Notes from
Hyperbolic Functions
(Topics in Mathematics Translated from the Russian)
by V. G. Shervatov
(translated by A. Gordon Foster and Coley Mills, Jr.)

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Reflection 1 (Introduction). These are the author's highly personal notes based on this great book. They may occasionally sound like author's conversation with the reader (or even himself). Even if they are personal and appear in the public domain, they may help the reader, although the author isn't exactly sure how. He wrote them for 0) the love of mathematics and writing 1) LaTeX practice (of writing hopefully readable mathematics, albeit longer than appearing in standard texts), and 2) reliable archival (paper, on which most math is created, tends to get lost) that incurs minimal overhead.

These notes are interspersed with "reflection boxes" like this one. They are meant to express author's 'rough' thinking process, of which frustration is often an inseparable part. The author writes mostly in first person for an authentic conversational tone.

I was lucky to find this book at the time I was discussing the hyperbolic sine and cosine functions with my daughter (who was a high school student then).

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1 The Hyperbolic Rotation

1.1 Homothetic Transformations

Reflection 2 (The Mathematical Language). My computer dictionary flags 'homothetic' as a typo, although our understanding of 'similarity' in geometry is based on these transformations. Agreed, it is a somewhat esoteric word of a Greek origin. Such transformation means 'scaling' in everyday Modern English. A good definition follows.

Definition 1 (Homothetic Transformation). If we pick a point O in a plane and a positive real number k, and *transform* any other point A to a point A' on the half-line OA, such that $\frac{OA'}{OA} = k$ while leaving the point O fixed, then we have carried out a Homothetic Transformation of A. See Figure 1.

The point O is called the *homothetic center* or *center of similitude*.

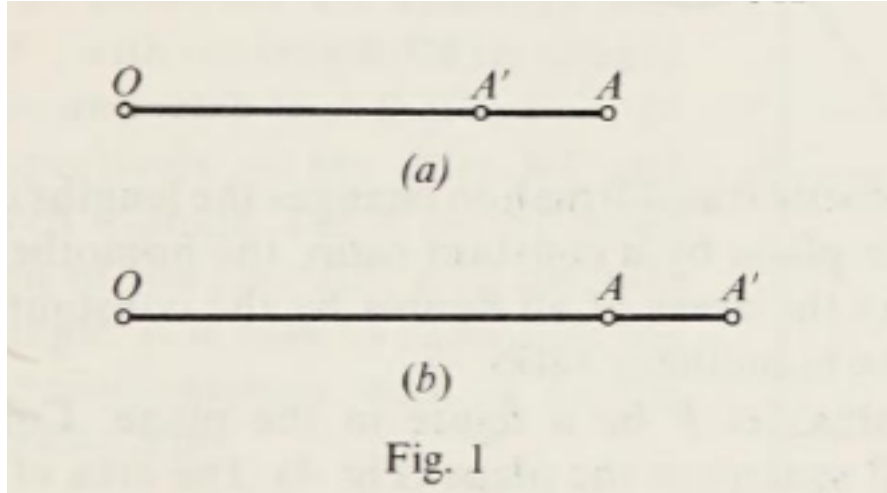k is called the *homothetic ratio* or *ratio of similitude*.

Figure 1: Homothetic Transformation of A

In arguably simpler terms, we have *scaled OA* by a factor k to OA'. Note that O, A, A' are collinear, by definition.

Since $l(OA) = l(OA') \cdot k$, if $k < 1$, $l(OA) < l(OA')$ and if $k > 1$, $l(OA) > l(OA')$.

> **Reflection 3** (Homothetic Transformation of a Straight Line). What do we mean by homothetic transformation of a straight line?
>
> We simply transform every point on that line using the same O and k.
>
> Do we get a straight line by doing that?
>
> It appears so, but let's prove that it is so.

Problem 1. Prove that the homothetic transformation of a straight line is a straight line.

