THERON J HITCHMAN

ELEMENTS OF LINEAR ALGEBRA, VOLUME II CAHIER DES LIGNES

I MADE IT MYSELF.

Contents

This is the dedication page. You should

know that dedication is an important part

of being successful in mathematics. Imagination

plays a role, but mostly mathematicians

are people who like the work just enough

to fail to give up when things become challenging.

Mathematics challenges everyone at one

point or another. Being confused and stuck

is a mathematician's natural state. The

professionals just got used to it.

So, stick with dedication, persistence,

and perseverance. Those you can build.

Don't give up. Talk about math with whoever

will join in. All the cool ideas are buried

under hard work.

Introduction

This is a *workbook*. It is a collection of tasks that you should do to try to learn linear algebra for yourself.

It is not a *textbook*. We don't have a textbook, as such things are understood today. We have a primer, the first volume of this set: *Livre des Lignes*, which should serve as a basic source for reading about the main ideas, and we have this workbook. The books are separate so that you may have them both open at once. I hope this works.

It is important to focus on your own understanding when working on these tasks. At every stage possible, you should ask yourself, "Self, how do I know this?" and maybe also, "So, Self, do I know this for sure? Or do I still have doubts?"

If you are feeling at a loss for how to explain your thinking, try relying on simple geometric models and reasoning. It turns out that most of even the fanciest linear algebra is done that way. So, "Self, can you draw the relevant picture? Does that help?"

Good luck. I'll see you in class, and I welcome your visits to office hours to talk about mathematics.

Vectors and Lines the Plane

Points and Vectors

Task 1. Write down three distinct points in the plane in proper notation. Plot those three points on a single diagram.

Task 2. Write down three distinct vectors in the plane in proper notation. Your three points from this task should NOT match any of the three points from the previous task. Plot those three vectors on a single diagram.

Definition. If $u_1, u_2, \ldots, u_n$ is a collection of vectors, and $\lambda_1, \lambda_2, \ldots$ is a collection of scalars, then the vector formed below is called a *linear combination* of the u_i's.

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots \lambda_n u_n$$

Task 3. Find the sum of your three vectors from the last exercise. Then, choose some order of those three vectors so that they are u_1, u_2 and u_3, and compute the linear combination

$$3u_1 - 2u_2 + (1/2)u_3.$$

Task 4. Let's consider the vectors $u = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)$. Compute all of the vectors in this list:

$$\frac{u+v}{2}, v - u, v - w, u + \left(\frac{v-u}{3}\right), u + \left(\frac{3(v-u)}{4}\right)$$

Then make a single diagram which contains u, v, w and all of those vectors from the list, plotted as accurately as you can.

What do you notice? Is anything interesting going on?

Task 5. Consider the vector $u = \left(\begin{smallmatrix} -4 \\ 2 \end{smallmatrix}\right)$. Find a vector v which has the property that $u + v$ is the zero vector, or explain why this is not possible.

Definition. An equation of the form $\lambda_1 u_1 + \ldots \lambda_n u_n = w$, where all of the vectors u_i and w are known, but the scalars λ_i are unknown, is called a *linear combination of vectors equation*. A *solution* to such an equation is a collection of scalars which make the equation true.

Task 6. For now, keep $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$. Let $v = \left(\begin{smallmatrix}3\\5\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Here, treat 0 as the zero vector.)

Task 7. We still use the notation u for the vector $u = \left(\begin{smallmatrix}-4\\2\end{smallmatrix}\right)$, but now use v for the vector $v = \left(\begin{smallmatrix}28\\-14\end{smallmatrix}\right)$. How many solutions does the linear combination of vectors equation $\lambda u = v$ have?

How many solutions does the linear combination of vectors equation $\lambda u + \mu v = 0$ have? (Again, treat 0 as the zero vector.)

Task 8. Find the midpoint between the points $P = (4, -2)$ and $Q = (3, 5)$. Then find the two points which divide the segment PQ into thirds.

How can vectors make this simpler than it first appears?

Challenge 9. Suppose you are given three points in the plane. Let's call them P, Q, and R. How can you use vectors to (quickly) determine if these three points are collinear?

Parametric Lines in the Plane

Challenge 10. How many different solutions to the equation

$$x \begin{pmatrix}3\\7\end{pmatrix} + y \begin{pmatrix}6\\14\end{pmatrix} = \begin{pmatrix}-3\\-7\end{pmatrix}$$

can you find? How many are they? Can you find a good way to describe all of the solutions? Is there a natural geometric way to describe all of the solutions? Is there an algebraic way to describe all of the solutions?

Task 11. Write down a parametric description for the line which passes through the origin $O = (0, 0)$ and the point $S = (-5, 5)$.

Is this the *only* way to write down such a parametric description for that line?

Task 12. Write down a parametric description for the line which passes through the points below.

$$T = (\pi, 0), \qquad J = (0, -\pi)$$

Now find a different parametric description for that line. (*Hint: Can you find a way to write a parametric description that doesn't use the number π?*)

Task 13. Compare the lines from the tasks 11 and 12. Do you note anything interesting? How do you know your observation is true?

Task 14. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$t \begin{pmatrix} 3 \\ -1 \end{pmatrix} = Y$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 15. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 16. For each of the conditions below, either find an example of a 2-vector Y so that the equation

$$\begin{pmatrix} -2/5 \\ 2 \end{pmatrix} + tY = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 17. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x \begin{pmatrix} -5/3 \\ 1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 7 \end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Task 18. For each of the conditions below, either find an example of a 2-vector Z so that the equation

$$x \begin{pmatrix} -5/3 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -3/5 \end{pmatrix} = Z$$

has the given number of solutions, or explain why such an example is not possible.

a) exactly zero solutions;
b) exactly one solution;
c) exactly two solutions.

Challenge 19. Find an example of four 2-vectors X, Y, Z, and W so that the equation

$$aX + bY + cZ = W$$

has at exactly two solutions, or explain why such an example is not possible.

Task 20.