Limits of foliations in quasi-Fuchsian manifolds

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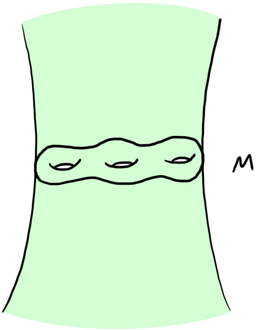

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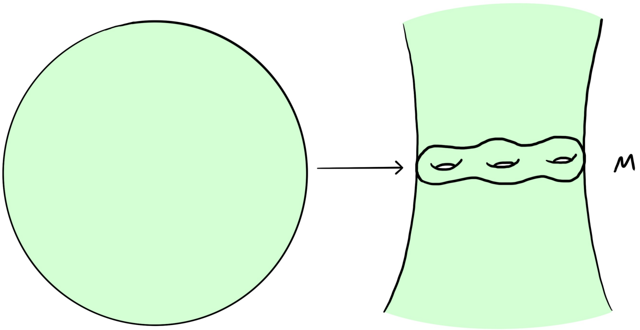
$\tilde{M} \simeq \mathbb{H}^3 \simeq \mathbb{B}^3$



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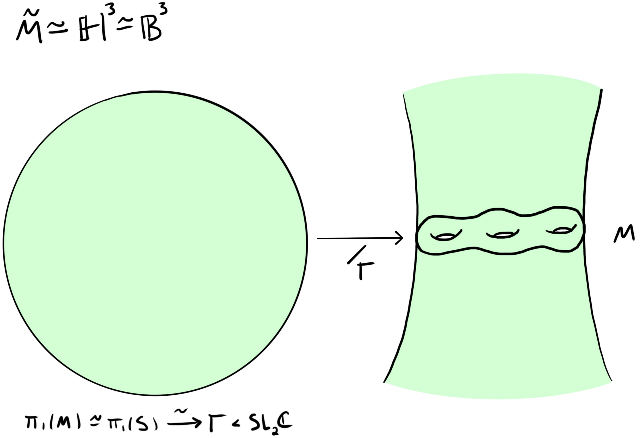

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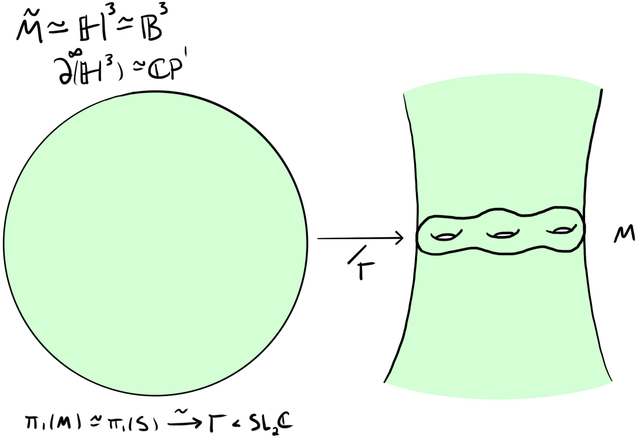

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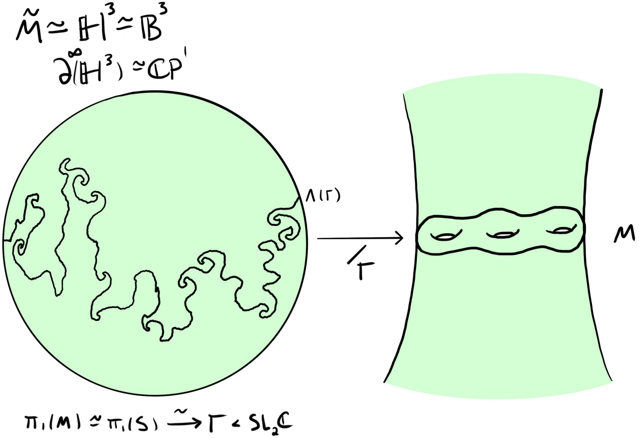

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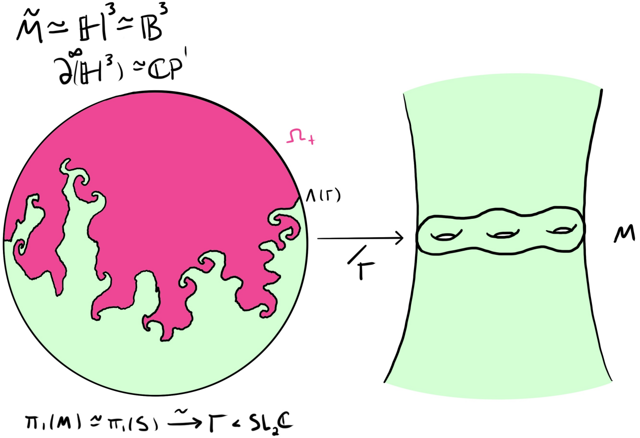

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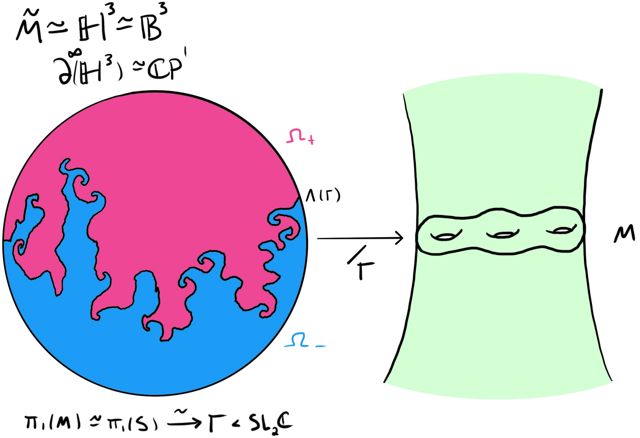

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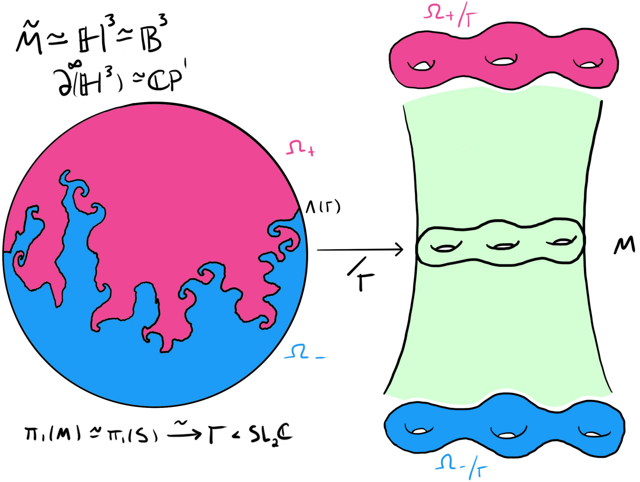

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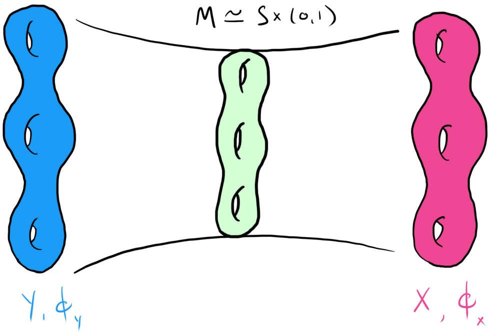
$X = \Omega_+/\Gamma$ and $Y = \Omega_-/\Gamma$ are called the surfaces at infinity and they inherit both conformal structures and complex projective structures.

