Fiscal Paradoxes in a Liquidity Trap

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Abstract

We provide a non-linear analysis of the dynamics of a Representative Agent New Keynesian Model following an unanticipated discount factor shock. As is standard in this class of models when monetary policy is described by a Taylor rule , for a sufficient high discount factor shock, the economy enter a liquidity trap with a zero nominal interest rate, where the fiscal multiplier is large, and the effects of the shock is amplified the more flexible prices are. We also show that for a sufficiently large, but finite shock, or when prices are sufficiently flexible, the equilibrium with stable long-run prices fails to exist. Related, we show that the fiscal cost of implementing the Taylor rule become arbitrarily large when the economy approaches these finite limits. We propose a simple modification of the Taylor rule in which we add a limit to the fiscal cost of implementing the interest rate rule. With the alternative simple rule an equilibrium exist for any size shock and degree of price stickiness. Moreover, with the alternative rule the model features a milder contraction, a fiscal multiplier lower than 1, and non-paradoxical comparative statics with respect to price flexibility.

1 Introduction

What are the consequences of shocks driving the economy to a liquidity traps? What policy responses can ameliorate these shocks? A recent literature stresses the paradoxical economics operating at the liquidity trap, e.g., fiscal multipliers are greater than one, price flexibility tend to exacerbate the effect of shocks (Christiano et al., 2011; Eggertsson and Krugman, 2012). Desired policy responses are to use fiscal stimulus and for the monetary authority to commit to maintain low interest rate interest rates after the shock. (Eggertsson and Woodford, 2003; Werning, 2011). Notwithstanding the rich interactions between fiscal and monetary policy in a liquidity trap, the fiscal implications of implementing a Taylor rule have been mostly ignored. A passive fiscal policy is often assumed, which is rendered to be inconsequential given that the economy is assumed to operate at, or very closed to, the cashless limit. The goal of this paper is to analyze the fiscal consequences of implementing a Taylor rule at a liquidity Trap, and to characterize the implications of adding a simple restriction to the standard Taylor rule: to bound the fiscal costs of implementing the rule.

In particular, we provide a non-linear analysis of the dynamics of a Representative Agent New Keynesian Model following an unanticipated shock to the discount factor. The benchmark model features sticky prices à la Calvo, a cash-credit monetary environment, and a standard Taylor rule describing the behavior of the monetary authority. As is standard in this class of models, for a sufficient high discount factor shock, the economy enter a liquidity trap with a zero nominal interest rate, where the fiscal multiplier is large and the effects of the shock is amplified the more flexible prices are. We show two new results in this benchmark model.

First, We show that for a sufficiently large, but finite shock, or when prices are sufficiently flexible, the equilibrium with stable long-run prices fails to exist. In particular, we show that in any of these two finite limit cases, and for any rate of deflation, the representative agent is willing to substitute money across periods at a lower rate than the zero nominal interest rate. Thus, in both of these limit cases, there is always an excess demand for savings for any rate of deflation at the zero lower bound.

Related, We also show that the fiscal cost of implementing the Taylor rule become arbitrarily large as the economy approaches these limits. As a response to the shock, the Taylor rule calls for a monetary expansion in the period of the shock, and a monetary contraction in the following period. The monetary expansion results in a surplus for the monetary authority, or the consolidated balance sheet of the government. The opposite is true for the monetary contraction in the period following the shock. As the shock to the discount rate approaches the aforementioned limit, the future policy response induces an arbitrarily large deflation which requires an equally large monetary contractions in the second period. To implement this policy the consolidate government needs to impose arbitrarily large lump sum taxes on the representative agent.

Motivated by these results, we propose a simple modification to the Taylor rule consisting on adding a limit to the fiscal costs of implementing the rule. With the alternative rule a unique stable equilibrium exist for any size of the shock and degree of price stickiness. Moreover, with the alternative rule the model features a small fiscal multiplier, non-paradoxical comparative statics with respect to price flexibility, and a milder contraction.

The analytical results are obtained in a simplify version of the model in which prices are only sticky in the initial period and the economy is in the cashless limit. In this case, we calculate the fiscal consequences of implementing the Taylor rule as a fraction of the steady state consumption of cash goods, which is bounded away from zero in the cashless limit we consider in this paper. In addition, we numerically solve for calibrated versions of the model with standard sticky prices à la Calvo and in which the economy is away from the cashless limit. We find qualitatively similar results. Importantly, in these numerical examples the fiscal consequences of implementing a Taylor rule become large relate to GDP.

2 Model Economy

We consider a standard representative agent new-Keynesian model. The model features a standard representative household, monopolistic competitive firms, and nominal price frictions *à* la Calvo (1983). In addition, to analyze the fiscal consequence of the interest rule at zero lower bound, we consider an explicit monetary friction, a cash-in-advance constraints on a subset of the goods, and assume that the government uses lump-sum taxes and subsidies to control the money supply in order to implement an interest rate policy rule (Taylor rule).

2.1 Households

The representative household has preferences over sequences of a consumption aggregate C_t and leisure $1-N_t$ represented by the following utility function:

$$\sum_{t=1}^{\infty} \beta^{t-1} \xi_t \left[\gamma \log C_t + (1-\gamma) \log (1-N_t) \right], \tag{1}$$

where the consumption aggregate C_t is a Cobb-Douglas function of the consumption of cash and credit goods C_{1t} and C_{2t}, respectively,

$$C_t = C_{1t}^{\nu} C_{2t}^{1-\nu},$$

and ξ_t is a preference shock.

The representative household can trade indexed and nominal bonds. We denote by D_t the indexed bonds sold in period t, promising to pay a unit of the consumption aggregate at the beginning of period $t+1$. The availability of a nominal bond give rise to an arbitrage condition between the real interest rate, expected inflation, and the nominal interest rates, i.e., the Fisher equation. To simplify the exposition, we abstract from the demand for nominal bonds by the households when writing the budget constraint. The budget constraint is given by

$$C_{1t} + C_{2t} + D_{t-1} + \frac{M_t}{P_t} + T_t = \frac{D_t}{1+r_t} + \frac{M_{t-1}}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}, \tag{2}$$

where M_{t-1} is the money the household carries form $t-1$ to t, T_t the lump-sum taxes paid at t, W_t is the nominal wage, and Π_t are the aggregate of profits from the producers of differentiated intermediate goods. Finally, the consumption of cash good C_{1t} is constrained by the money holding carries from $t-1$:

$$P_t C_{1t} \leq M_{t-1}. \tag{3}$$

The representative agent maximizes (1) subject to the budget constraint (2) and the cash-in-advance constraint (3). The first order conditions of this problem imply standard intra and inter-temporal optimality conditions.

The intratemporal optimality condition takes the followoing form

$$\frac{1-\gamma}{1-N_t} = \frac{W_t}{P_t} \frac{\gamma (1-\nu)}{C_{2t}}, \tag{4}$$

implying that the labor supply N_t is negatively related to the real wage $\frac{W_t}{P_t}$ and the consumption of credit goods C_{2t}.

The Euler equations in terms of the consumption of credit goods is given by

$$\frac{1}{C_{2t}} = \frac{\xi_{t+1}}{\xi_t} \beta (1+r_t) \frac{1}{C_{2t+1}}. \tag{5}$$

Related, using the arbitrage condition between an indexed and a nominal bond (the Fisher equation), $1+i_t = (1+r_t) P_{t+1}/P_t$, we obtain the following Euler equation in terms of the return of a nominal bond and expected inflation

$$\frac{1}{C_{2t}} = \frac{\xi_{t+1}}{\xi_t}\beta\left(1+i_t\right)\frac{P_t}{P_{t+1}}\frac{1}{C_{2t+1}}. \tag{6}$$

Finally, the inter-temporal condition relating the consumption of credit goods at t, the consumption of cash goods at $t+1$, and the rate of return of money P_t/P_t is given by

$$\frac{1-\nu}{C_{2t}} = \frac{\xi_{t+1}}{\xi_t}\beta\frac{P_t}{P_{t+1}}\frac{\nu}{C_{1t+1}}. \tag{7}$$

From the last two conditions it follows that the cash-in-advance constraint in period $t+1$ is binding when the nominal interest rate is strictly positive, $(1+r_t)\, P_{t+1}/P_t > 0$.

2.2 Firms

On the production side, we assume that there is a continuum of firms with measure 1. Firm j produces a differentiated intermediate good $Y_t(j)$ with a linear technology

$$Y_t(j) = N_t(j).$$

There is a representative final producer combining the differentiated intermediate goods $Y_t(j)$, $j \in [0,1]$, into final good Y_t with the following CES production function:

$$Y_t = \left[\int Y_t(j)^{1-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{8}$$

Denoting by P_t the aggregate price index and $P_t(j)$ the price of good j, the first order conditions of the final good producer implies that the demand for good j takes the following familiar form:

$$Y_t(j) = \left(\frac{P_t(j)}{P_{t+k}}\right)^{-\epsilon} Y_t. \tag{9}$$

We assume that monopolistic producers are only able to adjust the price of their product in period t with probability $1-\theta_t$. We allow the degree of price stickiness to be time varying. We later use this flexibility to obtain an analytic characterization of simple examples where the economy is fully flexible from the second period onward, $\theta_1 > 0$, $\theta_t = 0$, $t \geq 2$. We also analyze numerical solutions of more standard cases in which $\theta_t = \theta \in (0,1)$, for all $t \geq 1$.

