LINEAR ALGEBRA KNOWLEDGE SURVEY

Instructions: This is a *Knowledge Survey*. For this assignment, I am only interested in your *level of confidence* about your ability to do the tasks on the following pages.

So, for each question, rather than do the task, please circle the number which indicates your level of confidence in your ability. There are three options each time, which represent the following feelings:

(1) I am confident that I cannot do this task, even with access to reference material.
(2) I need to be reminded of something, so by consulting a reference, I can do this task.
(3) I am confident that I can do this task right now without a reference.

Since you are not going to do the computations and figuring, this should go relatively quickly, even given the length of this paper. Simply indicate you confidence and move to the next item.

By the way, the questions below are grouped as they are for our course learning goals. Each section represents a bundle of questions that could be asked to check your understanding in one of our fifteen target topics:

- (core) Vector Algebra
- (core) Matrix Algebra
- (core) The Dot Product
- The Geometry of Lines and (Hyper)planes
- (core) Gauss-Jordan Elimination
- The Three Viewpoints
- (core) Solving Systems of Equations
- Implicit and Explicit Descriptions for Subspaces
- The Four Subspaces
- Approximate Solutions
- Matrices as Transformations
- Orthonormal Bases, Gram-Schmidt, and the QR Decomposition
- (core) The Determinant
- (core) Eigenvalues and Eigenvectors
- The Singular Value Decomposition

So I can match these later, please put your student ID number here: ____________________________

Date: Spring 2017.

1. Vector Algebra

Task 1.1. Write down an interesting linear combination of 3-vectors. Compute this linear combination.

1 2 3

Task 1.2. On a single set of axes, plot the vectors a and b below, along with their sum and their difference.

$$a = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

1 2 3

Task 1.3. Pick some empty space on a page and draw two dots. Label these dots as points v and w. The origin in this picture is somewhere off the edge of the page. (It doesn't really matter where.) Show where in the picture you can find the vectors (a) $v - w$, and (b) $(v + w)/2$.

1 2 3

Task 1.4. What shape is the collection of all linear combinations of the two vectors below?

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

1 2 3

2. Matrix Algebra

Task 2.1. Compute the following:

$$2 \begin{pmatrix} 7 & 0 & 2 \\ 3 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 20 \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

1 2 3

Task 2.2. Compute the following:

$$\begin{pmatrix} 7 & 6 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}^T, \qquad \begin{pmatrix} 7 & 6 \\ 0 & 5 \end{pmatrix}^T.$$

1 2 3

Task 2.3. There are several ways that multiplication of 2×2 matrices can be "not as convenient" as multiplication of numbers. Describe two of them with examples.

1 2 3

Task 2.4. There are two ways to think about the operation of multiplication of a matrix with a vector. *Discuss how they work.* Use each method to compute

$$\begin{pmatrix} 7 & 2 \\ 4 & 6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and verify that the two computations give the same result.

1 2 3

Task 2.5. Describe the three different ways to multiply matrices. Illustrate each method by multiplying these two matrices in the way that makes sense.

$$X = \begin{pmatrix} 7 & 2 \\ 4 & 6 \\ 1 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 10 & 9 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

1 2 3

3. The Dot Product

Task 3.1. Compute each of these:

$$\begin{pmatrix}2\\-7\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}, \qquad \begin{pmatrix}7\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\10\\6\end{pmatrix}.$$

1 2 3

Task 3.2. Compute

$$\left\|\begin{pmatrix}3\\2\\\sqrt{3}\end{pmatrix}\right\|.$$

1 2 3

Task 3.3. Find an expression which gives the angle between these two vectors:

$$u=\begin{pmatrix}1\\0\\5\end{pmatrix}, \qquad v=\begin{pmatrix}0\\-1\\8\end{pmatrix}.$$

1 2 3

Task 3.4. Find a unit vector which points in the same direction as

$$w=\begin{pmatrix}-2\\-3\end{pmatrix}.$$

1 2 3

Task 3.5. Suppose that $\mathcal{S}$ is the collection of all vectors in $\mathbb{R}^4$ which are perpendicular to both

$$a=\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \text{ and } b=\begin{pmatrix}0\\1\\0\\1\end{pmatrix}.$$

What shape is $\mathcal{S}$? How do you know?

1 2 3

4. The Geometry of Lines and (Hyper)planes

Task 4.1. The equation $-4x+3y=5$ defines a hyperplane $\mathcal{P}_1$ in $\mathbb{R}^3$. Give the equation of the hyperplane $\mathcal{P}_2$ which is parallel to $\mathcal{P}_1$ and passes through the point $p=(1,1,1)$. **Explain** how you know your answer is correct. **Make** a sketch of this situation.

1 2 3

Task 4.2. Write down the normal vector to the line $2x-3y=-10$ in the plane $\mathbb{R}^2$.

