Systems of Equations

For the next seven tasks, find the solution of the system of linear equations using the forward pass of Gaussian Elimination and back-solving. For each, identify m and n, write down the triangular form of the system, and write down the solution in vector form. Keep track of which row operations you use, and the order in which you use them. Then, make a plot of the system of hyperplanes involved and use it to verify your solution. (The Computer might be helpful.)

Task 1.
$$\begin{cases} 7x & + & 3y & = & 0 \\ & & 4y & = & 2 \end{cases}$$

Task 2.
$$\begin{cases} & & 2y & = & 1 \\ 6x & - & 3y & = & 5 \end{cases}$$

Task 3.
$$\begin{cases} 2x & + & y & = & 7 \\ x & + & y & = & 5 \end{cases}$$

Task 4.
$$\begin{cases} -x & - & y & + & 3z & = & 8 \\ & & 2y & - & z & = & 3 \\ & & 5y & + & 3z & = & 0 \end{cases}$$

Task 5.
$$\begin{cases} x & + & y & + & 3z & = & 8 \\ 2x & - & y & - & z & = & 3 \\ & & 5y & + & 3z & = & 0 \end{cases}$$

Task 6.
$$\begin{cases} & & 5y & + & 3z & = & 0 \\ 3x & + & y & + & 3z & = & 5 \\ 2x & + & y & + & 2z & = & 7 \end{cases}$$

Task 7.
$$\begin{cases} x & + & 2y & + & 3z & = & 4 \\ 4x & + & 5y & + & 6z & = & 10 \\ 7x & + & 8y & + & 10z & = & 17 \end{cases}$$

For the next four tasks, find the solution of the system of linear equations using the forward pass of Gaussian Elimination and back-solving. For each, identify m and n, and write down the solution in vector form. Keep track of which row operations you use, and the order in which you use them. Describe in a sentence what the row picture should be for each situation.

Task 8.
$$\begin{cases} 3x_1 + 10x_2 = 2/5 \\ 5x_1 + 17x_2 = -3 \end{cases}$$

Task 9.
$$\begin{cases} x_1 + x_2 - x_3 = -1 \\ 2x_1 + \frac{1}{2}x_2 + x_3 = 3 \\ -2x_1 - x_2 + \frac{3}{4}x_3 = 2 \end{cases}$$

Task 10.
$$\begin{cases} 3x_2 + 4x_3 - x_4 = 1 \\ x_2 + 6x_3 + x_4 = 1 \\ 2x_1 + 3x_3 - x_4 = 1 \\ 5x_1 - x_2 + x_3 + 3x_4 = 1 \end{cases}$$

Task 11.
$$\begin{cases} 3x_1 + 17x_2 - x_3 + 3x_4 + x_5 = 1 \\ x_1 + 6x_2 - 2x_3 + x_4 + x_5 = 0 \\ 2x_1 + 2x_2 + x_3 - 5x_4 + x_5 = 1 \\ 3x_3 + x_4 - 3x_5 = 1 \\ -2x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 0 \end{cases}$$

Challenging Systems

Task 12. Find all of the solutions to this system, if you can. If this is impossible, say why.

$$\begin{cases} 14x + 7y = 21 \\ 6x + 3y = 9 \end{cases}$$

Task 13. Find all of the solutions to this system, if you can. If this is impossible, say why.

$$\begin{cases} 14x + 7y - z = 21 \\ 6x + 4y + z = 9 \end{cases}$$

Task 14. Find all of the solutions to this system, if you can. If this is impossible, say why.

$$\begin{cases} 14x + 7y = 21 \\ 6x + 3y = 15 \\ x + 4y = 0 \end{cases}$$

Task 15. Find all of the solutions to this system, if you can. If this is impossible, say why.

$$\begin{cases} 2x_1 & + & x_2 & - & 3x_3 & = & 0 \\ 4x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 6x_1 & + & 3x_2 & + & x_3 & = & 0 \end{cases}$$

Task 16. What choices of the numbers b_i will make this equation inconsistent? What choices of these numbers will allow us to find a solution?

$$\begin{cases} x & + & 2y & = & b_1 \\ 2x & + & 4y & = & b_2 \\ 2x & + & 5y & = & b_3 \\ 3x & + & 9y & = & b_4 \end{cases}$$

Challenge 17. Choose the coefficients a_{ij} for a system of equations of this form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}.$$

Your coefficients should have the properties that
1. for some choice of b_i's the resulting system of equations has no solutions, and
2. for some (other) choice of b_i's the resulting system has at least two solutions.

Write down the different choices of b_i's that you use, and the solutions that correspond to the second case.

Row Operations and Equivalence of Systems

Definition. Given a system of m linear equations in n unknowns, the *solution set* for the system is the collection of all points P in $\mathbb{R}^n$ whose coordinates $P = (x_1, \ldots, x_n)$ satisfy all of the equations simultaneously.

Fix a counting number n. We say that two systems of linear equations in n variables are *equivalent* if they have the same solution set.

Task 18. Suppose that λ is a non-zero number. Show that the system

$$ax + by + cz = d$$

is equivalent to the system

$$\lambda ax + \lambda by + \lambda cz = \lambda d.$$

You will want to do two things: first show that any solution of the first system is a solution of the second system; then show that any solution of the second system is a solution of the first.

Task 19. Show that the system

$$\begin{cases} ax &+& by &+& cz &=& d \\ \alpha x &+& \beta y &+& \gamma z &=& \delta \end{cases}$$

is equivalent to the system

$$\begin{cases} \alpha x &+& \beta y &+& \gamma z &=& \delta \\ ax &+& by &+& cz &=& d \end{cases}$$

You will want to do two things: first show that any solution of the first system is a solution of the second system; then show that any solution of the second system is a solution of the first.

Task 20. Show that the system

$$\begin{cases} ax &+& by &+& cz &=& d \\ \alpha x &+& \beta y &+& \gamma z &=& \delta \end{cases}$$

is equivalent to the system

$$\begin{cases} ax &+& by &+& cz &=& d \\ (a+\alpha)x &+& (b+\beta)y &+& (c+\gamma)z &=& d+\delta \end{cases}$$

You will want to do two things: first show that any solution of the first system is a solution of the second system; then show that any solution of the second system is a solution of the first.

Task 21. Suppose that λ is a non-zero number. Show that the system

$$\begin{cases} ax &+& by &+& cz &=& d \\ \alpha x &+& \beta y &+& \gamma z &=& \delta \end{cases}$$

is equivalent to the system

$$\begin{cases} ax &+& by &+& cz &=& d \\ (\lambda a+\alpha)x &+& (\lambda b+\beta)y &+& (\lambda c+\gamma)z &=& d+\delta \end{cases}$$

You will want to do two things: first show that any solution of the first system is a solution of the second system; then show that any solution of the second system is a solution of the first.

Task 22. Think about the tasks above. Discuss why applying each of our three types of row operations to a system of equations leads to an equivalent system of equations.

That is, discuss why Gaussian Elimination works.