STA 601 - Lab 6

Kedar Prabhudesai

October 25, 2013

Normalizing Constant:

Given distribution,

$$\pi(\theta) \propto exp\left[-\frac{\theta^2}{2}\right] + \frac{1}{2}exp\left[-\frac{(\theta-3)^2}{2}\right]$$

To find the Normalizing constant (κ), we can integrate the above equation.

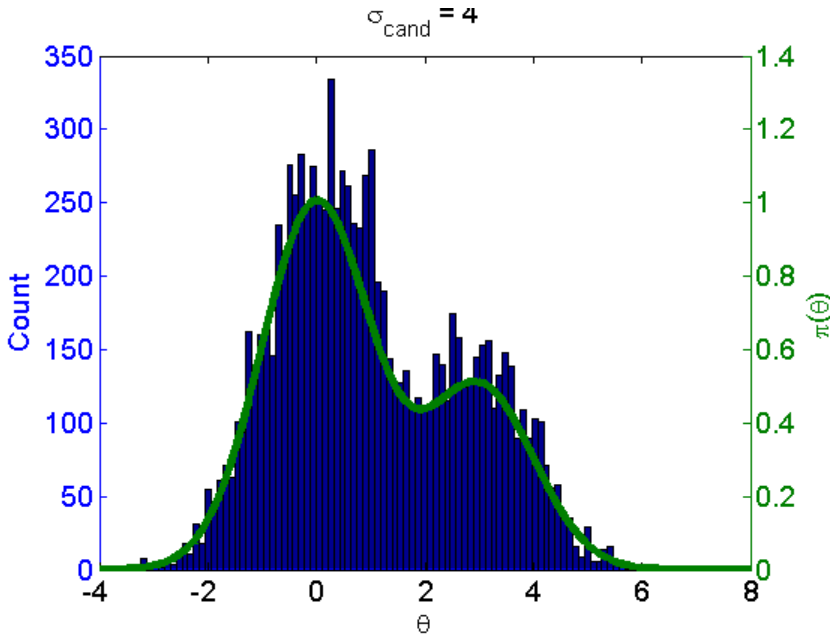
$$\begin{aligned}
\kappa &= \int_0^\infty exp\left[-\frac{\theta^2}{2}\right] + exp\left[-\frac{(\theta-3)^2}{2}\right] d\theta \\
&= \int_0^\infty exp\left[-\frac{\theta^2}{2}\right] d\theta + \frac{1}{2}\int_0^\infty exp\left[-\frac{(\theta-3)^2}{2}\right] d\theta \\
&= \sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{2\pi}} exp\left[-\frac{\theta^2}{2}\right] d\theta + \frac{1}{2}\sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{2\pi}} exp\left[-\frac{(\theta-3)^2}{2}\right] d\theta \\
&= \sqrt{2\pi} + \frac{1}{2}\sqrt{2\pi} \\
&= \sqrt{2\pi}\left(1 + \frac{1}{2}\right) \\
\therefore \kappa &= \frac{3\sqrt{2\pi}}{2}
\end{aligned}$$

Metropolis-Hastings Algorithm:

