Cross-correlation misfit measurement and adjoint source for full-waveform inversion

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(March 21, 2019)

GEO-Example

Running head: ***C*ross-correlation misfit**

ABSTRACT

INTRODUCTION

THEORY

Cross-correlation

The cross-correlation of observed and synthetic signal is defined as

$$C(\tau) = \int_{t1}^{t2} u(t-\tau)u^{obs}(t)dt = \int_{t1}^{t2} u(t)u^{obs}(t+\tau)dt \tag{1}$$

where $u(t)$ is synthetic waveform data using model parameter m and elastic wave equation.

The peak time shift is defined as the time shift that maximize the cross-correlation function

$$\delta\tau = \max_{\tau} C(\tau) = \max_{\tau} \int_{t1}^{t2} u(t)u^{obs}(t+\tau)dt \tag{2}$$

To obtain the connection between the peak time shift $\delta\tau$ and model parameter m, one needs to first estimate the relation between the peak time shift $\delta\tau$ and synthetic data $u(t)$. The derivative of $C(\tau)$ with respect to τ is given by

$$\begin{aligned} \frac{\partial C(\tau)}{\partial \tau} &= \int_{t1}^{t2} u(t)\frac{\partial u^{obs}(t+\tau)}{\partial \tau}dt = \int_{t1}^{t2} u(t)\frac{\partial u^{obs}(t+\tau)}{\partial (t+\tau)}\frac{\partial (t+\tau)}{\partial \tau}dt \\ &= \int_{t1}^{t2} u(t)\dot{u}^{obs}(t+\tau)dt \end{aligned} \tag{3}$$

where over-dot means time derivative. The derivative $\partial C(\tau)/\partial\tau$ vanishes at $\tau = \delta\tau$.

$$\frac{\partial C(\tau)}{\partial \tau}\,|_{\tau=\delta\tau} = \int_{t1}^{t2} u(t)\dot{u}^{obs}(t+\delta\tau)dt = 0 \tag{4}$$

Equation 4 indicates that the peak time shift $\delta\tau$ is an implicit function of synthetic data $u(t)$. It can be denoted as

$$f(u,\delta\tau) = \int_{t1}^{t2} u(t)\dot{u}^{obs}(t+\delta\tau)dt = 0. \tag{5}$$

Cross-correlation travel time misfit function

The cross-correlation travel time misfit function for full-waveform inversion is defined as the sum of the square of the peak time shift

$$J = \frac{1}{2}\sum_s\sum_r \delta\tau(x_r, x_s)^2 \tag{6}$$

Gradient

One of the key component of full-waveform inversion is the gradient of the misfit function

$$g = \frac{\partial J}{\partial m} = \sum_s\sum_r \frac{\partial \delta\tau(x_r, x_s)}{\partial m}\delta\tau(x_r, x_s) \tag{7}$$

The gradient is obtained by applying the Frechet derivative on the peak time shift. To get the Frechet derivative, we differentiate equation $f(u, \delta\tau) = 0$ with respect to model parameter m

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial m} + \frac{\partial f}{\partial \delta\tau}\frac{\partial \delta\tau}{\partial m} = 0 \tag{8}$$

The Frechet derivative is given by

$$\frac{\partial \delta\tau}{\partial m} = -\left(\frac{\partial f}{\partial \delta\tau}\right)^{-1}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial m}\right) \tag{9}$$

Using equation 5, the derivative $\partial f/\partial\delta\tau$ can be written as

$$\begin{aligned}\frac{\partial f}{\partial \delta\tau} &= \int_{t1}^{t2} u(t)\frac{\partial \dot{u}^{obs}(t+\delta\tau)}{\partial \delta\tau}dt = \int_{t1}^{t2} u(t)\frac{\partial \dot{u}^{obs}(t+\delta\tau)}{\partial (t+\delta\tau)}\frac{\partial (t+\delta\tau)}{\partial \delta\tau}dt \\ &= \int_{t1}^{t2} u(t)\ddot{u}^{obs}(t+\delta\tau)dt\end{aligned} \tag{10}$$

and

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial m} = \int_{t1}^{t2} \frac{\partial u(t)}{\partial m}\dot{u}^{obs}(t+\delta\tau)dt. \tag{11}$$

The Frechet derivative can be expressed as (Luo and Schuster, 1991)

$$\frac{\partial \delta\tau}{\partial m} = -\frac{\int_{t1}^{t2} \frac{\partial u(t)}{\partial m}\dot{u}^{obs}(t+\delta\tau)dt}{\int_{t1}^{t2} u(t)\ddot{u}^{obs}(t+\delta\tau)dt} \tag{12}$$

If assuming that $u(t)^{obs}$ and $u(t)$ are purely time shifted (Marquering et al., 1999; Dahlen et al., 2000; Tromp et al., 2005; Tape et al., 2007), using $u^{obs}(t+\delta\tau) \approx u(t)$, the Frechet derivative can be rewritten as

$$\frac{\partial \delta\tau}{\partial m} \approx -\frac{\int_{t1}^{t2} \frac{\partial u(t)}{\partial m} \dot{u}(t) dt}{\int_{t1}^{t2} u(t)\ddot{u}(t) dt} \tag{13}$$

Now, the gradient of misfit function (equation 7) changes to

$$\begin{aligned} g = \frac{\partial J}{\partial m} &= \sum_s \sum_r \frac{\partial \delta\tau}{\partial m} \delta\tau \approx -\sum_s \sum_r \delta\tau \frac{\int_{t1}^{t2} \frac{\partial u(t)}{\partial m} \dot{u}(t) dt}{\int_{t1}^{t2} u(t)\ddot{u}(t) dt} \\ &= -\sum_s \sum_r \int_{t1}^{t2} \frac{\partial u(t)}{\partial m} \left[\dot{u}(t) \frac{\delta\tau}{N} \right] dt \end{aligned} \tag{14}$$

where normalizer $N = \int_{t1}^{t2} u(t)\ddot{u}(t)dt$ and $\partial u(t)/\partial m$ is the Frechet derivative of waveform data $u(t)$ with respect to m. This formulation is similar to the gradient of the ℓ_2 norm waveform misfit function

$$g_{\ell_2} = -\sum_s \sum_r \int_{t1}^{t2} \frac{\partial u(t)}{\partial m} \left[\delta u(t) \right] dt \tag{15}$$

where $\delta u(t) = u^{obs}(t) - u(t)$.

