. It follows from the $\mathrm{SL}_2\mathbb{C}$-frame definition of the Epstein map that in the upper half space model $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}^+$, the family map is given by

$$\mathrm{Ep}_{\tilde\sigma}(z,\epsilon) = (z,0) + \frac{2}{e^{2\eta} + 4|\eta_z|^2}\left(2\eta_{\bar z}, e^{\eta}\right).$$

Writing $\tilde\sigma(\epsilon) = e^{2\eta(z,\epsilon)}|dz|^2$ and $\tilde\gamma(\epsilon) = e^{2\lambda(z,\epsilon)}|dz|^2$, the condition $\tilde\gamma(\epsilon) = f(\epsilon)\tilde\sigma(\epsilon)$ becomes $\eta(z,\epsilon) = \lambda(z,\epsilon) - (1/2)\ln(f(\epsilon))$. Note that $\lambda(z,\epsilon) \to \rho(z)$ as $\epsilon \to 0$, uniformly in z, where ρ is the log density of the Poincaré metric of Ω. Hence we can rewrite $\mathrm{Ep}_{\tilde\sigma}$ as

$$\mathrm{Ep}_{\tilde\sigma}(z,\epsilon) = (z,0) + \frac{2}{e^{2\lambda} + 4f(\epsilon)|\lambda_z|^2}\left(2f(\epsilon)\lambda_{\bar z}, \sqrt{f(\epsilon)}e^{\lambda}\right)$$

and see that

$$\lim_{\epsilon\to 0} \mathrm{Ep}_{\tilde\sigma}(z,\epsilon) = (z,0).$$

So we may extend $\mathrm{Ep}_{\tilde\sigma}$ to a map $\Omega \times [0,1) \to \mathbb{H}^3 \sqcup \Omega$, which is the identity on the boundary $\Omega \times \{0\} \to \Omega$.

While this map is not differentiable at $\epsilon = 0$ (due to the $\sqrt{f(\epsilon)}$), the map $F : \Omega \times [0,1) \to \mathbb{H}^3 \sqcup \Omega$ given by $F(z,\epsilon) = \mathrm{Ep}_{\tilde\sigma}(z,\epsilon^2)$ satisfies $F(z,0) = (z,0)$, is differentiable at $\epsilon = 0$, and has derivative

$$dF_{(z,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\sqrt{f'(0)}e^{-\rho(z)} \end{pmatrix}.$$

Since this is invertible, F is a local C^1-diffeomorphism at the boundary of $\Omega \times [0,1) \to \mathbb{H}^3 \sqcup \Omega$.

Define $\bar M = (\mathbb{H}^3 \sqcup \Omega)/\Gamma = M \sqcup X$. Then $\bar M$ is a smooth manifold with compact boundary X. By Γ-equivariance of $\mathrm{Ep}_{\tilde\sigma}$, F descends to a map $X \times [0,1) \to \bar M$ that is the identity on the boundary $\partial(X \times [0,1)) \to \partial\bar M = X$ and that is a local diffeomorphism there. We now show this implies the restriction of F to $X \times [0,\delta)$, for some small enough δ, is a diffeomorphism onto a collar neighborhood of $\partial\bar M = X$.

Given $(z,0)$ there exists neighborhoods $U_{(z,0)}$ and $V_{(z,0)}$ such that $F : U_{(z,0)} \to V_{(z,0)}$ is a diffeomorphism. By compactness of X we may take a finite number U_i and V_i such that $X \times \{0\} \subset \cup U_i$. Call $\cup_i U_i = U$ and $\cup_i V_i = V$. Then $F : U \to V$ is a local diffeomorphism and $X \times \{0\} \subset U, V$. In fact, we have

Lemma. *There exists $\epsilon, \delta > 0$ such that $X \times [0,\delta] \subset U$, and $X \times [0,\epsilon] \subset V$ and such that $X \times [0,\epsilon) \subset F(X \times [0,\delta))$.*

Proof. By construction of U there exists a δ' such that $X \times [0,\delta') \subset U$. So choose $\delta = \delta'/2$. Then $X \times [0,\delta] \subset U$. We show that there is an $\epsilon > 0$ so that $X \times [0,\epsilon) \subset F(X \times [0,\delta))$, as if this is true, we may replace ϵ with $\epsilon/2$ to get $X \times [0,\epsilon] \subset V$.

Now, suppose no such ϵ exists. Take a sequence $\epsilon_n \to 0$. Then for each n there exists $(w_n, t_n) \in X \times [0, \epsilon_n)$ such that $(w_n, t_n) \notin F(X \times [0, \delta))$. But $t_n < \epsilon_n \to 0$. Moreover, by compactness of X there exists a subsequence (which we still call) w_n that converges to, say, $w \in X$. Then $(w_n, t_n) \to (w, 0)$. Since F is a local diffeomorphism, there is a neighborhood W of $(w, 0)$, which we may assume is a subset of $X \times [0, \delta)$ by taking intersections if needed, that is diffeomorphic to a neighborhood $F(W)$ of $(w, 0)$. Since $(w_n, t_n) \to (w, 0)$, for large enough n we get $(w_n, t_n) \in F(W) \subset F(X \times [0, \delta))$. This is a contradiction. Hence, we get an $\epsilon > 0$ so that $X \times [0, \epsilon) \subset F(X \times [0, \delta))$. ☐

