Elementary Calculus–An Infinitesimal Approach, Chapters 01, 02

Calculus Based on Nonstandard Analysis

Jerome Keisler[1]

[1]Original Author

September 2025 / Free Learner's School Conversations

Outline

- These Are My Notes from H. Jerome Keisler's Book by The Same Name.
- While Teaching Elementary Calculus to My Daughter, I Realized That I Better Learn The Infinitesimal Approach Better.

- These Are My Notes from H. Jerome Keisler's Book by The Same Name.
- While Teaching Elementary Calculus to My Daughter, I Realized That I Better Learn The Infinitesimal Approach Better.

- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.

Infinitesimal Calculus
JK,KM
Introduction
Chapter 1

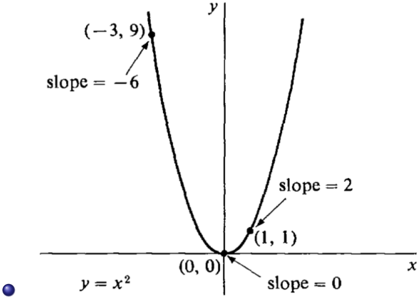
- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.

- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.

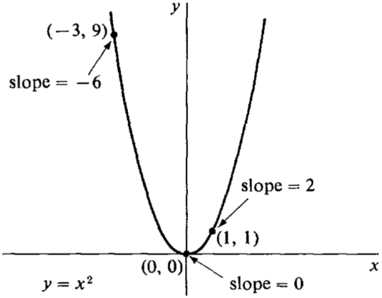
Infinitesimal Calculus
JK,KM
Introduction
Chapter 1

- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.

- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.



- For Example, The Slope Is $2x_0 = 2 \cdot 0 = 0$ at $(0, 0)$, $2 \cdot 1 = 2$ at $(1, 1)$, And $2 \cdot -3 = -6$ at $(-3, 9)$.

- 

- For Example, The Slope Is $2x_0 = 2 \cdot 0 = 0$ at $(0,0)$, $2 \cdot 1 = 2$ at $(1,1)$, And $2 \cdot -3 = -6$ at $(-3,9)$.

- We Can Visualize Slope as Velocity.
- If The Horizontal Axis Represents Time And Vertical Axis Position, Is The Average Velocity between (y_0, t_0) and $(y_0 + \Delta y, t_0 + \Delta t)$ = The Velocity *at*[1] Either of Those Points?
- Since in This Case The Velocity Constantly Increases, The Answer Is NO.
- $v_{ave} = 2t_0 + \Delta t$ And, After We Treat Δt Negligible, $v_{ave} = 2t_0$.

[1] In Physics, We Call It The *Instantaneous Velocity*.

- We Can Visualize Slope as Velocity.
- If The Horizontal Axis Represents Time And Vertical Axis Position, Is The Average Velocity between (y_0, t_0) and $(y_0 + \Delta y, t_0 + \Delta t)$ = The Velocity *at*[1] Either of Those Points?
- Since in This Case The Velocity Constantly Increases, The Answer Is NO.
- $v_{ave} = 2t_0 + \Delta t$ And, After We Treat Δt Negligible, $v_{ave} = 2t_0$.

[1] In Physics, We Call It The *Instantaneous Velocity*.

- We Can Visualize Slope as Velocity.
- If The Horizontal Axis Represents Time And Vertical Axis Position, Is The Average Velocity between (y_0, t_0) and $(y_0 + \Delta y, t_0 + \Delta t)$ = The Velocity *at*[1] Either of Those Points?
- Since in This Case The Velocity Constantly Increases, The Answer Is NO.
- $v_{ave} = 2t_0 + \Delta t$ And, After We Treat Δt Negligible, $v_{ave} = 2t_0$.

[1] In Physics, We Call It The *Instantaneous Velocity*.

- We Can Visualize Slope as Velocity.
- If The Horizontal Axis Represents Time And Vertical Axis Position, Is The Average Velocity between (y_0, t_0) and $(y_0 + \Delta y, t_0 + \Delta t)$ = The Velocity *at*[1] Either of Those Points?
- Since in This Case The Velocity Constantly Increases, The Answer Is NO.
- $v_{ave} = 2t_0 + \Delta t$ And, After We Treat Δt Negligible, $v_{ave} = 2t_0$.

[1] In Physics, We Call It The *Instantaneous Velocity*.