Given the pricing friction, a monopolistic producer that is able to adjust the price at t chooses the price that maximize expected discounted profit given by:

$$\sum_{k=0}^{\infty}\left(\prod_{i=0}^{k}\theta_{t+i}\right)\beta^k\frac{\xi_{t+k}}{\xi_t}\frac{C_{2t}}{C_{2t+k}}\frac{1}{P_{t+k}}\left(P_t(j)\,Y_{t+k|t}(j) - W_{t+k}Y_{t+k|t}(j)\right). \tag{10}$$

subject to equation (9).

The optimal price of a flexible firms is given by:

$$\frac{P_t^*(j)}{P_t} = \frac{\epsilon}{\epsilon-1} \frac{\frac{W_t}{P_t}\frac{Y_t}{C_{2t}} + \sum_{k=1}^{N} \beta^k \frac{\xi_{t+k}}{\xi_t}\left(\prod_{i=1}^{k}\theta_{t+i}\right)\left(\frac{P_{t+k}}{P_t}\right)^{\epsilon}\frac{W_{t+k}}{P_{t+k}}\frac{Y_{t+k}}{C_{2t+k}}}{\frac{Y_t}{C_{2t}} + \sum_{k=1}^{N} \beta^k \frac{\xi_{t+k}}{\xi_t}\left(\prod_{i=1}^{k}\theta_{t+i}\right)\left(\frac{P_{t+k}}{P_t}\right)^{\epsilon-1}\frac{Y_{t+k}}{C_{2t+k}}}. \tag{11}$$

Given the symmetric choice of P_t^* and P_{t-1}, the dynamic of the price level is given by:

$$P_t = \left[\theta_t P_{t-1}^{1-\epsilon} + (1-\theta_t) P_t^{*1-\epsilon}\right]^{\frac{1}{1-\epsilon}}. \tag{12}$$

Finally, aggregate profits are given by:

$$\Pi_t = P_t Y_t - W_t N_t. \tag{13}$$

2.3 Government Policy

To close the model we specify the behavior of fiscal and monetary policies in terms of simple rules. We consider two alternative policies: (i) a standard Taylor rule, implemented with a passive fiscal policy; (ii) a constrained Taylor rule, implemented with a partially passive fiscal policy and a simple upper bound on the lump sum taxes. In both cases we assume that government expenditures follow an exogenous path $\{G_t\}_{t=1}^{\infty}$.

2.3.1 Taylor Rule

We first consider the case where policy is given by an active monetary policy rule and a passive fiscal policy rule. In particular, we assume monetary policy is given by a standard Taylor Rule, specifying the behavior of the nominal rate as a function of realized inflation:

$$1 + i_t = \max\left\{1, \frac{1}{\beta} + \phi_\Pi\left(\frac{P_t}{P_{t-1}} - 1\right)\right\}. \tag{14}$$

To implement the Taylor Rule, the government implicitly chooses a sequence of money supply and lump-sum taxes, i.e., it follows a passive fiscal policy. In particular, the sequence of money supply $\{M_t^s\}_{t=1}^{\infty}$ and lump-sum taxes $\{T_t\}_{t=1}^{\infty}$ must satisfy the government budget constraint:

$$\frac{M_{t-1}^s - M_t^s}{P_t} + G_t = T_t. \tag{15}$$

As it is standard, to prevent local indeterminacy, we assume that the Taylor rule features a sufficiently strong response to inflation:

$$\beta\phi_\Pi > 1.$$

2.3.2 Constrained Taylor Rule

We also consider an alternative policy rule which we label the *constrained Taylor Rule*. The alternative rule is defined by equations (14) and (15) as long as the lump-sum taxes required to implement the rule satisfy the following constraint

$$\frac{M_{t-1}^s - M_t^s}{P_t} + G_t = T_t \leq \bar{T}. \tag{16}$$

If the lump-sum taxes required to implement the Taylor rule violate the limit in (16), the evolution of the money supply is described by the following simple money rule

$$\frac{M_{t-1}^s - M_t^s}{P_t} + G_t = \bar{T}, \tag{17}$$

and the nominal interest rate must satisfy the following constrained:

$$1 + i_t \leq \min\left\{1, \frac{1}{\beta} + \phi_\Pi \left(\frac{P_t}{P_{t-1}} - 1\right)\right\}.$$

Again, to prevent local indeterminacy, we assume that

$$\beta\phi_\Pi > 1.$$

2.4 Definition of an Equilibrium

Given exogenous preference shocks and probability of price adjustment $\{\xi_t, \theta_t\}_{t=1}^\infty$, a sequence of government expenditure $\{G_t\}_{t=1}^\infty$, a competitive equilibrium is given by sequences of allocations $\{C_{1,t}, C_{2,t}, N_t, D_t, M_t, \Pi_t\}_{t=1}^\infty$, prices $\{r_t, W_t, P_t\}_{t=1}^\infty$, and policies $\{i_t, M_t^s, T_t\}_{t=1}^\infty$ such that:

1. Given the sequences of prices $\{r_t, W_t, P_t\}_{t=1}^\infty$, profits $\{\Pi_t\}_{t=1}^\infty$, and lump-sum transfers $\{T_t\}_{t=1}^\infty$, households' choices $\{C_{1,t}, C_{2,t}, N_t, D_t, M_t\}_{t=1}^\infty$ maximize (1) subject to the budget constraint (2) and cash-in-advance constraint (3);

2. The price of intermediate good producers that are able to adjust the price maximize the present discount value of profit (10) subject to the demand (9), the evolution of the aggregate price index is given by (12), and aggregate profits equals (13);

3. The sequences of nominal rate $\{i_t\}_{t=1}^\infty$, money supply $\{M_t^s\}_{t=0}^\infty$, and lump-sum taxes $\{T_t\}_{t=0}^\infty$ follow either: (i) equations (14) and (15) where the policy is described by a standard Taylor rule and a passive fiscal policy; or (ii) equations (14), (15), (16), and (17) in which case policy is described by a constrained Taylor rule and a partially passive fiscal policy;

4. Goods, labor and money markets clearing

$$C_{1t} + C_{2t} + G_t = Y_t, \tag{18}$$

and

$$\frac{Y_t}{A_t} = N_t, \tag{19}$$

where A_t is a total factor productivity term defined recursively by

$$A_t = \left[\theta_t \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} \frac{1}{A_{t-1}} + (1 - \theta_t)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon}\right]^{-1}, \tag{20}$$

and

$$M_t = M_t^s.$$

To limit the set of possible equilibria, we focus on the equilibria with long run stable price level, i.e.,

$$\lim_{t\to\infty} \frac{P_{t+1}}{P_t} = 1.$$

In addition, when considering the case with a constrained Taylor rule and the equilibrium requires that the money supply is constrained by equation (17), we also require that

$$1 + i_t \leq \min\left\{1, \frac{1}{\beta} + \phi_\Pi \left(\frac{P_t}{P_{t-1}} - 1\right)\right\}.$$

That is, we only consider equilibrium with a constrained Taylor rule where the cental bank is unable to lower rates by taking money out of the system.

3 Dynamics Following an Unanticipated Preference Shock

In this section we characterize the dynamics following an unanticipated preference shock. We assume that the economy starts at an steady state with $\xi_t = 1$, $t \leq 0$, and is hit by an unanticipated preference shock at $t = 1$, $\xi_1 < 1$, that permanently reverts to the initial value from period $t = 2$ onward, i.e.,

$$\xi_t = \begin{cases} 1 & t \leq 0 \\ \xi_1 < 1, & t = 1 \\ 1, & t > 1 \end{cases}.$$

The fraction ξ_{t+1}/ξ_t captures the effect of the shock on a household's intertemporal marginal rate of substitution. When ξ_1 drops, households are inclined to save more in the first period. As we show below, the drop of ξ_1 results in a lower real and nominal interest rates. Furthermore, with sufficiently low ξ_1, the nominal rate hits a zero lower bound and the economy enters a liquidity trap.

To keep the analytical tractability of the model, we make the following two additional assumptions. First, we assume that prices as only sticky in the first period, i.e.,

$$\theta_t = \begin{cases} \theta \in (0,1), & t = 1 \\ 0, & t \geq 2. \end{cases}$$

This assumption simplifies the optimal pricing rule of flexible firms in equation (11). Since the price would be fully flexible for all $t \geq 2$, the optimal price in the first period is given by the price that maximizes static profits:

$$P_t^*(j) = \frac{\epsilon}{\epsilon - 1} W_t, \text{ all } t \geq 0. \tag{21}$$

Second, as is common in the literature, we analyze the cashless limit. In particular, we assume that $\nu, C_{1t}, M_t \to 0$, but $C_{t1}/\nu, M_t^s/\nu \to \tilde{C}_{1t}, \tilde{M}_t^s > 0$. [1]

When $\nu, M_t \to 0$ the fiscal consequences of implementing a Taylor rule are arbitrarily small, as $T_t - G_t \to 0$. We therefore illustrate the taxes required to implement the Taylor rule as a share of the real balances

[1]If $\nu > 0$, the equilibrium dynamic following the unanticipated shocks as more complicated as the economy only converges asymptotically to the steady state. We provide numerical solutions of the general case.

(consumption of cash goods) in the steady state, i.e.,

$$\lim_{\nu \to 0} \frac{T_t - G_t}{C_1^*} = \frac{\frac{\tilde{M}_{t-1} - \tilde{M}_t}{P_t}}{\frac{\tilde{M}}{P}} > 0.$$

In addition, we assume $G_t = 0$. The only role of government expenditures is to analyze the fiscal multiplier, which we define as the derivative of output with respect to G_t evaluated at $G_t = 0$.

However, it worth mentioning that the main mechanisms that we analyze do not depend on these simplifying assumptions. We also report numerical simulations of cases with more standard assumptions of price stickiness, $\theta_t = \theta > 0$, all $t \geq 1$, and a calibrated value for $\nu > 0$.

