Linear Algebra Problem Book: Notes and Problem Solutions

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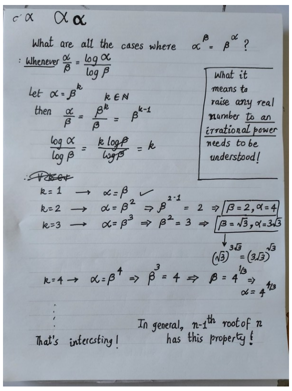
Abstract

THIS is an objective, yet personal, narrative of the author's odyssey in the enchanted land of linear algebra. It contains his notes and solutions to problems from Professor Paul Halmos's [1] *Linear Algebra Problem Book*.

I like to write. I don't have an established audience, but that does not deter me from writing. However, I have often wondered why I should carefully typeset my mathematical writing using a comprehensive system like LaTeX. First, my mathematical writing is not 'research' yet, but mainly problem-solving (which does involve at least some dogged pursuit, if not research). Second, I love writing by hand!

Figure 1: Handwriting is fun!



Freehand writing on a good paper with a good pen is fun. It's quick. It's rewarding.

Typesetting is, on the other hand, time-consuming and feels like *Yak-shaving* [3]. However, all good life is controlled Yak-shaving. When I typeset LaTeX documents, I tend to minimize Yak-shaving and focus on having a conversation with myself. I suspect that I understand the subject matter better that way. I like the following quote in this regard:

> I don't know what I think unless I read what I write. –Unknown

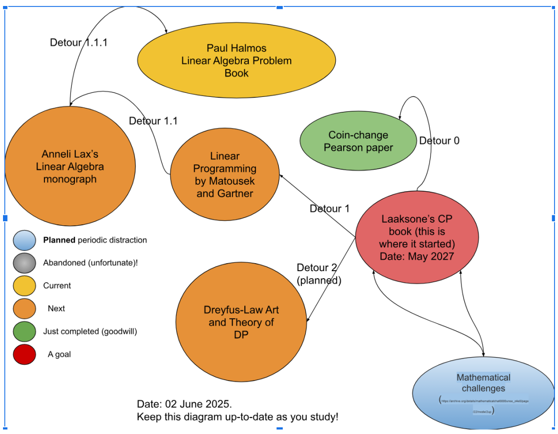
Of course, you cannot begin solving problems on a computer. A pencil and papers are a must. In this respect, typesetting is favoring form over content. We strive to present beautifully what we have thought well but scribbled hastily. Fortunately, I don't mind taking the time to do that; it at least keeps me busy fine-tuning my thoughts. It sometimes even helps to find flaws.

However, the biggest advantage of typesetting my writing is keeping a record of beautifully typeset account of something, anything. If I could ever make a case to study under a stalwart like Professor Yaser Abu-Mostafa (https://work.caltech.edu/), or Professor Avrim Blum (https://home.ttic.edu/~avrim/), perhaps I can demonstrate what I have done in my scarce free time.

I have gone back and forth between handwritten pages and typeset manuscripts. However, I am resorting to typesetting for this work despite the overhead incurred. I hope I follow through. It's one thing to be motivated but another to be determined and disciplined.

Perhaps a picture (Figure [2]) can explain better than words how I started reading Halmos's book.

Figure 2: A June 2025 Mindmap



I will avoid answering "Why Halmos's Linear Algebra book?" It suffices to say that his style resonates with me, a teaser of which can be enjoyed from what follows:

> Is it obvious that
> $$63 + 48 = 27 + 84$$
> ? It is a true and thoroughly uninteresting mathematical statement that can be verified in a few seconds–but is it *obvious*? If calling it obvious means that the reason for its truth is clearly understood, without even a single second's verification, then most people would probably say no.
> What about
> $$(27 + 36) + 48 = 27 + (36 + 84)$$
> ? –is that obvious? Yes it is, for most people; the instinctive (and correct) reaction is that the way the terms of a sum are bunched together cannot affect the answer. The approved technical term is not "bunch together" but "associate"; the instinctive reaction is a readiness to accept what is called the **associative law** of addition for real numbers. (Surely every reader has noticed by now that the non-obvious statement and the obvious one are in some sense the same:
> $$63 = 27 + 36 \quad \textit{and} \quad 84 = 36 + 48$$

>)
> **Linear algebra is concerned with several different kinds of operations (such as addition) on several different kinds of objects (not necessarily real numbers)**. To prepare the ground for the study of strange operations and to keep the associative law from being unjustly dismissed as a triviality, a little effort to consider some good examples and some bad ones is worthwhile.
> Some of the examples will be useful in the sequel, and some won't–some are here to show that associativity can fail, and others are here to show that even when it holds it may be far from obvious. In the world of linear algebra non-associative operations are rare, but associative operations whose good behavior is not obvious are more frequently met.

