STA 601 - Lab 3

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Data Model:
$y|N, \beta \sim Binomial(N, \beta)$
Priors:
$N \sim Beta(1, 1)$
$\beta \sim Poisson(\lambda)$
Given:
$y = 20$, $\lambda = 25$.

1. Joint Posterior:

$$\begin{aligned}
p(N, \beta|y) &\propto p(y|N, \beta)p(N, \beta) \\
&\propto p(y|N, \beta)p(N)p(\beta) \\
&\propto \binom{N}{20}\beta^{20}(1-\beta)^{N-20} \times \frac{25^N}{N!}exp(-25) \\
&\propto \frac{N!}{(N-20)!20!}\beta^{20}(1-\beta)^{N-20} \times \frac{25^N exp(-25)}{N!} \times \frac{25^{-20}}{25^{-20}} \\
\therefore p(N, \beta|y) &\propto \frac{\beta^{20}[25(1-\beta)]^{N-20}}{(N-20)!}
\end{aligned}$$

Since this is a non-standard distribution, we will find the Full Conditionals.

2. Posterior Full Conditionals:

$$\begin{aligned}
p(N|\beta, y) &\propto \frac{[25(1-\beta)]^{N-20}}{(N-20)!} \times \frac{exp^{-25(1-\beta)}}{exp^{-25(1-\beta)}} \\
&\propto \frac{[25(1-\beta)]^{N-20}exp^{-25(1-\beta)}}{(N-20)!} \\
\therefore N|\beta, y &\sim Poisson(25(1-\beta))
\end{aligned}$$

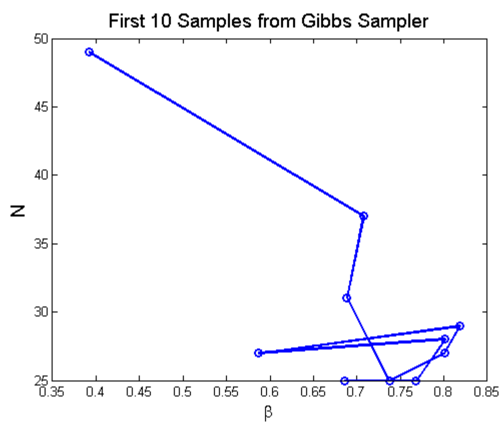
This is a shifted Poisson. Hence, we need to add 20 to the samples drawn from this distribution.

$$\begin{aligned}
p(\beta|N, y) &\propto \frac{\beta^{20}[25(1-\beta)]^{N-20}}{(N-20)!} \\
&\propto \beta^{20}(1-\beta)^{N-20} \\
\therefore \beta|N, y &\sim Beta(21, N-19).
\end{aligned}$$

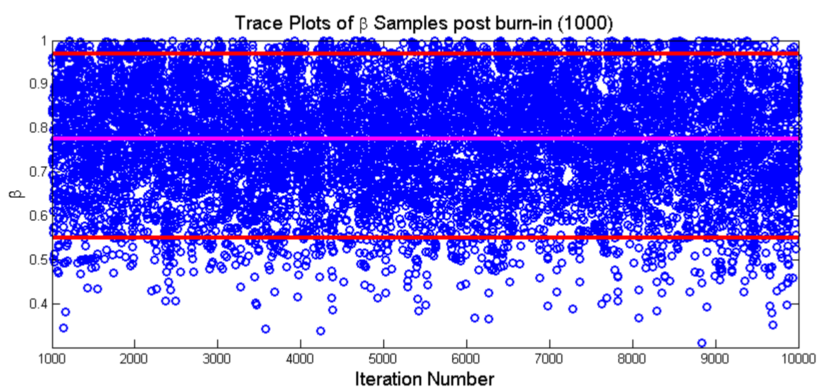
3. To sample from the Joint Posterior we can do Gibbs Sampling:

   - Select, $\beta^{(0)} = 0.05$,
   - Draw, $N^{(1)} \sim p(N|\beta^{(0)}, y)$
   - Draw, $\beta^{(1)} \sim p(\beta|N^{(1)}, y)$
   - Hence, we get $\{N^{(1)}, \beta^{(1)}\}$.
   - Repeat.

4. Trace Plot for first 10 Samples of Gibbs Sampler.



5. Credible Interval:
   90% Posterior Credible Interval for $\beta : [0.54, 0.97]$. Magenta line is the mean and red lines represent 90% Credible Limits.



6. $P(N = 20) = 0.07$ (post burn-in).

Appendix:

```
%% STA 601: Lab 3
% Author: Kedar S Prabhudesai
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close all;
clear all;

%% Full Conditionals
% Initial values
Beta0 = 0.05;

% N|beta,y ~ Poisson(25(1-beta))
NGivenBetaAndY = makedist('Poisson','lambda',25*(1-Beta0));
N1 = NGivenBetaAndY.random() + 20;
% beta|N,y ~ Beta(21,N-19)
BetaGivenNAndY = makedist('Beta','a',21,'b',N1-19);
Beta1 = BetaGivenNAndY.random();
% Samples from Full Conditionals
nSamples = 10000;
NSamples = zeros(1,nSamples);
BetaSamples = zeros(1,nSamples);

NSamples(1) = N1;
BetaSamples(1) = Beta1;

for iSample = 2:nSamples
    NGivenBetaAndY.lambda = 25*(1-BetaSamples(iSample-1));
    NSamples(iSample) = NGivenBetaAndY.random()+20;

    BetaGivenNAndY.b = NSamples(iSample)-19;
    BetaSamples(iSample) = BetaGivenNAndY.random();
end
% figure;plot(BetaSamples,NSamples,'bo-','Linewidth',2);
% title('First 10 Samples from Gibbs Sampler','FontSize',14);
% xlabel('\beta','FontSize',14);
% ylabel('N','FontSize',14);

% Burn-In
BetaSamples(1:1000) = [];
NSamples(1:1000) = [];

PostCredIntrval = quantile(BetaSamples,[0.05 0.95]);

figure;plot(1:numel(BetaSamples),BetaSamples,'bo','Linewidth',2);hold on;
plot(1:numel(BetaSamples),repmat(mean(BetaSamples),1,numel(BetaSamples)),'m-','Linewidth',3);
plot(1:numel(BetaSamples),repmat(PostCredIntrval(1),1,numel(BetaSamples)),'r-','Linewidth',3);
plot(1:numel(BetaSamples),repmat(PostCredIntrval(2),1,numel(BetaSamples)),'r-','Linewidth',3);hold off;
title('Trace Plots of \beta Samples post burn-in (1000)','FontSize',14);
xlabel('Iteration Number','FontSize',14);
ylabel('\beta','FontSize',14);

% P(N=20)
ProbOfInterest = sum(NSamples==20)/numel(NSamples);
```