Corollary. $X \times [0, \epsilon] \subset F(X \times [0, \delta])$

Lemma. $F : X \times [0, \delta] \to F(X \times [0, \delta])$ *is a covering.*

Proof. F is a local diffeomorphism since $X \times [0, \delta] \subset U$ and $X \times [0, \epsilon] \subset V$ and since $F : U \to V$ is a local diffeomorphism. Since $X \times [0, \delta]$ is compact, F is a proper mapping and proper local diffeomorphisms are coverings. ☐

Lemma. *The covering is trivial.*

Proof. Note that the cardinality of the fibers of $F : X \times [0, \delta] \to F(X \times [0, \delta])$ are constant since everything is connected. Also note that $F(X \times (0, \delta]) \subset X \times (0, 1)$ and $F : X \times \{0\} \to X \times \{0\}$ is the identity. In particular $F^{-1}(\{(w, 0)\}) = \{(w, 0)\}$. Hence, $F : X \times [0, \delta] \to F(X \times [0, \delta])$ is injective. ☐

Corollary. $F : X \times [0, \delta] \to F(X \times [0, \delta])$ *is injective and contains a neighborhood of* $X \times \{0\}$.

Unraveling, we get that Ep_σ is a diffeomorphism from a collar neighborhood $X \times (0, \sqrt{\delta})$ to a neighborhood of infinity of M. In particular, each Epstein surface $\mathrm{Ep}_\sigma(\cdot, \epsilon) = \mathrm{Ep}_{\sigma(\epsilon)}$, for $\epsilon < \sqrt{\delta}$, is an immersion and injective with compact domain X. Hence each Epstein surface, for $\epsilon < \sqrt{\delta}$, is embedded. To complete the proof take $\epsilon_0 = \sqrt{\delta}$.

☐

The Poincaré family foliates the end of M since they are parallel. In the preceding proof, we have the map $F : X \times [0, \delta) \to \bar{M}$ is a diffeomorphism onto its image. Hence we have the following result for asymptotically Poincaré families.

Corollary 4.4. *If* S_ϵ *is an asymptotically Poincaré family of surfaces, then there exists an* $\epsilon_0 > 0$ *such that for* $\epsilon < \epsilon_0$ *the surfaces* S_ϵ *form a foliation of the end of* M *whose surface at infinity is* X. ☐

We soon turn to our main result, but first note that the co-orientation on an Epstein surface we are using is that induced by the lift $\widehat{\mathrm{Ep}}_\sigma$, which points towards the surface at infinity X. This implies that $I\!I(\sigma)$ is negative definite (for small enough ϵ), and so $-I\!I(\sigma)$ is a smooth Riemannian metric. Also recall we are using the Riemannian model of Teichmüller space. That is, we use

$$\mathcal{T}(X) = \mathrm{Met}^\infty(X)/\mathrm{Diff}_0^\infty(X) \rtimes P^\infty(X),$$

where $P^\infty(X)$ is the set of smooth positive functions on X and $\mathrm{Diff}_0^\infty(X)$ is the group of smooth diffeomorphisms isotopic to the identity (see [**?**] for details). The smooth topology, however, is difficult to work with directly. So, we work in the Sobolev setting tensors and functions.

4.2. **Sobolev Spaces on Manifolds.** Smooth classes of functions and tensors are difficult to work with because the C^∞ setting lacks a robust inverse and implicit function theorem (among other reasons). The Sobolev setting is useful because they form Banach spaces, which enjoy the standard tools of analysis.

To define the relevant Sobolev spaces on the closed surface S we need to fix a background Riemannian metric, the natural choice being the hyperbolic metric h. Let ∇ be its Levi-Civita connection. Both h and ∇ extend to metrics and connections on all the tensor bundles on S made from TS and T^*S. Hence all tensor bundles have a norm induced by h. Let $\mathcal{T}^{(k,l)}(S)$ be the space of smooth (k,l)-tensor fields on S. The space $W^{s,p}(\mathcal{T}^{(k,l)}(S))$ will denote the space of Sobolev tensors. It is the completion of $\mathcal{T}^{(k,l)}(S)$ under the norm

$$\|T\|_{s,p} = \left(\sum_{j=0}^{s}\int_S |\nabla^j T|^p \ dVol(h)\right)^{1/p}.$$

Here ∇^j is the iterated covariant derivative (j-times).

Compactness of S is vital here as it guarantees this norm is finite. These Sobolev spaces are Banach spaces and when $p=2$ they are Hilbert spaces. We will denote them by $H^s(\mathcal{T}^{(k,l)}(S))$ (note: this is not cohomology). The spaces in which we will be most interested are the Sobolev spaces of real-valued functions $H^s(S,\mathbb{R})$, metrics $\mathrm{Met}^s(S)$, and conformal metrics $\mathrm{Conf}^s(X)$.

Sobolev functions (or tensors) that satisfy an equation in the sense of distributions are called weak solutions to the equation. On S, where the dimension is 2, the Sobolev Embedding Theorem on smooth surfaces guarantees that weak solutions in H^s for $s \geq 3$ are actually strong solutions, being $C^{2,\alpha}$-regular [**?**]. We will therefore assume $s \geq 3$ from now on. We also note that $C^\infty = \cap_s H^s$.

