0.1 The Algebra of Matrices, Basics

Now that we are considering matrices as objects in their own right, we should think through their properties more carefully.

The Additive Structure

It is possible to add matrices, as long as they have the same shape. We simply add them component-by-component. If the matrices have different shapes, then addition is not defined. Fortunately for us, this kind of addition behaves just like regular addition of numbers.

Definition 1. Let A and B be $m \times n$ matrices. The *sum* $C = A + B$ of A and B is the matrix whose ij entry is the sum of the ij entries of A and B.

Theorem 2 (Properties of Matrix Addition)**.** Matrix addition behaves much like addition for numbers. In particular, these properties hold for any three matrices A, B, and C of the same shape $m \times n$:
closure: $A + B$ is also an $m \times n$ matrix.
associative law: $(A + B) + C = A + (B + C)$.
existence of an identity: There is a matrix 0 so that $A + 0 = 0 + A = A$.
additive inverses: There is a matrix $-A$ so that $A+(-A) = (-A)+A = 0$.
commutative law: $A + B = B + A$.

Remark 3. A set of things which has an operation satisfying this list of properties is usually called an *abelian group* or a *commutative group*. The name is chosen to honor Norwegian mathematician Niels Henrik Abel.

In fact, all of the structure of addition is basically just recognizing that matrices are vectors that are stacked in a weird way — they are not just one vertical column, but several of them. That includes the fact that there is a scalar multiplication.

Definition 4. If A is a matrix and λ is a scalar, then we define the *scalar product* λA to be the matrix obtained by multiplying component-by-component all of the entries of A by λ.

Since we have both an addition and a scalar product, it makes sense to form linear combinations of matrices which a have the same shape.

Theorem 5 (Properties of Scalar Multiplication)**.** Suppose that A and B are vectors, and λ and μ are numbers. Scalar multiplication on matrices has the following properties:

- Scalar multiplication distributes over vector addition: $\lambda(A+B) = \lambda A + \lambda B$;
- Scalar multiplication distributes over scalar addition: $(\lambda + \mu)A = \lambda A + \mu A$;
- Scalar multiplication and regular multiplication can be done in either order: $\lambda(\mu A) = (\lambda\mu)A$;
- if $\lambda = 0$, then $\lambda A = 0A = 0$ is the zero matrix.
- if n is a counting number, then nA is the same thing as adding together n copies of A. In particular, $1A = A$.

Remark 6. With addition and scalar multiplication, the set of all matrices of shape $m \times n$ is an example of a vector space. You can think of it as a weird way to write down $\mathbb{R}^{mn}$. When you study modern abstract algebra, this set will likely be called something like $M_{m,n}(\mathbb{R})$.

The Transpose

There is a truly new operation for matrices, the *transpose*. The funny thing is that it changes a matrix from one shape to a different shape.

Definition 7. Let A be an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix A^T obtained by exchanging the rows and the columns. That is, the first row of A becomes the first column of A^T, etc.

Theorem 8. Suppose that u and v are vectors in $\mathbb{R}^n$. Then the dot product of u and v can be computed as the matrix product $u^T v$, where we think of u and v as $n \times 1$ matrices.

Remark 9. This trick of writing a vector as an $n \times 1$ matrix, i.e. a matrix with a single column, is really useful for writing things out. It gives us a convenient shorthand for writing those nasty columns as simpler rows (English is written across, not up or down). We can just write a row, and slap a transpose symbol on it!

Definition 10. A matrix is called *symmetric* if it is equal to its own transpose, that is, when $A = A^T$. A matrix is called *skew-symmetric* when it is the opposite of its transpose, that is, when $A = -A^T$.

Two Views on Matrix-vector Multiplication

Now that we have an extra structure on matrices (the transpose), we have another way to think about the action of a matrix on a vector. First, recall that we defined the matrix-vector product in a way that emphasizes columns. If A is written as a bundle of columns $v_1, v_2, \dots v_n$, and the vector in question is $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$, the product Ax is

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

But now we can re-organize things using the rows! Instead write A as a bundle of rows. Here we think of each row as the transpose of a vector r_i. Then the matrix product Ax is put together using the dot product!

$$Ax = \begin{pmatrix} — & r_1^T & — \\ — & r_2^T & — \\ & \vdots & \\ — & r_m^T & — \end{pmatrix} x = \begin{pmatrix} r_1^T x \\ r_2^T x \\ \vdots \\ r_m^T x \end{pmatrix} = \begin{pmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{pmatrix}$$

Exercises

Exercise 1. Choose some scalars. (Make a list.) Make up three different examples of pairs of $m \times n$ matrices, for different values of m and n. Use all of this to compute some linear combinations of matrices.

Exercise 2. Write down five different examples of matrices with a variety of different shapes. Compute the transposes of these matrices.

Exercise 3. Writing the vectors u and v as below, give an argument for why Theorem 8 is true.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Exercise 4. Make an example of a 3×3 matrix A with no entries equal to zero. Compute the matrix vector product Av, where v is the vector $v = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T$.

Exercise 5. Make an example of a 5×2 matrix B with no entries equal to zero. Carefully work out the action of B on the vector $w = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$ using both methods! Be sure to write out enough detail so that it is clear you know how to use both methods.

Exercise 6. Can a matrix which is not a square matrix be a symmetric matrix? Why or why not?

Suppose you have a matrix A. What is $(A^T)^T$? how do you know?

Challenge 7. Consider a generic 2×2 matrix:

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Find a way to write D as a sum $D = S_1 + S_2$, where S_1 is a symmetric matrix and S_2 is a skew-symmetric matrix.