STA 601 - Homework 4

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Model for Data:

$$y_i \sim Poisson(\lambda\gamma^{x_i})$$

Likelihood:

$$\begin{aligned}
L(\mathbf{y};\lambda,\gamma) &= \prod_{i=1}^{n} \frac{(\lambda\gamma^{x_i})^{y_i}\, exp(-\lambda\gamma^{x_i})}{y_i!} \\
&= \prod_{i=1}^{n} \frac{\lambda^{y_i}\gamma^{x_i y_i}\, exp(-\lambda\gamma^{x_i})}{y_i!} \\
&= C(\mathbf{y})\lambda^{\sum_{i=1}^{n} y_i}\gamma^{\sum_{i=1}^{n} y_i x_i}\prod_{i=1}^{n} exp(-\lambda\gamma^{x_i})
\end{aligned}$$

Priors for Parameters: $\lambda \sim Gamma(1,1), \gamma \sim Gamma(1,1)$. The Joint Distribution $p(\lambda,\gamma) = p(\lambda)p(\gamma)$, because λ and γ are conditionally independent given x_i.

Posterior Joint Distribution: $p(\lambda,\gamma|y)$.

$$\begin{aligned}
p(\lambda,\gamma|\mathbf{y}) &\propto L(\mathbf{y};\lambda,\gamma)p(\lambda,\gamma) \\
&\propto \lambda^{\sum_{i=1}^{n} y_i}\gamma^{\sum_{i=1}^{n} y_i x_i}\prod_{i=1}^{n} exp(-\lambda\gamma^{x_i}) \times exp(-\lambda) \times exp(-\gamma) \\
p(\lambda,\gamma|\mathbf{y}) &\propto \lambda^{\sum_{i=1}^{n} y_i}\gamma^{\sum_{i=1}^{n} y_i x_i}\prod_{i=1}^{n} exp(-\lambda\gamma^{x_i} - \lambda - \gamma)
\end{aligned}$$

This expression does not look like a Gamma Distribution, hence the Joint Posterior is not conjugate. However, if we get the full conditionals we will get Conjugacy.

Full Conditionals:

$$
\begin{aligned}
p(\lambda|\gamma, \mathbf{y}) &\propto \lambda^{\sum_{i=1}^{n} y_i} exp\left[-\lambda\left(\sum_{i=1}^{n}\gamma^{x_i} + n\right)\right] \\
&\propto \lambda^{\left(\sum_{i=1}^{n} y_i + 1\right) - 1} exp\left[-\lambda\left(\sum_{i=1}^{n}\gamma^{x_i} + n\right)\right] \\
\lambda|\gamma, \mathbf{y} &\sim Gamma\left(\sum_{i=1}^{n} y_i + 1, \sum_{i=1}^{n}\gamma^{x_i} + n\right).
\end{aligned}
$$

Now,

$$
p(\gamma|\lambda, \mathbf{y}) \propto \gamma^{\sum_{i=1}^{n} y_i x_i} exp\left(-\lambda\sum_{i=1}^{n}\gamma^{x_i} + \gamma\right)
$$

To solve for this, we will assume that m out of n subjects are treated. Since, x_i is 1 for treated subjects, and 0 for untreated, the above expression simplifies as,

$$
\begin{aligned}
p(\gamma|\lambda, \mathbf{y}) &\propto \gamma^{\sum_{i=1}^{n} y_i x_i} exp\left[-\lambda\left(n - m + m\gamma\right) + \gamma\right] \\
&\propto \gamma^{\sum_{i=1}^{n} y_i x_i} exp\left[-\gamma\left(\lambda m + 1\right)\right] \\
\gamma|\lambda, \mathbf{y} &\sim Gamma\left(\sum_{i=1}^{n} y_i x_i + 1, \lambda m + 1\right)
\end{aligned}
$$

where, m is the number of treated subjects.

Therefore, to sample from the Joint Posterior we can do Gibbs Sampling.
Select, $\gamma^{(0)}$,
Draw, $\lambda^{(1)} \sim p(\lambda|\gamma^{(0)}, \mathbf{y})$
Then Draw, $\gamma^{(1)} \sim p(\gamma|\lambda^{(1)}, \mathbf{y})$
Hence, we get $\{\lambda^{(1)}, \gamma^{(1)}\}$.
Repeat.