4.3. **Main Results.** So, we work in the Sobolev setting of functions $H^s(X)$ and tensors $\mathrm{Met}^s(X)$ and $\mathrm{Conf}^s(X)$. Since K and B are smooth functions of σ and its derivatives we have that they both extend to functions on Sobolev classes of metrics. Hence if $\sigma \in \mathrm{Conf}^s(X)$ then $I(\sigma)$ and $-I\!I(\sigma)$ belongs to $\mathrm{Met}^{s-2}(X)$. We will obtain results with these extensions and then argue our results are independent of the chosen s.

Proposition 4.5. *Suppose S_ϵ is an asymptotically Poincaré family of surfaces. Let $\gamma : [0,1) \to \mathrm{Met}^s(X)$ be the extension of $f\sigma$ thought of as taking values in a class of Sobolev metrics for a fixed $s > 3$. Then the first and second fundamental forms $I \circ \sigma : (0,1) \to \mathrm{Met}^{s-2}(X)$ and $I\!I \circ \sigma : (0,1) \to \mathrm{Met}^{s-2}(X)$ satisfy*

$$I_\epsilon = 4f'(0)\epsilon I(\sigma(\epsilon)) \to h \quad \textit{and} \quad I\!I_\epsilon = -4f'(0)\epsilon I\!I(\sigma(\epsilon)) \to h \quad \textit{as } \epsilon \to 0.$$

Moreover, I_ϵ and $I\!I_\epsilon$ are differentiable at $\epsilon = 0$ and their tangent vectors are given by

$$\dot{I}_\epsilon = \dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi) \quad \textit{and} \quad \dot{I\!I}_\epsilon = \dot{\gamma}$$

Proof. We have that $\gamma : [0,1) \to \mathrm{Met}^\infty(X)$ is continuous and differentiable at $\epsilon = 0$. Therefore, as $\mathrm{Met}^\infty(X) = \cap_{s>3}\mathrm{Met}^s(X)$ we also have that γ is continuous to Met^s and differentiable at $\epsilon = 0$.

Then, we have $I(\sigma(\epsilon)) = I(\frac{1}{f(\epsilon)}\gamma(\epsilon))$ is equal to

$$4f(\epsilon)\frac{|B(\gamma(\epsilon))|^2}{\gamma(\epsilon)} + \frac{1}{4f(\epsilon)}(1 - f(\epsilon)K(\gamma(\epsilon)))^2\gamma(\epsilon) + 2(1 - f(\epsilon)K(\gamma(\epsilon)))\mathrm{Re}(B(\gamma(\epsilon))),$$

which is a smooth tensor independent of s. Since $f : [0,1) \to \mathbb{R}$ and $\gamma : [0,1) \to \mathrm{Met}^s(X)$ are differentiable at 0 and since $K : \mathrm{Met}^s(X) \to H^{s-2}(X)$ and $B : \mathrm{Conf}^s(X) \to \Gamma^{s-2}(\Sigma^2(X))$ are differentiable at the hyperbolic metric h, we can write

$$f(\epsilon) = \epsilon f'(0) + O(\epsilon^2), \quad K(\gamma(\epsilon)) = -1 + O(\epsilon), \quad B(\gamma(\epsilon)) = \frac{1}{2}\phi + O(\epsilon).$$

Substitution and some simplification gives

$$I(\sigma(\epsilon)) = \frac{1}{4\epsilon f'(0)}h + \frac{1}{4f'(0)}\dot{\gamma} + \frac{1}{2}h + \mathrm{Re}(\phi) + O(\epsilon).$$

Consequently,

$$I_\epsilon = h + \epsilon(\dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi)) + O(\epsilon^2).$$

The same reasoning will give

$$I\!I_\epsilon = h + \epsilon\dot{\gamma} + O(\epsilon^2).$$

The result then follows. □

Recall the tangent space to $\mathrm{Met}^s(X)$ at g may be decomposed as the orthogonal sum

$$T_g\mathrm{Met}^s(X) = \{\dot{g} \mid \mathrm{tr}_g(\dot{g}) = 0 = \mathrm{div}_g(\dot{g})\} \oplus \{\mathcal{L}_X g + fg \mid f \in H^s \text{ and } X \in \Gamma(TX)\},$$

where $\mathrm{div}_g(\dot{g})$ is the divergence of the 1-1 tensor $g^{-1}\dot{g}$. The second summand is tangent to the $\mathrm{Diff}_0^s(X) \rtimes P^s(X)$ orbit of g and the first summand is the transverse traceless tensors consisting of elements that are smooth, trace-free, and divergence-free (see [?] for more details). Since the transverse traceless tensors are orthogonal to the group orbit they may be identified with the tangent space $T_{[g]}\left(\mathrm{Met}^s(X)/\mathrm{Diff}_0^s(X) \rtimes P^s(X)\right)$ to the quotient at $[g]$. Moreover transverse traceless tensors are a set of smooth tensors that are L^2 orthogonal to the $\mathrm{Diff}_0^s(X) \rtimes P^s(X)$ orbit *for any* s. Consequently, they may be identified with the tangent space to the quotient $\mathrm{Met}^\infty(X)/(\mathrm{Diff}_0^\infty(X) \rtimes P^\infty(X))$. That is, they may be naturally identified with the tangent space to Teichmüller space at $[g]$:

$$T_{[g]}\mathcal{T}(X) = \{\dot{g} \mid \mathrm{tr}_g(\dot{g}) = 0 = \mathrm{div}_g(\dot{g})\}.$$

Under this identification, the action of the derivative of the projection $\pi : \mathrm{Met}^\infty(X) \to \mathcal{T}(X)$ at g is given by orthogonal projection onto the transverse traceless tensors $T_g\mathrm{Met}^\infty(X) \to \{\dot{g} \mid \mathrm{tr}_g(\dot{g}) = 0 = \mathrm{div}_g(\dot{g})\}$.