- In Either Case (Slope, Velocity), The Intuitive Reasoning Fails to Clarify When Something Is to Be Treated Negligible.
- We Need a Sharp Distinction between Which Numbers Are Small Enough to Ignore And Which Aren't.
- Actually, No Real Number Except 0 Is Small Enough To Ignore[2].
- To Address This Difficulty, We Take The Bold Step[3] of Introducing a New *Kind*[4] *of Number* Which Is Infinitely Small And Yet $\neq 0$.

---

[2] Does This Hint at a New *Kind* of Number?
[3] A Rigorous Treatment Is Provided by Keisler in His *Foundations*.
[4] Unreal?

- In Either Case (Slope, Velocity), The Intuitive Reasoning Fails to Clarify When Something Is to Be Treated Negligible.
- We Need a Sharp Distinction between Which Numbers Are Small Enough to Ignore And Which Aren't.
- Actually, No Real Number Except 0 Is Small Enough To Ignore[2].
- To Address This Difficulty, We Take The Bold Step[3] of Introducing a New *Kind*[4] *of Number* Which Is Infinitely Small And Yet $\neq 0$.

[2] Does This Hint at a New *Kind* of Number?
[3] A Rigorous Treatment Is Provided by Keisler in His *Foundations*.
[4] Unreal?

- In Either Case (Slope, Velocity), The Intuitive Reasoning Fails to Clarify When Something Is to Be Treated Negligible.
- We Need a Sharp Distinction between Which Numbers Are Small Enough to Ignore And Which Aren't.
- Actually, No Real Number Except 0 Is Small Enough To Ignore[2].
- To Address This Difficulty, We Take The Bold Step[3] of Introducing a New *Kind*[4] *of Number* Which Is Infinitely Small And Yet $\neq 0$.

[2] Does This Hint at a New *Kind* of Number?

[3] A Rigorous Treatment Is Provided by Keisler in His *Foundations*.

[4] Unreal?

- In Either Case (Slope, Velocity), The Intuitive Reasoning Fails to Clarify When Something Is to Be Treated Negligible.
- We Need a Sharp Distinction between Which Numbers Are Small Enough to Ignore And Which Aren't.
- Actually, No Real Number Except 0 Is Small Enough To Ignore[2].
- To Address This Difficulty, We Take The Bold Step[3] of Introducing a New *Kind*[4] *of Number* Which Is Infinitely Small And Yet $\neq 0$.

[2] Does This Hint at a New *Kind* of Number?
[3] A Rigorous Treatment Is Provided by Keisler in His *Foundations*.
[4] Unreal?

Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal*, If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only Real Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And Infinitesimals That Are Not Zero.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

[5] M. Gardner Calls It The *Number Neverland*.

Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal*, If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only Real Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And Infinitesimals That Are Not Zero.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

[5] M. Gardner Calls It The *Number Neverland*.

Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal,* If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only Real Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And Infinitesimals That Are Not Zero.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

[5] M. Gardner Calls It The *Number Neverland*.

Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal,* If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only <u>Real</u> Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And <u>Infinitesimals That Are Not Zero</u>.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

[5] M. Gardner Calls It The *Number Neverland*.

Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal*, If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only Real Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And Infinitesimals That Are Not Zero.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

[5] M. Gardner Calls It The *Number Neverland*.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \dots$ Denote Reals. $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\dots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- $\mathbb{R}^*$: The Set of All Hyperreal Numbers.
- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
  - If $\Delta x = (x_0 + \Delta x) - (x_0)$ Is Infinitesimal, Then $x_0 + \Delta x$ And x_0 Are "Infinitely Close" to Each Other.
- If ε Is Positive Infinitesimal, Then
  - $-\varepsilon$ Is Negative Infinitesimal.
  - $\frac{1}{\varepsilon}$ Is Infinite Positive, $>$ All Real Numbers.
  - $\frac{-1}{\varepsilon}$ Is Infinite Negative, $<$ All Real Numbers.

- Hyperreal Numbers Which Are Not Infinite, Are Finite Numbers.
- About each $c \in \mathbb{R}$ Is A Portion of Hyperreal Line Composed of The Numbers Infinitely Close to c.
- Numbers Infinitely Close to 0 Are Infinitesimals.

- Hyperreal Numbers Which Are Not Infinite, Are Finite Numbers.
- About each $c \in \mathbb{R}$ Is A Portion of Hyperreal Line Composed of The Numbers Infinitely Close to c.

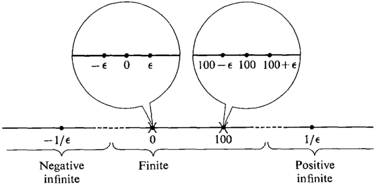


Figure: Finite And Infinite Parts of The Hyperreal Line (*Infinitesimal Microscope* about $c = 0$, $c = 100$)

- Numbers Infinitely Close to 0 Are Infinitesimals.

