LINEAR ALGEBRA EXAM TWO: FALL '14

Instructions: Be sure your name is on each sheet of paper. Explain your thinking clearly and in complete sentences. This examination has is broken into clearly defined sections, each corresponding to one of our learning goals. There is likely too much here to accomplish in an hour, so use your time wisely by focusing on those things you think you are best prepared to demonstrate.

The learning goals assessed on this examination paper are:

- Solving Systems of Equations
- Implicit and Explicit Descriptions for Subspaces
- The Four Subspaces
- Approximate Solutions
- (Advanced) Matrices as Transformations
- (Advanced) Orthonormal Bases, Gram-Schmidt, and the QR Decomposition

1. Solving Systems of Equations

Task 1.1. Find the complete solution to the following system of linear equations. Label all the important parts of the process.

$$\left\{\begin{array}{rcrcrcrcl} x & + & 3y & + & z & + & t & = & 1 \\ 2x & + & 6y & + & 4z & + & 8t & = & 3 \\ & & & & 2z & + & 4t & = & 1 \end{array}\right.$$

Task 1.2. Suppose that the matrix B has the reduced row echelon form given below. What can you say about solving an equation of the form $Bx = c$?

$$B = \begin{pmatrix} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

What parts of your answer depend upon the choice of the vector c? What parts do not depend upon the choice of c?

Task 1.3. In Strang's terms, what are the *special solutions*, and why are they important? How do they fit into our understanding of matrices?

Task 1.4. Give an example of a system of equations meeting the following conditions, or describe why it is impossible to do so.

- A system of 5 equations in 3 variables which has a solution.

- A system of 3 equations in 5 variables having a solution set which is a 4-dimensional hyperplane.

2. The Four Subspaces

Task 2.1. Below is the LU decomposition of a matrix A. Give a basis or the dimension for each of the four subspaces. Do only what you can without performing row operations!

$$X = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

Describe why you know what you know.

Task 2.2. Suppose that a matrix B has the data below associated to it. Describe as fully as you can the 4 subspaces of B.

$$\mathrm{rref}(B) = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{rref}(B^T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Task 2.3. Continue with the matrix B from the last task. What can you say about the four subspaces of the matrix $B^T B$?

3. Implicit vs Explicit Descriptions

Task 3.1. Consider the set of 4-vectors below. Is this set linearly dependent or linearly independent? How do you know?

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 3 \end{pmatrix}.$$

Task 3.2. Let S be the subspace spanned by the vectors v_1, v_2, v_3 in the last task. Find equations which describe the subspace S as their nullspace.

Task 3.3. Choose vectors w_3 and w_4 which will make the set below into a basis of $\mathbb{R}^4$. Be sure to say how you know your solution works.

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad w_3, \quad w_4.$$

Task 3.4. Let W be the subspace of $\mathbb{R}^5$ described as the set of all points $(x_1, x_2, x_3, x_4, x_5)$ which satisfy the equations below. Find a basis for W.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 - x_5 = 0 \\ x_1 + x_2 + x_3 - x_4 - x_5 = 0 \\ x_1 + x_2 - x_3 - x_4 - x_5 = 0 \end{cases}$$

4. Approximate Solutions

Task 4.1. Write an expression which gives the closest point to $(1,1,1,1)$ in the subspace of $\mathbb{R}^4$ described by the equations

$$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 0 \\ 3x_1 - 5x_4 = 0 \end{cases}$$

You don't have to actually compute the point in question, but you should be able to compute it easily from your expression.

Task 4.2. We wish to fit a curve of the form $p(x) = ax + bx^2$ as best we can to the data below. Find the curve of best fit.

x	p
1	6
3	7
4	9
4	8

Task 4.3. What is *orthogonal projection onto a subspace*, and how does it work? Be sure to draw a good picture to go along with your explanation.

5. Orthonormal Bases, etc.

Task 5.1. The matrix A below has linearly independent columns, so the columns form a basis for $T = \mathrm{col}(A)$. Use the Gram-Schmidt algorithm to find an orthonormal basis for the subspace T.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$$

Task 5.2. Give an example of a 2×2 matrix which is not orthogonal. Explain how you know the matrix is not orthogonal.

Task 5.3. Find the QR decomposition of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Task 5.4. The Gram-Schmidt process can break so that it is impossible to construct a QR decomposition of a matrix. Give an example of a 2×2 matrix for which this happens, and explain why it happens.

6. Matrices as Transformations

Task 6.1. Draw a complete schematic diagram of the four subspaces, and what information about them one can compute from a matrix A.

What role do each of the four subspaces play when understanding how A behaves as a function?

Be sure to include in your diagram the way to see what happens to particular points by looking at their relationship to the four subspaces.

Task 6.2. Suppose that A and B are matrices so that $AB = 0$ is the zero matrix. Which of the following are definitely true, and which are not? Explain how you know which is which.

- $\mathrm{col}(A) \subseteq \mathrm{null}(B)$

- $\mathrm{null}(A^T B^T) \subseteq \mathrm{null}(A^T)$

- $\mathrm{col}(B^T)$ is orthogonal to $\mathrm{col}(A)$.