Paul Halmos has been one of my favorite authors. His insistence on problem-solving is admirable. I hope I can painstakingly (and gleefully at the same time) solve a number of problems from this book, and understand at least some of linear algebra. The problems have been solved by Halmos himself and the answers appear at the back of this book (Thank You!), but these solutions are mine. I have also felt free to think aloud, write willfully about questions that came to my mind as I solved the stated problems. That part of writing appears like 'personal reflections':

Reflection 1. *Problems come in various levels of difficulty. And, unless you are George Dantzig (who solved an open problem written on a blackboard assuming his instructor had given a homework assignment) or the like, you have to toil through them. Some you are able to solve quickly, perhaps because you are experiencing "flow"*[a]*, but many are challenging and you suffer (albeit purposefully) through them. They are distinct from exercises, which are also essential for fluency and emotional well-being, but expected to be easier to answer.*

[a] A highly focused mental state, defined and popularized by the psychologist Mihaly Csikszentmihalyi, conducive to productivity

Feel free to skip them. As Halmos urges us in his preface to this book, I have read his solutions too. He considers solutions an integral part of exposition. The ■ (the QED symbol) appears after every solution.

Readable and flawless mathematical typesetting is hard. It's ironic that the epitome of exact sciences breeds a degree of inexactness in notation. Like in literature, meaning sometimes depends on context not captured in notation. And yet, an encyclopedic treatment of notation at the beginning of a work like this tends to bore readers[1]. Here too, a balance needs to be sought. A few conventions are therefore in order:

- A roman letter in italics, like, for example, p, denotes an integer, unless specified otherwise (sometimes a real number).
- The so-called `\cdot`: $\cdot$ to denote multiplication is sometimes omitted. Thus, ab is equivalent (and often even preferred and ubiquitous, thanks to Euler!) to $a \cdot b$.

[1] What disappointments them even more are typos and errors.

Contents

Chapter 1

Scalars

Problem 1. *If a new addition for real numbers, denoted by the temporary symbol* $\boxplus$*, is defined by*

$$\alpha \boxplus \beta = 2\alpha + 2\beta$$

, is $\boxplus$ *associative?*

Note: The + sign on the right denotes ordinary addition.

Note: The new operation $\boxplus$ *is commutative:* $\alpha \boxplus \beta = 2\alpha + 2\beta$ *indeed equals* $\beta \boxplus \alpha = 2\beta + 2\alpha$*.*

Solution.

No. We can easily demonstrate

$$\alpha \boxplus (\beta \boxplus \gamma) \neq (\alpha \boxplus \beta) \boxplus \gamma$$

■

Problem 2. *If a new addition for real numbers, denoted by the temporary symbol* $\boxplus$*, is defined by*

$$\alpha \boxplus \beta = 2\alpha + \beta$$

, is $\boxplus$ *associative?*

Solution.

No. We can easily demonstrate

$$\alpha \boxplus (\beta \boxplus \gamma) \neq (\alpha \boxplus \beta) \boxplus \gamma$$

■

Problem 3. *If an operation for* positive integers*, denoted by the temporary symbol* $*$*, is defined by*

$$\alpha * \beta = \alpha^\beta$$

, is it commutative? Is it associative?

Solution.

Although $\alpha^\beta = \beta^\alpha$ when $\alpha = \beta$, in general, α^β doesn't seem to equal β^α. A simple counterexample is $\alpha = 1, \beta = 2$. Therefore, $*$ is not a commutative operation on two positive integers. ■

Reflection 2. *I asked myself, "For which real numbers α, β (although in Problem [3] they are positive integers) are α^β and β^α equal?"*

The following exploration amazed me.

$$\alpha^\beta = \beta^\alpha$$

$$\therefore \beta \log \alpha = \alpha \log \beta \tag{1.1}$$

$$\therefore \frac{\alpha}{\beta} = \frac{\log \alpha}{\log \beta} \tag{1.2}$$

A general solution of equation [1.2] (a Diophantine equation for we seek integer solutions) is perhaps hard, but somehow I asked, "What if $\alpha = \beta^k$ for some $k \in \mathbb{N}$?" I don't know why I thought of that. Is that intuition? Maybe.

Equation [1.2] then gives

$$\frac{\beta^k}{\beta} = \frac{k\cancel{\log \beta}}{\cancel{\log \beta}}$$

which simplifies to

$$\beta^{k-1} = k \tag{1.3}$$

k	*What follows from* $\alpha = \beta^k$	α, β
1	$\alpha = \beta$	*Any real numbers*
2	$\beta^{2-1} = \beta = 2$	$\alpha = 4, \beta = 2$
3	$\beta^{3-1} = \beta^2 = 3$	$\alpha = 3\sqrt{3}, \beta = \sqrt{3}$
4	$\beta^{4-1} = \beta^3 = 4$	$\alpha = 4\sqrt[3]{4}, \beta = \sqrt[3]{4}$

That was interesting!