We now prove our main result, Theorem **??** from the Introduction. Recall that $[I\!I]$ denotes the point in Teichmüller space corresponding to the conformal class of $-I\!I$.

Theorem 4.6. *Let S_ϵ for $\epsilon \in (0,1)$ be an asymptotically Poincaré family of surfaces with metrics at infinity $\sigma(\epsilon)$. If h is the hyperbolic metric of X and ϕ the holomorphic quadratic differential at infinity, then in Teichmüller space $\mathcal{T}(X)$ we have*

$$[I(\sigma(\epsilon))] \to [h] \quad \text{and} \quad [I\!I(\sigma(\epsilon))] \to [h] \quad \text{as } \epsilon \to 0.$$

Moreover, the tangent vectors in $T_{[h]}\mathcal{T}(X)$ are given by

$$[I(\dot{\sigma}(\epsilon))] = 4f'(0)\mathrm{Re}(\phi) \quad \text{and} \quad [I\!I(\dot{\sigma}(\epsilon))] = 0.$$

Proof. Using the notation from Proposition 4.5, for each $s > 3$ we have that I_ϵ and $I\!I_\epsilon$, which are paths through smooth tensors, converge in $\mathrm{Met}^s(X)$ to the hyperbolic metric h. Since $\mathrm{Met}^\infty(X) = \cap \mathrm{Met}^s(X)$ we know that I_ϵ and $I\!I_\epsilon$ converge to h in $\mathrm{Met}^\infty(X)$. Moreover, since the projection $\mathrm{Met}^\infty(X) \to \mathcal{T}(X)$ is continuous, and since $[I(\sigma(\epsilon))] = [I_\epsilon]$ and $[I\!I(\sigma(\epsilon))] = [I\!I_\epsilon]$, we have that

$$[I(\sigma(\epsilon))] \to [h] \quad \text{and} \quad [I\!I(\sigma(\epsilon))] \to [h] \quad \text{as } \epsilon \to 0.$$

Since both paths converge to $[h]$ at $\epsilon = 0$ we can extend them to continuous paths $[0,1) \to \mathcal{T}(S)$. Since $[I(\sigma(\epsilon))]$ and $[I\!I(\sigma(\epsilon))]$ agree with $[I_\epsilon]$ and $[I\!I_\epsilon]$, respectively, we also have they are differentiable at $\epsilon = 0$.

From Proposition 4.5 we know that

$$\dot{I}_\epsilon = \dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi) \quad \text{and} \quad \dot{I\!I}_\epsilon = \dot{\gamma}.$$

Since ϕ is holomorphic, $4f'(0)\mathrm{Re}(\phi)$ is trace-free and divergence-free. On the other hand, $2f'(0)h$ is pure trace and so belongs to the $\mathrm{Diff}_0^s(X) \rtimes P^s(X)$ orbit of h. Furthermore, for any s it belongs to the group orbit of h. Hence, the derivative of the projection at h removes this term. Similarly, γ is a path in the group orbit of h and so $\dot{\gamma}$ projects to 0. We then have

$$[I(\dot{\sigma(\epsilon)})] = [\dot{I}_\epsilon] = d\pi_h(\dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi)) = 4f'(0)\mathrm{Re}(\phi)$$

and

$$[I\!I(\dot{\sigma(\epsilon)})] = \dot{I\!I}_\epsilon = d\pi_h(\dot{\gamma}) = 0,$$

as claimed. □

5. k-SURFACES

In this section we discuss some applications of Theorem 4.6.

5.1. A Conjecture of Labourie.

Labourie proved in [?] that geometrically finite ends of hyperbolic 3-manifolds admit foliations by surfaces of constant Gaussian curvature. These he called k-surfaces since for each $k \in (-1, 0)$ there exists a unique surface in the foliation whose curvature is identically k. Both ends of a quasi-Fuchsian manifold are geometrically finite, hence this foliation result applies in the quasi-Fuchsian case.

Focusing on one end of the quasi-Fuchsian manifold with surface at infinity X, in [?] Labourie discusses how the conformal classes $[I_k]$ and $[I\!I_k]$ of the first and second fundamental forms of the k-surfaces behave as paths in the Teichmüller space of X. When $k \to -1$, he proves that $[I\!I_k]$ approaches the point at infinity of $\mathcal{T}(X)$ corresponding to the measured geodesic lamination on the convex core of the quasi-Fuchsian manifold. When $k \to 0$, he showed both $[I_k]$ and $[I\!I_k]$ converge to the same point: the complex structure X on the surface at infinity; or in metric terms, converge to the class of the hyperbolic metric $[h]$. He conjectures that the tangent vector to these paths is related to the holomorphic quadratic differential at infinity ϕ.

