Learning Calculus from the Masters

How to Build on Elementary Ideas

A. De Morgan[1] T. McCormack[2]

[1]Original Author

[2]First Maintainer

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Outline

Learn from the Masters

WE LEARN FROM THE MASTERS.

Learn from the Masters

WHY?

Learn from the Masters

- Masters May Seem Austere And Sound Cryptic, But They Have Two Essential Traits
  - Kindness
  - Dependable and Thorough Knowledge of Something
- They Want You to Know What They Do (Know)
- They Understand Your Difficulties and Sympathize with You
- They Are Gentle, But *Don't* Dumb Down The Subject Matter

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THAT'S WHY.

Learn from the Masters

Figure: Augustus De Morgan Was A Master

He Was A Great Mathematician Interested in Math Education, And He Wrote Several Books to Help Students Learn Math on Their Own!
Here Is a Tiny Story about Him:

Someone asked De Morgan, who lived in the 1800s, "Dear Mr. De Morgan, in which year were you born?"
De Morgan replied, "I was x years old in the year x^2. In which year was I born?"

Which Year? :-)

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DON'T WE WANT TO LEARN FROM SUCH A MASTER?

Lest We Forget . . .

- Reminder: In Mathematics, As in Everyday Speech, Words are Important. We Must Understand Them Precisely in The Given Context.
- Problem: Mathematics Uses Common Words and Gives Them Meanings That May Be Confusing at First.

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- Be Patient.
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Commensurable

Here Is A Definition!

Definition (Commensurable)

Two Magnitudes or Lengths or Measures Are Commensurable If They Can Be Expressed in Whole Numbers of Some Common Unit.

To Recap [25] To Modern Language [30]

Note: The Common Unit Can Be Made As Small or As Big As We Please.

Note: Mensurable Means Measurable.

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But We Can Deconstruct it Slowly . . .

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Example: Commensurable

Consider An Example!

Example

'Foot' and 'Yard' Are Units of Length.

1 Foot is **Commensurable** with 1 Yard.

1 Foot = 12 Inches and 1 Yard = 36 Inches

- If the "Common Unit" is 3 (Inches), then 1 Foot Is **4** Common Units and 1 Yard Is **12** Common Units. Both 4 and 12 are Whole Numbers.
- If the "Common Unit" is 6 (Inches), then 1 Foot Is **2** Common Units and 1 Yard = **6** Common Units. Both 2 and 6 are Whole Numbers.
- If the "Common Unit" is 1 (Inch), then 1 Foot Is **12** Common Units and 1 Yard Is **36** Common Units. Both 12 and 36 are Whole Numbers.

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SFO $\rightarrow$ NYC is a 6-Hour Flight.
SFO $\rightarrow$ DTW is a $5\frac{1}{2}$-Hour Flight.
Are These Two Periods, or Lengths of Time:
$T_1 = 6, T_2 = 5\frac{1}{2}$, Commensurable?

- What is Your Common Unit?
- How Many Whole Numbers of That Common Unit Are T_1, T_2?

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Very Well Then!

To Show That Two Lengths Are Commensurable, We Find A Common Unit and Demonstrate That Each Length Equals Some Whole Number Times It.

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You Are Convinced That $T_1 = 6, T_2 = 5\frac{1}{2}$ Are Commensurable, Right?

Why Lengths?

We Say 'Lengths' Because Humanity Measured Things Since Antiquity.

We Also Say 'Magnitudes'.

Like In Everyday Language, We Use 'Length' and 'Magnitude' *Synonymously*.

Geometry Played A Big Role in Our Understanding of Mathematics . . .

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Examples And Counterexamples

We Saw 2 'Examples' of Commensurable Magnitudes.
Sometimes, However, Counterexamples Are Helpful Too.
Are There Lengths That Are Not Commensurable?

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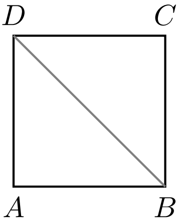
Commensurable Or Not?

Figure: A Unit Square

In the Unit Square ABCD,
Are Its Side (e.g. AB) and Its Diagonal (e.g. BD) Commensurable?

Commensurable Or Not?



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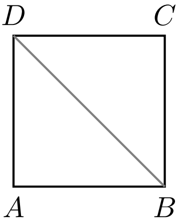

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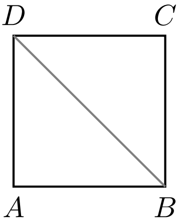

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In The Square ABCD [2],

$$l(AB) = 1, l(BD) = \sqrt{1^2 + 1^2} = \sqrt{2}$$

(From The Pythagorean Theorem).