1 2 3

Task 4.3. Find an equation for the plane in $\mathbb{R}^3$ which passes through the points $P_1=(1,-1,1)$, $P_2=(0,2,5)$, and $P_3=(-1,1,0)$.

1 2 3

5. Gauss-Jordan Elimination

Task 5.1. Consider the matrix-vector equation below. Find the solution wihtout performing any row operations.

$$\begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$$

1 2 3

Task 5.2. Find the LU-decomposition of the matrix X:

$$X = \begin{pmatrix} 7 & 1 & 0 \\ 14 & 3 & 1 \\ 21 & 2 & 6 \end{pmatrix}$$

1 2 3

Task 5.3. Use the result of the last task to compute the determinant of X. What simple, but important, thing does this tell you about the matrix X?

1 2 3

Task 5.4. Describe how the LU decomposition process can break. Give an example.

1 2 3

Task 5.5. The Gauss-Jordan Elimination algorithm can break when the matrix under consideration doesn't have an inverse. What happens in such a case? That is, how do you know when to give up when computing an inverse via Gauss-Jordan Elimination?

1 2 3

Task 5.6. Use Gauss-Jordan Elimination to find the inverse of the matrix B:

$$B = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 2 & 1 \\ 5 & 4 & 2 \end{pmatrix}$$

1 2 3

6. The Three Viewpoints

Task 6.1. We have studied two other ways to express a system of linear equations as algebraic expressions. Translate

$$\begin{cases} 2x - 3y + z = 1 \\ 2y - w = 0 \\ 5y + z + 2w = -2 \end{cases}$$

into the two other forms.

1 2 3

Task 6.2. For each of the algebraic forms in the last task, describe the proper geometric interpretation and what it means, from the perspective of the geometry, to find a solution.

1 2 3

Task 6.3. Consider the equation

$$x\begin{pmatrix} 6 \\ * \end{pmatrix} + y\begin{pmatrix} 9 \\ * \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use the rows to describe all of the possible ways to choose the $*$'s so that this equation will have a solution.

1 2 3

Task 6.4. Staying with the system in the last task, use the columns to discuss all of the possible ways to choose the $*$'s so that the equation does NOT have a solution.

1 2 3

7. Solving Systems of Equations

Task 7.1. Find the complete solution of the following system of equations. Label the important parts of the process as you go.

$$\begin{cases} 2x - y - z = -1 \\ x + y + w = 1 \\ -x + 2y + z + w = 2 \end{cases}$$

1 2 3

Task 7.2. Describe how the column space of a matrix A helps us understand the process of solving an equation of the form $Ax = b$.

1 2 3

Task 7.3. Let B be a 3×5 matrix and d a 3-vector. Suppose the equation $Bx = d$ has complete solution

$$\begin{pmatrix} -2 \\ 0 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

What is the reduced row-echelon form of the matrix B?

1 2 3

Task 7.4. Is it possible to have a system of 35 equations in 10 unknowns whose solution set is 5-dimensional? If not, say why not. If so, describe how to construct an example.

1 2 3

8. Implicit and Explicit Descriptions for Subspaces

Task 8.1. Give an example of a set of 4-vectors that is linearly independent but is not a basis for $\mathbb{R}^4$, or explain why that is impossible.

1 2 3

Task 8.2. Give an example of a set of 4-vectors which spans $\mathbb{R}^4$ but is not a basis for $\mathbb{R}^4$, or explain why that is impossible.

1 2 3

Task 8.3. Let $\mathcal{U}$ be the subspace of $\mathbb{R}^5$ which is described by the equations

$$\begin{cases} -x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 0 \\ 10x_1 + 3x_2 + x_3 + x_4 + 2x_5 = 0 \end{cases}$$

Give a basis for $\mathcal{U}$.

1 2 3

Task 8.4. Let $\mathcal{T}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors below. Find equations which describe $\mathcal{P}$ as an intersection of hyperplanes.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 5 \\ 1 \end{pmatrix}$$

1 2 3

9. The Four Subspaces

Task 9.1. Without resorting to performing matrix multiplication or row operations, give a basis (if you can) or the dimension (if it is not feasible to find a basis) for each of the four subspaces associated to the matrix below. Be sure to explain how you know what you know.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 7 & 6 & 5 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

1 2 3

Task 9.2. Find bases for each of the four subspaces of the matrix B, given the data below:

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & -3 & -1 & 6 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{rref}(B^T) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1 2 3

Task 9.3. What interesting things can be said about the four subspaces of a symmetric matrix?

1 2 3

10. Approximating Solutions

Task 10.1. We wish to fit a curve of the form

$$f(x) = a + bx^2 + cx^3$$

to the data below as best we can. Show how to set up a simple system of equations which can be solved exactly to find the coefficients a, b, c which make $y = f(x)$ best fit the data.

x	$f(x)$
-1	0
1	2
2	3
2	4
3	2

1 2 3

Task 10.2. Draw a clear diagram which shows how orthogonal projection onto a subspace works. Be sure to label all of the important parts, and to exclude anything that is not essential.

