STA 601 - Homework 7

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1. Data:
   We generate, $y_i \sim \mathcal{N}_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0.8\\0.8 & 1\end{bmatrix}\right)$, $i = 1, 2, \ldots, 100$.

   We can calculate μ_{MLE} and Σ_{MLE}, from this generated data.

   Then we run Gibbs Sampler and estimate μ_{Bayes} and Σ_{Bayes}.

2. Predictive Density:
   For prediction, we will generate 50 new samples (y_1, y_2), and predict $y_1|y_2$, using the given density, $(y_{j1}|y_{j2} = a) \sim \mathcal{N}\left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho(a - \mu_2), (1-\rho^2)\sigma_1^2\right)$.

   Prediction Estimate, $E_{True}(y_1|y_2) = 0.0608$.
   95% Credible Intervals: [-1.8264 2.0562].

3. Mean Square Errors:
   We will then use MLE and Bayes estimates, and predict $(y_{j1}|y_{j2} = a)$. Then we will compute $E(y_1|y_2)$, for each of the two predictors and compare their performance with the Truth predictor using Mean Squared Error.

   $[E_{True}(y_1|y_2) - E_{MLE}(y_1|y_2)]^2 = 0.0807$.
   $[E_{True}(y_1|y_2) - E_{Bayes}(y_1|y_2)]^2 = 0.0039$.

4. Comparison:
   $E_{MLE}(y_1|y_2) = 0.3448$.
   $E_{Joint}(y_1|y_2) = 0.0667$. Here, $(y_1, y_2) \sim \mathcal{N}_2(\mu_{MLE}, \Sigma_{MLE})$.

   Hence, the results are indeed quite different.

5. Confidence Intervals: Following are the Predictive Intervals for the Bayes and MLE Predictors.
   Interval for $E_{MLE}(y_1|y_2)$:[-1.7668 2.9977].
   Interval for $E_{Bayes}(y_1|y_2)$:[-2.1768 2.5306].

Appendix:

```matlab
%% STA 601 - Homework 7
% Author: Kedar Prabhudesai
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close all;
clear all;

%% Setup Data and Distributions
% Create Bivariate Distribution
nSamples = 100;
rho = 0.8;
mu = [0 0];
SIGMA = [1 rho; rho 1];
y1 = -3:0.1:3;y2 = -3:0.1:3;
[Y1,Y2] = meshgrid(y1,y2);
yPDF = mvnpdf([Y1(:) Y2(:)],mu,SIGMA);
yPDF = reshape(yPDF,length(Y2),length(Y1));

% Get Data
rng('shuffle');
rSamples = mvnrnd(mu,SIGMA,nSamples);

% Find MLE
muMLE = mean(rSamples);
sigmaMLE = cov(rSamples);

%% Gibbs Sampler
nGibbs = 10000;
ySamples = rSamples';
ybar = mean(rSamples)';
mu0 = [0.2 0.2]';
L0 = [1.25 0.6;0.6 1.25];

nu0 = 4;
S0 = [1.2 0.4;0.4 1.3];

thetaSamples = zeros(2,nGibbs);
sigmaSamples = zeros(2,2,nGibbs);
sigmaSamples(:,:,1) = S0;

for iSample = 2:nGibbs
    % Update theta
    Ln = inv(inv(L0) + nSamples.*inv(sigmaSamples(:,:,iSample-1)));
    mun = Ln*(inv(L0)*mu0 + nSamples.*inv(sigmaSamples(:,:,iSample-1))*ybar);
    thetaSamples(:,iSample) = mvnrnd(mun,Ln);

    % Update Sigma
    Sn = S0 + (bsxfun(@minus,ySamples,thetaSamples(:,iSample)))*(bsxfun(@minus,ySamples,...
        thetaSamples(:,iSample)))';
    Z = mvnrnd([0 0],inv(Sn),nu0+nSamples);
    sigmaSamples(:,:,iSample) = inv(Z'*Z);
end

% Burn-In
thetaSamples(:,1:1000) = [];
sigmaSamples(:,:,1:1000) = [];

% Estimate mu and Sigma
muBayes = mean(thetaSamples,2)';
sigmaBayes = mean(sigmaSamples,3);
```

```
%% Prediction
nPred = 50;
rng('shuffle');

muPred = mu;
sigmaPred = SIGMA;

yTruth = mvnrnd(mu,SIGMA,nPred);

y1PredTruth = zeros(nPred,1);
y1PredMLE = zeros(nPred,1);
y1PredBayes = zeros(nPred,1);

% Estimate (y1|y2) using truth
for iPred = 1:nPred
    y1PredTruth(iPred,1) = normrnd(muPred(1) + (sigmaPred(1,1)*sigmaPred(1,2)*(yTruth(iPred...
        ,2)-muPred(2)))/sigmaPred(2,2),sqrt((1-sigmaPred(1,2)^2)*sigmaPred(1,1)^2));
end
Ey1Giveny2Truth = mean(y1PredTruth)

% Estimate (y1|y2) using MLE
for iPred = 1:nPred
    y1PredMLE(iPred,1) = normrnd(muMLE(1) + (sigmaMLE(1,1)*sigmaMLE(1,2)*(yTruth(iPred,2)-...
        muMLE(2)))/sigmaMLE(2,2),sqrt((1-sigmaMLE(1,2)^2)*sigmaMLE(1,1)^2));
end
Ey1Giveny2MLE = mean(y1PredMLE)

% Estimate (y1|y2) using Bayes Estimates
for iPred = 1:nPred
    y1PredBayes(iPred,1) = normrnd(muBayes(1) + (sigmaBayes(1,1)*sigmaBayes(1,2)*(yTruth(...
        iPred,2)-muBayes(2)))/sigmaBayes(2,2),sqrt((1-sigmaBayes(1,2)^2)*sigmaBayes(1,1)^2)...
        );
end
Ey1Giveny2Bayes = mean(y1PredBayes);

MSPredErrMLE = (Ey1Giveny2Truth - Ey1Giveny2MLE).^2
MSPredErrBayes = (Ey1Giveny2Truth - Ey1Giveny2Bayes).^2

%% MLE Comparison
rng('shuffle');
yTruthMLEJoint = mvnrnd(muMLE,sigmaMLE,nPred);
EyTruthMLEJoint = mean(yTruthMLEJoint(:,1))

%% Confidence intervals for different predictors
ConfIntervalsTruth = quantile(y1PredTruth,[0.025 0.975])
ConfIntervalsMLE = quantile(y1PredMLE,[0.025 0.975])
ConfIntervalsBayes = quantile(y1PredBayes,[0.025 0.975])
```