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- It Turns out That 1 and $\sqrt{2}$ Are Not Commensurable!
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Incommensurable?

- We Repeat, To Show That Commensurable Lengths (e.g. 2.1, 17.8) Are Indeed So …
  - We *Creatively* Find A Common Unit (e.g. 0.1), And
  - Demonstrate That The Lengths Are Its Whole Numbers (e.g. $21 \in \mathbb{Z}$ and $178 \in \mathbb{Z}$ respectively).
- Will That Approach Work To Show That Incommensurable Lengths (e.g. $\sqrt[3]{3}$, $\sqrt[4]{5}$) Are Indeed So?
  - That Will Be An Unending Task! You See Why, Right?
  - And This Is Where We Need Proofs!

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Commensurable Recap

Let's Recap!

Reread the Definition: Commensurable [1].

- 12 and 36 Are Commensurable. So Are 6 and $5\frac{1}{2}$. So Are Countless Other Pairs of Numbers.
- 1 and $\sqrt{2}$ Are Not Commensurable. So Are Not 2 and $\sqrt{5}$. So Are Countless Other Pairs of Numbers.
- Not Commensurable (Incommensurable) Pairs Are Far Far More Than Commensurable Pairs!
- When We Discovered Incommensurable Magnitudes Back in $\sim$ 500 BC, A Few Decades Passed in A Confusion ...

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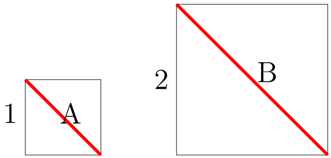

Figure: Diagonals of A 1x1 Square And A 2x2 Square

Are The Two Diagonals: A, B Commensurable?

- If Not, Why Not?
- If Yes, What Are The Common Unit And Whole Numbers for A, B?

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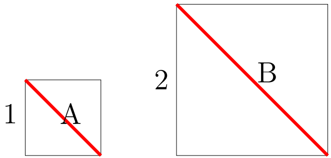

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Commensurable Or Not?

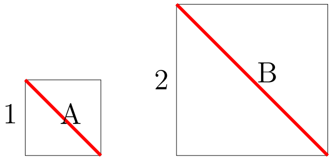

Figure: Diagonals of A 1x1 Square And A 2x2 Square

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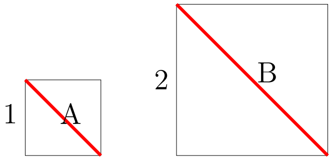

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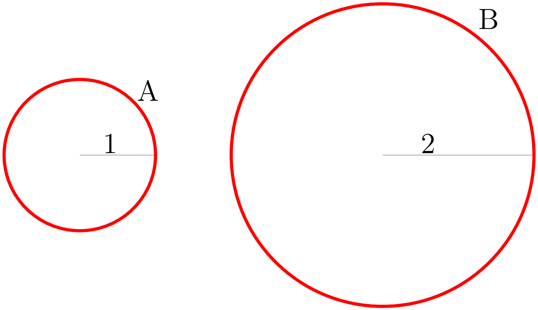

Figure: Circumferences of Circles of Radii 1 and 2

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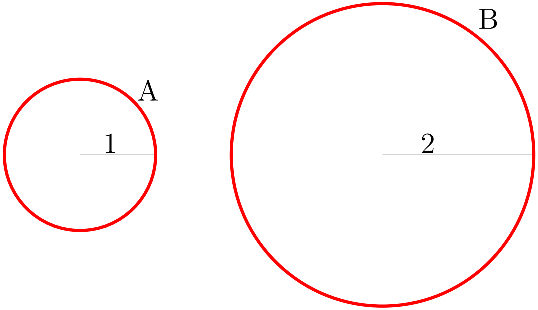

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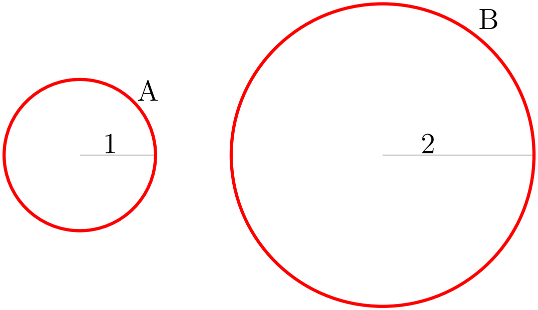

