Elementary Calculus–An Infinitesimal Approach, Chapters 01, 02

Calculus Based on Nonstandard Analysis

Jerome Keisler[1]

[1]Original Author

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Outline

- These Are My Notes from H. Jerome Keisler's Book by The Same Name.
- While Teaching Elementary Calculus to My Daughter, I Realized That I Better Learn The Infinitesimal Approach Better.

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- Consider Two Points on The Parabola $y = x^2$: x_0, y_0, $x_0 + \Delta x, y_0 + \Delta y$.
- Its Average Slope between These Points Is Calculated as The Slope of The *Secant*: $\frac{\Delta y}{\Delta x} = \frac{(x_0+\Delta x)^2-(x_0)^2}{\Delta x} = 2x_0 + \Delta x$.
- This Computation Makes Sense Only When $\Delta x \neq 0$ Because Otherwise $\frac{\Delta y}{\Delta x}$ Is Undefined.
- Intuitively, If Non Rigorously, We Consider Δx Negligible.
- We Therefore State That The Average Slope Equals $2x_0$.

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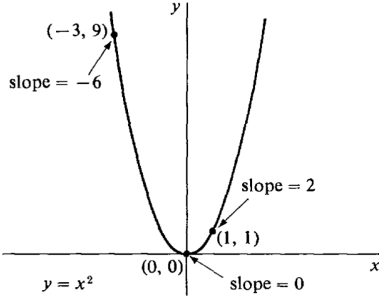
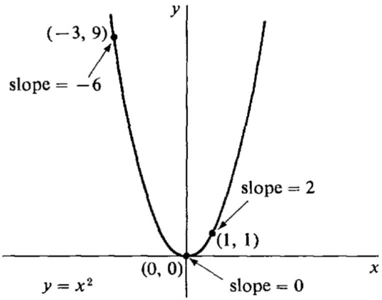
- 

- For Example, The Slope Is $2x_0 = 2 \cdot 0 = 0$ at $(0, 0)$, $2 \cdot 1 = 2$ at $(1, 1)$, And $2 \cdot -3 = -6$ at $(-3, 9)$.

- 

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- We Can Visualize Slope as Velocity.
- If The Horizontal Axis Represents Time And Vertical Axis Position, Is The Average Velocity between (y_0, t_0) and $(y_0 + \Delta y, t_0 + \Delta t)$ = The Velocity *at*[1] Either of Those Points?
- Since in This Case The Velocity Constantly Increases, The Answer Is NO.
- $v_{ave} = 2t_0 + \Delta t$ And, After We Treat Δt Negligible, $v_{ave} = 2t_0$.

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- In Either Case (Slope, Velocity), The Intuitive Reasoning Fails to Clarify When Something Is to Be Treated Negligible.
- We Need a Sharp Distinction between Which Numbers Are Small Enough to Ignore And Which Aren't.
- Actually, No Real Number Except 0 Is Small Enough To Ignore[2].
- To Address This Difficulty, We Take The Bold Step[3] of Introducing a New *Kind*[4] *of Number* Which Is Infinitely Small And Yet $\neq 0$.

[2] Does This Hint at a New *Kind* of Number?
[3] A Rigorous Treatment Is Provided by Keisler in His *Foundations*.
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Definition (Infinitely Small Or Infinitesimal Number)

A *Number* ε Is Said to Be *Infinitely Small, Or Infinitesimal,* If $-a < \varepsilon < a \quad \forall a \in \mathbb{R}^+$.

- Note: All Infinitesimal Numbers Exist Between *Every* Positive Real Number And Its Negative[5].
- The Only Real Infinitesimal Number is 0.
- We Introduce A New Number System, The Hyperreal Numbers, Which Contains All The Real Numbers And Infinitesimals That Are Not Zero.
- Integers Create Rationals. Rationals Create Reals. Reals Create Hyperreals.
- Right Now, We Study Properties of Hyperreals Needed for The Calculus. We'll Study Their Creation Later.

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- $x \in \mathbb{R} \implies x \in \mathbb{R}^* \implies \mathbb{R} \subseteq \mathbb{R}^*$. However, $\mathbb{R}^*$ Has Other Elements Too, $\therefore \mathbb{R} \subset \mathbb{R}^*$.
- Infinitesimals in $\mathbb{R}^*$ Are of Three Kinds: Positive, Negative, And The Real Number 0.
- $x, x_0, x_1, y, \ldots$ Denote Reals.
  $\Delta x, \Delta y, \varepsilon$ (epsilon), δ (delta), $\ldots$ Denote Infinitesimals.
- If $a, b \in \mathbb{R}^*$ And $a - b$ Is Infinitesimal, Then We Say a Is "Infinitely Close" to b.
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- Hyperreal Numbers Which Are Not Infinite, Are Finite Numbers.
- About each $c \in \mathbb{R}$ Is A Portion of Hyperreal Line Composed of The Numbers Infinitely Close to c.
- Numbers Infinitely Close to 0 Are Infinitesimals.

