Nonstandard Analysis, The 1972-SciAm Article

Ideas That Common People Brand Nonstandard

Martin Davis[1] Reuben Hersh[2]

[1]Original Author

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Outline

- This Is A Review of A Davis-Hersh[1] Article in Scientific American, June 1972.
    - You Perhaps Haven't Even Heard "Nonstandard Analysis" in The Context of Calculus-1.
    - Analysis Refers to Foundations of Calculus.
    - Nonstandard Only Because The Word 'Standard' Was Taken to Mean Formalization of Calculus Using Some Other (Beautiful) Way!
- These Slides Are A Gentle Introduction to Nonstandard Analysis, Are for Personal Benefit, Only Indirectly for Actual Presentation.
- There's No Need to Get Intimidated by Big Words.
- Let's Be Fearless in Guided Imagination, and Make Inquiry A Habit.

[1] Both Were Mathematicians with a Gift of Writing Extraordinary Expository Articles.

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Common Words, Uncommon Meanings, Yet Again!

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A Bit of History

Everything Has A History!

- Influential Ancient Greeks, Euclid, Aristotle, & Archimedes, Avoided The *Unthinkable* Infinity (Infinitely Large) & The *Strange* Infinitesimals (Infinitely Small).
- Ideas of Calculus Over Time Culminated into Leibniz's and Newton's Formulation of Calculus in The 1680s.
- Both Used The *Banned* Infinitesimals:
  - Small, Infinitely Small Non-Archimedean Numbers!
  - Smaller Than Any Finite Number, Yet Bigger Than 0!
- Several Philosophers/Mathematicians Criticized[2] Leibniz and Newton for These *Infinitesimal Numbers* That Exist in A Sort of Neverland!
- Infinitesimals Defied Logic Prevalent at The Time!
- And So, Karl Weierstrass et. al. Banished Infinitesimals from Calculus by The 1890s!

[2] Prominent Among Them Was Bishop George Berkeley, The Famous Clergyman for Whom the University of Berkeley, California is Named!

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- Weierstrass Formalized Calculus Using The Allegedly Non-intuitive Idea of Limits.
- However, In 1966, The Logician Abraham Robinson Reintroduced Infinitesimals And Provided The Necessary Rigor to Them!
- Several Mathematicians Criticized Robinson's Work!
- But in 2020s, It Is Considered Original Body of Work That Is Equivalent To Standard Analysis; Strangely, Logic Prevalent Now Necessitates Infinitesimals!
- In The End, Your Approach to Calculus Reduces to Your Philosophy of Mathematics: Choice Is Yours!
- The War between The Continuous And The Discrete Is Reignited.

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Does That Make You Uncomfortable?

Be My Guest!

- On The First Encounter, Nonstandard Analysis Feels, Well, *Nonstandard*!
- It Makes Us Ponder The Nature of Mathematics.
- Although It Stretches Our Imagination, Its Logic And Equivalence with Standard Analysis Is Fun!
  - After All, When *Applied to Practical Problems*, Both Approaches Produce Exactly The Same Results!

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Geometry

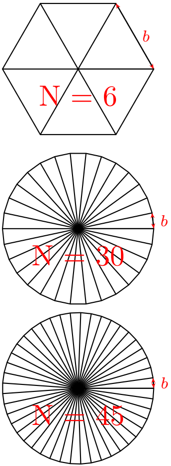
Let's Introduce Infinitesimals in Geometry.

The Area And Circumference of The Unit Circle

Argument 1: A *Logically Unacceptable* Argument Based on Infinitesimals

Area of Unit Circle = $\frac{1}{2}$ Its Circumference (Would Euclid Accept It?)

Any circle can be thought of as composed of infinitely many straight-line segments, all equal to each other and infinitely short. It is then the sum of infinitesimal triangles, all of which have altitude 1. Area of a triangle = half the base times the altitude. Therefore, the sum of the areas of the triangles is half the sum of the bases. But the sum of the areas of the triangles is the area of the circle, and the sum of the bases of the triangles is its circumference. Therefore, the area of the unit circle = half its circumference. $\square$



Seriously?

Is That (Argument 1 [8]) A Valid Mathematical Proof?

- That *Argument* [8] Was Published by Nicholas of Cusa in The 15^{th} Century!
- We Wonder If It Is Even Logical! Clearly, Objections Galore!
  - The Very Notion of A Triangle with An Infinitely Small Base Is Illusive!
  - The Base of The Triangle, b, However Small, Must Clearly Be ≥ 0, Right?
    1. If $b = 0$, No Number of Triangles Can Make A Circle with Positive Circumference!
    2. If $b > 0$, Infinitely Many Triangles Make An Infinitely Large Circumference!
  - In Either Case, How Can We Ever Get A Finite Circle from Infinite Infinitely Small Pieces?
    1. Every Real Number Must Be Archimedean!
    2. Since No Infinitesimal Is Archimedean, They Were Rejected by Euclid, Aristotle, Archimedes.

