STA 601 - Homework 13

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Linear Regression:

Model Specification:

This is the general form of Linear Regression Model:

$$\begin{aligned} Y_i &= X_i\beta + \epsilon_i, i = 1, 2, \ldots, n \\ \beta &= [\beta_1, \beta_2, \ldots, \beta_p] \\ \epsilon_i &\sim \mathcal{N}(0, \sigma^2) \end{aligned}$$

X is $(n \times p)$ matrix of predictor variables and β is a $(p \times 1)$ vector of regression co-efficients.

Prior Specification:

$$\begin{aligned} \beta_j &\sim (1-\pi_0)\delta_0 + \pi_0\mathcal{N}(0, c) \\ \pi_0 &\sim Beta(a, b) \\ 1/\sigma^2 = \tau &\sim Gamma(c, d) \end{aligned}$$

Posterior Computation:

The posterior is given as,

$$\begin{aligned} p(\beta_1, \beta_2, \ldots, \beta_p, \pi_0, \tau \mid y^n, x^n) \propto & \left[\prod_{j=1}^{p} (1-\pi_0)\delta_0(\beta_j) + \pi_0\mathcal{N}(\beta_j; 0, c)\right] \times \left[\pi_0^{a-1}(1-\pi_0)^{b-1}\right] \\ & \times \left[\tau^{c-1}exp(-\tau d)\right] \times \left\{\prod_{i=1}^{n} \tau^{1/2} exp\left[-\frac{\tau}{2}(y_i - x_i\beta)^2\right]\right\} \end{aligned}$$

Full Conditionals:

To compute this posterior we can use Gibbs Sampling, for which we need to compute full conditionals. Assume that out of p predictors, we have p_γ that are not equal to zero.

$$\begin{aligned}
p(\pi_0 \mid \beta_1, \beta_2, \ldots, \beta_p, \tau, y^n, x^n) &\propto \left[\prod_{j=1}^{p} (1-\pi_0)\delta_0(\beta_j) + \pi_0 \mathcal{N}(\beta_j; 0, c)\right] \times \left[\pi_0^{a-1}(1-\pi_0)^{b-1}\right] \\
&\propto \left[\prod_{j:\beta_j=0} (1-\pi_0)\right] \times \left[\prod_{j:\beta_j\neq 0} \pi_0 \mathcal{N}(\beta_j; 0, c)\right] \times \left[\pi_0^{a-1}(1-\pi_0)^{b-1}\right] \\
&\propto \left[(1-\pi_0)^{p-p_\gamma}\pi_0^{p_\gamma}\right] \times \left[\pi_0^{a-1}(1-\pi_0)^{b-1}\right] \\
&\propto \pi_0^{a+p_\gamma-1}(1-\pi_0)^{b+p-p_\gamma-1} \\
\therefore (\pi_0 \mid \beta_1, \beta_2, \ldots, \beta_p, \tau, y^n, x^n) &\propto Beta(a+p_\gamma, b+p-p_\gamma)
\end{aligned}$$

$$\begin{aligned}
p(\beta_j \mid \beta_{\sim j}, \pi_0, \tau, y^n, x^n) &\propto [(1-\pi_0)\delta_0(\beta_j) + \pi_0 \mathcal{N}(\beta_j; 0, c)] \times \left\{\prod_{i=1}^{n} exp\left[-\frac{\tau}{2}(y_i - x_i\beta)^2\right]\right\} \\
&\propto [(1-\pi_0)\delta_0(\beta_j) + \pi_0 \mathcal{N}(\beta_j; 0, c)] \times \left\{exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(y_i - x_i\beta)^2\right]\right\}
\end{aligned}$$

$$\begin{aligned}
p(\tau \mid \beta_1, \beta_2, \ldots, \beta_p, \pi_0, y^n, x^n) &\propto \left[\tau^{c-1}exp(-\tau d)\right] \times \left\{\prod_{i=1}^{n} exp\left[-\frac{\tau}{2}(y_i - x_i\beta)^2\right]\right\} \\
&\propto \left[\tau^{c-1}exp(-\tau d)\right] \times \left\{\tau^{n/2} exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(y_i - x_i\beta)^2\right]\right\} \\
&\propto \tau^{n/2+c-1} exp\left\{-\tau\left[\frac{1}{2}\sum_{i=1}^{n}(y_i - x_i\beta)^2 + d\right]\right\} \\
\therefore p(\tau \mid \beta_1, \beta_2, \ldots, \beta_p, \pi_0, y^n, x^n) &\propto Gamma\left(n/2 + c, \left[\frac{1}{2}\sum_{i=1}^{n}(y_i - x_i\beta)^2 + d\right]\right)
\end{aligned}$$

We can perform Gibbs Sampling using these full conditionals. I could not simplify the full conditional for $p(\beta_j \mid \beta_{\sim j}, \pi_0, \tau, y^n, x^n)$. We can still solve this by doing Metropolis-Hastings to update the β_j's, similar to the way we did it in lab.