- Hyperreal Numbers Which Are Not Infinite, Are Finite Numbers.
- About each $c \in \mathbb{R}$ Is A Portion of Hyperreal Line Composed of The Numbers Infinitely Close to c.

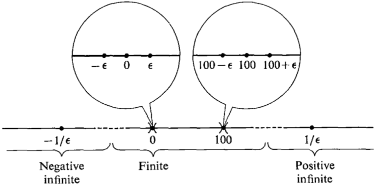


Figure: Finite And Infinite Parts of The Hyperreal Line (*Infinitesimal Microscope* about $c = 0$, $c = 100$)

- Numbers Infinitely Close to 0 Are Infinitesimals.

The Nature of Physical Space

Euclid Struggled to Define a 'Point': Something That Has a Position But No Magnitude[a].

We Have No Way of Knowing What a Line in Physical Space Is Really Like ("What Is It Composed of?"). It Might Be Like the Hyperreal Line (with Infinitesimals Surrounding Every 'Real Point'), The Real Line (without Any Infinitesimals), Or Neither. However, in Applications of The Calculus It Is Helpful to Imagine a Line in Physical Space as A Hyperreal (Rather Than Real) Line. The Hyperreal Line Is, Like The Real Line, A Useful Mathematical Model[b] for A Line in Physical Space.

[a] Isn't This 'Definition' (*Accepted* for Centuries) *Meaningless*?

[b] And, Here's A Stark Reminder: All Models Are Wrong; Some Are Useful!

Definition (The Slope of A Curve at (x_0, y_0))

Let $y = f(x)$ Be a Certain Function. Let $P(x_0, y_0)$ Be Any Point on The Curve Representing y. Let Δx Be a Positive Or Negative Infinitesimal. Consider A Point $(x_0 + \Delta x, y_0 + \Delta y)$ Infinitely Close to P.
Then,

$$\text{Slope of } f \text{ at}(x_0, y_0) = \text{Real Number Infinitely Close to} \frac{\Delta y}{\Delta x}$$

Note: The Slope is *Defined to Be* A Real Number.

Example (Slope of $y = x^2$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ \therefore \Delta y &= 2x\Delta x + (\Delta x)^2 \\ \therefore \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\ \therefore \text{Slope} &= 2x \text{ (From Definition [2])} \end{aligned} \tag{1}$$

$2x$ Is the Real Number Infinitely Close to The Hyperreal Number $2x + \Delta x$.

In This Example It Was Rather Straightforward to Show $2x + \Delta x$ And $2x$ Are Infinitely Close To Each Other.

Example (Slope of $y = x^2$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ \therefore \Delta y &= 2x\Delta x + (\Delta x)^2 \\ \therefore \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\ \therefore \text{Slope} &= 2x \text{ (From Definition [2])} \end{aligned} \tag{1}$$

$2x$ Is the Real Number Infinitely Close to The Hyperreal Number $2x + \Delta x$.

In This Example It Was Rather Straightforward to Show $2x + \Delta x$ And $2x$ Are Infinitely Close To Each Other.

Example (Slope of $y = x^2$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ \therefore \Delta y &= 2x\Delta x + (\Delta x)^2 \\ \therefore \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\ \therefore \text{Slope} &= 2x \text{ (From Definition [2])} \end{aligned} \tag{1}$$

$2x$ Is the Real Number Infinitely Close to The Hyperreal Number $2x + \Delta x$.

In This Example It Was Rather Straightforward to Show $2x + \Delta x$ And $2x$ Are Infinitely Close To Each Other.

Example (Slope of $y = x^3$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^3 \\ \therefore \Delta y &= \cancel{x^3} + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - \cancel{x^3} \\ \therefore \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \end{aligned} \tag{2}$$

If (Because Δx Is Infinitesimal), $3x\Delta x + (\Delta x)^2$ Is Infinitesimal, Then

$$\text{Slope} = 3x^2 \tag{3}$$

- Δx Is An Infinitesimal. That Brings $x + \Delta x$ Infinitely Close to x.
- However, It Is Not Immediately Clear That $3x^2 + 3x\Delta x + (\Delta x)^2$ Is Infinitely Close to $3x^2$, Is It?
- It Depends on Whether $3x\Delta x + (\Delta x)^2$ Is Infinitesimal. We Only Know Δx to Be Infinitesimal. Does That Make $3x^2 + 3x\Delta x + (\Delta x)^2$ Infinitesimal?
- Thus, Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

