STA 601 - Lab 5

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Multivariate Normal Density:

Any p-Variate Normal Distribution can be expressed as follows:

$y \sim \mathcal{N}_p(\mu, \Sigma)$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$.

The dimensions are given as follows:

$\mu_1 \to q \times 1$
$\mu_2 \to (p-q) \times 1$
$\Sigma_{11} \to q \times q$
$\Sigma_{12} \to q \times (p-q)$
$\Sigma_{21} \to (p-q) \times q$
$\Sigma_{22} \to (p-q) \times (p-q)$

The Conditional Distribution can be given as:

$(y_1 \mid y_2 = a) \sim \mathcal{N}_q \left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$

Given Posterior:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim \mathcal{N}_3 \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.9 & 0.1 \\ 0.9 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{pmatrix} \right]$$

1. **Complete Conditionals:** Using the above equations we get the following parameters for the full conditionals:
   $\mu_1 = 0$, $\mu_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma_{11} = 1$.

   - $X \mid Y, Z$
     $\Sigma_{12} = \begin{pmatrix} 0.9 & 0.1 \end{pmatrix}$, $\Sigma_{21} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}$.

   - $Y \mid X, Z$
     $\Sigma_{12} = \begin{pmatrix} 0.9 & 0.1 \end{pmatrix}$, $\Sigma_{21} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}$.

   - $Z \mid X, Y$
     $\Sigma_{12} = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}$, $\Sigma_{21} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$.

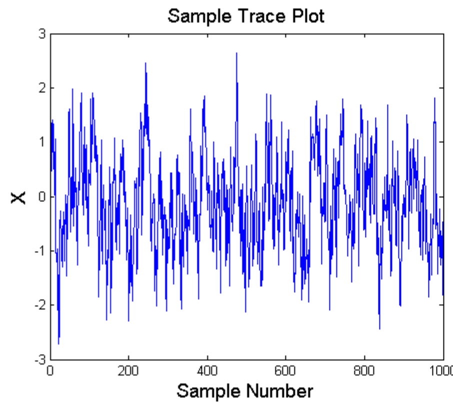
2. **Gibbs Sampler:** Algorithm:

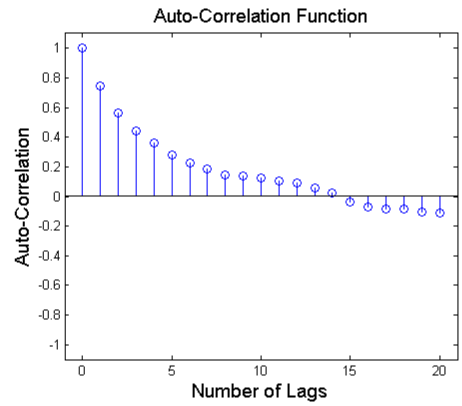
   Start with $\{Y^{(0)}, Z^{(0)}\}$,

   - Draw $X^{(s+1)} \mid Y^{(s)}, Z^{(s)}$,

   - Draw $Y^{(s+1)} \mid X^{(s+1)}, Z^{(s)}$,

   - Draw $Z^{(s+1)} \mid X^{(s+1)}, Y^{(s+1)}$.





3. **Gibbs Sampler with Block Updates:** The following are the parameters for conditional distributions:

- $X, Y \mid Z$
  $\mu_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mu_2 = 0$.
  $\Sigma_{11} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$, $\Sigma_{12} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\Sigma_{21} = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}$, $\Sigma_{22} = 1$.

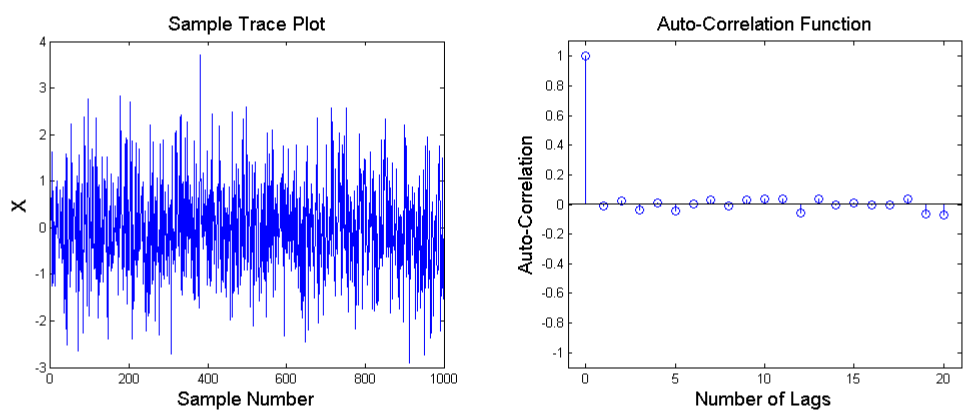
- $Z \mid X, Y$
  $\mu_1 = 0$ $\mu_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
  $\Sigma_{11} = 1$, $\Sigma_{12} = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}$, $\Sigma_{21} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$.