His conjecture is correct as shown in Theorem 5.5. Indeed, Labourie's k-surfaces form an asymptotically Poincaré family (as we will show). Therefore, our main result Theorem 4.6 applies, describing the asymptotic behavior of the k-surfaces as $k \to 0$. In the process, our work gives an alternative proof to Labourie's theorem on the existence of k-surfaces (at least for k near 0), in this case for quasi-Fuchsian manifolds: see Theorem 5.2.

5.2. **The k-surface equation.** We now prove that k-surfaces form an asymptotically Poincaré family of surfaces. To do this we derive an equation for a conformal metric that implies its Epstein surface is a k-surface and we show this equation has a unique smooth solution for each k near zero. In the proof of existence of solutions we will see that these conformal metrics satisfy the hypothesis of Proposition 4.3.

Finding a metric σ whose Epstein surface has constant Gaussian curvature k is finding a metric σ that solves $K(I(\sigma)) = k$, which from Lemma 3.4 is solving

$$\frac{4K(\sigma)}{(1 - K(\sigma))^2 - \frac{16}{\sigma^2}|B(\sigma)|^2} = k.$$

For now we will focus on solving

$$4K(\sigma) = k\left((1 - K(\sigma))^2 - \frac{16}{\sigma^2}|B(\sigma)|^2\right), \tag{1}$$

as we will see that for small enough k, $K(I(\sigma)) = k$.

We are interested in obtaining solutions to (1) for k near zero. This is hampered by the fact that there are no solutions to (1) when $k = 0$. Indeed we would be asking for $K(\sigma) = 0$, which is impossible on a surface with genus bigger than 1. In an attempt to obtain better asymptotics we consider the case when Γ is Fuchsian. Here we have explicit solutions to the k-surface equation (1). Indeed, the k-surfaces are the Poincaré family. Working in the universal covers, $\mathrm{Ep}_h : \Omega \to \mathbb{H}^3$ gives a totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$ with constant curvature -1. For $-1 < k < 0$ the k-surfaces are given by equidistant copies of this $\mathbb{H}^2$. The conformal metric whose Epstein surface is the k-surface is then $c(k)h$ for some function of k satisfying $K(I(c(k)h)) = k$. Since $B(g_{\mathbb{CP}^1}, c(k)h) = 0$ we get the defining equation of c as $4K(c(k)h) = k(1 - K(c(k)h))^2$. More explicitly c is given by

$$c(k) = \frac{1 + \sqrt{1+k}}{1 - \sqrt{1+k}}.$$

Suppose we have solution metrics σ_k on X that give Epstein surfaces with constant Gaussian curvature k. As we just saw, in the Fuchsian case we have $\sigma_k = c(k)h$. In the general quasi-Fuchsian case, define $f(k)$ by

$$f(k) = c(k)^{-1} = \frac{1 - \sqrt{1+k}}{1 + \sqrt{1+k}}$$

and τ_k by $\tau_k = f(k)\sigma_k$, so that in the Fuchsian case the metric τ_k is the hyperbolic metric for all k. Since σ_k solves (1), by substitution and some simplification we get that $\tau = \tau_k$ solves the equation

$$(2+k)(1+K(\tau))^2 + 2\sqrt{1+k}\left(1 - K(\tau)^2\right) + 16\left(2\sqrt{1+k} - 2 - k\right)\frac{|B(\tau)|^2}{\tau^2} = 0. \tag{2}$$

In the limiting case $k = 0$ we see that the hyperbolic metric h on X solves (2). Solutions actually exists in a neighborhood of $(k, \tau) = (0, h)$ as we will show. To apply PDE theory we define the function $F : U \times \mathrm{Conf}^\infty(X) \to C^\infty(X)$ given by

$$F(k,\tau) = (2+k)(1+K(\tau))^2 + 2\sqrt{1+k}\left(1 - K(\tau)^2\right) + 16\left(2\sqrt{1+k} - 2 - k\right)\frac{|B(\tau)|^2}{\tau^2}.$$

Here, U is an open interval around 0 small enough not to contain -1 so that F is smooth on its domain. Points (k, τ) where $F(k, \tau) = 0$ are solutions to the scaled equation (2); we have already found $F(0, h) = 0$.

5.3. **Solutions to the k-Surface Equation.** We will use the Implicit Function Theorem in the Banach space setting to obtain solutions to $F(k,\tau)=0$ and so we first work with $\mathrm{Conf}^s(X)=\{ph \mid p\in H^s(X,\mathbb{R}^+)\}$, where $H^s(X,\mathbb{R}^+)$ is the Sobolev space of functions on X taking positive values. With the norm $\|\tau\|_s := \|\tau/h\|_s$, the set $\mathrm{Conf}^s(X)$ is naturally identified with an open subset of the Banach space $H^s(X)=H^s(X,\mathbb{R})$. As stated above, we work with Sobolev spaces of a fixed regularity $s>3$. Recall the function F is defined on $U\times\mathrm{Conf}^\infty(X)$ and maps to $C^\infty(X)$. Extend F to a function $U\times\mathrm{Conf}^s(X)\to H^{s-2}(X)$ so that F is then defined on an open subset of a Banach space.

We still have $F(0,h)=0$ and we can now get solutions near 0 as well.