- Δx Is An Infinitesimal. That Brings $x + \Delta x$ Infinitely Close to x.
- However, It Is Not Immediately Clear That $3x^2 + 3x\Delta x + (\Delta x)^2$ Is Infinitely Close to $3x^2$, Is It?
- It Depends on Whether $3x\Delta x + (\Delta x)^2$ Is Infinitesimal. We Only Know Δx to Be Infinitesimal. Does That Make $3x^2 + 3x\Delta x + (\Delta x)^2$ Infinitesimal?
- Thus, Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

- Δx Is An Infinitesimal. That Brings $x + \Delta x$ Infinitely Close to x.
- However, It Is Not Immediately Clear That $3x^2 + 3x\Delta x + (\Delta x)^2$ Is Infinitely Close to $3x^2$, Is It?
- It Depends on Whether $3x\Delta x + (\Delta x)^2$ Is Infinitesimal. We Only Know Δx to Be Infinitesimal. Does That Make $3x^2 + 3x\Delta x + (\Delta x)^2$ Infinitesimal?
- Thus, Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

- Δx Is An Infinitesimal. That Brings $x + \Delta x$ Infinitely Close to x.
- However, It Is Not Immediately Clear That $3x^2 + 3x\Delta x + (\Delta x)^2$ Is Infinitely Close to $3x^2$, Is It?
- It Depends on Whether $3x\Delta x + (\Delta x)^2$ Is Infinitesimal. We Only Know Δx to Be Infinitesimal. Does That Make $3x^2 + 3x\Delta x + (\Delta x)^2$ Infinitesimal?
- Thus, Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

- Δx Is An Infinitesimal. That Brings $x + \Delta x$ Infinitely Close to x.
- However, It Is Not Immediately Clear That $3x^2 + 3x\Delta x + (\Delta x)^2$ Is Infinitely Close to $3x^2$, Is It?
- It Depends on Whether $3x\Delta x + (\Delta x)^2$ Is Infinitesimal. We Only Know Δx to Be Infinitesimal. Does That Make $3x^2 + 3x\Delta x + (\Delta x)^2$ Infinitesimal?
- Thus, Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

- Our Choice of The Mathematical Model of A Line in The Physical Space, The Hyperreal Line, Contains Points Representing Real And Hyperreal Numbers.
- Surrounding Each Real Number $r \in \mathbb{R}$, There Are Hyperreal Numbers Infinitely Close to r.
- Hyperreal Numbers Infinitely Close to 0 Are Called Infinitesimals.
- 0 Is The Only *Real* Infinitesimal Number. There Are Many Nonzero Hyperreal Infinitesimals.
- In This Section, We Describe Hyperreal Numbers More Precisely And Develop a Facility for Computation with Them.
- We Must Undertake This Effort, Because, After All, We Took The Bold Step of Conceiving *Hyperreal Number: A New Kind of Number*; We Must Follow Through!

- Our Choice of The Mathematical Model of A Line in The Physical Space, The Hyperreal Line, Contains Points Representing Real And Hyperreal Numbers.
- Surrounding Each Real Number $r \in \mathbb{R}$, There Are Hyperreal Numbers Infinitely Close to r.
- Hyperreal Numbers Infinitely Close to 0 Are Called Infinitesimals.
- 0 Is The Only *Real* Infinitesimal Number. There Are Many Nonzero Hyperreal Infinitesimals.
- In This Section, We Describe Hyperreal Numbers More Precisely And Develop a Facility for Computation with Them.
- We Must Undertake This Effort, Because, After All, We Took The Bold Step of Conceiving *Hyperreal Number: A New Kind of Number*; We Must Follow Through!

- Our Choice of The Mathematical Model of A Line in The Physical Space, The Hyperreal Line, Contains Points Representing Real And Hyperreal Numbers.
- Surrounding Each Real Number $r \in \mathbb{R}$, There Are Hyperreal Numbers Infinitely Close to r.
- Hyperreal Numbers Infinitely Close to 0 Are Called Infinitesimals.
- 0 Is The Only *Real* Infinitesimal Number. There Are Many Nonzero Hyperreal Infinitesimals.
- In This Section, We Describe Hyperreal Numbers More Precisely And Develop a Facility for Computation with Them.
- We Must Undertake This Effort, Because, After All, We Took The Bold Step of Conceiving *Hyperreal Number: A New Kind of Number*; We Must Follow Through!

