STA 601 - Lab 9

Kedar Prabhudesai

November 27, 2013

We have an additional parameter to do inference on. Let, $W_i = 1$, denote weekday and $W_i = 0$ denote weekend.

$$W_i \sim Bernoulli(\theta)$$

If $W_i = 1$, $X_i \sim log\mathcal{N}(\mu_1, \tau_1)$, else $X_i \sim log\mathcal{N}(\mu_2, \tau_2)$.

Data Likelihood:

Let $\tau = 1/\sigma^2$, and k is the number of weekdays.

$$L(X^n, W^n \mid \mu_1, \tau_1, \mu_2, \tau_2, \theta) = \prod_{i=1}^{n} [\mathbb{1}(W_i = 1) log\mathcal{N}(\mu_1, \tau_1) + \mathbb{1}(W_i = 0) log\mathcal{N}(\mu_2, \tau_2)] \times \prod_{i=1}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k}$$

Priors:

We use the same priors we defined on μ and τ from last time.

$$\begin{aligned} \mu_j &\sim \mathcal{N}(\mu_{j0}, \tau_{j0}) \therefore p(\mu_j) \propto exp\left[\frac{-\tau_{j0}(\mu_j - \mu_{j0})^2}{2}\right] \\ \tau_j &\sim Gamma(\alpha_j, \beta_j) \therefore p(\tau_j) \propto \tau_j^{\alpha_j - 1} exp(-\beta_j \tau_j) \\ \theta &\sim Beta(5, 2) \therefore p(\theta) \propto \theta^{5-1}(1-\theta)^{2-1} \end{aligned}$$

Posterior:

$$p(\mu_1, \tau_1, \mu_2, \tau_2, \theta \mid X^n, W^n) \propto L(X^n, W^n \mid \mu_1, \tau_1, \mu_2, \tau_2, \theta) \times p(\mu_1) \times p(\tau_1) \times p(\mu_2) \times p(\tau_2) \times p(\theta)$$

Full Conditionals:

$$\begin{aligned} p(\theta \mid \ldots) &\propto \theta^{5+k-1}(1-\theta)^{2+n-k-1} \\ \therefore p(\theta \mid \ldots) &\propto Beta(5+k, 2+n-k) \end{aligned}$$

$$\begin{aligned} p(\mu_1 \mid \ldots) &\propto \prod_{i=1}^{n} \left\{ \mathbb{1}(W_i = 1) exp\left[\frac{-\tau_1 (lnx_i - \mu_1)^2}{2}\right]\right\} \times exp\left[\frac{-\tau_{10}(\mu_1 - \mu_{10})^2}{2}\right] \\ &\propto \left\{ \mathbb{1}(W_i = 1) exp\left[\frac{-\tau_1}{2} \sum_{i=1}^{n} (lnx_i - \mu_1)^2\right]\right\} \times exp\left[\frac{-\tau_{10}(\mu_1 - \mu_{10})^2}{2}\right] \\ \therefore p(\mu_1 \mid \ldots) &\propto exp\left[\frac{-\tau_1}{2} \sum_{i:W_i=1} (lnx_i - \mu_1)^2 - \frac{\tau_{10}(\mu_1 - \mu_{10})^2}{2}\right] \end{aligned}$$

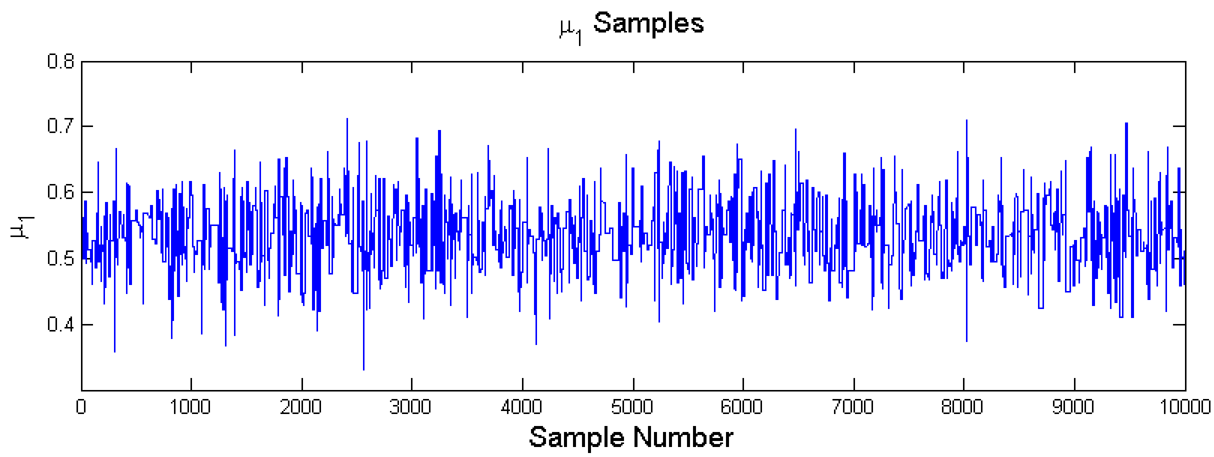
$$p(\mu_2 \mid \ldots) \propto exp\left[\frac{-\tau_2}{2} \sum_{i:W_i=0} (lnx_i - \mu_2)^2 - \frac{\tau_{20}(\mu_2 - \mu_{20})^2}{2}\right]$$