In the following two subsection we characterize the equilibrium dynamics following an unanticipated preference shocks under the two alternative policy rules: (i) a standard Taylor rule and, (ii) the constrained Taylor rule.

3.1 Dynamics with a Standard Taylor Rule

In this section we characterize the equilibrium with a standard Taylor rule. We show that an equilibrium does not exist when the value of the discount factor shock is sufficiently low, or prices are sufficiently flexible (for any value of $\xi_1 < 1$). Subsequently, to shed lights on the non-existence result, we characterize the fiscal consequences of implementing the Taylor rule. We should that the taxes required to implement the Taylor rule become arbitrarily large when the value of the discount factor shock is sufficiently low, or prices are sufficiently flexible (for any value of $\xi_1 < 1$). We conclude that an equilibrium does not exist because it is not feasible to implement a Taylor rule in these cases.

3.1.1 Characterization of the Economy

Given the simplifying assumptions about the shock, $\xi_t = 1$, $t \neq 1$, and the nature of the nominal frictions, $\theta_t = 0$, $t \geq 2$, it is straightforward to show that the real variables of the economy are back to the steady state in period $t = 2$. In particular, from equation (21) and the assumption that prices are flexible, $P_t^* = P_t$, we obtain that the real wage $W_t/P_t = (\epsilon - 1)/\epsilon$, for all $t \geq 2$. From the intratemporal condition (4) and the aggregate resource constraint, $C_t = Y_t = N_t$, we can solved for time invariant values of aggregate consumption and the labor supply. Finally, the Euler equation (5) implies that the real rate equals the reciprocal of the discount factor, $r_t = 1/\beta - 1$.

Furthermore, since we are focusing on equilibrium with long run stable price, i.e., $\lim_{t\to\infty} P_{t+1}/P_t = 1$, the Taylor rule and the Euler equation of the representative agent imply that inflation is zero from period $t \geq 2$ onward.[2]

[2]Using the real rate for $t \geq 2$, $r_t = 1/\beta - 1$, the arbitrage condition between real and nominal bonds (the Fisher equation), the Taylor rule, and assuming away deflationary paths, we obtain the following difference equation describing the evolution of the inflation rate

$$\frac{1}{\beta}\frac{P_{t+1}}{P_t} = \frac{1}{\beta} + \phi_\Pi \left(\frac{P_t}{P_{t-1}} - 1\right).$$

This difference equation can be solved forward to obtain

$$\frac{P_{t+1}}{P_t} = (\beta\phi_\Pi)^{-j}\left(\frac{P_{t+j}}{P_{t+j-1}} - 1\right) + 1.$$

Since $\beta\phi_\Pi > 1$, long term stable prices, $\lim_{j\to\infty} P_{t+j}/P_{t+j-1} < \infty$, implies that

$$\frac{P_t}{P_{t-1}} = 1, \text{ all } t \geq 2.$$

The the allocations and prices for $t \geq 2$ as summarized in the following lemma.

Lemma 1 *For all $t \geq 2$, the allocation and prices are given by their steady state values:*

$$C_t = N_t = \frac{1}{1 + \frac{1-\gamma}{\gamma}\frac{\epsilon}{\epsilon-1}}, \tag{22}$$

$$r_t = \frac{1}{\beta} - 1; \tag{23}$$

$$\frac{W_t}{P_t} = \frac{\epsilon - 1}{\epsilon}; \tag{24}$$

$$\frac{P_t}{P_{t-1}} = 1. \tag{25}$$

To characterize the equilibrium in the period of the shock, $t = 1$, we manipulate the equilibrium conditions to express allocations and prices as simple functions of the gross inflation rate in this period, P_1/P. Using these relationships, an equilibrium can be expressed as the solution of a single equation in the the gross inflation rate, P_1/P, given by the Euler equation of a nominal bond (6). This characterization is summarized in the following Lemma.

Lemma 2 *The equilibrium gross inflation rate in period $t = 1$ is the solutions to the following equation:*

$$\underbrace{\frac{\xi_1}{\beta}\frac{C_2}{C_1}\frac{P_2}{P_1}}_{NMRS} = \max\left\{1, \frac{1}{\beta} + \phi_\Pi\left(\frac{P_1}{P} - 1\right)\right\}, \tag{26}$$

where the consumption C_1 and real wage W_1/P_1 are the following functions of the gross inflation rate P_1/P

$$C_1 = \left[\frac{1-\gamma}{\gamma}\left(\frac{W_1}{P_1}\right)^{-1} + \underbrace{\theta\left(\frac{P}{P_1}\right)^{-\epsilon} + (1-\theta)\left(\frac{\epsilon}{\epsilon-1}\frac{W_1}{P_1}\right)^{-\epsilon}}_{A_1^{-1}}\right]^{-1}, \tag{27}$$

and

$$\frac{W_1}{P_1} = \frac{\epsilon - 1}{\epsilon}\left[\frac{1-\theta}{1-\theta\left(\frac{P_1}{P}\right)^{\epsilon-1}}\right]^{\frac{1}{\epsilon-1}}, \tag{28}$$

and C_2 and P_2/P_1 are values independent of P_1/P given by equations (22) and (25).

The right-hand-side of (26) is the nominal interest rate targeted by the Taylor policy rule, i.e., the units of the numeraire in the second period that the representative agents obtains from an unit of the numeraire in the first period, which is a non-decreasing function of P_1/P. The policy rate equals 1 when $P_1/P \leq (1 - 1/\beta)/\phi_\Pi - 1 < 1$ and it is an strictly increasing function of P_1/P otherwise.

The left-hand-side of (26) is the intertemporal marginal rate of substitution in nominal terms (NMRS), i.e., the units of the numeraire in the second period required by the representative agents to be willing to give up a unit of the numeraire in the first period. Notice that the consumption in the second period and the expected inflation are independent of P_1/P (see Lemma 1, equations (22) and (25)). Therefore the left-hand-side is a decreasing function of consumption C_1. In turn, the consumption in period 1 is affected by the inflation rate in the first period through two channels: an intratemporal substitution channel and a TFP channel.

The intratemporal substitution channel is given by the first term inside of the brackets in equation (27). As the inflation rate increases, the real wage increases, and this leads to an increase in the consumption in the first period. As a result, the nominal interest rate required by the representative agent declines. The positive relationship between the real wage and the inflation rate is shown in equation (28). Intuitively, the evolution of the price level P_1 is a geometric average of the price level in the steady state P and the price of flexible firms P_1^*. The price chosen by flexible firms is proportional to the nominal wage W_1. Thus, a higher inflation P_1/P requires a more than proportional increase of the nominal wage, i.e., a rise in W_1/P_1.[3]

The TFP channel is given by the second and third terms inside of the brackets in equation (27). Any deviation of the price level from the steady state value, $P_1 = P$, is associated with more price dispersion and, as a consequence, lower aggregate TFP and consumption in the first period. Thus, the effect of gross inflation in the first period on TFP and consumption is positive (negative) for $P_1/P < 1 (> 1)$.[4]

Notice that the two channels imply that when $P_1/P \leq 1$ the net effect of gross inflation on the consumption in the first period is unambiguously positive and, therefore, the effect of gross inflation on the intertemporal marginal rate of substitution is unambiguously negative. This is the relevant range for the discussion that follows.[5]

Of particular interest for the discussion of existence of equilibria is the behavior of consumption as the price level in the initial period approaches 0. Evaluating (27) at $P_1/P = 0$ we obtain

$$C_1|_{\frac{P_1}{P}=0} = \frac{(1-\theta)^{\frac{1}{\epsilon-1}}}{1 + \frac{1-\gamma}{\gamma}\frac{\epsilon}{\epsilon-1}}. \tag{29}$$

Naturally, the effect of an arbitrarily large deflation is less pronounced the more flexible prices are. Indeed, if prices are fully flexible then it can be easily seen from (27) that the level of consumption in the first period is independent of the inflation rate. A counterpart of this result is that the deflation required to clear the asset markets when implementing a Taylor rule is larger the more flexible prices are. Related, for sufficiently large shocks or degree of price flexibility an equilibrium would not exist.

Summarizing the previous discussion, when $P_1/P \leq 1$ the left-hand-side of equation (26) is a strictly decreasing function of P_1/P. The right-hand-side of equation (26) is a weakly increasing function of P_1/P

[3] Equation (28) follows from substituting (21) into (12). From this equation it also follows that there is an upper bound on the gross inflation consistent with a finite real wage, i.e., $W_1/P_1 \to \infty$ as $P_1/P \to \theta^{-1/(\epsilon-1)}$. The lower possible value for the real wage, $W_1/P_1 = \frac{\epsilon-1}{\epsilon}(1-\theta)^{1/(\epsilon-1)}$, is attained with an arbitrarily large deflation, i.e., $P_1/P = 0$.

[4] This can easily be seen by differentiating the TFP in the first period: $A_1 = \left[\theta\left(\frac{P}{P_1}\right)^{-\epsilon} + (1-\theta)^{-\frac{1}{\epsilon-1}}\left[1-\theta\left(\frac{P_1}{P}\right)^{\epsilon-1}\right]^{\frac{\epsilon}{\epsilon-1}}\right]^{-1}$,

$$\frac{\partial A_1}{\partial \frac{P_1}{P}} = -\epsilon (A_1)^2 \theta \left(\frac{P}{P_1}\right)^{1-\epsilon}\left[1 - \left[\frac{1-\theta\left(\frac{P_1}{P}\right)^{\epsilon-1}}{(1-\theta)\left(\frac{P_1}{P}\right)^{\epsilon-1}}\right]^{\frac{1}{\epsilon-1}}\right].$$

[5] For calibrations featuring featuring low values of γ, the positive effect of the substitution effect on the consumption in the first period will tend to dominate the negative TFP effect when $P_1/P > 1$. The opposite is true for calibrations featuring high values of γ.