Theorem 5.1. *There is a neighborhood V of 0 such that for each $k\in V$, there exists a unique $\tau\in\mathrm{Conf}^s(X)$ such that $F(k,\tau)=0$.*

Proof. Note that since the constituent parts of F are smooth on U, F is smooth. Furthermore, $D_2F_{(0,h)}: H^s(X)\to H^{s-2}(X)$ is an isomorphism, where D_2F is the partial derivative of F with respect to its second argument. Indeed, a direct computation of the derivative of F at $(0,h)$ yields

$$DF_{(0,h)}(\dot{k},\dot{\tau}) = 4\,DK_h(\dot{\tau}).$$

Note that $D_1F_{(0,h)}=0$ and that $D_2F_{(0,h)}=4\,DK_h$. The differential of the curvature function DK_h is given by a formula of Lichnerowicz:

$$4\,DK_h(\dot{\tau}) = -2(\Delta_h - Id)\frac{\dot{\tau}}{h},$$

which is an isomorphism $H^s(X)\to H^{s-2}(X)$ (see [**?**, Page 33]).

Consequently, by the Banach Implicit Function Theorem (see [**?**, Theorem 17.6]) there exists a neighborhood V of 0 and a curve $\gamma: V\to\mathrm{Conf}^s(X)$ with $\gamma(0)=h$ and $F(k,\gamma(k))=0$. Moreover, these are the only solutions to $F(k,\tau)=0$ in V, and γ is smooth since F is. □

5.4. **Regularity of Solutions.** Theorem 5.1 furnishes weak solutions $\gamma(k)$ that vary smoothly as a map $V\to\mathrm{Conf}^s(X)$. The Sobolev Embedding Theorem immediately strengthens the regularity of the individual $\gamma(k)$ to at least $C^{2,\alpha}$, for each k (see [**?**]), and so we have strong solutions $F(k,\gamma(k))=0$.

A nonlinear equation $A(u)=0$ is said to be elliptic at u if the derivative of A at u, DA_u, is an elliptic linear operator. When A is smooth, solutions u where A is elliptic are also smooth ([**?**, Lemma 17.16]). In our case, since $D_2F_{(0,h)}=-2(\Delta_h-Id)$ is an elliptic operator we have $F(0,\tau)=0$ is elliptic at $\tau=h$. Moreover, ellipticity is an open condition in $\mathbb{R}\times C^2(X)$, so there exists an open interval $(-\delta,\delta)$ such that the linearization $D_2F(k,\gamma(k))$ is elliptic for all $k\in(-\delta,\delta)$. This has two consequences. First, it shows the solutions given by Theorem 5.1 are individually smooth conformal metrics. We conclude the following theorem.

Theorem 5.2. *There exists an $\delta>0$ such that for all $-\delta<k<0$ there exists a unique smooth metric σ_k whose Epstein surface is a k-surface.*

Proof. We have a curve γ defined on an interval $(-\delta,\delta)$ such that the smooth metric $\gamma(k)$ satisfies $F(k,\gamma(k))=0$. This means that $\gamma(k)$ solves the scaled equation (2) and so

$$\sigma_k = f(k)^{-1}\gamma(k) = \frac{1+\sqrt{1+k}}{1-\sqrt{1+k}}\,\gamma(k),$$

defined for $k \in (-\delta, 0)$, solves the k-surface equation (1).

Finally, we must show that $(1 - K(\sigma))^2 - \frac{16}{\sigma^2}|B(\sigma)|^2$ in (1) is everywhere nonzero so that (1) is equivalent to the desired Guassian curvature condition. But this follows from Nehari's Theorem (see [**?**, Theorem 1.3], or the original paper [**?**]), which gives an a priori bound on $|B(h)|^2/h^2$ (recall $B(h)$ is the Schwarzian derivative of an injective holomorphic map), implying that $|B(\sigma_k)|^2/\sigma_k^2 \to 0$ as $k \to 0$. Hence by shrinking δ if necessary to make k small enough, this expression is nonzero and so we may rearrange (1) to get that σ_k produces a k-surface: $K(I(\sigma_k)) = k$. $\square$

Second, ellipticity also implies the family $\gamma(k)$ varies smoothly in k in the C^∞ topology.

Proposition 5.3. *There exists a neighborhood U of 0 and a smooth path $\gamma : U \to \mathrm{Conf}^\infty(X)$ such that $F(k, \gamma(k)) = 0$ for all $k \in U$.*

Proof. Since $\mathrm{Conf}^\infty(X) = \cap \mathrm{Conf}^s(X)$, it suffices to show there exits a neighborhood U of 0 and a function $\gamma : U \to \mathrm{Conf}^\infty(X)$ such that for all sufficiently large s, the function $\gamma : U \to \mathrm{Conf}^s(X)$ is a smooth path.

To this end, apply The Implicit Function Theorem to the smooth function $F : (-1, 1) \times \mathrm{Conf}^s(X) \to H^{s-2}(X)$ at $(0, h)$ to get a solution interval U_s around 0 and a unique smooth path $\gamma_s : U_s \to \mathrm{Conf}^s(X)$ such that $F(k, \gamma_s(k)) = 0$ for all $k \in U_s$. The partial derivative $D_2F(0, h)$ is elliptic, so there exists a neighborhood U' of 0 such that for all $k \in U'$, $D_2F(k, \gamma_s(k))$ is elliptic. Define $U = U_s \cap U'$.