$$\begin{aligned} p(\tau_1 \mid \ldots) &\propto \prod_{i=1}^{n} \left\{ \mathbb{1}(W_i = 1) \tau_1^{1/2} exp\left[\frac{-\tau_1 (lnx_i - \mu_1)^2}{2}\right]\right\} \times \tau_1^{\alpha_1 - 1} exp(-\beta_1 \tau_1) \\ &\propto \mathbb{1}(W_i = 1) \tau_1^{k/2} exp\left[\frac{-\tau_1}{2} \sum_{i=1}^{n} (lnx_i - \mu_1)^2\right] \times \tau_1^{\alpha_1 - 1} exp(-\beta_1 \tau_1) \\ &\propto \tau_1^{\alpha_1 + k/2 - 1} exp\left\{-\tau_1 \left[\frac{\sum_{i:W_i=1} (lnx_i - \mu_1)^2}{2} + \beta_1\right]\right\} \\ \therefore p(\tau_1 \mid \ldots) &\propto Gamma\left(\alpha_1 + \frac{k}{2}, \frac{\sum_{i:W_i=1} (lnx_i - \mu_1)^2}{2} + \beta_1\right) \end{aligned}$$

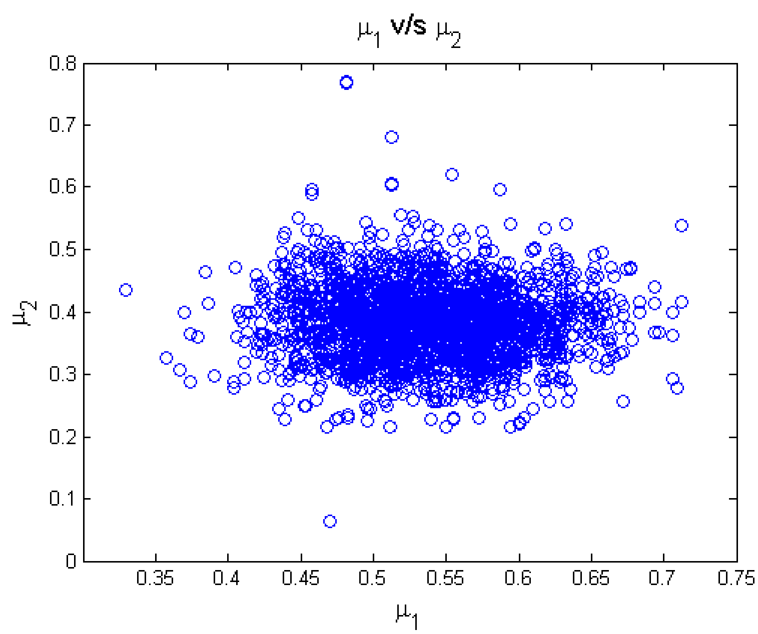
$$p(\tau_2 \mid \ldots) \propto Gamma\left(\alpha_2 + \frac{n-k}{2}, \frac{\sum_{i:W_i=0} (lnx_i - \mu_2)^2}{2} + \beta_2\right)$$

Results from Gibbs Sampler:

Following plot shows posterior μ_1 samples. We can see that our Gibbs Sampler has converged.



Next plot is that of μ_1 v/s μ_2.



Point and interval estimates of number of days that were weekdays (k):

$k \rightarrow 72.4366$ $[36, 97]$.

Probability that proportion of weekends is less than $2/7 = 0.5579$

Point and interval estimates of all parameters:

$\mu_1 \to 0.5403$ $[0.4481, 0.6373]$.
$\sigma_1^2 \to 1.2155$ $[0.7776, 1.8222]$.
$\mu_2 \to 0.3831$ $[0.2859, 0.4805]$.
$\sigma_2^2 \to 1.1263$ $[0.3057, 2.3891]$.

Point and interval estimates from last time:

$\mu_1 \to 0.4102$ $[0.3037, 0.5219]$.
$\sigma_1^2 \to 0.8585$ $[0.5750, 1.2491]$.
$\mu_2 \to 0.1533$ $[0.0462, 0.2561]$.
$\sigma_2^2 \to 2.4503$ $[1.4273, 4.1320]$.

