Learning Calculus from the Masters

How to Build on Elementary Ideas

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Outline

Introducing Next Section

In Our Calculus Journey

- We Continue to Learn from The Masters.
- We Started Thinking about Changing Magnitudes And Their Ratios.
- You May Wish to Review What We Learned Before.

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Two *Diminishing* Magnitudes

- Consider Two Points A, B on A Curve Below.

- A Approaches B Along The Curve.
- Consider Two Lengths: Chord AB, $\overline{AB}$, and Arc AB, $\widehat{AB}$.
- As A Approaches B Along The Curve ...
  - Isn't It Clear That Both Magnitudes $\overline{AB}$, $\widehat{AB}$ **Diminish**?
  - What Happens to The Ratio $\frac{\overline{AB}}{\widehat{AB}}$? Does It Decrease, Increase, or Remain Constant?

Ratio of Constantly Diminishing Magnitudes

Does It Also Diminish?

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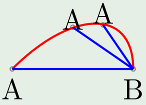

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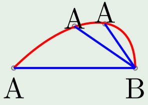

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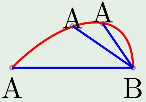

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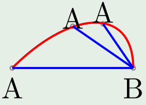

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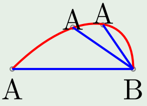

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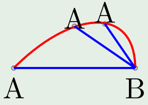

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- How Could We Ascertain?
- Right, We Analyze!
  - Let's Start with Concrete Cases.
  - We Consider Discrete Values of The Two Magnitudes as *Corresponding Values of Two Sequences*[1].
  - Observe How The Values of The Two Magnitudes Change ...
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How The Two Magnitudes Diminish

Do They Diminish Haphazardly?

- We Are Studying The Ratio of Diminishing Magnitudes.
- However, The Magnitudes Are Not Diminishing Haphazardly, Whimsically, Or Randomly.
- In The Previous Example, A Is Constrained to Move along The Predefined Curve as It Approaches B.
  - And A Curve Can Be Expressed As A Function Or Formula.
  - Functions[2], with Many of Which You Are Familiar, Are Key!
- We Use Sequences of *Corresponding Numbers* to Aid The Discovery Of Patterns Of Change Or Functions.

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A Few Words on Sequences of Numbers

Mostly A Matter of Notation ...

- Recall Arithmetic and Geometric Sequences?
- A Number in A Sequence Is Its Term.
- The Order of Terms of A Sequence Is Crucial.
- A Sequence May Have A Finite Number of Terms, e.g.,
  - $1, 2, 3, 4, 5$ Represents The Sequence of Exactly 5 First Counting Numbers.
- A Sequence May Have Infinite Terms! Such An Unending Sequence Is Shown by The Trailing $\ldots$, e.g.,
  - $1, 3, 5, 7, \ldots$ Represents The Sequence of *All* Odd Counting Numbers!
  - It's Fun to Guess The *Pattern of Its Terms*.
  - A Given Infinite Sequence Should Have Sufficient Number[3] of Terms for Us to Guess The Pattern, If Any!

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Infinite Sequences Are Challenging

And Fun Too!

- In Calculus, We Mainly Study Infinite Sequences.
- The Idea of Infinity, Denoted by ∞, Intrigues, or, Baffles Us.
- We Usually Need to Determine The *General Term* of a Sequence as A Closed-Form Formula[4] in n.
  - We Denote The General Term of Sequence S by $s_n, n \in \mathbb{N}$.

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Another Thing We Study in Calculus

Among Other Things, of Course!

- One Magnitude (B) Is Changing. So Is Another (A).
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- Therefore, Think of Rate of Change of One Thing wrt Another!

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A Slight Digression about Rate of Change

Coordinate Plane, Plots, Functions, ...

- The Words 'Rate' And 'Ratio' Are Related.
- The Coordinate Planes (XY, $r\theta$, Etc.) Facilitate The Same: Visualization of Two Simultaneously Changing Magnitudes!
- You Are Of Course Familiar with 'Slope' of A Straight Line
    - The Ratio $\frac{\text{Rise}}{\text{Run}}$.
    - It Is Constant for A Straight Line.
    - Can Be Thought of As The Rate at Which Y Changes wrt X.

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Back to Sequences of

Discrete, Simultaneously Diminishing Magnitudes . . .

Our First Sequences of Diminishing Magnitudes

As A And B Diminish, What Happens to $\frac{A}{B}$?

- We Represent Two Diminishing Magnitudes as Sequences A, B:
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