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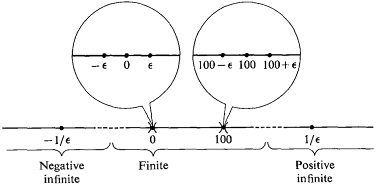

Figure: Finite And Infinite Parts of The Hyperreal Line (*Infinitesimal Microscope* about $c = 0$, $c = 100$)

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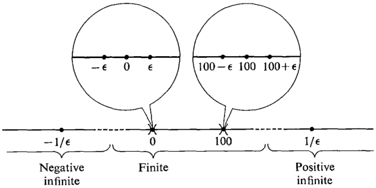

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- Numbers Infinitely Close to 0 Are Infinitesimals.

The Nature of Physical Space

Euclid Struggled to Define a 'Point': Something That Has a Position But No Magnitude[a].

We Have No Way of Knowing What a Line in Physical Space Is Really Like ("What Is It Composed of?"). It Might Be Like the Hyperreal Line (with Infinitesimals Surrounding Every 'Real Point'), The Real Line (without Any Infinitesimals), Or Neither. However, in Applications of The Calculus It Is Helpful to Imagine a Line in Physical Space as A Hyperreal (Rather Than Real) Line. The Hyperreal Line Is, Like The Real Line, A Useful Mathematical Model[b] for A Line in Physical Space.

[a] Isn't This 'Definition' (*Accepted* for Centuries) *Meaningless*?

[b] And, Here's A Stark Reminder: All Models Are Wrong; Some Are Useful!

Definition (The Slope of A Curve at (x_0, y_0))

Let $y = f(x)$ Be a Certain Function. Let $P(x_0, y_0)$ Be Any Point on The Curve Representing y. Let Δx Be a Positive Or Negative Infinitesimal. Consider A Point $(x_0 + \Delta x, y_0 + \Delta y)$ Infinitely Close to P.
Then,

$$\text{Slope of } f \text{ at}(x_0, y_0) = \text{Real Number Infinitely Close to} \frac{\Delta y}{\Delta x}$$

Note: The Slope is *Defined to Be* A Real Number.

Example (Slope of $y = x^2$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$
\begin{aligned}
y + \Delta y &= (x + \Delta x)^2 \\
\therefore \Delta y &= 2x\Delta x + (\Delta x)^2 \\
\therefore \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\
\therefore \text{Slope} &= 2x \text{ (From Definition [2])}
\end{aligned}
\tag{1}
$$

$2x$ Is the Real Number Infinitely Close to The Hyperreal Number $2x + \Delta x$.

In This Example It Was Rather Straightforward to Show $2x + \Delta x$ And $2x$ Are Infinitely Close To Each Other.

Example (Slope of $y = x^2$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ \therefore \Delta y &= 2x\Delta x + (\Delta x)^2 \\ \therefore \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\ \therefore \text{Slope} &= 2x \text{ (From Definition [2])} \end{aligned} \tag{1}$$

$2x$ Is the Real Number Infinitely Close to The Hyperreal Number $2x + \Delta x$.

In This Example It Was Rather Straightforward to Show $2x + \Delta x$ And $2x$ Are Infinitely Close To Each Other.

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Example (Slope of $y = x^3$)

The Definition [2] Defines Slope as The Real Number Infinitely Close to $\frac{\Delta y}{\Delta x}$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^3 \\ \therefore \Delta y &= \cancel{x^3} + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - \cancel{x^3} \\ \therefore \frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + (\Delta x)^2 \end{aligned} \tag{2}$$

If (Because Δx Is Infinitesimal), $3x\Delta x + (\Delta x)^2$ Is Infinitesimal, Then

$$\text{Slope} = 3x^2 \tag{3}$$

$x + \Delta x$ And x Are Infinitely Close, But ...

Δx Is An Infinitesimal. That Makes $x + \Delta x$ And x Infinitely Close to Each Other.
But Unless We Have Precise Rules To Determine Which Hyperreal Numbers Are Infinitely Close to Which Real Numbers, We Won't Be Able to Go Too Far.

That's What We Study Next.

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