- In Our Quest, We Must Satisfactorily (Firmly, Rigorously) Answer Questions Such as:
  - Δx Denotes An Infinitesimal. Do Familiar Functions (e.g. Polynomial, Rational, ...) Map Infinitesimals on to Infinitesimals? Thus, If Δx Infinitesimal, Is A Polynomial Function of Infinitesimals Alone, e.g. $(\Delta x)^2 - (\Delta x)$, Infinitesimal Too?
  - How Do We Think of Functions That *Combine Infinitesimals with Reals*? For Example, Given That Δx Is Infinitesimal, Is $\underline{2x\Delta x} + (\Delta x)^2$ Infinitesimal Too?

Shouldn't We Get The Taste of The Pain That Robinson et.al. Had While Conceiving A New Kind of Number (Hyperreal) That Must Live Amicably with The Kind of Number (Real) That Already Existed And Was Widely Understood?

- In Our Quest, We Must Satisfactorily (Firmly, Rigorously) Answer Questions Such as:
  - Δx Denotes An Infinitesimal. Do Familiar Functions (e.g. Polynomial, Rational, ...) Map Infinitesimals on to Infinitesimals? Thus, If Δx Infinitesimal, Is A Polynomial Function of Infinitesimals Alone, e.g. $(\Delta x)^2 - (\Delta x)$, Infinitesimal Too?
  - How Do We Think of Functions That *Combine* Infinitesimals with Reals? For Example, Given That Δx Is Infinitesimal, Is $\underline{2x\Delta x} + (\Delta x)^2$ Infinitesimal Too?

Shouldn't We Get The Taste of The Pain That Robinson et.al. Had While Conceiving A New Kind of Number (Hyperreal) That Must Live Amicably with The Kind of Number (Real) That Already Existed And Was Widely Understood?

- In Our Quest, We Must Satisfactorily (Firmly, Rigorously) Answer Questions Such as:
    - Δx Denotes An Infinitesimal. Do Familiar Functions (e.g. Polynomial, Rational, ...) Map Infinitesimals on to Infinitesimals? Thus, If Δx Infinitesimal, Is A Polynomial Function of Infinitesimals Alone, e.g. $(\Delta x)^2 - (\Delta x)$, Infinitesimal Too?
    - How Do We Think of Functions That *Combine* Infinitesimals with Reals? For Example, Given That Δx Is Infinitesimal, Is $\underline{2x\Delta x} + (\Delta x)^2$ Infinitesimal Too?

Shouldn't We Get The Taste of The Pain That Robinson et.al. Had While Conceiving A New Kind of Number (Hyperreal) That Must Live Amicably with The Kind of Number (Real) That Already Existed And Was Widely Understood?

- In Our Quest, We Must Satisfactorily (Firmly, Rigorously) Answer Questions Such as:
  - Δx Denotes An Infinitesimal. Do Familiar Functions (e.g. Polynomial, Rational, …) Map Infinitesimals on to Infinitesimals? Thus, If Δx Infinitesimal, Is A Polynomial Function of Infinitesimals Alone, e.g. $(\Delta x)^2 - (\Delta x)$, Infinitesimal Too?
  - How Do We Think of Functions That *Combine* Infinitesimals with Reals? For Example, Given That Δx Is Infinitesimal, Is $\underline{2x\Delta x} + (\Delta x)^2$ Infinitesimal Too?

Shouldn't We Get The Taste of The Pain That Robinson et.al. Had While Conceiving A New Kind of Number (Hyperreal) That Must Live Amicably with The Kind of Number (Real) That Already Existed And Was Widely Understood?

What might Have inspired Robinson to take the effort? Humans look at the historical context of developments. We rejoice in history. We respect those who came before us. After Karl Weierstrass 'banished' infinitesimals with his rigorous reformulation of calculus (he also must have felt the burden of creation while conceiving "this baby" in the 1870s; he defied Newton, Euler, . . .), mathematical community thought that calculus was 'solved'.
Why would Abraham Robinson, a young mathematician in the 1940s and 50s, "bring infinitesimals back"? Did he want to become famous? Could he not banish infinitesimals convincingly? Why conceive a new kind of number and put it on a rigorous pedestal? Why solve a "solved problem"? These questions on "Foundations of Mathematics" are deep.

I haven’t read sufficiently about Robinson’s inspiration to conceive hyperreal numbers.
However, my rather mundane opinion on this is that, after all, infinitesimals had been practically successful. Many brilliant mathematicians used them to solve practical problems before Weierstrass’s reformulation of calculus. What was missing was the rigor. Perhaps Robinson felt dearly about providing a rigorous treatment to something that worked effectively. That became his life’s work.
We don’t want to get completely lost in questions like “Is Mathematics Invented or Discovered?”, but can we ignore them?
Alright, Back to Answering Questions on Slide [17]: Computing with Hyperreals.