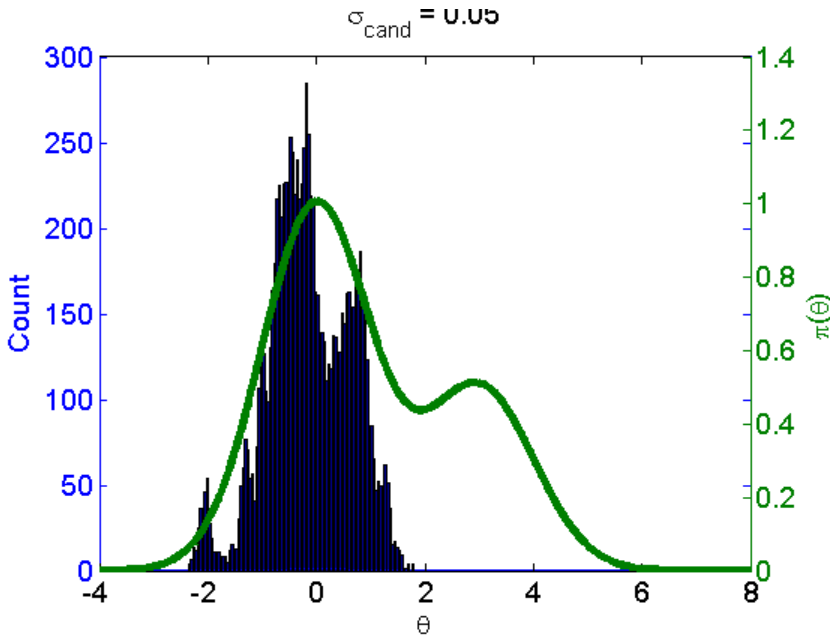
Start with $\theta^{(0)} = 0$

- Sample, $\theta' \sim \mathcal{N}(\theta^{(s)}, \sigma_{cand})$
- Compute acceptance ration, $r = \pi(\theta')/\pi(\theta^{(s)})$
- Draw, $u \sim \text{Uniform}[0, 1]$. If $u < r$ $\theta^{(s+1)} = \theta'$, else $\theta^{(s+1)} = \theta^{(s)}$.

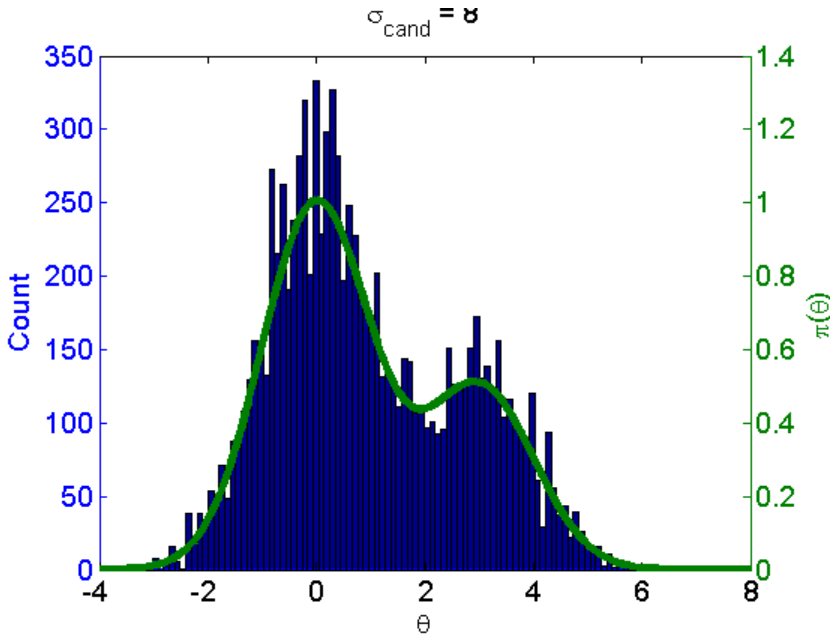
I used $\sigma_{cand} = 4$, and I got an acceptance ratio of 43.54%. Given below is the Histogram and a plot of the function. Since, my starting value was 0, which is in the sample range, I did not need Burn-In.