If we compare our current estimates to estimates from last time we observe that the separation between μ_1 and μ_2 is much smaller. In fact there is a slight overlap in our credible intervals for μ_1 and μ_2 despite using priors such that $\mu_1 > \mu_2$. This is because this time we have added uncertainty about what type of day it is.

Appendix:

```
%% STA 601: Lab 8
% Author: Kedar S Prabhudesai
% Created on: 11/22/2013
function sta601_ksp6_lab9

close all;
clear all;

%% Get Data
X = importdata('data.txt');
X = X.data;

%% Prior Parameters
mu10 = 0.6;mu20 = 0.4;
t10 = 400;t20 = 400;
a1 = 1;a2 = 1;
% Remember that 'b' parameter in matlab's gamma function is in fact '1/b'
b1 = 0.05;b2 = 0.05;
n = numel(X);

% Target Distribution for Mu1 and Mu2
Mu1FullCond =  @(Xwd,t1,mu1) exp(-0.5*t1*sum((log(Xwd)-mu1).^2) - 0.5*t10*(mu1-mu10)^2);
Mu2FullCond =  @(Xwe,t2,mu2) exp(-0.5*t2*sum((log(Xwe)-mu2).^2) - 0.5*t20*(mu2-mu20)^2);

%% Gibbs Sampler
nGibbs = 12000;
nBurnIn = 2000;
% Proposal Distribution Std Dev
SCand = 0.5;

% Initialize
ThetaSamples = zeros(1,nGibbs);
ThetaSamples(1) = 5/7;
Mu1Samples = zeros(1,nGibbs);
Mu2Samples = zeros(1,nGibbs);

Tau1Samples = zeros(1,nGibbs);
Tau2Samples = zeros(1,nGibbs);
Tau1Samples(1) = 0.05;
Tau2Samples(1) = 0.05;

kVals = zeros(1,nGibbs);

for iGibbs = 2:nGibbs
    home;disp(iGibbs);
    % Draw W from Binomial(n,Theta)
    W = rand(n,1);
    W = W < ThetaSamples(iGibbs-1);

    % Split data according to weekdays/weekends
    XWkDays = X(W == 1);
    XWkEnds = X(W == 0);

    kVals(iGibbs-1) = sum(W);
    %% Update Theta | ---
    ThetaSamples(iGibbs) = betarnd(5+kVals(iGibbs-1),2+n-kVals(iGibbs-1));

    %% Update Mu1 | --- using M-H
    % Step 1: Sample from Mu1'|Mu1(s)
    Mu1Prime = normrnd(Mu1Samples(iGibbs-1),SCand);
```

```
    % Step 2: Compute Acceptance Ratio
    r = Mu1FullCond(XWkDays,Tau1Samples(iGibbs-1),Mu1Prime)/Mu1FullCond(XWkDays,Tau1Samples...
        (iGibbs-1),Mu1Samples(iGibbs-1));

    % Step 3: Accept/Reject
    u = rand;
    if u < r
        Mu1Samples(iGibbs) = Mu1Prime;
    else
        Mu1Samples(iGibbs) = Mu1Samples(iGibbs-1);
    end

    %% Update Mu2 | --- using M-H
    % Step 1: Sample from Mu2'|Mu2(s)
    Mu2Prime = normrnd(Mu2Samples(iGibbs-1),SCand);

    % Step 2: Compute Acceptance Ratio
    r = Mu2FullCond(XWkEnds,Tau2Samples(iGibbs-1),Mu2Prime)/Mu2FullCond(XWkEnds,Tau2Samples...
        (iGibbs-1),Mu2Samples(iGibbs-1));

    % Step 3: Accept/Reject
    u = rand;
    if u < r
        Mu2Samples(iGibbs) = Mu2Prime;
    else
        Mu2Samples(iGibbs) = Mu2Samples(iGibbs-1);
    end

    %% Update Tau1 | ---
    Tau1Samples(iGibbs) = gamrnd(a1+kVals(iGibbs-1)/2,1/(0.5*sum((log(XWkDays)-Mu1Samples(...
        iGibbs)).^2)+b1));

    %% Update Tau2 | ---
    Tau2Samples(iGibbs) = gamrnd(a2+(n-kVals(iGibbs-1))/2,1/(0.5*sum((log(XWkEnds)-...
        Mu2Samples(iGibbs)).^2)+b2));
end

%% Burn-In
ThetaSamples(1:nBurnIn) = [];
Mu1Samples(1:nBurnIn) = [];
Mu2Samples(1:nBurnIn) = [];
Tau1Samples(1:nBurnIn) = [];
Tau2Samples(1:nBurnIn) = [];
kVals(1:nBurnIn) = [];

% Convert Taus to sigma^2
S1Samples = 1./Tau1Samples;
S2Samples = 1./Tau2Samples;
keyboard
end
```