- We Start with The Extension Principle, Which Gives Us Hyperreal Numbers, And Extends All Real Functions to Them.
- $f : \mathbb{R} \to \mathbb{R}$ Defines a Real-Valued Function. When We Say $f(a) = b$, f *relates* Some $a \in \mathbb{R}$ to Some $b \in \mathbb{R}$. f May Fail to Relate some $a \in \mathbb{R}$ with Any Real Number. Then We Say $f(a)$ is *Undefined*.
- Similarly, F, A Hyperreal Function, *relates* A Hyperreal Number H with Another Hyperreal Number K, Or Is *Undefined*. Symbolically, $F : \mathbb{R}^* \to \mathbb{R}^*$. We say $K = F(H)$.
- We Can, Like Real-Valued Functions, Easily Define Hyperreal Functions of More Than 1 Hyperreal Number.

- We Start with The Extension Principle, Which Gives Us Hyperreal Numbers, And Extends All Real Functions to Them.
- $f : \mathbb{R} \to \mathbb{R}$ Defines a Real-Valued Function. When We Say $f(a) = b$, f *relates* Some $a \in \mathbb{R}$ to Some $b \in \mathbb{R}$. f May Fail to Relate some $a \in \mathbb{R}$ with Any Real Number. Then We Say $f(a)$ is *Undefined*.
- Similarly, F, A Hyperreal Function, *relates* A Hyperreal Number H with Another Hyperreal Number K, Or Is *Undefined*. Symbolically, $F : \mathbb{R}^* \to \mathbb{R}^*$. We say $K = F(H)$.
- We Can, Like Real-Valued Functions, Easily Define Hyperreal Functions of More Than 1 Hyperreal Number.

- We Start with The Extension Principle, Which Gives Us Hyperreal Numbers, And Extends All Real Functions to Them.
- $f : \mathbb{R} \to \mathbb{R}$ Defines a Real-Valued Function. When We Say $f(a) = b$, f *relates* Some $a \in \mathbb{R}$ to Some $b \in \mathbb{R}$. f May Fail to Relate some $a \in \mathbb{R}$ with Any Real Number. Then We Say $f(a)$ is *Undefined*.
- Similarly, F, A Hyperreal Function, *relates* A Hyperreal Number H with Another Hyperreal Number K, Or Is *Undefined*. Symbolically, $F : \mathbb{R}^* \to \mathbb{R}^*$. We say $K = F(H)$.
- We Can, Like Real-Valued Functions, Easily Define Hyperreal Functions of More Than 1 Hyperreal Number.

- We Start with The Extension Principle, Which Gives Us Hyperreal Numbers, And Extends All Real Functions to Them.
- $f : \mathbb{R} \to \mathbb{R}$ Defines a Real-Valued Function. When We Say $f(a) = b$, f *relates* Some $a \in \mathbb{R}$ to Some $b \in \mathbb{R}$. f May Fail to Relate some $a \in \mathbb{R}$ with Any Real Number. Then We Say $f(a)$ is *Undefined*.
- Similarly, F, A Hyperreal Function, *relates* A Hyperreal Number H with Another Hyperreal Number K, Or Is *Undefined*. Symbolically, $F : \mathbb{R}^* \to \mathbb{R}^*$. We say $K = F(H)$.
- We Can, Like Real-Valued Functions, Easily Define Hyperreal Functions of More Than 1 Hyperreal Number.

- a) $\mathbb{R} \subset \mathbb{R}^*$. The Order Relation '$<$' for $\mathbb{R}$ Is A Subset of The Order Relation '$<$' for $\mathbb{R}^*$. This Part Asserts That The Real Line *Is A Part of* The Hyperreal Line.
- b) There's A Hyperreal Number Greater Than Zero But Less Than Every Positive Real Number. This Part Begets A Precise Definition of Infinitesimals.
- c) For Every $f : \mathbb{R} \to \mathbb{R}$ (of One Or More Variables), We Are Given A Hyperreal Function, f^*, Called The *Natural Extension of f*. This Part Allows Us to Apply Real Functions to Hyperreal Numbers.

Infinitesimal Calculus
JK,KM
Introduction
Chapter 1

- a) $\mathbb{R} \subset \mathbb{R}^*$. The Order Relation ‘$<$’ for $\mathbb{R}$ Is A Subset of The Order Relation ‘$<$’ for $\mathbb{R}^*$. This Part Asserts That The Real Line *Is A Part of* The Hyperreal Line.
- b) There’s A Hyperreal Number Greater Than Zero But Less Than Every Positive Real Number. This Part Begets A Precise Definition of Infinitesimals.
- c) For Every $f : \mathbb{R} \to \mathbb{R}$ (of One Or More Variables), We Are Given A Hyperreal Function, f^*, Called The *Natural Extension of f*. This Part Allows Us to Apply Real Functions to Hyperreal Numbers.

