Problem Settings we explore:-

1) $X \rightarrow Y \rightarrow Z$

(We expect message from Bob to Charlie to be independent of U)

2)
$$Y \leftarrow X \rightarrow Z$$

This is a difficult proposition.

i) If R_1 is sufficient then Bob → Charlie message will be independent of Y.

But, in general it won't ($\because$ We can learn from Y more about X & then send it to Z).

3)
$$Y \leftarrow X' \quad X'' \rightarrow Z$$

$X = (X', X'')$.

This is slightly simpler (intuitively) as Alice needs to independently send correlated messages for compressing Y, Z (one of which is forwarded.)

$$g_1(R_b) = \min_{\substack{V_1 :\ V_1 - Y - Z \\ H(Z|V_1) \le R_b}} I(Y;V_1)$$

(we don't care about X). 1

$$g_2(R_b) = \min \min \left(I(Y,U;V_2),\ R_1 + I(Y;V_2|U) \right)$$

$$\begin{tikzcd} U \arrow[dr] & X \arrow[l] & Y \arrow[l] \arrow[r] \arrow[dl] & Z \\ & V_2 & & \end{tikzcd}$$

$$H(Z|V_2) \le R_b$$
$$I(X;U) \le R_1$$

i) $g_1(R_b) \le g_2(R_b)$

Let V_2^* minimize $g_2(R_b)$, then :-

Now, $V_2^* - Y - Z$, thus :-

$g_1(R_b) \le I(Y;V_2^*) \longrightarrow$ 1

- $I(Y,U;V_2^*) \ge I(Y;V_2^*) \longrightarrow$ 2

- $R_1 \ge I(X;U)$
$\ge I(Y;U)$ $(U - X - Y)$

- $R_1 + I(Y;V_2|U) \ge I(Y;U) + I(Y;V_2|U)$
$= I(Y;U,V_2)$
$\ge I(Y;V_2^*) \longrightarrow$ 3

From 2 & 3

$g_2(R_b) \ge I(Y;V_2^*) \longrightarrow$ 4

& form 1, 4

$g_2(R_2) \ge g_1(R_b)$

ii) $g_1(R_b) \geq g_2(R_b)$ (2)

Let V_1^* minimizes $g_1(R_b)$, then :-

(we can always find a $V_1^* \perp X | Y$ (can we?))

~~g~~ V_1^* satisfies the requirements for $g_2(R_b)$,

thus we can go ahead with that :-

$$I(Y, U ; V_1^*) = I(Y ; V_1^*) + \underbrace{I(U ; V_1^* | Y)}_{\Downarrow \; 0}$$

$$\therefore \; g_2(R_b) \leq I(Y ; V_1^*) \longrightarrow (i),$$

$$\Rightarrow \; g_2(R_b) \leq g_1(R_b).$$

Converse :-

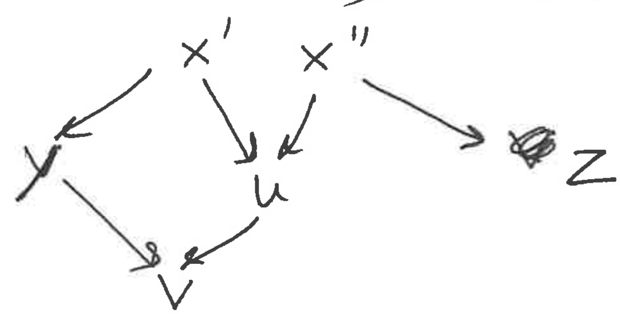
~~$H(Z^n | W_2)$~~

$$\begin{aligned}
nR_2 &= H(W_2) \\
&\geq I(Y^n ; W_2) \\
&= \sum_i I(Y_i ; Y^{i-1}, W_2) \\
&\geq \sum_i g_1(H(Z_i | Y^{i-1}, W_2)) \\
&= \sum_i g_1(H(Z_i | Y^{i-1}, Z^{i-1}, W_2)) \\
&\geq \sum_i g_1(H(Z_i | Z^{i-1}, W_2)) \\
&\geq n g\left(\frac{H(Z^n | W_2)}{n}\right) \\
&\geq n g(R_b).
\end{aligned}$$

3) Consider

$$g_1(R_b) = \min I(X'';U)$$
$$I(X'';U) \le R_1$$
$$H(Z|U) \le R_b$$
$$U - X'' - Z$$

$$g_2(R_b) = \min \min\big(I(Y,U;V),\ I(Y;V|U) + I(X;U) \big)$$
$$I(X;U) \le R_1$$
$$H(Z|V) \le R_b$$



i) $g_2(R_b) \le g_1(R_b)$

Let U^* minimize g_1,

$U^* = V$ satisfies the constraints.

Also.

$$g_2(R_b) \le I(X'';U)$$

ii) $g_2(R_b) \ge g_1(R_b)$

$\bar{V} - \bar{U} - Z$, thus,

$R_b \ge H(Z|\bar{V}) \ge H(Z|\bar{U})$. (& $\bar{U} - X'' - Z$)

Thus $\bar{U}$ satifies conditions for g_2