Exponentiation is associative only when $\beta\gamma = \beta^\gamma$. There are specific cases when that is true (e.g. $\beta = \gamma = 2$), but it is not true in general. A simple counterexample is $\alpha = 2, \beta = 1, \gamma = 3$.

Problem [3] shows that "natural" operations can fail to be associative.

Halmos introduces complex numbers simply as a "pair of real numbers":$\langle \alpha, \beta \rangle \mid \alpha, \beta \in \mathbb{R}$. We can then *define* operations of interest on them, and examine if they are commutative and associative.

An operation $\boxplus$ defined on complex numbers $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ as:

$$\langle \alpha, \beta \rangle \boxplus \langle \gamma, \delta \rangle = \langle \alpha + \gamma, \beta + \delta \rangle$$

is commutative and associative because the addition of real numbers is so. The result of this operation is also a complex number.

Problem 4. *If an operation for the ordered pairs of real numbers, denoted by the temporary symbol $\boxdot$, is defined by*

$$\langle \alpha, \beta \rangle \boxdot \langle \gamma, \delta \rangle = \langle \alpha\gamma - \beta\delta, \alpha\delta + \beta\gamma \rangle$$

, is it commutative? Is it associative?

Solution.

The $\boxdot$ operation is indeed commutative because the addition and multiplication of real numbers are.

$$\begin{aligned}
\langle \gamma, \delta \rangle \boxdot \langle \alpha, \beta \rangle &= \langle \gamma\alpha - \delta\beta, \gamma\beta + \delta\alpha \rangle \\
&= \langle \alpha\gamma - \beta\delta, \alpha\delta + \beta\gamma \rangle \\
&= \langle \alpha, \beta \rangle \boxdot \langle \gamma, \delta \rangle
\end{aligned}$$

Still, we are lucky! It wouldn't be commutative had it been defined just a little differently as: $\langle \alpha, \beta \rangle \boxdot \langle \gamma, \delta \rangle = \langle \alpha\gamma + \beta\delta, \alpha\delta - \beta\gamma \rangle$

Associativity: (We'll use a for α, b for β, and so on for easier typesetting.)

$$\begin{aligned}
(\langle a, b \rangle \boxdot \langle c, d \rangle) \boxdot \langle m, n \rangle &= \langle ac - bd, ad + bc \rangle \boxdot \langle m, n \rangle \\
&= \langle (ac - bd)m - (ad + bc)n, (ac - bd)n + (ad + bc)m \rangle \\
&= \langle a(cm - dn) - b(cn + dm), a(cn + dm) + b(cm - dn) \rangle \\
&= \langle a, b \rangle \boxdot (\langle c, d \rangle \boxdot \langle m, n \rangle)
\end{aligned}$$

$\therefore$ $\boxdot$ is associative and commutative. ■

Reflection 3. *The discussion of complex numbers (and their representation as just a pair of real numbers) so far is algebraic. There is also a geometric equivalent. Addition of two complex numbers (that generates a new complex number whose real and imaginary parts are the sums of those of the addends) is perhaps straightforward. Why is multiplication defined this way ($\langle a, b \rangle \boxdot \langle c, d \rangle = \langle ac - bd, ad + bc \rangle$)? A geometric interpretation is:*

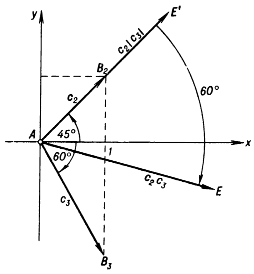
> *To multiply (a complex number represented by) a vector $\vec{v_1}$ by $\vec{v_2}$ we scale $\vec{v_1}$ by $\mid \vec{v_2} \mid$ i.e. the length of $\vec{v_2}$ and then rotate the scaled vector by an angle that is same as the argument of $\vec{v_2}$ (which is the angle it makes with the positive $X-axis$). See Figure [1.1].*

Given the algebraic definition of multiplication of two complex numbers ($\langle a, b \rangle \boxdot \langle c, d \rangle = \langle ac - bd, ad + bc \rangle$), one should be able to devise the geometric construction (and vice versa).

