STA 360/601: Homework 4 Answers

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1. The joint posterior of λ, γ is:

$$\begin{aligned}
p(\lambda, \gamma \mid X, Y) &\propto \prod_{i=1}^{n} \left(\frac{(\lambda \gamma^{x_i})^{y_i} \, e^{-\lambda \gamma^{x_i}}}{y_i!} \right) e^{-\lambda} e^{-\gamma} \\
&\propto \prod_{i=1}^{n} \left(\lambda^{y_i} \gamma^{x_i y_i} e^{-\lambda \gamma^{x_i}} \right) e^{-\lambda} e^{-\gamma} \\
&\propto \lambda^{\sum_{i=1}^{n} y_i} \gamma^{\sum_{i=1}^{n} x_i y_i} e^{-\lambda \sum_{i=1}^{n} (\gamma^{x_i}) - \lambda - \gamma}
\end{aligned}$$

   Because λ and γ are not easily separated in the exponential term, the joint posterior cannot be written as the product of two gamma distributions. Consequently, the joint posterior cannot be written in the same form as the priors and is not conjugate.

2. Fortunately, the full conditional distributions have known forms. For λ, the full conditional distribution is:

$$\begin{aligned}
p(\lambda \mid \gamma, X, Y) &\propto \lambda^{\sum_{i=1}^{n} y_i} e^{-\lambda \sum_{i=1}^{n} \gamma^{x_i} - \lambda} \\
&\propto Ga\left(\sum_{i=1}^{n} y_i + 1, \sum_{i=1}^{n} \gamma^{x_i} + 1 \right)
\end{aligned}$$

   For γ, the full conditional distribution is:

$$\begin{aligned}
p(\gamma \mid \lambda, X, Y) &\propto \gamma^{\sum_{i=1}^{n} x_i y_i} e^{-\lambda \sum_{i=1}^{n} \gamma^{x_i} - \gamma} \\
&\propto Ga\left(\sum_{i=1}^{n} x_i y_i + 1, \lambda \sum_{i=1}^{n} x_i + 1 \right)
\end{aligned}$$

   Note that $\sum_{i=1}^{n} \gamma^{x_i} = \gamma \sum_{i=1}^{n} x_i + (1 - \sum_{i=1}^{n} x_i)$. Since the $(1 - \sum_{i=1}^{n} x_i)$ part does not have γ in it, we can shove that part into the integration constant. To run our Gibbs sampler, we need to do the following:

   (a) Set initial values (λ^0, γ^0).
   (b) For iterations $t = 1, ..., T$, sample λ^t from $p(\lambda \mid \gamma^{t-1}, X, Y)$.
   (c) Sample γ^t from $p(\gamma \mid \lambda^t, X, Y)$.