LINEAR ALGEBRA EXAM ONE: FALL 14

Instructions: Be sure your name is on each sheet of paper. Explain your thinking clearly and in complete sentences. This examination has 17 questions on 6 pages.

1. Basic Computational Fluency

For most of this section, no work is required. Just write down the correct outcome from the computation.

Task 1. Add the two vectors

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

Task 2. Add the two matrices

$$\begin{pmatrix} 3 & 4 & 7 \\ 1 & -1 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & 0 & 3 \\ 1 & 2 & 6 \end{pmatrix}.$$

Task 3. Write down the transpose of each of these two matrices:

$$A = \begin{pmatrix} 6 & 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

Task 4. Find the product of these two matrices in the order that makes sense:

$$C = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Task 5. Give an example of a pair of 2×2 matrices X and Y which do not commute.

Task 6. Compute this linear combination of vectors

$$5 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 7 \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Task 7. Compute the dot product of the vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Task 8. Compute the norm of the vector

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Task 9. Compute the angle between the vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(Write an exact expression that gives the angle. Do not write a decimal approximation.)

2. Interpretations

The next few tasks ask for careful translations between our different viewpoints. Describe your understanding clearly.

Task 10. We have seen two ways to compute the product below, which involes multiplying a matrix times a vector. Describe them both briefly, and show that they give the same result.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Task 11. We are given a situation where the unknown vector $X = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ is perpendicular to each of the vectors $U = \left(\begin{smallmatrix} -3 \\ 0 \\ 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 4 \\ 4 \\ 0 \end{smallmatrix}\right)$. Write these conditions as a system of linear equations on the components of X.

What shape do you expect the set of solutions X to this problem to take, and why?

Task 12. We have seen that a system of linear equations can be written in two other algebraic forms involving such things as matrices or vectors. For the system below, write down those alternate forms. (Don't solve the system of equations. Just give the other forms.)

$$\begin{cases} 2x + 3y - z = 7 \\ 4x - y + 2z = 0 \end{cases}$$

Task 13. In the last task, we found there to be three different but equivalent representations for the information. Each of the three forms has a picture associated with it. Describe each of the three pictures (it need not be exact, we are interested in what kind of thing each picture is) and describe what a solution to the problem means in each picture.

3. Gauss-Jordan Elimination & Matrix Factorization

This last section asks you to show you understand the basics of Gauss-Jordan Elimination. Be sure to show your work.

Task 14. Consider the matrix H below.

$$H = \begin{pmatrix} 2 & 1 & 6 \\ 1 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

Use Gauss-Jordan elimination to find

- The LU decomposition of H, and
- The inverse of H.

Task 15. Let H be the matrix given in the last task. Let b be the vector

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Use the LU decomposition of H to solve the equation $Hx = b$ by solving two triangular systems with the back-substitution technique.

4. A challenge

Task 16. Find a 3-vector b so that the equation

$$x \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + y \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = b \tag{*}$$

has no solution.

Task 17. Continue with the set-up from Task 16. It is also possible to find a vector b so that the equation (*) has more than one solution. In fact, there are lots of ways to choose such a vector b. Use the kinds of pictures we have been developing to discuss what all of the possible choices of b are that make (*) have more than one solution.