However, the question remains. The multiplication of positive integers can be thought *of as repeated addition. But the multiplication of complex numbers ($\langle a, b\rangle \boxdot \langle c, d\rangle = \langle ac - bd, ad + bc\rangle$) seems much different from their addition($\langle a, b\rangle \boxplus \langle c, d\rangle = \langle a + b, c + d\rangle$). Why is it defined this way?*

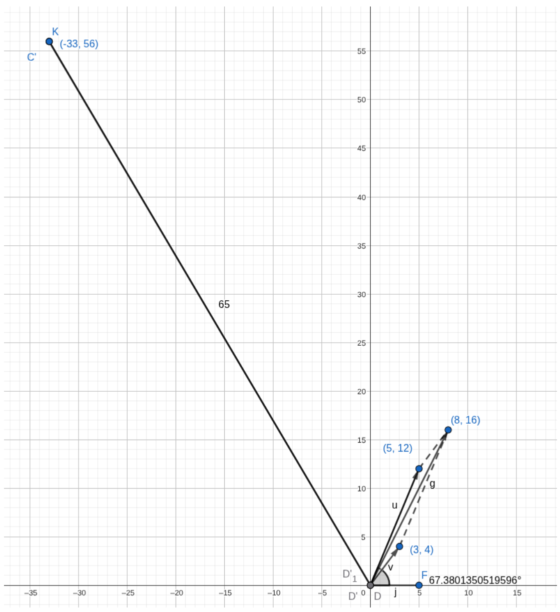
If we resort to a 'proper' definition of a complex number: $z = a + bi, i = \sqrt{-1}$, then everything evaluates correctly. The product $(a + bi) \cdot (c + di)$ does evaluate to a complex number $ac - bd + (ad + bc)i$.

Figure 1.1: Geometric Interpretation of Complex Number Multiplication (Reproduced from [2])



Here is the multiplication of two vectors: $\langle 3, 4\rangle$ and $\langle 5, 12\rangle$ to produce the vector $\langle -33, 56\rangle$. Note: $\langle 33, 56, 65\rangle$ is a Pythagorian triple.

Figure 1.2: Another Geometric Interpretation of Complex Number Multiplication (Drawn to Scale using Geogebra)



Problem 5. *If an operation for the ordered pairs of real numbers, denoted by $\boxdot$ again, is defined by*

$$\langle a, b\rangle \boxdot \langle c, d\rangle = \langle ac, ad + b\rangle$$

, is it commutative? Is it associative?

Note: Looking strange is not necessarily a sign of being artificial or useless.

Solution.

$$\langle a, b\rangle \boxdot \langle c, d\rangle = \langle ac, ad + b\rangle$$

$$\langle c, d\rangle \boxdot \langle a, b\rangle = \langle ca, cb + d\rangle$$

$\therefore \boxdot$ is not commutative in general; only commutative if $\frac{a-1}{b} = \frac{c-1}{d}$.

$$\begin{aligned}(\langle a, b\rangle \boxdot \langle c, d\rangle) \boxdot \langle m, n\rangle &= \langle ac, ad + b\rangle \boxdot \langle m, n\rangle \\ &= \langle acm, acm + ad + b\rangle\end{aligned}$$

and

$$\begin{aligned}\langle a, b\rangle \boxdot (\langle c, d\rangle \boxdot \langle m, n\rangle) &= \langle a, b\rangle \boxdot \langle cm, cn + d\rangle \\ &= \langle acm, acn + ad + b\rangle\end{aligned}$$

$\therefore \boxdot$ is associative.

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Problem 6. *If an operation for the ordered quadruples of real numbers, denoted by* $\boxdot$*, is defined by*

$$\langle a, b, c, d\rangle \boxdot \langle a', b', c', d'\rangle = \langle aa' + bc', ab' + bd', ca' + dc', cb' + dd'\rangle$$

, is it commutative? Is it associative?

Solution. Since the multiplication of real numbers is commutative ($aa' + bc' = a'a + c'b$, and so on), the *matrix multiplication* defined here is commutative.

The visual symmetry in this operation feels useful in proving its associativity, although one should carry out the operation rigorously to believe that it is associative. That is left out here (we wouldn't say "as an exercise to the reader!").

It's easier to visualize this as a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

■

How is Problem [5] a "special case" of Problem [6]?

Consider

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix}$$

The first row of the result is the affine tranform of the first rows of the two matrices on the left-hand side. Halmos, being an algebraist, does not describe it so, however.

Also, is Problem [5] a "special case" of Problem [4]? Consider

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$$

I still don't understand how matrix multiplication is a "more general" case of vector multiplication or affine transform; it might become clear later.

References

[1] Paul R. Halmos. *Linear Algebra Problem Book. The Dolciani Mathematical Expositions Number 16*. 2nd ed. Mathematical Association of America, 2008 (cit. on p. 1).

[2] A.I. Markushevich. *Complex Numbers and Conformal Mappings*. Trans. by Irene Aleksanova. Moscow: Mir Publishers, 1982 (cit. on p. 8).

[3] *What exactly is Yak shaving?* Mar. 5, 2019. URL: https://softwareengineering.stackexchange.com/q/388092/148975 (visited on 06/03/2025) (cit. on p. 1).