These induce $[X]$ and $[Y]$ in $\mathcal{T}(S)$, the Teichmüller space of S and we get holomorphic quadratic differentials ϕ_X and ϕ_Y that parametrize the projective structures.



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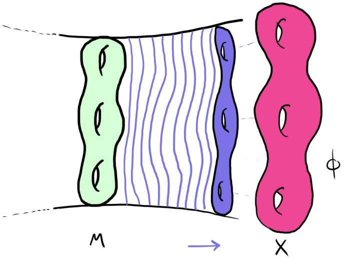
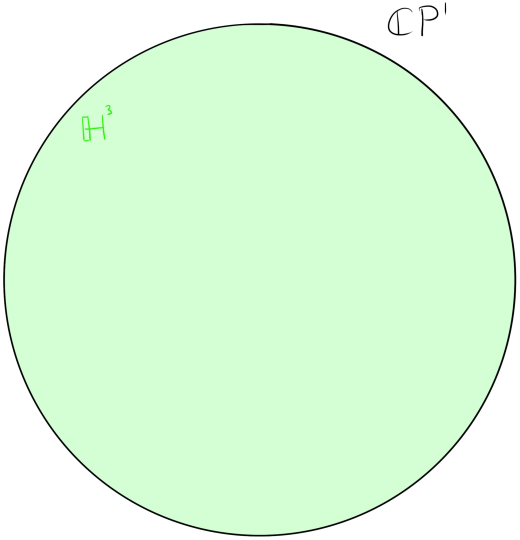
We will consider certain foliations of the ends of M and investigate their limits as the foliations leave the ends.



We will focus on just one end of M. From now on let X denote its surface at infinity and ϕ its holomorphic quadratic differential.

Visual Metrics



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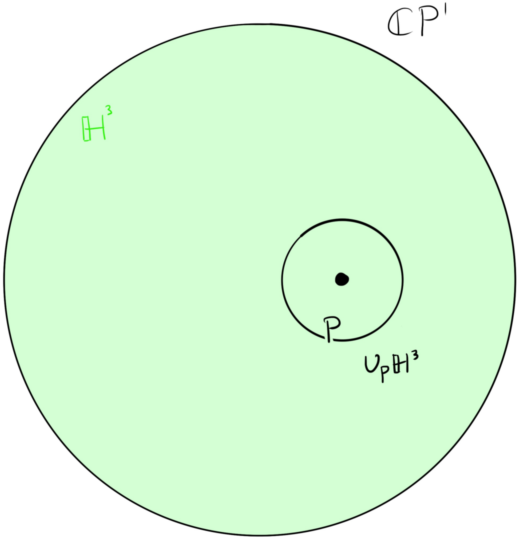

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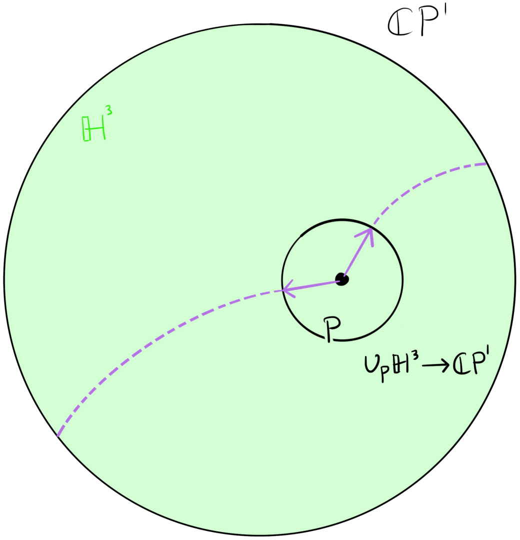

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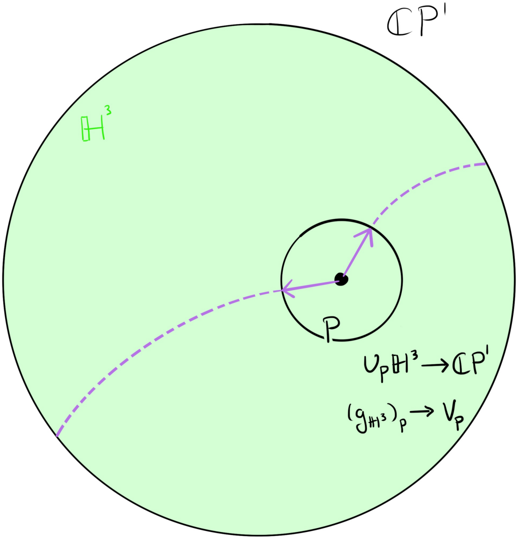

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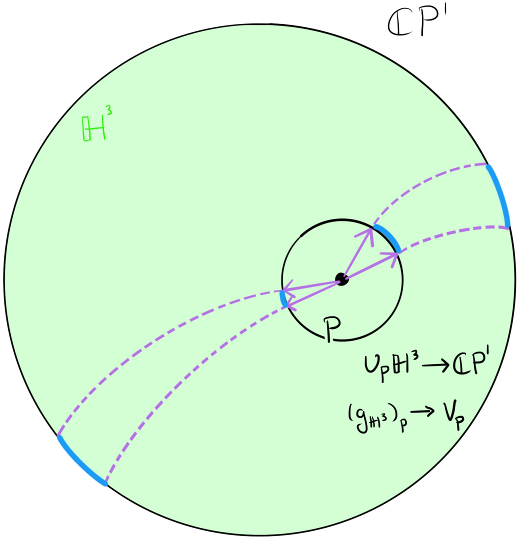
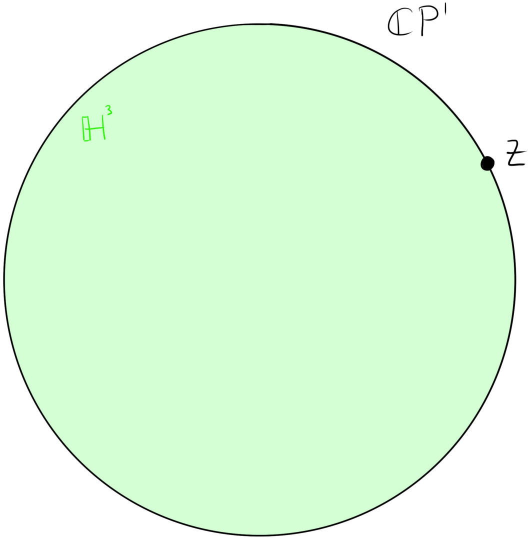


Epstein Surfaces



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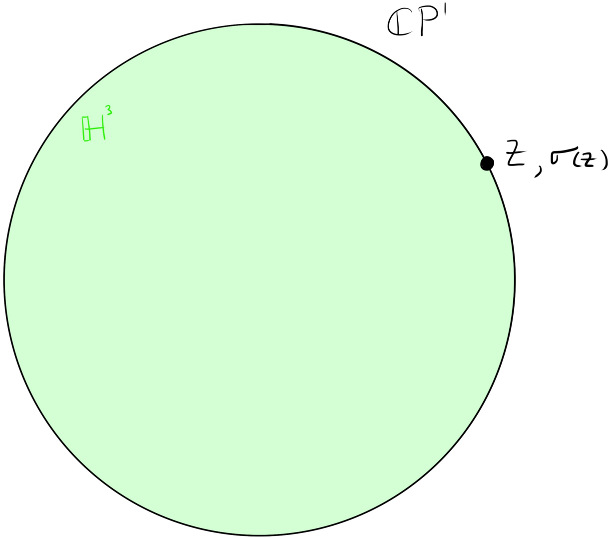