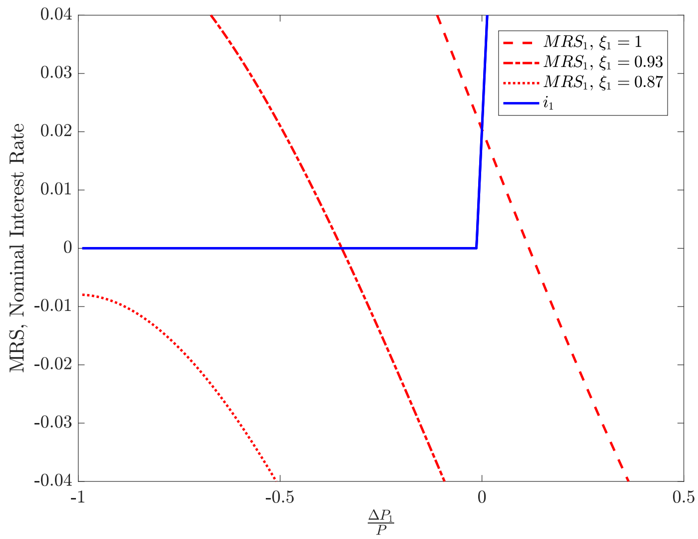

Figure 1: Equilibrium Determination for alternative values of ξ_1.

(it equals 1 when $P_1/P \leq (1 - 1/\beta)/\Phi_\Pi - 1 < 1$). Furthermore, when $\xi_1 = 1$ the unique solution of equation (26) is given by $P_1/P = 1$. Thus, we derive the following proposition for the existence and uniqueness of an equilibrium when there is an unanticipated shock to the discount factor, $\xi_1 \leq 1$, under a Taylor rule.

Proposition 1 *A necessary and sufficient condition for the existence of an equilibrium in the interval $P_1/P \in [0, 1]$ for any $\xi_1 \leq 1$ is that the value of the NMRS evaluated at $P_1/P = 0$ is greater than one, or, equivalently,*

$$\frac{\xi_1}{\beta} \frac{1}{(1-\theta)^{\frac{1}{\epsilon - 1}}} \geq 1. \tag{30}$$

In other words, there is no equilibrium with $P_1/P \leq 1$ for a sufficiently large negative shock, low ξ_1, or for sufficiently flexible prices, low θ.

Figure 1 illustrates the equilibrium determination, i.e., equation (26), for three alternative values of the preference shock $\xi_1 = 1, 0.93, 0.87$. The solid line shows the nominal interest rate implied by the Taylor policy rule, the right-hand-side of equation (26). The three downward sloping curves illustrate the NMRS for the three alternative values of the preference shock ξ_1. The dashed line is the NMRS in the initial steady state, in which case the equilibrium features zero inflation. The dashed-dotted line is the NMRS for intermediate value of the preference shock $\xi_1 = 0.93$. In this case, there is an equilibrium with deflation. Finally, the dotted line displays the NMRS for the shock $\xi_1 = 0.87$, where the marginal rate of substitution is always lower than the nominal rate implied by the Taylor rule when $P_1/P \leq 1$: given the market nominal return to savings, the representative agent is unwilling to substitute a unit of the numeraire in period two for a unit of the numeraire in period one, i.e., to save less. This inequality would only be consistent with a case where the representative agent is unable to save, yet this is not consistent with the assumptions in the model!

Intuitively, the preference shock leads to a decline in the NMRS curve for all possible values of the inflation rate in the initial period P_1/P. In particular, for the nominal rate that is implied by the inflation rate in the initial steady state, the price of consumption in the second period is low relative to the marginal rate of substitution between consumption in the two periods. Importantly, the expected inflation rate in the first period P_2/P_1 is determined by the equilibrium from period $t = 2$ onwards, including the behavior of policy implied by the Taylor rule, as shown by equation (25). Therefore, any adjustment in the price level in the first period is accommodated by policy with an equiproportional adjustment in the price level in the second period. This puts upwards pressure on the price of money in period one, i.e., downward pressure on the price level P_1. As seen in equation (29), this results in a downward adjustment in the consumption in the first period, leading to an upward movement along the NMRS curve.

3.1.2 Fiscal Consequence of the Taylor Rule

To further clarify the equilibrium adjustment and the non-existence result, in this section we unpack the underlying changes in money balances and lump-sum taxes required to implement the Taylor rule when an equilibrium exists. Proposition 1 shows that when the economy satisfies equation (30), given any shock ξ_1, there exists a unique equilibrium gross inflation P_1/P. We now derive the fiscal cost to implement the Taylor rule as a function of the equilibrium inflation P_1/P. We show that the fiscal consequences become arbitrarily large relative to the initial consumption of cash goods as the equilibrium gross inflation approaches zero, $P_1/P \to 0$. To make the point more consequential, we then show a numerical example away from the cashless limit with $\nu > 0$. Corresponding to the condition (30) of the benchmark model, the equilibrium does not exist with sufficiently large negative shocks or sufficiently flexible prices. Likewise, the fiscal consequences become arbitrarily large relative to aggregate consumption as we increase the shock or of the flexibility of prices.

From the intertemporal condition for money holding in equation (7) and the cash in advance constraint in equation (3) we obtain the following lower bound on the money balances carried by the representative agent in an equilibrium.

Lemma 3 *For all $t \geq 0$, there exists a lower bound for the cash demanded by households given by*

$$\tilde{M}_t \geq \frac{\xi_{t+1}}{\xi_t} \beta P_t C_t, \tag{31}$$

where the equality is strict when the economy is away from zero lower bound, $i_t > 0$.

When the economy is away from the zero lower bound, real return on bonds is strictly higher than the return on money. Therefore, households only hold the cash needed for transactions in the following period. In this case, equation (31) holds with equality and the money demand is unambiguously determined. This is the relevant case in the initial steady state as well as in the second period, i.e., $\tilde{M} = \beta P C^*$ and $\tilde{M}_2 = \beta P_2 C^*$. In the first period, depending on the value of the shock, the zero lower bound could be binding. Consequently, households might be indifferent to hold excessive cash balances and equation (31) could hold with strict inequality.

Because of the possibility of the indeterminacy of the money demand at $t = 1$, the timing of the lump-sum taxes could be also indeterminate. Yet, the present value of lump-sum taxes in the first two periods is well-defined independently of the value of the nominal interest rate. In particular, by combining the budget constraint of the government in equation (15) and the Fisher equation, i.e. $(1 + r_1)P_2/P_1 = 1 + i_1$, we obtain the following expression for the present value of lump-sum taxes relative to the real balances in the steady state:

$$\frac{\tilde{T}_1 + \frac{1}{1+r_1}\tilde{T}_2}{\frac{\tilde{M}}{P}} = \frac{\frac{P}{P_1}\frac{\tilde{M}}{P} + \left(\frac{1}{1+i_1} - 1\right)\frac{\tilde{M}_1}{P_1} - \frac{1}{1+i_1}\frac{P_2}{P_1}\frac{\tilde{M}_2}{P_2}}{\frac{\tilde{M}}{P}}. \tag{32}$$

Notice that when the zero lower bound is binding, i.e. $i_1 = 0$, the term involving the money balances in the initial period drops from this expression. Thus, the present value of the taxes over the first two periods is well-defined for all cases.

Building on these observations, the following proposition describes the equilibrium fiscal cost to implement the Taylor policy rule as a function of the equilibrium gross inflation P_1/P, i.e, the inflation satisfying equation (26).

Proposition 2 *Let P_1/P be a value of the gross inflation satisfying equation (26). Then, the lump-sum taxes required to implement the Taylor rule are given by:*

1.

$$\frac{\tilde{T}_1}{\frac{\tilde{M}}{P}} = \frac{(1-\beta\phi_\Pi)\left(1-\frac{P_1}{P}\right)}{1+\beta\phi_\Pi\left(\frac{P_1}{P}-1\right)} < 0, \tag{33}$$

$$\frac{\tilde{T}_2}{\frac{\tilde{M}}{P}} = \frac{\beta\phi_\Pi\left(1-\frac{P_1}{P}\right)}{1+\beta\phi_\Pi\left(\frac{P_1}{P}-1\right)} > 0, \tag{34}$$

if the nominal interest rate is strictly positive at $t = 1$, $i_1 > 0$;

2. $\tilde{T}_1/(\tilde{M}/P) \leq P/P_1 - 1/\beta$ and $\tilde{T}_2/(\tilde{M}/P) \geq 1/\beta - 1$ satisfying

$$\frac{\tilde{T}_1 + \frac{1}{1+r_1}\tilde{T}_2}{\frac{\tilde{M}}{P}} = \frac{P}{P_1} - 1, \tag{35}$$

if the nominal interest rate is zero at $t = 1$, $i_1 = 0$.

