STA 601 - Lab 2

Kedar Prabhudesai

September 12, 2013

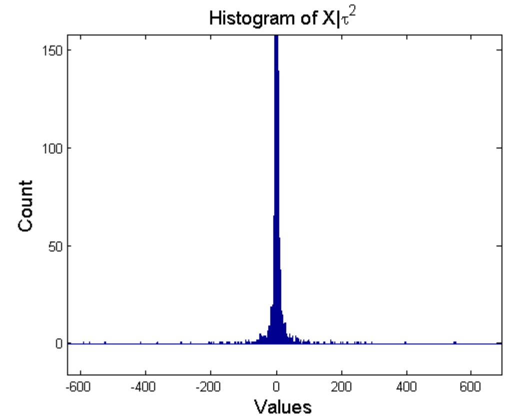
1. Normal-Gamma Model:

   $X|\tau^2 \sim Normal(0, 1/\tau^2)$ and $\tau^2 \sim Gamma(\nu/2, \nu/2)$. To find the Marginal Distribution, $p(X)$

$$
\begin{aligned}
p(X) &= \int_0^\infty p(X|\tau^2)p(\tau^2)d\tau^2 \\
&= \int_0^\infty \frac{\tau}{\sqrt{2\pi}} exp\left(\frac{-x^2\tau^2}{2}\right)\frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2)}(\tau^2)^{(\nu/2-1)} exp\left(\frac{-\nu\tau^2}{2}\right)d\tau^2 \\
&= \frac{(\nu/2)^{(\nu/2)}}{\sqrt{2\pi}\Gamma(\nu/2)}\int_0^\infty \left(\tau^2\right)^{\frac{\nu+1}{2}-1} exp\left[-\tau^2\left(\frac{x^2+\nu}{2}\right)\right]d\tau^2 \\
&= \frac{(\nu/2)^{(\nu/2)}}{\sqrt{2\pi}\Gamma(\nu/2)}\frac{\Gamma(\frac{\nu+1}{2})}{\left(\frac{x^2+\nu}{2}\right)^{\frac{\nu+1}{2}}}\int_0^\infty \frac{\left(\frac{x^2+\nu}{2}\right)^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu+1}{2})}\left(\tau^2\right)^{\frac{\nu+1}{2}-1} exp\left[-\tau^2\left(\frac{x^2+\nu}{2}\right)\right]d\tau^2 \\
&= \frac{(\nu/2)^{(\nu/2)}\Gamma(\frac{\nu+1}{2})}{\sqrt{2\pi}\Gamma(\nu/2)}\left(\frac{x^2+\nu}{2}\right)^{-\frac{\nu+1}{2}} \\
p(X) &\propto \left(\frac{x^2+\nu}{2}\right)^{-\frac{\nu+1}{2}} \\
p(X) &\propto \left(\frac{x^2/\nu+1}{2/\nu}\right)^{-\frac{\nu+1}{2}} \\
p(X) &\propto \left[\left(1+\frac{x^2}{\nu}\right)\left(\frac{\nu}{2}\right)\right]^{-\frac{\nu+1}{2}} \\
p(X) &\propto \left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}
\end{aligned}
$$

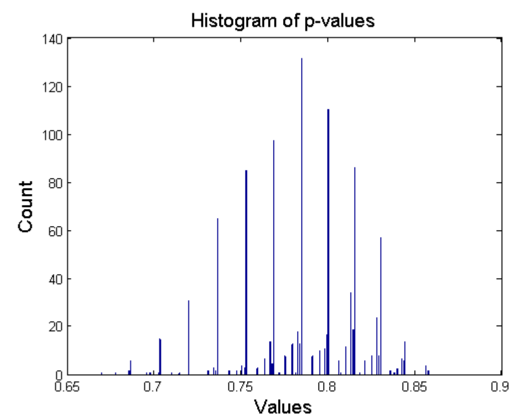
   Therefore, we can say that $X \sim$ T-Distribution with ν Degrees of Freedom.

2. Given is histogram of samples from $X|\tau^2$. The samples do look like they are drawn from a T-Distribution with $\nu = 1$. Because, it was a heavy tailed distribution, I did have extreme values. I have provided here a zoomed plot of the histogram. T-Distribution with $\nu = 1$ is also called 'The Cauchy Distribution'.



3. Running KS-Test in Matlab I got, $p = 0.7528$.

4. This is a histogram of 1000 p-values. For me it looks more or less (but not quite) like a normal. I know it is supposed to look like a Uniform which would be $Beta(1, 1)$. But looking at this one I don't think I can say that. Not sure why it looks like this.



5. The Central Limit Theorem (CLT) states that, the arithmetic mean of a sufficiently large number of independent samples of a random variable, will be approximately normally distributed.

   For $\nu = 1$, the Mean and Variance of the t-distribution is 'Not Defined'. Hence the CLT will not hold.

   For $\nu = 2$, the mean is 0, but the variance is ∞. ('Infinity' which is different from 'Not Defined'). This also imples that because of large variance in our samples, we will not converge to a normal distribution.

   For $\nu = 3$, the mean is 0 and variance is $\frac{\nu}{\nu-2}$. In this case we can say that we will converge to a normal distribution if we draw a large number of samples from the t-distribution.