Fix $t > s$. The partial derivative $D_2F(0, h) : H^t(X) \to H^{t-2}(X)$ is still an isomorphism and so the Implicit Function Theorem may be applied again to get a solution interval and path. Let U_t be the maximal interval around 0 such that a solution path $\gamma_t : U_t \to \mathrm{Conf}^t(X)$ exists and is smooth, and such that $D_2F(k, \gamma_t(k))$ is elliptic. We have $U_t \subset U$ with $\gamma_t = \gamma_s$ on U_t by uniqueness. We want $U_t = U$.

Assume towards a contradiction that U_t is a proper subset of U and that the supremum of U_t is an element of U, call it k. Then *if* we can show $D_2F(k, \gamma_s(k)) : H^t(X) \to H^{t-2}(X)$ is an isomorphism, then we may apply the Implicit Function Theorem to get an interval V around k and a smooth curve $\alpha : V \to \mathrm{Conf}^t(X)$ that agrees with γ_t to the left of k. By uniqueness we may extend γ_t using α to $U_t \cup V$. This contradicts maximality of U_t. Hence the supremum of U_t does not belong to U. A similar argument can be made regarding the infimum of U_t to show it does not belong to U. Hence, since both U_t and U are intervals, we must have $U_t = U$. Again, this is provided we can show $D_2F(k, \gamma_s(k)) : H^t(X) \to H^{t-2}(X)$ is an isomorphism. We do this now.

Since $k \in U$ we know $D_2F(k, \gamma_s(k))$ is an elliptic operator. Hence it is a Fredholm operator from $H^t(X) \to H^{t-2}(X)$. If a Fredholm operator has trivial kernel and index 0, then it is an isomorphism. Note that the index of a family of Fredholm operators is constant on connected components of the domain of the parameter of the family [**?**]. Since U is connected, the index of $D_2F(k, \gamma_s(k))$ is the same as the index of $D_2F(0, h)$. The latter is an isomorphism and so has index 0. Thus $D_2F(k, \gamma_s(k))$ has index zero. To show it has trivial kernel, suppose there exists a function $f \in H^t(X)$ such that $D_2F(k, \gamma_s(X))f = 0$. Then since $f \in H^s(X)$ we also have f is in the kernel of the map $D_2F(k, \gamma_s(k)) : H^s(X) \to H^{s-2}(X)$. But k is in U so $D_2F(k, \gamma_s(k))$ is an isomorphism from $H^s(X) \to H^{s-2}(X)$. Thus, we get $D_2F(k, \gamma_s(k))$ has trivial kernel and index 0 as a map $H^t(X) \to H^{t-2}(X)$ and therefore it is an isomorphism.

□

Corollary 5.4. *The family of k-surfaces forms an asymptotically Poincaré family of surfaces.*

Proof. We show that $\tilde{\sigma}(\epsilon) = \sigma_{-\epsilon}$ satisfies the hypotheses of Proposition 4.3. Since $\mathrm{Conf}^{\infty}(X) = \cap \mathrm{Conf}^{s}(X)$ and since γ is smooth on $(-\delta, \delta)$ into $\mathrm{Conf}^{s}(X)$ for all $s > 3$, we have the γ is smooth into $\mathrm{Conf}^{\infty}(X)$. Now, define $\tilde{f}(\epsilon) = f(-\epsilon)$ and $\tilde{\gamma}(\epsilon) = \gamma(-\epsilon)$ for $\epsilon \in (0, \delta)$. Then the Corollary follows from Proposition 4.3 since $\tilde{f}$ is smooth on $[0, \delta)$ with derivative $\tilde{f}'(0) = 1/4$, since $\tilde{\gamma}$ is differentiable, and since $\tilde{f}(\epsilon)\tilde{\sigma}(\epsilon) = \tilde{\gamma}(\epsilon) = \gamma(k) \to h$ as $\epsilon \to 0$. □

Our Theorem 5.2 is another proof of the existence of these k-surfaces, at least for k close to zero. By the uniqueness result of Theorem 1.10 in [**?**], given an end of M and $k \in (-1, 0)$, there exists a unique immersed incompressible k-surface. Since Epstein surfaces are incompressible, the k-surfaces produced by our Theorem 5.2 are the k-surfaces Labourie obtains. Our approach here is more concrete than the methods used by Labourie. In return for sacrificing the generality of Labourie's pseudoholomorphic methods, a proof of Labourie's conjecture follows easily. Corollary 5.4 gives that, for k near zero, these k-surfaces form an asymptotically Poincaré family, and so by Theorem 4.6 we know how the tangent vectors to the families relate to the holomorphic quadratic differential at infinity.