When the economy is away from the zero lower bound, we have expressions for the money growth and taxes in each period given by equations (33) and (34). In particular, there is an expansion of money in the first period and a subsequent contraction in the second period, which is financed with subsidies and taxes in the first and second periods, respectively. When the economy is at the zero lower bound, the timing of taxes over the first two periods is not determined, only the present value of taxes is.[6]

As ξ_1 or θ approaches the bound in (30), and $P_1/P \to 0$, the present value of the taxes required to implement the Taylor rule following the unanticipated shock become arbitrarily large (see equation 35). Intuitively, the Taylor rule requires that prices are stabilized in the second period after the shock. The shock itself results in a deflationary pressure in the first period. In particular, the deflation is exacerbated when the zero lower bound it binding because the excessive cash holding of households increases the demand for money. To stabilize the price, the deflation requires a large contraction of the money supply, which is

[6] In the first period taxes are bound by the requirement that agents should have enough money balances to consume cash goods in the second period. Therefore, there exists an upper bound of tax as shown in proposition 2. Notice that when the zero lower bound is just binding, or equivalently, the equilibrium gross inflation is given by $P_1/P = 1 - (1/\phi_\Pi)(1/\beta - 1)$, the upper bound of the tax $P/P_1 - 1$ is identical to the imposed tax when zero lower bound is not binding as shown by equation (33). This bound approaches infinity as $P_1/P \to 0$.

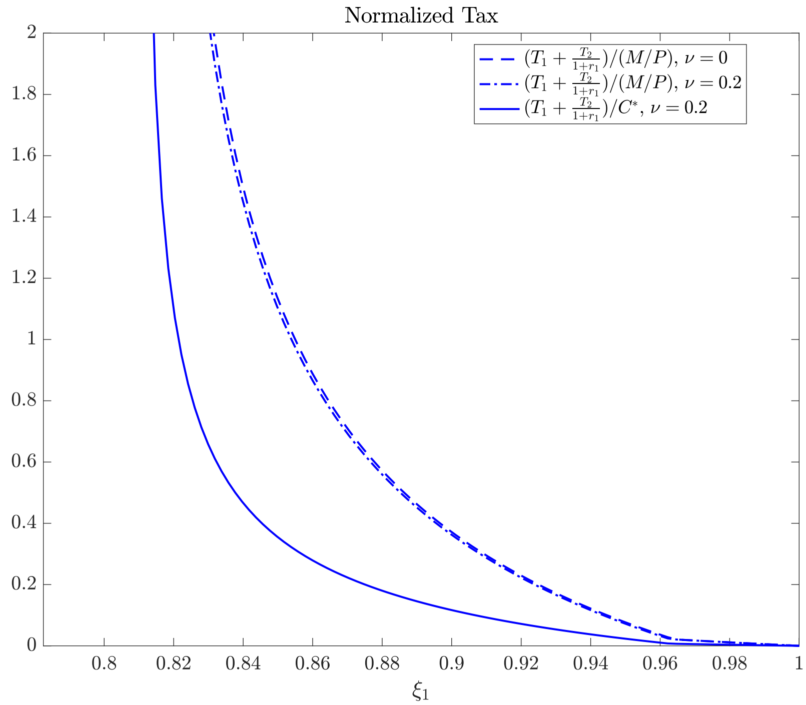

Figure 2: Tax to Implement the Taylor Rule.

financed by a large discounted taxes over the first two periods. In the limit, the equilibrium fail to exist because it is not feasible to implement the Taylor rule.

Although the assumption of the cashless limit allows us to characterize the fiscal cost analytically, in this limit the fiscal consequences are negligible. To better grasp the quantitative relevance of the characterization in Proposition 2, we also numerically trace the fiscal consequences of the Taylor rule away from the cashless limit. Figure 2 compare the implied taxes as a function of the shock for the cashless limit and the case with real balances bounded away from zero ($\nu = 0.2$.). In both cases, as the absolute size of the negative preference shock increases, the present value of taxes become arbitrarily large, although for any value of the shock the required taxes in the case with positive cash is slightly less than the required tax at cashless limit. Intuitively, the convergence to the steady state is smoother in the case with cash. In the first period, the preference shock motivates households to save and consume more cash goods tomorrow. Therefore, the money demand increases causing expected inflation to be larger, $P_2/P_1 > 1$. Consequently, the present discounted value of taxes is somewhat smaller in the case with cash, $\nu > 0$.

3.1.3 Real Consequences of the Taylor Rule

In this section we illustrate the real consequences of implementing a Taylor rule following a shock that brings the economy into a liquidity trap. These are examples of the well-documented paradoxes of liquidity traps: the negative effect of shocks are greatly amplified, price flexibility do not ameliorate these real effects, and the fact that the fiscal multiplier is particularly large when the economy is in a liquidity trap (e.g., Christiano et al., 2011; Eggertsson and Krugman, 2012; Werning, 2011).

Figure 3 presents the equilibrium prices as functions of preference shock, for two values of the degree of

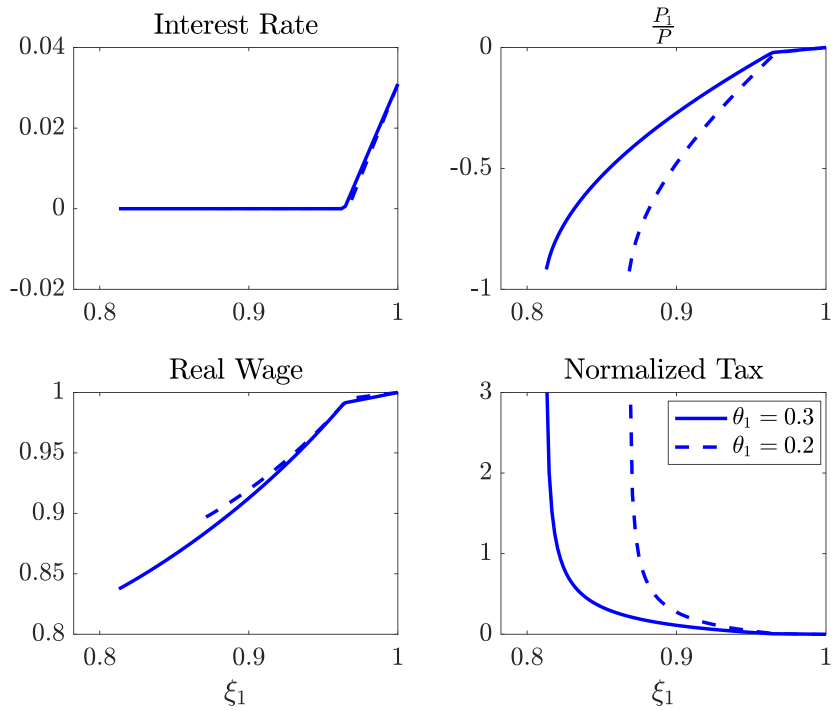

Figure 3: Equilibrium Prices as a Function of the Preference Shock, ξ_1, for Alternative Values of the Degree of Nominal Frictions, θ.

nominal frictions $\theta = 0.2, 0.3$. The nominal and real interest rates are declining with the magnitude of the drop in ξ_1 up to the point where the economy hits the zero lower bound. When the nominal rate hits the lower bound, the real interest rate also remains at zero. This is due to the fact that the Taylor rule stabilizes the expected inflation in the second period, $P_2/P_1 = 1$. The top right panel plots the equilibrium inflation P_1/P. The most interesting aspect of this graph is that when the zero lower bound is binding, inflation is more sensitive to the shock. Eventually P_1/P becomes arbitrarily close to zero. Moreover, the deflation becomes more pronounced when prices are more flexible. As shown in the bottom left panel, the real wage declines as the negative preference shock becomes more severe, although the decline in the real wage is less pronounced when prices are more flexible. Finally, in the bottom right panel we show the present value of taxes relative to the cash balances in the steady state. When prices are more flexible, the fiscal cost of implementing the Taylor rule are larger and grow faster.

What's the intuition behind the excess sensitivity of inflation to the shock at the ZLB? The negative discount factor shock leads to an increase in the demand for safe assets. The greater demand results in a decline in the return of the safe asset. Meanwhile, the Taylor Rule fixes the return on money, $P_2/P_1 = 1$. As a result, with a sufficient severe preference shock, the economy is at zero lower bound. When the nominal interest rate is bounded at 0, money is a perfect substitute for the safe asset. This fuels the demand of the money, which rises the price of money and thus exacerbates the deflation.

Figure 4 shows the equilibrium allocations as a function of the preference shock. Consumption, top left panel, declines as the negative shock becomes more severe. In particular, the decline of the consumption is faster when the economy is at the zero lower bound. Away from the liquidity trap the decline in the interest rate partially absorbs the negative preference shock. At the zero lower bound consumption is a steeper,

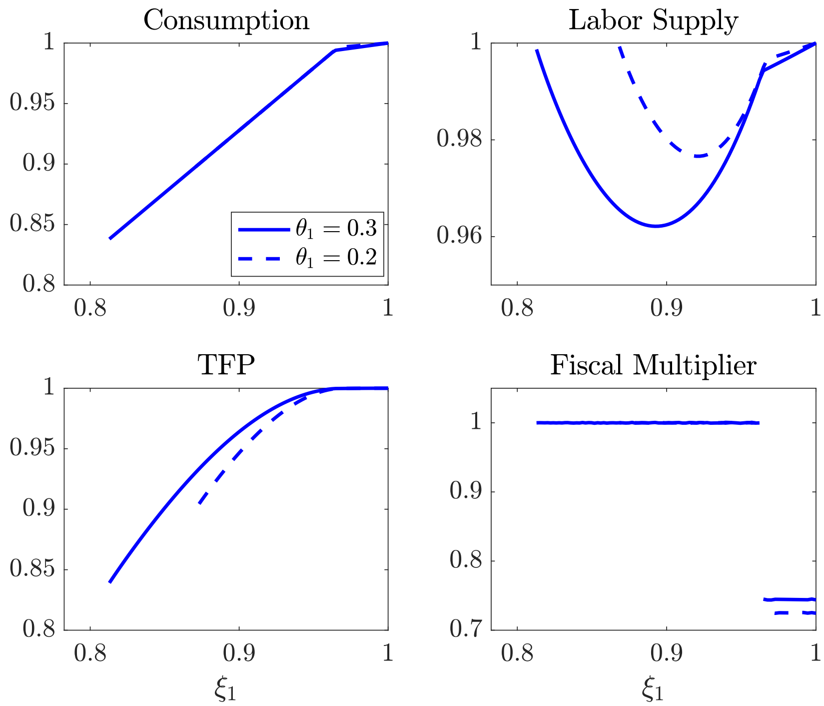

Figure 4: Equilibrium Quantities as a Function of the Preference Shock, ξ_1, for Alternative Values of the Degree of Nominal Frictions, θ.

linear function of the preference shock. Moreover, at the ZLB the value of consumption is independent of the degree of price stickiness, or the value of the government expenditure. Thus, as shown in the lower right panel, the liquidity trap features a large fiscal multiplier (equal to one).[7]

Yet, the effect of the shock on the labor supply is non-monotonic. There exists a trade-off between the price effect and the income effect in household's decision to supply labor. Notice that the equilibrium real wage is always a decreasing function with the increase of the shock. At the beginning the price-effect dominates: the drop of the real wage leads to a decrease in the labor supply. Yet, with the increase in the size of shocks and the decrease of the equilibrium consumption of households, the income-effect dominates. Households choose to supply more labor as the shocks becomes more negative.

