STA 601 - Homework 11

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Bayesian Model Selection:

Data:

$$y^n \sim Binomial(n, \theta).$$

Hypotheses:

$$H_0 : \theta = 0.5$$
$$H_1 : \theta \neq 0.5.$$

Prior on Hypotheses/Models:

$$p(H_0) = 0.5$$
$$p(H_1) = 0.5.$$

Prior on Parameter of the model: Let's use a Uniform prior, $\alpha = \beta = 1$.

$$\theta \sim Beta(\alpha, \beta).$$

Likelihood under each hypotheses: k denotes number of heads.

$$\begin{aligned}
L(y^n \mid H_0) &= \binom{n}{k} 0.5^k 0.5^{(n-k)} = \binom{n}{k} 0.5^n. \\
L(y^n \mid H_1) &= \int_0^1 \binom{n}{k} \theta^k (1-\theta)^{(n-k)} \pi(\theta) d\theta \\
&= \int_0^1 \binom{n}{k} \theta^k (1-\theta)^{(n-k)} \times \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} d\theta \\
&= \frac{\binom{n}{k}}{B(\alpha,\beta)} \int_0^1 \theta^{\alpha+k-1}(1-\theta)^{(n+\beta-k-1)} d\theta \\
&= \frac{\binom{n}{k}}{B(\alpha,\beta)} \times B(\alpha+k, n+\beta-k) \int_0^1 \frac{\theta^{\alpha+k-1}(1-\theta)^{(n+\beta-k-1)}}{B(\alpha+k, n+\beta-k)} d\theta \\
\therefore L(y^n \mid H_1) &= \frac{\binom{n}{k} B(\alpha+k, n+\beta-k)}{B(\alpha,\beta)}
\end{aligned}$$

Posteriors:

Assuming, $p(H_0) = p(H_1) = 0.5$.

$$\begin{aligned}
P(H_1 \mid y^n) &= \frac{L(y^n \mid H_1)p(H_1)}{L(y^n)} \\
&= \frac{L(y^n \mid H_1)p(H_1)}{L(y^n \mid H_0)p(H_0) + L(y^n \mid H_1)p(H_1)} \\
&= \frac{L(y^n \mid H_1)}{L(y^n \mid H_0) + L(y^n \mid H_1)} \\
P(H_1 \mid y^n) &= \frac{1}{1 + \frac{L(y^n \mid H_0)}{L(y^n \mid H_1)}}
\end{aligned}$$

$\frac{L(y^n|H_0)}{L(y^n|H_1)}$ is Bayes' Factor in favor of H_0.

$$\therefore \frac{L(y^n \mid H_0)}{L(y^n \mid H_1)} = \frac{0.5^n B(\alpha, \beta)}{B(\alpha + k, n + \beta - k)}$$

Similarly we can prove,

$$P(H_0 \mid y^n) = \frac{1}{1 + \frac{L(y^n|H_1)}{L(y^n|H_0)}}$$

$\frac{L(y^n|H_1)}{L(y^n|H_0)}$ is Bayes' Factor in favor of H_1.

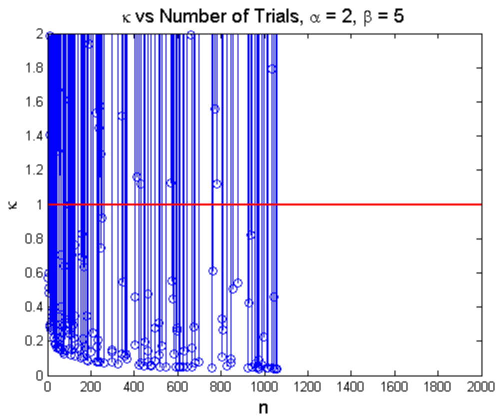
$$\therefore \kappa = \frac{L(y^n \mid H_1)}{L(y^n \mid H_0)} = \frac{B(\alpha + k, n + \beta - k)}{0.5^n B(\alpha, \beta)}$$

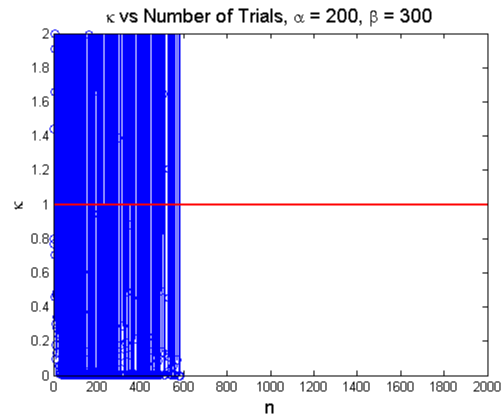
Based on value of κ we can choose an appropriate model. If $\kappa > 1$, we are more in favor of H_1, whereas if $\kappa < 1$, we are more in favor of H_0.