Theorem 5.5. *Let I_k and $I\!I_k$ be the first and second fundamental forms of the k-surface. Let ϕ be the holomorphic quadratic differential at infinity of M. Then, as $k \to 0$, Then, as $k \to 0$, the tangent vectors to $[I_k]$ and $[I\!I_k]$ in Teichmüller space are given by*

$$[\dot{I_k}] = -\mathrm{Re}(\phi) \quad \textit{and} \quad [\dot{I\!I_k}] = 0.$$

Proof. We take $\epsilon = -k$. The derivative $f'(0) = -1/4$ can be computed directly. Theorem 4.6 now gives the theorem. □

5.5. **Constant Mean Curvature Surfaces.** As another application of our asymptotically Poincaré families, we obtain results regarding the asymptotic behavior of the constant mean curvature family constructed by Mazzeo and Pacard in [**?**]. More specifically, we prove that there exists an asymptotically Poincaré family of surfaces S_k for k near zero, such that the mean curvature of the surface S_k is $-\sqrt{1+k}$. This is done similarly to the previous section: we derive an equation for a conformal metric that implies its Epstein surface has mean curvature $-\sqrt{1+k}$ and we show this equation has a unique smooth solution for each k near zero. The proof of existence of solutions will show that these conformal metrics satisfy the hypothesis of Proposition 4.3.

Recall from Lemma 3.4 that the mean curvature of $\mathrm{Ep}_{\sigma} : X \to M$ is given by

$$H(\mathrm{Ep}_{\sigma}) = \frac{K(\sigma)^2 - 1 - 16\frac{|B(\sigma)|^2}{\sigma^2}}{(1 - K(\sigma))^2 - 16\frac{|B(\sigma)|^2}{\sigma^2}},$$

which, again, is an equation on the compact Riemann surface X. To find a metric whose Epstein surface has constant mean curvature $-\sqrt{1+k}$ we must solve the equation $H(\mathrm{Ep}_{\sigma}) = -\sqrt{1+k}$, which simplifies to

$$1 - K(\sigma)^2 - \sqrt{1+k}(1 - K(\sigma))^2 + (1 + \sqrt{1+k})\frac{16}{\sigma^2}|B(\sigma)|^2 = 0. \tag{3}$$

As in the k-surfaces case it suffices to solve (3) since $(K(\sigma)-1)^2 - \frac{16}{\sigma^2}|B(\sigma)|^2$ will eventually be nonzero. Furthermore, we scale the equation by assuming σ_k solves (3) and defining $\tau_k = f(k)\sigma_k$ for $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$ the function defined above. If σ_k solves (3) then $\tau = \tau_k$ solves $G(k,\tau) = 0$ for $G : U \times \mathrm{Conf}^\infty(X) \to C^\infty(X)$ defined by

$$G(k,\tau) = 1 + \sqrt{1+k} + 2\sqrt{1+k}K(\tau) + (-1+\sqrt{1+k})(K(\tau)^2 - \frac{16}{\tau^2}|B(\tau)|^2).$$

Here U is a small enough open set around zero not containing -1 so that G is smooth on its domain. To find solutions to (3) we find solutions to $G(k,\tau) = 0$ and then scale them by $f(k)^{-1}$. Notice that $G(0,h) = 0$ is a solution.

Theorem 5.6. *There exists a neighborhood W of 0 so that for each $k \in W$, there exists a unique $\tau \in \mathrm{Conf}^\infty(X)$ so that $G(k,\tau) = 0$.*

Proof. Extend G to a map $U \times \mathrm{Conf}^s(X) \to H^{s-2}(X)$ for $s > 3$. When $k = 0$ we have the hyperbolic metric h as solution $G(0,h) = 0$. The map G is smooth. The derivative of G at $(0,h)$ is given by

$$dG_{(0,h))}(\dot{k},\dot{\tau}) = -4\frac{|\phi|^2}{h^2}\dot{k} + 2DK_h(\dot{\tau}).$$

Notice that $D_2G_{(0,h)} = DK_h$, which—as in the proof of Theorem 5.1—is an isomorphism $H^s(X) \to H^{s-2}(X)$. Hence, by the Banach Implicit Function Theorem there exists an open set W and a smooth curve $\gamma : W \to \mathrm{Conf}^s(X)$ such that $G(k,\gamma(k)) = k$. Moreover, these are the only solutions to $G(k,\tau) = 0$ in W.

The same regularity arguments apply here as in the k-surface case and imply the existence of an $\delta > 0$ so that when $k \in (-\delta,\delta)$, each metric $\gamma(k)$ is smooth and the family γ varies smoothly in k in the C^∞ topology. □

This implies that the mean curvature surfaces form an asymptotically Poincaré family of surfaces.

Corollary 5.7. *There exists an $\delta > 0$ so that for $k \in (-\delta, 0)$ the family of surfaces S_k where S_k has constant mean curvature $-\sqrt{1+k}$ forms an asymptotically Poincaré family of surfaces.*

Proof. Since $\gamma(k)$ solves $G(k,\gamma(k)) = 0$ for $k \in (-\delta,\delta)$, the metric $\gamma(k)$ solves the scaled mean curvature equation. So, defining $\sigma_k = \frac{1+\sqrt{1+k}}{1-\sqrt{1+k}}\,\gamma(k)$ for $k \in (-\delta, 0)$, we see that σ_k solves the mean curvature equation (3), which implies that Ep_{σ_k} has constant mean curvature $-\sqrt{1+k}$.

The same arguments as in Corollary 5.4 show that these metrics satisfy the hypothesis of Proposition 4.3. So, the surfaces $S_k = \mathrm{Ep}_{\sigma_k}(X)$ form an asymptotically Poincaré family of surfaces. □

Consequently, we obtain Theorem **??**.

6. Future Directions