3.2 Dynamics with A Constrained Taylor Rule

The extreme fiscal consequence associated with a standard Taylor Rule in a liquidity trap motivates us to consider a simple alternative policy which we label as the constrained Taylor Rule. We add an upper bound to the taxes that can be used to implement the Taylor rule. With the upper bound on the taxes, the government cannot stabilize prices when the shock is sufficiently large or prices are sufficiently flexible (given a value of the shock). Importantly, the equilibrium always exists with the constrained Taylor rule, independently of the size of the shock, or the degree of price flexibility. Moreover, the economy features a low fiscal multiplier and the consequences of negative shocks are less dramatic.

[7]Indeed, it follows from equation (6) and the fact that the Taylor rule stabilizes prices in the second period, $P_2/P_1 = 1$, that at the zero lower bound consumption equals $C_1 = \xi_1 C_2/\beta$. Thus, consumption is independent of the degree of price flexibility, θ_1, and of the value of government expenditure, G_1.

In particular, We impose the following simple bound on the taxes that can be used to implement the Taylor Rule:

$$\frac{\tilde{M}_{t-1} - \tilde{M}_t}{P_t} + G_t \leq \overline{T} \tag{36}$$

for some $\overline{T} > 0$. Similar to the case with standard Taylor rule, we assume that $G_t = 0$ and evaluate the fiscal multiplier as the derivatives of output with respective to G_t. All the other assumptions are identical to the case with the standard Taylor rule.

3.2.1 Characterization of the Economy

As discussed in Section 3.1.2, in a period when the economy is at the ZLB, and constraint (36) is not binding, any money supplied that is greater than the one needed by the representative agent to transact in cash goods in the following period is consistent with the equilibrium conditions in the first period.[8] Constraint (36) makes the lower bound on the money supply tighter for sufficiently low levels of P_1, but do not constrained the money supply from being larger that this value. To simplify the analysis, we assume that when the inflation rate is such that the Taylor rule is constrained by the ZLB the government sets the money supply equals to maximum between the amount required by the representative agent to transact in cash goods in the following period, and the lowest value consistent with constraint (36). This is stated formally in Assumption 1:[9]

Assumption 1 *If $P_t/P_{t-1} < 1 - \frac{1-\beta}{\beta\phi_\Pi}$, then*

$$\frac{\tilde{M}_t}{P_t} = \begin{cases} \frac{P_{t-1}}{P_t}\frac{\tilde{M}_{t-1}}{P_{t-1}} - \overline{T}, & \text{if } P_t \leq \tilde{M}_{t-1}/(\beta\xi_{t+1}C_t/\xi_t + \bar{T}), \\ \beta\frac{\xi_{t+1}}{\xi_1}C_t, & \text{otherwise.} \end{cases}$$

As before, Lemma 1 still applies. That is, the assumption that prices are fully flexible after the first period and that the economy is at the cashless limit, implies the real allocations are in the steady state after the first period. Similarly, the real allocation in the first period is still characterized by simple functions of the initial gross inflation P_1/P given by equations (27) and (28). The addition of constraint 36 results instead on alternative equilibrium paths for inflation, $\{P_{t+1}/P_t\}_{t=1}^{\infty}$, given a value of the initial inflation, P_1/P. Therefore, the key to characterize the economy with the constrained Taylor rule is to pin down the sequence of inflation for a given initial inflation, P_1/P. We do this next.

For $t \geq 2$, given stable expected inflation $P_{t+1}/P_t = 1$ and the path for the real allocation, the Euler equation for a nominal bond (6) and the (unconstrained) Taylor rule implies the following value for the inflation in period t:

$$\frac{P_t^u}{P_{t-1}} = 1. \tag{37}$$

We refer to this value as the unconstrained inflation rate. This is the equilibrium inflation provided the

[8]Related, the associated lump-sum taxes are only required to be lower than those needed to implement the minimum money supply.

[9]Alternative assumptions would implied a larger inflation in the following period. These assumptions would reinforce the conclusions obtained in the analysis that follows.

lump-taxes required to implement the Taylor rule are feasible, i.e.,

$$\begin{aligned}\tilde{T}_t &= \frac{P_{t-1}}{P_t^u}\frac{\tilde{M}_{t-1}}{P_{t-1}} - \frac{\tilde{M}_t}{P_t} \\ &= \frac{P_{t-1}}{P_t^u}\frac{\tilde{M}_{t-1}}{P_{t-1}} - \beta C^* \\ &\leq \bar{T}, \end{aligned} \tag{38}$$

where the second equality uses that real balances in period t are given by equation (7), the facts that the cash-in-advance constraint in period $t+1$ is binding and that the real allocations are in the steady state for $t \geq 2$, i.e., $M_t/P_t = \beta C^*$. If the unconstrained inflation satisfied (38), then $P_t/P_{t-1} = P_t^u/P_{t-1}$.

When the lump-sum taxes implied by the unconstrained inflation P_t^u/P_{t-1} violates condition (38), then the current inflation is given by the value that is consistent with budget balance, i.e., the value that solves

$$\frac{P_{t-1}}{P_t}\frac{\tilde{M}_{t-1}}{P_{t-1}} - \beta C^* = \bar{T}.$$

In this case, the nominal interest rate is given by

$$\begin{aligned} 1 + i_t &= \frac{1}{\beta} \\ &< \frac{1}{\beta} + \phi_\Pi\left(\frac{P_t}{P_{t-1}} - 1\right), \end{aligned}$$

where the inequality follows from $P_t/P_{t-1} > P_t^u/P_{t-1}$. Intuitively, the Taylor rule cannot implement a contractionary monetary policy, i.e., the nominal interest rate is too low relative to what it is prescribed by the Taylor rule.

The previous discussion implies that for $t \geq 2$ the inflation is a piece-wise linear function of the initial money balances, as stated in the following lemma.

Lemma 4 *Assume $t \geq 2$, $P_{t+1}/P_t = 1$, and let $\tilde{M}_{t-1}/P_{t-1}$ be given. Then, the unique inflation rate in period t consistent with the constrained Taylor rule is given by*

$$\frac{P_t}{P_{t-1}} = \begin{cases} 1 & \text{if } \frac{\tilde{M}_{t-1}}{P_{t-1}} - \beta C^* \leq \bar{T} \\ \frac{\frac{\tilde{M}_{t-1}}{P_{t-1}}}{\bar{T}+\beta C^*} > 1, & \text{otherwise.} \end{cases} \tag{39}$$

Furthermore, a corollary of Lemma 4 is that the Taylor rule is unconstrained for $t \geq 3$, as for these periods the real allocation in the previous period implied that the initial real balances equal $M_{t-1}/P_{t-1} = \beta C^*$.

Corollary 1 *The Taylor rule is unconstrained after the second period and the equilibrium features a zero-inflation path, i.e. $\tilde{T}_t = 0 < \bar{T}$ and $\frac{P_t}{P_{t-1}} = 1$ for all $t \geq 3$.*

Figure 5 illustrates the determination of inflation for $t \geq 3$. The top panel shows the determination of inflation when the Taylor rule is unconstrained. The dashed line gives the value of the nominal marginal rate of substitution, a value independent of the inflation rate, while the solid line is the nominal rate implied by the (unconstrained) Taylor rule. The lower panel shows the taxes required to implement the Taylor rule,