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Archimedes's Work: Lost And Found

Riddle of The Quadrature of The Parabola

- Archimedes's Work Came in Two Streams of Tradition: 1) Continuous, 2) After A Gap of 1000 Years.
- Of Present Interest Is His Method of Exhaustion.
  - Discovered in Constantinople in 1906!
  - Relies on An "Indirect Argument" And Purely Finite Constructions.
  - Finds The Volumes of Surfaces of Revolution (e.g., A Paraboloid).
  - Since Infinitesimals Don't Exist[3], It Gives A Logically Acceptable, Rigorous Proof of His Results!
    1. No References Are Made to Infinitesimals in It.
    2. Ironically, You May Find Its Logic Akin to Weierstrass's Formulation.

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  - Since Infinitesimals Don't Exist[3], It Gives A Logically Acceptable, Rigorous Proof of His Results!
    1. No References Are Made to Infinitesimals in It.
    2. Ironically, You May Find Its Logic Akin to Weierstrass's Formulation.

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Nonstandard Analysis
MD,RH,KM
Introduction
Summary

- Archimedes's Work Came in Two Streams of Tradition: 1) Continuous, 2) After A Gap of 1000 Years.
- Of Present Interest Is His Method of Exhaustion.
  - Discovered in Constantinople in 1906!
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Archimedes's Work: Lost And Found

Riddle of The Quadrature of The Parabola

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Area of Unit Circle = $\frac{1}{2}$ Its Circumference (Flawless Logic?)

Let S Be The Proposition[a] That The Area of A Unit Circle ($\mathbb{A}_C$) = Its Half-circumference ($\mathbb{H}_C$).
If S Is `false`, Then

$$\text{Either} \quad \mathbb{A}_C > \mathbb{H}_C \quad \text{Or} \quad \mathbb{H}_C > \mathbb{A}_C \tag{1}$$

Let $\mathbb{D}$ Be The *Positive Difference* between $\mathbb{A}_C$ And $\mathbb{H}_C$.
Therefore, If $\mathbb{H}_C > \mathbb{A}_C$, Then

$$\mathbb{D} = \mathbb{H}_C - \mathbb{A}_C \tag{2}$$

Otherwise,

$$\mathbb{D} = \mathbb{A}_C - \mathbb{H}_C \tag{3}$$

[a]Therefore, S Must Be Either `true` Or `false`.

The Area And Circumference of The Unit Circle

Argument 2: Continued ...

Area of Unit Circle = $\frac{1}{2}$ Its Circumference (Flawless Logic?)

...

Now We Use Proof by Contradiction to Show that Both Eq. [2] and Eq. [3] Lead to Contradiction with Eq. [1].

Let's start with Eq. [2] first.

We Can Circumscribe @ The Circle A Regular Polygon with as Many Sides as We Wish.

Since The Polygon Is Composed of Finite Number of Finite Triangles with Altitude = 1, Area of The Polygon equals Its Half-perimeter.

$$\mathbb{A}_P = \mathbb{H}_P \tag{4}$$

As The Number of Sides of The Polygon Circumscribing The Circle Increases, The Difference in Their Areas, $\mathbb{A}_P - \mathbb{A}_C$, Reduces.

The Area And Circumference of The Unit Circle

Argument 2: Continued ...

Area of Unit Circle = $\frac{1}{2}$ Its Circumference (Flawless Logic?)

...

We Can Increase The Number of Sides of The Polygon Such That $\mathbb{A}_P - \mathbb{A}_C < \frac{\mathbb{D}}{2}$

Then, from Equations [2] and [4],

$$\begin{aligned} \mathbb{H}_P - (\mathbb{H}_C - \mathbb{D}) &< \frac{\mathbb{D}}{2} \\ \therefore \mathbb{H}_P - \mathbb{H}_C + \mathbb{D} &< \frac{\mathbb{D}}{2} \\ \therefore \mathbb{H}_P &< \mathbb{H}_C - \frac{\mathbb{D}}{2} \end{aligned} \tag{5}$$

This Is A Contradiction; The Polygon Circumscribes The Circle, How Can Its Perimeter Be Smaller Than That of The Circle?

The Area And Circumference of The Unit Circle

Argument 2: Continued ...

Area of Unit Circle = $\frac{1}{2}$ Its Circumference (Flawless Logic?)