Gibbs Sampling Algorithm:

Start with $\{Z^{(0)}\}$,

- Draw $\{X^{(s+1)}, Y^{(s+1)}\} \mid Z^{(s)}$,

- Draw $Z^{(s+1)} \mid \{X^{(s+1)}, Y^{(s+1)}\}$.

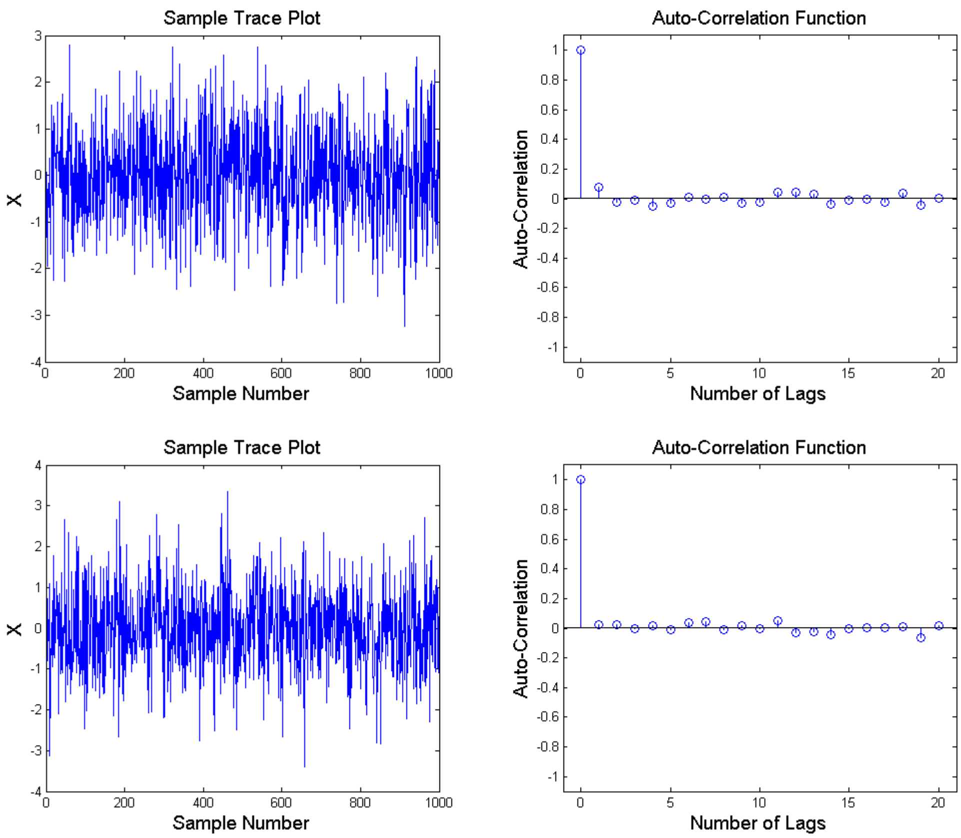


4. **Comparison between the two Samplers:** From the autocorrelation plots, we can see that the second chain clearly mixes better than the first one. From the covariance matrix we see that X, Y and very highly correlated ($\rho = 0.9$) which are both poorly correlated with Z ($\rho = 0.1$). We know that consecutive samples from Gibbs Sampler are highly correlated.

   In the first sampler we draw X and Y from their conditional distributions. Since, they are highly correlated, each X sample is now not only correlated with its previous sample but also with its Y sample. This is why it takes time for the first chain to converge.