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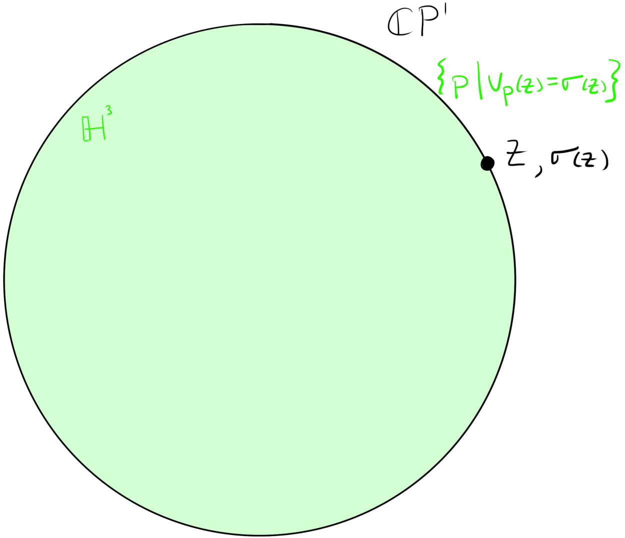

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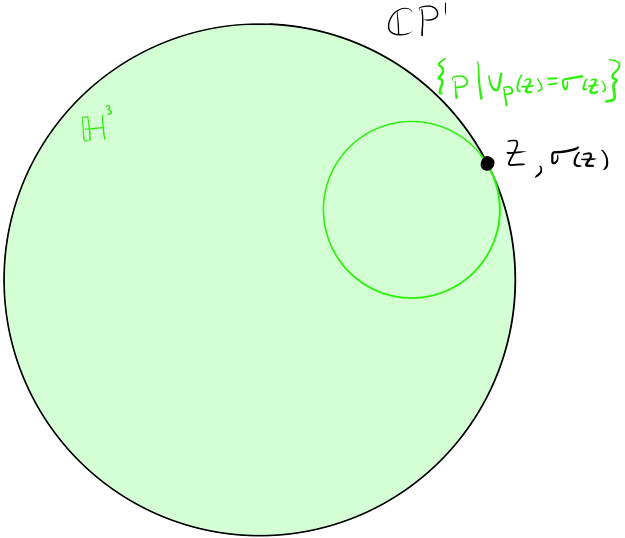

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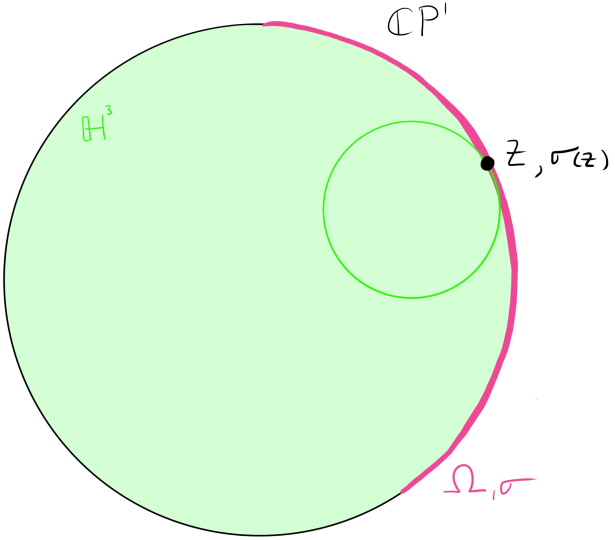

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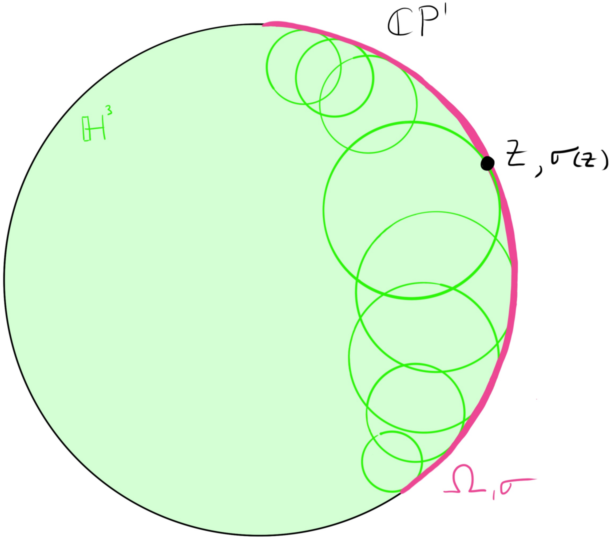

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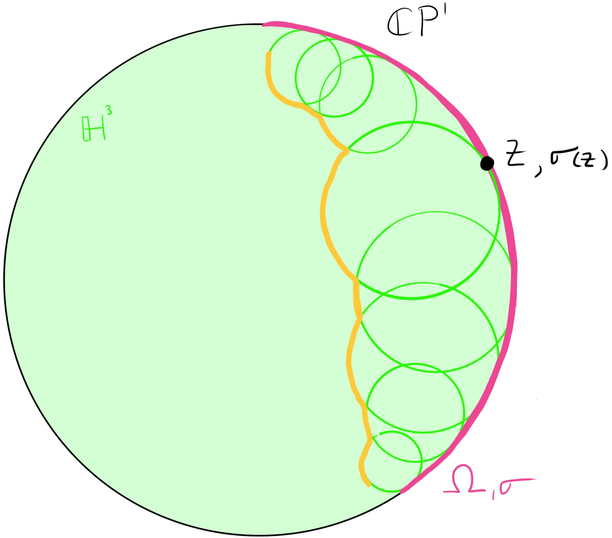

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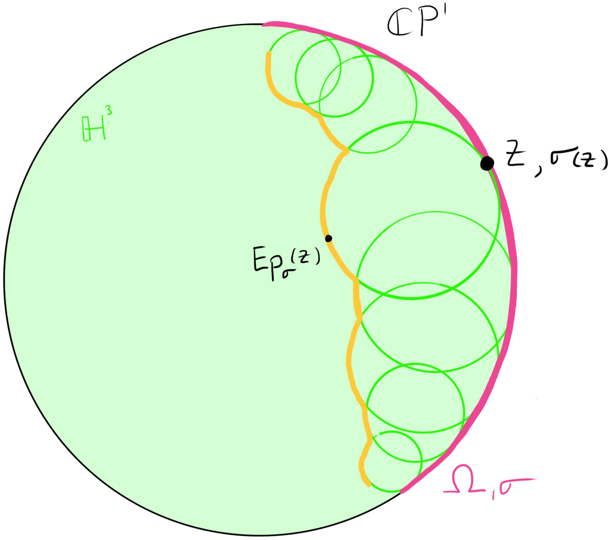


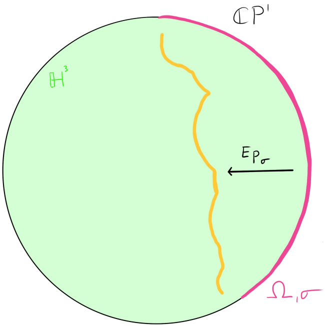
Epstein Surfaces

Theorem (Epstein 1984, Thurston)