- a) $\mathbb{R} \subset \mathbb{R}^*$. The Order Relation '$<$' for $\mathbb{R}$ Is A Subset of The Order Relation '$<$' for $\mathbb{R}^*$. This Part Asserts That The Real Line *Is A Part of* The Hyperreal Line.
- b) There's A Hyperreal Number Greater Than Zero But Less Than Every Positive Real Number. This Part Begets A Precise Definition of Infinitesimals.
- c) For Every $f : \mathbb{R} \to \mathbb{R}$ (of One Or More Variables), We Are Given A Hyperreal Function, f^*, Called The *Natural Extension of f*. This Part Allows Us to Apply Real Functions to Hyperreal Numbers.

Restating Part b):

- There's A Hyperreal Number Greater Than Zero But Less Than Every Positive Real Number. This Part Begets A Precise Definition of Infinitesimals.

Definition (Infinitesimal)

A Hyperreal Number b Is Said To Be Positive Infinitesimal If $b > 0$, But Less Than Every Positive Real Number.
A Hyperreal Number b Is Said To Be Negative Infinitesimal If $b < 0$, But Greater Than Every Negative Real Number.
A Hyperreal Number b Is Said To Be Infinitesimal If It Is Either Positive Infinitesimal, Negative Infinitesimal, Or Zero.

- We *Naturally* Extend Real-Valued Functions to Functions That Operate on Hyperreal Numbers.
  - For Example, If x, y Are Hyperreal Numbers, $x + y$ Adds Them Just Like $+$ Adds Two Real Numbers.
  - If x, y Are Hyperreal Numbers, A *Real Expression* Such as $\sin(x + \cos(y))$ Implies Natural Extensions of the $\sin, \cos$ Functions to Them.

- We *Naturally* Extend Real-Valued Functions to Functions That Operate on Hyperreal Numbers.
  - For Example, If x, y Are Hyperreal Numbers, $x + y$ Adds Them Just Like $+$ Adds Two Real Numbers.
  - If x, y Are Hyperreal Numbers, A *Real Expression* Such as $\sin(x + \cos(y))$ Implies Natural Extensions of the $\sin, \cos$ Functions to Them.

- We *Naturally* Extend Real-Valued Functions to Functions That Operate on Hyperreal Numbers.
    - For Example, If x, y Are Hyperreal Numbers, $x + y$ Adds Them Just Like $+$ Adds Two Real Numbers.
    - If x, y Are Hyperreal Numbers, A *Real Expression* Such as $\sin(x + \cos(y))$ Implies Natural Extensions of the $\sin, \cos$ Functions to Them.

Definition (Transfer Principle)

Every Real Statement That Holds for One Or More Particular Real Functions Holds for The Hyperreal Natural Extensions of These Functions.

Example (Real Statements)

If x, y Are Real Numbers, The Following Hold:

1. Closure Law of Addition: $x + y$ Is Defined.
2. Commutative Law of Addition: $x + y = y + x$.
3. A Rule for Order: If $0 < x < y$, Then $0 < \frac{1}{y} < \frac{1}{x}$.
4. Division by Zero ($\frac{x}{0}$ or $\frac{0}{0}$) Is Undefined.
5. An Algebraic Identity: $(x - y)^2 = x^2 - 2xy + y^2$.
6. A Trigonometric Identity: $\sin^2 x + \cos^2 x = 1$.
7. A Rule for Logarithms: If $x > 0, y > 0$, Then $\log xy = \log x + \log y$.

The Transfer Principle States That These Statements @ Sophisticated Real-Valued Functions Also Hold When We Apply Them to Hyperreal Numbers.

Example (More Real Statements)

If x, y Are Real Numbers, The Following Hold:

1. The Square-root Function:
   $y = \sqrt{x} \iff y^2 = x$ and $x \geq 0$.
2. The Absolute Value Function: $y = | x | \iff y = \sqrt{x^2}$.
3. The Common Logarithm Function:
   $y = \log_{10} x \iff 10^y = x$.

The Transfer Principle States That The Natural Extensions of These Functions Hold for Hyperreal Numbers.

We Now Logically Proceed to Doing Computations Involving Hyperreal Numbers.

