

Conic section Assignment

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Problem Statement - The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a

Solution

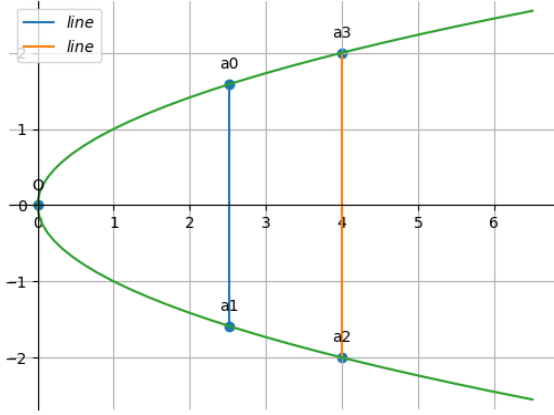


Figure 1: The parabola formed by the curve $y^2 = x$ and the line $x=4$

The given equation of parabola $y^2 = x$ can be written in the general quadratic form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}, \quad (3)$$

$$f = 0 \quad (4)$$

The point of intersection of the lines $x=1$ and $x=4$ to the parabola is given by

The points of intersection of the line

$$L: \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \quad (5)$$

with the conic section are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (6)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \right]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (7)$$

From the line $x-a=0$ the vectors \mathbf{q}, \mathbf{m} are taken,

$$\mathbf{q}_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (8)$$

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (9)$$

by substituting eq(2),(3),(4),(8),(9) in eq(7)

$$\mu_i = a, -a \quad (10)$$

substituting eq(8),(9),(10) in eq(6) the intersection points on the parabola are

$$\mathbf{a}_0 = \begin{pmatrix} a \\ a \end{pmatrix}, \quad (11)$$

$$\mathbf{a}_1 = \begin{pmatrix} a \\ -a \end{pmatrix} \quad (12)$$

From the line $x-4=0$ the vectors \mathbf{q}, \mathbf{m} are taken,

$$\mathbf{q}_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (13)$$

$$\mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (14)$$

by substituting eq(2),(3),(4),(13),(14) in eq(7)

$$\mu_i = 2, -2 \quad (15)$$

substituting eq(13),(14),(15) in eq(6) the intersection points on the parabola are

$$\mathbf{a}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (16)$$

$$\mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (17)$$

Area between parabola and the line $x=4$ is divided equally by the line $x=a$

$$\Rightarrow A_1 = \int_0^a \sqrt{x} dx \quad (18)$$

$$\Rightarrow A_2 = \int_a^4 \sqrt{x} dx \quad (19)$$

$$\implies A_1 = A_2 \tag{20}$$

$$\int_0^a \sqrt{x} \, dx = \int_a^4 \sqrt{x} \, dx \tag{21}$$

$$\implies a = 4^{\frac{2}{3}} \tag{22}$$

Construction

Points	intersection points
a0	$\begin{pmatrix} a \\ a \end{pmatrix}$
a1	$\begin{pmatrix} a \\ -a \end{pmatrix}$
a3	$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$
a2	$\begin{pmatrix} 4 \\ -2 \end{pmatrix}$