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We Reason Similarly for Eq. [3]: Increase The Number of Sides of The Polygon Such That $\mathbb{H}_P - \mathbb{H}_C < \frac{\mathbb{D}}{2}$ (Note: Half-Perimeters). Then, from Equations [3] and [4],

$$\begin{aligned} \mathbb{A}_P - (\mathbb{A}_C - \mathbb{D}) &< \frac{\mathbb{D}}{2} \\ \therefore \mathbb{A}_P - \mathbb{A}_C + \mathbb{D} &< \frac{\mathbb{D}}{2} \\ \therefore \mathbb{A}_P &< \mathbb{A}_C - \frac{\mathbb{D}}{2} \end{aligned} \tag{6}$$

A Contradiction Again; How Can The Area of A Circle Be Greater Than That of The Circumscribing Polygon?

The Area And Circumference of The Unit Circle

Argument 2 Ends

Area of Unit Circle $= \frac{1}{2}$ Its Circumference (Flawless Logic?)

...

Thus, When $\mathbb{A}_C \neq \mathbb{H}_C$, We Reach A Contradiction.

Therefore, $\mathbb{A}_C = \mathbb{H}_C$. $\square$

We Have Two Proofs That $\mathbb{A}_C = \mathbb{H}_C$: The Direct Proof of Nicholas of Cusa [8] Using Infinitesimals And The Indirect Argument of Archimedes [15] Avoiding Infinitesimals!

- The Direct Proof Is Strange, The Indirect Proof Pedantic!
- They Agree And Reflect the 'Beliefs' of Their Creators: Infinitesimals Made Archimedes Uncomfortable, But Inspired Nicholas of Cusa[4].
- Nicholas of Cusa, Johannes Kepler, Blaise Pascal Reveled Mysticism of ∞. Newton, Leibniz, Bernoulli Brothers, L'Hôpital, Euler Vowed to Unravel It.
    - Not Everyone Who Contributed to Formalizing Calculus Using Infinitesimals Believed Their Existence!
    - Newton Accepted But Avoided Them in *Principia*, Leibniz Accepted But Didn't Claim Their *Existence*!

[4] Who Was A Cardinal of The Church

Comparing The Two 'Proofs'

One Direct And One Indirect

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Since Antiquity, Geometry Played A Crucial Role in Providing Analysis Problems.

Dynamics Was a Relatively Recent Experience. All Kinds of Moving Bodies Started to Appear After Newton. Tools of Calculus Help Tremendously to Analyze Motion.

Motion Was, After All, Difficult to Ignore. Since The Time of Zeno[5] of Elea And, His Teacher, Parmenides, We Were Confused about Motion. Zeno Asserted That The Universe Was Static And All Motion Was Illusion. He Postulated Several Paradoxes to Confound (Illuminate?) People.

But After Newton, We Had to Study Motion Systematically, Which Would Remove Zeno's Paradoxes about The Ubiquitous Motion (If Not Erase His Philosophy) . . .

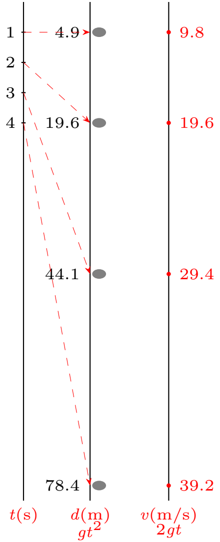
Let's Analyze An Everyday Motion by The Two Approaches (Standard: Weierstrass, Nonstandard: Robinson).

[5] Around 450 BC in Ancient Greece.

Analysis of The Motion of A Falling Body

1: The Basic Setup

Consider a Falling Stone.
Its Motion Is Described by Giving Its Position (Displacement from A Reference) as A *Function* of Time. As It Falls, Its Velocity Increases, So That The Velocity at Each Instant Is Also A Variable *Function* of Time. Newton Called The *Instantaneous Position* Function The 'Fluent' And The *Instantaneous Velocity* Function the 'Fluxion'.
If One Is Given, The Other Can Be Determined; This Connection Is The Heart of The Infinitesimal Calculus Fashioned by Newton and Leibniz.



Analysis of The Motion of A Falling Body

2: The Concept of *Instantaneous* Velocity

- We Take This Routine Experience of A Falling Body for Granted. However, Its Motion is Complicated...
  - First, We Must Be Able to Tell (Somehow) When An Exact Time Interval (e.g., 1 Second) Elapses[6].
  - We Can Then Think of Measuring The Precise Distance Traveled by A Body Falling Freely (Under Gravity).
  - Then We Can Tell *Where in Air* The Body Was When $t = 1s, 2s, \ldots$, i.e., We Can Describe Its Instantaneous Position as A *Function* of Time.
  - We Understand The Idea of The Average Speed of A Moving/Moved Body over The Entire Duration of Its Travel as $\frac{\text{Total Distance Traveled}}{\text{Total Time Elapsed since The Beginning}}$.
  - But Do We Really Comprehend How Fast A Falling Body Was Moving *at* $t = 1s$, *or* $t = 2s$?
  - Determining This Velocity Is Challenging When We Realize that The Body <u>Constantly</u> Moves Faster and Faster ...