- The Extension Principle Tells Us That At Least One Positive Infinitesimal, ε, Exists. By Definition, ε Is *Infinitely Close to 0*.
- Readily, The Transfer Principle Suggests $0 < \varepsilon^2 < \varepsilon$ (Follows from the 'Real Statement': $0 < r^2 < r \; \forall r \in \mathbb{R}^+ \mid 0 < r < 1$).
- It Follows That We Can Now Construct Infinitely Many Infinitesimals. Here Are Some Examples Listed In An Order of Increasing Magnitude: $\varepsilon^3, \varepsilon^2, \frac{\varepsilon}{100}, \varepsilon, 75\varepsilon, \sqrt{\varepsilon}, \varepsilon + \sqrt{\varepsilon}$.

We Now Logically Proceed to Doing Computations Involving Hyperreal Numbers.

- The Extension Principle Tells Us That At Least One Positive Infinitesimal, ε, Exists. By Definition, ε Is *Infinitely Close to 0*.
- Readily, The Transfer Principle Suggests $0 < \varepsilon^2 < \varepsilon$ (Follows from the 'Real Statement': $0 < r^2 < r \; \forall r \in \mathbb{R}^+ \mid 0 < r < 1$).
- It Follows That We Can Now Construct Infinitely Many Infinitesimals. Here Are Some Examples Listed In An Order of Increasing Magnitude: $\varepsilon^3, \varepsilon^2, \frac{\varepsilon}{100}, \varepsilon, 75\varepsilon, \sqrt{\varepsilon}, \varepsilon + \sqrt{\varepsilon}$.

We Now Logically Proceed to Doing Computations Involving Hyperreal Numbers.

- The Extension Principle Tells Us That At Least One Positive Infinitesimal, ε, Exists. By Definition, ε Is *Infinitely Close to 0*.
- Readily, The Transfer Principle Suggests $0 < \varepsilon^2 < \varepsilon$ (Follows from the 'Real Statement': $0 < r^2 < r \, \forall r \in \mathbb{R}^+ \mid 0 < r < 1$).
- It Follows That We Can Now Construct Infinitely Many Infinitesimals. Here Are Some Examples Listed In An Order of Increasing Magnitude: $\varepsilon^3, \varepsilon^2, \frac{\varepsilon}{100}, \varepsilon, 75\varepsilon, \sqrt{\varepsilon}, \varepsilon + \sqrt{\varepsilon}$.

We Defined Infinitesimal in Definition [5].
We Have a Corresponding Definition Pertaining The Other Hyperreal Numbers.

Definition (Finite And Infinite Hyperreal Number)

A Hyperreal Number b Is

1. *Finite*, If b Is between Two Real Numbers.
2. *Positive Infinite* If b Is Greater Than Every Real Number.
3. *Negative Infinite* If b Is Less Than Every Real Number.

Since Every Infinitesimal Is between 0 And, Say, 1 (Which Are Both Real Numbers), It Is Finite. Clearly, Not Every Finite Is Infinitesimal.

Now That The Definitions of Real Numbers And Infinitesimal [5], Finite, And Infinite Hyperreal Numbers [9] Are Established, We Are Ready for Rules for Their Computations. These Rules, Which Are Intuitive, Can Be Rigorously Proved[6] From Those Definitions Using Standard Proof Techniques.

Definition (Denoting Hyperreal Numbers of Various Kinds)

We Assume The Following:

1. ε, δ Are Infinitesimal.
2. b, c Are Finite But Not Infinitesimal.
3. H, K Are Infinite.

[6] TBD!

Hyperreal Computation Rule (Real Numbers)

1. 0 Is The Only Real Infinitesimal Number.
2. Every Real Number is Finite.

Hyperreal Computation Rule (Negatives)

1. $-\varepsilon$ Is Infinitesimal.
2. $-b$ Is Finite But Not Infinitesimal.
3. $-H$ Is Infinite.

Hyperreal Computation Rule (Reciprocals)

1. $\frac{1}{\varepsilon}$ Is Infinite.
2. $\frac{1}{b}$ Is Finite But Not Infinitesimal.
3. $\frac{1}{H}$ Is Infinitesimal.

Hyperreal Computation Rule (Sums)

1. Addition of Hyperreal Numbers Is Commutative (Extension Principle).
2. $\varepsilon + \delta$ Is Infinitesimal.
3. $\varepsilon + b$ Is Finite But Not Infinitesimal.
4. $\varepsilon + H$ Is Infinite.
5. $b + c$ Is Finite (Could Be Infinitesimal Because c May Be $-b + \varepsilon$).
6. $b + H$ Is Infinite.
7. $H + K$ Is Indeterminate.