Asymptotic Behavior:

If $n = 1$, $\kappa = \frac{L(y^n|H_1)}{L(y^n|H_0)} = \frac{B(\alpha+k,1+\beta-k)}{0.5B(\alpha,\beta)}$. Hence, κ will depend on the parameters of our prior, and the outcome k that we get for the one coin flip. Using, $\alpha = \beta = 1$, we always get $\kappa = 1$, no matter what the outcome of the flip is! However, using different values of α, β will change what model we select based on the outcome of the flip.

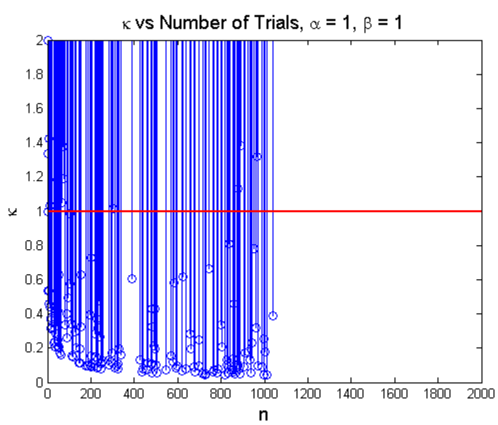
As $n \to \infty$, we have $0.5^n \to 0$, as a result of which $\kappa \to \infty$, and does not depend on the outcomes of flips. This means that we always accept H_1. The prior parameters α, β will only determine at what asymptotic value κ comes very large. This is demonstrated in the following figures.

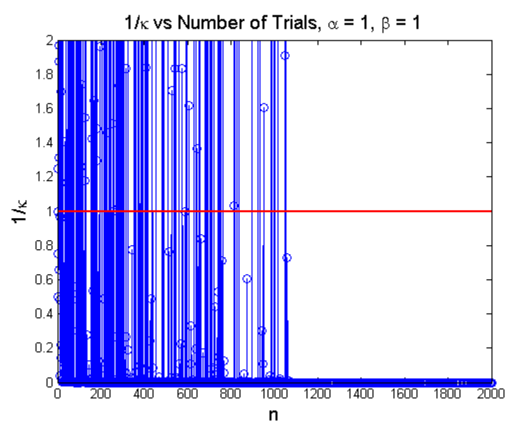




Simulate Data:

Assume, under H_0, $\theta = 0.5$, and H_1, $\theta \neq 0.5$. To simulate H_1, we draw θ randomly between $[0, 1]$. We get the expected behavior, we are always in favor of H_1 after around 1100 trials. This is confirmed by plotting both κ and $1/\kappa$.





Appendix:

```
%% STA 601 - Homework 11
% Author: Kedar Prabhudesai
% Created on: 10/22/2013

close all;
clear all;

% Beta Prior Parameters
a = 1;
b = 1;
% Number of Trials
nTrials = 1:2000;
% We draw theta at random in [0,1]
H1Theta = rand(1,numel(nTrials));
% Get number of successes
k = binornd(nTrials,H1Theta);
% Calculate Bayes Factor
BayesFactor = beta(a+k,nTrials-k+b)./((0.5.^nTrials).*beta(a,b));

% Manage Plotting
figure;plot(nTrials,BayesFactor,'b-o');ylim([0 2]);hold on;
plot(nTrials,ones(numel(nTrials),1),'r','LineWidth',2);hold off;
xlabel('n','FontSize',14);
ylabel('\kappa','FontSize',14);
title(['\kappa vs Number of Trials, \alpha = ',num2str(a),', \beta = ',num2str(b)],'...
    FontSize',14);
```