AMAZING THESIS

KEATON QUINN

DRAFT VERSION 1 — June 18, 2019

1. Introduction

2. Preliminaries

2.1. **The Geometry of Surfaces.** Let S be a closed, oriented, smooth surface. A Riemannian metric g on S is a smooth section of the symmetric tensor product of the cotangent bundle of S, i.e., $g \in \Gamma(\Sigma^2 T^* S)$, such that $g(p) : T_p S \times T_p S \to \mathbb{R}$ is an inner product. The non-degeneracy of g at each point allows us to identity TS with T^*S via the map that sends $v \in T_p S$ to $g_p(\cdot, v) \in T_p^* S$ and this extends to an identification of vector fields and 1-forms. The Levi-Civita connection ∇ of g on S is the unique torsion-free, metric connection on S

$$T^{\nabla} = 0 \text{ and } \nabla g = 0.$$

Suppose $(M, \tilde{g})$ is a 3-dimensional Riemannian manifold. When $f : S \to M$ is an immersion, the pullback tensor $I = f^* \tilde{g}$ is a Riemannian metric on S that we call the First Fundamental Form. If we regard f as an identification of S with its image in M, then we may also identify $T_p S$ with its image in $T_{f(p)} M$. We therefore have a $\tilde{g}$-orthogonal splitting

$$T_p M = T_p S \oplus N_p S.$$

Here, $N_p S$ is the normal space to the surface in M given by all vectors orthogonal to S at p. The disjoint union of the normal spaces forms a vector bundle (in this case a line bundle) called the Normal Bundle of S. By our assumptions, there is a unit normal vector field n so that S is co-oriented in M.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$ on M. By extending vector fields X and Y on S to a neighborhood of S in M, we may take $\widetilde{\nabla}_X Y$. This resulting vector field is not necessarily also tangent to S and so we may decompose it into its tangential part $(\widetilde{\nabla}_X Y)^{\top}$ and its normal part $(\widetilde{\nabla}_X Y)^{\perp}$. The Gauss Formula tells us $(\widetilde{\nabla}_X Y)^{\top} = \nabla_X Y$ and since the normal bundle is spanned by n we have $(\widetilde{\nabla}_X Y)^{\perp} = I\!I(X, Y) n$ for a symmetric 2-tensor field $I\!I \in \Gamma(\Sigma^2 T^* S)$. This $I\!I$ we call the Second Fundamental Form.

Given a torsion-free connection ∇ on S and a q-form ω with values on TS, the exterior covariant derivative of ω is

$$d^{\nabla} \omega = \mathrm{Alt}(\nabla \omega).$$

By considering vector fields as 0-forms with values in TS we define the Riemann Curvature Endomorphism of ∇ as the 2-form with values in $\mathrm{End}(TS)$ given by

$$R^{\nabla}(X, Y) Z = (d^{\nabla} \circ d^{\nabla} Z)(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemann Curvature Tensor is then

$$Rm(X,Y,Z,W) = g(R^{\nabla}(X,Y)Z,W).$$

and the Gaussian curvature of S at a point p is the function $K(g)$ defined by

$$K(g)(p) = \frac{Rm(v,w,w,v)}{\|v\|^2\|w\|^2 - g(v,w)}$$

where v, w is any basis for T_pS. One dimension upwards in M, the Sectional Curvature of $\tilde{g}$ is a function on 2-planes in the tangent bundle $sec : \mathrm{Gr}_2(TM) \to \mathbb{R}$ whose value on $\Pi \leq T_pM$ is the Gaussian curvature at p of the image of Π in M under the exponential map.

The Gauss Equation relates the Gaussian curvature K of S to the sectional curvature of the ambient manifold M via the first and second fundamental forms. Since $I\!I$ is symmetric and since I is non-degenerate, we may form the shape operator $I^{-1}I\!I : TM \to TM$. The Gauss Equations states

$$K(I) = sec(TS) + \det(I^{-1}I\!I).$$

3. Asymptotically Poincaré Families

4. k-surfaces

5. Constant Mean Curvature Surfaces

6. Future Directions