Subspaces

The Idea of a Subspace

Task 1. Use the definition of the word *subspace* to decide if each subset is a subspace of $\mathbb{R}^2$:

a. The set consisting of only the zero vector $\mathcal{Z} = \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\}$.

b. The set consisting of the vectors whose heads lie on the circle of radius of radius 7 about the origin.

c. The set consisting of those vectors which have their heads on either the x-axis or the y-axis.

Task 2. Use the definition of the word *subspace* to decide if each subset is a subspace of $\mathbb{R}^3$:

a. The set of all vectors which are linear combinations of the vectors e_1, e_2, e_3:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

b. The set of solutions of the equation $2x + 3y = 6$.

c. The set of vectors which are solutions of the equation $x^2 + y^2 - z^2 = 0$.

Task 3. Experiment with some possibilities. What kinds of shapes can the subspaces of $\mathbb{R}^2$ be?

Task 4. Experiment with some possibilities. What kinds of shapes can the subspaces of $\mathbb{R}^3$ be?

Task 5. Run through the argument for Theorem 37 in the particular example of

$$\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Do you recognize this subspace? Is there an easier way to describe it?

Task 6. Show that the set of solutions to the system of equations below is not a subspace of $\mathbb{R}^3$, by showing that it actually fails *both* conditions in the definition.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_1 - 2x_2 + 3x_3 = 0 \end{cases}$$

How could we repair this task? Find a way to make a small change to the system of equations so that the set of solutions is a subspace. How are the two solution sets related?

Spans vs. Homogeneous Systems

In the next three tasks, you are given a subspace described as the solution set of a homogeneous system of linear equations. Find a way to describe that same subspace as a span.

Task 7. The subspace $\mathcal{S}_1$ is the set of solutions in $\mathbb{R}^2$ to the system of equations

$$\begin{cases} 2x + y = 5 \\ x + y = 7 \end{cases}.$$

Task 8. The subspace $\mathcal{S}_2$ is the set of solutions in $\mathbb{R}^3$ to the system of equations

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases}.$$

Task 9. The subspace $\mathcal{S}_3$ is the set of solutions in $\mathbb{R}^4$ to the system of equations

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 - x_3 - 2x_4 = 0 \\ x_2 + 3x_3 = 0 \end{cases}.$$

In the next three tasks, you are given a subspace described as a span. Find a homogeneous system of linear equations whose solution set is the given subspace.

Task 10. The subspace $\mathcal{S}_4$ is the span of the 2-vectors below.

$$u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For what value of n is $\mathcal{S}_4$ a subspace of $\mathbb{R}^n$?

Task 11. The subspace $\mathcal{S}_5$ is the span of the 3-vectors below.

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

For what value of n is $\mathcal{S}_5$ a subspace of $\mathbb{R}^n$?

Task 12. The subspace $\mathcal{S}_6$ is the span of the 4-vectors below.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}$$

For what value of n is $\mathcal{S}_6$ a subspace of $\mathbb{R}^n$?