[6] Exact 'Time-keeping' Is As Fascinating As It Is Challenging!

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  - The Velocity Remains Constant During A Tiny (Infinitesimal) Period of Time.
  - The Continuous Change in Velocity Actually Comes in The Form of Tiny, Discrete Jumps.
  - Robinson Formalized This Change as A Number That Behaves Differently from Real Numbers.
- We'll First Study What Newton Proposed. Then We'll Go through Bishop Berkeley's Objections And Weierstrass's 'Limit'ed Remedy Resulting in Standard Analysis. Finally, We Will Encounter Robinson's Revival of Infinitesimals Resulting in Nonstandard Analysis!
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- The Instantaneous Position of A Falling Body (in Meters) is Given by A Function of Time: $s = 4.9t^2$ (Where Time t Is Measured in Seconds).
- Therefore, at $t = 1$, $s = 4.9$.
- An Infinitesimal Time Interval dt Later, Its Instantaneous Position, $s' = 4.9(1 + dt)^2$.
- Correspondingly, The Change in Instantaneous Position, $ds = s' - s = 4.9((1 + dt)^2 - 1)$.
- $\therefore ds = 9.8dt + 4.9dt^2$.
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8: *Limits* Achieve Limitless Rigor!

- Weierstrass Illustrates The Idea of A *Limit* Before Defining It Precisely . . .
- He Tries to Show That However Close to Zero Δt Goes, $\frac{\Delta s}{\Delta t} = 9.8 + 4.9(\Delta t)$ May Go Closer to A Real Number. If He Succeeds, His Rigor Holds Strong!
- First, We Give Weierstrass A Positive Real Number, However Small, ϵ. Thus, $\epsilon > 0$.
- Weierstrass Is Free To Choose Another Positive Real Number, δ. In This Case, He Chooses $\delta = \frac{\epsilon}{4.9}; \delta > 0$.
- Then, Weierstrass Asserts That For Any Positive Value of Δt That Is Less Than δ, $(\frac{\Delta s}{\Delta t} - 9.8)$ Which Equals $4.9\Delta t$ Must Be Less Than $\cancel{4.9}\frac{\epsilon}{\cancel{4.9}}$, That Is, ϵ.
- IOW, He Asserts That as δ *Approaches* 0, $(\frac{\Delta s}{\Delta t} - 9.8)$ Also Approaches 0, i.e., $\frac{\Delta s}{\Delta t}$ *Approaches* No Other Number But 9.8: $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = 9.8$.

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8: *Limits* Achieve Limitless Rigor!

- Weierstrass Illustrates The Idea of A *Limit* Before Defining It Precisely ...
- He Tries to Show That However Close to Zero Δt Goes, $\frac{\Delta s}{\Delta t} = 9.8 + 4.9(\Delta t)$ May Go Closer to A Real Number. If He Succeeds, His Rigor Holds Strong!
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- On A First Reading, This Mere Replacement of dt, ds by $\Delta t, \Delta s$ Feels Strange.
- Granted, Weierstrass Achieved Two Important Things:
  - Removal of All Non-finite Quantities; Sticking to Reals.
  - Avoiding Division by Zero; He Never Sets Δt in $\frac{\Delta s}{\Delta t}$ to 0.
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  - An Everyday, Intuitive (If Paradoxical) Quantity, Instantaneous Velocity, Is Subjected to Unrelated Real Numbers, ϵ, δ, A Surprisingly Subtle Notion of *Limits*.
  - Not Knowing Limits Didn't Keep The Bernoullis, Euler, ... From Finding Velocities in Complicated Cases.
  - We Knew The Answers Using Infinitesimals That Matched Answers Found By Limits. And Yet, For "Logical Consistency", We Embraced a Subtle Definition That Mathematicians Appreciate[11] Much More Than Common People. Did We Treat Infinitesimals Fairly?

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Here's That Rigorous Definition of The Limit of A Function in Its Modern Grandeur:

Definition (Limit of A Real-Valued Function)

Consider The Function, $f : \mathbb{R} \to \mathbb{R}$, And Two Real Numbers p, L. We Say, The Limit of f of x, as x Approaches p, Exists And Equals L, And Write, $\lim_{x \to p} f(x) = L$, If The Following Property Holds:
For Every Real $\epsilon > 0$, There Exists A Real $\delta > 0$, Such That For All Real $x, 0 <\mid x - p \mid< \delta$ Implies $\mid f(x) - L \mid< \epsilon$.

It Is More Cryptic When Expressed Symbolically:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(0 <\mid x - p \mid< \delta) \implies \mid f(x) - L \mid< \epsilon)$$

Summary of Part 1

Introducing Nonstandard Analysis

References

[Goldbach, 1742] Christian Goldbach.
A problem we should try to solve before the ISPN 43 deadline,
Letter to Leonhard Euler, 1742.