LINEAR ALGEBRA EXAM THREE: FALL '14

Instructions: Be sure your name is on each sheet of paper. Explain your thinking clearly and in complete sentences. This examination has is broken into clearly defined sections, each corresponding to one of our learning goals. There is likely too much here to accomplish in an hour, so use your time wisely by focusing on those things you think you are best prepared to demonstrate.

The learning goals assessed on this examination paper are:

- Determinants
- Eigenvectors, Eigenvalues, & diagonalization
- (Advanced) The Singular Value Decomposition

1. The Determinant

Task 1.1. Suppose that A is an $n \times n$ square matrix. Give *five* statements which are equivalent to $\det(A) \neq 0$.

Task 1.2. How does the determinant of a matrix change when one of our standard row operations is performed on that matrix? (Remember that we have more than one type of row operation. Discuss each.)

Task 1.3. Compute each of the following:

$$\det\begin{pmatrix} 7 & 2 \\ 6 & 1 \end{pmatrix}, \qquad \det\begin{pmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ 9 & 1 & 1 \end{pmatrix}, \qquad \det\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad \det\begin{pmatrix} 3 & 1 & 2 & -5 & -1 \\ 6 & 2 & 4 & -10 & -2 \\ 2 & 6 & 4 & 93 & 1 \\ 0 & -10 & 0 & -2 & 10 \\ 9 & 6 & 1 & 0 & 7/4 \end{pmatrix}$$

2. Eigenvalues and Eigenvectors

Task 2.1. Give an example of a 3×3 matrix which does not have enough eigenvectors to make a basis of $\mathbb{R}^3$. Explain why you know your example works, or explain why such an example is impossible.

Task 2.2. Show how to construct a matrix which has the vectors

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 7 \\ 8 \\ 10/3 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

as eigenvectors with corresponding eigenvalues

$$\lambda_1 = \sqrt{3}, \qquad \lambda_2 = 4, \qquad \lambda_3 = 1/3,$$

respectively, or explain why this is impossible.

Task 2.3. Give an example of a 2×2 matrix which has enough eigenvectors to make a basis for $\mathbb{R}^2$, but not enough eigenvectors to make an orthonormal basis, or explain why this is not possible.

Task 2.4. Find the eigenvalues and eigenvectors of

$$X = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

3. Singular Value Decomposition

Task 3.1. Design a rank two 3×4 matrix B having singular values $\sigma_1 = 3$ and $\sigma_2 = 2$ so that

$$B \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad B \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Task 3.2. Find the SVD of the matrix
$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$
Discuss how the "geometrically good" bases of the four subspaces of C compare with the regular choices we get from looking at reduced row echelon forms.