Proof. We use step-by-step geometric constructions based on what we are given. First we pick O, k, and two points P_1, Q_1 on a given line ℓ_1. Let $k > 1$. Then, using the definition (1), we transform P_1 to P_2, Q_1 to Q_2.
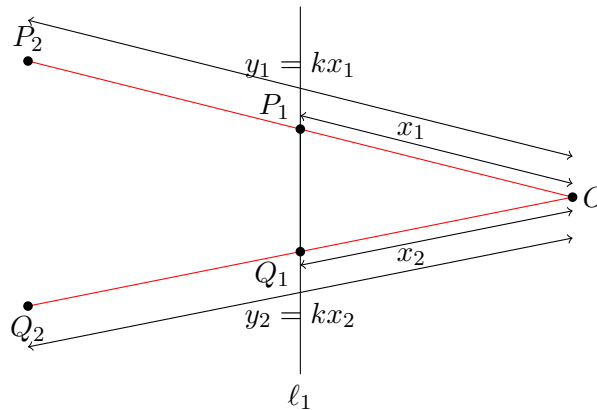

Figure 2: Construction 1 for Problem [1]

We place arbitrarily a point R_1 on line ℓ_1 and transform it using the same k to a point R_2. Join P_2Q_2 to make ℓ_2.

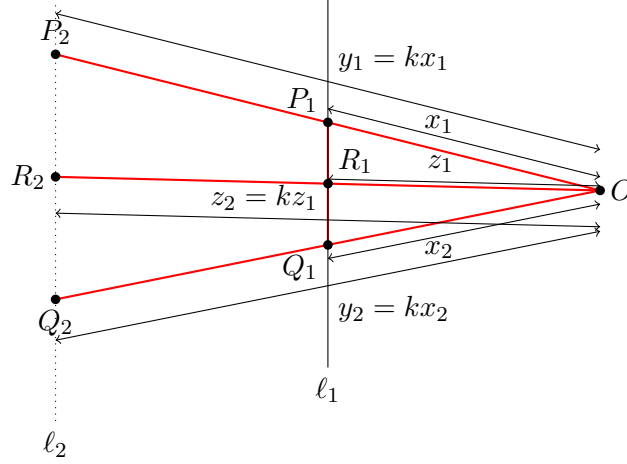We then ask, "Is R_2 on ℓ_2 (shown as a dotted line)?"

Figure 3: Construction 2 for Problem [1]

Reflection 4 (On Collinearity). I have been intrigued about proving collinearity. How do you *prove* whether or not the given points in a plane are collinear? I have clearly forgotten 'Elements.'

Perhaps the easiest approach is to use 'similarity of triangles.'

Or, should I first show that line P_2Q_2 (ℓ_2) is parallel to the line P_1Q_1 (ℓ_1)?

Should I resort to the Cartesian plane? Here, the idea of 'slope' makes it easier, doesn't it? If two segments formed by any three points have the same slope, the points are collinear. Alternatively, with three points, we can make a triangle and argue that if its area is zero, then they are collinear.

And suddenly an insight came, was it because I recalled the S-A-S test of similarity?

Consider $\triangle OP_1Q_1$ and $\triangle OP_2Q_2$. By construction, 1) OP_1 and OP_2 are proportional and OQ_1 and OQ_2 are proportional; the ratio of proportionality is the same, k, and 2) The included angle, $\angle P_1OQ_1 \cong \angle P_2OQ_2$, because they are the same angle.

Therefore, $\triangle OP_1Q_1 \sim \triangle OP_2Q_2$.

Therefore, $\angle OP_1Q_1 = \angle OP_2Q_2, \angle OQ_1P_1 = \angle OQ_2P_2$.

Therefore, $\ell_1 \parallel \ell_2$.

Similarly, $\triangle OP_1R_1 \sim \triangle OP_2R_2, \triangle OQ_1R_1 \sim \triangle OQ_2R_2$.

Therefore, $\ell_1 \parallel P_2R_2 \parallel R_2Q_2$. This is possible only if R_2 is on ℓ_2. ∎