Let Ω be a domain in $\mathbb{CP}^1$ and σ a C^k conformal metric on Ω, then there exists a unique C^{k-1} map $\mathrm{Ep}_\sigma : \Omega \to \mathbb{H}^3$, called the Epstein map of Ω for the metric σ, such that for all $z \in \Omega$,

$$V_{\mathrm{Ep}_\sigma(z)}(z) = \sigma(z).$$

Moreover, the image of a point z depends only on the 1-jet of σ at z. □



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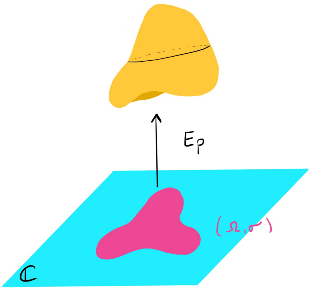

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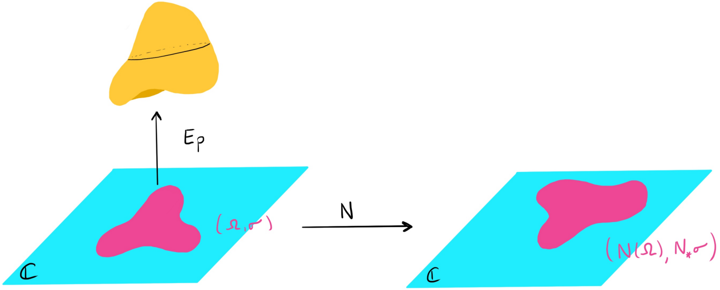

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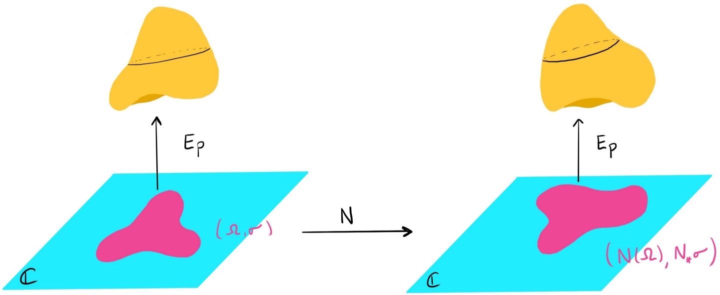

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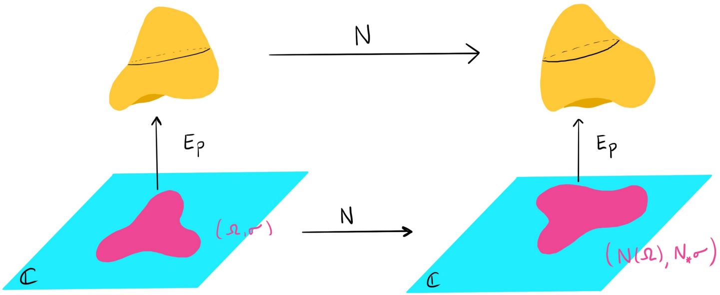
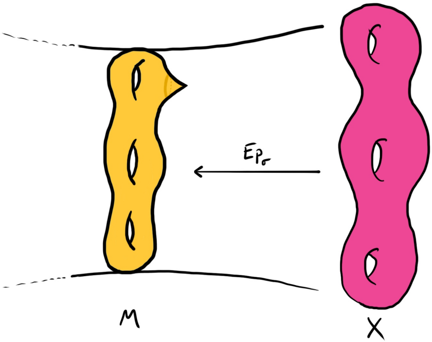


Properties

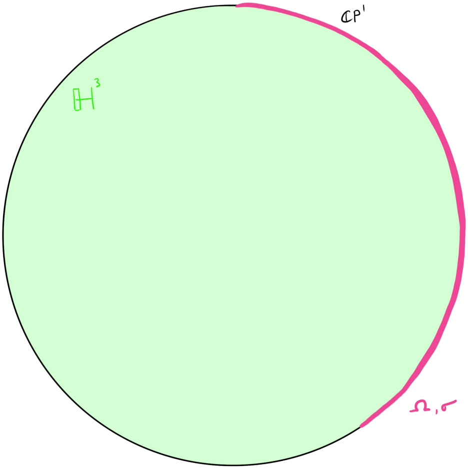
From $\mathrm{Ep}_{N_*\sigma}(N \cdot z) = N \cdot \mathrm{Ep}_\sigma(z)$ we get a variant of this Epstein construction for quotients.

In the quasi-Fuchsian case this means if we take a conformal metric σ on the surface at infinity X, we get an Epstein surface $\mathrm{Ep}_\sigma : X \to M$. (Caution: we say surface even though it may fail to be immersed)



Properties

The Epstein construction behaves well under a normal flow.
Flowing Ep_σ for time t in the normal direction gives a parallel copy of the Epstein surface at distance t from the original copy. This parallel surface is also obtained as the Epstein surface $\mathrm{Ep}_{e^{2t}\sigma}$.



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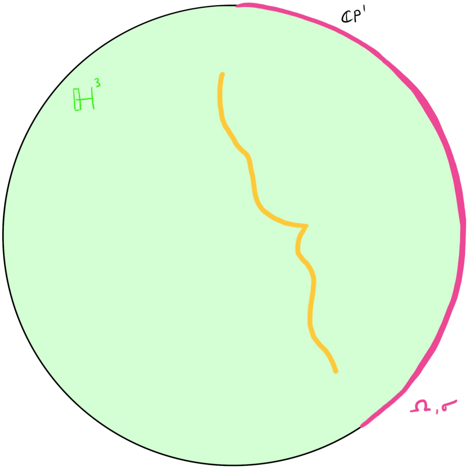

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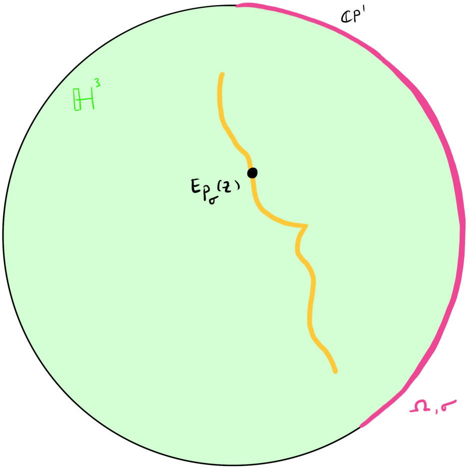

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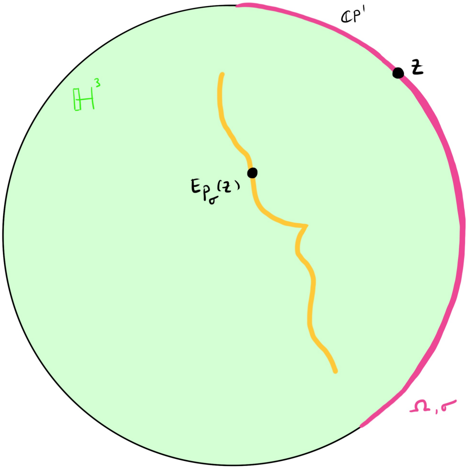

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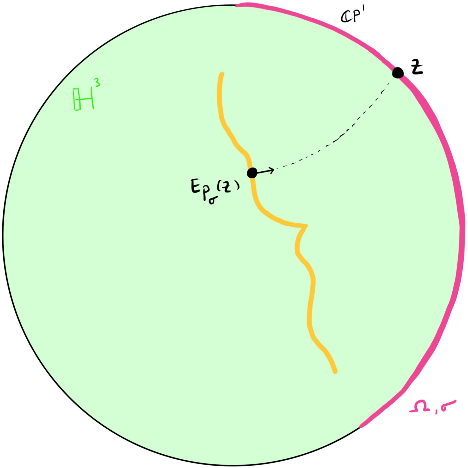