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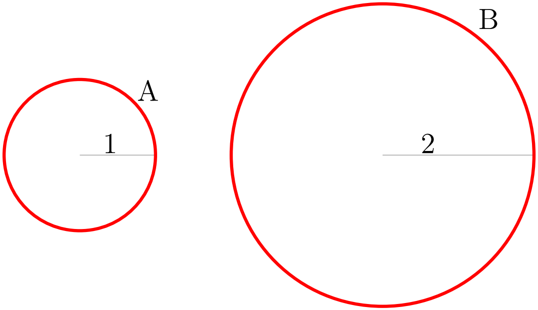

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God Made The Integers; All Else Is The Work of Man[a].

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They Can Only Be Approximated

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- That Means, When Expressed as A Ratio, There's Nothing Common in Them.
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Definition (Rational Number)

If Magnitudes A, B Are Commensurable, Then The Ratio $\frac{A}{B}$ Is Called *A Rational Number*.

Definition (Irrational Number)

If A Number is Not Rational, Then It Is *Irrational*.

Note: This Is a *Negative* Definition. We Specify What Irrational Number Is Not.

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Our Vocabulary

A Growing List of New Words And Symbols

- Commensurable.
- Incommensurable.
- Rational Number.
- Irrational Number.
- Integer.
- Ratio.
- Fraction, Numerator, Denominator[2].
- $\approx$: Approximately Equals.

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Commensurable Recap

Was That Fun?

Is the Meaning of 'Commensurable' Clearer Now?

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Does This Matter?

There Are Definitions, Theorems, Proofs, Problems. Period.

It's A Matter of Perspective

Perhaps None of This Matters.
Commensurable Magnitudes, Irrational Numbers, . . . Are All Meaningless Without The Fun.
Having Fun Matters.

Let's Play Mindfully

This Playful Introduction Is Meant to Be . . .
Well, Play!
Slowly, We Understand More, And Play More!

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The Ever-Changing Magnitudes

Things Change And That Changes Everything in Turn!

- The Ever-Changing Universe Fascinates Us.
- The Study of Changing, Rather Than Fixed, Magnitudes Fascinates Us.
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- Let's Start with The Ratio of Magnitudes That Are Capable of Changing . . .

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Ratio $r = \frac{A}{B}$

Both A and B Can Change ...

- Let's Play with Money! Let A Represent Your Dollars and B Mine.
- Let's Say You Have $A = \$1$ And I $B = \$2$.
- $B - A = \$1$, $r = \frac{A}{B} = \frac{1}{2}$.
- Which Is Right?
  - I Have Much More Money Than You.
  - I Have Almost as Much Money as You.
- Let's Say We Both Got \$98 (Things Changed!). You Now Have \$99 and I \$100.
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Equality of Changing Quantities

Their Difference Or Ratio?

- Although in Both The Cases ...
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- Therefore,
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- Let's Recap!
- We Observe That
  - Two Quantities[3], Both Capable of Changing, Approach Each Other . . .
  - When Their Ratio Approaches 1 *Regardless* of Their Initial Difference.
- Is Our Observation Reliable Or Illusory?
  - After All, We Only Tried One Case: Increasing A and B by 98.
- In Other Words, Does Our Observation Hold *In General?*

[3] Or Numbers, Magnitudes, Lengths, . . .

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Observation, Reflection, Analysis . . .

Hypothesis, Thinking and Experimentation, Generalization . . .

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Informally . . .

We Want to Show That, In General, Two Different Magnitudes Approach Equality When Both Are Increased While Preserving The Difference.

- We Need to Be Precise.
- But We Have Got to Start Somewhere and Proceed Logically.
- We Use Symbols Or Notation for This Purpose.

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Equality of Changing Quantities: Ratio or Difference?

Everyday Language → Language of Math

Informally . . .

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Helpful Symbols?

- Let A, B *Stand for* Initial Values of The Magnitudes.
- Let $A < B$. In Other Words, A *Stands for* The Lesser of The Two Magnitudes.
- Let d *Stand for* The Difference between A, B.

$$B = A + d \tag{1}$$

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- What Does r Now Become?

$$r \to \frac{A_{new}}{B_{new}} = \frac{A + m}{A + m + d} \tag{3}$$

- What Is The Question Now?
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- Commensurable Magnitudes (e.g. $2.5, 3.7$) Can Be Expressed As Exact Ratios (e.g. *Exactly* $\frac{25}{37}$).
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