   In the second chain, we draw samples from the X, Y joint distribution, we avoid the inter variable correlation. Also since Z is poorly correlated with X and Y, we get good mixing.

   This can be further explained using the following figures. Here I dropped the correlation between X and Y to $\rho = 0.15$. In this case both samplers show good mixing.



Appendix:

```
%% STA 601: Lab 5
% Author: Kedar S Prabhudesai
% Created on: 10/03/2013

close all;
clear all;

%% Gibbs Sampler with single variable update
% Initialize
mu1 = 0;
mu2 = [0;0];
S11 = 1;

nGibbs = 5000;
XSamples = zeros(1,nGibbs);
YSamples = zeros(1,nGibbs);
ZSamples = zeros(1,nGibbs);
% Starting values of Y, Z
YSamples(1) = 2;
ZSamples(1) = 3;

for iGibbs = 1:nGibbs-1
    % Update X
    S12 = [0.9 0.1];
    S21 = [0.9;0.1];
    S22 = [1 0.1;0.1 1];
    XSamples(iGibbs) = normrnd(mu1 + S12*inv(S22)*([YSamples(iGibbs);ZSamples(iGibbs)] - ...
        mu2),sqrt(S11 - S12*inv(S22)*S21));

    % Update Y
    YSamples(iGibbs+1) = normrnd(mu1 + S12*inv(S22)*([XSamples(iGibbs);ZSamples(iGibbs)] - ...
        mu2),sqrt(S11 - S12*inv(S22)*S21));

    % Update Z
    S12 = [0.1 0.1];
    S21 = [0.1;0.1];
    S22 = [1 0.9;0.9 1];
    ZSamples(iGibbs+1) = normrnd(mu1 + S12*inv(S22)*([XSamples(iGibbs);YSamples(iGibbs+1)] ...
        - mu2),sqrt(S11 - S12*inv(S22)*S21));
end

[XSamplesACF,lags] = autocorr(XSamples(1:1000),20);

figure('Position',[125 490 1175 400]);
subplot(1,2,1);
plot(XSamples(1:1000));
xlabel('Sample Number','Fontsize',14);
ylabel('X','Fontsize',14);
title('Sample Trace Plot','Fontsize',14);
subplot(1,2,2);
stem(lags,XSamplesACF);
xlim([-1 21]);
ylim([-1.1 1.1]);
xlabel('Number of Lags','Fontsize',14);
ylabel('Auto-Correlation','Fontsize',14);
title('Auto-Correlation Function','Fontsize',14);

%% Gibbs Sampler with Block Updates
XYSamples = zeros(2,nGibbs);
ZBlockSamples = zeros(1,nGibbs);
```

```
ZBlockSamples(1) = 5;

for iGibbs = 1:nGibbs-1
    % Update X,Y|Z
    mu1 = [0;0];
    mu2 = 0;
    S11 = [1 0.9;0.9 1];
    S12 = [0.1;0.1];
    S21 = [0.1 0.1];
    S22 = 1;
    XYSamples(:,iGibbs) = mvnrnd(mu1 + S12*inv(S22)*(ZBlockSamples(iGibbs) - mu2),sqrt(S11 ...
        - S12*inv(S22)*S21));

    % Update Z|X,Y
    mu1 = 0;
    mu2 = [0;0];
    S11 = 1;
    S12 = [0.1 0.1];
    S21 = [0.1;0.1];
    S22 = [1 0.9;0.9 1];
    ZBlockSamples(iGibbs+1) = normrnd(mu1 + S12*inv(S22)*(XYSamples(:,iGibbs) - mu2),sqrt(...
        S11 - S12*inv(S22)*S21));
end

[XYSamplesACF,lags] = autocorr(XYSamples(1,1:1000),20);

figure('Position',[125 490 1175 400]);
subplot(1,2,1);
plot(XYSamples(1,1:1000));
xlabel('Sample Number','Fontsize',14);
ylabel('X','Fontsize',14);
title('Sample Trace Plot','Fontsize',14);
subplot(1,2,2);
stem(lags,XYSamplesACF);
xlim([-1 21]);
ylim([-1.1 1.1]);
xlabel('Number of Lags','Fontsize',14);
ylabel('Auto-Correlation','Fontsize',14);
title('Auto-Correlation Function','Fontsize',14);
```