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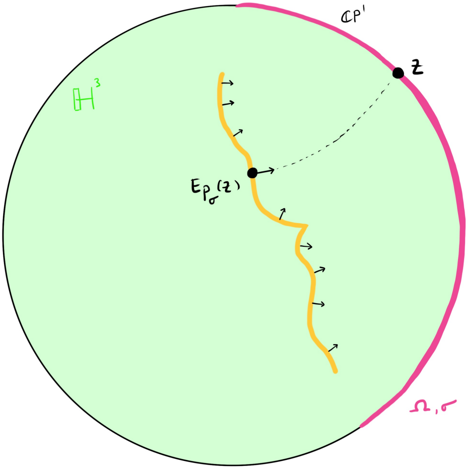

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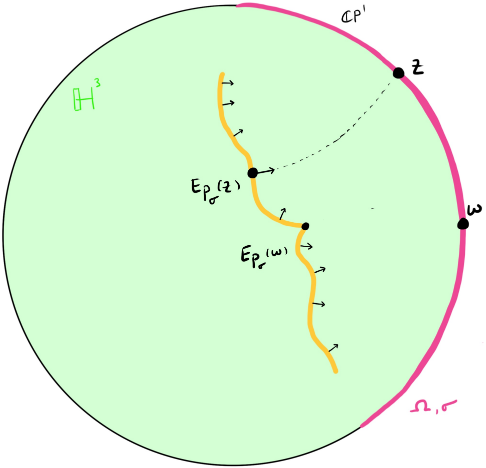

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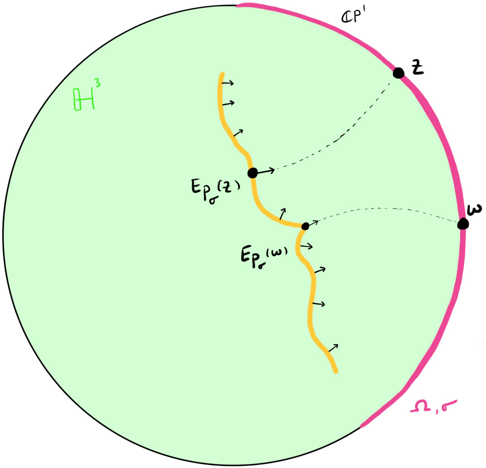

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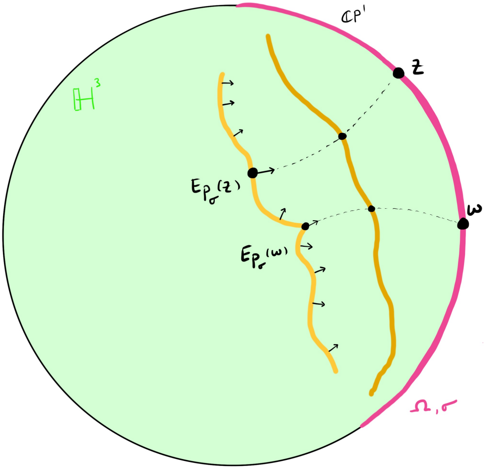

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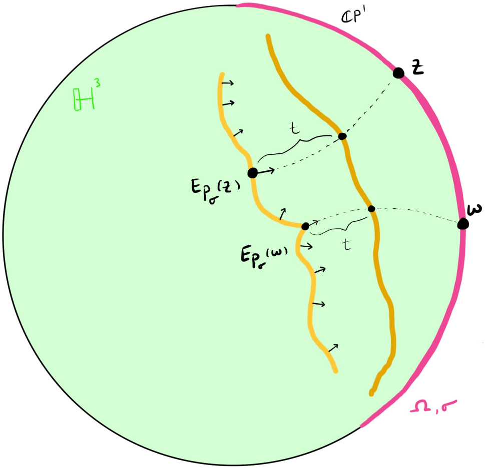

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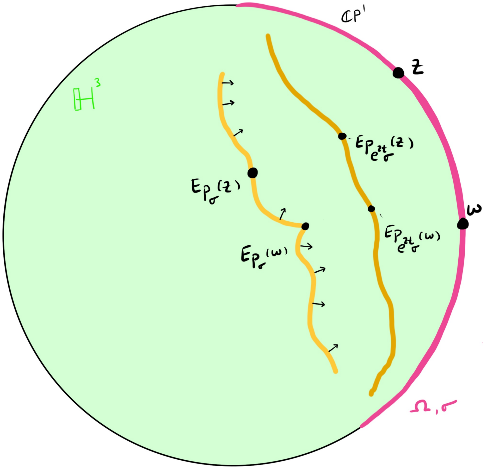
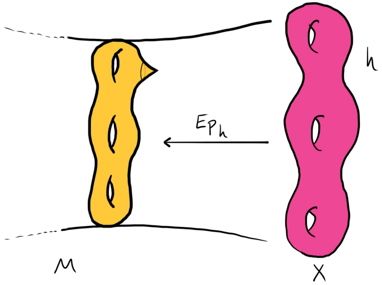


The Poincaré Family

We call the unique hyperbolic conformal metric h on X the Poincaré metric and its Epstein surface $\mathrm{Ep}_h : X \to M$ the Poincaré surface of X.



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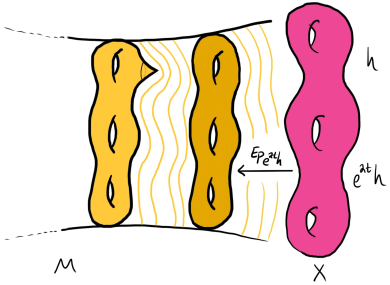
The family obtained by parallel flowing Ep_h we call the Poincaré Family, and this family is given as $(\mathrm{Ep}_{e^{2t}h})_{t\geq 0}$.



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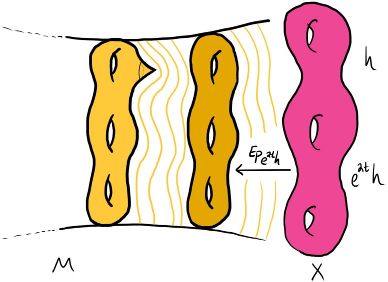

For $t > c = c(\phi, h)$, the surfaces $\mathrm{Ep}_{e^{2t}h}$ are embedded, and so for t sufficiently large, this family forms a foliation of the end of M.

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The families of surfaces we consider are in this sense asymptotic to the Poincaré family.

Asymptotically Poincaré Families

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$$f\sigma : (0,1) \to \mathrm{Met}^\infty(X)$$

is differentiable and converges to h, as $\epsilon \to 0$, in the space $\mathrm{Met}^\infty(X)$ *of smooth metrics on X,*

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2. *the function f is smooth, $f(0) = 0$, and $f'(0) \neq 0$,*
3. *the continuous extension $\gamma : [0,1) \to \mathrm{Met}^\infty(X)$ of $f\sigma$ is differentiable.*

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ *be an asymptotically Poincaré family for the conformal metrics* $\sigma(\epsilon)$. *There exists a* $\delta > 0$ *(depending on the family) such that for* $\epsilon \in (0, \delta)$ *the surfaces* S_ϵ *foliate the end of* M.

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Proof.

- The 1-parameter family of associated Epstein maps gives a map $\mathrm{Ep}_\sigma : [0, 1) \times X \longrightarrow M \cup X$ that restricts to the identity on the boundary $\{0\} \times X \to X$.