1 2 3

Task 10.3. Suppose that the subspace T of $\mathbb{R}^4$ is described as a solution set of the equations

$$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 0 \\ 4x_1 - 2x_2 + x_3 - 2x_4 = 0 \end{cases}$$

Find the point on T which is nearest to $b = (1, 0, 1, 0)$. It is not necessary to actually go as far as to compute the point exactly. But do give an expression from which the point can be computed easily.

1 2 3

11. Matrices as Transformations

Task 11.1. How do the four subspaces help us understand how a matrix behaves as a function? Discuss how each of the four subspaces gives us information about how x turns into Ax. Draw a helpful diagram.	**1 2 3**
Task 11.2. Suppose that A and B are non-square matrices so that AB is a 5×5 symmetric square matrix of rank 5. What can you say about the four subspaces of A and B, and the relationships between them?	**1 2 3**
Task 11.3. Suppose that C and D are both 2×2 matrices and that $CD = DC$. What can you say about the four subspaces of C and the four subspaces of D and the relationships between them?	**1 2 3**

12. Gram-Schmidt, etc.

Task 12.1. Give an example of a 3×3 orthogonal matrix which is not the identity matrix.	**1 2 3**
Task 12.2. Give an example of a 2×2 matrix which is not orthogonal, and say how you know you are correct.	**1 2 3**
Task 12.3. Describe an example of a set of three vectors in $\mathbb{R}^3$ for which the Gram-Schmidt algorithm breaks, and fails to produce an orthonormal basis of $\mathbb{R}^3$. What feature of your example makes this happen?	**1 2 3**
Task 12.4. Find the QR decomposition of X: $$X = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$	**1 2 3**
Task 12.5. Find an orthonormal basis for $\mathcal{T}$: $$\mathcal{T} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$	**1 2 3**

13. The Determinant

Task 13.1. Suppose that A is a square matrix. Give five conditions on the matrix A which are equivalent to $\det(A) \neq 0$.

1 2 3

Task 13.2. Using only row operations and their effects on the determinant, and the fact that

$$\det \begin{pmatrix} 2 & 7 & 9 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} = x,$$

find each of these determinants in terms of x:

$$a = \det \begin{pmatrix} 2 & 7 & 9 \\ 0 & -13 & -16 \\ 3 & 0 & 1 \end{pmatrix}, \quad b = \det \begin{pmatrix} 4 & 1 & 2 \\ 2 & 7 & 9 \\ 3 & 0 & 1 \end{pmatrix}, \quad c = \det \begin{pmatrix} 2 & 7 & 9 \\ 4 & 1 & 2 \\ 9 & 0 & 3 \end{pmatrix}.$$

1 2 3

Task 13.3. Compute each of these determinants:

$$\det \begin{pmatrix} 2 & 6 \\ 6 & 14 \end{pmatrix}, \det \begin{pmatrix} 3 & 1 & 1 \\ 9 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix}, \det \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

$$\det \begin{pmatrix} -2 & 1 & 4 & 0 & 7 \\ 2 & 2 & 2 & 2 & 2 \\ -6 & 3 & 12 & 0 & 21 \\ 34 & 8 & 1 & -11 & 0 \\ 1 & -1 & 2 & 1 & -2 \end{pmatrix}.$$

1 2 3

14. Eigenvalues and Eigenvectors

Task 14.1. Find the eigenvalues and eigenvectors of A:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 6 & 2 & 8 \end{pmatrix}.$$

1 2 3

Task 14.2. Show how to construct a matrix which has the following vectors as eigenvectors

$$v_1 = \begin{pmatrix} 3 \\ 7 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 9 \\ 0 \\ -1 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 3 \end{pmatrix}$$

with corresponding eigenvalues $2, 3, 4$ and -5, or explain why such a thing is not possible.

1 2 3

Task 14.3. Give an example of a 2×2 matrix with enough eigenvectors to make a basis of $\mathbb{R}^2$, but not enough eigenvectors to make an orthonormal basis, or explain why no such example is possible.

1 2 3

Task 14.4. Give an example of a 3×3 matrix which does not have enough eigenvectors to make a basis of $\mathbb{R}^3$, or explain why such a thing is impossible. Either way, be sure to explain how you know.

1 2 3

15. The Singular Value Decomposition

Task 15.1. Design a 3×4 matrix B with rank 2 and singular values $\sigma_1 = \sqrt{10}$ and $\sigma_2 = 1$ such that

$$B \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \sqrt{10} \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \text{ and } B \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

or explain why this is not possible.

1 2 3

Task 15.2. Find the Singular Value Decomposition of the matrix

$$C = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

1 2 3