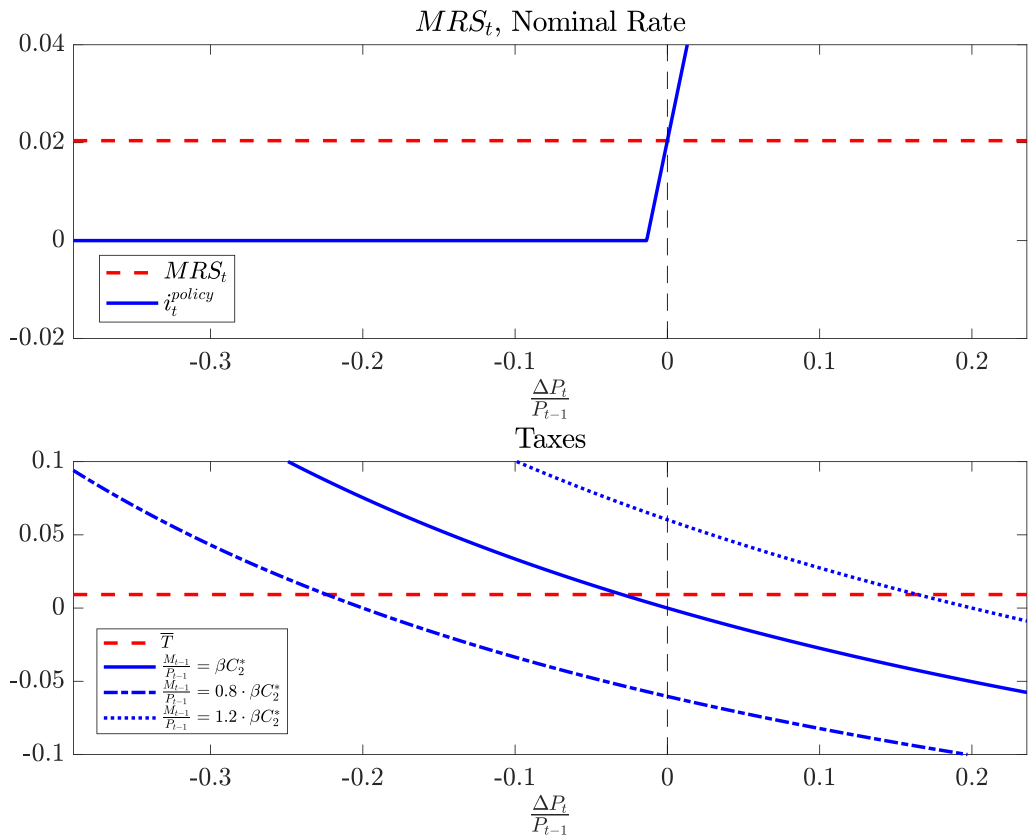

Figure 5: Determination of the inflation rate for $t \geq 2$ given $P_{t+1}/P_t = 1$, alternative values of M_{t-1}/P_{t-1}.

for three alternative values of the initial real balances, $M_{t-1}/P_{t-1} = 0.8\beta C^*, \beta C^*$, and $1.2\beta C^*$. The dashed line gives the upper limit to lump-sum taxes $\bar{T}$. The equilibrium value of inflation is given by either (i) the intersection between the dashed and solid lines in the upper panel and a value for lump-sum taxes that are lower than the upper limit, or (ii) an intersection between the downward slopping curve in the lower panel and the upper limit to the lump-sum taxes, together with the nominal rate $1+i_t = 1/\beta \le 1/\beta + \phi_\Pi(P_t/P_{t-1}-1)$. When $M_{t-1}/P_{t-1} = 0.8\beta C^*$ and βC^* the equilibrium features an unconstrained Taylor rule, while when $M_{t-1}/P_{t-1} = 1.2\beta C^*$ the equilibrium features a constrained Taylor rule.

We next characterize the inflation rate in the second period as a function of the initial inflation rate. Lemma 4 gives a characterization of the inflation rate in period two in terms of the initial money balances, $\tilde{M}_1/P_1$. Thus, in order to understand the behavior inflation in period two, we need to characterize the behavior of real cash balances in this period as a function of the initial inflation rate.

For low values of inflation in the first period, the real balances are given by Assumption 1. If the Taylor rule implies a feasible positive nominal rate, real balances are given by the demand of the representative agent which follows from (3) and (7). Thus, the real balances in the first period can be written as a function of the gross inflation in this period as follows

$$\frac{\tilde{M}_1}{P_1} = \begin{cases} \frac{P}{P_1}\frac{\tilde{M}}{P} - \bar{T}, & P_1 \le \hat{P}_1 \le P \\ \frac{\beta}{\xi_1} C_1, & \hat{P}_1 < P_1 \le P, \end{cases} \tag{40}$$

where C_1 is the value of consumption in period one as a function of the gross inflation rate in this period given by (27), and $\hat{P}$ is the value of the price level in the first period for which the upper bound of taxes is just binding in the first period.[10] The real balances in period $t = 1$ are an u-shaped function of the inflation rate in this period, with a minimum at $P_1/P = \hat{P}/P$. For any value of gross inflation greater than $\hat{P}/P$, the real balances are a strictly decreasing function of ξ_1. The threshold $\hat{P}$ is an increasing function of ξ_1.

Equations (39) and (40) imply the following Lemma characterizing the relationship between the gross inflation rate in the first and second periods:

Lemma 6: *There exist threshold values for the price level and preference shock in the first period,* $0 < \underline{P} \le \bar{P} \le 1$, $0 < \underline{\xi} < \bar{\xi} < 1$, *such that the gross inflation rate in the second period can be written as the following function of inflation rate in the first period:*

$$\frac{P_2}{P_1} = \begin{cases} \frac{\frac{P}{P_1}\beta C_2^* - \bar{T}}{\beta C_2^* + \bar{T}} > 1, & P_1 \in (0, \underline{P}] \\ 1, & P_1 \in \left(\underline{P}, \bar{P}\right] \\ \frac{\frac{\beta}{\xi_1} C_1}{\beta C_2^* + \bar{T}} > 1, & P_1 \in \left(\bar{P}, 1\right], \end{cases} \tag{41}$$

where C_1 is a function of P_1/P given in (29). Moreover, depending on the size of the shock ξ_1, the threshold values for the prices level are related as follows: $0 < \underline{P} < \bar{P} = 1$ *if* $\xi_1 \in (\bar{\xi}, 1]$, $0 < \underline{P} < \bar{P} < 1$ *if* $\xi_1 \in (\underline{\xi}, \bar{\xi}]$, *and* $0 < \underline{P} = \bar{P} < 1$ *if* $\xi \in (0, \underline{\xi}]$.

In the first region the Taylor rule is constrained in the first two periods. In the intermediate region the Taylor rule is unconstrained in the second period, while it might be constrained in the first period. In the last region, the Taylor rule is only constrained in the second period. Depending on the size of the shock ξ_1 the second and third cases might not exist.

[10] The threshold $\hat{P}$ solves $\tilde{M}/\hat{P} - (\beta/\xi_1)\hat{C} = \bar{T}$, where $\hat{C}$ is the value of consumption in (27) evaluated at $P_1/P = \hat{P}/P$.

Importantly, as the price level in the initial period approaches zero, the expected inflation in the second period becomes arbitrarily large. As the price level in the initial period goes to zero, the value of the real money balances carried to the second period becomes arbitrarily large and, therefore, the taxes required to finance the monetary contraction that implement stable prices in the second period becomes arbitrarily large (see Proposition 2). This implies that the lump-sum taxes required to implement the Taylor rule violate the constraint in (36), for any finite value of $\bar{T}$. As a consequence, the government is not able to adjust the supply of real balances, resulting in higher inflation in the second period. This result would be crucial to guarantee the existence of equilibrium with a constrained Taylor rule, and to understand why other paradoxes are not present in this case.

As in the Lemma 2, the equilibrium inflation rate in the first period is the solution to a single equation that is obtained by combining the Euler equation of a nominal bond (6), the Taylor rule in the first period (14), and the expression for the consumption in the first period as a function of inflation (29). The constrained on the Taylor rule only alters this equation by changing the expected inflation as shown in (41).

Lemma 7 *The equilibrium gross inflation rate in the first period P_1/P is the solution to the following equation:*

$$\underbrace{\frac{\xi_1}{\beta}\frac{C_2}{C_1}\frac{P_2}{P_1}}_{NMRS} = \max\left\{1, \frac{1}{\beta} + \phi_\Pi\left(\frac{P_1}{P} - 1\right)\right\}, \tag{42}$$

where

$$NMRS \quad = \quad \begin{cases} \frac{\xi_1}{\beta}\frac{C_2^*}{C_1}\frac{\frac{P}{P_1}\beta C_2^* - \bar{T}}{\beta C_2^* + \bar{T}}, & P_1 \in (0, \underline{P}] \\ \frac{\xi_1}{\beta}\frac{C_2^*}{C_1}, & P_1 \in (\underline{P}, \bar{P}] \\ \frac{C_2^*}{\beta C_2^* + \bar{T}}, & P_1 \in (\bar{P}, 1] \end{cases}, \tag{43}$$

the consumption C_1 as a function of P_1/P is given by (29), and the threshold values for the prices level in the initial period, $\underline{P}$ and $\bar{P}$, are given in Lemma 6.

Notice that when $P_1/P \to 0$, the policy is constrained in both the first and the second periods, while the expected inflation $P_2/P_1 \to \infty$. Intuitively, if the price level approaches zero in the first period, and the zero lower bound is binding, the value of households' initial money holds become arbitrarily large. Since the policy is constrained, the government cannot tax households' money holdings in the second period to contract the money supply and implement stable inflation. Therefore, the equilibrium in the money market in the second period would imply an arbitrarily large expected inflation, $\lim_{P_1/P \to 0} NMRS \to \infty$, guaranteeing that an equilibrium with a constrained Taylor rule always exists.