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- The function given by $E(\epsilon, x) = \mathrm{Ep}_\sigma(\epsilon^2, x)$ has invertible derivative on the boundary and so it is a local C^1-diffeomorphism from a neighborhood of the boundary into $M \cup X$.

Asymptotically Poincaré Families Foliate

Proposition (Q.)

Let S_ϵ be an asymptotically Poincaré family for the conformal metrics $\sigma(\epsilon)$. There exists a $\delta > 0$ (depending on the family) such that for $\epsilon \in (0, \delta)$ the surfaces S_ϵ foliate the end of M.

Proof.

- The 1-parameter family of associated Epstein maps gives a map $\mathrm{Ep}_\sigma : [0, 1) \times X \longrightarrow M \cup X$ that restricts to the identity on the boundary $\{0\} \times X \rightarrow X$.
- The function given by $E(\epsilon, x) = \mathrm{Ep}_\sigma(\epsilon^2, x)$ has invertible derivative on the boundary and so it is a local C^1-diffeomorphism from a neighborhood of the boundary into $M \cup X$.
- Therefore, the restriction of E to $[0, \delta^2) \times X$, for some small enough δ, is a diffeomorphism onto a collar neighborhood of X in $M \cup X$.

□

Main Results

Theorem (Q.)

Let S_ϵ for $\epsilon \in (0,1)$ be an asymptotically Poincaré family of surfaces for the conformal metrics $\sigma(\epsilon)$. If h is the Poincaré metric of X and ϕ the holomorphic quadratic differential at infinity, then in Teichmüller space we have

$$[I(\sigma(\epsilon))] \to [h] \quad \textit{and} \quad [II(\sigma(\epsilon))] \to [h] \quad \textit{as } \epsilon \to 0,$$

where $[g]$ denotes the point in $\mathcal{T}(X)$ represented by the Riemannian metric g.

Moreover, the tangent vectors at $\epsilon = 0$ in $T_{[h]}\mathcal{T}(X)$ are given by

$$[I(\dot{\sigma}(\epsilon))] = 4f'(0)\mathrm{Re}(\phi) \quad \textit{and} \quad [II(\dot{\sigma}(\epsilon))] = 0.$$

Riemannian Model of $\mathcal{T}(X)$

We model $\mathcal{T}(X)$ as the quotient

$$\mathrm{Met}^{\infty}(X)/(\mathrm{Diff}_0^{\infty}(X) \ltimes P^{\infty}(X))$$

of smooth metrics on X by the semi-direct product of $\mathrm{Diff}_0^{\infty}(X)$, the group of smooth diffeomorphisms isotopic to the identity, and $P^{\infty}(X)$, the group of smooth positive functions on X.

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However, we will first work with Sobolev tensors.

Denote by $\mathrm{Met}^s(X)$ the set of Riemannian metrics of Sobolev class H^s for a fixed $s > 3$ and note that $\mathrm{Met}^\infty(X) = \cap_{s>3}\mathrm{Met}^s(X)$

With these regularity assumptions $T_h\mathrm{Met}^s(X)$ is a Hilbert space.

Riemannian Model of $\mathcal{T}(X)$

$T_h \mathrm{Met}^s(X)$ decomposes as the orthogonal direct sum

$$\{g \mid \mathrm{tr}_h(g) = 0 = \mathrm{div}_h(g)\} \oplus \{\mathcal{L}_V h + uh \mid u \in H^s, V \in \Gamma(TX)\}.$$

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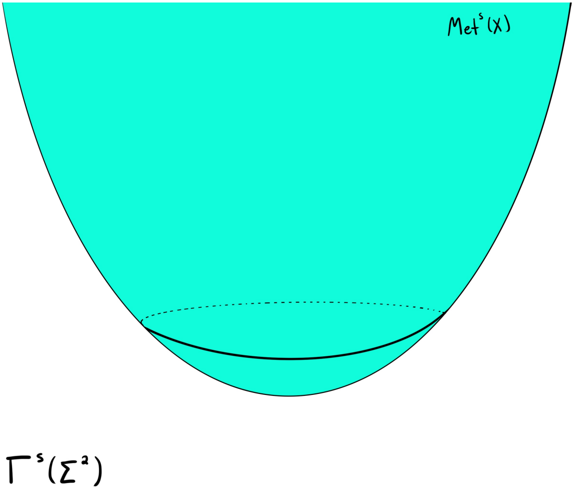

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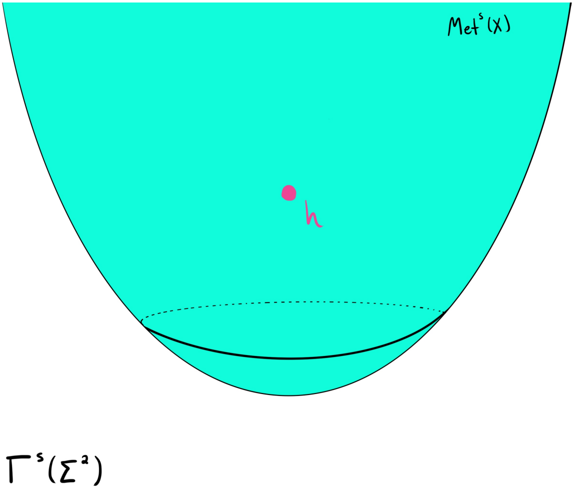

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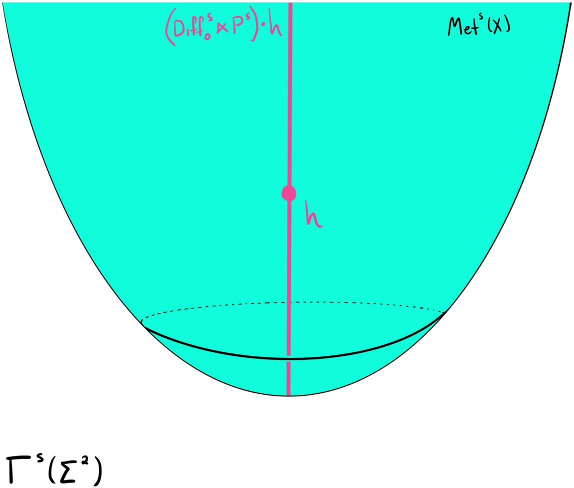

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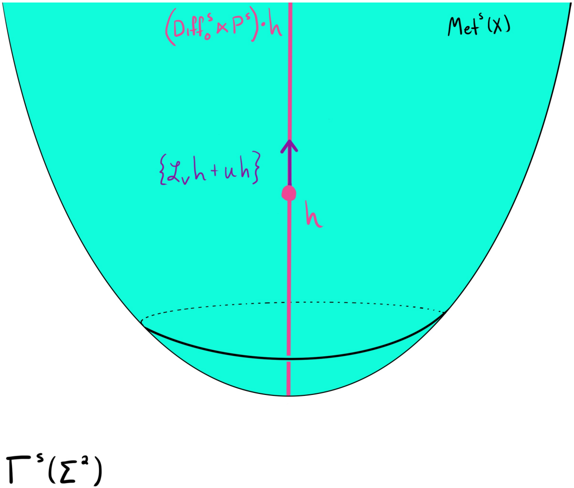

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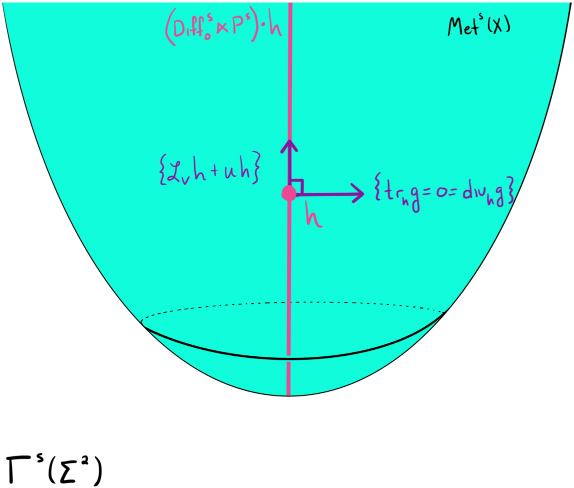

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The set $\{g \mid \mathrm{tr}_h(g) = 0 = \mathrm{div}_h(g)\}$ is a model for the tangent space to the quotient $\mathrm{Met}^s(X)/(\mathrm{Diff}_0^s(X) \ltimes P^s(X))$ at $[h]$.