Adjoint source

Instead of explicitly construct the Frechet derivative, the adjoint-state method is usually used to compute the action of Frechet derivative on data residuals.

EXAMPLES

CONCLUSIONS

ACKNOWLEDGMENTS

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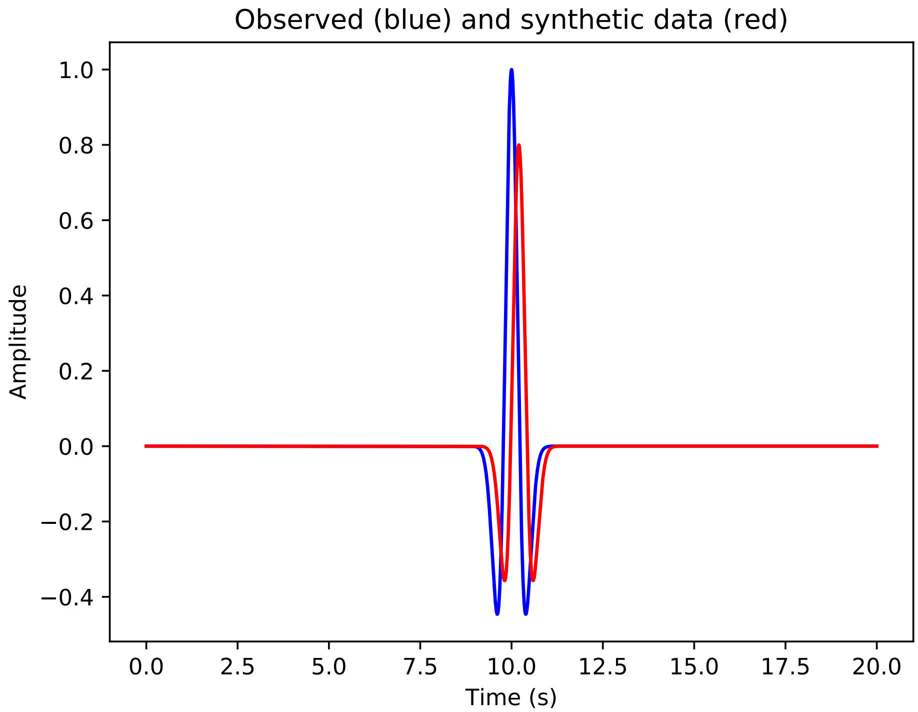
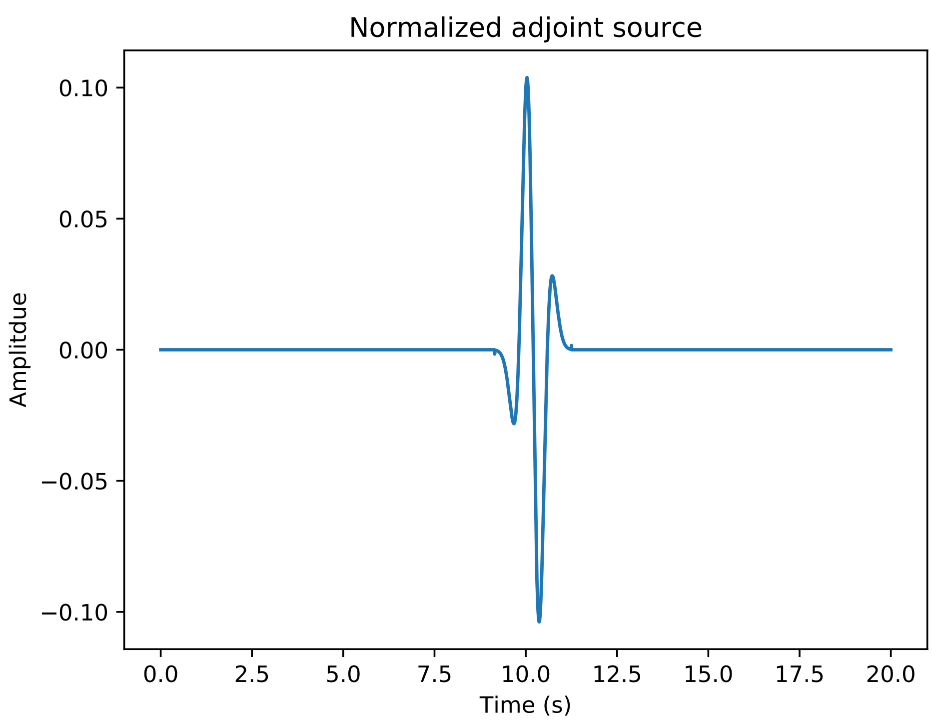
LIST OF FIGURES


Observed (blue) and synthetic data (red)
Amplitude
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (s)

(a)


Normalized adjoint source
Amplitdue
0.10
0.05
0.00
−0.05
−0.10
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (s)

(b)