Proposition 3 *If $\bar{T} < \infty$ and $\bar{T} \neq (1-\beta)C^*$, then there exist a unique equilibrium with $\frac{P_1}{P} \in [0, 1]$. Moreover, if $\bar{T} > (1-\beta)C^*$, then there exist a $\xi_{ZLB} < 1$ such that $i_1 = 0$ for all $\xi_1 \leq \xi_{ZLB}$; while if $\bar{T} < (1-\beta)C^*$, then the equilibrium interest rate $i_1 > 0$, for all ξ_1.*

Figure 6 illustrates equation (43) for two values of the preference shock. The solid line shows the nominal interest rate prescribed by the Taylor rule in the first period. The dashed and dotted lines are corresponding to the nominal marginal rate of substitution in the cases of a constrained and unconstrained Taylor rule, respectively. Given a value for P_1/P, the real allocations C_1 and C_2 are independent of the specific policy. The difference in the nominal marginal rate of substitution stems from the difference in the expected inflation rate, P_2/P_1. Notice that with sufficient large shocks, the equilibrium does not exist for

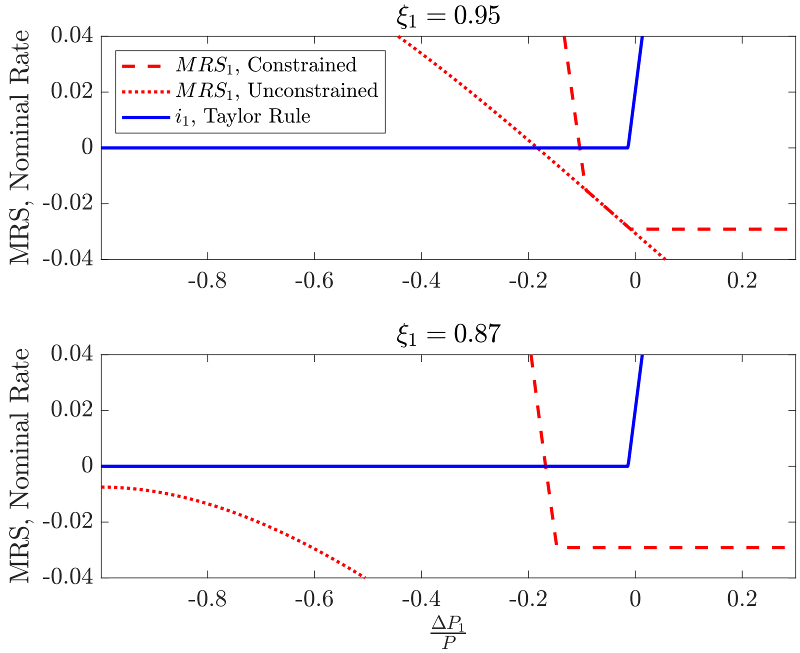

Figure 6: Equilibrium Determination with Constrained Taylor Rule for Alternative values ξ_1.

the case of an unconstrained Taylor rule as the MRS_1 is strictly less than zero for all values of P_1/P. In contrast, with a constrained Taylor rule the nominal marginal rate of substitution goes to infinity when inflation approaches zero guaranteeing the existence of the equilibrium.

3.2.2 Non-Paradoxical Allocations with a Constrained Taylor Rule

Figure 7 contrasts the equilibrium consequences for prices of following constrained and unconstrained Taylor rules. While the nominal rate is identical (top left panel), the equilibrium real rate in the first period is a strictly increasing function of the shock with the constrained Taylor rule. An important consequence of the constrained rule is that the deflation in the initial period is substantially alleviated (bottom left panel), because higher expected inflation results in a higher nominal marginal rate of substitution as shown in figure 6.

With a constrained Taylor Rule the negative consequences of the preference shock are substantially ameliorated, as shown in the top panel of Figure 8. Related, the large fiscal multiplier (bottom panel) and the paradox of price flexibility disappear. The effects of shocks are more benign when prices are more flexible.

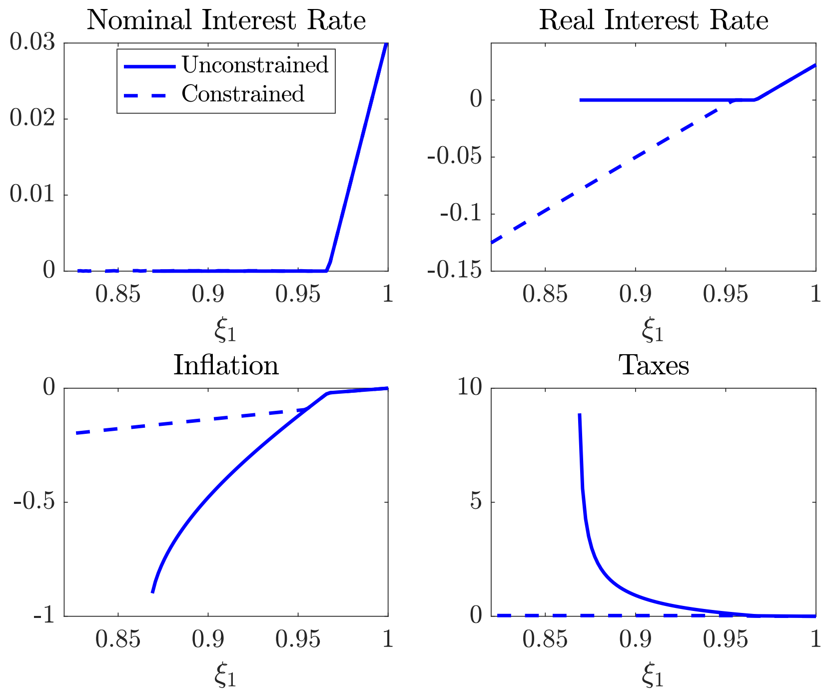

Figure 7: Equilibrium Prices and Taxes for Alternative values ξ_1 and Policies.

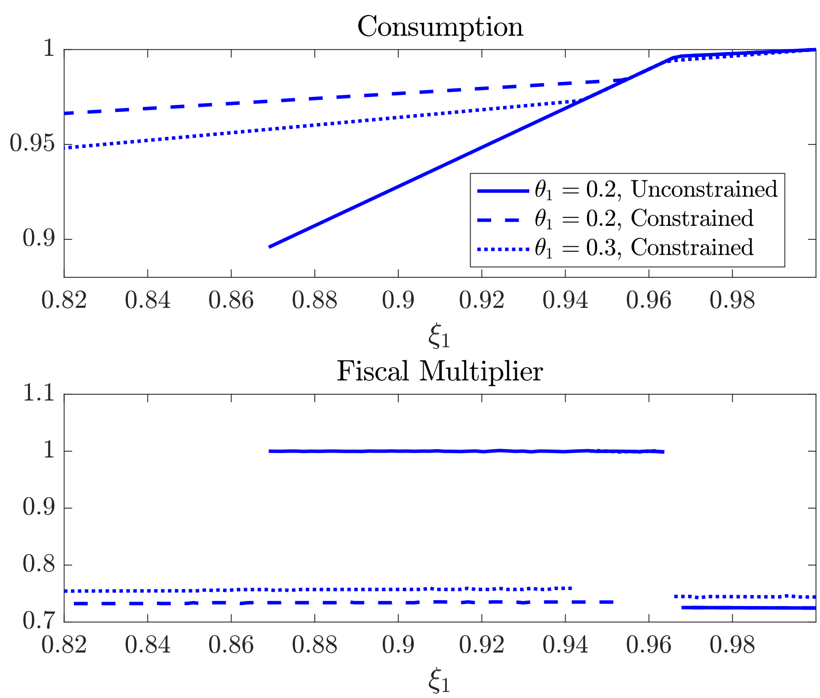

Figure 8: Consumption and Fiscal Multiplier for Alternative Price Stickiness θ_1 and Policies.

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Appendix

A Proofs of the Results in the Paper

Proof of Proposition 2 Equation (35) is implied by equation (32) after setting $i_1 = 0$. To derive the upper bound on the lump-sum taxes at $t = 1$ we start from the government budget constraint at $t = 1$

$$\begin{aligned}
\frac{\tilde{T}_1}{\frac{\tilde{M}}{P}} &= \frac{\frac{\tilde{M}-\tilde{M}_1}{P_1}}{\frac{\tilde{M}}{P}} \\
&\leq \frac{P}{P_1} - \frac{C^* \frac{\beta}{\xi_1} \frac{C_1}{C^*}}{\frac{\tilde{M}}{P}} \\
&\leq \frac{P}{P_1} - \frac{1}{\beta}.
\end{aligned}$$

where the inequality follows from the bound in (31) and the last equality uses equation (26) specialized to the case $i_1 = 0$ and the fact that $\tilde{M} = \beta P C^*$. The lower bound on the taxes in the second period follows from the upper bound on the taxes in the first period and the constraint on the present value of the taxes.

When the nominal interest rate in period $t = 1$ is strictly positive, we use the government budget constraint for $t = 2$ and condition (31) holding with equality to write

$$\begin{aligned}
\frac{\tilde{T}_1}{\frac{\tilde{M}}{P}} &= \frac{1}{\beta C^*} \frac{\beta P C^* - P_1 C^* \frac{\beta}{\xi_1} \frac{P_1}{P_2} \frac{C_1}{C^*}}{P_1} \\
&= \frac{\frac{P-P_1}{P_1}\left(1 - \beta\phi_\Pi\right)}{1 + \beta\phi_\Pi \left(\frac{P_1}{P} - 1\right)} \\
&< 0,
\end{aligned}$$

where the second equality uses equation (26) for the case $i_1 > 0$ and the inequality follows from the assumption that the Taylor rule features a strong response to inflation, i.e., $\beta\phi_\Pi > 1$. Similar arguments can be use to derive an expression for the lump-sum taxes in period $t = 2$ as follows

$$\begin{aligned}
\frac{\tilde{T}_2}{\frac{\tilde{M}}{P}} &= \frac{1}{\frac{\tilde{M}}{P}} \frac{P_1 \frac{\beta}{\xi_1} \frac{C_1}{C^*} C^* - \beta P_2 C^*}{P_1} \\
&= \frac{\beta \phi_\Pi \left(1 - \frac{P_1}{P}\right)}{1 + \beta \phi_\Pi \left(\frac{P_1}{P} - 1\right)} \\
&> 0.
\end{aligned}$$