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$$\begin{aligned} T_{[h]}\mathcal{T}(X) &= \{g \mid \mathrm{tr}_h(g) = 0 = \mathrm{div}_h(g)\} \\ &= \{\mathrm{Re}(\psi) \mid \psi \text{ a holomorphic quadratic differential on } X\}. \end{aligned}$$

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The projection $\mathrm{Met}^\infty(X) \to \mathcal{T}(X)$ is continuous and its derivative at h is given by orthogonal projection onto $\{\mathrm{Re}(\psi)\}$

Proof

Returning to our theorem, we are given that $\gamma : [0, 1) \longrightarrow \mathrm{Met}^{\infty}(X)$ is continuous and differentiable at $\epsilon = 0$. Therefore, we also have that γ is continuous to $\mathrm{Met}^{s}(X)$, for each s, and is differentiable at $\epsilon = 0$.

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In $\mathrm{Met}^{s}(X)$ we can write

$$I(\sigma(\epsilon)) = \frac{1}{4\epsilon f'(0)} h + \frac{1}{4f'(0)} \dot{\gamma} + \frac{1}{2} h + \mathrm{Re}(\phi) + O(\epsilon),$$

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so that

$$I_{\epsilon} := 4\epsilon f'(0) I(\sigma(\epsilon)) = h + \epsilon(\, \dot{\gamma} + 2 f'(0) h + 4 f'(0) \mathrm{Re}(\phi)\,) + O(\epsilon^{2}).$$

Proof

From the expression

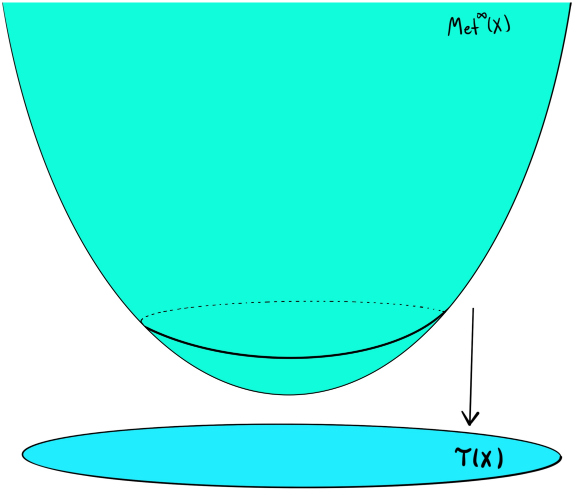
$$I_\epsilon = h + \epsilon(\,\dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi)\,) + O(\epsilon^2)$$

we can see that $I_\epsilon \to h$ in $\mathrm{Met}^s(X)$ for all s, implying $I_\epsilon \to h$ in $\mathrm{Met}^\infty(X)$.

Moreover, the derivative at $\epsilon = 0$ is

$$\dot{I}_\epsilon = \dot{\gamma} + 2f'(0)h + 4f'(0)\mathrm{Re}(\phi).$$

Proof



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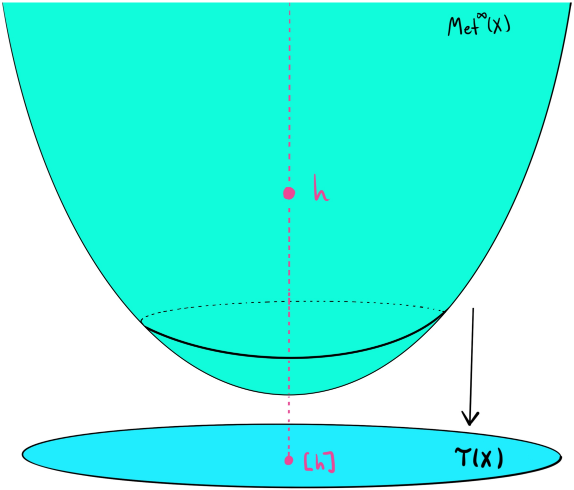

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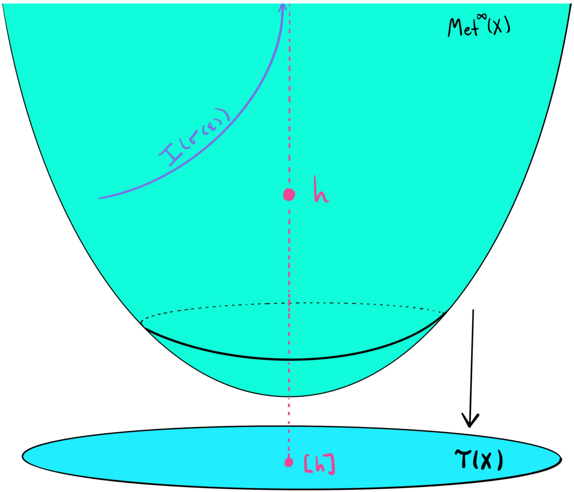

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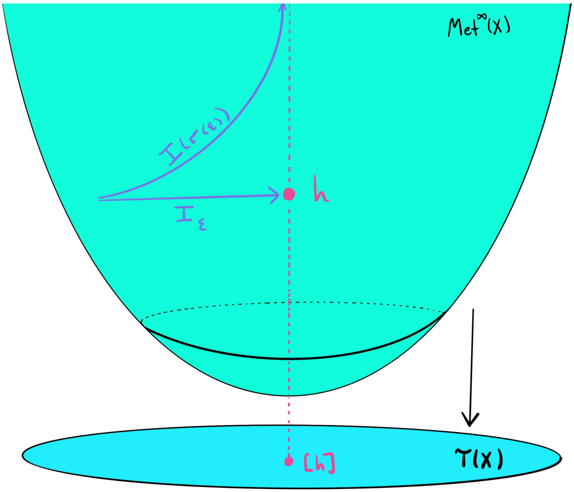

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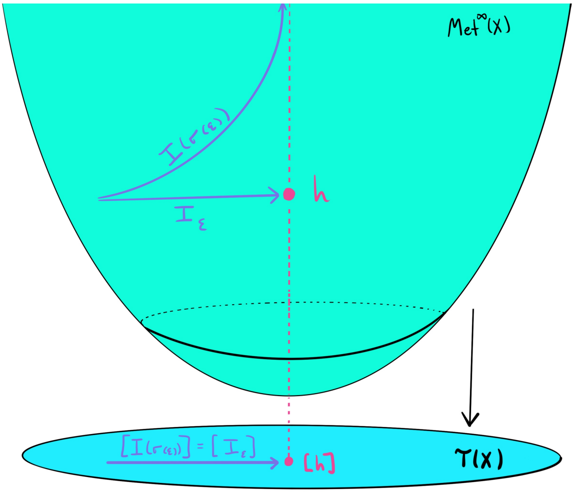

Therefore, as $I_\epsilon \longrightarrow h$ in $\mathrm{Met}^\infty(X)$ we have that $[I(\sigma(\epsilon))] = [I_\epsilon] \longrightarrow [h]$ in $\mathcal{T}(X)$.

