STA 601 - Lab 7

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Data Likelihood:

$$L(x;\mu,\sigma) = \prod_{i=1}^{n} \frac{1}{x_i \sigma \sqrt{2\pi}} exp\left[\frac{-(lnx_i - \mu)^2}{2\sigma^2}\right]$$

Let $\tau = 1/\sigma^2$

$$\begin{aligned} L(x;\mu,\tau) &= \prod_{i=1}^{n} \frac{\sqrt{\tau}}{x_i\sqrt{2\pi}} exp\left[\frac{-\tau(lnx_i - \mu)^2}{2}\right] \\ \therefore L(x;\mu,\tau) &= \frac{\tau^{n/2}}{x_i^n 2\pi^{n/2}} exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2\right] \end{aligned}$$

Priors:

$$\begin{aligned} \mu &\sim \mathcal{N}(\mu_0, \tau_0) \\ \therefore p(\mu) &= \frac{\sqrt{\tau_0}}{\sqrt{2\pi}} exp\left[\frac{-\tau_0(\mu - \mu_0)^2}{2}\right] \\ \tau &\sim Gamma(\alpha, \beta) \\ \therefore p(\tau) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} exp(-\beta\tau) \end{aligned}$$

Posterior:

$$\begin{aligned} p(\mu,\tau \mid x) &\propto L(x;\mu,\tau) \times p(\mu) \times p(\tau) \\ \therefore p(\mu,\tau \mid x) &\propto \frac{\tau^{n/2}}{x_i^n} exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2\right] \times \sqrt{\tau_0} exp\left[\frac{-\tau_0(\mu - \mu_0)^2}{2}\right] \times \tau^{\alpha-1} exp(-\beta\tau) \end{aligned}$$

Full Conditionals:

$$\begin{aligned}
p(\mu \mid \tau, x) &\propto exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2\right] \times exp\left[\frac{-\tau_0(\mu-\mu_0)^2}{2}\right] \\
\therefore p(\mu \mid \tau, x) &\propto exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2 - \frac{\tau_0}{2}(\mu-\mu_0)^2\right]
\end{aligned}$$

$$\begin{aligned}
p(\tau \mid \mu, x) &\propto \tau^{n/2} exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2\right] \times \tau^{\alpha-1} exp(-\beta\tau) \\
&\propto \tau^{\alpha+n/2-1} exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(lnx_i - \mu)^2 - \beta\tau\right] \\
&\propto \tau^{\alpha+n/2-1} exp\left[-\tau\left(\frac{\sum_{i=1}^{n}(lnx_i - \mu)^2}{2} + \beta\right)\right] \\
\therefore p(\tau \mid \mu, x) &\sim Gamma\left(\alpha + \frac{n}{2}, \frac{\sum_{i=1}^{n}(lnx_i - \mu)^2}{2} + \beta\right)
\end{aligned}$$

In our Gibbs Sampler, we will update μ using Metropolis-Hastings and τ using the Gamma Distribution. I used the following prior parameters,

$$\begin{aligned}
\mu_0 &= 0, \\
\tau_0 &= 0.05, \\
\alpha &= 1, \\
\beta &= 20.
\end{aligned}$$

Gibbs Sampler:

Start with $\{\mu^{(0)}\}$

- Sample, $\mu' \sim \mathcal{N}(\mu^{(s)}, \sigma_{cand})$
- Compute acceptance ratio, $r = p(\mu' \mid \tau^{(s)}, x)/p(\mu \mid \tau^{(s)}, x)$
- Draw, $u \sim$ Uniform$[0, 1]$. If $u < r$, $\mu^{(s+1)} = \mu'$, else $\mu^{(s+1)} = \mu^{(s)}$.
- Sample, $\tau^{(s+1)} \sim p(\tau \mid \mu^{(s+1)}, x)$

Sampling Results:

I used a Burn-In of 2000 trials. After much tinkering, I got good mixing for μ updates with $\sigma_{cand} = 0.5$. We have to convert τ to σ^2. Following are trace plots for μ and σ^2.

Estimates and 95% Credible Intervals for Mean and Variance:

Mean $= e^{\mu+\sigma^2/2} \rightarrow 0.3904\ [0.2990, 0.5199]$.

Variance $= \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2} \rightarrow 0.4041\ [0.1541, 0.9669]$.

Appendix:

```
%% STA 601: Lab 7
% Author: Kedar S Prabhudesai
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close all;
clear all;

% Get Data
X = importdata('data.txt');
% Prior Parameters
mu0 = 0;
t0 = 0.05;
a = 1;
% Remember that 'b' parameter in matlab's gamma function is in fact '1/b'
b = 0.05;
n = numel(X);

% Target Distribution for mu
muFullCond =  @(t,mu) exp(-0.5*t*sum((log(X)-mu).^2) - 0.5*t0*(mu-mu0)^2);

% Number of Trials
nTrials = 500;
% Burn-In
nBurnIn = 100;
% Proposal Distribution Std Dev
SCand = 0.5;

% Initialize
muSamples = zeros(1,nTrials);
tSamples = zeros(1,nTrials);

% Full Conditional distribution for tau
tFullCond = makedist('Gamma','a',a+n/2,'b',1/(0.5*sum((log(X)-muSamples(1)).^2) + b));
tSamples(1) = tFullCond.random();

% Gibbs Sampling
for iTrial = 2:nTrials
    home;
    disp(iTrial);
    % Update mu | tau,x using M-H

    % Step 1: Sample from mu'|mu(s)
    muPrime = normrnd(muSamples(iTrial-1),SCand);

    % Step 2: Compute Acceptance Ratio
    r = muFullCond(tSamples(iTrial-1),muPrime)/muFullCond(tSamples(iTrial-1),muSamples(...
        iTrial-1));

    % Step 3: Accept/Reject
    u = rand;
    if u < r
        muSamples(iTrial) = muPrime;
    else
        muSamples(iTrial) = muSamples(iTrial-1);
    end

    % Update tau | mu,x
    tFullCond.b = 1/(0.5*sum((log(X)-muSamples(iTrial)).^2) + b);
    tSamples(iTrial) = tFullCond.random();
end
```

```

% Burn-In
muSamples(1:nBurnIn) = [];
tSamples(1:nBurnIn) = [];

% Convert to Sigma^2
s2Samples = 1./tSamples;

% Manage Plotting
figure('Position',[67   304   922   345]);
plot(muSamples,'b-');
xlabel('Iterations','FontSize',14);
ylabel('Samples','FontSize',14);
title('\mu Samples','FontSize',14);

figure('Position',[67   304   922   345]);
plot(s2Samples,'b-');
xlabel('Iterations','FontSize',14);
ylabel('Samples','FontSize',14);
title('\sigma^2 Samples','FontSize',14);

% Find estimates of mean and variance
MeanFromSamples = exp(muSamples + s2Samples./2);
VarFromSamples = (exp(s2Samples) - 1).*exp(2.*muSamples + s2Samples);

EstMean = mean(MeanFromSamples);
MeanConfInts = quantile(MeanFromSamples,[0.025 0.975]);

EstVar = mean(VarFromSamples);
VarConfInts = quantile(VarFromSamples,[0.025 0.975]);
```