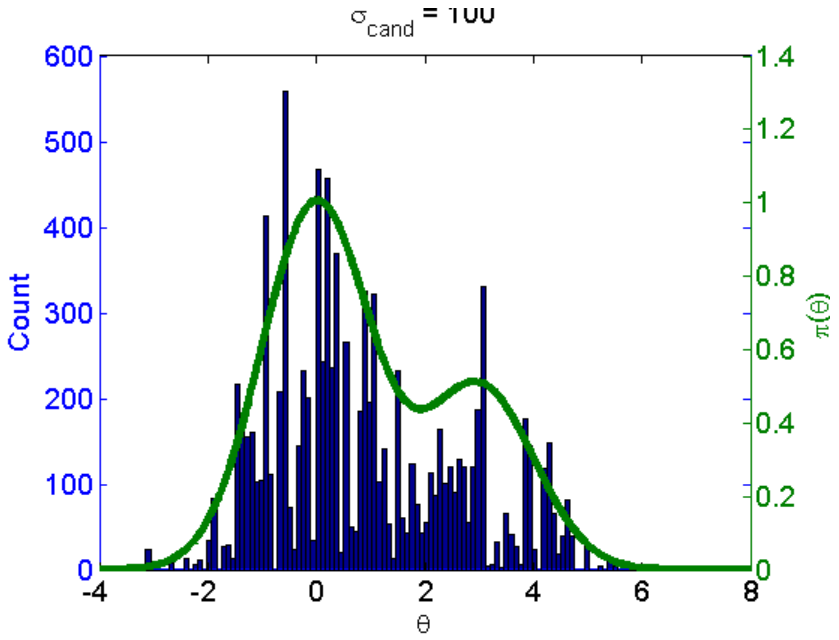
Using $\sigma_{cand} = 0.05$, we need lot more samples to cover the entire distribution, since we take very small steps. Hence we end up not sampling from the entire distribution.



Using $\sigma_{cand} = 8$, makes our candidate distribution very wide and end up rejecting a lot of samples. Note that here we do cover our target distribution but it is very inefficient. For this chain my acceptance probability was 20.49%



Using $\sigma_{cand} = 100$, makes our candidate distribution even wider and it is the worst case scenario. Now, we are drawing from a candidate distribution which has a huge variance, hence we are going to reject most of the samples. For the following plot, I got an acceptance probability of 1.83%



Appendix:

```
%% STA 601: Lab 6
% Author: Kedar S Prabhudesai
% Created on: 10/23/2013

close all;
clear all;

% Target Distribution
TarDist =  @(t) exp(-0.5.*(t.^2)) + 0.5*exp(-0.5.*((t-3).^2));
% Number of Trials
nTrials = 10000;
% Burn-In
nBurnIn = 2000;
% Proposal Distribution Std Dev
SCand = 100;
% Find number of Accepted Samples
nAccept = 0;

% Initialize
ThetaSamples = zeros(1,nTrials);

% Run Metropolis-Hastings
for iTrial = 2:nTrials
    home;
    disp(iTrial);
    % Step 1: Sample from theta'|theta(s)
    ThetaPrime = normrnd(ThetaSamples(iTrial-1),SCand);

    % Step 2: Compute Acceptance Ratio
    r = TarDist(ThetaPrime)/TarDist(ThetaSamples(iTrial-1));

    % Step 3: Accept/Reject
    u = rand;
    if u < r
        ThetaSamples(iTrial) = ThetaPrime;
        if iTrial ≥ nBurnIn
            nAccept = nAccept + 1;
        end
    else
        ThetaSamples(iTrial) = ThetaSamples(iTrial-1);
    end
end

% Compute Acceptance Ratio
AccRat = nAccept/numel(ThetaSamples);

% Theta Support to find analytic distribution
ThetaSupport = -4:0.01:8;
ThetaAnalytic = TarDist(ThetaSupport);

disp(AccRat);
figure;
axes = plotyy(ThetaSupport,ThetaAnalytic,ThetaSupport,ThetaAnalytic);
hold on
hist(axes(1),ThetaSamples,100);
ylim(axes(1),'Auto');
set(axes(1),'YTickMode','auto');
set(axes(2),'YTickMode','auto');
set(axes,'FontSize',14);

```

```
61 XLabelHandles = get(axes,'XLabel');
62 set(XLabelHandles{1},'String','\theta','FontSize',14);
63 YLabelHandles = get(axes,'YLabel');
64 set(YLabelHandles{1},'String','Count','FontSize',14);
65 set(YLabelHandles{2},'String','\pi(\theta)','FontSize',14);
66 LineHandle = get(axes(2),'Children');
67 set(LineHandle,'LineWidth',4);
68 title(['\sigma_{cand} = ',num2str(SCand)]);
69 hold off
```