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And since
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we see

$$[I(\dot{\sigma}(\epsilon))] = 4f'(0)\mathrm{Re}(\phi).$$

If we apply the same arguments to the Taylor series for $II(\sigma(\epsilon))$ then we see $[II(\sigma(\epsilon))] \longrightarrow [h]$ and $[II(\dot{\sigma}(\epsilon))] = 0$.

$\square$

Applications

We apply this result to two foliations by constant curvature surfaces.

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Theorem (Labourie, 1992)

Let E be an end of a quasi-Fuchsian manifold M, then for each $k \in (-1, 0)$, there exists a unique (incompressible) surface embedded in E with constant Gaussian curvature k. Moreover, this family of surfaces foliates the end E.

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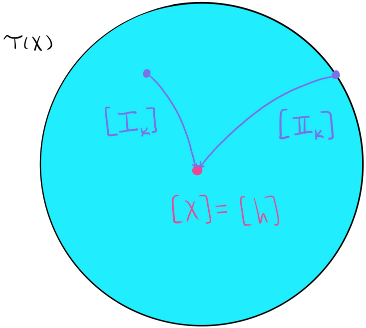
Theorem (Mazzeo-Pacard, 2011)

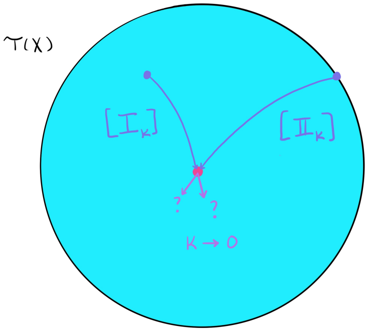
Each end of a quasi-Fuchsian manifold admits a unique foliation by constant mean curvature surfaces.

A Conjecture of Labourie

Labourie called the constant Gaussian curvature surfaces k-surfaces and he discusses how their first and second fundamental forms may be considered as paths in Teichmüller space.



A Conjecture of Labourie



He shows that as $k \to 0$, the paths $[I_k]$ and $[II_k]$ converge to $[X] = [h]$ and he asks after the tangent vectors to these paths at $k = 0$.

He conjectures $[\dot{I}_k]$ is related to the holomorphic quadratic differential ϕ.

A Conjecture of Labourie

Theorem (Q.)

Let I_k and II_k be the first and second fundamental forms of the family of k-surfaces in an end of a quasi-Fuchsian manifold. Let ϕ be the holomorphic quadratic differential at infinity of M. Then, as $k \to 0$, the tangent vectors to $[I_k]$ and $[II_k]$ in Teichmüller space are given by

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This will follow from our main theorem if we can show the k-surfaces form an asymptotically Poincaré family. To do this we must present the k-surfaces as Epstein surfaces.

Proof

The Gaussian curvature of an Epstein surface for a conformal metric σ can be calculated as

$$K(I(\sigma)) = \frac{4K(\sigma)}{(1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}}.$$

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- $B(\sigma)$ is the Schwarzian tensor of σ. This is a quadratic differential (not necessarily holomorphic) depending on σ and its first two derivatives.

To find a k-surface, we therefore need to solve $K(I(\sigma)) = k$ for σ.

Proof

We focus on solving the equivalent equation,

$$4K(\sigma) = k\left((1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}\right) \qquad (*)$$

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$$4K(\sigma) = k\left((1 - K(\sigma))^2 - 16|B(\sigma)|^2\sigma^{-2}\right) \qquad (*)$$

Our technique is to use the Implicit Function Theorem to obtain solutions for each k near 0.

Notice though, that when $k = 0$ this reads $K(\sigma) = 0$, which has no solutions on X.

Proof

Since there are no solutions for $k = 0$ we rescale the equation by considering the function $f(k) = \frac{1-\sqrt{1+k}}{1+\sqrt{1+k}}$.

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If σ solves $(*)$ then $\tau = f(k)\sigma$ solves

$$(2+k)(1+K(\tau))^2 + 2\sqrt{1+k}\left(1 - K(\tau)^2\right) + 16\left(2\sqrt{1+k} - 2 - k\right)\frac{|B(\tau)|^2}{\tau^2} = 0$$

or

$$F(k,\tau) = 0.$$

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Note that $F(0,\tau) = 0$ is equivalent to $K(\tau) = -1$, which has the solution $\tau = h$.

Proof

Since we have a solution for $k = 0$, we can then use the Implicit Function Theorem to get solution for k near zero as well.

The function F, while defined on smooth conformal metrics $\mathrm{Conf}^\infty(X)$, extends to one of Sobolev conformal metrics $\mathrm{Conf}^s(X)$. We have

$$F : (-1, 0) \times \mathrm{Conf}^s(X) \to H^{s-2}(X).$$

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$$F : (-1, 0) \times \mathrm{Conf}^s(X) \to H^{s-2}(X).$$

We compute

$$DF_{(0,h)}(\dot{k}, \dot{\tau}) = 4DK_h(\dot{\tau}) = -2(\Delta_h - Id)\frac{\dot{\tau}}{h}$$

and see that $D_2F_{(0,h)} = 4DK_h : H^s(X) \to H^{s-2}(X)$ is an isomorphism.

Proof

Consequently, there exists a neighborhood V of 0 and a curve $\gamma : V \to \mathrm{Conf}^s(X)$ with $\gamma(0) = h$ and $F(k, \gamma(k)) = 0$.

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Elliptic regularity theory applied to DK_h tells us that there is a $\delta > 0$ such that

- $\gamma(k)$ is a smooth conformal metric for each $k \in (-\delta, \delta)$,

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- the path $\gamma : (-\delta, \delta) \to \mathrm{Conf}^\infty(X)$ is a smooth path.

To complete the proof we take $\sigma(k) := f(k)^{-1}\gamma(k)$ for $k \in (-\delta, 0)$.

The family of Epstein surfaces for $\sigma(k)$ then satisfy the definition of an asymptotically Poincaré family and our main theorem applies. $\square$

Constant Mean Curvature Surfaces

Theorem (Mazzeo-Pacard, 2011)

The ends of a quasi-Fuchsian manifold admit unique foliations by constant mean curvature surfaces.

Theorem (Q.)

Let I_k and II_k be the first and second fundamental forms of the Epstein surface with constant mean curvature $-\sqrt{1+k}$. Let ϕ be the holomorphic quadratic differential at infinity. Then, as $k \to 0$, the tangent vectors to $[I_k]$ and $[II_k]$ in Teichmüller space are given by

$$[\dot{I}_k] = -\mathrm{Re}(\phi) \quad \textit{and} \quad [\dot{II}_k] = 0.$$

Proof

The proof proceeds the same as in the constant Gaussian curvature case. However, here we wish to solve the equation

$$H(\mathrm{Ep}_\sigma) = \frac{K(\sigma)^2 - 1 - 16|B(\sigma)|^2\sigma^{-2}}{(K(\sigma) - 1)^2 - 16|B(\sigma)|^2\sigma^{-2}} = -\sqrt{1+k}.$$

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Using the same scaling function $f(k)$ we are led to consider solutions to a function $G(k,\tau) = 0$, which has partial derivative $D_2G_{(0,h)} = 2DK_h(\dot{\tau})$.

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The Implicit Function Theorem then gives solutions for k near zero. □

Thank you!