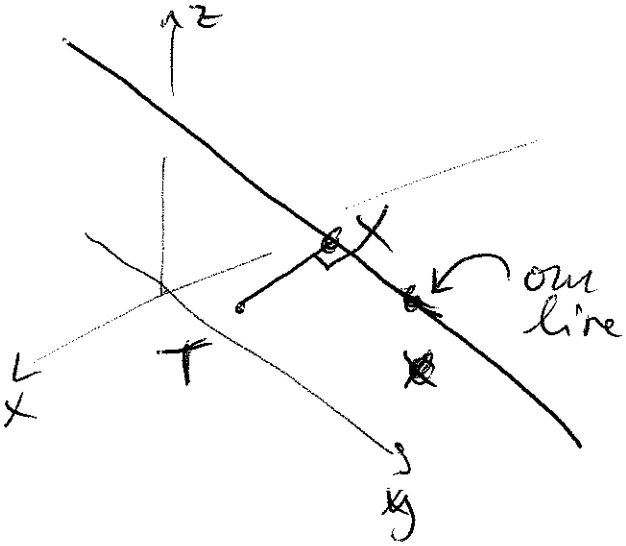
# Challenge 52 – Prof Hitchman's Soln

We have the parametric line $t \mapsto \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$, and we want the point on this line closest to $T = (1,1,1)$.

Suppose $X$ is the point we seek.




Then if $X$ is closest to $T$, $X - T$ is orthogonal to the line.

So if $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

$X - T$ is $\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix} = 0$

that is $3(x-1) - \frac{1}{2}(y-1) + (z-1) = 0$

for convenience, we rewrite this

$$3x - \frac{1}{2}y + z = 7/2 \qquad (\star)$$

this describes the plane which

(1) is orthogonal to the line; and

(2) passes through $T$.

Now our point X lies on that plane, and it lies on the original line. That means X has to both satisfy (✫) and be of the form

$$\begin{cases} x = -6 + 3t \\ y = -2 - \frac{1}{2} t \\ z = 1 + t \end{cases}$$

when it is

$$X = \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}$$

for some $t$.
if we unbundle that we get the equations on the left.

So now we combine all of this by putting the expressions involving $t$ into (✫). We obtain

$$3(-6+3t) - \frac{1}{2}\left(-2 - \frac{1}{2}t\right) + (1+t) = 7/2$$

which simplifies to

$$\frac{41}{4} t = \frac{49}{2}$$

so $t = \dfrac{98}{41}$.

So, we can now find the point

$$X = \begin{pmatrix} -6 \\ -2 \\ 1 \end{pmatrix} + \frac{98}{411}\begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}$$

which is messy, but done.

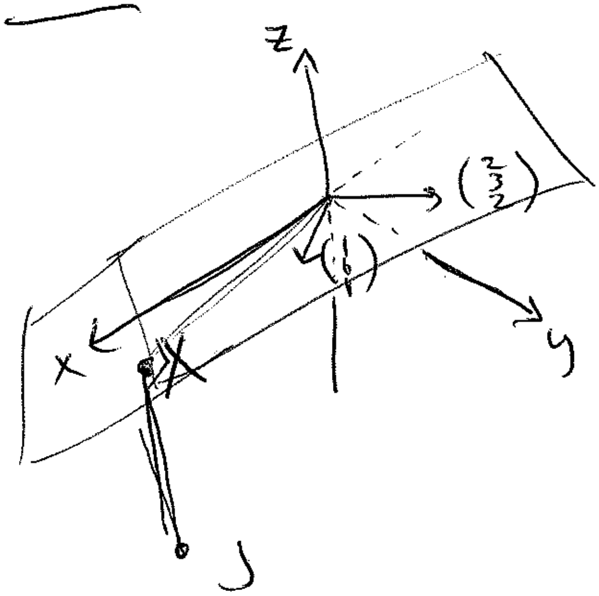
~~Task~~ Challenge 53 – Prof Hitchman's solution

Find the point on the plane

$$(s,t) \longmapsto s\begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}2\\3\\2\end{pmatrix}$$

which is closest to $J = \begin{pmatrix}12\\3\\-9\end{pmatrix}$

Soln:




Suppose $X = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is the point we seek. then $X - J$ is orthogonal to our plane so

$$(X-J)\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \quad \text{and}$$

$$(X-J)\cdot\begin{pmatrix}2\\3\\2\end{pmatrix} = 0.$$

This gives us a pair of equations

$$(*) \quad \begin{cases} (x-12)+(y-3)+(z+9)=0 \\ 2(x-12)+3(y-3)+2(z+9)=0 \end{cases}$$

these equations represent the line through J which is orthogonal to the plane.

But since the point X is on the plane we must also have

$$\begin{cases} x = s + 2t \\ y = s + 3t \\ z = s + 2t \end{cases}$$

We subsitute these into ★ to get

$$\begin{cases} (s+2t-12)+(s+3t-3)+(s+2t+9)=0 \\ 2(s+2t-12)+3(s+3t-3)+2(s+2t+9)=0 \end{cases}$$

which is 2 equations in the 2 unknowns $s, t$.

Let's clean up...

$-24$
$-9$
$18$
$-15$

$$\begin{cases} 3s + 7t = 6 \\ 7s + 17t = 15 \end{cases}$$

and we can solve this!

eliminate $s$: $-7(i)+3(ii) \longrightarrow (-49+51)t = -42+45$

$2t = 3 \Rightarrow t = 3/2$

~~eliminate t: 2 [illegible]~~ $3s + 7\left(\frac{3}{2}\right) = 6 \to s + \frac{7}{2} = 2$

$\to s = \frac{1}{2}$

So our point X is

$$X = \frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\3\\2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\end{pmatrix} + \begin{pmatrix}3\\9/2\\3\end{pmatrix}$$

$$= \begin{pmatrix}7/2\\5\\7/2\end{pmatrix}$$